Received: 14 October 2022 | Accepted: 21 February 2023

DOI: 10.1112/jlms.12746

RESEARCH ARTICLE

Journal of the London Mathematical Society

The Diophantine equation $f(x) = g(y)$ for polynomials with simple rational roots

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Funding information
Eötvös Loránd Research Network (ELKH); NKFIH, Grant/Award Numbers: 128088, 130909; European Union, Grant/Award Number: EFOP-3.6.1-16-2016-00022; European Social Fund

Abstract

In this paper we consider Diophantine equations of the form $f(x) = g(y)$ where f has simple rational roots and g has rational coefficients. We give strict conditions for the cases where the equation has infinitely many solutions in rationals with a bounded denominator. We give examples illustrating that the given conditions are necessary. It turns out that such equations with infinitely many solutions are strongly related to Prouhet–Tarry–Escott tuples. In the special, but important case when g has only simple rational roots as well, we can give a simpler statement. Also we provide an application to equal products with terms belonging to blocks of consecutive integers of bounded length. The latter theorem is related to problems and results of Erdős and Turk, and of Erdős and Graham.

MSC 2020
11N32, 11B75, 11D41, 11P05 (primary)

1 | INTRODUCTION

Let $a_1, \dots, a_k$ be distinct rationals and $a_0 \in \mathbb{Q}$ with $a_0 \neq 0$. Put

$$f(x) = a_0(x - a_1) \cdots (x - a_k) \tag{1}$$

and let $g(y) \in \mathbb{Q}[y]$. In this paper we investigate for which f, g equation

$$f(x) = g(y) \tag{2}$$

has infinitely many solutions. Moreover, we study for which f, g this holds if g is of the form

$$g(y) = b_0(y - b_1) \cdots (y - b_\ell), \tag{3}$$

where $b_1, \ldots, b_\ell$ are distinct elements of $\mathbb{Q}$ and $b_0 \in \mathbb{Q}$ with $b_0 \neq 0$. We say that an equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator if there exists a positive integer Δ such that $f(x) = g(y)$ has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with $(\Delta x, \Delta y) \in \mathbb{Z}^2$. Our focus is the question for which f, g Equation (2) has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator.

Using results of Bilu and Tichy [11] and of Davenport, Lewis and Schinzel [24], both based on a theorem of Siegel [75], we prove the following theorem.

Theorem 1.1. *Let $f(x) \in \mathbb{Q}[x]$ have only simple rational roots and let $g(x) \in \mathbb{Q}[x]$. Suppose the equation $f(x) = g(y)$ has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator.*

Then there exist positive integers m, n, s with $m \in \{1, 2, 3, 4, 6\}$ or $n \in \{1, 2\}$ such that $\deg(f) = ms$, $\deg(g) = ns$.

If g has also only simple rational roots and $\deg(f) \leqslant \deg(g)$, then there exist $m \in \{1, 2\}$, $n, s \in \mathbb{Z}_{>0}$ such that $\deg(f) = ms$, $\deg(g) = ns$.

The first statement will be proved in Section 7. After the proof we shall argue that if $m \in \{1, 2, 3, 4, 6\}$, then for every such m, n, s a pair of polynomials (f, g) can be constructed, f having only simple integral roots, such that $f(x) = g(y)$ has infinitely many integral solutions (x, y). For the remaining cases, see Section 11.

The second statement will be proved in Section 9. Observe that it follows that $\deg(f) \mid 2\deg(g)$.

As illustration of Theorem 1.1 we present some non-trivial examples. Later more examples will follow.

Example 1.1 ([cf. Example 5.2]). An example of the second statement where $\deg(f)$ does not divide $\deg(g)$. Let

$$f(x) = (x - 6)(x + 6), \quad g(y) = (y - 1)(y - 4)(y - 9).$$

Then $f(x) = g(y)$ has solution

$$(x, y) = (X(X^2 - 7), X^2) \quad \text{for every } X \in \mathbb{Z}.$$

Example 1.2 ([cf. Example 5.3]). An example of the second statement where $\deg(f)$ divides $\deg(g)$. Let

$$f(x) = (x - 7)(x - 1)(x + 1)(x + 7), \quad g(y) = 4(y - 5)(y - 1)(y + 1)(y + 5).$$

Consider the Pell equation $x^2 = 2y^2 - 1$. It has solutions $(X_i, Y_i)_{i=1}^{\infty}$ given by $(X_1, Y_1) = (1, 1)$, $(X_2, Y_2) = (7, 5)$ and

$$X_{i+1} = 6X_i - X_{i-1}, \quad Y_{i+1} = 6Y_i - Y_{i-1} \quad (i = 2, 3, \dots).$$

The equation $f(x) = g(y)$ has a solution

$$(x, y) = (X_i, Y_i) \quad \text{for every } i \in \mathbb{Z}_{>0}.$$

Example 1.3 ([cf. Example 7.3]). An example of the first statement for $m = 3$, $n = 4$ and $s = 1$. Let

$$f(x) = (x + 286)(x + 13)(x - 299), \quad g(y) = y^4 - 8788y^2 + 8\,541\,936.$$

For every $X \in \mathbb{Z}$ there is a solution

$$(x, y) = (X^4 - 52X^2 + 338, X^3 - 39X).$$

In Section 2 we give a historical overview of the literature on equations $f(x) = g(y)$ where f has only simple rational roots. In Section 3 we present the Bilu–Tichy decomposition [11] which is fundamental for our treatment. Bilu and Tichy attach a standard pair of polynomials (F, G) to each equation $f(x) = g(y)$ which has infinitely many solutions with a bounded denominator. They distinguish five kinds of standard pairs. We exclude the fifth kind and rephrase Theorem 1.1 as Lemma 3.2. In Section 4 we present Prouhet–Tarry–Escott (PTE) sets, an extension of ideal PTE pairs. In Section 5 we consider standard pairs of the first and second kinds where g need not satisfy (3). In the next section we assume that g satisfies (3) too. Section 7 deals with standard pairs of the third and fourth kinds where g need not satisfy (3). In particular, we prove here the first statement of Theorem 1.1. This part of our argument is the most involved. Here we need to give a complete description of shifts of Dickson polynomials having only rational roots. For this, we need to combine certain identities for such polynomials with various (both theoretical and computational) tools from algebraic number theory. Section 8 restricts the cases with standard pairs of the third and fourth kinds if g has only simple rational roots, too. In Section 9 we give a more precise statement than Theorem 1.1 under (3) which completes the proof of Theorem 1.1. We give an application of our results to equal products with terms belonging to blocks of consecutive integers of bounded lengths in Section 10. We finish the paper with some open problems.

2 | HISTORICAL OVERVIEW

There are numerous publications on the title equation where f has only simple rational roots. In many of them the roots of f, and often also of g, are well structured and all solutions are found. In other papers only finiteness of the number of such solutions is considered. The present paper deals with the finiteness of the number of solutions for a wide class of equations covering the equations cited in this overview. Subsections 2.1 and 2.2 correspond to Section 5, Subsections 2.3 and 2.4 to Sections 6, 8 and 9, Subsection 2.5 to Section 10.

2.1 | The roots of f form an arithmetic progression and g is almost a perfect power

First we consider the case that the roots of f form an arithmetic progression and g is almost a perfect power, more precisely:

$$x(x+d)\cdots(x+(k-1)d) = b_0y^{\ell} + b_{\ell}, \tag{4}$$

where b_0, b_{ℓ}, d, k and ℓ are integers with $k > 1, \ell > 1, k\ell > 4, b_0 \neq 0$, ℓth power free, the greatest prime factor of b_0 is at most k and solutions $(x, y) \in \mathbb{Z}^2$ satisfy $\gcd(x, d) = 1, |y| > 1$. (If $k = \ell = 2$, then we may have a Pell equation which has infinitely many solutions.) If $b_{\ell} = 0$, then there are only finitely many solutions according to a theorem of Siegel [74] if $\ell > 2$ and by a result of Schinzel [71], Corollary 7 if $\ell = 2$.

Let $d = 1$. In 1975 Erdős and Selfridge [30] proved that Equation (4) has no solutions when $b_0 = 1, b_{\ell} = 0$. Erdős [28] and Győry [34] showed that the equation $\binom{x+k-1}{k} = y^{\ell}$, which agrees with the case $b_0 = k!, b_{\ell} = 0$, has only the solution $\binom{50}{3} = 140^2$. Saradha [60] and Győry [35] dealt with Equation (4) with $b_0 > 1, b_{\ell} = 0$. Bilu, Kulkarny and Sury [10] proved that Equation (4) has only finitely many solutions (k, ℓ, m, n) if b_{ℓ} is not a perfect power and that all solutions can be explicitly determined. For more results with $d = 1$ see [21, 38, 81].

Next let $d > 1, b_0 = 1, b_{\ell} = 0$. A famous result due to Euler is that the product of $k = 4$ distinct positive integers in arithmetic progression cannot be a square. For a generalization of this result to $4 \leqslant k \leqslant 109$, see [7, 36, 56] and finally [47]. Similar results for $\ell = 3$ and $\ell = 5$ can be found in [44] and [39], respectively. Euler's result has been extended for arbitrary powers ℓ and $k \leqslant 34$ by Győry, Hajdu and Saradha [37], Bennett, Bruin, Győry and Hajdu [7] and Győry, Hajdu and Pintér [36]. Bennett [6] obtained the following strong finiteness result: There exist at most finitely many integer tuples d, k, ℓ, x, y, with $4 \leqslant k \leqslant 15\,177$ for which Equation (4) is satisfied. Bennett and Siksek [8] proved that there exists an effectively computable k_0 such that for fixed $k > k_0$ there are only finitely many integers d, ℓ, x, y satisfying Equation (4). For related papers see [32, 60, 61, 72].

Case $d > 1$, $b_0 > 1$. Saradha and Shorey [66] proved that for d at most some explicitly given $d_0 = d_0(\ell)$ and $b_{\ell} = 0$ Equation (4) has no solutions. It follows from Yuan [81] that if $k \geqslant 8$ then all solutions of (4) satisfy $\max(|x|, |y|, \ell) < C$ where C is an effectively computable constant depending only on k, b_0, b_{ℓ}. For other results with $b_{\ell} = 0$ see [32, 49, 55, 65, 66], and for general b_{ℓ} the survey [73].

2.2 | The roots of f form almost an arithmetic progression and g is almost a perfect power

First we turn to the case that the roots of f form an arithmetic progression with some terms missing, more precisely, to the equation

$$(x+d_1d)\cdots(x+d_kd) = b_0y^{\ell} + b_{\ell}, \tag{5}$$

where $0 \leqslant d_1 < d_2 < \cdots < d_k < K, d, b_0, b_{\ell}$ and ℓ are integers with $k > 2, \ell > 1, b_0$ is ℓth power free, the greatest prime factor of b_0 is at most k and solutions $(x, y) \in \mathbb{Z}^2$ satisfy $\gcd(x, d) = 1$.

Several papers deal with the case $K - k = 1$. Saradha and Shorey [63], Hanrot, Saradha and Shorey [46] and Bennett [5] together proved that for $d = K - k = b_0 = 1, b_\ell = 0$ the only solutions of (5) are given by $4!/3 = 2^3$, $6!/5 = 12^2$, $10!/7 = 720^2$. For other papers with $K - k = 1$, $b_\ell = 0$ see [23, 64, 65, 67, 69]. Hajdu and Papp [40] proved that Equation (5) with $K - k = 1$, $K \geqslant 8$ has only finitely many solutions x, y, ℓ.

All solutions of Equation (5) with $K - k = 2$, $k \geqslant 4$, $\ell \geqslant 3$ have been given in [55] and [68]. For papers with $K - k \geqslant 2, b_\ell = 0$ see [3] and [22]. Hajdu, Papp and Tijdeman [41] provided effective upper bounds for $\max(|x|, |y|, \ell)$ in (5) under the assumption that $K - k < cK^{2/3}$ for some explicit $c > 0$.

For results and history concerning the case when instead of omitting terms from an arithmetic progression we have an extra term, see [45] and the references there.

2.3 | Both f and g have simple rational roots almost in arithmetic progressions

In the literature many papers deal with special cases of the equation

$$a_0 x(x + d_1) \cdots (x + (k-1)d_1) = b_0 y(y + d_2) \cdots (y + (\ell - 1)d_2), \tag{6}$$

where k, ℓ, a_0, b_0 are integers with $1 < k \leqslant \ell$, $a_0 b_0 \neq 0$, and d_1, d_2 are positive integers with $d_1 \neq d_2$ if $k = \ell$.

