



ON THE PROJECTIVE GEOMETRY AND METRIZABILITY OF SPRAY MANIFOLDS

doktori (Ph.D.) értekezés

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Debreceni Egyetem
Debrecen, 2004.

Ezen értekezés a Debreceni Egyetem TTK Matematika Doktori Iskola Differenciálgeometria programja keretében készítettem a Debreceni Egyetem TTK doktori (Ph.D.) fokozatának elnyerése céljából.

Debrecen, 2004. április 24.

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Debrecen, 2004. április 24.

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KÖSZÖNETNYILVÁNÍTÁS

A szerző ezúton mond köszönetet Dr. Szilasi Józsefnek lelkiismeretes témavezetői munkájáért. Közreműködése és tanácsai nélkül ez a disszertáció nem készülhetett volna el.

ACKNOWLEDGEMENT

The author would like to thank Dr. József Szilasi, his supervisor, for his invaluable help and encouragement during the years of Ph.D course. We also grateful to him for his suggestions and contribution to preparation of this Ph.D dissertation.

TABLE OF CONTENTS

INTRODUCTION	3
I. THE FUNDAMENTAL PROJECTIVE INVARIANTS OF A SPRAY MANIFOLD	13
1. Basic tools	13
2. Horizontal endomorphisms and semisprays	16
3. Finsler connections of Berwald types	23
4. The mixed curvature and Ricci tensor of a Berwald-type connection	27
5. Yano-type connection	32
6. The Douglas tensor of a Berwald endomorphism	36
7. The Weyl tensor of a spray manifold	42
II. FINSLER-METRIZABILITIES OF SPRAY MANIFOLDS	46
8. Hilbert 1-forms and Lagrangians	46
9. Projectively Finsler spray manifolds	51
10. Helmholtz-type condition	54
III. PROJECTION ONTO THE INDICATRIX BUNDLE OF A FINSLER MANIFOLD	66
11. Finslerian Berwald and Cartan connection	66
12. General remarks on Landsberg and Berwald manifolds	68
13. Projection onto the indicatrix bundle	71
14. Landsberg manifolds with vanishing Douglas tensor	78
IV. APPLICATIONS TO TWO-DIMENSIONAL FINSLER MANIFOLDS ..	83
15. Basic constructions	83
16. Two-dimensional Landsberg manifolds with vanishing Douglas tensor .	90
17. Two-dimensional Finsler manifolds admitting a special conformal change	95
V. A COORDINATE FORMULARY	105
SUMMARY	113
ÖSSZEFOGLALÓ (Hungarian summary)	119
REFERENCES	123
APPENDIX	127
List of publications	127
List of citations	128
Participation in conferences	129

INTRODUCTION

Our PhD dissertation treats some current problems, as well as some old problems, from a new point of view of spray and Finsler geometry in the framework of the theory initiated by JOSEPH GRIFONE in the early seventies of the last century ([25],[26]). We apply mainly the calculus of vector-valued differential forms elaborated by A. FRÖLICHER and A. NIJENHUIS [24] combining it with (and simplifying at the same time) a systematic use of a moving frame field consisting of vertically and completely (or vertically and horizontally) lifted vector fields.

Spray geometry grew out the theory of 'affinely connected manifolds', i.e. theory of manifolds endowed with a torsion-free linear connection. HERMANN WEYL was the first to point out the importance of this structure, independently of the metric [60]. To convey a more precise preliminary picture, consider a manifold M , and its tangent bundle $\pi : TM \rightarrow M$. Let $(U, (u^i)_{i=1}^n)$ be a chart on M and $(\pi^{-1}(U), (x^i, y^i)_{i=1}^n)$ the induced chart on TM . Suppose that ∇ is a torsion-free linear connection on M , with Christoffel symbols Γ_{ij}^k with respect to $(U, (u^i))$. The so-called *geodesic spray* S_∇ of ∇ can locally be given by

$$S_\nabla \upharpoonright \pi^{-1}(U) = y^i \frac{\partial}{\partial x^i} - (\Gamma_{ij}^k \circ \pi) y^i y^j \frac{\partial}{\partial y^k}.$$

Here the functions $(\Gamma_{ij}^k \circ \pi) y^i y^j$ ($k \in \{1, \dots, n\}$), the 'forces' of S_∇ with respect to $(U, (u^i))$, are quadratic in the velocities, so S_∇ is a prototype of the so-called *quadratic* or *affine sprays*. To define this, seemingly more general, concept, identify TM locally with $u(U) \times \mathbb{R}^n \subset \mathbb{R}^{2n}$. Then an affine spray over M is a vector field on TM which has a local expression of form

$$(p, v) \in u(U) \times \mathbb{R}^n \rightarrow (p, v, v, f(p, v)) \in u(U) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n,$$

where the map $f : \varphi(U) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is of class C^2 , smooth on $u(U) \times (\mathbb{R}^n \setminus \{0\})$, and has the homogeneity property

$$f(p, tv) = t^2 f(p, v); \quad (p, v) \in u(U) \times \mathbb{R}^n, \quad 0 < t \in \mathbb{R}.$$

Then it follows that f is *quadratic* in its second variable, namely

$$f(p, v) = \frac{1}{2} D_2 D_2 f(p, 0)(v, v) \quad \text{for all } (p, v) \in u(U) \times \mathbb{R}^n$$

(D_2 is the 'second' partial derivative operator). A curve $c : I \rightarrow M$ defined on an interval $I \subset \mathbb{R}$ is said to be a *geodesic* of an affine spray $S : TM \rightarrow TTM$, if

$\dot{c} : I \rightarrow TM$ is an integral curve of S , i.e., if $S \circ \dot{c} = \ddot{c}$. This relation for curves in M is called the *second-order differential equation* for the curve c , determined by S .

