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Square values of Littlewood polynomials

L. Hajdu[1,2] · O. Herendi[1] · Sz. Tengely[1] · N. Varga[1,2]

Dedicated to Rob Tijdeman on the occasion of his 80th birthday.

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Abstract
We study the square values of Littlewood polynomials. Using various methods we give all these values for the degrees $n = 3, 5$ and $n \leq 24$ even. Beside this, we gather computational data (by providing all solutions in a certain range) for n odd with $n \leq 17$. We propose some striking problems for further research, as well.

Keywords Littlewood polynomials · Square values · Runge's method · Elliptic curves · Hyperelliptic curves

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1 Introduction

Littlewood polynomials, that is polynomials with only ± 1 coefficients, have an extensive literature. Their aggregated set of zeroes, that is the set

$$\mathcal{L} = \{\alpha \in \mathbb{C} : \alpha \text{ is a root of a Littlewood polynomial}\},$$

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✉ L. Hajdu
hajdul@science.unideb.hu

O. Herendi
herendi.orsolya@science.unideb.hu

Sz. Tengely
tengely@science.unideb.hu

N. Varga
nvarga@science.unideb.hu

1 Institute of Mathematics, University of Debrecen, P. O. Box 400, Debrecen 4002, Hungary

2 HUN-REN DE Equations, Functions Curves and their Applications Research Group, Debrecen, Hungary

has a lot of interesting properties, and has attracted a lot of attention. We only mention a few related recent papers, and suggest the interested reader to study them and their references. Peled et al. [22] studied double roots of random Littlewood polynomials. Recently, Balister et al. [2] solved an old conjecture of Littlewood, by showing that there exist so-called flat Littlewood polynomials of any degree $n \geq 2$. Han and Schied [16] (among others) investigated so-called step roots of such polynomials, and provided some applications. Hare and Jankauskas [17] and Yakir [34] investigated the roots of Littlewood polynomials inside the unit disk. Beside this, certain divisibility properties of Littlewood polynomials are also of interest; see e.g. Dubickas and Jankauskas [8] who studied the problem of Newman polynomials not dividing any Littlewood polynomial, or Mossinghoff [20] for a study of Littlewood polynomials with prescribed cyclotomic factors. Recently, Diophantine properties of Littlewood polynomials have also been investigated. Hajdu and Varga [14] and Hajdu et al. [15] provided various finiteness results for the power values, shifted power values and polynomial values of such polynomials. It is important to mention that the famous Nagell–Ljunggren equation

$$\frac{x^n - 1}{x - 1} = y^\ell, \tag{1}$$

in integers x, y, n, ℓ with $|x| > 1$, $|y| > 1$, $n > 2$, $\ell \geq 2$ is an important, particular case of this problem, studied by many mathematicians. Indeed, the polynomial on the left hand side of (1) is a particular Littlewood polynomial, with all coefficients equal to one. Equation (1) has a huge literature. We only mention a classical result of Ljunggren [18], stating that the only solution of (1) with $\ell = 2$ is given by $(x, y, n) = (7, \pm 20, 4)$. Since we shall be concerned with square values of Littlewood polynomials, this result is of particular interest for us. We mention already at this point that based upon our new results, the set of solutions seems to be rather restricted in case of general Littlewood polynomials, as well. For further related results and surveys on the Nagell–Ljunggren equation we refer the interested reader to the book Shorey and Tijdeman [26] and the recent paper Bennett and Levin [3], and the references there.

In this paper we explicitly give all square values of Littlewood polynomials of degrees $n = 3, 5$ and $n \leq 24$ even. For this, we need to combine several tools, including elliptic- and higher genus curves, Chabauty's method and Runge's method. To be able to handle the higher degree cases (say with $n \geq 14$), because of the huge number of polynomials to be studied, we need careful considerations and a delicate approach. Beside this, we gather computational data (by providing all solutions in a certain range) for n odd with $n \leq 17$. Based upon our results, we formulate some striking problems for further research, as well.

2 Main results

Our main theorem is the following.

Table 1 All solutions of Eq. (2) with $|x| > 2$ and $y \geq 0$ for $n = 3, 5$

$f(x)$	(x, y)
$\pm x^3 + x^2 \pm x - 1$	$(\pm(t^2+1), t(t^2+2))$ $(t \in \mathbb{Z}_{>1})$
$\pm x^3 - x^2 \pm x + 1$	$(\pm(t^2-1), t(t^2-2))$ $(t \in \mathbb{Z}_{>1})$
$\pm x^3 + x^2 \pm x + 1$	$(\pm 7, 20)$
$\pm x^5 + x^4 \pm x^3 - x^2 \mp x - 1$	$(\pm(t^2+1), t(t^4+3t^2+3))$ $(t \in \mathbb{Z}_{>1})$
$\pm x^5 - x^4 \pm x^3 + x^2 \mp x + 1$	$(\pm(t^2-1), t(t^4-3t^2+3))$ $(t \in \mathbb{Z}_{>1})$

The $\pm$ and $\mp$ signs change together in every row

Theorem 2.1 *Let $f(x)$ be a Littlewood polynomial of degree n with $n = 3, 5$ or $2 \leq n \leq 24$ even. Then all solutions of the equation*

$$f(x) = y^2, \tag{2}$$

in integers x, y with $|x| > 2$ and $y \geq 0$ are precisely those appearing in Tables 1 and 2.

For the sake of completeness, and in particular, to gather more substantial numerical data also in case of odd exponents, we provide a similar statement for n odd. Note however, that in this case we are not able to find all solutions of (2), our purpose is only to get some computational insight in this case, as well.

Proposition 2.1 *Let $f(x)$ be a Littlewood polynomial of degree n with $7 \leq n \leq 17$ odd. Suppose further that $f(x)$ does not belong to any of the families*

$$\pm(x^{2k+1} + \cdots + x^{k+1} - x^k - \cdots - 1) = \pm(x-1)(x^k + \cdots + x + 1)^2, \tag{3}$$

$$\begin{aligned} &\pm(x^{2k+1} - x^{2k} + \cdots + (-1)^{k+2}x^{k+1} + (-1)^k x^k + \cdots + 1) \\ &= \pm(x+1)(x^k - x^{k-1} + \cdots + (-1)^k)^2, \end{aligned} \tag{4}$$

$$\begin{aligned} &\pm(x^{4k+3} + x^{4k+2} - x^{4k+1} - x^{4k} + \cdots + (-1)^k x^{2k+3} + (-1)^k x^{2k+2} \\ &\quad +(-1)^k x^{2k+1} + (-1)^k x^{2k} + \cdots + x + 1) \\ &= \pm(x+1)(x^2+1)(x^{2k} - x^{2k-2} + \cdots + (-1)^k)^2, \end{aligned} \tag{5}$$

$$\begin{aligned} &\pm(x^{4k+3} - x^{4k+2} - x^{4k+1} + x^{4k} + \cdots + (-1)^k x^{2k+3} - (-1)^k x^{2k+2} \\ &\quad +(-1)^k x^{2k+1} - (-1)^k x^{2k} + \cdots + x - 1) \\ &= \pm(x-1)(x^2+1)(x^{2k} - x^{2k-2} + \cdots + (-1)^k)^2. \end{aligned} \tag{6}$$

Then all solutions of Eq. (2) in integers x, y with $100 \geq |x| > 2$ and $y \geq 0$ are precisely those appearing in Table 3.

Note that it is obvious that if $f(x)$ is of the shape (3) or (4) then (2) has infinitely many solutions. Further, by the procedure `IntegralPoints` of Magma [5], we get that the solutions of the equations

$$\pm(x+1)(x^2+1) = y^2 \quad \text{and} \quad \pm(x-1)(x^2+1) = y^2$$

Table 2 All solutions of Eq. (2) with $|x| > 2$ and $y \geq 0$ for n even with $2 \leq n \leq 24$