First the case $a_0 = b_0 = d_1 = d_2 = 1$ attracted attention. For these values Mordell [54], Boyd and Kisilevsky [14] and Hajdu and Pintér [42] computed all positive solutions for $(k, \ell) = (2, 3), (3, 4)$ and $(4,6)$, respectively. Saradha and Shorey [62] proved that the only solution with $\ell = 2k$ is given by $(k, \ell, x, y) = (3, 6, 8, 1)$. They, together with Mignotte (see [53]) determined all solutions in case $\ell/k \in \{3, 4, 5, 6\}$.

Saradha, Shorey and Tijdeman [70] studied the cases $a_0 = b_0 = 1$, $d_1 = 1$, $d_2 > 1$, ℓ/k is integral. Beukers, Shorey and Tijdeman [9] proved that Equation (6) with $a_0 = b_0 = 1$ admits only finitely many positive integral solutions x, y except for the infinite class of solutions $x = y^2 + 3d_2 y$ when $k = 2$, $\ell = 4$ and $d_1 = 2d_2^2$. By a similar reasoning the restriction $a_0 = b_0 = 1$ can be replaced by $\ell > 2$.

Brindza and Pintér [17] showed that the equation

$$x(x + 1) \cdots (x + k - 1) = \binom{y}{\ell}$$

for $k > 2, \ell > 2$ has only finitely many solutions in positive integers x, y. This corresponds to the choice $a_0 = 1, b_0 = k!, d_1 = d_2 = 1$.

By taking $a_0 = \ell!, b_0 = k!, d_1 = d_2, m = x + k - 1, n = y + \ell - 1$ in (6) the question becomes which binomial coefficients $\binom{m}{k}$ and $\binom{n}{\ell}$ are equal. Without loss of generality we assume $1 < k < \ell, k \leqslant m/2, \ell \leqslant n/2$. For several pairs (k, ℓ) all solutions were found, see, for example, [1, 54, 78] and [18]. Gallegos-Ruiz, Katsipis, Tengely and Ulas [33] described all binomial coefficients $\binom{m}{k}$, $\binom{n}{\ell}$ with $(k, \ell) = (2, 3), (2,4), (2,6), (2,8), (3,4), (3,6), (4,6), (4,8)$ whose difference is at most three. Surveys on (almost) equal binomial coefficients can be found in Blokhuis, Brouwer, de Weger [12][†] and Gallegos-Ruiz *et al.* [33].

[†] In their list on page 2 the sporadic solution $n = 78, k = 2, m = 15, \ell = 5$ is missing.

For a generalization related to figurate numbers see Hajdu, Pintér, Tengely and Varga [43] and the references there.

2.4 | The roots of f are simple and rational and $g(y) \in \mathbb{Q}[y]$

Consider the equation

$$f(x) := (x + d_1 d) \cdots (x + d_k d) = g(y) \tag{7}$$

in integers x, y where $d, k, K, d_1, d_2, \dots, d_k$ are integers with $0 \leqslant d_1 < d_2 < \cdots < d_k < K$ and $k > 2$, $g(y) \in \mathbb{Q}[y]$ of degree $\ell \geqslant 2$. Kulkarni and Sury [48] for $d = 1, k = K, \ell > 2$ completely described all cases where (7) has infinitely many solutions. Hajdu, Papp and Tijdeman [41] proved the finiteness of the number of solutions of (7) under the assumption that $K - k \leqslant cK^{2/3}$ with c an explicit constant, except for two explicitly given classes of gs. Both papers are based on a theorem of Bilu and Tichy [11], which will also play an important role in our present study and is formulated in the next section. For other papers related to (7) see [3, 16, 58, 59, 77]. For finiteness results on similar equations related to figurate numbers, see [43, 45], and the references there.

2.5 | Power values and equal values of products with terms coming from an interval

Finally, we recall some papers and results from the literature concerning products of terms coming from blocks of consecutive integers.

Erdős and Turk [31] studied the existence of terms from a 'short' interval I having a power product, and also the existence of two distinct sets of integers in I with equal product. Roughly speaking, they proved that these properties never hold for 'very short' intervals; that for 'medium-sized' intervals they hold in infinitely many cases and fail in infinitely many cases, and that they always hold if the size of I is 'large enough'. They gave precise formulas for these sizes.

Another problem of somewhat similar flavor is due to Erdős and Graham [29] who asked when the product of two or more disjoint blocks of consecutive integers can be a power. Ulas [80] exhibited families of blocks of precisely four integers whose product gives perfect squares. Bauer and Bennett [4] described the 'minimal examples' yielding perfect square products. For related results, see [76, 79] and the references there.

3 | THE BILU–TICHY THEOREM

We say that a polynomial f as in (1) is symmetric, if there exists an $a \in \mathbb{Q}$ such that the set $\{a_1, \dots, a_k\}$ is symmetric around a.

We call polynomials $f, \tilde{f} \in \mathbb{Q}[x]$ similar if there exist $a, b \in \mathbb{Q}$, $a \neq 0$ such that $f(x) = \tilde{f}(ax + b)$. Notation $f \simeq \tilde{f}$. Obviously this induces an equivalence relation on $\mathbb{Q}[x]$. Observe that if f has only simple rational roots, then $\tilde{f}$ has only simple rational roots too. In every equivalence class there are polynomials with sum of roots equal to 0. Moreover, if the roots of f are all rational, then there exists a similar polynomial $\tilde{f}(x) \in \mathbb{Z}[x]$ of which the roots are integers with sum 0. If

TABLE 1 Standard pairs. Here α, β are non-zero rational numbers, μ, ν, q are positive integers, p is a non-negative integer, $v(x) \in \mathbb{Q}[x]$ is a non-zero, but possibly constant polynomial.

Kind	Standard pair (unordered)	Parameter restrictions
First	$(x^q, \alpha x^p v(x)^q)$	$0 \leqslant p < q$, $\gcd(p, q) = 1$,
		$p + \deg(v) > 0$
Second	$(x^2, (\alpha x^2 + \beta)v(x)^2)$	-
Third	$(D_\mu(x, \alpha^\nu), D_\nu(x, \alpha^\mu))$	$\gcd(\mu, \nu) = 1$
Fourth	$(\alpha^{-\mu/2} D_\mu(x, \alpha), -\beta^{-\nu/2} D_\nu(x, \beta))$	$\gcd(\mu, \nu) = 2$
Fifth	$((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$	-

the polynomial equation $f(x) = g(y)$ has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator and $f \simeq \tilde{f}$, $g \simeq \tilde{g}$, then the equation $\tilde{f}(x) = \tilde{g}(y)$ has also infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator. We call equations $f(x) = g(y)$ and $\tilde{f}(x) = \tilde{g}(y)$ with $f \simeq \tilde{f}$, $g \simeq \tilde{g}$ similar equations.

We call $f(x) \in \mathbb{Q}[x]$ decomposable over $\mathbb{Q}$ if there exist $G(x), H(x) \in \mathbb{Q}[x]$ with $\deg(G) > 1$, $\deg(H) > 1$ such that $f = G(H)$, and otherwise indecomposable. Since $\deg(f) = \deg(G) \cdot \deg(H)$, f is indecomposable if $\deg(f)$ is prime.

Let δ be a non-zero rational number and μ be a positive integer. Then the μth Dickson polynomial is defined by

$$D_\mu(x, \delta) := \sum_{i=0}^{\lfloor \mu/2 \rfloor} d_{\mu,i} x^{\mu-2i} \quad \text{where } d_{\mu,i} = \frac{\mu}{\mu - i} \binom{\mu - i}{i} (-\delta)^i.$$

For properties of Dickson polynomials see, for example, [52].

In this section we prove a variant of Theorem 1.1. In the proof the following result of Bilu and Tichy [11] on Equation (2) is crucial. Here the polynomials $F, G \in \mathbb{Q}[x]$ form a standard pair over $\mathbb{Q}$ if either $(F(x), G(x))$ or $(G(x), F(x))$ appears in Table 1.

Lemma 3.1 Bilu, Tichy [11, Theorem 1.1]. *Let $f(x), g(x) \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following two statements are equivalent.*

(I) *The equation $f(x) = g(y)$ has infinitely many rational solutions x, y with a bounded denominator.*

(II) *We have $f = \varphi(F(\kappa))$ and $g = \varphi(G(\lambda))$, where $\kappa(x), \lambda(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $F(x), G(x)$ form a standard pair over $\mathbb{Q}$ such that the equation $F(x) = G(y)$ has infinitely many rational solutions with a bounded denominator.*

Observe that $F(\kappa) \simeq F$ and $G(\lambda) \simeq G$. The Bilu–Tichy theorem implies that if (I) holds then the equation $F(\kappa(x)) = G((\lambda(y))$ has infinitely many rational solutions with a bounded denominator. The converse is obvious.

In Theorem 1.1 one may read $m = \deg(F), n = \deg(G), s = \deg(\varphi)$.

An interesting result in connection with Lemma 3.1 is due to Avanzi and Zannier [2]. Namely, [2, Theorem 1] implies that if the equation $f(x) = g(y)$ with $f(x), g(x) \in \mathbb{Q}[x]$, $\gcd(k, \ell) = 1$ and $k, \ell > 6$ has infinitely many rational solutions, then infinitely many of them have a bounded denominator (cf. Bilu's MathSciNet review MR1845348 of that paper).

We start with investigating when the equation

$$F(x) = G(y) \tag{8}$$

for standard pairs (F, G) has infinitely many solutions (x, y) with a bounded denominator in our settings. Lemma 3.2 shows that condition (1) restricts the possibilities.

Lemma 3.2. *Suppose f is of the form (1) and Equation (2) has infinitely many rational solutions with a bounded denominator. Let (F, G) be a corresponding standard pair. Then one of the following cases holds:*

1) *(F, G) is of the first or second kind,* $\min(\deg(F), \deg(G)) \leqslant 2$,
2) *(F, G) is of the third or fourth kind.*

Proof. Without loss of generality we may assume $f = \varphi(F)$, $g = \varphi(G)$. Since f has only simple rational roots, $f' = \varphi'(F)F'$ has only simple real roots. Hence F' has only simple real roots. If (F, G) is of the fifth kind, then F' has a multiple root and so the fifth kind is excluded. Therefore, if we are not in case 2), we have a pair (F, G) of the first or second kind. By case 1) we may assume that $\deg(F) \geqslant 3$ and $\deg(G) \geqslant 3$. Then (F, G) is not of the second kind. If (F, G) is of the first kind, then $q \geqslant 3$ and if $\deg(v) = 0$ then $p \geqslant 3$. However, then F' has a multiple root, which is not the case. □

Remark 3.1. It follows that if (F, G) is a standard pair of the first or second kind, then $\deg(f) \mid 2\deg(g)$ or $\deg(g) \mid 2\deg(f)$.

Remark 3.2. In Examples 1.1, 1.2, 1.3 we may take

$$F(x) = x^2, \ \ G(y) = y(y-7)^2, \ \ \varphi(x) = x - 36, (\text{first kind})$$

$$F(x) = x^2, \ \ G(y) = 2y^2 - 1, \ \ \varphi(x) = (x-1)(x-49), (\text{second kind})$$

$$F(x) = D_3(x, 13^4), \ \ G(y) = D_4(y, 13^3), \ \ \varphi(x) = x - 1\,111\,682, (\text{third kind})$$

respectively.

4 | PTE$_{m,s}$ SETS

An ideal PTE pair is a pair of sets of an equal number of distinct integers, $\alpha_1, \alpha_2, \ldots, \alpha_m$ and $\beta_1, \beta_2, \ldots, \beta_m$ say, such that

$$\sum_{i=1}^{m} \alpha_i^j = \sum_{i=1}^{m} \beta_i^j$$

for $j = 1, 2, \ldots, m-1$. Ideal PTE pairs are known for $2 \leqslant m \leqslant 10$ and for $m = 12$. For general information on such pairs we refer to [57]. In this section we study the case of s tuples of m distinct integers having the same sums of jth powers for $1 \leqslant j \leqslant m-1$, a so-called PTE$_{m,s}$ set. An ideal

PTE pair of each m integers is therefore a $\mathrm{PTE}_{m,2}$ set. Observe that a $\mathrm{PTE}_{m,s}$ set remains a $\mathrm{PTE}_{m,s}$ set if all elements are multiplied by the same constant and also if a constant is added to all numbers. We call such $\mathrm{PTE}_{m,s}$ sets equivalent. If there exists a $\mathrm{PTE}_{m,s}$ set of rationals, then there exists an equivalent $\mathrm{PTE}_{m,s}$ set of integers. The following result will be useful in the sequel.

