

A FUNCTIONAL EQUATION AND ITS APPLICATION TO THE CHARACTERIZATION OF GAMMA DISTRIBUTIONS

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Dedicated to the 70th birthday of Professor Zoltán Daróczy

ABSTRACT. The functional equation

$$f(x)g(y) = p(x+y)q\left(\frac{x}{y}\right)$$

is investigated for almost all $(x, y) \in \mathbb{R}_+^2$ and for the measurable functions $f, g, p, q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. This equation is related to the Lukács characterization of the gamma distribution.

1. INTRODUCTION

Let \mathbb{R} be the set of real numbers and \mathbb{R}_+ be the set of positive real numbers. The functional equation

$$f(x)g(y) = p(x+y)q\left(\frac{x}{y}\right), \quad (x, y) \in \mathbb{R}_+^2 \quad (1)$$

with $f, g, p, q : \mathbb{R}_+ \rightarrow \mathbb{R}$, was investigated by J. A. Baker in [1] and by K. Lajkó in [10], where the results of [2] and [3] were applied.

The purpose of this paper is to determine all measurable solutions $f, g, p, q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of equation (1) satisfied almost everywhere. This result can be used for a characterization of gamma distributions. Our proof is based on the following theorem of A. Járai (see [5], [6]):

Theorem 1 (Járai). *Let Z be a regular topological space, Z_i ($i = 1, 2, \dots, n$) be topological spaces and T be a first countable topological space. Let Y be an open subset of \mathbb{R}^k , X_i an open subset of \mathbb{R}^{r_i} , ($i = 1, 2, \dots, n$) and D an open subset of $T \times Y$. Let furthermore $T' \subset T$ be a dense subset, $F : T' \rightarrow Z$, $g_i : D \rightarrow X_i$ and $h : D \times Z_1 \times \dots \times Z_n \rightarrow Z$. Suppose that the function f_i is almost everywhere defined on X_i (with respect to the r_i -dimensional Lebesgue measure) with values in Z_i ($i = 1, 2, \dots, n$) and the following conditions are satisfied:*

(1) *for all $t \in T'$ and for almost all $y \in D_t = \{y \in Y : (t, y) \in D\}$*

$$F(t) = h(t, y, f_1(g_1(t, y)), \dots, f_n(g_n(t, y))); \quad (2)$$

(2) *for each fixed y in Y , the function h is continuous in the other variables;*

(3) *f_i is Lebesgue measurable on X_i ($i = 1, 2, \dots, n$);*

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- (4) g_i and the partial derivative $\frac{\partial g_i}{\partial y}$ are continuous on D ($i = 1, 2, \dots, n$);
 (5) for each $t \in T$ there exist a y such that $(t, y) \in D$ and the partial derivative $\frac{\partial g_i}{\partial y}$ has the rank r_i at $(t, y) \in D$ ($i = 1, 2, \dots, n$).

Then there exists a unique continuous function \tilde{F} such that $F = \tilde{F}$ almost everywhere on T , and if F is replaced by \tilde{F} then equation (2) is satisfied almost everywhere on D .

2. THE MEASURABLE SOLUTION OF (1) FOR A.E. $(x, y) \in \mathbb{R}_+^2$

First we prove the following

Lemma 1. *If the measurable functions $f, g, p, q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy equation (1) for almost all $(x, y) \in \mathbb{R}_+^2$, then there exist unique continuous functions $\tilde{f}, \tilde{g}, \tilde{p}, \tilde{q} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\tilde{f} = f$, $\tilde{g} = g$, $\tilde{p} = p$ and $\tilde{q} = q$ almost everywhere, and if f, g, p, q are replaced by $\tilde{f}, \tilde{g}, \tilde{p}, \tilde{q}$ respectively, then (1) is satisfied everywhere on \mathbb{R}_+^2 .*

Proof. First, by the help of J  rai's Theorem, we prove that there exist unique continuous function \tilde{p} which is almost everywhere equal to p on \mathbb{R}_+ and replacing p by \tilde{p} , equation (1) is satisfied almost everywhere.

With the substitution $t = x + y$ we get from (1) the equation

$$p(t) = \frac{f(t-y)g(y)}{q\left(\frac{t}{y} - 1\right)} \quad (3)$$

which is satisfied for almost all $(t, y) \in D$, where $D = \{(t, y) \in \mathbb{R}_+^2 \mid y < t\}$. By Fubini's Theorem it follows that there exists $T' \subseteq \mathbb{R}_+$ of full measure such that for all $t \in T'$ equation (3) is satisfied for almost every $y \in D_t = \{y \in \mathbb{R}_+ \mid (t, y) \in D\} = (0, t)$.

Let us define the functions g_1, g_2, g_3, h in the following way:

$$\begin{aligned} g_1(t, y) &= t - y, & g_2(t, y) &= y, \\ g_3(t, y) &= \frac{t}{y} - 1, & h(t, y, z_1, z_2, z_3) &= \frac{z_1 z_2}{z_3}, \end{aligned}$$

and let us now apply Theorem 1 of J  rai to (3) with a suitable casting.

