

Research Article

Scale-Free Property for Degrees and Weights in a Preferential Attachment Random Graph Model

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A random graph evolution mechanism is defined. The evolution studied is a combination of the preferential attachment model and the interaction of four vertices. The asymptotic behaviour of the graph is described. It is proved that the graph exhibits a power law degree distribution; in other words, it is scale-free. It turns out that any exponent in $(2, \infty)$ can be achieved. The proofs are based on martingale methods.

1. Introduction

During the last 15 years the behaviour of many types of real-world networks was investigated. Such networks are the WWW, the Internet, and social and biological networks (see [1] for an overview). The main common characteristic of such networks is their scale-free nature, in other words the power law degree distribution, that is, $p_k \sim Ck^{-\gamma}$, as $k \rightarrow \infty$. Using real-life data, the exponents were determined for several cases. For the WWW the in-degree and the out-degree of web pages follow power law with $\gamma_{\text{in}} = 2.1$ and $\gamma_{\text{out}} = 2.7$, for the Internet $\gamma = 2.3$, for the movie actor network $\gamma = 2.3$, and for the collaboration graph of mathematicians $\gamma = 2.4$ (see [1] for details). To describe the phenomenon, in [2] the preferential attachment model was introduced. In that model the growing procedure of the random graph is the following. At every time $t = 2, 3, \dots$ a new vertex with m edges is added so that the edges link the new vertex to m existing vertices. The probability π_i that the new vertex will be connected to the old vertex i depends on the degree k_i of vertex i , so that $\pi_i = k_i / \sum_j k_j$. Then several papers were devoted to the proof of the power law in the preferential attachment model (see, e.g., [3]).

There are versions of the preferential attachment model (see [4, 5]). It turned out that besides the degrees of the vertices other characteristics of the graph can be important (see [5]). In [6] a model based on the interaction of three vertices

was introduced. Then the power law degree distribution in that model was obtained in [7].

In this paper, we extend the model and the results of [6, 7] to interactions of four vertices. Our model is the following. For short, a complete graph with four vertices we will call tetragon. The starting point at time $n = 0$ is one tetragon. The initial weight of this graph is one. This graph contains 4 vertices, 6 edges, and 4 triangles. Each of these objects has initial weight 1. After the initial step we start to increase the size of the graph. The main feature of the procedure is that at each step $n = 1, 2, \dots$ we consider four vertices and draw all nonexisting edges between these vertices. So we obtain a tetragon. The weight of this tetragon and the weights of all the objects in the tetragon are increased by 1. (That is, we increase the weights of 4 vertices, 6 edges, 4 triangles, and 1 tetragon.) The choice of the four vertices is the following.

There are two possibilities at each step. With probability p we add a new vertex that interacts with three old vertices, on the other hand, with probability $(1 - p)$ four old vertices interact. Here $0 < p \leq 1$ is fixed.

When we add a new vertex, then we choose 3 old vertices and they together will form a tetragon. However, to choose the three old vertices we have two possibilities. With probability r we choose a triangle from the existing triangles according to the weights of the triangles. It means that a triangle of weight w_t is chosen with probability $w_t / \sum_h w_h$ (preferential attachment rule). On the other hand, with

probability $1 - r$, we choose among the existing vertices uniformly; that is, all three vertices have the same chance.

When four old vertices interact, we have again two possibilities. With probability q , we choose among the existing tetragons according to their weights. It means that a tetragon of weight \tilde{w}_t is chosen with probability $\tilde{w}_t / \sum_h \tilde{w}_h$ (preferential attachment rule). On the other hand, with probability $1 - q$, we choose among the existing vertices uniformly; that is, all four vertices have the same chance.

Our aim is to prove that the above mechanism produces a scale-free graph. We follow the lines of [6, 7]. Let $X(n, d, w)$ denote the number of vertices of weight w and degree d after the n th step. Let V_n denote the number of vertices after the n th step. Let \mathcal{F}_{n-1} denote the σ -algebra of observable events after the $(n - 1)$ th step. First we calculate the conditional expectation $\mathbb{E}\{X(n, d, w) \mid \mathcal{F}_{n-1}\}$, see Lemma 2. Then we prove (Theorem 3) that $X(n, d, w)/V_n \rightarrow x_{d,w}$ almost surely (a.s.) as $n \rightarrow \infty$, where $x_{d,w}$ are fixed nonnegative numbers. The main tool of the proof is the Doob-Meyer decomposition of submartingales. We remark that in the 3-interaction model of [6] the limit $x_{d,w}$ is always positive (see [7]). As in our case the limit can be zero, we should modify the proof presented in [7].

We show that $x_{d,w}$, $d = 3, 4, \dots, 3w$, $w = 1, 2, \dots$, is a proper two-dimensional discrete probability distribution (Lemma 4). Then we turn to the scale-free property for the weights. Let $X(n, w)$ denote the number of vertices of weight w after the n th step. Then for all $w = 1, 2, \dots$ we have

$$\frac{X(n, w)}{V_n} \rightarrow x_w = x_{3,w} + x_{4,w} + \dots + x_{3w,w} \quad (1)$$

almost surely and $x_w \sim Cw^{-(1+1/\alpha)}$, as $w \rightarrow \infty$ (Theorem 5). To derive the above results from Theorem 3, we need only some known facts about the Γ -function, see Lemma 4 and Theorem 5. Finally, we obtain the scale-free property for the degrees. Let us denote by $U(n, d)$ the number of vertices of degree d after the n th step. For any $d \geq 3$ we have

$$\frac{U(n, d)}{V_n} \rightarrow u_d \quad (2)$$

almost surely as $n \rightarrow \infty$, where u_d are positive numbers. Furthermore,

$$u_d \sim \frac{\Gamma(1 + (\beta + 1)/\alpha)}{\alpha_2 \Gamma(1 + \beta/\alpha)} \left(\frac{\alpha d}{\alpha_2} \right)^{-(1+1/\alpha)}, \quad (3)$$

as $d \rightarrow \infty$, where α , β , and α_2 are appropriate constants (see Theorem 8). In both cases the exponent is $1 + 1/\alpha$. We can see that its value can be any number in $(2, \infty)$.

If we compare the results and methods of the present paper with the ones in [6, 7], we can see that the calculations for the 4-vertex model are much longer than those of the 3-vertex model (see the proofs of Lemma 2 and Theorem 3). One can think that when we want to extend our model to the interaction of N vertices, then formulae will be burdening. But it is not the case. Analysing the proof of Theorem 3, it turns out that several terms are asymptotically negligible

and this phenomenon remains true for the N -vertices model, too. Therefore the asymptotic results of the 3- and 4-vertex models can be extended to the N -vertices model. Moreover, several versions of the N -vertices model can be constructed and their fit to real-life data can be studied.

2. The Evolution of the Graph

Throughout the paper $0 < p \leq 1$, $0 \leq r \leq 1$, $0 \leq q \leq 1$ are fixed numbers. Let $X(n, d, w)$ denote the number of vertices of weight w and degree d after the n th step. Let V_n denote the number of vertices after the n th step.

Remark 1. Each vertex has initial weight 1 and initial degree 3. When a vertex takes part in an interaction, then its weight is increased by 1 and its degree may be increased by 0, 1, 2, or 3. Therefore $X(n, d, w)$ can be positive only for $1 \leq w$ and $3 \leq d \leq 3w$.

Let \mathcal{F}_{n-1} denote the σ -algebra of observable events after the $(n - 1)$ th step. We compute the conditional expectation of $X(n, d, w)$ with respect to \mathcal{F}_{n-1} for $w \geq 1$.

Let

$$\begin{aligned} \alpha_1 &= (1 - p)q, & \alpha_2 &= \frac{3}{4}pr, \\ \alpha &= \alpha_1 + \alpha_2, & (4) \\ \beta &= 3(1 - r) + \frac{4(1 - p)(1 - q)}{p}. \end{aligned}$$

The following lemma contains the basic equation of the paper.