Lemma 4.1. *For $m \in \{3, 4, 6\}$ and every positive integer s there exist infinitely many equivalence classes of $\mathrm{PTE}_{m,s}$ sets.*

To prove the case $m = 4$ we use the following result.

Lemma 4.2 [51, Theorem 7.5]. *Let M be the product of ρ distinct primes of the form $\equiv 1 \pmod 4$. Then the number of representations of M as $\alpha_1^2 + \alpha_2^2$ with $\alpha_1, \alpha_2 \in \mathbb{Z}$, $\alpha_1 > \alpha_2 > 0$, $\gcd(\alpha_1, \alpha_2) = 1$ equals $2^{\rho-1}$.*

Proof of Lemma 4.1 for $m = 4$. Choose a ρ with $2^{\rho-1} \geqslant s$ and ρ primes $\equiv 1 \pmod 4$. Call their product M. Obviously infinitely many choices of the primes are possible. According to Lemma 4.2 there exist s pairs of integers $\alpha_{1,i}, \alpha_{2,i}$ with $\alpha_{1,i} > \alpha_{2,i} > 0$, $\gcd(\alpha_{1,i}, \alpha_{2,i}) = 1$ such that $\alpha_{1,i}^2 + \alpha_{2,i}^2 = M$ for $i = 1, 2, \dots, s$. Put $\alpha_{3,i} = -\alpha_{1,i}, \alpha_{4,i} = -\alpha_{2,i}$ for all i. Then $\sum_{h=1}^4 \alpha_{h,i} = 0$, $\sum_{h=1}^4 \alpha_{h,i}^2 = 2M$, $\sum_{h=1}^4 \alpha_{h,i}^3 = 0$ for $i = 1, 2, \dots, s$. □

Example 4.1. We apply Lemma 4.2 with $\rho = 3$ and primes 5,13,17. We have

$$5 \cdot 13 \cdot 17 = 1105 = x^2 + y^2 \text{ for } (x, y) = (33, 4), (32, 9), (31, 12), (24, 23).$$

Therefore the sets

$$\{-33, -4, 4, 33\}, \{-32, -9, 9, 32\}, \{-31, -12, 12, 31\}, \{-24, -23, 23, 24\}$$

form a $\mathrm{PTE}_{4,4}$ set.

For the cases $m = 3$ and 6 we use the following analogue of Lemma 4.2.

Lemma 4.3 [26, para 48, item 4]. *Let M be the product of ρ distinct primes of the form $\equiv 1 \pmod 6$. Then the number of representations of M as $\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2$ with $\alpha_1, \alpha_2 \in \mathbb{Z}$, $\alpha_1 > \alpha_2 > 0$, $\gcd(\alpha_1, \alpha_2) = 1$ equals $2^{\rho-1}$.*

Proof of Lemma 4.1 for $m = 6$. Choose a ρ with $2^{\rho-1} \geqslant s$. Let M be the product of ρ distinct primes of the form $\equiv 1 \pmod 6$. Clearly, we may choose such primes in infinitely many ways. The number of representations of M as $x^2 + xy + y^2$ with coprime integers x, y with $x > y > 0$ equals $2^{\rho-1}$. Choose s such pairs, (x_i, y_i) for $i = 1, 2, \dots, s$. Then $x_i^2 + y_i^2 + (x_i + y_i)^2 = 2M$ and

$$x_i^4 + y_i^4 + (x_i + y_i)^4 = 2(x_i^2 + x_i y_i + y_i^2)^2 = 2M^2$$

(cf. [19, Section 4]). Put $\alpha_{3,i} = \alpha_1 + \alpha_2, \alpha_{4,i} = -\alpha_{1,i}$, $\alpha_{5,i} = -\alpha_{2,1}$, $\alpha_{6,i} = -\alpha_{3,i}$ for $i = 1, 2, \dots, s$. Then $\sum_{h=1}^6 \alpha_{h,i} = 0$, $\sum_{h=1}^6 \alpha_{h,i}^2 = 4M$, $\sum_{h=1}^4 \alpha_{h,i}^3 = 0$, $\sum_{h=1}^4 \alpha_{h,i}^4 = 4M^2$, $\sum_{h=1}^4 \alpha_{h,i}^5 = 0$ for $i = 1, 2, \dots, s$. Thus this yields a $\mathrm{PTE}_{6,s}$ set. □

Example 4.2. We have

$$7 \cdot 13 \cdot 19 = 1729 = x^2 + xy + y^2 \text{ for } (x, y) = (40, 3), (37, 8), (32, 15), (25, 23).$$

Thus the sets

$$\{\pm 40, \pm 3, \pm 43\}, \{\pm 37, \pm 8 \pm 45\}, \{\pm 32, \pm 15, \pm 47\}, \{\pm 25, \pm 23, \pm 48\}$$

each have sum 0, sum of squares $4 \cdot 1729$, sum of cubes 0, sum of biquadrates $4 \cdot 1729^2$, sum of fifth powers 0.

Proof of Lemma 4.1 for $m = 3$. We use the notation introduced in the proof of the case $m = 6$. Consider the triples

$$(M + x_i(y_i - x_i), -M + y_i(y_i - x_i), x_i^2 - y_i^2) \quad (i = 1, 2, \dots, s).$$

Each triple has sum 0 and sum of squares

$$2M^2 - 2M(x_i^2 - 2x_i y_i + y_i^2) + 2x_i^4 - 2x_i^3 y_i - 2x_i y_i^3 + 2y_i^4.$$

Using that $M = x_i^2 + x_i y_i + y_i^2$, we obtain that the sums of squares equal $2M^2$. Of course, this is also true for the opposite triples

$$-(M + x_i(y_i - x_i)), \quad M - y_i(y_i - x_i), \quad y_i^2 - x_i^2 \quad (i = 1, 2, \dots, s)$$

and for $\{-M, 0, M\}$. Thus we have a $\mathrm{PTE}_{3,s+1}$ set, and maybe even a $\mathrm{PTE}_{3,2s+1}$ set. □

Example 4.3. We start again from the pairs

$$(x, y) = (40, 3), \quad (37, 8), \quad (32, 15), \quad (25, 23)$$

from Example 4.2 which each satisfy $M = x^2 + xy + y^2 = 1729$. According to the above rules they lead to the nine triples

$$\pm(249, -1840, 1591), \ \pm(656, -1961, 1305),$$

$$\pm(1185, -1984, 799), \ \pm(1679, -1775, 96), \ (-1729, 0, 1729),$$

which each have sum 0 and sum of squares $2 \cdot 1729^2$. We obtain a $\mathrm{PTE}_{3,9}$ set.

Let $f(x) \in \mathbb{Q}[x]$ with only simple rational zeros be decomposable over $\mathbb{Q}$ as $\varphi(F(x))$ (cf. Lemma 3.1). Let

$$\varphi(x) = p_0(x - p_1)(x - p_2) \cdots (x - p_s)$$

with $s > 0$, $p_0 \in \mathbb{Q}$ ($p_0 \neq 0$) and $p_i \in \mathbb{C}$ ($i = 1, \dots, s$). Then

$$f(x) = p_0(F(x) - p_1)(F(x) - p_2) \cdots (F(x) - p_s).$$

From this, we see that $p_i \in \mathbb{Q}$ ($i = 1, \dots, s$), and that these numbers are distinct. Further, writing $F_i(x) = F(x) - p_i$ for $i = 1, 2, \dots, s$ we obtain that $F_1(x), F_2(x), \dots, F_s(x) \in \mathbb{Q}[x]$ are such that

$F_i(x)/F_j(x) \notin \mathbb{Q}$, $F_i(x) - F_j(x) \in \mathbb{Q}$ for $1 \leqslant i < j \leqslant s$ and, moreover, $F_i(x)$ has only simple rational roots for $1 \leqslant i \leqslant s$. These polynomials have the same degree, m say. It follows that there are rationals $r_1, r_2, \dots, r_m$ independent of i such that $F_i(x) = r_m x^m + r_{m-1}x^{m-1} + \cdots + r_1 x + f_i$ for all i with $f_1, f_2, \dots, f_s \in \mathbb{Q}$ distinct. Then, by the formulas of Newton–Girard, the roots of $F_1, F_2, \dots, F_s$ form a $\mathrm{PTE}_{m,s}$ set. We call f a $\mathrm{PTE}_{m,s}$-polynomial, $\{F_1, F_2, \dots, F_s\}$ a $\mathrm{PTE}_{m,s}$ polynomial set and F a $\mathrm{PTE}_{m,s}$ base of f. Of course, $\deg(f) = ms$. In the literature $\mathrm{PTE}_{m,2}$ polynomial sets are mentioned; see [57].

We apply Lemma 4.1 in the following way.

Corollary 4.1. *For $m \in \{2, 3, 4, 6\}$ and every positive integer s there exists a polynomial $F(x) \in \mathbb{Z}[x]$ of degree m and s integers $f_1, f_2, \dots, f_s$ such that $F(x) + f_i$ has only simple integer roots for $i = 1, 2, \dots, s$.*

Proof. For $m = 2$ we choose $F(x) = x^2$ and $f_i = -i^2$ for $i = 1, 2, \dots, s$.

According to Lemma 4.1 there exists a $\mathrm{PTE}_{m,s}$ set of integers $\{H_1, H_2, \dots, H_s\}$ for $m \in \{3, 4, 6\}$ and all $s \in \mathbb{Z}_{>0}$. Consider the s monic polynomials $P_i(x)$ with the elements of H_i as roots for $i = 1, 2, \dots, s$. Then they differ only by a constant. So we can take $F(x) = P_1(x) - P_1(0)$ and $f_i = P_i(x)$ $(i = 1, \dots, s)$. □

Examples 4.1–4.3 continued. Let $m = s = 4$. The polynomial $P(x) = x^4 - 1105x^2$ has simple rational roots when 17 424, 82 944, 138 384 or 304 704 is added, since the corresponding polynomials equal

$$(x^2 - 33^2)(x^2 - 4^2), (x^2 - 32^2)(x^2 - 9^2), (x^2 - 31^2)(x^2 - 12^2), (x^2 - 24^2)(x^2 - 23^2),$$

respectively.

Let $m = 6, s = 4$. The polynomial $P(x) = x^6 - 2 \cdot 1729x^4 + 1729^2x^2$ has simple integer roots when 26 625 600, 177 422 400, 508 953 600 or 761 760 000 is subtracted, since the corresponding polynomials equal

$$(x^2 - 3^2)(x^2 - 40^2)(x^2 - 43^2), \quad (x^2 - 8^2)(x^2 - 37^2)(x^2 - 45^2),$$

$$(x^2 - 15^2)(x^2 - 32^2)(x^2 - 47^2), \quad (x^2 - 23^2)(x^2 - 25^2)(x^2 - 48^2),$$

respectively.

Let $m = 3, s = 9$. The polynomial $P(x) = x^3 - 1729^2x$ has simple integer roots when one from

$$0, \quad \mp 728\,932\,560, \quad \mp 1\,678\,772\,880, \quad \mp 1\,878\,480\,960, \quad \mp 286\,101\,600$$

is added, as we get the polynomials $(x - 1729)x(x + 1729)$,

$$(x \pm 249)(x \mp 1840)(x \pm 1591), \quad (x \pm 656)(x \mp 1961)(x \pm 1305),$$

$$(x \pm 1185)(x \mp 1984)(x \pm 799), \quad (x \pm 96)(x \mp 1775)(x \pm 1679),$$

respectively. (Either triple upper signs or triple lower signs.) □

5 | STANDARD PAIRS OF THE FIRST OR SECOND KIND

In this section we return to the original problem on Equation (2) subject to (1) and show by the help of examples that all cases of the first or second kind which are not excluded may indeed occur. Suppose the equation $f(x) = g(y)$ with $f(x), g(x) \in \mathbb{Q}[x]$ has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator. According to Lemma 3.1 we have $f = \varphi(F(\kappa))$ and $g = \varphi(G(\lambda))$, where $\kappa(x), \lambda(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $F(x), G(x)$ form a standard pair over $\mathbb{Q}$ such that the equation $F(x) = G(y)$ has infinitely many rational solutions with a bounded denominator. In the sequel we suppose that $f = \varphi(F)$ and $g = \varphi(G)$. The results then extend to all equations similar to the equation $\varphi(F(x)) = \varphi(G(y))$, in particular to the original equation $f(x) = g(y)$.