Hence the first assumption in Theorem 1 with respect to (3) holds. It is obvious that the conditions (2)–(5) of Theorem 1 are satisfied. So we get from Theorem 1 that there exists unique continuous function \tilde{p} which is almost everywhere equal to p on \mathbb{R}_+ and f, g, \tilde{p}, q satisfy equation (1) almost everywhere, which is equivalent to the equation

$$f(x)g(y) = \tilde{p}(x+y)q\left(\frac{x}{y}\right) \quad (4)$$

for almost all $(x, y) \in \mathbb{R}_+^2$.

By a similar argument we can prove the same for the function q . From equation (4) with the substitution $t = \frac{x}{y}$ we get the equation

$$q(t) = \frac{f(ty)g(y)}{\tilde{p}(y(t+1))},$$

which with a suitable casting

$$g_1(t, y) = ty, \quad g_2(t, y) = y,$$

$$g_3(t, y) = y(t+1), \quad h(t, y, z_1, z_2, z_3) = \frac{z_1 z_2}{z_3},$$

by Fubini's Theorem, and the fact that the assumptions of Theorem 1 are fulfilled again, gives us that there exists unique continuous function \tilde{q} which is almost everywhere equal to q on \mathbb{R}_+ and $f, g, \tilde{p}, \tilde{q}$ satisfy equation (1) almost everywhere, i.e.

$$f(x)g(y) = \tilde{p}(x+y)\tilde{q}\left(\frac{x}{y}\right) \quad (5)$$

for almost all $(x, y) \in \mathbb{R}_+^2$.

There exist such x_0 and y_0 so that with the substitutions $x = x_0$ and $y = y_0$, respectively, we get from equation (5) that

$$f(x) = \frac{1}{g(y_0)}\tilde{p}(x+y_0)\tilde{q}\left(\frac{x}{y_0}\right)$$

holds for almost all $x \in \mathbb{R}_+$, and

$$g(y) = \frac{1}{f(x_0)}\tilde{p}(x_0+y)\tilde{q}\left(\frac{x_0}{y}\right)$$

holds for almost all $y \in \mathbb{R}_+$. Since \tilde{p}, \tilde{q} are continuous, therefore there exist unique continuous functions \tilde{f} and \tilde{g} , defined by the right-hand side of the last two equality, which are almost everywhere equal to f and g on \mathbb{R}_+ , respectively, and if we replace f and g by \tilde{f} and \tilde{g} , respectively, then the functional equation

$$\tilde{f}(x)\tilde{g}(y) = \tilde{p}(x+y)\tilde{q}\left(\frac{x}{y}\right) \quad (6)$$

is satisfied almost everywhere on \mathbb{R}_+^2 .

Both side of (6) define continuous functions on \mathbb{R}_+^2 , which are equal to each other on a dense subset of \mathbb{R}_+^2 , therefore we obtain that (6) is satisfied everywhere on \mathbb{R}_+^2 .

Further $f = \tilde{f}$, $g = \tilde{g}$, $p = \tilde{p}$, $q = \tilde{q}$ almost everywhere on their domains, respectively. \square

Hence, by the help of the measurable (continuous) solutions of equation (1) satisfied everywhere, we can give the solutions of the almost everywhere satisfied functional equation.

Theorem 2. *Suppose that the measurable (continuous) functions $f, g, p, q : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy functional equation (1) almost everywhere, then*

$$\begin{aligned} f(x) &= A \exp[ax + b \ln x] & a.e. & \quad x \in \mathbb{R}_+, \\ g(x) &= B \exp[ax + (c-b) \ln x] & a.e. & \quad x \in \mathbb{R}_+, \\ p(x) &= C \exp[ax + c \ln x] & a.e. & \quad x \in \mathbb{R}_+, \\ q(x) &= D \exp[b \ln x - c \ln(x+1)] & a.e. & \quad x \in \mathbb{R}_+, \end{aligned}$$

where $a, b, c \in \mathbb{R}$ and $A, B, C, D \in \mathbb{R}_+$ are arbitrary constants with $AB = CD$.

Proof. The measurable (continuous) solutions of equation (6) satisfied everywhere (see [1] and [10]) and the previous lemma immediately gives our statement. \square

3. ON CHARACTERIZATION OF GAMMA DISTRIBUTIONS

Let X be a gamma-distributed random variable with parameters λ and p , where λ and p are fixed positive numbers.

It is known that its density function is

$$f(x) = \begin{cases} \frac{\lambda^p}{\Gamma(p)} x^{p-1} e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0, \end{cases}$$

where

$$\Gamma(p) = \int_0^\infty u^{p-1} e^{-u} du$$

is the gamma function.

The well-known Lukács characterization of gamma distributions is the following: The independent random variables X and Y are gamma distributed (with the same scale parameter) if and only if $X + Y$ and $\frac{X}{X+Y}$ are independent (see [9] and [11]).