Suppose that $\gamma : I \rightarrow TM$ is an integral curve of S . Then, in the above local decomposition of π , γ has two components:

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in u(U) \times \mathbb{R}^n, \quad t \in \mathbb{R}.$$

In terms of the local representation f of S , the condition $S \circ \gamma = \dot{\gamma}$ leads to the relation

$$(\gamma'_1(t), \gamma'_2(t)) = (\gamma_2(t), f(\gamma_1(t), \gamma_2(t))),$$

therefore our differential equation can be rewritten as follows:

$$\begin{cases} \gamma'_1(t) = \gamma_2(t), \\ \gamma''_1(t) = \gamma'_2(t) = f(\gamma_1(t), \gamma'_1(t)). \end{cases}$$

In this way we arrive at the familiar second order ordinary differential equation of form

$$x'' = f(x, x').$$

In fact, the concept of a 'generic' affine spray is not more general than the concept of the geodesic spray of a torsion-free linear connection. It was proved by Ambrose, Palais and Singer [1] that there is a canonical correspondence between the set of torsion-free linear connections on a manifold M and the set of affine sprays over M in such a way that the corresponding objects have the same geodesics. The term 'spray' was also introduced in the cited paper. Following the recommendation of Richard Palais, Serge LANG was the first who systematically used sprays in his important book *Differential manifold* (Addison-Wesley, 1962). This book has been revised and essentially enlarged at least three times by Lang himself until now. The last edition under the title *Fundamentals of Differential Geometry* (Springer-Verlag, 1999), provides the most systematic treatment of infinite dimensional Riemannian geometry as well.

The strong consequences of the quadratic character of the local representation f of an affine spray is due to the C^2 -differentiability of f . If we weaken the differentiability hypothesis and require only the 1-times continuous differentiability of f on its whole domain, we arrive at the concept of (general) **sprays**. These ideas will gain an elegant, concise, intrinsic formulation in Chapter 1 of our Thesis (Definition 2.5); the coordinate aspects will only briefly be discussed in the present work; see the brief last chapter.

The seemingly mild generalization of the concept of an affine spray leads us to another exciting world: the *general geometry of paths*, or in modern terms, the *geometry of spray manifolds*. The local study of sprays is a classical field in differential geometry, whose first golden age was in the twenties-thirties of the last century. Several outstanding mathematicians worked in this area, e.g. L. BERWALD, E. CARTAN, J. DOUGLAS, M. S. KNEBELMANN, T. Y. THOMAS,

O. VEBLEN and others. L. P. EISENHART's excellent book 'Non-Riemannian Geometry' (Amer. Math. Soc. Colloq. Publ. **8**, 1927) gives a good picture of the spirit and techniques of these classical investigations. One of the Hungarian geometers, ANDRÁS RAPCSÁK, developed the theory to a significant extent in the 1960's. We owe a lot to his scientific heritage.

A renaissance of spray geometry began in the 1970's, by recognizing the fundamental role of sprays in the geometrical background of Lagrangian mechanics [16, 18], and, in particular in the foundation of Finsler geometry [25, 26]. Indeed, in differential-geometric terms, the dynamics of a time-independent Lagrangian dynamical system is determined by a spray acting on the tangent manifold of the configuration space of the system. The 'canonical spray' of a Finsler manifold arises from the energy determined by a suitable Lagrangian (see 8.4 and 8.6). J. KLEIN, J. GRIFONE and M. CRAMPIN did pioneering work in this field, and our Thesis follows the path which was opened by them.

One of the main topics of our dissertation belongs to the territory of *projective geometry of sprays*. Roughly speaking, two sprays over the same manifold are said to be projectively equivalent if they have the same geodesics as point sets, i.e., if they have common *pregeodesics*. (Recall: a curve is called a pregeodesic of a spray if it has a reparametrization as a geodesic). A diffeomorphism of a manifold endowed with a spray is said to be a *projective transformation* if it sends geodesics to pregeodesics. Roughly speaking again, in the projective geometry of sprays we are interested in those properties which are invariant under projective transformations. The main ingredient of Part I of our thesis is a projectively invariant tensor, discovered by Jesse DOUGLAS [23] and called *Douglas tensor* after him. In the context of affine sprays, a fundamental projectively invariant tensor, the *Weyl projectively curvature tensor* (briefly Weyl tensor) has already been constructed [61]. It is well-known that an affine spray is projectively flat, i.e. projectively equivalent to a spray whose local expression is of form

$$(p, v) \in u(U) \times \mathbb{R}^n \rightarrow (p, v, v, 0) \in u(U) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n,$$

if and only if the projective Weyl tensor vanishes, provided that $n \geq 3$. (In the two-dimensional case the Weyl tensor is identically 0.) The counterpart of the Weyl curvature tensor in the more general context of non-affine (i.e. not necessarily affine) sprays was intrinsically constructed by L. DEL CASTILLO [20] in 1976, using the Frölicher-Nijenhuis formalism; see also [34].