Let $\varphi(x) = p_0(x - p_1) \cdots (x - p_s)$ with $p_0 \in \mathbb{Q}$ $(p_0 \neq 0)$, $p_1, \dots, p_s \in \mathbb{C}$. Then $f = p_0 F_1 \cdots F_s$ with $F_i(x) = F(x) - p_i$ for $i = 1, 2, \dots, s$. As we have seen in the previous section, $\{F_1, F_2, \dots, F_s\}$ form a $\mathrm{PTE}_{m,s}$ polynomial set. In this section we assume that (F, G) is a standard pair of the first or second kind and consider successively the cases $\deg(F) = 1$, $\deg(F) = 2$ and $\deg(F) > 2$. As we shall see, in each case $\deg(\varphi)$ can attain any positive integer value, hence $\deg(f), \deg(g)$ can be arbitrarily large.

Case $\deg(F) = 1$. The standard pair is of the first kind and we may assume that $F(x) = x$. Then $f = \varphi$. Hence for every $X \in \mathbb{Q}$ equation $F(x) = G(y)$ has as solution $(x, y) = (G(X), X)$. Thus equation $f(x) = g(y)$ has also solution $(x, y) = (G(X), X)$ for every $X \in \mathbb{Q}$.

Example 5.1. First kind, $F(x) = x, G$ is arbitrary, $\varphi = f$.

For every set of non-zero rationals $\{a_1, a_2, \dots, a_k\}$ the equation

$$(x - a_1)(x - a_2) \cdots (x - a_k) = (G(y) - a_1)(G(y) - a_2) \cdots (G(y) - a_k),$$

has solution $(x, y) = (G(X), X)$ $(X \in \mathbb{Q})$.

Case $\deg(F) = 2$. Then either $F(x) = x^2$, or $F(x) = \alpha x^2 + \beta x + \gamma$. In the latter case we use that $F(x) \simeq x^2 + c$ for some $c \in \mathbb{Q}$. Here $p = 0$, $q = 1$, $\deg(v) = 2$ if (F, G) is of the first kind and $\deg v = 0$ if (F, G) is of the second kind. Next we replace $\varphi(x)$ by $\varphi(x - c)$ so that we get $F(x) = x^2$. Thus we may choose $F(x) = x^2$ anyhow.

We obtain that $f(x)$ is of the form

$$\varphi(F(x)) = p_0(x^2 - p_1) \cdots (x^2 - p_s)$$

has only simple rational roots. It follows that $p_1, p_2, \dots, p_s$ are squares of distinct rational numbers and that the roots $\pm b_1, \pm b_2, \dots, \pm b_s$ of f are symmetric around 0. Further, $g(y) = p_0(G(y) - b_1^2) \cdots (G(y) - b_s^2)$. By Lemma 3.1 the equation $x^2 = G(y)$ has to have infinitely many rational solutions (x, y) with a bounded denominator. Let (X_i, Y_i) $(i = 1, 2, \dots)$ be such solutions. By the main result of LeVeque [50] (for the effective version, see Brindza [15]) we obtain that the polynomial G can have at most two roots of odd multiplicities. It follows that the equation $f(x) = g(y)$ has infinitely many rational solutions $(x, y) = (X_i, Y_i)$ $(i = 1, 2, \dots)$ with a bounded denominator. Writing $n = \deg(G), s = \deg(\varphi)$, we have $\deg(f) = 2s \mid 2ns = 2\deg(g)$. In this case s and n can be arbitrary, and hence $\deg(f), \deg(g)$ may be arbitrarily large.

Example 5.2. First kind, $F(x) = x^2$, $G(y) = yv^2(y)$, for some $v(y) \in \mathbb{Q}[y]$, $\varphi(x) = (x - b_1^2) \cdots (x - b_s^2)$ with distinct positive rationals $b_1, b_2, \dots, b_s$. We have

$$f(x) = (x - b_1)(x + b_1) \cdots (x - b_s)(x + b_s), g(y) = (G(y) - b_1^2) \cdots (G(y) - b_s^2),$$

and $f(x) = g(y)$ has solutions $(Xv(X^2), X^2)$ for every $X \in \mathbb{Z}$.

Example 5.3. Second kind, $F(x) = x^2$, $G(y) = (2y^2 - 1)v^2(y)$ for some $v(y) \in \mathbb{Q}[y]$, $\varphi(x) = (x - b_1^2) \cdots (x - b_s^2)$ for distinct positive rationals $b_1, b_2, \dots, b_s$.

Let $(X_i)_{i=1}^{\infty}$ be distinct integers such that $2Y_i^2 - 1 = X_i^2$ for integers Y_i. Then

$$f(x) = (x - b_1)(x + b_1) \cdots (x - b_s)(x + b_s), g(y) = (G(y) - b_1^2) \cdots (G(y) - b_s^2),$$

and $f(x) = g(y)$ has solutions $(X_i v(Y_i), Y_i)$ for $i = 1, 2, \dots$

Examples 5.2 and 5.3 are generalizations of Examples 1.1 and 1.2, respectively; see also Remark 3.2.

Case $\deg(F) > 2$. Here either $F(x) = x^q$ for some $q > 2$ or $G(x) = x^q$ for some positive integer q.

If $F(x) = x^q$, then $f(x) = p_0(x^q - p_1)(x^q - p_2) \cdots (x^q - p_s)$ has simple rational roots which implies $q \leqslant 2$, but since $\deg(F) > 2$ this is not possible. If $G(y) = y^q$, then from Table 1 we see that either $F(x) = \alpha x^p v(x)^q$ with $0 \leqslant p < q$, $\gcd(p, q) = 1$ (if (F, G) is of the first kind), or $q = 2$ and $F(x) = (\alpha x^2 + \beta)v(x)^2$ (if (F, G) is of the second kind). Since f has simple rational roots, $f' = \varphi'(F)F'$ has only simple real roots and therefore F' has only simple real roots. So $q \leqslant 2$ and in view of $\deg(F) > 2$ and the gcd-condition, we have only the following possibilities, with $\alpha \neq 0$.

(a) $G(y) = y$ and $F(x) = \alpha v(x)$ has only simple rational roots.
(b) $G(y) = y^2$ and $F(x)$ is $\alpha x v(x)^2$.
(c) $G(y) = y^2$ and $F(x)$ is $(\alpha x^2 + \beta)v(x)^2$.

We have $\deg(g) \mid \deg(f)$ in cases (a) and (c), while $\deg(g) \mid 2\deg(f)$ in case (b). The degree of φ can be arbitrary, hence the degrees of f, g can be arbitrarily large.

We give an example for each case.

Example 5.4. First kind, case (a), $F(x) = x^3 - 1729^2 x, G(y) = y, \varphi(x) = x(x - 728\,932\,560)$. We start from two triples from Example 4.3, $(-1729, 0, 1729)$, $(249, -1840, 1591)$, both having sum 0 and sum of squares $2 \cdot 1729^2$. This gives

$$f(x) = x(x - 1729)(x + 1729)(x + 249)(x - 1840)(x + 1591),$$

and $g(y) = y(y - 249 \cdot 1840 \cdot 1591)$. The equation $f(x) = g(y)$ has solution $(x, y) = (X, F(X))$ for every $X \in \mathbb{Q}$. Both f and g have only simple integer roots.

Example 5.5. First kind, case (b), $F(x) = x(x - 1729^2)^2, G(y) = y^2, \varphi(x) = (x - 728\,932\,560^2)(x - 1\,678\,772\,880^2)$. This example is based on the same triples as the previous example. We obtain

$$f(x) = (x - 249^2)(x - 1840^2)(x - 1591^2)(x - 656^2)(x - 1961^2)(x - 1305^2)$$

and

$$g(y) = (y^2 - (249 \cdot 1840 \cdot 1591)^2)(y^2 - (656 \cdot 1961 \cdot 1305)^2).$$

The equation $f(x) = g(y)$ has infinitely many solutions given by $(x, y) = (X^2, X(X^2 - 1729^2))$ $(X \in \mathbb{Q})$. Again f and g have only simple rational roots.

Example 5.6. Second kind, case (c), $F(x) = 26x^2(x^2 - 1105), G(y) = y^2, \varphi(x) = (x + 26 \cdot (33 \cdot 4)^2)(x + 26 \cdot (32 \cdot 9)^2)$.

We use data from Example 4.1, viz. $1105 = 33^2 + 4^2 = 32^2 + 9^2$. Thus

$$f(x) = 26^2(x^2 - 33^2)(x^2 - 4^2)(x^2 - 32^2)(x^2 - 9^2), g(y) = \varphi(y^2).$$

A Magma [13] calculation shows that $\varepsilon = 5 + \sqrt{26}$ is a fundamental unit of $\mathbb{Q}(\sqrt{26})$ of norm -1, and that $\alpha = -1248 + 247\sqrt{26}$ is an algebraic integer of this number field of norm $-28\,730 = -26 \cdot 1105$. From this we obtain that the equation

$$26(x^2 - 1105) = y^2$$

has solutions $(x, y) = (X_i, Y_i)$ $(i \in \mathbb{Z})$, with $(X_1, Y_1) = (247, -1248)$, $(X_2, Y_2) = (117, 572)$, and

$$(X_i, Y_i) = 102(X_{i-1}, Y_{i-1}) - (X_{i-2}, Y_{i-2}) \quad (i \in \mathbb{Z}_{>2}).$$

This, after some simple calculations, follows from the fact that $\varepsilon^{2t}\alpha$ is of norm $-26 \cdot 1105$, for any positive integer t. So the equation $f(x) = g(y)$ has infinitely many integral solutions $(X, Y) = (X_i, X_i Y_i)$. (Note that the above Pell equation has more solutions. One can, for example, take $(x, y) = (39, 104)$, which belongs to the algebraic integer $104 + 39\sqrt{26}$ of norm $-26 \cdot 1105$ not being associate of α. However, for our present purposes it is sufficient to exhibit an infinite family of solutions of the Pell equation.)

6 | BOTH f AND g HAVE ONLY SIMPLE RATIONAL ROOTS

In this section we consider Equation (2) with both f and g having only simple rational roots and (F, G) is of the first or second kind. Without loss of generality we may assume $\deg(f) \leqslant \deg(g)$, hence $\deg(F) \leqslant \deg(G)$. We again assume $f = \varphi(F)$, $g = \varphi(G)$.

Theorem 6.1. *Let $f(x), g(x) \in \mathbb{Q}[x]$, both having only simple rational roots. Suppose that the equation $f(x) = g(y)$ has infinitely many rational solutions x, y with a bounded denominator and that the corresponding standard pair $(F(x), G(x)) \in \mathbb{Q}[x]$ is of the first or second kind. Then we can choose F, G, φ such that one of the following items holds:*

1. $\deg(f) \mid \deg(g)$, *there exist $p_0 \in \mathbb{Q}, p_0 \neq 0$ and distinct $p_1, p_2, \ldots, p_s \in \mathbb{Q}$ such that*

$$f(x) = p_0 \prod_{i=1}^{s}(x - p_i), \quad g(y) = p_0 \prod_{i=1}^{s}(G(y) - p_i), \tag{9}$$

$F(x) = x$ and $G(y)$ is a $PTE_{n,s}$ base where $n = \deg(G)$. For every $X \in \mathbb{Z}$ the equation $f(x) = g(y)$ has solution $(x, y) = (G(X), X)$.