Lemma 2. *We have*

$$\begin{aligned} \mathbb{E}\{X(n, d, w) \mid \mathcal{F}_{n-1}\} &= X(n - 1, d, w) \left[1 - \left(\frac{w}{n} \alpha + \frac{p}{V_{n-1}} \beta \right) \right] \\ &\quad + X(n - 1, d, w - 1) \\ &\quad \times \left[(1 - p) \left(q \frac{w - 1}{n} + (1 - q) \frac{\binom{d}{3}}{\binom{V_{n-1}}{4}} \right) \right] \\ &\quad + X(n - 1, d - 1, w - 1) \\ &\quad \times \left[p \left(r \frac{3(w - 1)}{4n} + (1 - r) \frac{\binom{d-1}{2}}{\binom{V_{n-1}}{3}} \right) \right. \\ &\quad \left. + (1 - p)(1 - q) \frac{\binom{d-1}{2} (V_{n-1} - d)}{\binom{V_{n-1}}{4}} \right] \\ &\quad + X(n - 1, d - 2, w - 1) \end{aligned}$$

$$\begin{aligned}
& \times \left[p(1-r) \frac{(d-2)(V_{n-1}-d+1)}{\binom{V_{n-1}}{3}} \right. \\
& \quad \left. + (1-p)(1-q) \frac{(d-2)\binom{V_{n-1}-d+1}{2}}{\binom{V_{n-1}}{4}} \right] \\
& + X(n-1, d-3, w-1) \\
& \times \left[p(1-r) \frac{\binom{V_{n-1}-d+2}{2}}{\binom{V_{n-1}}{3}} \right. \\
& \quad \left. + (1-p)(1-q) \frac{\binom{V_{n-1}-d+2}{3}}{\binom{V_{n-1}}{4}} \right] + p\delta_{d,3}\delta_{w,1}
\end{aligned} \tag{5}$$

for $w \geq 1$ and $3 \leq d \leq 3w$. Here $\delta_{k,l}$ denotes the Dirac-delta.

Proof. The total weight of tetragons after $(n-1)$ steps is n . The total weight of triangles after $(n-1)$ steps is $4n$. The total weight of the triangles having a fixed common vertex of weight w is $3w$. Moreover, after $(n-1)$ steps, we have the following. When we choose three vertices randomly, then the probability that a given vertex is chosen is

$$\frac{\binom{V_{n-1}-1}{2}}{\binom{V_{n-1}}{3}} = \frac{3}{V_{n-1}}. \tag{6}$$

When we choose four vertices randomly, then the probability that a given vertex is chosen is

$$\frac{\binom{V_{n-1}-1}{3}}{\binom{V_{n-1}}{4}} = \frac{4}{V_{n-1}}. \tag{7}$$

Therefore the probability that an old vertex of weight w takes part in the interaction at step n is

$$\begin{aligned}
& p \left(r \frac{3w}{4n} + (1-r) \frac{3}{V_{n-1}} \right) + (1-p) \left(q \frac{w}{n} + (1-q) \frac{4}{V_{n-1}} \right) \\
& = \frac{w}{n} \alpha + \frac{p}{V_{n-1}} \beta,
\end{aligned} \tag{8}$$

where α and β are defined by (4). A new vertex always takes part in the interaction. At each step with probability p a new vertex with weight 1 and with degree 3 is born. This explains term $p\delta_{d,3}\delta_{w,1}$ in (5).

Consider a fixed vertex with weight w and degree d . The probability of the event that in the n th step

(i) neither its degree d nor its weight w change is

$$1 - \left(\frac{w}{n} \alpha + \frac{p}{V_{n-1}} \beta \right); \tag{9}$$

(ii) its degree does not change but its weight is increased by 1 is

$$(1-p) \left(q \frac{w-1}{n} + (1-q) \frac{\binom{d}{3}}{\binom{V_{n-1}}{4}} \right); \tag{10}$$

(iii) both its degree and its weight are increased by 1 is

$$\begin{aligned}
& p \left(r \frac{3(w-1)}{4n} + (1-r) \frac{\binom{d-1}{2}}{\binom{V_{n-1}}{3}} \right) \\
& + (1-p)(1-q) \frac{\binom{d-1}{2}(V_{n-1}-d)}{\binom{V_{n-1}}{4}};
\end{aligned} \tag{11}$$

(iv) its degree is increased by 2 and its weight is increased by 1 is

$$\begin{aligned}
& p(1-r) \frac{(d-2)(V_{n-1}-d+1)}{\binom{V_{n-1}}{3}} \\
& + (1-p)(1-q) \frac{(d-2)\binom{V_{n-1}-d+1}{2}}{\binom{V_{n-1}}{4}};
\end{aligned} \tag{12}$$

(v) its degree is increased by 3 and its weight is increased by 1 is

$$p(1-r) \frac{\binom{V_{n-1}-d+2}{2}}{\binom{V_{n-1}}{3}} + (1-p)(1-q) \frac{\binom{V_{n-1}-d+2}{3}}{\binom{V_{n-1}}{4}}. \tag{13}$$

Using the above formulae, we obtain (5). \square

The following theorem is an extension of Theorem 3.1 in [7].

Theorem 3. Let $0 < p < 1$, $q > 0$, $r > 0$ and $(1-r)(1-q) > 0$. For any fixed w and d with $1 \leq w$ and $3 \leq d \leq 3w$ one has

$$\frac{X(n, d, w)}{V_n} \rightarrow x_{d,w} \tag{14}$$

almost surely as $n \rightarrow \infty$, where $x_{d,w}$ are fixed nonnegative numbers. Furthermore, the numbers $x_{d,w}$ satisfy the following recurrence:

$$\begin{aligned}
& x_{3,1} = \frac{1}{\alpha + \beta + 1}, \quad x_{d,1} = 0, \quad \text{for } d \neq 3, \\
& x_{d,w} = \frac{1}{\alpha w + \beta + 1} [\alpha_1 (w-1) x_{d,w-1} + \alpha_2 (w-1) x_{d-1,w-1} \\
& \quad + \beta x_{d-3,w-1}],
\end{aligned} \tag{15}$$

for $w \geq 2$, $3 \leq d \leq 3w$, where α_1 , α_2 , α , and β are given by (4). One has $x_{d,w} > 0$ if $w \geq 1$, $3 \leq d \leq 3w$ unless $d = 3w - 1$. If $d = 3w - 1$, then one has $x_{d,w} = 0$; moreover, in this case

$$\frac{X(n, d, w)}{V_n} = o(n^{-a}), \tag{16}$$

where a is a positive number which may depend on w and d . If $3 \leq d \leq 3w$ does not satisfied, then $x_{d,w} = 0$.

Proof. During the proof we take care of the case $x_{d,w} = 0$ which does not appear in [7]. Introduce notation

$$c(n, w) = \prod_{i=w-1}^n \left(1 - \frac{\alpha w}{i} - \frac{\beta p}{V_{i-1}} \right)^{-1}, \quad n \geq w-1, \quad w \geq 1. \quad (17)$$

$c(n, w)$ is an \mathcal{F}_{n-1} -measurable random variable. Applying the Marcinkiewicz strong law of large numbers to the number of vertices, we have

$$V_n = pn + o(n^{1/2+\varepsilon}) \quad (18)$$

almost surely, for any $\varepsilon > 0$.

Using (18) and the Taylor expansion for $\log(1+x)$, we obtain

$$\begin{aligned} \log c(n, w) &= - \sum_{i=w-1}^n \log \left(1 - \frac{\alpha w}{i} - \frac{\beta}{i + o(i^{1/2+\varepsilon})} \right) \\ &= (\alpha w + \beta) \sum_{i=w-1}^n \frac{1}{i} + O(1), \end{aligned} \quad (19)$$

where the error term is convergent as $n \rightarrow \infty$. Therefore

$$c(n, w) \sim a_w n^{\alpha w + \beta} \quad (20)$$

almost surely, as $n \rightarrow \infty$, where a_w is a positive random variable.