A panoramic view of the thesis

Our dissertation is divided into four chapters. In *part I*, after summarizing the necessary conceptual and calculational background, we intrinsically construct the above mentioned *Douglas tensor*. This tensor always vanishes in affinely connected manifolds (hence, in particular, in Riemannian manifolds), so, in the phrase of L. P. EISENHART, it has a typically "non-Riemannian" character.

To build up a geometric theory, we cannot start from scratch. Our most elementary geometric structure will be a *nonlinear connection*, represented by a projector, called a *horizontal map*, whose kernel is the vertical subbundle. To obtain a real nonlinear theory, differentiability on the zero section will be dropped, of course. Any horizontal endomorphism gives rise to a special Finsler connection, called a *Berwald-type connection*. A Berwald-type connection is said to be a *Berwald connection* if the horizontal endomorphism is generated by a spray. In this case the horizontal endomorphism will be mentioned as a *Berwald endomorphism*. Any Berwald-type connection has two surviving “partial curvatures”, the horizontal and the mixed curvature. A careful analysis of the behavior of the mixed curvature of a Berwald connection under a projective change of the associated spray yields the Douglas tensor. However, in order to identify the Douglas tensor we shall follow a slightly different path. As a generalization of Berwald-type connections, we introduce the so-called *Yano-type connections*. If, in particular, we start from a Berwald endomorphism, then the construction results in a *Yano connection*, called also – unfortunately – a projective connection in the literature. Indeed, the parameters of this connection were originally found by K. YANO, motivated by purely technical considerations. (For the quite curious story of this connection, see MATSUMOTO’s instructive remarks in [4].) Thus in our present approach the definition, as well as the proof of the projective invariance of the Douglas tensor will be given in terms of a Yano connection. Although the central object of this part is the Douglas tensor, a very brief section will also be devoted to the projective Weyl tensors. This includes a simple proof of the following, quite recently discovered (see [5]) important fact: *the vanishing of the Douglas tensor implies that one of the Weyl tensors is a vertical lift*, i.e., its components depend only on the position.

In *part II* we start with the definition of a *Hilbert 1-form* and introduce the concept of a *Finsler manifold*. Simply put, a general Finsler structure on a differentiable manifold M is a function

$$L : TM \longrightarrow \mathbb{R}$$

satisfying appropriate differentiability, homogeneity and regularity conditions. In conformity with the demands of Finsler geometry, the smoothness is not required or assured on the whole tangent manifold TM . With the help of a so-called *Finsler-Lagrangian* L we can introduce the *canonical spray* of a Finsler manifold and the canonical horizontal endomorphism generated by the spray which is called the *Barthel endomorphism*. Since every Finsler manifold is a spray manifold, we can also speak of the projective equivalence of a spray manifold and a Finsler manifold, and of that of two Finsler manifolds (which have, of course, a common carrier manifold). A spray manifold (M, S) is said to be *Finsler-metrizable in a broad sense* or — following SHEN’s terminology [48] — *projectively Finsler*, if there exists a Finsler-Lagrangian $L : TM \rightarrow \mathbb{R}$ such that the Finsler manifold (M, L) is projectively equivalent to (M, S) . If, in particular, the canonical spray of (M, L) coincides with the given spray S , then we say that (M, S) is *Finsler-metrizable in a natural sense* or that S is a *Finsler-variational spray*. The latter concept is

a faithful analogue of the variationality of a spray (or a semispray) used in the classical inverse problem of the calculus of variations ([19], [28], [29], [33]).

The following problems naturally arise:

- (1) *Find criteria for the Finsler-metrizability of a spray both in a broad sense and in a natural sense.*
- (2) *Search for an “algorithm” to decide whether or not a given spray is projectively Finsler /Finsler-variational.*

In our thesis we would like to contribute to a better geometrical understanding of problem (1), hoping that our results (combined e.g. with the technique applied in [17] and [18]) will serve as a good starting point for the attack on the much harder problem (2). As for problem (1), the key ingredients will be *the fundamental equations of projective equivalence*. These provide equivalent (and, in our presentation, intrinsically formulated) second order partial differential equations for the Finsler-Lagrangian to be determined. Their coordinate version was discovered by A. RAPCSÁK in the early sixties (see [42], [43], [44], [45] and, for a recent account, [48]), hence we call them *Rapcsák equations*.

In his influential paper [16] M. CRAMPIN presented a stimulating intrinsic reformulation of the famous *Helmholtz conditions* from the classical inverse problem of the calculus of variations through the existence of a 2-form on the tangent manifold. Prescribing some extra condition on this 2-form concerning the homogeneity and regularity *we derive necessary and sufficient conditions for metrizability of a spray in both senses*. Next we investigate the question of the metrizability from another point of view. We present equivalent conditions for a spray to be Finsler-variational assuming the existence of a symmetric, non-degenerate (0,2) tensor field on the vertical bundle with some further properties. Similarly, the *alternative theorem* (10.9) gives a characterization of the projectively Finsler spray manifolds through the existence of a (0,2) tensor field on the tangent manifold, satisfying certain algebraic conditions, some of them are quite complicated. Nevertheless, the real significance of our alternative theorem lies in the fact that it reduces the problem of Finsler-metrizability in a broad sense to a *first order* partial differential equation. Finally we give its integrability condition and, supplementing by some conditions on the regularity, we derive through the Rapcsák equation necessary and sufficient conditions for a spray to be projectively Finsler.