$f(x)$	(x, y)
$x^4 \pm x^3 - x^2 \mp x + 1$	$(\mp 3, 7)$
$x^4 \pm x^3 - x^2 \pm x - 1$	$(\pm 5, 27)$
$x^4 \pm x^3 + x^2 \mp x - 1$	$(\mp 5, 23)$
$x^4 \pm x^3 + x^2 \pm x + 1$	$(\pm 3, 11)$
$x^6 \pm x^5 - x^4 \mp x^3 - x^2 \mp x + 1$	$(\mp 9, 683)$
$x^6 \pm x^5 - x^4 \mp x^3 + x^2 \pm x + 1$	$(\pm 7, 363)$
$x^6 \pm x^5 + x^4 \pm x^3 - x^2 \pm x + 1$	$(\mp 3, 23)$
$x^{12} \pm x^{11} - x^{10} \mp x^9 - x^8 \pm x^7 - x^6 \mp x^5 + x^4 \mp x^3 + x^2 \pm x + 1$	$(\mp 3, 553)$
$x^{12} \pm x^{11} - x^{10} \pm x^9 + x^8 \mp x^7 + x^6 \mp x^5 - x^4 \pm x^3 + x^2 \pm x + 1$	$(\pm 3, 821)$
$x^{12} \pm x^{11} + x^{10} \mp x^9 - x^8 \mp x^7 + x^6 \pm x^5 - x^4 \pm x^3 - x^2 \mp x + 1$	$(\mp 3, 655)$
$x^{14} \pm x^{13} - x^{12} \mp x^{11} - x^{10} \pm x^9 + x^8 \pm x^7 - x^6 \pm x^5 - x^4 \pm x^3 + x^2 \pm x + 1$	$(\mp 3, 1661)$
$x^{14} \pm x^{13} - x^{12} \pm x^{11} - x^{10} \mp x^9 - x^8 \pm x^7 - x^6 \mp x^5 + x^4 \mp x^3 - x^2 \pm x + 1$	$(\pm 3, 2437)$
$x^{14} \pm x^{13} - x^{12} \pm x^{11} + x^{10} \mp x^9 - x^8 \pm x^7 + x^6 \pm x^5 - x^4 \pm x^3 - x^2 \pm x + 1$	$(\mp 3, 1597)$
$x^{14} \pm x^{13} + x^{12} \mp x^{11} - x^{10} \pm x^9 + x^8 \pm x^7 - x^6 \pm x^5 + x^4 \mp x^3 + x^2 \mp x + 1$	$(\mp 3, 1955)$
$x^{14} \pm x^{13} + x^{12} \pm x^{11} - x^{10} \pm x^9 - x^8 \pm x^7 + x^6 \pm x^5 - x^4 \mp x^3 - x^2 \pm x + 1$	$(\mp 3, 1859)$
$x^{14} \pm x^{13} + x^{12} \pm x^{11} + x^{10} \mp x^9 - x^8 \pm x^7 + x^6 \mp x^5 - x^4 \pm x^3 + x^2 \mp x + 1$	$(\mp 3, 1901)$
$x^{16} \pm x^{15} - x^{14} \mp x^{13} - x^{12} \mp x^{11} - x^{10} \mp x^9 - x^8 \mp x^7 - x^6 \pm x^5 - x^4 \mp x^3 + x^2 \mp x + 1$	$(\mp 3, 5011)$
$x^{16} \pm x^{15} - x^{14} \mp x^{13} - x^{12} \mp x^{11} - x^{10} \mp x^9 - x^8 \pm x^7 - x^6 \mp x^5 - x^4 \mp x^3 + x^2 \mp x + 1$	$(\pm 3, 7087)$

Table 2 continued

$f(x)$	(x, y)
$x^{16} \pm x^{15} - x^{14} \mp x^{13} - x^{12} \pm x^{11} + x^{10} \mp x^9 + x^8 \mp x^7 + x^6 \pm x^5 - x^4 \mp x^3 - x^2 \pm x + 1$	$(\pm 3, 7121)$
$x^{16} \pm x^{15} - x^{14} \mp x^{13} + x^{12} \mp x^{11} - x^{10} \mp x^9 + x^8 \pm x^7 + x^6 \mp x^5 - x^4 \mp x^3 + x^2 \pm x + 1$	$(\mp 3, 5117)$
$x^{16} \pm x^{15} - x^{14} \mp x^{13} + x^{12} \pm x^{11} + x^{10} \pm x^9 - x^8 \mp x^7 - x^6 \mp x^5 - x^4 \mp x^3 + x^2 \pm x + 1$	$(\mp 3, 5089)$
$x^{16} \pm x^{15} - x^{14} \pm x^{13} + x^{12} \mp x^{11} + x^{10} \pm x^9 - x^8 \pm x^7 - x^6 \pm x^5 + x^4 \pm x^3 + x^2 \pm x + 1$	$(\pm 5, 422409)$
$x^{16} \pm x^{15} + x^{14} \pm x^{13} + x^{12} \mp x^{11} - x^{10} \pm x^9 - x^8 \mp x^7 + x^6 \pm x^5 - x^4 \pm x^3 + x^2 \mp x + 1$	$(\pm 3, 8005)$
$x^{16} \pm x^{15} + x^{14} \pm x^{13} + x^{12} \mp x^{11} + x^{10} \pm x^9 + x^8 \pm x^7 - x^6 \mp x^5 + x^4 \pm x^3 + x^2 \mp x + 1$	$(\mp 3, 5713)$
$x^{16} \pm x^{15} + x^{14} \pm x^{13} + x^{12} \pm x^{11} + x^{10} \mp x^9 + x^8 \pm x^7 - x^6 \pm x^5 - x^4 \pm x^3 + x^2 \mp x + 1$	$(\pm 3, 8033)$
$x^{18} \pm x^{17} - x^{16} \mp x^{15} - x^{14} \mp x^{13} - x^{12} \mp x^{11} + x^{10} \mp x^9 - x^8 \pm x^7 + x^6 \pm x^5 + x^4 \mp x^3 + x^2 \mp x + 1$	$(\pm 3, 21263)$
$x^{18} \pm x^{17} - x^{16} \mp x^{15} - x^{14} \pm x^{13} - x^{12} \mp x^{11} - x^{10} \mp x^9 + x^8 \pm x^7 - x^6 \mp x^5 - x^4 \pm x^3 - x^2 \pm x + 1$	$(\mp 3, 14927)$
$x^{18} \pm x^{17} - x^{16} \mp x^{15} + x^{14} \mp x^{13} + x^{12} \mp x^{11} - x^{10} \mp x^9 + x^8 \mp x^7 - x^6 \pm x^5 - x^4 \pm x^3 - x^2 \mp x + 1$	$(\mp 3, 15383)$
$x^{18} \pm x^{17} - x^{16} \mp x^{15} + x^{14} \pm x^{13} - x^{12} \pm x^{11} + x^{10} \pm x^9 + x^8 \pm x^7 - x^6 \pm x^5 - x^4 \pm x^3 - x^2 \mp x + 1$	$(\mp 3, 15235)$
$x^{18} \pm x^{17} - x^{16} \pm x^{15} - x^{14} \mp x^{13} + x^{12} \mp x^{11} - x^{10} \pm x^9 + x^8 \pm x^7 + x^6 \pm x^5 + x^4 \mp x^3 + x^2 \mp x + 1$	$(\mp 3, 14083)$