Let

$$Z(n, d, w) = c(n, w) X(n, d, w) \quad \text{for } 1 \leq w, \quad 3 \leq d \leq 3w. \quad (21)$$

Using (5), we can see that $\{Z(n, d, w), \mathcal{F}_n, n = w-1, w, w+1, \dots\}$ is a nonnegative submartingale for any fixed $1 \leq w, 3 \leq d \leq 3w$. Define $Z(n, d, w) = 0$ for $n = 1, 2, \dots, w-2$. Applying the Doob-Meyer decomposition to $Z(n, d, w)$, we can write

$$Z(n, d, w) = M(n, d, w) + A(n, d, w), \quad (22)$$

where $M(n, d, w)$ is a martingale and $A(n, d, w)$ is a predictable increasing process. The general form of $M(n, d, w)$ and $A(n, d, w)$ is the following:

$$M(n, d, w) = \sum_{i=1}^n [Z(i, d, w) - \mathbb{E}(Z(i, d, w) | \mathcal{F}_{i-1})], \quad (23)$$

$$A(n, d, w)$$

$$= \mathbb{E}Z(1, d, w) \quad (24)$$

$$+ \sum_{i=2}^n [\mathbb{E}(Z(i, d, w) | \mathcal{F}_{i-1}) - Z(i-1, d, w)],$$

where \mathcal{F}_0 is the trivial σ -algebra. Using (5) and (24), we obtain

$$\begin{aligned} A(n, d, w) &= \mathbb{E}Z(1, d, w) \\ &+ \sum_{i=2}^n \left[c(i, w) X(i-1, d, w-1) (1-p) \right. \\ &\quad \times \left(q \frac{w-1}{i} + (1-q) \frac{\binom{d}{3}}{\binom{V_{i-1}}{4}} \right) \\ &\quad + c(i, w) X(i-1, d-1, w-1) \\ &\quad \times \left(p \left(r \frac{3(w-1)}{4i} + (1-r) \frac{\binom{d-1}{2}}{\binom{V_{i-1}}{3}} \right) \right. \\ &\quad \quad \left. + (1-p)(1-q) \frac{\binom{d-1}{2} (V_{i-1} - d)}{\binom{V_{i-1}}{4}} \right) \\ &\quad + c(i, w) X(i-1, d-2, w-1) \\ &\quad \times \left(p(1-r) \frac{(d-2)(V_{i-1} - d + 1)}{\binom{V_{i-1}}{3}} \right. \\ &\quad \quad \left. + (1-p)(1-q) \frac{(d-2) \binom{V_{i-1}-d+1}{2}}{\binom{V_{i-1}}{4}} \right) \\ &\quad + c(i, w) X(i-1, d-3, w-1) \\ &\quad \times \left(p(1-r) \frac{\binom{V_{i-1}-d+2}{2}}{\binom{V_{i-1}}{3}} \right. \\ &\quad \quad \left. + (1-p)(1-q) \frac{\binom{V_{i-1}-d+2}{3}}{\binom{V_{i-1}}{4}} \right) \\ &\quad \left. + c(i, w) p \delta_{d,3} \delta_{w,1} \right]. \end{aligned} \quad (25)$$

Let $B(n, d, w)$ denote the sum of the conditional variances of the process $Z(n, d, w)$. Now we will give an upper bound of $B(n, d, w)$ as follows:

$$B(n, d, w)$$

$$\begin{aligned} &= \sum_{i=2}^n \mathbb{D}^2(Z(i, d, w) | \mathcal{F}_{i-1}) \\ &= \sum_{i=2}^n \mathbb{E} \left\{ (Z(i, d, w) - \mathbb{E}(Z(i, d, w) | \mathcal{F}_{i-1}))^2 | \mathcal{F}_{i-1} \right\} \\ &= \sum_{i=2}^n c(i, w)^2 \\ &\quad \times \mathbb{E} \left\{ (X(i, d, w) - \mathbb{E}(X(i, d, w) | \mathcal{F}_{i-1}))^2 | \mathcal{F}_{i-1} \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{i=2}^n c(i, w)^2 \\
&\quad \times \mathbb{E} \left\{ (X(i, d, w) - X(i-1, d, w))^2 \mid \mathcal{F}_{i-1} \right\} \\
&\leq 16 \sum_{i=2}^n c(i, w)^2 = O(n^{2(\alpha w + \beta + 1)}).
\end{aligned} \tag{26}$$

Above we used that $c(i, w)$ is \mathcal{F}_{i-1} -measurable and at each step four vertices can interact. Jensen's inequality implies that $M^2(n, d, w)$ is a (nonnegative) submartingale if $M(n, d, w)$ is a martingale. Now we can apply the Doob-Meyer decomposition to $M^2(n, d, w)$. It is known that $B(n, d, w)$, that is the sum of the conditional variances of terms $Z(n, d, w)$ from formula (26), is the same (up to an additive constant) as the increasing predictable process in the Doob-Meyer decomposition of the nonnegative submartingale $M^2(n, d, w)$. Therefore the Doob-Meyer decomposition is

$$M^2(n, d, w) = Y(n, d, w) + B(n, d, w), \tag{27}$$

where $Y(n, d, w)$ is a martingale and the predictable increasing process $B(n, d, w)$ is given by (26).

We use induction on w . Let $w = 1$. We can see that a vertex of weight 1 could take part in an interaction when it was born. Therefore its degree must be equal to 3. By (25) and (20),

$$A(n, 3, 1) \sim p \sum_{i=2}^n c(i, 1) \sim p \sum_{i=2}^n a_1 i^{\alpha+\beta} \sim pa_1 \frac{n^{\alpha+\beta+1}}{\alpha + \beta + 1} \tag{28}$$

almost surely as $n \rightarrow \infty$. By (26), $B(n, 3, 1) = O(n^{2(\alpha+\beta+1)})$ and therefore $(B(n, 3, 1))^{1/2} \log B(n, 3, 1) = O(A(n, 3, 1))$. It follows from Proposition VII-2-4 of [8] (see Proposition 10 in the Appendix) that

$$Z(n, 3, 1) \sim A(n, 3, 1) \tag{29}$$

a.s. on the event $\{A(n, 3, 1) \rightarrow \infty\}$ as $n \rightarrow \infty$.

As, by (28), $A(n, 3, 1) \rightarrow \infty$ almost surely, therefore using (18) and (20), relation (29) implies

$$\begin{aligned}
\frac{X(n, 3, 1)}{V_n} &= \frac{Z(n, 3, 1)}{c(n, 1) V_n} \sim \frac{A(n, 3, 1)}{c(n, 1) V_n} \\
&\sim \frac{pa_1 (n^{\alpha+\beta+1} / (\alpha + \beta + 1))}{a_1 n^{\alpha+\beta} pn} \\
&= \frac{1}{\alpha + \beta + 1} = x_{3,1} > 0
\end{aligned} \tag{30}$$

almost surely. So (14) is valid for $w = 1$.

Now let $w = 2$. In this case the degree of the vertex must be $3 \leq d \leq 6$. If $w = 2$ and $d = 3, 4$, or 6 , then we will show

$x_{d,w} > 0$. By (18), (20), and (25), we can find the asymptotic behaviour of $A(n, d, 2)$ as

$$A(n, 3, 2) \sim pa_2 \frac{n^{2\alpha+\beta+1}}{2\alpha + \beta + 1} (1-p) qx_{3,1} \rightarrow \infty,$$

$$A(n, 4, 2) \sim pa_2 \frac{n^{2\alpha+\beta+1}}{2\alpha + \beta + 1} \frac{3}{4} prx_{3,1} \rightarrow \infty,$$

$$\begin{aligned}
A(n, 6, 2) &\sim pa_2 \frac{n^{2\alpha+\beta+1}}{2\alpha + \beta + 1} \\
&\quad \times \left[3(1-r) + \frac{4(1-p)(1-q)}{p} \right] x_{3,1} \rightarrow \infty.
\end{aligned} \tag{31}$$