In *part III* we present an intrinsic description of the *projection onto the indicatrix bundle* of a Finsler manifold and give a new, coordinate-free proof for the following theorem: *a Landsberg manifold with vanishing Douglas tensor is a Berwald manifold*. In the two-dimensional case this was shown by L. BERWALD in the third part [9] of his famous series on Finsler and Cartan geometries. More than 40 years later it was announced by H. IZUMI [32] that this is also true in higher dimensions. IZUMI’s statement was proved by S. BÁCSÓ, F. ILOSVAY and B. KIS in [3], using the machinery of classical tensor calculus and utilizing an important observation of T. SAKAGUCHI [47] on the projected Douglas tensor. Finsler manifolds with vanishing Douglas tensor were baptized *Douglas manifolds* by M. MATSUMOTO

and S. BÁCSÓ [4]. Thus, briefly speaking, *any Landsberg-Douglas manifold is a Berwald manifold*.

To realise our program, first we introduce some basic metrical data arising from a Finsler-Lagrangian on a manifold M , including the *Riemann-Finsler metric*, which is a pseudo-Riemannian metric on TM , the *first* and *second Cartan tensors* C, C' and the classical *Cartan connection* of a Finsler manifold. Next we recall some well-known characterizations of two special Finsler manifolds, the Landsberg and the Berwald manifolds. After this preparation we present an intrinsic construction of the projection onto the indicatrix bundle initiated by H. IZUMI and systematized by M. MATSUMOTO (see e.g. [37]). One of the canonical geometrical objects on the tangent bundle, the Liouville vector field, is a normal vector field to the “unit sphere bundle” or “indicatrix bundle” $U(M)$ of any Finsler manifold (M, L) . Due to this fact an orthogonal projection into $U(M)$ can be given by the operator

$$\tau := I - \frac{1}{2E} dE \otimes C,$$

first introduced by J. GRIFONE [27]. Two new operators, the operator

$$\kappa := J^* \circ \tau = J - \frac{1}{2E} d_J E \otimes C,$$

and its lowered tensor κ_b , will also play an important role in our constructions. (J stands for the canonical vertical endomorphism of TTM .) If TM is the deleted tangent manifold of M , then we define the *projected tensor* A^* of a tensor $A \in \mathcal{T}_r^1(TM)$ as follows:

$$A^* := \tau \circ \tau^* A.$$

The behavior of the Douglas tensor becomes much more transparent after it has been projected onto the indicatrix bundle. We show that the projected tensor $\mathbb{D}^*$ can be represented in the form

$$\mathbb{D}^* = \overset{\circ}{\mathbb{P}} - \frac{1}{n+1} \overset{\sim}{\mathbb{P}} \odot \kappa + \frac{1}{E} C'_b \otimes C,$$

where $\overset{\circ}{\mathbb{P}}$ is the mixed curvature of the Berwald connection, C'_b is the type $(0, 3)$ second Cartan tensor, and the sign $\sim$ means semibasic trace. Then we elaborate a new, coordinate-free proof of SAKAGUCHI’s important observation that $\mathbb{D}^* = 0$ *if and only if* $\mathbb{D} = 0$. We note that in local coordinates our formulas, including the coordinate expression of $\mathbb{D}^*$, do not coincide word by word with those of SAKAGUCHI. The reason for this is the difference in the underlying bundle structures, which implies that the notions of “Finsler tensor fields” are also different. Most of the “essential” tensors are semibasic in our framework, while the vector field variables of MATSUMOTO’s theory [38] used by SAKAGUCHI correspond to the vertical vector fields in our approach.

Finally the reader will find the deduction of the main result of this part. We show that in any “Landsberg-Douglas manifold” the mixed curvature of the Berwald connection has the form

$$\overset{\circ}{\mathbb{P}} = \frac{1}{n^2 - 1} \overset{\circ}{\mathbb{P}} \overset{\sim}{\circ}^{\#} \kappa_{\flat} \odot \kappa,$$

where $\#$ and $\flat$ are the musical isomorphisms with respect to the metric tensor arising from the Finsler structure; and the relation

$$(n - 2) \overset{\circ}{\mathbb{P}} \overset{\sim}{\circ}^{\#} \tilde{\mathcal{C}} = 0$$

holds. Izumi’s statement follows immediately from these formulae.

Unfortunately, our elegant method fails in two dimensions. There the Douglas tensor has components only along the Liouville vector field. Since the Liouville vector field is orthogonal to the unit sphere bundle, it follows immediately that *the projected Douglas tensor of a two-dimensional Finsler manifold vanishes identically*. In *part IV* we complete the story discussing the previously cited paper of BERWALD on the two-dimensional Finsler manifolds from our present day standpoint. In BERWALD’s own words: “*The Landsberg spaces, the extremals of which form a quasigeodesic system of curves, are identical with the two-dimensional affinely connected Finsler spaces.*” ([9], p. 110.) The notion of a “quasigeodesic system of curves” was introduced by W. BLASCHKE and G. BOL in their book [12]. In modern language, the condition on the extremals states that the geodesics of a Landsberg manifold coincide with those of a linear connection on the underlying manifold. Since by the Douglas-Shen theorem (6.8) this property is equivalent to the vanishing of the Douglas tensor, our formulation is indeed a translation of BERWALD’s theorem.