We shall use the transformation

$$\psi : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2, \quad \psi(x, y) = \left(x + y, \frac{x}{y}\right). \quad (7)$$

Let X, Y be absolutely continuous and independent random variables with range in \mathbb{R}_+ . Let us denote the densities by f_X, f_Y respectively. Then, by the Transformation Theorem (see [4]), the random variable $(U, V) = \psi(X, Y)$ is absolutely continuous with density function g defined by

$$g(u, v) := \frac{u}{(v+1)^2} f_X\left(\frac{uv}{v+1}\right) f_Y\left(\frac{u}{v+1}\right) \quad (8)$$

for almost all $(u, v) \in \mathbb{R}_+^2$.

If U and V are independent with density functions f_U, f_V respectively, then from (8) we get the functional equation

$$f_U(u) f_V(v) = \frac{u}{(v+1)^2} f_X\left(\frac{uv}{v+1}\right) f_Y\left(\frac{u}{v+1}\right) \quad \text{a.e. } (u, v) \in \mathbb{R}_+^2 \quad (9)$$

for unknown density functions $f_X, f_Y, f_U, f_V : \mathbb{R}_+ \rightarrow \mathbb{R}_+$.

This equation can be transformed easily to equation

$$f(x) g(y) = p(x+y) q\left(\frac{x}{y}\right)$$

for almost all $(x, y) \in \mathbb{R}_+^2$, which was investigated before.

From Theorem 2, we get that

$$f(x) = A \exp[ax + b \ln x] \quad \text{a.e. } x \in \mathbb{R}_+,$$

and

$$g(x) = B \exp[ax + (c-b) \ln x] \quad \text{a.e. } x \in \mathbb{R}_+.$$

Hence

$$f_X(x) = Ax^b e^{ax} \quad \text{a.e. } x \in \mathbb{R}_+,$$

and

$$f_Y(x) = Bx^{c-b-2} e^{ax} \quad \text{a.e. } x \in \mathbb{R}_+,$$

where $-a, b+1, c, A, B \in \mathbb{R}_+$ are arbitrary constants. That is, X is gamma distributed with parameters $-a$ and $b+1$, Y is gamma distributed with parameters $-a$ and $c-b-1$.

Remark 1. Moreover, from Theorem 2, we get that

$$f_U(x) = Cx^{c-1}e^{ax} \quad \text{a.e. } x \in \mathbb{R}_+,$$

where $-a, c, C \in \mathbb{R}_+$ are arbitrary constants. Hence $U = X + Y$ is gamma distributed with parameters $-a$ and c . This provides us the known property, that is, the sum of two gamma distributed independent random variables (with a common scale parameter) is gamma distributed as well.

Remark 2. The density function of $V = X/Y$ is

$$f_V(x) = Dx^b(1+x)^{-c} \quad \text{a.e. } x \in \mathbb{R}_+,$$

where $b+1, c, D \in \mathbb{R}_+$ are arbitrary constants. Hence V has beta distribution of the second kind with parameters $b+1$ and $c-b-1$. (The density function of a beta distribution of the second kind with parameters α and β is $f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1+x)^{-\alpha-\beta}$, where α, β are fixed positive numbers and B is the beta function, see [7].)

This yields again a property of the gamma distributed random variables: If X and Y are independent gamma variates with shape parameters α_1 and α_2 , respectively, then $\frac{X}{Y}$ has a beta distribution of the second kind with shape parameters α_1 and α_2 .

REFERENCES

- [1] Baker, J. A., *On the functional equation $f(x)g(y) = p(x+y)q\left(\frac{x}{y}\right)$* , Aequationes Math. 14 (1976), 493–506.
- [2] Daróczy, Z., Lajkó, K. and Székelyhidi, L., *Functional equations on ordered fields*, Publ. Math. Debrecen 24 (1977), 173–179.
- [3] Daróczy, Z. and Losonczi, L., *Über die Erweiterung der auf einer Punktmenge additive Funktionen*, Publ. Math. Debrecen 14 (1967), 239–245.
- [4] Giri, N. C., *Introduction to probability and statistics*, Marcel Dekker, New York, 1974.
- [5] Járαι, A., *Measurable solutions of functional equations satisfied almost everywhere*, Mathematica Pannonica 10/1 (1999), 103–110.
- [6] Járαι, A., *Regularity Properties of Functional Equations in Several Variables*, Springer, Adv. Math. Dordrecht, Vol. 8, 2005.
- [7] Johnson, N. L., Kotz, S. and Balakrishnan, N., *Continuous univariate distributions*, Vol. 2, 2nd Edition., John Wiley, New York, 1994.
- [8] Kotlarski, I. I., *Una caratterizzazione della distribuzione gamma per mezzo di statistiche indipendenti*, Rendiconti di Matematica 3–4/2 (1969), 671–675.
- [9] Lajkó, K., *A characterization of generalized normal and gamma distributions*, Colloq. Math. Soc. J. Bolyai 21. Analytic Function Methods in Prob. Theory (1979), 199–225.
- [10] Lajkó, K., *Remark to a paper by J. A. Baker*, Aequationes Math. 19 (1979), 227–231.
- [11] Lukács, E., *A characterization of the gamma distribution*, Ann. Math. Statist. 26 (1955), 319–324.
- [12] Olkin, I., *Problem (P128)*, Aequationes Math. 12 (1975), 290–292.

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