Moreover, by (26), $B(n, d, 2) = O(n^{2(2\alpha+\beta+1)})$; thus, $(B(n, d, 2))^{1/2} \log B(n, d, 2) = O(A(n, d, 2))$ for $d = 3, 4$, and 6 . Therefore in these cases $Z(n, d, 2) \sim A(n, d, 2)$ almost surely on $\{A(n, d, 2) \rightarrow \infty\}$ as $n \rightarrow \infty$. It implies that

$$\begin{aligned}
\frac{X(n, d, 2)}{V_n} &= \frac{Z(n, d, 2)}{c(n, 2) V_n} \sim \frac{A(n, d, 2)}{c(n, 2) V_n} \\
&\sim \frac{pa_2 (n^{2\alpha+\beta+1} / (2\alpha + \beta + 1))}{a_2 n^{2\alpha+\beta} pn} T_{d,2} \\
&= \frac{T_{d,2}}{2\alpha + \beta + 1}
\end{aligned} \tag{32}$$

with appropriate $T_{d,2}$. In particular, we have

$$\begin{aligned}
\frac{X(n, 3, 2)}{V_n} &\rightarrow \frac{(1-p) qx_{3,1}}{2\alpha + \beta + 1} = x_{3,2} > 0, \\
\frac{X(n, 4, 2)}{V_n} &\rightarrow \frac{(3/4) prx_{3,1}}{2\alpha + \beta + 1} = x_{4,2} > 0, \\
\frac{X(n, 6, 2)}{V_n} &\rightarrow \left[3(1-r) + \frac{4(1-p)(1-q)}{p} \right] \frac{x_{3,1}}{2\alpha + \beta + 1} \\
&= x_{6,2} > 0,
\end{aligned} \tag{33}$$

as $n \rightarrow \infty$. So (14) is valid for $w = 2$ and $d = 3, 4, 6$.

However, the case $d = 5, w = 2$ is different from the previous cases. As $x_{5,2} = 0$, the relation $X(n, 5, 2)/V_n \rightarrow x_{5,2}$ would not be enough to proceed the induction. By (25) and using Remark 1, we have

$$A(n, 5, 2) = \mathbb{E} Z(1, 5, 2)$$

$$+ \sum_{i=2}^n \left[c(i, 2) X(i-1, 3, 1) \right]$$

$$\times \left(p(1-r) \frac{3(V_{i-1}-4)}{\binom{V_{i-1}}{3}} + (1-p)(1-q) \frac{3\binom{V_{i-1}-4}{2}}{\binom{V_{i-1}}{4}} \right) \Bigg]. \quad (34)$$

Using (18), (34), and the limit of $X(i-1, 3, 1)/V_{i-1}$, we obtain

$$\begin{aligned} A(n, 5, 2) &\sim \sum_{i=2}^n a_2 i^{2\alpha+\beta} x_{3,1} i p \\ &\times \left[\frac{18p(1-r)}{(pi)^2} + \frac{36(1-p)(1-q)}{(pi)^2} \right] \\ &\sim a_2 x_{3,1} \frac{18}{p} \sum_{i=2}^n i^{2\alpha+\beta-1} (p(1-r) + 2(1-p)(1-q)) \\ &\sim a_2 x_{3,1} \frac{18}{p} (p(1-r) + 2(1-p)(1-q)) \\ &\times \frac{n^{2\alpha+\beta}}{2\alpha+\beta} \rightarrow \infty, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (35)$$

Now we have

$$\frac{X(n, 5, 2)}{V_n} = \frac{Z(n, 5, 2)}{c(n, 2) V_n} = \frac{M(n, 5, 2) + A(n, 5, 2)}{c(n, 2) V_n}. \quad (36)$$

We denoted by $B(n, d, w)$ the increasing predictable process in the Doob-Meyer decomposition of $M^2(n, d, w)$. We know, by (26), that $B(n, 5, 2) = O(n^{4\alpha+2\beta+1})$ and so $(B(n, 5, 2))^{1/2} \log B(n, 5, 2) = O(n^{2\alpha+\beta+1/2+\varepsilon})$ with arbitrary small positive ε .

Applying Propositions VII-2-3 and VII-2-4 of [8] (see Proposition 9 in the Appendix), we have

$$\begin{aligned} M(n, 5, 2) &= o((B(n, 5, 2))^{1/2} \log B(n, 5, 2)) \\ &= o(n^{2\alpha+\beta+1/2+\varepsilon}) \quad \text{a.e. on } \{B(n, 5, 2) \rightarrow \infty\}. \end{aligned} \quad (37)$$

Moreover, on the set $\{B(\infty, 5, 2) < \infty\}$, the sequence $M(n, 5, 2)$ is almost surely convergent. So $M(n, 5, 2) = o(n^{2\alpha+\beta+1/2+\varepsilon})$ almost surely. Therefore, using (18), (20), and (35), we obtain

$$\begin{aligned} \frac{X(n, 5, 2)}{V_n} &= \frac{M(n, 5, 2) + A(n, 5, 2)}{c(n, 2) V_n} \leq C \frac{n^{2\alpha+\beta+1/2+\varepsilon}}{n^{2\alpha+\beta} n} \\ &= C \frac{1}{n^a} \rightarrow 0, \end{aligned} \quad (38)$$

if $n \rightarrow \infty$ where $1/4 < a < 1/2$. So the proposition is valid for $w = 1$ and $w = 2$.

Now, suppose that the statement is true for all weights less than w and for all possible degrees. First we study the positive

limits. Consider $A(n, d, w)$ in (25) and assume that at least one of the coefficients $x_{d,w-1}, x_{d-1,w-1}, x_{d-3,w-1}$ is positive. Then by (18), (20), and using the induction hypothesis, we see that

$$\begin{aligned} A(n, d, w) &\sim \sum_{i=2}^n \left[c(i, w) x_{d,w-1} p i (1-p) q \frac{w-1}{i} \right. \\ &\quad + c(i, w) x_{d-1,w-1} p i p r \frac{3(w-1)}{4i} \\ &\quad + c(i, w) x_{d-3,w-1} p i \\ &\quad \times \left(\frac{3p(1-r)}{pi} + \frac{4(1-p)(1-q)}{pi} \right) \Bigg] \\ &\sim \sum_{i=2}^n a_w i^{\alpha w + \beta} \\ &\quad \times [x_{d,w-1} p(1-p) q(w-1) \\ &\quad + x_{d-1,w-1} \frac{3}{4} p^2 r(w-1) \\ &\quad + x_{d-3,w-1} (3p(1-r) + 4(1-p)(1-q))] \\ &\sim p a_w \frac{n^{\alpha w + \beta + 1}}{\alpha w + \beta + 1} \\ &\quad \times \left[(1-p) q(w-1) x_{d,w-1} \right. \\ &\quad + \frac{3}{4} p r(w-1) x_{d-1,w-1} \\ &\quad + \left(3(1-r) + \frac{4(1-p)(1-q)}{p} \right) x_{d-3,w-1} \Bigg]. \end{aligned} \quad (39)$$

In the above computation we deleted all terms having asymptotically smaller degree than the largest one.

Consider the case when $3 \leq d \leq 3w$ and $d \neq 3w - 1$. In this case, by the induction hypothesis, there is at least one positive term in (39). Therefore (39) implies $A(n, d, w) \sim p a_w n^{\alpha w + \beta + 1} x_{d,w} \rightarrow \infty$ (because $x_{d,w} > 0$). In this case $(B(n, d, w))^{1/2} \log B(n, d, w) = O(A(n, d, w))$. So, using Proposition VII-2-4 of [8] (see Proposition 10 in the Appendix), we have $Z(n, d, w) \sim A(n, d, w)$. Therefore

$$\begin{aligned} \frac{X(n, d, w)}{V_n} &= \frac{Z(n, d, w)}{c(n, w) V_n} \sim \frac{A(n, d, w)}{c(n, w) V_n} \sim \frac{p a_w n^{\alpha w + \beta + 1} x_{d,w}}{a_w n^{\alpha w + \beta} p n} \\ &= x_{d,w} \quad \text{a.s. as } n \rightarrow \infty, \end{aligned} \quad (40)$$

where, by (39),

$$\begin{aligned} x_{d,w} &= \frac{1}{\alpha w + \beta + 1} \\ &\quad \times [\alpha_1 (w-1) x_{d,w-1} + \alpha_2 (w-1) x_{d-1,w-1} + \beta x_{d-3,w-1}], \end{aligned} \quad (41)$$

with $\alpha_1, \alpha_2, \alpha$, and β defined by (4).