We emphasize that our approach is based on BERWALD’s original ideas, and it may be considered as an intrinsic formulation of them. In this section we restrict our investigation to the *positive definite two-dimensional Finsler manifold*. (There is a generalization of the theorem for two-dimensional Finsler manifolds endowed with a nondegenerate, not necessarily positive definite Riemann-Finsler metric. The proof needs a slight modification of Berwald’s method as turns out from the paper [7], where the machinery of classical tensor calculus is applied.) First we sketch a general, intrinsically formulated theory of the two-dimensional (positive definite) Finsler manifolds, introducing the *Berwald frame*, the *main scalar* and the *Gaussian curvature*, and derive E. CARTAN’s “*permutation formulas*” as well as the “*Bianchi identity*”. Next we discuss the two-dimensional Landsberg-Douglas manifolds proving the statement mentioned above.

In the concluding section of part IV we investigate the following problem: *what kind of (two-dimensional) Finsler manifolds do admit a conformal change leaving the mixed curvature of the Berwald connection invariant?* Two Finsler structures $\tilde{L}$ and L on the same manifold are said to be *conformally equivalent* if there exists a positive smooth function $\varphi : TM \rightarrow \mathbb{R}$ such that $\tilde{g} = \varphi g$, where $\tilde{g}$ and g are the

Riemann-Finsler metrics of $\tilde{L}$ and L , respectively. φ can always be written in the form $\varphi = \exp \circ \alpha \circ \pi$, $\alpha \in C^\infty(M)$ [46]. If the mixed curvature of the Berwald connection remains invariant, then the manifold admits an associated Riemannian metric [59] and we can form the Riemannian gradient $\text{grad}_R \alpha$ of α . Fix a nonzero tangent vector v such that $v\alpha = 0$ and consider the integral curve of the vertical lift of $\text{grad}_R \alpha$ given by $c(t) := v + t \text{grad}_R \alpha(p)$. Along this integral curve the energy function $E = \frac{1}{2}L^2$ satisfies the following differential equation:

$$2(a^2 + t^2 b^2(1 - b^2))y(t)y''(t) - (a^2 + t^2 b^2)(y'(t))^2 + 4tb^4 y(t)y'(t) - 4b^4(y(t))^2 = 0,$$

where $a^2 = 2E_R(v)$, $b^2 = 2E_R(\text{grad}_R \alpha(p))$ and E_R is the energy of the Riemannian metric. To find the general solution seems to be very complicated, so we restrict ourselves here to the case $b^2 = 1$. Then it turns out that the solutions essentially are the singular Finsler metrics with constant main scalar determined by Berwald [9]. We show that in these manifolds the mixed curvature of the Berwald connection remains invariant under *any* conformal change. Fixing the initial conditions we get the following (singular) Finslerian energy:

$$E \circ c(t) = 4K^* \left(E_R \circ c(t) + KL_R(v) \frac{t}{4} \right) \cdot \\ \cdot \exp \frac{2K}{\sqrt{16 - K^2}} \left(\arctan \frac{4t + KL_R(v)}{L_R(v)\sqrt{16 - K^2}} - \arctan \frac{2K}{\sqrt{16 - K^2}} \right),$$

where $-4 < K(p) < 4$ and K^* are constant. We determine the vertical members of the Berwald frame and conclude that the main scalar is $\lambda = \frac{K(p)}{2L}$. This Finsler structure has constant main scalar in Berwald's sense, since Berwald's main scalar is $L\lambda$. Finally we show that if the Douglas tensor vanishes, then we arrive at the Berwald manifolds again.

The concluding *Part V* is devoted to the coordinate description of the geometric objects and basic relations (identities) we were dealing with in an invariant form.

A more detailed insight: section by section

In *section 1* we recall the rudiments of the Frölicher–Nijenhuis theory of vector-valued differential forms. Useful identities concerning the Frölicher–Nijenhuis bracket are collected here which will be extensively used later on.

Section 2 deals with semisprays and horizontal endomorphisms, their fundamental relation is established by the Crampin–Grifone's theorem. We introduce the Jacobi endomorphism and deduce some of its basic properties.

Section 3 is devoted to a summarization of the basic data of Berwald-type Finsler connections.

In *section 4* we give explicit expressions for the curvatures of the Berwald-type connections. We derive some useful identities (Ricci formulas) of the covariant

differentials of a semibasic 1-form with respect to a Berwald connection. Further we compute the mixed curvature tensor and show that it is symmetric, (-1) -homogeneous and has no potential. Similar properties concerning the mixed Ricci tensor are also discussed here.

As a slight generalization of the Berwald-type connection, in the following *section 5* we present an axiomatic description of the Yano-type connection and derive the basic rules of calculation with respect to this connection. We establish the relation between the corresponding mixed curvature tensors and the mixed Ricci tensors.

In *section 6* we define the Douglas tensor in terms of the Yano connection and also present it with the data of the Berwald connection. We show that it has no potential and its semibasic trace also vanishes. We prove that it is invariant under any projective change of the associated spray. We give a new, coordinate-free proof for the Douglas-Shen theorem.