Table 2 continued

$f(x)$	(x, y)
$x^{18} \pm x^{17} + x^{16} \pm x^{15} - x^{14} \mp x^{13} + x^{12} \pm x^{11} + x^{10} \mp x^{9} + x^{8} \pm x^{7} - x^{6} \pm x^{5} - x^{4} \mp x^{3} + x^{2} \mp x + 1$	$(\mp 3, 16859)$
$x^{18} \pm x^{17} + x^{16} \pm x^{15} + x^{14} \pm x^{13} + x^{12} \mp x^{11} - x^{10} \pm x^{9} + x^{8} \mp x^{7} - x^{6} \mp x^{5} - x^{4} \pm x^{3} + x^{2} \mp x + 1$	$(\mp 3, 17053)$
$x^{20} \pm x^{19} - x^{18} \mp x^{17} - x^{16} \pm x^{15} + x^{14} \pm x^{13} - x^{12} \mp x^{11} - x^{10} \pm x^{9} + x^{8} \mp x^{7} + x^{6} \mp x^{5} - x^{4} \pm x^{3} - x^{2} \pm x + 1$	$(\mp 3, 44851)$
$x^{20} \pm x^{1}9 - x^{18} \mp x^{17} + x^{16} \mp x^{15} + x^{14} \mp x^{13} + x^{12} \mp x^{11} - x^{10} \pm x^{9} + x^{8} \mp x^{7} - x^{6} \pm x^{5} + x^{4} \mp x^{3} + x^{2} \mp x + 1$	$(\mp 3, 46159)$
$x^{20} \pm x^{19} - x^{18} \pm x^{17} + x^{16} \pm x^{15} - x^{14} \mp x^{13} + x^{12} \mp x^{11} - x^{10} \pm x^{9} - x^{8} \mp x^{7} - x^{6} \mp x^{5} + x^{4} \pm x^{3} + x^{2} \mp x + 1$	$(\pm 3, 66649)$
$x^{20} \pm x^{19} + x^{18} \mp x^{17} + x^{16} \mp x^{15} + x^{14} \mp x^{13} + x^{12} \mp x^{11} - x^{10} \mp x^{9} + x^{8} \mp x^{7} - x^{6} \pm x^{5} - x^{4} \pm x^{3} + x^{2} \mp x + 1$	$(\mp 3, 53903)$
$x^{20} \pm x^{19} + x^{18} \pm x^{17} + x^{16} \mp x^{15} + x^{14} \mp x^{13} + x^{12} \pm x^{11} + x^{10} \mp x^{9} - x^{8} \pm x^{7} - x^{6} \pm x^{5} + x^{4} \mp x^{3} + x^{2} \pm x + 1$	$(\mp 3, 51449)$
$x^{20} \pm x^{19} + x^{18} \pm x^{17} + x^{16} \pm x^{15} + x^{14} \mp x^{13} + x^{12} \pm x^{11} + x^{10} \pm x^{9} - x^{8} \mp x^{7} - x^{6} \pm x^{5} - x^{4} \pm x^{3} - x^{2} \pm x + 1$	$(\mp 3, 51169)$
$x^{22} \pm x^{21} - x^{20} \mp x^{19} - x^{18} \pm x^{17} + x^{16} \mp x^{15} + x^{14} \mp x^{13} + x^{12} \pm x^{11} + x^{10} \mp x^{9} - x^{8} \pm x^{7} + x^{6} \pm x^{5} + x^{4} \mp x^{3} - x^{2} \pm x + 1$	$(\mp 3, 134699)$
$x^{22} \pm x^{21} - x^{20} \mp x^{19} + x^{18} \mp x^{17} - x^{16} \pm x^{15} - x^{14} \mp x^{13} + x^{12} \pm x^{11} + x^{10} \pm x^{9} + x^{8} \pm x^{7} - x^{6} \pm x^{5} + x^{4} \pm x^{3} - x^{2} \mp x + 1$	$(\mp 3, 138031)$
$x^{22} \pm x^{21} - x^{20} \mp x^{19} + x^{18} \mp x^{17} - x^{16} \pm x^{15} - x^{14} \pm x^{13} - x^{12} \mp x^{11} + x^{10} \mp x^{9} - x^{8} \mp x^{7} + x^{6} \mp x^{5} + x^{4} \pm x^{3} - x^{2} \mp x + 1$	$(\mp 3, 138017)$
$x^{22} \pm x^{21} - x^{20} \mp x^{19} + x^{18} \pm x^{17} + x^{16} \mp x^{15} + x^{14} \mp x^{13} + x^{12} \mp x^{11} + x^{10} \pm x^{9} - x^{8} \mp x^{7} + x^{6} \mp x^{5} + x^{4} \pm x^{3} - x^{2} \mp x + 1$	$(\mp 3, 137545)$
$x^{22} \pm x^{21} - x^{20} \mp x^{19} + x^{18} \pm x^{17} + x^{16} \mp x^{15} + x^{14} \pm x^{13} - x^{12} \mp x^{11} + x^{10} \mp x^{9} + x^{8} \pm x^{7} - x^{6} \pm x^{5} + x^{4} \pm x^{3} - x^{2} \mp x + 1$	$(\mp 3, 137531)$
$x^{22} \pm x^{21} - x^{20} \pm x^{19} - x^{18} \pm x^{17} + x^{16} \pm x^{15} + x^{14} \pm x^{13} - x^{12} \mp x^{11} + x^{10} \pm x^{9} - x^{8} \mp x^{7} - x^{6} \pm x^{5} - x^{4} \mp x^{3} + x^{2} \pm x + 1$	$(\mp 3, 125645)$
$x^{22} \pm x^{21} - x^{20} \pm x^{19} + x^{18} \mp x^{17} + x^{16} \pm x^{15} + x^{14} \pm x^{13} - x^{12} \mp x^{11} - x^{10} \mp x^{9} + x^{8} \pm x^{7} - x^{6} \pm x^{5} - x^{4} \pm x^{3} + x^{2} \pm x + 1$	$(\pm 3, 199595)$
$x^{22} \pm x^{21} - x^{20} \pm x^{19} + x^{18} \pm x^{17} + x^{16} \pm x^{15} - x^{14} \pm x^{13} + x^{12} \pm x^{11} + x^{10} \pm x^{9} + x^{8} \mp x^{7} - x^{6} \pm x^{5} + x^{4} \pm x^{3} + x^{2} \pm x + 1$	$(\pm 3, 200221)$
$x^{22} \pm x^{21} + x^{20} \mp x^{19} - x^{18} \mp x^{17} - x^{16} \mp x^{15} - x^{14} \mp x^{13} - x^{12} \pm x^{11} + x^{10} \pm x^{9} + x^{8} \mp x^{7} - x^{6} \pm x^{5} - x^{4} \pm x^{3} - x^{2} \pm x + 1$	$(\pm 3, 208771)$
$x^{22} \pm x^{21} + x^{20} \pm x^{19} - x^{18} \pm x^{17} - x^{16} \pm x^{15} + x^{14} \pm x^{13} + x^{12} \mp x^{11} + x^{10} \mp x^{9} - x^{8} \pm x^{7} - x^{6} \mp x^{5} + x^{4} \mp x^{3} - x^{2} \pm x + 1$	$(\mp 3, 150583)$
$x^{22} \pm x^{21} + x^{20} \pm x^{19} + x^{18} \pm x^{17} - x^{16} \pm x^{15} - x^{14} \mp x^{13} + x^{12} \mp x^{11} - x^{10} \mp x^{9} - x^{8} \mp x^{7} + x^{6} \pm x^{5} + x^{4} \pm x^{3} - x^{2} \pm x + 1$	$(\mp 3, 153113)$
$x^{24} \pm x^{23} - x^{22} \mp x^{21} - x^{20} \mp x^{19} - x^{18} \mp x^{17} - x^{16} \mp x^{15} + x^{14} \pm x^{13} - x^{12} \mp x^{11} - x^{10} \mp x^{9} - x^{8} \pm x^{7} - x^{6} \mp x^{5} + x^{4} \pm x^{3} + x^{2} \mp x + 1$	$(\pm 3, 574033)$

Table 2 continued

$f(x)$	(x, y)
$x^{24} \pm x^{23} - x^{22} \mp x^{21} + x^{20} \pm x^{19} - x^{18} \pm x^{17} + x^{16} \pm x^{15} + x^{14} \pm x^{13} + x^{12} \mp x^{11} + x^{10} \pm x^{9} - x^{8} \mp x^{7} + x^{6} \mp x^{5} - x^{4} \mp x^{3} - x^{2} \pm x + 1$	$(\pm 3, 582397)$
$x^{24} \pm x^{23} - x^{22} \mp x^{21} + x^{20} \pm x^{19} + x^{18} \mp x^{17} - x^{16} \pm x^{15} + x^{14} \pm x^{13} - x^{12} \mp x^{11} - x^{10} \mp x^{9} - x^{8} \pm x^{7} - x^{6} \mp x^{5} + x^{4} \pm x^{3} + x^{2} \mp x + 1$	$(\mp 3, 412495)$
$x^{24} \pm x^{23} - x^{22} \pm x^{21} - x^{20} \mp x^{19} + x^{18} \mp x^{17} - x^{16} \pm x^{15} - x^{14} \mp x^{13} - x^{12} \pm x^{11} - x^{10} \mp x^{9} - x^{8} \mp x^{7} - x^{6} \mp x^{5} - x^{4} \pm x^{3} - x^{2} \pm x + 1$	$(\pm 3, 592643)$
$x^{24} \pm x^{23} - x^{22} \pm x^{21} - x^{20} \mp x^{19} + x^{18} \pm x^{17} + x^{16} \mp x^{15} - x^{14} \pm x^{13} + x^{12} \pm x^{11} + x^{10} \pm x^{9} + x^{8} \mp x^{7} + x^{6} \pm x^{5} + x^{4} \mp x^{3} - x^{2} \pm x + 1$	$(\pm 3, 592913)$
$x^{24} \pm x^{23} - x^{22} \pm x^{21} + x^{20} \pm x^{19} - x^{18} \mp x^{17} - x^{16} \mp x^{15} - x^{14} \mp x^{13} + x^{12} \mp x^{11} - x^{10} \mp x^{9} + x^{8} \mp x^{7} + x^{6} \pm x^{5} + x^{4} \mp x^{3} + x^{2} \pm x + 1$	$(\mp 3, 385331)$
$x^{24} \pm x^{23} - x^{22} \pm x^{21} + x^{20} \pm x^{19} + x^{18} \mp x^{17} + x^{16} \mp x^{15} + x^{14} \mp x^{13} + x^{12} \pm x^{11} - x^{10} \mp x^{9} - x^{8} \mp x^{7} + x^{6} \mp x^{5} - x^{4} \mp x^{3} + x^{2} \mp x + 1$	$(\pm 3, 600493)$
$x^{24} \pm x^{23} - x^{22} \pm x^{21} + x^{20} \pm x^{19} + x^{18} \pm x^{17} - x^{16} \mp x^{15} - x^{14} \mp x^{13} - x^{12} \pm x^{11} + x^{10} \mp x^{9} - x^{8} \mp x^{7} + x^{6} \pm x^{5} + x^{4} \mp x^{3} - x^{2} \mp x + 1$	$(\mp 3, 385999)$
$x^{24} \pm x^{23} + x^{22} \mp x^{21} - x^{20} \pm x^{19} - x^{18} \pm x^{17} + x^{16} \pm x^{15} - x^{14} \pm x^{13} - x^{12} \mp x^{11} + x^{10} \mp x^{9} - x^{8} \mp x^{7} + x^{6} \mp x^{5} + x^{4} \mp x^{3} + x^{2} \mp x + 1$	$(\mp 3, 474325)$
$x^{24} \pm x^{23} + x^{22} \mp x^{21} + x^{20} \mp x^{19} - x^{18} \mp x^{17} + x^{16} \mp x^{15} + x^{14} \pm x^{13} - x^{12} \pm x^{11} + x^{10} \pm x^{9} + x^{8} \pm x^{7} - x^{6} \mp x^{5} - x^{4} \mp x^{3} - x^{2} \mp x + 1$	$(\mp 3, 484333)$
$x^{24} \pm x^{23} + x^{22} \mp x^{21} + x^{20} \mp x^{19} + x^{18} \mp x^{17} - x^{16} \pm x^{15} + x^{14} \mp x^{13} - x^{12} \mp x^{11} - x^{10} \mp x^{9} - x^{8} \pm x^{7} + x^{6} \pm x^{5} - x^{4} \pm x^{3} - x^{2} \pm x + 1$	$(\pm 3, 632495)$
$x^{24} \pm x^{23} + x^{22} \pm x^{21} - x^{20} \mp x^{19} - x^{18} \pm x^{17} - x^{16} \mp x^{15} - x^{14} \mp x^{13} + x^{12} \mp x^{11} + x^{10} \mp x^{9} + x^{8} \pm x^{7} + x^{6} \pm x^{5} - x^{4} \pm x^{3} - x^{2} \mp x + 1$	$(\mp 3, 454241)$
$x^{24} \pm x^{23} + x^{22} \pm x^{21} - x^{20} \mp x^{19} + x^{18} \mp x^{17} + x^{16} \pm x^{15} + x^{14} \mp x^{13} - x^{12} \mp x^{11} - x^{10} \mp x^{9} + x^{8} \mp x^{7} - x^{6} \mp x^{5} + x^{4} \mp x^{3} + x^{2} \pm x + 1$	$(\mp 3, 455449)$
$x^{24} \pm x^{23} + x^{22} \pm x^{21} - x^{20} \pm x^{19} - x^{18} \pm x^{17} - x^{16} \mp x^{15} - x^{14} \mp x^{13} + x^{12} \pm x^{11} + x^{10} \pm x^{9} - x^{8} \mp x^{7} - x^{6} \mp x^{5} + x^{4} \mp x^{3} + x^{2} \mp x + 1$	$(\pm 3, 644801)$
$x^{24} \pm x^{23} + x^{22} \pm x^{21} + x^{20} \pm x^{19} + x^{18} \mp x^{17} - x^{16} \pm x^{15} + x^{14} \pm x^{13} + x^{12} \pm x^{11} + x^{10} \mp x^{9} - x^{8} \pm x^{7} + x^{6} \pm x^{5} - x^{4} \mp x^{3} - x^{2} \mp x + 1$	$(\pm 3, 650615)$
$x^{24} \pm x^{23} + x^{22} \pm x^{21} + x^{20} \pm x^{19} + x^{18} \pm x^{17} + x^{16} \pm x^{15} - x^{14} \pm x^{13} + x^{12} \pm x^{11} + x^{10} \pm x^{9} - x^{8} \pm x^{7} + x^{6} \pm x^{5} + x^{4} \mp x^{3} + x^{2} \mp x + 1$	$(\mp 3, 460231)$