To handle the case when the limit is 0, we argue as follows. Consider the case when $d = 3w - 1$. By (25) and using the induction hypothesis, we have

$$\begin{aligned}
& A(n, 3w - 1, w) \\
&= \mathbb{E}Z(1, 3w - 1, w) \\
&+ \sum_{i=2}^n \left[c(i, w) X(i - 1, 3w - 3, w - 1) \right. \\
&\quad \times \left(p(1 - r) \frac{(3w - 3) \binom{V_{i-1} - 3w + 2}{3}}{\binom{V_{i-1}}{3}} \right. \\
&\quad \left. \left. + (1 - p)(1 - q) \frac{(3w - 3) \binom{V_{i-1} - 3w + 2}{2}}{\binom{V_{i-1}}{4}} \right) \right. \\
&\quad \left. + c(i, w) X(i - 1, 3w - 4, w - 1) \right. \\
&\quad \times \left(p(1 - r) \frac{\binom{V_{i-1} - 3w + 3}{2}}{\binom{V_{i-1}}{3}} \right. \\
&\quad \left. \left. + (1 - p)(1 - q) \frac{\binom{V_{i-1} - 3w + 3}{3}}{\binom{V_{i-1}}{4}} \right) \right] \\
&\sim \sum_{i=2}^n a_w i^{\alpha w + \beta} \left[x_{3w-3, w-1} C_1 \frac{1}{p i} + O\left(\frac{1}{i^a}\right) \right] \\
&\leq C_1 \frac{n^{\alpha w + \beta}}{\alpha w + \beta} + C_2 \frac{n^{\alpha w + \beta + 1 - a}}{\alpha w + \beta + 1 - a} \\
&= O\left(n^{\alpha w + \beta + 1 - a}\right). \tag{42}
\end{aligned}$$

On the other hand, $B(n, d, w) = O(n^{2(\alpha w + \beta) + 1})$. Therefore, using (18), (20), and (42), by Propositions VII-2-3 and VII-2-4 of [8] (see Proposition 9 in the Appendix), we obtain

$$\begin{aligned}
\frac{X(n, d, w)}{V_n} &= \frac{Z(n, d, w)}{c(n, w) V_n} = \frac{M(n, d, w) + A(n, d, w)}{c(n, w) V_n} \\
&= \frac{O\left(n^{\alpha w + \beta + 1 - a}\right)}{n^{\alpha w + \beta} p n} = O\left(n^{-a}\right) \longrightarrow 0 \\
&= x_{3w-1, w}, \quad \text{a.s.} \tag{43}
\end{aligned}$$

So we have obtained the desired result for the case of 0 limit as well. \square

3. The Scale-Free Property for the Weights and Degrees

Lemma 4. Let $p > 0$ and define

$$x_w = x_{3, w} + x_{4, w} + \cdots + x_{3w, w} \tag{44}$$

for $w = 1, 2, \dots$. Then x_w , $w = 1, 2, \dots$, are positive numbers satisfying the following recurrence:

$$\begin{aligned}
x_1 &= \frac{1}{\alpha + \beta + 1}, \\
x_w &= \frac{\alpha(w - 1) + \beta}{\alpha w + \beta + 1} x_{w-1}, \quad \text{if } w > 1, \tag{45}
\end{aligned}$$

where α and β are defined by (4). x_w , $w = 1, 2, \dots$, is a discrete probability distribution. Moreover, $x_{d, w}$, $d = 3, 4, \dots, 3w$, $w = 1, 2, \dots$, is a two-dimensional discrete probability distribution.

Proof. If $\alpha = 0$, then the statement is obvious. Now assume $\alpha \neq 0$. As $x_{d, w}$ is defined as $x_{d, w} = 0$ for $d \notin \{3, 4, \dots, 3w\}$, therefore $x_w = \sum_d x_{d, w}$. From the recurrence (15) for $x_{d, w}$, we obtain

$$\begin{aligned}
x_w &= \sum_{d=3}^{3w} x_{d, w} = \sum_d x_{d, w} \\
&= \frac{1}{\alpha w + \beta + 1} \left[\alpha_1 (w - 1) \sum_d x_{d, w-1} \right. \\
&\quad \left. + \alpha_2 (w - 1) \sum_d x_{d-1, w-1} \right. \\
&\quad \left. + \beta \sum_d x_{d-3, w-1} \right] \\
&= \frac{\alpha(w - 1) + \beta}{\alpha w + \beta + 1} x_{w-1}. \tag{46}
\end{aligned}$$

Using this recursive formula for x_w , we obtain

$$\begin{aligned}
x_w &= x_1 \prod_{j=2}^w \frac{\alpha(j - 1) + \beta}{\alpha j + \beta + 1} \\
&= \frac{1}{\alpha + \beta + 1} \frac{\alpha + \beta}{2\alpha + \beta + 1} \frac{2\alpha + \beta}{3\alpha + \beta + 1} \cdots \frac{(w - 1)\alpha + \beta}{w\alpha + \beta + 1} \\
&= \frac{1}{\alpha w + \beta + 1} \prod_{j=1}^{w-1} \frac{\beta/\alpha + j}{(\beta + 1)/\alpha + j} \\
&= \frac{1}{\alpha w + \beta + 1} \frac{\Gamma(w + \beta/\alpha)}{\Gamma(1 + \beta/\alpha)} \frac{\Gamma(1 + (\beta + 1)/\alpha)}{\Gamma(w + (\beta + 1)/\alpha)} \\
&= \frac{\Gamma(1 + (\beta + 1)/\alpha)}{\alpha \Gamma(1 + \beta/\alpha)} \frac{\Gamma(w + \beta/\alpha)}{\Gamma(w + (\beta + 1)/\alpha + 1)}. \tag{47}
\end{aligned}$$

Moreover, by [9], we have the following formula:

$$\sum_{k=0}^n \frac{\Gamma(k + a)}{\Gamma(k + b)} = \frac{1}{a - b + 1} \left[\frac{\Gamma(n + a + 1)}{\Gamma(n + b)} - \frac{\Gamma(a)}{\Gamma(b - 1)} \right]. \tag{48}$$

Therefore, by some calculation, we obtain $\sum_{w=1}^n x_w \rightarrow 1$, as $n \rightarrow \infty$. So $\sum_{w=1}^{\infty} x_w = 1$. As $\sum_d x_{d, w} = x_w$, so

$\sum_{w=1}^{\infty} \sum_{d=3}^{3w} x_{d,w} = 1$ and therefore $x_{d,w}$, $d = 3, 4, \dots, 3w$, $w = 1, 2, \dots$, is a (proper) two-dimensional discrete probability distribution. \square

Let $X(n, w)$ denote the number of vertices of weight w after n steps. Next theorem is the scale-free property for the weights. It is an extension of Theorem 3.1 in [6].

Theorem 5. Let $0 < p < 1$, $q > 0$, $r > 0$ and $(1-r)(1-q) > 0$. Then for all $w = 1, 2, \dots$ one has

$$\frac{X(n, w)}{V_n} \longrightarrow x_w = x_{3,w} + x_{4,w} + \dots + x_{3w,w} \quad (49)$$

almost surely, as $n \rightarrow \infty$, where x_w , $w = 1, 2, \dots$, are positive numbers satisfying the recurrence (45). Moreover,

$$x_w \sim Cw^{-(1+1/\alpha)}, \quad (50)$$

as $w \rightarrow \infty$, with $C = \Gamma(1 + (\beta + 1)/\alpha) / (\alpha\Gamma(1 + \beta/\alpha))$.

Proof. We have

$$X(n, w) = X(n, 3, w) + X(n, 4, w) + \dots + X(n, 3w, w). \quad (51)$$

Therefore, by Theorem 3,

$$\frac{X(n, w)}{V_n} \longrightarrow x_{3,w} + \dots + x_{3w,w} = x_w \quad (52)$$

almost surely, as $n \rightarrow \infty$. Here each x_w is positive.