In *section 7* we restate del Castillo's result on the Weyl tensor of a Berwald endomorphism. We introduce the Weyl tensor of a Berwald connection which is just the vertical covariant derivatives of del Castillo's Weyl tensor with respect to the Berwald (or the Yano) connection. We prove: if the Douglas tensor vanishes then the Weyl tensor of the Berwald connection is a vertical lift. We also present the Weyl tensor in terms of the Yano connection.

In *section 8* we recall the definition of Hilbert 1-forms and Finsler manifolds. We summarize their elementary properties and discuss conditions for a Lagrange function to be a Finsler-Lagrangian in detail. The fundamental relation (8.10) provides a special formula representing the canonical spray of a Finsler manifold.

In the following *section 9* we give an intrinsic reformulation of Rapcsák's equations and present a necessary and sufficient condition for the metrizability in both senses (see above), prescribing the existence of a Hilbert 1-form with appropriate additional properties.

Section 10 deals with a deeper analysis of the metrizability of a spray in both senses. First we present a criterion supplementing Crampin's theorem by some extra condition, next we give equivalent conditions through the existence of a symmetric $(0, 2)$ tensor on the vertical bundle.

In *section 11* we introduce the Riemann-Finsler metric of a Finsler manifold and define the first and the second Cartan tensors. Some basic data of the Finslerian Berwald and Cartan connection are also collected here. We reformulate the fifth Bianchi identity and present (without proof) a formula for the h -covariant differential of the vertical curvature of the Cartan connection for later use.

Section 12 starts with J.-G. DIAZ's characterization of Landsberg manifolds. We derive a useful identity concerning the mixed curvature tensor of the Berwald connection and show that the lower mixed curvature (which is metrically equivalent to the mixed curvature) of the Berwald connection is totally symmetric in any Landsberg manifold. We also recall the concept of Berwald manifold and present a simple coordinate-free proof for the well-known observation: a Finsler manifold

is a Berwald manifold if and only if the mixed curvature tensor of the Berwald connection vanishes.

In the following *section 13* we intrinsically construct the operator of the projection onto the indicatrix bundle. We compute the projected Douglas tensor and present a new coordinate-free proof of the theorem: *the Douglas tensor and the projected Douglas tensor vanish at the same time*.

In *section 14* we derive formulas for the mixed curvature and the mixed Ricci tensor of the Berwald connection in a Landsberg–Douglas manifold. After these preparations, we elaborate our new proof of the theorem mentioned a moment ago.

In *section 15* we introduce the Berwald frame of a two-dimensional (positive definite) Finsler manifold. We give an intrinsic definition of the main scalar and of the Gaussian curvature and derive Cartan’s important permutation formulas.

In the following *section 16* we compute the Douglas tensor of a two-dimensional Landsberg manifold and prove: if a positive definite Landsberg manifold has vanishing Douglas tensor, then it is a Berwald manifold.

In the concluding *section 17* we turn to the conformal equivalence of Finsler manifolds. As mentioned above, our aim is to determine those two-dimensional Finsler manifolds which admit a conformal change leaving the mixed curvature tensor of the Berwald connection invariant. We prove that manifolds with constant main scalar are just of this type. We also show: the vanishing of the Douglas tensor yields Berwald manifolds in this case again.

References

A large part of the work presented here has already been published by the author. Chapter I is based on the paper [52]. It is extended here by a description of the Jacobi endomorphism in section 2 and by a presentation of the Weyl tensor in terms of the Yano connection in section 7. The next chapter is built upon the work [53], however its last section, except for the alternative theorem, is new. Chapter III follows the exposition of our article [56]. The first two sections in Chapter IV are based on [57], the last section is ready for publication [58].

I. THE FUNDAMENTAL PROJECTIVE INVARIANTS OF A SPRAY MANIFOLD

1. Basic tools

1.1. Throughout the Dissertation, M will denote a connected, smooth (i.e., C^∞), sometimes orientable manifold of dimension $n \geq 2$. $C^\infty(M)$ is the ring of real-valued smooth functions on M , $\mathfrak{X}(M)$ denotes the $C^\infty(M)$ -module of vector fields on M . For $(r, s) \in \mathbb{N} \times \mathbb{N}$, $\mathcal{T}_s^r(M)$ is the module over $C^\infty(M)$ of smooth tensor fields (briefly tensors) of type (r, s) , contravariant of order r and covariant of order s . We define the symmetric product $\omega_1 \odot \omega_2$ of the covariant tensors $\omega_1 \in \mathcal{T}_{s_1}^\circ(M)$, $\omega_2 \in \mathcal{T}_{s_2}^\circ(M)$ by the formula

$$(1.1) \quad \omega_1 \odot \omega_2 := \frac{(s_1 + s_2)!}{s_1!s_2!} \text{Sym}(\omega_1 \otimes \omega_2).$$

The symmetric product of the tensors $\omega \in \mathcal{T}_s^\circ(M)$ and $L \in \mathcal{T}_r^1(M)$ is the $(1, s+r)$ tensor $\omega \odot L$ given by the formula

$$\omega \odot L(X_1, \dots, X_{s+r}) := \frac{1}{s!r!} \sum_{\sigma \in S_{s+r}} \omega(X_{\sigma(1)}, \dots, X_{\sigma(s)}) L(X_{\sigma(s+1)}, \dots, X_{\sigma(s+r)}).$$