The $\pm$ and $\mp$ signs change together in every row

Table 3 All solutions of equation (2) excluding polynomials $f(x)$ with (3), (4), (5), (6), with $100 \geq |x| > 2$ and $y \geq 0$ for $7 \leq n \leq 17$ odd

$f(x)$	(x, y)
$\pm x^7 - x^6 \mp x^5 - x^4 \pm x^3 - x^2 \pm x + 1$	$(\pm 3, 34)$
$\pm x^9 - x^8 \pm x^7 + x^6 \pm x^5 + x^4 \pm x^3 - x^2 \pm x + 1$	$(\pm 3, 128)$
$\pm x^9 + x^8 \mp x^7 + x^6 \pm x^5 - x^4 \pm x^3 - x^2 \mp x + 1$	$(\pm 3, 158)$
$\pm x^9 + x^8 \pm x^7 - x^6 \mp x^5 + x^4 \pm x^3 - x^2 \mp x + 1$	$(\pm 3, 166)$
$\pm x^{11} - x^{10} \mp x^9 + x^8 \mp x^7 - x^6 \pm x^5 + x^4 \pm x^3 - x^2 \mp x + 1$	$(\pm 3, 320)$
$\pm x^{11} - x^{10} \mp x^9 + x^8 \pm x^7 + x^6 \mp x^5 - x^4 \pm x^3 - x^2 \mp x + 1$	$(\pm 3, 328)$
$\pm x^{11} - x^{10} \pm x^9 - x^8 \mp x^7 - x^6 \mp x^5 + x^4 \pm x^3 - x^2 \pm x + 1$	$(\pm 3, 358)$
$\pm x^{11} - x^{10} \pm x^9 + x^8 \pm x^7 - x^6 \pm x^5 - x^4 \mp x^3 - x^2 \mp x + 1$	$(\pm 3, 382)$
$\pm x^{13} - x^{12} \mp x^{11} + x^{10} \pm x^9 + x^8 \pm x^7 - x^6 \mp x^5 - x^4 \pm x^3 + x^2 \mp x + 1$	$(\pm 3, 986)$
$\pm x^{13} - x^{12} \pm x^{11} - x^{10} \mp x^9 - x^8 \pm x^7 + x^6 \pm x^5 - x^4 \mp x^3 - x^2 \mp x + 1$	$(\pm 3, 1076)$
$\pm x^{13} - x^{12} \pm x^{11} - x^{10} \pm x^9 + x^8 \pm x^7 + x^6 \mp x^5 + x^4 \pm x^3 - x^2 \pm x + 1$	$(\pm 3, 1100)$
$\pm x^{13} + x^{12} \mp x^{11} - x^{10} \mp x^9 - x^8 \pm x^7 + x^6 \mp x^5 - x^4 \pm x^3 + x^2 \pm x + 1$	$(\pm 3, 1366)$
$\pm x^{13} + x^{12} \pm x^{11} + x^{10} \mp x^9 + x^8 \mp x^7 + x^6 \mp x^5 - x^4 \mp x^3 - x^2 \pm x + 1$	$(\pm 3, 1532)$
$\pm x^{15} - x^{14} \pm x^{13} - x^{12} \mp x^{11} - x^{10} \pm x^9 - x^8 \pm x^7 - x^6 \mp x^5 + x^4 \pm x^3 + x^2 \mp x + 1$	$(\pm 3, 3226)$
$\pm x^{15} - x^{14} \pm x^{13} - x^{12} \pm x^{11} - x^{10} \mp x^9 - x^8 \mp x^7 + x^6 \mp x^5 + x^4 \pm x^3 - x^2 \pm x + 1$	$(\pm 3, 3274)$
$\pm x^{15} - x^{14} \pm x^{13} - x^{12} \pm x^{11} + x^{10} \pm x^9 - x^8 \mp x^7 + x^6 \pm x^5 - x^4 \mp x^3 - x^2 \mp x + 1$	$(\pm 3, 3298)$
$\pm x^{15} - x^{14} \pm x^{13} + x^{12} \mp x^{11} - x^{10} \pm x^9 + x^8 \mp x^7 - x^6 \mp x^5 - x^4 \pm x^3 + x^2 \mp x + 1$	$(\pm 3, 3388)$
$\pm x^{17} - x^{16} \mp x^{15} - x^{14} \pm x^{13} + x^{12} \mp x^{11} - x^{10} \mp x^9 - x^8 \mp x^7 + x^6 \pm x^5 - x^4 \pm x^3 - x^2 \pm x + 1$	$(\pm 3, 8296)$
$\pm x^{17} - x^{16} \mp x^{15} + x^{14} \mp x^{13} + x^{12} \mp x^{11} - x^{10} \pm x^9 - x^8 \mp x^7 - x^6 \mp x^5 - x^4 \mp x^3 - x^2 \pm x + 1$	$(\pm 3, 8674)$
$\pm x^{17} - x^{16} \mp x^{15} + x^{14} \pm x^{13} + x^{12} \pm x^{11} - x^{10} \pm x^9 - x^8 \mp x^7 + x^6 \pm x^5 + x^4 \pm x^3 - x^2 \pm x + 1$	$(\pm 3, 8876)$
$\pm x^{17} - x^{16} \pm x^{15} - x^{14} \pm x^{13} - x^{12} \mp x^{11} - x^{10} \pm x^9 + x^8 \mp x^7 + x^6 \pm x^5 - x^4 \mp x^3 - x^2 \mp x + 1$	$(\pm 3, 9824)$
$\pm x^{17} - x^{16} \pm x^{15} - x^{14} \pm x^{13} - x^{12} \pm x^{11} + x^{10} \pm x^9 + x^8 \mp x^7 + x^6 \mp x^5 + x^4 \pm x^3 - x^2 \pm x + 1$	$(\pm 3, 9848)$

Table 3 continued

$f(x)$	(x, y)
$\pm x^{17} - x^{16} \pm x^{15} - x^{14} \pm x^{13} + x^{12} \pm x^{11} - x^{10} \pm x^{9} + x^{8} \pm x^{7} - x^{6} \mp x^{5} + x^{4} \pm x^{3} + x^{2} \mp x + 1$	$(\pm 3, 9896)$
$\pm x^{17} - x^{16} \pm x^{15} + x^{14} \mp x^{13} - x^{12} \pm x^{11} + x^{10} \pm x^{9} - x^{8} \mp x^{7} + x^{6} \pm x^{5} - x^{4} \mp x^{3} + x^{2} \mp x + 1$	$(\pm 3, 10166)$
$\pm x^{17} + x^{16} \mp x^{15} + x^{14} \pm x^{13} - x^{12} \pm x^{11} + x^{10} \mp x^{9} - x^{8} \mp x^{7} - x^{6} \pm x^{5} + x^{4} \pm x^{3} - x^{2} \mp x + 1$	$(\pm 3, 12802)$
$\pm x^{17} + x^{16} \pm x^{15} - x^{14} \mp x^{13} - x^{12} \pm x^{11} - x^{10} \pm x^{9} + x^{8} \pm x^{7} + x^{6} \pm x^{5} - x^{4} \mp x^{3} + x^{2} \pm x + 1$	$(\pm 3, 13408)$
$\pm x^{17} + x^{16} \pm x^{15} - x^{14} \mp x^{13} + x^{12} \pm x^{11} + x^{10} \mp x^{9} - x^{8} \pm x^{7} + x^{6} \mp x^{5} - x^{4} \pm x^{3} - x^{2} \mp x + 1$	$(\pm 3, 13450)$
$\pm x^{17} + x^{16} \pm x^{15} + x^{14} \pm x^{13} - x^{12} \pm x^{11} - x^{10} \mp x^{9} + x^{8} \pm x^{7} - x^{6} \mp x^{5} + x^{4} \mp x^{3} - x^{2} \mp x + 1$	$(\pm 3, 13874)$

The $\pm$ and $\mp$ signs change together in every row are given by $(x, y) = (0, \pm 1), (\pm 1, 0), (\pm 7, \pm 20)$. So if $f(x)$ is of the shape (5) or (6) then (2) has a solution with $x = \pm 7$ (with $+$ and $-$ signs on the left hand side, respectively) for any n with $n \equiv 3 \pmod 4$.