Using formula (47) and the Stirling-formula for the Gamma function, we obtain

$$\begin{aligned} x_w &= \frac{\Gamma(1 + (\beta + 1)/\alpha)}{\alpha\Gamma(1 + \beta/\alpha)} \frac{\Gamma(w + \beta/\alpha)}{\Gamma(w + (\beta + 1)/\alpha + 1)} \\ &\sim C_0 \frac{(w + \beta/\alpha)^{(w+\beta/\alpha)}}{(w + \beta/\alpha + 1/\alpha + 1)^{(w+\beta/\alpha+1/\alpha+1)}} \\ &= C_0 \left(\frac{(w + \beta/\alpha)}{(w + \beta/\alpha + 1/\alpha + 1)} \right)^{(w+\beta/\alpha)} \\ &\quad \times \frac{1}{(w + \beta/\alpha + 1/\alpha + 1)^{1/\alpha+1}} \sim Cw^{-(1+1/\alpha)}, \end{aligned} \quad (53)$$

where $C_0 = \Gamma(1 + (\beta + 1)/\alpha) / (\alpha\Gamma(1 + \beta/\alpha))(1/(1/e)^{1+1/\alpha})$ and $C = \Gamma(1 + (\beta + 1)/\alpha) / (\alpha\Gamma(1 + \beta/\alpha))$. \square

Now, following [7], we construct a representation of the limiting joint distribution of degrees and weights.

Let W be a random variable with distribution $\mathbb{P}(W = w) = x_w$, $w = 1, 2, \dots$. Let $\xi_1 \equiv 3$ and ξ_2, ξ_3, \dots be

independent random variables being independent of W , too. For $w \geq 2$ let ξ_w have the following distribution:

$$\begin{aligned} \mathbb{P}(\xi_w = 0) &= \frac{\alpha_1(w-1)}{\alpha(w-1) + \beta}, \\ \mathbb{P}(\xi_w = 1) &= \frac{\alpha_2(w-1)}{\alpha(w-1) + \beta}, \\ \mathbb{P}(\xi_w = 3) &= \frac{\beta}{\alpha(w-1) + \beta}. \end{aligned} \quad (54)$$

Introduce notation $S_w = \xi_1 + \xi_2 + \dots + \xi_w$.

The following representation of the joint distribution of degrees and weights is useful to obtain scale-free property for degrees.

Theorem 6. $\mathbb{P}(S_W = d, W = w) = x_{d,w}$ for all $w = 1, 2, \dots$, $d = 3, 4, \dots, 3w$.

Proof. It is easy to see that if $w = 1$ and $d = 3$, then we have

$$\begin{aligned} \mathbb{P}(S_W = 3, W = 1) &= \mathbb{P}(S_1 = 3, W = 1) \\ &= \mathbb{P}(\xi_1 = 3, W = 1) \\ &= \mathbb{P}(W = 1) = x_1 = x_{3,1}. \end{aligned} \quad (55)$$

If $w = 2$ and $d = 5$ we have

$$\begin{aligned} \mathbb{P}(S_W = 5, W = 2) &= \mathbb{P}(S_2 = 5, W = 2) \\ &= \mathbb{P}(\xi_1 = 3, \xi_2 = 2, W = 2) \\ &= 0 = x_{5,2}. \end{aligned} \quad (56)$$

Generally, for all $w = 1, 2, \dots$ and $d = 3w - 1$ we have

$$\begin{aligned} \mathbb{P}(S_W = 3w - 1, W = w) &= \mathbb{P}(\xi_1 = 3, \xi_2 = k_2, \dots, \xi_w = k_w, W = w) \\ &= 0 = x_{3w-1,w}, \end{aligned} \quad (57)$$

because one of k_2, \dots, k_w should be equal to 2, which is of zero probability. Furthermore, using the recursion (45) and the assumption that ξ_i are independent random variables which are independent of W , we have

$$\begin{aligned} \mathbb{P}(S_W = d, W = w) &= \mathbb{P}(S_w = d, W = w) \\ &= \mathbb{P}(S_w = d) \mathbb{P}(W = w) \\ &= [\mathbb{P}(S_{w-1} = d) \mathbb{P}(\xi_w = 0) \\ &\quad + \mathbb{P}(S_{w-1} = d - 1) \mathbb{P}(\xi_w = 1) \\ &\quad + \mathbb{P}(S_{w-1} = d - 3) \mathbb{P}(\xi_w = 3)] \\ &\quad \times \mathbb{P}(W = w - 1) \frac{x_w}{x_{w-1}} \end{aligned}$$

$$\begin{aligned}
&= [\mathbb{P}(S_{w-1} = d) \mathbb{P}(W = w-1) \alpha_1(w-1) \\
&\quad + \mathbb{P}(S_{w-1} = d-1) \mathbb{P}(W = w-1) \alpha_2(w-1) \\
&\quad + \mathbb{P}(S_{w-1} = d-3) \mathbb{P}(W = w-1) \beta] \\
&\quad \times \frac{1}{\alpha w + \beta + 1} \\
&= \frac{1}{\alpha w + \beta + 1} \\
&\quad \times [\alpha_1(w-1) \mathbb{P}(S_W = d, W = w-1) \\
&\quad + \alpha_2(w-1) \mathbb{P}(S_W = d-1, W = w-1) \\
&\quad + \beta \mathbb{P}(S_W = d-3, W = w-1)].
\end{aligned} \tag{58}$$

Therefore, the sequence $\mathbb{P}(S_W = d, W = w)$ satisfies the same recursion (15) as $x_{d,w}$. \square

Theorem 7. Suppose that $\alpha_1 > 0$ and $\alpha_2 > 0$. Then

$$\begin{aligned}
x_{d,w} &= x_w \frac{\alpha}{\sqrt{2\pi\alpha_1\alpha_2 w}} \\
&\quad \times \left[\exp\left(-\frac{(d - \mathbb{E}S_w)^2}{2\mathbb{D}^2 S_w}\right) + O(w^{-1/2}) \right], \tag{59} \\
&\quad \text{as } w \rightarrow \infty,
\end{aligned}$$

where the error term $O(w^{-1/2})$ does not depend on d .

Proof. We can follow the ideas of Theorem 4.2 in [7]. For $w \geq 1$ we have

$$\begin{aligned}
\mathbb{E}\xi_w &= \frac{\alpha_2(w-1)}{\alpha(w-1) + \beta} + 3\frac{\beta}{\alpha(w-1) + \beta} \\
&= \frac{\alpha_2}{\alpha} + \frac{(3\alpha - \alpha_2)\beta}{\alpha(\alpha(w-1) + \beta)}; \tag{60}
\end{aligned}$$

hence,

$$\mathbb{E}S_w = \mathbb{E}\xi_1 + \dots + \mathbb{E}\xi_w = w\frac{\alpha_2}{\alpha} + O(\log w), \tag{61}$$

as $w \rightarrow \infty$. Similarly, by simple computation, we have

$$\begin{aligned}
\mathbb{D}^2 \xi_w &= \frac{\alpha_1 \alpha_2}{\alpha^2} + O\left(\frac{1}{w}\right), \\
\mathbb{D}^2 S_w &= \frac{\alpha_1 \alpha_2}{\alpha^2} w + O(\log w), \tag{62}
\end{aligned}$$

as $w \rightarrow \infty$.

Now, we can apply Theorem VII.2.5 in [10] (see Proposition 11 in the Appendix) for S_w . The conditions of that theorem are satisfied; therefore, we have

$$\begin{aligned}
&\sup_{d \in \mathbb{Z}} \left| \mathbb{D}S_w \mathbb{P}(S_w = d) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d - \mathbb{E}S_w)^2}{2\mathbb{D}^2 S_w}\right) \right| \\
&= O\left(\frac{1}{\sqrt{w}}\right). \tag{63}
\end{aligned}$$

Using (62) and (63), we obtain $|\mathbb{D}S_w - \sqrt{\alpha_1 \alpha_2 w}/\alpha| \mathbb{P}(S_w = d) = O(w^{-1/2})$. Therefore, it follows from (63) that

$$\begin{aligned}
&\sup_{d \in \mathbb{Z}} \left| \frac{\sqrt{\alpha_1 \alpha_2 w}}{\alpha} \mathbb{P}(S_w = d) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(d - \mathbb{E}S_w)^2}{2\mathbb{D}^2 S_w}\right) \right| \\
&= O\left(\frac{1}{\sqrt{w}}\right). \tag{64}
\end{aligned}$$

The independence of W and ξ_i implies that $x_{d,w} = \mathbb{P}(S_W = d, W = w) = \mathbb{P}(S_w = d)x_w$. Using this in (64), we can obtain the desired result. \square

Our last theorem is an extension of Theorem 4.3 in [7] to the case of 4 interactions. The theorem shows the scale-free property for the degrees.