$\Omega^k(M)$ ($0 \leq k \leq n$) is the module of differential forms on M , $\Omega^\circ(M) := C^\infty(M)$. The differential forms constitute the graded algebra $\Omega(M) := \bigoplus_{k=0}^n \Omega^k(M)$, with multiplication given by the wedge product. The wedge product of the tensors $\omega \in \mathcal{T}_s^\circ(M)$ and $L \in \mathcal{T}_r^1(M)$ is the $(1, s+r)$ tensor $\omega \wedge L$ given by the formula

$$\omega \wedge L(X_1, \dots, X_{s+r}) := \frac{1}{s!r!} \sum_{\sigma \in S_{s+r}} \varepsilon(\sigma) \omega(X_{\sigma(1)}, \dots, X_{\sigma(s)}) L(X_{\sigma(s+1)}, \dots, X_{\sigma(s+r)}).$$

1.2 Vector forms and derivations. A vector k -form on the manifold M is a skew-symmetric $C^\infty(M)$ -multilinear map $[\mathfrak{X}(M)]^k \rightarrow \mathfrak{X}(M)$ if $k \in \mathbb{N}^+$, and a vector field on M if $k = 0$. The set of all vector k -forms on M is a $C^\infty(M)$ -module, denoted by $\Psi^k(M)$. In particular, the elements of $\Psi^1(M)$ are just the $(1, 1)$ tensor fields on M . To any vector k -form $K \in \Psi^k(M)$ two derivations of $\Omega(M)$, denoted by i_K and d_K , are associated. Now we briefly recall some of their basic properties.

- (i) i_K is a derivation of degree $k - 1$;

$$i_K \upharpoonright C^\infty(M) = 0, \quad i_K \omega := \omega \circ K, \quad \text{if } \omega \in \Omega^1(M).$$

- (ii) d_K is a derivation of degree k given by

$$d_K := [i_K, d] := i_K \circ d - (-1)^{k-1} d \circ i_K,$$

where d is the operator of the exterior derivative.

In particular,

$$\forall f \in C^\infty(M) : d_K f = i_K df \stackrel{(i)}{=} df \circ K,$$

and d_K is uniquely determined by this formula.

(iii) If $K \in \Psi^1(M)$, then we define the endomorphism

$$K^* : \Omega(M) \rightarrow \Omega(M), \quad \omega \in \Omega^\ell(M) \mapsto K^* \omega$$

by the formula

$$K^* \omega(X_1, \dots, X_\ell) := \omega(K(X_1), \dots, K(X_\ell)) \quad (X_i \in \mathfrak{X}(M), 1 \leq i \leq \ell).$$

If $\ell = 1$, $K^* \omega = i_K \omega$. If $\ell > 1$,

$$i_K \omega(X_1, \dots, X_\ell) = \sum_{i=1}^{\ell} \omega(X_1, \dots, K(X_i), \dots, X_\ell)$$

and $i_K \omega \neq K^* \omega$.

(iv) In case of a vector 0-form $X \in \Psi^0(M) = \mathfrak{X}(M)$ i_X means the usual insertion operator, while d_X reduces to the Lie derivative $\mathcal{L}_X$.

1.3 The Frölicher-Nijenhuis bracket. Suppose that $K \in \Psi^k(M)$, $L \in \Psi^\ell(M)$. The graded commutator of d_K and d_L is defined by the formula

$$[d_K, d_L] = d_K \circ d_L - (-1)^{k\ell} d_L \circ d_K.$$

A substantial result of the *Frölicher-Nijenhuis theory* [24] states that there exists a unique vector form $[K, L] \in \Psi^{k+\ell}(M)$ such that

$$[d_K, d_L] = d_{[K, L]}.$$

$[K, L]$ is said to be the *Frölicher-Nijenhuis bracket* of K and L . If K and L are vector 0-forms, i.e. vector fields on M , then $[K, L]$ reduces to the usual Lie bracket of vector fields. In our considerations we shall need the evaluation of the Frölicher-Nijenhuis bracket only in some special cases.

(1.3a) If $K \in \Psi^1(M)$, $Y \in \Psi^0(M) = \mathfrak{X}(M)$, then $\forall X \in \mathfrak{X}(M)$:

$$[K, Y](X) = [K(X), Y] - K[X, Y].$$

(1.3b) If $K, L \in \Psi^1(M)$, then $\forall X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned} [K, L](X, Y) &= [K(X), L(Y)] + [L(X), K(Y)] + K \circ L[X, Y] \\ &\quad + L \circ K[X, Y] - K[X, L(Y)] - K[L(X), Y] - L[X, K(Y)] - L[K(X), Y]. \end{aligned}$$