2. $\deg(f) \mid 2\deg(g)$, *there exist* $q_0 \in \mathbb{Q}, q_0 \neq 0$ *and distinct* $q_1, q_2, \dots, q_s \in \mathbb{Q}_{>0}$ *such that*

$$f(x) = q_0 \prod_{i=1}^{s} (x - q_i)(x + q_i), \quad g(y) = q_0 \prod_{i=1}^{s} (G(y) - q_i^2), \tag{10}$$

$F(x) = x^2$, $G(y) \in \mathbb{Q}[y]$ has at most two roots of odd multiplicities and is a $\mathrm{PTE}_{n,s}$ base, and the equation $x^2 = G(y)$ has infinitely many rational solutions $(x, y) = (X_i, Y_i)$ $(i = 1, 2, \dots)$ with a bounded denominator. Here the equation $f(x) = g(y)$ has solutions $(x, y) = (X_i, Y_i)$ $(i = 1, 2, \dots)$.

Proof. By Lemma 3.2 we know that $\deg(F) \leqslant 2$.

If $\deg(F) = 1$, then (using the notation of Table 1) we have $p = 0, q = 1, F(x) = x, G = v, f = \varphi$ and $G(y) - p_1, \dots, G(y) - p_s$ form a $\mathrm{PTE}_{n,s}$ polynomial set. Then f, g are as in (9). For every $X \in \mathbb{Q}$ there is a solution $(x, y) = (G(X), X)$. This is case **1**.

If $\deg(F) = 2$, then we may assume $F(x) = x^2$ according to the argument given in the preceding section. As before, we see that $p_1, p_2, \dots, p_s$ are squares in $\mathbb{Q}$. Let $p_i = q_i^2$ for $i = 1, 2, \dots, s$. Then f, g are as in (10) where $G(y) - q_1^2, \dots, G(y) - q_s^2$ form a $\mathrm{PTE}_{n,s}$ polynomial set. Further, by Lemma 3.1 we know that the equation $x^2 = G(y)$ has infinitely many solutions in rationals x, y with a bounded denominator. Clearly, these solutions will be solutions to the original equation, too. The main result of LeVeque [50] shows that $G(y)$ can have at most two roots of odd multiplicities. This is case 2. □

Case 1 corresponds with case (a) in the previous section, case 2 with cases (b) and (c). The following examples of case (a) illustrate that the results in Section 4 imply that there are instances of $\deg(F) = 1$ with $\deg(G) \in \{3, 4, 6\}$ and arbitrary $\deg(\varphi)$, (hence arbitrarily large $\deg(f), \deg(g)$ as well). This is obvious for $\deg(G) = 2$, cf. Example 5.1.

Example 6.1 (Cf., Examples 4.1–4.3). For $\deg(G) = 4$ choose $F(x) = x, G(y) = y^4 - 1105y^2$ and

$$\varphi(x) = (x + (33 \cdot 4)^2)(x + (32 \cdot 9)^2)(x + (31 \cdot 12)^2)(x + (24 \cdot 23)^2).$$

Then $f(x) = \varphi(x)$ and $g(y) = \varphi(G(y))$ is given by

$$(y^2 - 33^2)(y^2 - 4^2)(y^2 - 32^2)(y^2 - 9^2)(y^2 - 31^2)(y^2 - 12^2)(y^2 - 24^2)(y^2 - 23^2).$$

Similarly, for $\deg(G) = 6$ choose

$$F(x) = x, G(y) = y^6 - 2 \cdot 1729y^4 + 1729^2y^2,$$

$$\varphi(x) = (x - (40 \cdot 3 \cdot 43)^2)(x - (37 \cdot 8 \cdot 45)^2)(x - (32 \cdot 15 \cdot 47)^2)(x - (35 \cdot 23 \cdot 48)^2).$$

Then $f(x) = \varphi(x)$, $g(y) = \prod_{a \in T}(y^2 - a^2)$ with
$T = \{40, 3, 43, 37, 8, 45, 32, 15, 47, 25, 23, 48\}$.
Finally, for $\deg(G) = 3$ let $F(x) = x, G(y) = y^3 - 1729^2y$,

$$\varphi(x) = x(x^2 - (249 \cdot 1840 \cdot 1591)^2)(x^2 - (656 \cdot 1961 \cdot 1305)^2) \cdot$$
$$(x^2 - (1185 \cdot 1984 \cdot 799)^2)(x^2 - (1679 \cdot 1775 \cdot 96)^2).$$

Then $f(x) = \varphi(x)$ and $g(y) = y\prod_{a\in T}(y^2 - a^2)$ with

$$T = \{1729, 249, 1840, 1591, 656, 1961, 1305, 1185, 1984, 799, 1679, 1775, 96\}.$$

In all three cases, we obtain a solution $(x, y) = (G(X), X)$ of $f(x) = g(y)$ for every $X \in \mathbb{Q}$ and both f and g have only simple rational roots.

Now we turn to the case $\deg(F) = 2$. Again $\deg(\varphi)$ can be arbitrary (when $\deg(f)$ and $\deg(g)$ can be arbitrarily large). Note that $\deg(G) \geqslant 2$. An example of case (b) is given by Example 5.5. The following example illustrates case (c). It is another generalization of Example 1.2.

Example 6.2. Suppose that the equation $x^2 = ay^2 + b$ with $a, b \in \mathbb{Z}$, $ab \neq 0$ has solutions $(X_i, Y_i)_{i=1}^{\infty} \in \mathbb{Z}^2$. Let $s \geqslant 1$, $F(x) = x^2$, $G(y) = ay^2 + b$, $\varphi(x) = \prod_{i=1}^{s}(x - X_i^2)$. Then we have

$$f(x) = \prod_{i=1}^{s}(x^2 - X_i^2), \quad g(y) = \prod_{i=1}^{s}(ay^2 + b - X_i^2) = a^s\prod_{i=1}^{s}(y^2 - Y_i^2).$$

So both $f(x)$ and $g(y)$ have only simple rational roots. Further, the equation $f(x) = g(y)$ has as solutions (X_i, Y_i) for all i.

7 | STANDARD PAIRS F, G OF THE THIRD OR FOURTH KIND

To handle the cases corresponding to standard pairs of the third and fourth kinds, we apply the following result.

Lemma 7.1. *Let $a_1, \dots, a_N$ be distinct rationals, and assume that for some rational numbers u_1, u_2, v_1, v_2, b with $u_1v_1b \neq 0$ we have*

$$u_1D_N(x, b) + u_2 = (v_1x + v_2 - a_1)\cdots(v_1x + v_2 - a_N), \tag{11}$$

where $D_N(x, b)$ is the N-th Dickson polynomial with parameter b. Then $N \in \{1, 2, 3, 4, 6\}$.

For appropriate choices of the parameters the cases $N \in \{1, 2, 3, 4, 6\}$ are possible. In Theorem 7.1 we describe these cases completely. First we prove the first part of Theorem 1.1.

Proof of the first statement of Theorem 1.1. By Lemma 7.1 Equation (2) with (1) implies that $\deg(F) \in \{1, 2, 3, 4, 6\}$, if the corresponding standard pair (F, G) is of the third or fourth kind. This combined with Lemma 3.2 completes the proof of the first statement of Theorem 1.1. □

Lemma 7.1 has already been proved for $N \leqslant 12$ in [41] (see the proof of Theorem 2.3 there). However, in this paper we need a more precise statement. To keep the presentation self-contained, we include the complete proof.

Proof of Lemma 7.1. Writing $w_i = (v_2 - a_i)/v_1$ $(i = 1, \dots, N)$ and $u = u_2/v_1^N$, dividing both sides of (11) by v_1^N and using that D_N is monic, we get the similar equation

$$D_N(x, b) + u = (x + w_1)\dots(x + w_N). \tag{12}$$

Here $u \in \mathbb{Q}$ and $w_1, \dots, w_N$ are distinct rationals.

Applying the well-known identity

$$D_N\left(y + \frac{b}{y}, b\right) = y^N + \left(\frac{b}{y}\right)^N$$

to (12), we obtain

$$y^{2N} + uy^N + b^N = \prod_{i=1}^{N}(y^2 + w_i y + b)\,. \tag{13}$$

Write ζ, ξ for the roots of the polynomial $Y^2 + uY + b^N$. Clearly, ζ, ξ are algebraic numbers of degrees at most two. Further, $b \neq 0$ yields $\zeta\xi \neq 0$. Also observe that $\zeta \neq \xi$, since the numbers w_i in (12) are distinct. If $u = 0$, then the roots of the left-hand side of (13) are given by

$$\eta^j \sqrt{b} \quad (j = 0, 1, \ldots, 2N-1), \tag{14}$$

where $\sqrt{b}$ denotes one of the (complex) square roots of b, and η is a primitive $2N$-th root of unity. In view of the right-hand side of (13), we see that the numbers (14) are algebraic numbers of degrees at most two. Hence $\varphi(2N) = \deg(\eta) \leqslant 4$. This implies $N \in \{1, 2, 3, 4, 6\}$.

So from this point on, we assume $\zeta + \xi = -u \neq 0$. Then the roots of the polynomial on the left-hand side of (13) are given by

$$\zeta_0\varepsilon^i \text{ and } \xi_0\varepsilon^i \quad (i = 0, 1, \ldots, N-1),$$

where ζ_0 and ξ_0 are Nth roots of ζ and ξ, respectively, and ε is a primitive Nth root of unity. Since these are the roots of the polynomial on the right-hand side of (13), they are distinct algebraic numbers of degrees at most two. In particular, ζ_0 and $\zeta_0\varepsilon$ are at most quadratic algebraic numbers, so the degree of ε is at most four. Hence $\varphi(N) \leqslant 4$, and we obtain $N \in \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$.

To refine the restriction for N, we need a more careful consideration. Write $\zeta_1 := \zeta_0\varepsilon$. Then we see that

$$\varepsilon = \frac{\zeta_1}{\zeta_0} \tag{15}$$

belongs to the number field $K := \mathbb{Q}(\zeta_0, \zeta_1)$. Observe that if $\zeta_0 \in \mathbb{Q}(\zeta_1)$, then ε is (at most) quadratic, yielding $\varphi(N) \leqslant 2$, and our claim follows. So we may assume that $\deg(K) = 4$, and also that $K = \mathbb{Q}(\varepsilon)$ and that ζ_0 is quadratic. Denoting its algebraic conjugate by $\bar{\zeta}_0$, we have

$$(\overline{\zeta_0})^N = \overline{\zeta_0^N} = \bar{\zeta} = \xi.$$

Therefore, without loss of generality we may assume that $\xi_0 = \overline{\zeta_0}$ holds, in particular, that ζ_0 and ξ_0 belong to the same quadratic subfield of K. From this point on, we shall use this assumption. We deal with the remaining cases in turn. For the calculations we used Magma [13].

If $N = 5, 10$, then K is defined by $x^4 + x^3 + x^2 + x + 1$. The only quadratic subfield of K is given by $T_1 := \mathbb{Q}(\sqrt{5})$. So now $\zeta_0, \xi_0 \in T_1$. Recall that ζ_0, hence also ξ_0 is not rational. However, the (unique) factorization of

$$P(x) := x^{2N} + ux^N + b^N = (x^N - \zeta_0^N)(x^N - \xi_0^N) \tag{16}$$

(into irreducible factors) in $T_1[x]$ contains both for $N = 5$ and for $N = 10$ the factors

$$x^2 + (3 - \sqrt{5})\zeta_0 x + \zeta_0^2 \text{ and } x^2 + (3 - \sqrt{5})\xi_0 x + \xi_0^2.$$

Here the constant terms of the quadratic factors are not equal. Indeed, otherwise $\zeta_0^2 = \xi_0^2$ would imply $\zeta_0 = \pm\xi_0$, when $\zeta = \pm\xi$, which is excluded. Hence we see that (13) is not possible in these cases.

Let now $N = 8$. Then K is defined by $x^4 + 1$. The number field K has three quadratic subfields, namely $T_2 = \mathbb{Q}(i)$, $T_3 = \mathbb{Q}(\sqrt{2})$ and $T_4 = \mathbb{Q}(i\sqrt{2})$. Following the argument given above for the factorization of $P(x)$ defined by (16) we get that

- $x^2 + i\zeta_0^2$ and $x^2 + i\xi_0^2$ are factors of $P(x)$ in $T_2[x]$,
- $x^2 + \zeta_0^2$ and $x^2 + \xi_0^2$ are factors of $P(x)$ in $T_3[x]$ and $T_4[x]$,

assuming that $\zeta_0, \xi_0 \in T_2, T_3, T_4$, respectively. In all cases the constant terms of the quadratic factors are not the same. So $N = 8$ is also impossible.