Remark We can exclude the cases $|x| \leq 2$, even if we do not prescribe an upper bound for n. (Note that, clearly, the square values of Littlewood polynomials of any fixed degree at places x with $|x| \leq 2$ can be listed without any trouble.) The case $x = 0$ is trivial, and the cases $x = \pm 1$ are also easy. The cases $x = \pm 2$ require a little more attention. Since if $f(x)$ is a Littlewood polynomial then so is $f(-x)$, thus talking about the values of Littlewood polynomials at ± 2, it is sufficient to consider the values at 2. As one can readily check, any odd integer r with $-2^{n+1} + 1 \leq r \leq 2^{n+1} - 1$ can be represented in the form $f(2)$ with a unique Littlewood polynomial f of degree n, in precisely one way. Indeed, these numbers r are just given by

$$-2^n - 2^{n-1} - \cdots - 2 - 1, -2^n - 2^{n-1} - \cdots - 2 + 1, \ldots, 2^n + 2^{n-1} + \cdots + 2 + 1,$$

and obviously the representation (for fixed n) is unique. This shows that the solutions of (2) with $|x| = 2$ are also completely understood and described.

Based upon our results, we think that the solution set to (2) is very restricted. We propose the following problems. The first two problems ask about the existence of bounds for the solutions in a general sense. The third problem offers a possible complete description of all solutions with $|x| > 3$. Note that an affirmative answer to the first question in Problem 3 would certainly yield an affirmative answer to the first two problems, in an explicit form.

Problem 1 Is it true that there exists an absolute constant c_1 such that for all solutions x, y of (2) with any Littlewood polynomial f of degree $n \geq 2$ not of the shape (3) and (4) we have $|x| \leq c_1$?

Problem 2 Is it true that there exists an absolute constant c_2 such that if $n > c_2$ then for all solutions x, y of (2) with any Littlewood polynomial f of degree n not of the shape (3), (4), (5), (6), we have $|x| \leq 3$? On the other hand, is it true that there exist infinitely many Littlewood polynomials $f(x)$ not of the shape (4) such that $f(3)$ is a square? Even more, is it true that for every $n > 10$ there exists a Littlewood polynomial $f(x)$ of degree n not of the shape (4) such that $f(3)$ is a square?

Problem 3 Is it true that if (x, y) is a solution of (2) with $|x| > 3$ and $y \geq 0$ for some Littlewood polynomial f of degree $n \geq 2$ not of the form (3), (4), (5), (6), then

$$(x, y, n) = (\pm 5, 23, 4), (\pm 5, 27, 4), (\pm 7, 363, 6), (\pm 9, 683, 6), (\pm 5, 422409, 16),$$

holds? In particular, is it true that for all solutions of (2) with x even with any Littlewood polynomial f not of the shape (3), (4) we have $|x| \leq 2$?

Now we give some heuristics which shed some light on the problems, in particular, support that solutions with $|x| > 3$ should be extremely rare.

Some heuristics behind the problems. We have already argued that solutions with $|x| \le 2$ will appear infinitely often. So let z be an integer with $|z| \ge 3$, and consider the question whether $f(z)$ is a square for some Littlewood polynomial of degree n. Using symmetry as earlier, we may restrict our attention to positive integers z and Littlewood polynomials f of degree n with leading coefficient 1. Observe that for fixed n, we have

$$z^n - \frac{z^n - 1}{z - 1} \le f(z) \le z^n + \frac{z^n - 1}{z - 1}.$$

So $f(z)$ belongs to an interval of length approximately

$$\ell = \frac{2z^n}{z - 1}. \tag{7}$$

(It will be clear that the points where we work 'approximately', do not change the general argument.) Further, the number of squares S in this interval is given by

$$\sqrt{z^n + \frac{z^n - 1}{z - 1}} - \sqrt{z^n - \frac{z^n - 1}{z - 1}} = \frac{2z^n}{\sqrt{z^n + \frac{z^n-1}{z-1}} + \sqrt{z^n - \frac{z^n-1}{z-1}}},$$

so approximately

$$S = (\sqrt{z})^n. \tag{8}$$

The number of Littlewood polynomials of degree n (with leading coefficient 1) is 2^n. Considering the values of these polynomials at z as independent random variables, in view of (7) and (8) we get that the probability that none of them is a square, is approximately given by

$$P_n(z) := \left(1 - \frac{1}{(\sqrt{z})^n}\right)^{2^n} = \left(\left(1 - \frac{1}{(\sqrt{z})^n}\right)^{(\sqrt{z})^n}\right)^{\left(\frac{2}{\sqrt{z}}\right)^n}.$$

As

$$\lim_{n\to\infty} \left(1 - \frac{1}{(\sqrt{z})^n}\right)^{(\sqrt{z})^n} = \frac{1}{e},$$

we see that $P_n(3)$ tends to 0, while $P_n(z)$ tends to 1 for $z \ge 5$ as n tends to infinity. This suggests that for n 'large enough', for $z = 3$ there is a Littlewood polynomial with $f(z)$ being a square, while for $z \ge 5$ just the opposite statement is valid. We are left with the case $z = 4$. However, then (2) implies that $\pm 4 \pm 1 \equiv y^2 \pmod 8$ should be valid, which cannot hold. That is, $z = 4$ never yields a solution to (2).

Altogether, it seems that the above heuristics suggest that the answers to the problems might be affirmative.

Remark As we have seen, for n odd we have the identities (3), (4), (5), (6). The latter two are easy to handle; we did that already after Proposition 2.1. On the other hand, in case of (3), (4) there are infinitely many solutions for (2). Note that the existence of these solutions does not 'contradict' our heuristics above: special structures will certainly skip through such a probabilistic argument. There are no further identities similar to (3) and (4) (taken from [14]). In case of n even, any Littlewood polynomial of degree n is congruent to $(x^{n+1}-1)/(x-1)$ modulo 2, which has a square-free numerator. Further, the proof of Theorem 2.2 of [14] shows that a Littlewood polynomial of odd degree is of the form $g(x)(h(x))^2$ with $\deg(g)=1$ if and only if $f(x)$ appears in (3) or (4). So altogether, the above problems (after excluding the special polynomials in (3), (4), (5), (6)) might be answered to the affirmative.

3 The proof of Theorem 2.1

As in the proof of our theorem we use different tools in the different cases, we give our argument in different subsections. We made available the codes we used at https://shrek.unideb.hu/~tengely/LittleWood.html.

3.1 The proof of Theorem 2.1 for $n=3$

Consider the Littlewood polynomials $f(x)=\pm x^3\pm x^2\pm x\pm 1$, and investigate (2) for them. There are 16 equations to study, however, using the substitution $x\to -x$ it is sufficient to consider only those where the leading coefficient of $f(x)$ is positive. Indeed, (x,y) is a solution to $f(x)=y^2$ if and only if $(-x,y)$ is a solution to $f(-x)=y^2$. Two of the curves implied by (2) are singular. Namely, for

$$f(x)=x^3+x^2-x-1=(x-1)(x+1)^2,$$

and

$$f(x)=x^3-x^2-x+1=(x+1)(x-1)^2,$$

(2) has infinitely many solutions. These solutions are trivial to describe: they are given by

$$(x,y)=(t^2+1,\pm t(t^2+2))\quad\text{and}\quad(x,y)=(t^2-1,\pm t(t^2-2)),$$

respectively. By our assumptions $|x|>2$ and $y\geq 0$, we can omit the $\pm$ signs, and we may assume that $t>1$. In the other cases (2) is an elliptic curve, and we may apply the procedure `IntegralPoints` of Magma [5] (based upon methods of Gebel et al. [13] and Stroeker and Tzanakis [31]) to obtain the integral points (solutions). We get that (2) has an integral solution with $|x| > 2$ only for $f(x) = x^3 + x^2 + x + 1$, when the only such solutions are given by $(x, y) = (7, \pm 20)$. So our theorem follows in this case. The solutions (with $|x| > 2$ and $y \geq 0$, taking into consideration the substitution $x \to -x$ as well) are given in the first three rows of Table 1.

3.2 The proof of Theorem 2.1 for $n = 5$

We need to study Eq. (2) for $f(x) = \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x \pm 1$, which now is a hyperelliptic equation. Similarly as in case $n = 3$, we may restrict to polynomials $f(x)$ with positive leading coefficient, so we need to consider 32 equations. We shall use Chabauty's method [7] and the hyperelliptic logarithm method [11] to handle these equations.