Theorem 8. Let $0 < p < 1$, $q > 0$, $r > 0$ and $(1-r)(1-q) > 0$. Let us denote by $U(n, d)$ the number of vertices of degree d after n steps; that is, $U(n, d) = \sum_{w: d/3 \leq w \leq n+1} X(n, d, w)$. Then, for any $d \geq 3$ one has

$$\frac{U(n, d)}{V_n} \rightarrow u_d = \sum_w x_{d,w} \tag{65}$$

almost surely as $n \rightarrow \infty$, where u_d are positive numbers. Furthermore,

$$u_d \sim \frac{\Gamma(1 + (\beta + 1)/\alpha)}{\alpha_2 \Gamma(1 + \beta/\alpha)} \left(\frac{\alpha d}{\alpha_2}\right)^{-(1+1/\alpha)}, \tag{66}$$

as $d \rightarrow \infty$.

Proof. By Theorems 3 and 6, $X(n, d, w)/V_n$ converges almost surely to the distribution $x_{d,w} = \mathbb{P}(S_W = d, W = w)$. But the cardinalities of terms in $\sum_{w: d/3 \leq w \leq n+1} X(n, d, w)$ are not bounded when $n \rightarrow \infty$. However, using that $x_{d,w}$, $d = 3, 4, \dots, 3w$, $w = 1, 2, \dots$ is a proper two-dimensional discrete distribution, the convergence of the marginal distributions is a consequence of the convergence of the two-dimensional distributions. So we obtain (65).

To obtain (66), we can apply the method of Theorem 4.3 in [7]. Let

$$\begin{aligned}
f &= \frac{\alpha}{\alpha_2} d, \\
H &= H_d = \{w : f - f^{1/2+\varepsilon} \leq w \leq f + f^{1/2+\varepsilon}\}, \tag{67} \\
H^- &= H_d^- = \{w : w < f - f^{1/2+\varepsilon}\}, \\
H^+ &= H_d^+ = \{w : w > f + f^{1/2+\varepsilon}\}
\end{aligned}$$

with some fixed $0 < \varepsilon < 1/6$.

Using (61) and Hoeffding's exponential inequality (Theorem 2 in [11], see Proposition 12 in the Appendix) we obtain for $w \in H^-$

$$\begin{aligned} \mathbb{P}(S_w = d) &\leq \mathbb{P}(S_w \geq d) \\ &\leq \mathbb{P}\left(S_w - \mathbb{E}S_w \geq d - \frac{\alpha_2}{\alpha}w - O(\log w)\right) \\ &\leq \exp\left\{-\frac{2}{9w}\left(d - \frac{\alpha_2}{\alpha}w - O(\log w)\right)^2\right\} \\ &= \exp\left\{-\frac{2}{9}\left(\frac{\alpha_2}{\alpha}\right)^2 \frac{(f - w - O(\log w))^2}{w}\right\}. \end{aligned} \quad (68)$$

Here $w \in H^-$ implies that

$$\begin{aligned} (f - w - O(\log w))^2 &= (f - w)^2 - 2(f - w)O(\log w) + (O(\log w))^2 \\ &\geq f^{1+2\varepsilon} - O(f \log f). \end{aligned} \quad (69)$$

Therefore in the case when $w \in H^-$ we have

$$\begin{aligned} \mathbb{P}(S_w = d) &\leq \exp\left\{-\frac{2}{9}\left(\frac{\alpha_2}{\alpha}\right)^2 \frac{f^{1+2\varepsilon} - O(f \log f)}{f}\right\} \\ &= \exp\left\{-\frac{2}{9}\left(\frac{\alpha_2}{\alpha}\right)^2 f^{2\varepsilon} + O(\log f)\right\}. \end{aligned} \quad (70)$$

Using this, we can obtain that

$$\begin{aligned} \mathbb{P}(S_W = d, W \in H^-) &= \sum_{w \in H^-} \mathbb{P}(S_w = d, W = w) \\ &\leq \sum_{w \in H^-} \mathbb{P}(S_w = d) \\ &\leq f \exp\left\{-\frac{2}{9}\left(\frac{\alpha_2}{\alpha}\right)^2 f^{2\varepsilon} + O(\log f)\right\} \\ &= o\left(f^{-(1+1/\alpha)}\right). \end{aligned} \quad (71)$$

Similarly, if $w \in H^+$, again by Hoeffding's inequality, we have

$$\begin{aligned} \mathbb{P}(S_w = d) &\leq \mathbb{P}(S_w \leq d) \\ &\leq \mathbb{P}\left(S_w - \mathbb{E}S_w \leq d - \frac{\alpha_2}{\alpha}w\right) \\ &\leq \exp\left\{-\frac{2}{9w}\left(d - \frac{\alpha_2}{\alpha}w\right)^2\right\} \\ &= \exp\left\{-\frac{2}{9}\left(\frac{\alpha_2}{\alpha}\right)^2 \frac{(f - w)^2}{w}\right\}. \end{aligned} \quad (72)$$

Using that $w \in H^+$ and $1/2 + \varepsilon < 1$, we obtain $2(w - f) \geq f^{1/2+\varepsilon} + w - f \geq f^{1/2+\varepsilon} + (w - f)^{1/2+\varepsilon} \geq w^{1/2+\varepsilon}$. Therefore

$$\begin{aligned} \mathbb{P}(S_w = d) &\leq \exp\left\{-\frac{2}{9}\left(\frac{\alpha_2}{\alpha}\right)^2 \frac{w^{1+2\varepsilon}}{4w}\right\} \\ &= \exp\left\{-\frac{1}{18}\left(\frac{\alpha_2}{\alpha}\right)^2 w^{2\varepsilon}\right\}. \end{aligned} \quad (73)$$

Hence

$$\begin{aligned} \mathbb{P}(S_W = d, W \in H^+) &\leq \sum_{\{w: f < w\}} \exp\left\{-\frac{1}{18}\left(\frac{\alpha_2}{\alpha}\right)^2 w^{2\varepsilon}\right\} \\ &= o\left(f^{-(1+1/\alpha)}\right). \end{aligned} \quad (74)$$

Now consider the case when $w \in H = H_d$. First we need some general facts. Consider the set

$$B = \{(d, w) : w \geq 1, d \geq 3, w \in H_d\}. \quad (75)$$

It is easy to see that when $(d, w) \in B$ then $d \rightarrow \infty$ if and only if $w \rightarrow \infty$. More precisely,

$$\frac{w}{d} \rightarrow 1, \quad \text{if } d \rightarrow \infty, (d, w) \in B. \quad (76)$$

As $w \in H$, so we have $w = f + O(f^{1/2+\varepsilon})$. Then (with $\varepsilon_1 > 0$ arbitrarily small)

$$\begin{aligned} -\frac{(d - \mathbb{E}S_w)^2}{2\mathbb{D}^2S_w} &= -\frac{(d - w(\alpha_2/\alpha) - O(\log w))^2}{2(\alpha_1\alpha_2/\alpha^2)w + O(\log w)} \\ &= -\frac{\alpha_2}{\alpha_1} \frac{(f - w - O(\log w))^2}{2w + O(\log w)} \\ &= -\frac{\alpha_2}{\alpha_1} \frac{(f - w)^2 + O(f^{1/2+\varepsilon+\varepsilon_1})}{2w + O(\log w)} \\ &= -\frac{\alpha_2}{\alpha_1} \frac{(f - w)^2 + O(f^{1/2+\varepsilon+\varepsilon_1})}{2f} \\ &\quad \times \frac{2f}{2f + O(f^{1/2+\varepsilon})} \\ &= -\frac{\alpha_2}{\alpha_1} \frac{(f - w)^2 + O(f^{1/2+\varepsilon+\varepsilon_1})}{2f} \\ &\quad \times \left[1 - \frac{O(f^{1/2+\varepsilon})}{2f + O(f^{1/2+\varepsilon})}\right] \\ &= -\frac{\alpha_2}{\alpha_1} \frac{(f - w)^2}{2f} + O(f^{-1/2+3\varepsilon}), \end{aligned} \quad (77)$$