Finally, let $N = 12$. Then K is defined by $x^4 - x^2 + 1$. The number field K has three quadratic subfields, namely $T_5 = \mathbb{Q}(i)$, $T_6 = \mathbb{Q}(\sqrt{3})$ and $T_7 = \mathbb{Q}(i\sqrt{3})$. Now, similarly as before, for the factorization of $P(x)$ given by (16) we obtain that

- $x^2 + \zeta_0 x + \zeta_0^2$ and $x^2 + \xi_0 x + \xi_0^2$ are factors of $P(x)$ in $T_5[x]$,
- $x^2 + \zeta_0^2$ and $x^2 + \xi_0^2$ are factors of $P(x)$ in $T_6[x]$ and $T_7[x]$,

assuming that $\zeta_0, \xi_0 \in T_5, T_6, T_7$, respectively.

Again, in all cases we observe that the constant terms of the quadratic factors are not identical. So $N = 12$ is excluded, too. □

Theorem 7.1. *Let $N \in \{3, 4, 6\}$. For any $w_1, w_2 \in \mathbb{Q}$ we can choose $w_3, \dots, w_N, b, u \in \mathbb{Q}$ such that (12) holds. On the other hand, this provides the only solutions of Equation (12).*

Remark 7.1. The cases $N = 1$ and $N = 2$ are trivial. Indeed, for $N = 1$ we have $D_1(x, b) = x$, so $w_1 = u$ can be any rational number. Further, for $N = 2$ we have $D_2(x, b) = x^2 - 2b$, when $w_1 + w_2 = 0$, $w_1 w_2 = -2b + u$. Therefore all cases of (12) are given by

$$(x + w_1)(x - w_1) = D_2(x, b) + (2b - w_1^2),$$

that is, with $u = 2b - w_1^2$ for arbitrary $b, w_1 \in \mathbb{Q}$.

Proof of Theorem 7.1. We consider the possibilities in turn.

The case $N = 3$. We have

$$D_3(x, b) = x^3 - 3bx, \tag{17}$$

hence

$$w_1 + w_2 + w_3 = 0, w_1 w_2 + w_1 w_3 + w_2 w_3 = -3b, w_1 w_2 w_3 = u.$$

This gives

$$w_3 = -w_1 - w_2, b = (w_1^2 + w_1 w_2 + w_2^2)/3, u = -w_1^2 w_2 - w_1 w_2^2. \tag{18}$$

Thus we have for any $w_1, w_2 \in \mathbb{Q}$ that

$$(x + w_1)(x + w_2)(x - w_1 - w_2) = D_3(x, b) + u \tag{19}$$

and this provides all possibilities for (12).

The case $N = 4$. We have

$$D_4(x, b) = x^4 - 4bx^2 + 2b^2. \tag{20}$$

This implies

$$w_1 + w_2 + w_3 + w_4 = 0$$

and

$$w_1w_2w_3 + w_1w_2w_4 + w_1w_3w_4 + w_2w_3w_4 = 0.$$

It follows that

$$0 = w_1w_2w_3 - (w_1w_2 + w_1w_3 + w_2w_3)(w_1 + w_2 + w_3) = -(w_1 + w_2)(w_1 + w_3)(w_2 + w_3).$$

We assume, without loss of generality,

$$w_1 + w_3 = 0, \text{ hence } w_2 + w_4 = 0. \tag{21}$$

Further comparison of coefficients gives

$$b = -\frac{w_1w_2 + w_1w_3 + w_1w_4 + w_2w_3 + w_2w_4 + w_3w_4}{4} = \frac{w_1^2 + w_2^2}{4} \tag{22}$$

and

$$u = w_1w_2w_3w_4 - 2b^2 = w_1^2w_2^2 - \frac{1}{8}(w_1^2 + w_2^2)^2 = -\frac{1}{8}(w_1^4 - 6w_1^2w_2^2 + w_2^4). \tag{23}$$

For any $w_1, w_2 \in \mathbb{Q}$ and b, u chosen as above we have

$$(x + w_1)(x - w_1)(x + w_2)(x - w_2) = D_4(x, b) + u \tag{24}$$

and this provides all possibilities for (12).

The case $N = 6$ is the most involved one. We have

$$D_6(x, b) = x^6 - 6bx^4 + 9b^2x^2 - 2b^3. \tag{25}$$

On the other hand, the roots of the polynomial on the left-hand side of (13) are given by

$$\pm\zeta_0,\ \pm\zeta_0\varepsilon,\ \pm\zeta_0\varepsilon^2,\ \pm\xi_0, \pm\xi_0\varepsilon,\ \pm\xi_0\varepsilon^2,$$

where ε is a primitive sixth root of unity (that is, a root of $x^2 - x + 1$), and either $\zeta_0, \xi_0 \in \mathbb{Q}$, or they are conjugated quadratic algebraic numbers from the field $K = \mathbb{Q}(\varepsilon)$. Anyhow, the factorization of the polynomial on the right hand side of (13) over K reads as

$$(y - \zeta_0)(y + \zeta_0)(y - \zeta_0\varepsilon)(y + \zeta_0\varepsilon)(y - (1 - \varepsilon)\zeta_0)(y + (1 - \varepsilon)\zeta_0)$$
$$\cdot(y - \xi_0)(y + \xi_0)(y - \xi_0\varepsilon)(y + \xi_0\varepsilon)(y - (1 - \varepsilon)\xi_0)(y + (1 - \varepsilon)\xi_0).$$

Note that the (algebraic) conjugate of ε is $1-\varepsilon$. Hence we immediately get that the right hand side of (13) is given by

$$\begin{aligned}&(y^2-(\zeta_0+\xi_0)y+\zeta_0\xi_0)(y^2+(\zeta_0+\xi_0)y+\zeta_0\xi_0)\\&\cdot(y^2-(\zeta_0\varepsilon+\xi_0(1-\varepsilon))y+\zeta_0\xi_0)(y^2+(\zeta_0\varepsilon+\xi_0(1-\varepsilon))y+\zeta_0\xi_0)\\&\cdot(y^2-(\zeta_0(1-\varepsilon)+\xi_0\varepsilon)y+\zeta_0\xi_0)(y^2+(\zeta_0(1-\varepsilon)+\xi_0\varepsilon)y+\zeta_0\xi_0).\end{aligned}$$

Here all the above quadratic polynomials have rational coefficients. The coefficients of y are just the numbers w_i from (13) (and (12)). Observe that (by choosing an appropriate indexing) we have

$$w_3=w_1+w_2,\ w_4=-w_1,\ w_5=-w_2,\ w_6=-w_3 \tag{26}$$

in (12). Put $W=w_1^2+w_1w_2+w_2^2$. A simple calculation yields that

$$\begin{aligned}&(x+w_1)(x+w_2)(x+w_3)(x+w_4)(x+w_5)(x+w_6)\\&=x^6-2Wx^4+W^2x^2-w_1^2w_2^2(w_1+w_2)^2.\end{aligned}$$

Comparing the coefficients with $D_6(x,b)+u=x^6-6bx^4+9b^2x^2-2b^3+u$ we see that

$$b=\frac{W}{3},\ \ u=\frac{2W^3}{27}-w_1^2w_2^2(w_1+w_2)^2. \tag{27}$$

On the other hand, for any $w_1,w_2\in\mathbb{Q}$ we have, choosing b and u as in (27), that

$$(x+w_1)(x-w_1)(x+w_2)(x-w_2)(x+w_1+w_2)(x-w_1-w_2)=D_6(x,b)+u. \tag{28}$$

Thus this provides all possibilities for (12) if $N=6$. □

We give examples to show that for $\deg(F)=m\in\{3,4,6\}$ Equation (2) with f of the form (1) can have infinitely many solutions $(x,y)\in\mathbb{Q}^2$ with a bounded denominator. The examples are successively with $m=3,4,6$ for the third kind and with $m=4,6$ for the fourth kind. By the gcd condition in Table 1, $m=3$ cannot occur for the fourth kind.

Example 7.1. Third kind, case $m=3,n=4,b=7$, $F(x)=D_3(x,7^4), G(y)=D_4(y,7^3))$, $\varphi(x)=(x-14\cdot77\cdot91)(x-23\cdot71\cdot94)$.

We have

$$3\cdot7^4=14^2+14\cdot77+77^2=23^2+23\cdot71+71^2.$$

Hence, by (19) and (20),

$$\begin{aligned}D_3(x,7^4)&=(x+14)(x+77)(x-91)+14\cdot77\cdot91\\&=(x+23)(x+71)(x-94)+23\cdot71\cdot94,\\D_4(x,7^3)&=x^4-4\cdot7^3\cdot x^2+2\cdot7^6.\end{aligned}$$

According to [11, formula (5)] we have, for all coprime positive integers m,n and integers b,

$$D_m(D_n(x,b),b^n)=D_n(D_m(x,b),b^m). \tag{29}$$

So the equation $F(x) = G(y)$ has solutions $(x, y) = (D_4(X, 7), D_3(X, 7))$ for every $X \in \mathbb{Z}$. Then the equation $f(x) = g(y)$ with

$$f(x) = (D_3(x, 7^4) - 14 \cdot 77 \cdot 91)(D_3(x, 7^4) - 23 \cdot 71 \cdot 94),$$
$$g(y) = (D_4(y, 7^3) - 14 \cdot 77 \cdot 91)(D_4(y, 7^3) - 23 \cdot 71 \cdot 94)$$

has the same solutions. Note that f has only simple integral roots.

Example 7.2. Third kind, $m = 4, n = 3, b = 5$, $F(x) = D_4(x, 5^3), G(y) = D_3(y, 5^4)$, $\varphi(x) = (x + 4^2 \cdot 22^2 - 2 \cdot 5^6)(x + 10^2 \cdot 20^2 - 2 \cdot 5^6)$.

We have $4 \cdot 5^3 = 4^2 + 22^2 = 10^2 + 20^2$. Hence

$$D_4(x, 5^3) = (x + 4)(x - 4)(x + 22)(x - 22) - 4^2 \cdot 22^2 + 2 \cdot 5^6 =$$
$$= (x + 10)(x - 10)(x + 20)(x - 20) - 10^2 \cdot 20^2 + 2 \cdot 5^6.$$

By (29), the equation $F(x) = D_4(x, 5^3) = D_3(y, 5^4) = G(y)$ has solutions $(x, y) = (D_3(X, 5), D_4(X, 5))$ $(X \in \mathbb{Z})$. Thus the equation $f(x) = g(y)$ with

$$f(x) = (x + 4)(x - 4)(x + 22)(x - 22)(x + 10)(x - 10)(x + 20)(x - 20),$$
$$g(y) = (D_3(y, 5^4) + 4^2 \cdot 22^2 - 2 \cdot 5^6)(D_3(y, 5^4) + 10^2 \cdot 20^2 - 2 \cdot 5^6)$$

has the same solutions.

Example 7.3. Third kind, $m = 6, n = 5, b = 7$, $F(x) = D_6(x, 7^5), G(y) = D_5(y, 7^6))$, $\varphi(x) = (x + 7\,945\,347\,009\,886)(x + 3\,958\,608\,139\,486)$.

We have $3 \cdot 7^5 = 211^2 + 211 \cdot 25 + 25^2 = 196^2 + 196 \cdot 49 + 49^2$. Hence

$$D_6(x, 7^5) = (x + 211)(x + 25)(x + 236)(x - 211)(x - 25)(x - 236) - 7\,945\,347\,009\,886$$
$$= (x + 196)(x + 49)(x + 245)(x - 196)(x - 49)(x - 245) - 3\,958\,608\,139\,486.$$

By (29), the equation $F(x) := D_6(x, 7^5) = D_5(y, 7^6) =: G(y)$ has solutions $(x, y) = (D_5(X, 7), D_6(X, 7))$ for $(X \in \mathbb{Z})$. Thus the equation $f(x) = g(y)$ with

$$f(x) = (x + 211)(x + 25)(x + 236)(x - 211)(x - 25)(x - 236)$$
$$\cdot(x + 196)(x + 49)(x + 245)(x - 196)(x - 49)(x - 245),$$
$$g(y) = (D_5(y, 7^6) + 7\,945\,347\,009\,886)(D_5(y, 7^6) + 3\,958\,608\,139\,486)$$

has the same solutions.