We have two reducible cases that are not square-free, these are as follows:

$$x^5 + x^4 + x^3 - x^2 - x - 1 = (x-1)(x^2 + x + 1)^2,$$
$$x^5 - x^4 + x^3 + x^2 - x + 1 = (x+1)(x^2 - x + 1)^2.$$

In these cases we can easily describe all solutions of (2). These are given by

$$(x, y) = (t^2 + 1, t(t^4 + 3t^2 + 3)) \ (t \in \mathbb{Z}),$$

and

$$(x, y) = (t^2 - 1, t(t^4 - 3t^2 + 3)) \ (t \in \mathbb{Z}),$$

respectively.

In the remaining cases we need to consider genus 2 curves. By using Magma [5] we can compute the ranks of the Jacobians and determine generators of the Mordell–Weil groups based on Stoll's papers [27–29].

The rank of the Jacobian is 0 for the hyperelliptic curves related to the following polynomials:

$$x^5 + x^4 - x^3 - x^2 - x - 1,\ x^5 + x^4 - x^3 - x^2 + x - 1,\ x^5 - x^4 + x^3 - x^2 - x - 1,$$
$$x^5 - x^4 + x^3 - x^2 + x - 1,\ x^5 + x^4 - x^3 + x^2 - x - 1,\ x^5 - x^4 - x^3 + x^2 + x - 1,$$
$$x^5 + x^4 + x^3 + x^2 - x - 1,\ x^5 + x^4 - x^3 + x^2 + x - 1,\ x^5 - x^4 - x^3 - x^2 + x - 1.$$

Computing the rational points on the curves (2) by the procedure `Chabauty0` of Magma, we obtained only solutions with $x = \pm 1$ and $y = 0$ in these cases.

Consider now the polynomials that yield rank 1 Mordell–Weil groups. These are given by

$$x^5 - x^4 - x^3 - x^2 - x - 1,\ x^5 - x^4 + x^3 - x^2 - x + 1,\ x^5 - x^4 - x^3 + x^2 - x - 1,$$
$$x^5 + x^4 + x^3 - x^2 - x + 1,\ x^5 - x^4 + x^3 + x^2 - x - 1,\ x^5 - x^4 - x^3 + x^2 - x + 1,$$
$$x^5 + x^4 + x^3 - x^2 + x - 1,\ x^5 - x^4 - x^3 - x^2 + x + 1,\ x^5 - x^4 + x^3 + x^2 + x - 1,$$

$$x^5+x^4-x^3-x^2+x+1,\ x^5+x^4+x^3+x^2+x-1,\ x^5-x^4+x^3-x^2+x+1,$$
$$x^5-x^4-x^3-x^2-x+1,\quad x^5-x^4-x^3+x^2+x+1,$$
$$x^5+x^4-x^3-x^2-x+1,\quad x^5+x^4+x^3+x^2+x+1.$$

Now the Magma procedure `Chabauty` gives all the rational solutions of (2) in these cases. We obtain that only solutions with $|x| \le 2$ exist.

It remains to study the hyperelliptic curves defined by the polynomials

$$x^5+x^4-x^3+x^2-x+1,\ x^5+x^4-x^3+x^2+x+1,\ x^5+x^4+x^3+x^2-x+1,$$
$$x^5-x^4+x^3+x^2+x+1,\quad x^5+x^4+x^3-x^2+x+1.$$

In all these cases the curves (2) have rank 2 Mordell–Weil groups, therefore classical Chabauty's method cannot be applied. In these cases we shall use the hyperelliptic logarithm method from [11]. We provide details only in case of the polynomial $x^5+x^4+x^3+x^2-x+1$, all the other polynomials can be handled similarly. In this case the hyperelliptic curve (2) is given by

$$C:\quad y^2 = f(x) := x^5+x^4+x^3+x^2-x+1. \tag{9}$$

A Magma computation using the procedures `TorsionSubgroup` and `MordellWeilGroupGenus2` yields that the Jacobian of C has trivial torsion, and it has a Mordell–Weil basis given by

$$D_1 = (0,-1) - \infty, \quad D_2 = (1,-2) - \infty.$$

As a next step, by means of Baker's method [1] we derive a (large) bound for $\log|x|$. Here we use the following improved version from [11].

Lemma 3.1 (Proposition 2.1 in [11]) *Let α be a root of $f(x)$. Assuming the knowledge of explicit generators for the Mordell–Weil group $J(C)(\mathbb{Q})$, there is a finite computable set $\mathcal{K}$ consisting of integers of $\mathbb{Q}[\alpha]$ such that if (x, y) is an integral solution to C, then $x-\alpha = \kappa\xi^2$ for some $\kappa \in \mathcal{K}$ and $\xi \in \mathbb{Q}[\alpha]$.*

Moreover, suppose $\kappa \in \mathcal{K}$. Let $\alpha_1, \alpha_2, \alpha_3$ be different conjugates of α, and let $\kappa_1, \kappa_2, \kappa_3$ be the corresponding conjugates of κ. Let $K_1 = \mathbb{Q}(\alpha_1, \alpha_2, \sqrt{\kappa_1\kappa_2})$, $K_2 = \mathbb{Q}(\alpha_1, \alpha_3, \sqrt{\kappa_1\kappa_3})$, $K_3 = \mathbb{Q}(\alpha_2, \alpha_3, \sqrt{\kappa_2\kappa_3})$ and $L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \sqrt{\kappa_1\kappa_2}, \sqrt{\kappa_1\kappa_3})$. Then there is an explicitly computable constant B_κ depending on α and κ and the degrees, regulators, class numbers and unit ranks of the K_is, and the degree of L such that if $x \ne 0$ is an integer satisfying $x-\alpha = \kappa\xi^2$ for some $\xi \in \mathbb{Q}[\alpha]$, then $\log|x| \le B_\kappa$.

Hence, if (x, y) is an integral solution of C, then $\log|x| \le B := \max_{\kappa\in\mathcal{K}} B_\kappa$.

Importantly, the methods developed in [11] provide B in Lemma 3.1 explicitly indeed. To make the actual calculations, the worked out example [12] provided by Gallegos–Ruiz can be followed, as well. In case of our Eq. (9) we obtain the bound

$$\log|x| \le 4.56 \times 10^{319}.$$

If P is an integral point on C, then the image of P on the Jacobian of C can be expressed as

$$P - \infty = n_1 D_1 + n_2 D_2.$$

There are only finitely many integral points on C, so we may define M as

$$M = \max_{P \in C(\mathbb{Z})} \|n_P\| = \max_{P \in C(\mathbb{Z})} \sqrt{n_1^2 + n_2^2}.$$

To obtain an upper bound for M from the Baker bound we computed above we make use of Corollary 3.2 in [11]. To state the result we need to define μ_1, μ_2. From [9, Theorem 4] we can compute a lower bound for the height difference

$$\mu_1 \le h(D) - \hat{h}(D), \quad D \in J(C)(\mathbb{Q}).$$

(Here, as usual, h and $\hat{h}$ stand for the naive height and canonical height on $J(C)(\mathbb{Q})$, respectively.) Let us denote the eigenvalues of the height pairing matrix of the Mordell–Weil basis $\{D_1, D_2\}$ by λ_1, λ_2. Set $\mu_2 = \min\{\lambda_1, \lambda_2\}$. In our case the coefficient of x^5 is 1, so we have the following simplified version of Corollary 3.2 from [11].

Lemma 3.2 (simplified version of Corollary 3.2 in [11]) *Let B be an upper bound for the logarithmic height of integral points on C. Then*

$$M \le \sqrt{\mu_2^{-1}(2B - \mu_1)}.$$

The application of Lemma 3.2 to (9) yields that $M \le 1.28 \times 10^{160}$. It remains to reduce the bound following [11, Sect. 6] based on the LLL-algorithm (which results in many cases in a logarithmic improvement). To reduce the bound we set $K = 10^{600}$. The 6×6 matrix $\mathcal{A}_K$ mentioned in [11, Proposition 6.2] is given by (with 800 digits of precision—here only the first three digits after the decimal point are indicated)

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1.097 \times K & -4.253 \times K & -2.893 \times K & 2.620 \times K & -1.173 \times K & -2.346 \times K \\ 0.679 \times K & -1.590 \times K & -4.538 \times K & -0.679 \times K & 2.269 \times K & 1.358 \times K \\ 5.308 \times K & 0.345 \times K & -8.490 \times K & 1.122 \times K & 3.123 \times K & 6.246 \times K \\ -2.180 \times K & -0.847 \times K & 2.664 \times K & 2.180 \times K & -1.332 \times K & -4.360 \times K \end{pmatrix}.$$

Proposition 6.2 yields a new bound 70.25 for $\|n_P\|$. Further reductions are applied with $K = 10^{20}$, 10^{14} and 10^{12} to get bounds 13.73, 11.88 and 11.06, respectively. After this procedure, it remains to enumerate all integral points with

$$\|n_P\| \le 11.06,$$

which can be done easily. These points are as follows:

$$(x, y) \in \{(0, \pm 1), (1, \pm 2)\}.$$

Altogether, we obtained that (2) with $n = 5$ has only the solutions given in the last two rows of Table 1.