as $d \rightarrow \infty$. Here the error term does not depend on w . We will apply Theorem 7, that is, formula (59). The asymptotic

behaviour of x_w is known from (50). Using these facts and (77), we obtain

$$\begin{aligned}
 x_{d,w} &\sim Cw^{-(1+1/\alpha)} \frac{\alpha}{\sqrt{2\pi\alpha_1\alpha_2w}} \\
 &\times \left[\exp \left\{ -\frac{\alpha_2}{\alpha_1} \frac{(f-w)^2}{2f} + O(f^{-1/2+3\varepsilon}) \right\} + O(w^{-1/2}) \right] \\
 &\sim Cf^{-(1+1/\alpha)} \frac{\alpha}{\alpha_2} \frac{1}{\sqrt{2\pi(\alpha_1/\alpha_2)f}} \exp \left\{ -\frac{(f-w)^2}{2(\alpha_1/\alpha_2)f} \right\}
 \end{aligned} \quad (78)$$

as $d \rightarrow \infty$ and $w \in H$, where $C = \Gamma(1 + (\beta + 1)/\alpha)/(\alpha\Gamma(1 + \beta/\alpha))$. Therefore

$$\begin{aligned}
 \sum_{w \in H} x_{d,w} &\sim \sum_{f-f^{1/2+\varepsilon} < w < f+f^{1/2+\varepsilon}} Cf^{-(1+1/\alpha)} \frac{\alpha}{\alpha_2} \frac{1}{\sqrt{2\pi(\alpha_1/\alpha_2)f}} \\
 &\times \exp \left\{ -\frac{(f-w)^2}{2(\alpha_1/\alpha_2)f} \right\} \\
 &= Cf^{-(1+1/\alpha)} \frac{\alpha}{\alpha_2} \sum_{-f^{1/2+\varepsilon} < k < f^{1/2+\varepsilon}} \frac{1}{\sqrt{2\pi(\alpha_1/\alpha_2)f}} \\
 &\times \exp \left\{ -\frac{k^2}{2(\alpha_1/\alpha_2)f} \right\} \\
 &= A \sum_{-f^\varepsilon < k/\sqrt{f} < f^\varepsilon} \frac{1}{\sqrt{f}} \frac{1}{\sqrt{2\pi(\alpha_1/\alpha_2)}} \\
 &\times \exp \left\{ -\frac{(k/\sqrt{f})^2}{2(\alpha_1/\alpha_2)} \right\} \\
 &\rightarrow A \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi(\alpha_1/\alpha_2)}} \exp \left\{ -\frac{x^2}{2(\alpha_1/\alpha_2)} \right\} dx = A.
 \end{aligned} \quad (79)$$

Thus we have

$$\begin{aligned}
 \mathbb{P}(S_W = d, W \in H) &\sim A \\
 &= \frac{\Gamma(1 + (\beta + 1)/\alpha)}{\alpha_2\Gamma(1 + \beta/\alpha)} \left(\frac{\alpha d}{\alpha_2} \right)^{-(1+1/\alpha)},
 \end{aligned} \quad (80)$$

as $d \rightarrow \infty$. Finally, (71), (74), and (80) imply

$$\begin{aligned}
 u_d &= \sum_w x_{d,w} \\
 &= \sum_{w \in H^-} x_{d,w} + \sum_{w \in H} x_{d,w} \\
 &\quad + \sum_{w \in H^+} x_{d,w} \sim o(f^{-(1+1/\alpha)}) \\
 &\quad + C \frac{\alpha}{\alpha_2} f^{-(1+1/\alpha)} + o(f^{-(1+1/\alpha)}) \\
 &\sim \frac{\Gamma(1 + (\beta + 1)/\alpha)}{\alpha_2\Gamma(1 + \beta/\alpha)} ((\alpha/\alpha_2)d)^{-(1+1/\alpha)},
 \end{aligned} \quad (81)$$

as $d \rightarrow \infty$. The proof is complete. \square

Appendix

We use the following results on discrete time martingales. Let $\{Z_n, \mathcal{F}_n\}$ be a submartingale. Its Doob-Meyer decomposition is $Z_n = M_n + A_n$, where $\{M_n, \mathcal{F}_n\}$ is a martingale and $\{A_n, \mathcal{F}_n\}$ is an increasing predictable process. Here, up to an additive constant,

$$A_n = \mathbb{E}Z_1 + \sum_{i=2}^n (\mathbb{E}(Z_i | \mathcal{F}_{i-1}) - Z_{i-1}). \quad (A.1)$$

Now, $\{M_n^2, \mathcal{F}_n\}$ is again a submartingale. Let

$$M_n^2 = Y_n + B_n \quad (A.2)$$

be the Doob-Meyer decomposition of M_n^2 . Here, up to an additive constant,

$$\begin{aligned}
 B_n &= \sum_{i=2}^n \mathbb{D}^2(Z_i | \mathcal{F}_{i-1}) \\
 &= \sum_{i=2}^n \mathbb{E} \left\{ (Z_i - \mathbb{E}(Z_i | \mathcal{F}_{i-1}))^2 | \mathcal{F}_{i-1} \right\}.
 \end{aligned} \quad (A.3)$$

Proposition 9 (Propositions VII-2-3 and VII-2-4 of [8]). *Let $M_1 = 0$. On the set $\{B_\infty < \infty\}$ the martingale M_n almost surely converges to a finite limit. Moreover, $M_n = o(B_n^{1/2} \log B_n)$ almost surely on the set $\{B_n \rightarrow \infty\}$.*

A consequence of the above proposition is the following.

Proposition 10 (Proposition 2.3 of [12]). *Let $\{Z_n, \mathcal{F}_n\}$ be a square integrable nonnegative submartingale. If $B_n^{1/2} \log B_n = O(A_n)$, then $Z_n \sim A_n$, as $n \rightarrow \infty$, almost surely on the set $\{A_n \rightarrow \infty\}$.*

We need the following local limit theorem. Let X_1, X_2, \dots be independent, integer valued random variables. Let $p_{j,m} = \mathbb{P}(X_j = m)$ be the distribution, while $p_{j,m_j} = \max_m p_{j,m}$ be the maximal value of the distribution. Let $S_n = \sum_{i=1}^n X_i$ be the

partial sum, $P_n(N) = \mathbb{P}(S_n = N)$ be its distribution, $M_n = \sum_{i=1}^n \mathbb{E}X_i$ be the expectation, and $B_n = \sum_{i=1}^n \mathbb{E}(X_i - \mathbb{E}X_i)^2$ be the variance of S_n .

Proposition 11 (Theorem 5 and its consequence in Section VII, 2 of [10]). *Assume that the greatest common divisor of the values*

$$\left\{ m : \frac{1}{\log n} \sum_{j=1}^n p_{j,m_j} p_{j,m+m_j} \longrightarrow \infty \right\} \quad (\text{A.4})$$

is equal to 1; moreover,

$$\begin{aligned} \liminf \frac{B_n}{n} &> 0, \\ \limsup \frac{1}{n} \sum_{i=1}^n \mathbb{E}|X_i - \mathbb{E}X_i|^3 &< \infty. \end{aligned} \quad (\text{A.5})$$

Then

$$\sup_N \left| \sqrt{B_n} P_n(N) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(N - M_n)^2}{2B_n}\right) \right| = O\left(\frac{1}{\sqrt{n}}\right). \quad (\text{A.6})$$

The well-known Hoeffding's inequality is the following.

Proposition 12 (Theorem 2 of [11]). *Let X_1, X_2, \dots, X_n be independent random variables, $a_i \leq X_i \leq b_i$ ($i = 1, 2, \dots, n$). Let $\bar{X} = (X_1 + X_2 + \dots + X_n)/n$, $\mu = \mathbb{E}\bar{X}$. Then for any $t > 0$*

$$\mathbb{P}(\bar{X} - \mu \geq t) \leq \exp\left(\frac{-2n^2 t^2}{\sum_{i=1}^n (b_i - a_i)^2}\right). \quad (\text{A.7})$$

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