Example 7.4. Fourth kind, case $m = 4, n = 10, a = -10 \cdot 65^2, b = 65$, $F(x) = b^{-2}D_4(x, b)$, $G(y) = -a^{-5}D_{10}(y, a)$, $\varphi(x) = (x - 7426 \cdot b^{-2})(x + 4094 \cdot b^{-2})$.

We have $4b = 260 = 2^2 + 16^2 = 8^2 + 14^2$. Thus

$$D_4(x, b) = (x^2 - 2^2)(x^2 - 16^2) + 7426 = (x^2 - 8^2)(x^2 - 14^2) - 4094.$$

According to [11, formula (10)] with $m = 4$, $n = 10$,

$$b^2v_1^2 + av_2^2 = 4ab \tag{30}$$

with $v_1, v_2 \in \mathbb{Q}$ implies that

$$b^{-2}D_4(b^{-2}(v_2^5 - 5bv_2^3 + 5b^2v_2), b) = -a^{-5}D_{10}(v_1v_2, a). \tag{31}$$

Observe that (30) becomes the Pell equation $v_1^2 - 10v_2^2 = -2600$ with solutions $(v_1, v_2) = (X_i, Y_i)$ given by $(X_0, Y_0) = (-80, 30)$, $(X_1, Y_1) = (280, 90)$ and

$$(X_i, Y_i) = 38(X_{i-1}, Y_{i-1}) - (X_{i-2}, Y_{i-2}) \quad (i \geqslant 2).$$

Thus

$$F(x) = b^{-2}D_4(x, b) = -a^{-5}D_{10}(y, a) = G(y)$$

has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator. We conclude that the equation $f(x) = g(y)$ with

$$f(x) = b^{-4}(x-2)(x+2)(x-16)(x+16)(x-8)(x+8)(x-14)(x+14),$$
$$g(y) = (-a^{-5}D_{10}(y, a) - 7426b^{-2})(-a^{-5}D_{10}(y, a) + 4094b^{-2})$$

has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator.

Example 7.5. Fourth kind, case $m = 6, n = 10, a = -14 \cdot 91^3, b = 91$,
$F(x) = b^{-3}D_6(x, b), G(y) = -a^{-5}D_{10}(y, a)$,
$\varphi(x) = (x + 1\,433\,158 \cdot b^{-3})(x - 1\,288\,442 \cdot b^{-3})$.
We have $3b = 16^2 + 16 \cdot 1 + 1^2 = 11^2 + 11 \cdot 8 + 8^2$. Thus

$$D_6(x, 91) = (x^2 - 16^2)(x^2 - 1)(x^2 - 17^2) - 1\,433\,158$$
$$= (x^2 - 11^2)(x^2 - 8^2)(x^2 - 19^2) + 1\,288\,442.$$

By [11, formula (10)] with $m = 6$, $n = 10$,

$$b^3v_1^2 + av_2^2 = 4ab \tag{32}$$

with $v_1, v_2 \in \mathbb{Q}$ implies that

$$b^{-3}D_6(b^{-2}(v_2^5 - 5bv_2^3 + 5b^2), b) = -a^{-5}D_{10}(v_1(v_2^2 - b), a). \tag{33}$$

Observe that (32) becomes the Pell equation $v_1^2 - 14v_2^2 = -5096$, with solutions $(v_1, v_2) = (X_i, Y_i)$ given by $(X_0, Y_0) = (-140, 42)$, $(X_1, Y_1) = (252, 70)$ and

$$(X_i, Y_i) = 30(X_{i-1}, Y_{i-1}) - (X_{i-2}, Y_{i-2}) \quad (i \geqslant 2).$$

Thus, by (33),

$$F(x) = b^{-3}D_6(x, b) = -a^{-5}D_{10}(y, a) = G(y)$$

has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator. We conclude that the equation $f(x) = g(y)$ with

$$f(x) = b^{-6}(x^2 - 16^2)(x^2 - 1)(x^2 - 17^2)(x^2 - 11^2)(x^2 - 8^2)(x^2 - 19^2),$$
$$g(y) = (-a^{-5}D_{10}(y, a) + 1\,433\,158b^{-3})(-a^{-5}D_{10}(y, a) - 1\,288\,442b^{-3})$$

has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator.

Remark 7.2. It follows from Lemmas 4.2 and 4.3 that in all the above examples $\deg(\varphi)$ can equal any $s \in \mathbb{Z}_{>0}$, (hence $\deg(f)$ and $\deg(g)$ can be made arbitrarily large) by choosing suitable W with number of representations $\geqslant s$ and corresponding values $u = u_1, u_2, \ldots, u_s$.

Remark 7.3. Let $f(x) \in \mathbb{Q}[x]$ have only simple rational roots and let $g(x) \in \mathbb{Q}[x]$. Suppose the equation $f(x) = g(y)$ has infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator. By Lemma 7.1 we have $\deg(F) \in \{1, 2, 3, 4, 6\}$. Put $s = \gcd(\deg(f), \deg(g))$, $m = \deg(f)/s, n = \deg(g)/s$. Then $\gcd(m, n) = 1$ and $m \in \{1, 2, 3, 4, 6\}$ or $n \in \{1, 2\}$.

If $m = 1$ we refer to Example 5.1 to see that all pairs n, s are possible. For $m = 2$ Example 5.2 shows that all pairs (n, s) (with n odd since m and n are coprime) are possible. By (29) the equation $D_m(x, b^n) = D_n(y, b^m)$ with $\gcd(m, n) = 1$ has infinitely many solutions in integers (x, y) for any integer b. If $m = 4$, we proceed as in Example 7.2 (where $s = 2$) using a b which is the product of sufficiently many distinct primes $\equiv 1 \pmod 4$. If $m = 3$ or $m = 6$, then we proceed as in Example 7.1 or 7.3 (where $s = 2$ too) using a b which is the product of sufficiently many distinct primes $\equiv 1 \pmod 6$. Remark 7.2 underlines that this can be done for any s. Thus every pair $(\deg(f), \deg(g))$ with corresponding $m \in \{1, 2, 3, 4, 6\}$ can be represented.

8 | f AND g CANNOT HAVE ONLY SIMPLE RATIONAL ROOTS IF (F, G) IS OF THE THIRD OR FOURTH KIND

If both f and g have simple rational roots, then by symmetry we may assume that $\deg(f) \leqslant \deg(g)$. Throughout this chapter we shall do so without further mentioning. We show that if in this case the equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator and the corresponding standard pair (F, G) is of the third or fourth kind, then $\deg(F) \leqslant 2$. Note that $\deg(f) \leqslant \deg(g)$ implies $\deg(F) \leqslant \deg(G)$.

Theorem 8.1. *Suppose that f and g have only simple rational roots, and the equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator. If the corresponding standard pair (F, G) is of the third or fourth kind, then* $\deg(F) \leqslant 2$ *holds.*

In the proof we use the following lemmas.
We denote the discriminant of a polynomial P by $\operatorname{disc}(P)$.

Lemma 8.1 (Davenport, Lewis, Schinzel, [24, Theorem 1]). *Let $F(x) \in \mathbb{Z}[x]$ be of degree $m > 1$ and $G(y) \in \mathbb{Z}[x]$ of degree $n > 1$. Let*

$$D(z) = \operatorname{disc}(F(x) + z), \quad E(z) = \operatorname{disc}(G(y) + z).$$

Suppose there are at least $\lfloor \frac{1}{2}m \rfloor$ *roots of* $D(z) = 0$ *for which* $E(z) \neq 0$. *Then* $F(x) - G(y)$ *is irreducible over the complex field. Further, the genus of the equation* $F(x) - G(y) = 0$ *is strictly positive except possibly when* $n = 2$ *or* $m = n = 3$. *Apart from these possible exceptions, the equation has at most a finite number of integral solutions.*

Lemma 8.2. *Let* a, b, c *be rational numbers such that*

$$3a^2 + b^2 = c^2.$$

Then there exist rational numbers u, v, w *such that*

$$a = \pm w(2uv), \quad b = \pm w(3u^2 - v^2), \quad c = \pm w(3u^2 + v^2)$$

with independent choices of the $\pm$ *signs.*

Proof. Using the trivial solution $(a, b, c) = (0, 1, 1)$, the statement is a simple consequence of Corollary 6.3.6 of Cohen [20] (in the particular case $(A, B, C) = (3, 1, 1)$ there); see also Desboves [25] and Dickson [27, II p. 405]. We note that the parametrization given in [27] is not complete. □

Proof of Theorem 8.1. Suppose that the equation $f(x) = g(y)$ has infinitely many solutions $x, y \in \mathbb{Q}$ with a bounded denominator, and write (F, G) for the corresponding standard pair of the third or fourth kind. Assume that $\deg(F) \geqslant 3$. Then it follows from Lemma 7.1 that $\deg(F), \deg(G) \in \{3, 4, 6\}$. In view of the gcd restrictions on standard pairs of the third and fourth kinds it remains to consider $(m, n) := (\deg(F), \deg(G)) = (3, 4)$ for the third kind and $= (4, 6)$ for the fourth kind.

Standard pairs of the third kind. We have $(m, n) = (3, 4)$. Write

$$\varphi(x) = p_0(x - p_1) \cdots (x - p_s)$$

with $p_0 \in \mathbb{Q}$, $p_0 \neq 0$ and $p_i \in \mathbb{C}$ $(i = 1, \ldots, s)$. Since the roots of $f(x) = \varphi(F(x))$ are simple and rational, we see that $p_1, \ldots, p_s$ are distinct and rational. So we can write

$$F(x) - p_i = (x - a_1^{(i)})(x - a_2^{(i)})(x - a_3^{(i)}),$$

$$G(y) - p_i = (y - b_1^{(i)})(y - b_2^{(i)})(y - b_3^{(i)})(y - b_4^{(i)}) \ \ (i = 1, \ldots, s).$$

Here the $3s$ numbers a form the set of roots of f and are therefore distinct rationals. Similarly the $4s$ numbers b form the set of roots of g and are therefore distinct rationals. By Lemma 3.1 we know that the equation

$$F(x) - p_1 = G(y) - p_1 \tag{34}$$

has infinitely many solutions in rationals x, y with a bounded denominator. Since they are Dickson polynomials of degrees 3 and 4, respectively, the proof of Theorem 7.1 implies that the equation (after changing the indexing of the roots if it is necessary)

$$(x - a_1^{(1)})(x - a_2^{(1)})(x + a_1^{(1)} + a_2^{(1)}) = (y^2 - (b_1^{(1)})^2)(y^2 - (b_2^{(1)})^2)$$

has infinitely many solutions in rationals x, y with a bounded denominator. Then there exist positive integers Δ_1, Δ_2 such that, omitting the superscript (1) for simplicity and putting

$$A_i = \Delta_1 a_i \ (i = 1, 2, 3) \quad \text{and} \quad B_j = \Delta_2 b_j \ (j = 1, 2),$$

the equation

$$U(x) := (x - A_1)(x - A_2)(x + A_1 + A_2) = \Delta(y^2 - B_1^2)(y^2 - B_2^2) =: V(y) \tag{35}$$

with $\Delta = \Delta_1^3/\Delta_2^4$ has infinitely many solutions in integers x, y.

It follows from Lemma 8.1 that, writing

$$D(z) = \operatorname{disc}(U(x) + z) \quad \text{and} \quad E(z) = \operatorname{disc}(V(y) + z),$$

every root of $D(z)$ is a root of $E(z)$. A Maple calculation reveals that the roots of $D(z)$ are

$$-A_1^2 A_2 - A_1 A_2^2 \pm \frac{2}{9}\sqrt{3(A_1^2 + A_1 A_2 + A_2^2)^3} \tag{36}$$

and that the roots of $E(z)$,

$$-\Delta B_1^2 B_2^2, \quad \Delta\left(\frac{B_1^2 - B_2^2}{2}\right)^2 \tag{37}$$

(the latter one being a double root), are rational. So the roots of $D(z)$ have to be rational. Hence, for some $s \in \mathbb{Q}$,

$$3(A_1^2 + A_1 A_2 + A_2^2) = s^2. \tag{38}$$

We rewrite (38) as

$$3(2A_1 + A_2)^2 + (3A_2)^2 = (2s)^2.$$

By Lemma 8.2, we obtain

$$2A_1 + A_2 = \pm w(2uv), \quad 3A_2 = \pm w(3u^2 - v^2)$$

with some $u, v, w \in \mathbb{Q}$ and independent choice of the $\pm$ signs. This yields

$$(A_1, A_2) = w\left(\frac{-3u^2 \pm 6uv + v^2}{6}, \frac{3u^2 - v^2}{3}\right).$$

(Here in place of the factor $\pm w$ we can simply write w, since $w \in \mathbb{Q}$ is arbitrary.) Therefore the roots of $D(z)$ are given by

$$\frac{1}{2}w^3u^2(u - v)^2(u + v)^2, \quad -\frac{1}{54}w^3v^2(3u - v)^2(3u + v)^2.$$

Since, by (37), the products of any two roots (37) of $E(z)$ are $\pm$ squares and $2 \cdot 54 = 108$ is not a square in $\mathbb{Q}$, we see that one of the roots of $D(z)$ is zero. Then $E(z)$ has also a root 0. However, then either $B_1 B_2 = 0$ or $B_1 = \pm B_2$, which contradicts the distinctness of the roots B_1, B_2, B_3, B_4. This contradiction proves that (35) has only finitely many solutions $(x, y) \in \mathbb{Z}^2$, hence (34) and

thus also the equation $f(x) = g(y)$ has only finitely many rational solutions with a bounded denominator. So this case cannot occur.