3.3 The proof of Theorem 2.1 for n even with $2 \leq n \leq 24$

Let $f(x)$ be a Littlewood polynomial of even degree $n = 2k$ with $2 \leq n \leq 24$, and consider (2). In principle we have 2^{n+1} equations to solve. However, since $|x|^n > |x|^{n-1} + \cdots + |x| + 1$ when $|x| \geq 2$, we can immediately exclude the cases where the leading coefficient of $f(x)$ is -1, as otherwise there are no solutions with $|x| > 2$. Further, similarly as in cases $n = 3, 5$, using the substitution $x \to -x$ we can get rid of half of the equations. Namely, it is sufficient to study (2) only when $f(x)$ is of the shape

$$f(x) = x^n + x^{n-1} + e_2 x^{n-2} + \cdots + e_{n-1}x + e_n,$$

with $e_2, \ldots, e_n \in \{-1, 1\}$. This means that we need to consider 2^{n-1} equations only. To solve these equations, we shall follow Runge's method. We do not outline the general method here, however, we shall give all the details required to keep the presentation self-contained. Beside this, we shall illustrate our method by an example, too. For more details on Runge's method, its implementation and its applications, see e.g. [4, 32], [33, Chapter 2] and [19, Chapter 4] and the references there. We used the program package SageMath [24] for our computations.

First, we shall need the polynomial part of the Puiseux expansion of $\sqrt{f(x)}$ at ∞. The following statement, which we hope to be useful in later investigations as well, provides this in a general form.

Proposition 3.1 *Let*

$$f(x) = x^n + e_1 x^{n-1} + \cdots + e_{n-1}x + e_n, \tag{10}$$

be a Littlewood polynomial of even degree $n = 2k$. Then there exists a uniquely determined polynomial

$$u(x) = u_0 x^k + u_1 x^{k-1} + \cdots + u_{k-1}x + u_k, \tag{11}$$

with $u_0 = 1$ and $u_i \in \mathbb{Q}$ $(i = 1, \ldots, k)$ such that the coefficients of the terms x^i in $f(x)$ and $u^2(x)$ are the same for $i = k, k+1, \ldots, n$. Further, for the denominators d_i of the coefficients u_i of $u(x)$ we have

$$d_i = 2^{\nu_2((2i)!)} \quad (i = 0, 1, \ldots, k), \tag{12}$$

where $\nu_2(\ell)$ denotes the exponent of 2 in the prime factorization of a positive integer ℓ.

Proof By formula (3.1) on p. 481 of [25] the coefficients of $u(x)$ are uniquely determined, and are given by

$$u_i = \sum_{s=1}^{i} \binom{1/2}{s} \sum' \frac{s!}{j_1! \cdots j_n!} e_1^{j_1} \cdots e_n^{j_n} \quad (i = 0, 1, \ldots, k),$$

where $\sum'$ is taken over all tuples $(j_1, \ldots, j_n)$ such that $j_1 + \cdots + j_n = s$ and $j_1 + 2j_2 + \cdots + nj_n = i$. (Here, as usual, $\binom{r}{s}$ is defined as $r(r-1)\cdots(r-s+1)/s!$ for $r \in \mathbb{R}$.) This shows that the u_i are rational numbers. Observe that the terms in the inner sum $\sum'$ are integers. In particular, as for $s = i$ we must have $j_1 = i$ and $j_2 = \cdots = j_n = 0$, this integer is ± 1 in this case. Thus the denominator d_i of u_i is the same as that of $\binom{1/2}{i}$ $(i = 0, 1, \ldots, k)$. As one can easily check, this denominator is given by $2^{\nu_2(i!)+i} = 2^{\nu_2((2i)!)}$. Hence our claim follows. □

Observe that writing $f(x)$ as

$$f(x) = F(x) + g(x), \tag{13}$$

with

$$F(x) = x^n + x^{n-1} + e_2 x^{n-2} + \ldots + e_k x^k, \; g(x) = e_{k+1} x^{k-1} + \ldots + e_{n-1} x + e_n,$$

the polynomial $u(x)$ provided by Proposition 3.1 depends only on $F(x)$, it is independent of $g(x)$. This is extremely important for our purposes. Indeed, in this way we shall need to loop through only the possible choices of $F(x)$, which means that we can reduce the number of cases to be considered down to 2^{k-1}. (Though certainly, we also need to take care of the polynomials $g(x)$ appearing in (13). However, as we shall see, this can be done relatively easily.)

Let $t = 2^{-\nu_2(n!)}$. Observe that the polynomials $(u(x) - t)^2 - f(x)$ and $(u(x) + t)^2 - f(x)$ are of degree k, with leading coefficients $-2t$ and $2t$, respectively. This implies that there is a constant C (to be discussed later) such that for $|x| > C$ we have that either

$$(u(x) - t)^2 < f(x) < (u(x) + t)^2, \tag{14}$$

or

$$(u(x) + t)^2 < f(x) < (u(x) - t)^2, \tag{15}$$

is valid. Multiplying by $2^{2\nu_2(n!)-2}$, (14) and (15) yield

$$(2^{\nu_2(n!)-1} u(x) - 2^{\nu_2(n!)-1} t)^2 < 2^{2\nu_2(n!)-2} f(x) < (2^{\nu_2(n!)-1} u(x) + 2^{\nu_2(n!)-1} t)^2,$$

and

$$(2^{\nu_2(n!)-1} u(x) + 2^{\nu_2(n!)-1} t)^2 < 2^{2\nu_2(n!)-2} f(x) < (2^{\nu_2(n!)-1} u(x) - 2^{\nu_2(n!)-1} t)^2,$$

respectively. By Proposition 3.1 we see that $1/t$ is just the denominator of u_k, and that the denominators of all the other coefficients of $u(x)$ are powers of 2, with exponents strictly less than $\nu_2(n!)$. Thus the left and right hand sides of the above inequalities are integer polynomials of x. Further, we have

$$(2^{\nu_2(n!)-1}u(x) + 2^{\nu_2(n!)-1}t) - (2^{\nu_2(n!)-1}u(x) - 2^{\nu_2(n!)-1}t) = 1.$$

That is, $2^{2\nu_2(n!)-2}f(x)$ is between two consecutive (integer) squares in both cases, thus it cannot be a square. But then the same is true for $f(x)$. That is, $f(x)$ cannot be a square for $|x| > C$, or in other words, (2) has no solutions with $|x| > C$.

So we are left with the following tasks: find an appropriate C, and check the integer values of x with $2 < |x| \le C$. For this, we may assume that neither (14), nor (15) is valid. (Indeed, if either (14) or (15) holds, then as we have seen, x cannot yield a solution to (2).) Fix $F(x)$ in (13). Then, as we have pointed out earlier, $u(x)$ is also fixed. Think of $g(x)$ in (13) as an arbitrary, but fixed Littlewood polynomial of degree $k-1$. Put

$$h_1(x) = (u(x) - t)^2 - F(x) - g(x),\ h_2(x) = (u(x) + t)^2 - F(x) - g(x). \quad (16)$$

Then $h_1(x), h_2(x)$ are polynomials of degree k, with leading coefficients $-2t$ and $2t$, respectively. Since (14) is not valid, we have

$$h_1(x) \ge 0 \quad \text{or} \quad h_2(x) \le 0,$$

and as (15) is false,

$$h_1(x) \le 0 \quad \text{or} \quad h_2(x) \ge 0.$$

From these we easily see that $|x| \le \max(C_1, C_2)$, where C_i is the maximum of the absolute values of the roots of $h_i(x)$ for $i = 1, 2$. So we can take $C = \max(C_1, C_2)$. To get upper bounds for the values of C_1 and C_2, we use the following lemma.

Lemma 3.3 *Let $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients, with $a_n \ne 0$. The absolute values of all the roots of this polynomial can be bounded from above by*

$$1 + \max\left\{\left|\frac{a_{n-1}}{a_n}\right|, \left|\frac{a_{n-2}}{a_n}\right|, \ldots, \left|\frac{a_0}{a_n}\right|\right\}.$$

The bound above is due to Cauchy [6]. Certainly there exist other bounds for the roots of real zeros of polynomials like the one obtained by Fujiwara [10]. 'In general', Fujiwara's bound is better. However, our computations show that for our purposes the factor 2 in Fujiwara's bound makes the bound of Cauchy much better for us in the considered range. So to get C_1 and C_2 (and then C) we apply the following procedure. Fix $F(x)$; then t and $u(x)$ are also fixed, as well as the polynomials $(u(x)-t)^2 - F(x)$ and $(u(x) + t)^2 - F(x)$. Thus the absolute values of the coefficients of $h_1(x)$ and $h_2(x)$ defined by (16) can be easily bounded: the leading coefficients are $-2t$ and $2t$, respectively, while the absolute values of the coefficients of x^j with $0 \le j < k$ are at most one larger than those of the polynomials $(u(x)-t)^2-F(x)$ and $(u(x)+t)^2-F(x)$, respectively. So fixing $F(x)$, we can get an upper bound for the Cauchy bound both in case of h_1 and h_2. In this way we obtain a constant C such that if x, y is a solution to equation (2), then $|x| \le C$ must be valid. Importantly, observe that this C is uniformly valid for any $f(x)$ having the fixed $F(x)$ part in (13). We illustrate what we did so far with an example.

Example Let $n = 6$, and take a Littlewood polynomial $f(x)$ with $F(x) = x^6 + x^5 - x^4 - x^3$. Then by (13) we have

$$f(x) = x^6 + x^5 - x^4 - x^3 + g(x),$$

where $g(x)$ is a Littlewood polynomial of degree 2. Proposition 3.1 gives

$$u(x) = x^3 + \frac{1}{2}x^2 - \frac{5}{8}x - \frac{3}{16}.$$

So we have $t = 1/16$, while (14) and (15) read as

$$\left(x^3 + \frac{1}{2}x^2 - \frac{5}{8}x - \frac{2}{8}\right)^2 < x^6 + x^5 - x^4 - x^3 + g(x) < \left(x^3 + \frac{1}{2}x^2 - \frac{5}{8}x - \frac{1}{8}\right)^2,$$

and

$$\left(x^3 + \frac{1}{2}x^2 - \frac{5}{8}x - \frac{1}{8}\right)^2 < x^6 + x^5 - x^4 - x^3 + g(x) < \left(x^3 + \frac{1}{2}x^2 - \frac{5}{8}x - \frac{2}{8}\right)^2,$$

respectively. Expanding the above inequalities, we obtain

$$-\frac{1}{8}x^3 + \frac{9}{64}x^2 + \frac{5}{16}x + \frac{1}{16} < g(x) < \frac{1}{8}x^3 + \frac{17}{64}x^2 + \frac{5}{32}x + \frac{1}{64},$$

and

$$\frac{1}{8}x^3 + \frac{17}{64}x^2 + \frac{5}{32}x + \frac{1}{64} < g(x) < -\frac{1}{8}x^3 + \frac{9}{64}x^2 + \frac{5}{16}x + \frac{1}{16}.$$

By $\deg(g) = 2$ one of them is certainly valid if $|x| > C$ for some C. However, multiplying them by $(t/2)^2 = 8^2$, we get

$$(8x^3 + 4x^2 - 5x - 2)^2 < 64(x^6 + x^5 - x^4 - x^3 + g(x)) < (8x^3 + 4x^2 - 5x - 1)^2,$$

and

$$(8x^3 + 4x^2 - 5x - 1)^2 < 64(x^6 + x^5 - x^4 - x^3 + g(x)) < (8x^3 + 4x^2 - 5x - 2)^2,$$

respectively. This shows that $64f(x)$ cannot be a square, whence $f(x)$ cannot be a square—so (2) has no solutions with $|x| > C$. Now we need to find an appropriate C. For the polynomials h_1, h_2 defined by (16) we obtain

$$h_1(x) = -\frac{1}{8}x^3 + \frac{9}{64}x^2 + \frac{5}{16}x + \frac{1}{16} - g(x),$$

and

$$h_2(x) = \frac{1}{8}x^3 + \frac{17}{64}x^2 + \frac{5}{32}x + \frac{1}{64} - g(x).$$

For the Cauchy bounds of h_1, h_2 we get

$$11.5, \quad 11.125,$$

respectively. Hence

$$C = \max(11.5, 11.25) = 11.5.$$

This bound can be used for all $f(x)$ with $F(x)$ part $F(x) = x^6 + x^5 - x^4 - x^3$ in (13).

After establishing C, the only remaining task is to check the integers x with $2 < |x| \leq C$ whether they yield a solution to (2). For this, one may use Stoll's `ratpoints` code [30] to compute all "small" points on a hyperelliptic curve. We used the function `hyperellratpoints` the PARI [21] implementation of Stoll's code.

We mention that the total running times (and also the largest Cauchy's bound) on a HPC supercomputer using 128 cores parallel for each degree (between 12 and 24, we omit the low degree cases) were the following:

n (degree)	12	14	16	18	20	22	24
Running time	1.04 s	1.3 s	2.67 s	10.5 s	39.7 s	2 min 42 s	28 min 28 s
The largest bound C	5374	19,712	671,205	3,341,457	36,063,346	204,015,976	4,792,168,585

4 The proof of Proposition 2.1

Here we directly applied the previously mentioned PARI version of Stoll's `ratpoints` code on a HPC supercomputer using 128 cores. We mention that the total running times for each degree (between 7 and 17) were the following:

The code and the outputs can be downloaded from https://shrek.unideb.hu/~tengely/LittleWoodOdd.html.

n (degree)	7	9	11	13	15	17
running time	529 ms	2.19 s	9.55 s	41.9 s	3 min 32 s	24 min 17 s

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References

1. Baker, A.: Bounds for the solutions of the hyperelliptic equation. Proc. Camb. Philos. Soc. **65**, 439–444 (1969)
2. Balister, P., Bollobás, B., Morris, R., Sahasrabudhe, J., Tiba, M.: Flat Littlewood polynomials exist. Ann. Math. **19**, 977–1004 (2020)
3. Bennett, M.A., Levin, A.: The Nagell–Ljunggren equation via Runge's method. Monatsh. Math. **177**, 15–31 (2015)
4. Beukers, F., Tengely, Sz.: An Implementation of Runge's Method for Diophantine Equations. http://arxiv.org/abs/math.NT/0512418 (2005)
5. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language. J. Symb. Comput. **24**, 235–265 (1997)
6. Cauchy, A.-L.: Exercices de mathématique. Œuvres **2**(9), 122 (1829)
7. Chabauty, C.: Sur les points rationnels des courbes algébriques de genre supérieur à l'unité. C. R. Acad. Sci. Paris **212**, 882–885 (1941)
8. Dubickas, A., Jankauskas, J.: On Newman polynomials which divide no Littlewood polynomial. Math. Comp. **78**, 327–344 (2009)
9. Flynn, E.V., Smart, N.P.: Canonical heights on the Jacobians of curves of genus 2 and the infinite descent. Acta Arith. **79**(4), 333–352 (1997)
10. Fujiwara, M.: Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung. Tohoku Math. J. **10**, 167–171 (1916)
11. Gallegos-Ruiz, H.R.: Computing integral points on genus 2 curves estimating hyperelliptic logarithms. Acta Arith. **187**, 329–344 (2019)
12. Gallegos-Ruiz, H.R.: Worked Out Example for My Paper: Computing Integral Points on Genus 2 Curves Estimating Hyperelliptic Logarithms. http://personal.cimat.mx:8181/~hgallegos/programs/hyperlogs/WorkedExample.pdf
13. Gebel, J., Pethő, A., Zimmer, H.G.: Computing integral points on elliptic curves. Acta Arith. **68**, 171–192 (1994)
14. Hajdu, L., Varga, N.: Diophantine equations for polynomials with restricted coefficients, I (power values). Bull. Austral. Math. Soc. **106**, 254–263 (2022)
15. Hajdu, L., Tijdeman, R., Varga, N.: Diophantine equations for Littlewood polynomials. Acta Arith. **210**, 223–234 (2023)

16. Han, X., Schied, A.: Step roots of Littlewood polynomials and the extrema of functions in the Takagi class. Math. Proc. Camb. Philos. Soc. **173**, 591–618 (2022)
17. Hare, K.G., Jankauskas, J.: On Newman and Littlewood polynomials with a prescribed number of zeros inside the unit disk. Math. Comp. **90**, 831–870 (2021)
18. Ljunggren, W.: Noen Setninger om ubestemte likninger av formen $(x^n - 1)/(x - -1) = y^q$. Norsk. Mat. Tidsskr. **25**, 17–20 (1943)
19. Masser, D.: Auxiliary Polynomials in Number Theory. Cambridge University Press, Cambridge (2016)
20. Mossinghoff, M.J.: Polynomials with restricted coefficients and prescribed noncyclotomic factors. LMS J. Comp. Math. **6**, 314–325 (2003)
21. The PARI Group: PARI/GP Version `2.15.4`. Univ. Bordeaux. http://pari.math.u-bordeaux.fr/ (2023)
22. Peled, R., Sen, A., Zeitouni, O.: Double roots of random Littlewood polynomials. Israel J. Math. **213**, 55–77 (2016)
23. Runge, C.: Über ganzzahlige Lösungen von Gleichungen zwischen zwei Veränderlichen. J. Reine Angew. Math. **100**, 425–435 (1887)
24. SageMath: The Sage Mathematics Software System (Version 9.8). The Sage Developers. https://www.sagemath.org (2023)
25. Sankaranarayanan, A., Saradha, N.: Estimates for the solutions of certain Diophantine equations by Runge's method. Int. J. Number Theory **4**, 475–493 (2008)
26. Shorey, T.N., Tijdeman, R.: Exponential Diophantine Equations. Cambridge University Press, Cambridge (1986)
27. Stoll, M.: On the height constant for curves of genus two. Acta Arith. **90**, 183–201 (1999)
28. Stoll, M.: Implementing 2-descent for Jacobians of hyperelliptic curves. Acta Arith. **98**, 245–277 (2001)
29. Stoll, M.: On the height constant for curves of genus two, II. Acta Arith. **104**, 165–182 (2002)
30. Stoll, M.: Documentation for the Ratpoints Program. http://arxiv.org/abs/0803.3165 (2022)
31. Stroeker, R.J., Tzanakis, N.: Solving elliptic diophantine equations by estimating linear forms in elliptic logarithms. Acta Arith. **67**, 177–196 (1994)
32. Tengely, Sz.: On the Diophantine equation $F(x) = G(y)$. Acta Arith. **110**, 185–200 (2003)
33. Tengely, Sz.: Effective Methods for Diophantine Equations. Ph.D. thesis, Leiden University, The Netherlands (2005)
34. Yakir, O.: Approximately half of the roots of a random Littlewood polynomial are inside the disk. Studia Math. **261**, 227–240 (2021)