Standard pairs of the fourth kind. In this case the only possibility is $(m, n) = (4, 6)$, and Lemma 3.1 implies that the standard pair $(F(x), G(y))$ is of the form $(a^{-2}D_4(x, a), -b^{-3}D_6(y, b))$. Further, the equation

$$a^{-2}D_4(x, a) = -b^{-3}D_6(y, b) \tag{39}$$

should have infinitely many rational solutions x, y with bounded denominator. However, by Theorem 7.1 we know that here b is of the form $(w_1^2 + w_1w_2 + w_2^2)/3$ with some $w_1, w_2 \in \mathbb{Q}$, in particular, $b > 0$. However, since the signs of the leading coefficients of the even degree polynomials in (39) are different, this equation can have only finitely many solutions with a bounded denominator. So this case cannot occur either, and the proof of Theorem 8.1 is complete. □

9 | A SHARPENING OF THEOREM 1.1

We give a refinement of Theorem 1.1 in case both f and g have only simple rational roots. This completes the proof of Theorem 1.1.

Theorem 9.1. *Suppose that f and g have only simple rational roots, and that the equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator. Let $k = \deg(f), \ell = \deg(g)$. If $0 < k \leqslant \ell$, then $k \mid 2\ell$, f is a PTE$_{m,s}$-polynomial and g is a PTE$_{\ell m/k,s}$-polynomial with $m \in \{1, 2\}$ and $\ell m/k \in \mathbb{Z}$.*

Conversely, if k, ℓ are positive integers with $k \mid \ell$ and g is a PTE$_{\ell/k}$-polynomial of degree ℓ with only simple rational roots, then there exists a polynomial $f(x) \in \mathbb{Q}[x]$ with $\deg(f) = k$ and only simple rational roots such that the equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator.

Proof. Suppose that the equation $f(x) = g(y)$ has infinitely many solutions $x, y \in \mathbb{Q}$ with a bounded denominator. Write (F, G) for a corresponding standard pair. Combining Lemma 3.2 and Theorem 8.1 we see that $\deg(F) \leqslant 2$, hence $\deg(f) \mid 2\deg(g)$. Similarly as in the treatment of case $\deg(F) = 2$ in Section 5, without loss of generality we may assume $F(x) = x^m$ with $m \in \{1, 2\}$. If $\varphi(x) = p_0(x - p_1) \cdots (x - p_s)$, then the rationals $p_1, \dots, p_s$ are distinct, $f(x)$ is similar to $p_0(x^m - p_1) \cdots (x^m - p_s)$ and $g(y)$ is similar to $p_0(G(y) - p_1) \cdots (G(y) - p_s)$, which both have simple rational roots. Thus f is a PTE$_{m,s}$-polynomial, g is a PTE$_{\ell m/k}$-polynomial.

Conversely, let $k \mid \ell$ and g be a PTE$_{\ell/k,s}$-polynomial of degree ℓ with only simple rational roots. Then g is of the form

$$g(y) = p_0(G(y) - p_1)(G(y) - p_2) \cdots (G(y) - p_k) \tag{40}$$

for some $p_0, p_1, \dots, p_k \in \mathbb{Q}$ with $p_1, p_2, \dots, p_k$ distinct. Write $f(x) = p_0(x - p_1) \cdots (x - p_k)$. Then the equation $f(x) = g(y)$ has solutions $(x, y) = (G(X), X)$ for every $X \in \mathbb{Z}$. □

Remark 9.1. If in (40) $p_i = b_i^2$ for $b_i \in \mathbb{Q}, i = 1, \dots, k$, then we may choose $F(x) = x^2$, $f(x) = (x - b_1)(x + b_1) \cdots (x - b_k)(x + b_k)$. This is the case $m = 2$ in which f is both a PTE$_{1,2k}$-polynomial and, after replacing x^2 by x, a PTE$_{2,k}$-polynomial.

Remark 9.2. A remaining question is how large $\deg(v)$ in Table 1 can be, if both f and g have only simple rational roots. Similarly as earlier, without loss of generality we may assume $F(x) = x$ or $F(x) = x^2$. Below we treat the cases with the largest known degree of v. As before we distinguish between the cases (a), (b) and (c) (see Section 5). We have examples with $\deg(v) = 12$ for (a), $\deg(v) = 4$ for (b), $\deg(v) = 0$ for (c).

Example 9.1. Case $F(x) = x, G(y) = v(y), \varphi(x) = x^2 - A^2$.

It is known (see, for example, [57, p. 7]) that the sets

$$T_1 := \{\pm 22, \pm 61, \pm 86, \pm 127, \pm 140, \pm 151\},$$

$$T_2 := \{\pm 35, \pm 47, \pm 94, \pm 121, \pm 146, \pm 148\}$$

form an ideal $\mathrm{PTE}_{2,12}$ pair. Let

$$v(y) = \frac{\prod_{t\in T_1}(y-t) + \prod_{t\in T_2}(y-t)}{2} \quad \text{and} \quad A = \frac{\prod_{t\in T_1} t - \prod_{t\in T_2} t}{2}.$$

Then

$$g(y) = v(y)^2 - A^2 = (v(y) + A)(v(y) - A) = \prod_{t\in T_1\cup T_2}(y-t).$$

Thus $f(x) = x^2 - A^2$ and g have only simple rational roots and the equation $f(x) = g(y)$ has solutions $(x, y) = (\pm v(X), X)$ for every $X \in \mathbb{Z}$.

Example 9.2. Case $F(x) = x^2, G(y) = \alpha y v(y)^2, \varphi(x) = x - A^2$.

It is known (see, for example, [57, p. 7]) that the symmetric sets

$$T_3 := \{-98, -82, -58, -34, 13, 16, 69, 75, 99\} \text{ and } T_4 := \{t \in T_3 : -t\}$$

form an ideal $\mathrm{PTE}_{2,9}$ pair. Put $f(x) = x^2 - A^2$, $g(y) = \prod_{t\in T_3}(y - t^2)$, $A = \prod_{t\in T_3} t$ and $yT(y) = \prod_{t\in T_3}(y-t) + A$. Then

$$g(y^2) = \prod_{t\in T_3}(y-t) \cdot \prod_{t\in T_4}(y-t) = (yT(y) - A)(yT(y) + A).$$

Observe that $yT(y)$ is an odd polynomial (the coefficients of y^i with i odd are 0 in T), so $T(y)$ is an even polynomial. Then $T(y) = v(y^2)$ for some $v(y) \in \mathbb{Q}[y]$ and therefore $g(y) = yv(y)^2 - A^2$. Thus the equation $f(x) = g(y)$ has solutions $(x, y) = (Xv(X^2), X)$ for every $X \in \mathbb{Z}$.

Example 9.3. Case $F(x) = x^2, G(y)$ is of the form $(\alpha y^2 + \beta)v(y)^2$ with $\alpha\beta \neq 0$. An example with $\deg(v) = 0$ is given by Example 6.2.

10 | EQUAL PRODUCTS FROM BLOCKS

We give an application of Theorem 1.1 for equal products from blocks of integers of bounded size. By a block we mean a set of consecutive integers.

Theorem 10.1. *For every positive integer N there exist only finitely many pairs of disjoint blocks A and B of size at most N with the property that for some k, ℓ with $1 \leqslant k < \ell$ and $k \nmid 2\ell$, there exist distinct elements $a_1, \dots, a_k \in A$ and distinct elements $b_1, \dots, b_\ell \in B$ such that*

$$a_1 \cdots a_k = b_1 \cdots b_\ell. \tag{41}$$

Proof. Suppose the statement of the theorem is false for N. We may clearly assume that k and ℓ are fixed and that

$$a_1 < \cdots < a_k \quad \text{and} \quad b_1 < \cdots < b_\ell.$$

Then we may assume as well that the differences

$$c_i := a_i - a_1 \ (1 < i \leqslant k) \quad \text{and} \quad d_j := b_j - b_1 \ (1 < j \leqslant \ell)$$

are fixed. Therefore the equation

$$f(x) := x(x + c_1) \dots (x + c_{k-1}) = y(y + d_1) \dots (y + d_{\ell-1}) =: g(y)$$

would have infinitely many solutions in rationals x, y with a bounded denominator. By Theorem 9.1 the corresponding standard pair (F, G) satisfies $\deg(F) \leqslant 2$. This implies $k \mid 2\ell$, and the statement follows. □

Remark 10.1. Example 5.1 provides examples with $k \mid \ell$ such that (41) has infinitely many integral solutions. Here k can be arbitrarily large. Examples 5.5 and 9.2 provide examples with $k \mid 2\ell, k \nmid \ell$, and (41) has infinitely many integral solutions.

11 | OPEN PROBLEMS

Suppose Equation (2) for $f(x), g(x) \in \mathbb{Q}[x]$ admits infinitely many integral solutions (x, y) with f subject to (1). Put $s = \gcd(\deg(f), \deg(g))$, $m = \deg(f)/s, n = \deg(g)/s$. At the end of Section 7 we have proved that $m \in \{1, 2, 3, 4, 6\}$ or $n \in \{1, 2\}$. Moreover we have argued that every pair $(\deg(f), \deg(g))$ with corresponding $m \in \{1, 2, 3, 4, 6\}$ can be represented.

Problem 1. Which other possibilities are there for m, s, if $n = 1$ or 2 for Equation (2) under (1)?

Now let $f(x), g(x) \in \mathbb{Q}[x]$ both have only simple rational roots and Equation (1.1) have infinitely many integral solutions. We assume $\deg(f) \leqslant \deg(g)$, hence $m \leqslant n$. Theorem 1.1 implies $m \in \{1, 2\}$. Note that the cases $m = s = 1, n$ arbitrary, $m = n$, s arbitrary and $m = 1, n = 2$, s arbitrary are trivial, the latter in view of

$$(x - b_1^2) \cdots (x - b_s^2) = (y^2 - b_1^2) \cdots (y^2 - b_s^2), F(x) = x, G(y) = y^2$$

with solutions (X^2, X) for $X \in \mathbb{Z}$. Example 6.1 shows that the cases $m = 1$, $n \in \{3, 4, 6\}$, s arbitrary are possible, cf. Remark 7.3. Example 5.5 deals with the case $m = 2$, $n = 3$, s arbitrary. Using ideal PTE pairs Example 9.1 can be extended to the cases $m = 1$, $n \in \{5, 7, 8, 9, 10, 12\}$, $s = 2$ and Example 9.2 to the cases $m = 2$, $n \in \{5, 7, 9\}$, $s = 1$.

Problem 2. Which other possibilities for triples m, n, s exist for Equation (2) under (1) and (3)? In particular, which degrees of v are possible for standard pairs of the second kind? (Cf. Example 6.2.)

ACKNOWLEDGEMENTS

L. Hajdu would like to express his thanks to the Rényi Institute, where he was a visiting professor during this research. The authors thank Szabolcs Tengely for some useful remarks. The authors are grateful to the referee for the several insightful and helpful remarks. Research of L. Hajdu was supported in part by the Eötvös Loránd Research Network (ELKH), by the NKFIH grants 128088 and 130909, and the projects EFOP-3.6.1-16-2016-00022 co-financed by the European Union and the European Social Fund.

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