In particular, the *Nijenhuis torsion* $N_K := \frac{1}{2}[K, K]$ of a vector 1-form K can be given by the formula

$$(1.3c) \quad N_K(X, Y) = [K(X), K(Y)] + K^2[X, Y] - K[X, K(Y)] - K[K(X), Y].$$

A quite complete list of useful identities concerning the Frölicher-Nijenhuis bracket can be found in [64]. We shall explicitly need only the followings of them:

$$\forall K, L \in \Psi^1(M), \omega \in \Omega^k(M), N \in \Psi^l(M), X, Y \in \mathfrak{X}(M), f \in C^\infty(M) :$$

$$(1.3d,e) \quad [i_X, i_K] = i_{KX}, \quad [i_K, \mathcal{L}_X] = i_{[K, X]}.$$

$$(1.3f) \quad [i_K, i_L] = i_{L \circ K} - i_{K \circ L}.$$

$$(1.3g,h) \quad [i_X, d_K] = \mathcal{L}_{KX} + i_{[K, X]}, \quad [d_K, \mathcal{L}_X] = d_{[K, X]}.$$

$$(1.3i) \quad [i_K, d_L] = d_{L \circ K} - i_{[K, L]},$$

$$(1.3j) \quad [fX, N] = f[X, N] + df \wedge i_X N - d_N f \otimes X,$$

$$(1.3k) \quad [X, fL] = f[X, L] + (Xf)L,$$

$$(1.3l) \quad [X, \omega \otimes Y] = \mathcal{L}_X \omega \otimes Y + [X, Y] \otimes \omega,$$

$$(1.3m) \quad [K, fL] = f[K, L] + d_K f \wedge L - df \wedge K \circ L,$$

$$(1.3n) \quad [L, \omega \otimes X] = d_L \omega \otimes X - d\omega \otimes LX + (-1)^k \omega \wedge [L, X].$$

1.4. The tangent bundle of the manifold M will be denoted by $\pi : TM \rightarrow M$, while $\pi_0 : TM \rightarrow M$ stands for the subbundle of the nonzero tangent vectors. The kernel of the tangent map $T\pi : TTM \rightarrow TM$ is a distinguished subbundle of TTM , the *vertical subbundle*, whose total space will be denoted by $T^v TM$. The sections of this bundle constitute the $C^\infty(TM)$ -module $\mathfrak{X}^v(TM)$ of the *vertical vector fields*. In our calculations we shall frequently use the *vertical lift* X^v and the *complete lift* X^c of a vector field $X \in \mathfrak{X}(M)$. Their usefulness is established by the following simple observation.

1st local basis property. *If $(X_i)_{i=1}^n$ is a local basis for the module $\mathfrak{X}(M)$, then $(X_i^v, X_i^c)_{i=1}^n$ is a local basis for $\mathfrak{X}(TM)$.*

We also recall that

$$(1.4) \quad \forall X, Y \in \mathfrak{X}(M) : [X^v, Y^v] = 0, [X^v, Y^c] = [X, Y]^v, [X^c, Y^c] = [X, Y]^c.$$

1.5. Tangent bundle geometry is dominated by two canonical objects: the *Liouville vector field* $C \in \mathfrak{X}^v(TM)$ and the *vertical endomorphism* $J \in \Psi^1(TM) \cong \mathcal{T}_1^1(TM) \cong \text{End } \mathfrak{X}(TM)$. The following well-known relations will also be useful in

our considerations.

$$\begin{aligned}
(1.5a) \quad & \text{Im } J = \text{Ker } J = \mathfrak{X}^v(TM), & J^2 &= 0; \\
(1.5b) \quad & [J, C] = J, & [J, J] &= 0; \\
(1.5c) \quad & \forall X \in \mathfrak{X}(M) : JX^c = X^v, & [J, X^v] = [J, X^c] &= 0; \\
(1.5d) \quad & \forall X \in \mathfrak{X}(M) : [C, X^v] = -X^v, & [C, X^c] &= 0.
\end{aligned}$$

1.6 Semibasic tensors. A symmetric or skew-symmetric tensor $A \in \mathcal{T}_s^0(TM)$ ($s \neq 0$) is called *semibasic* if

$$\forall X \in \mathfrak{X}(TM) : i_{JX}A = 0.$$

Analogously, a symmetric or skew-symmetric tensor $L \in \mathcal{T}_s^1(TM)$ ($s \neq 0$) is said to be semibasic if

$$\forall X \in \mathfrak{X}(TM) : i_{JX}L = 0 \quad \text{and} \quad J \circ L = 0.$$

Owing to the local basis principle 1.4, the *semibasic forms are completely determined by their action on the complete lifts*.

1.7 Homogeneity. A function $f \in C^\infty(TM)$, a vector field $X \in \mathfrak{X}(TM)$, a differential form $\alpha \in \Omega(TM)$ and a vector form $A \in \Psi(TM)$ are called *positive homogeneous of degree r* (briefly *r -homogeneous*; $r \in \mathbb{Z}$) if the relations

$$Cf = rf, \quad [C, X] = (r-1)X, \quad \mathcal{L}_C\alpha = r\alpha, \quad [C, A] = (r-1)A$$

hold, respectively.

2. Horizontal endomorphisms and semisprays

2.1. In our approach the role of a “nonlinear connection” is played by the *horizontal endomorphisms*. A vector 1-form $h \in \Psi^1(TM) \cong \text{End } \mathfrak{X}(TM)$, *smooth only on TM* , is said to be a horizontal endomorphism on M if it is a *projector* (i.e., $h^2 = h$) and $\text{Ker } h = \mathfrak{X}^v(TM)$. $v := 1_{\mathfrak{X}(TM)} - h$ is the *vertical projector* belonging to h . $\mathfrak{X}^h(TM) := \text{Im } h$ is called the module of *horizontal vector fields*. It is a direct summand, namely

$$(*) \quad \mathfrak{X}(TM) = \mathfrak{X}^v(TM) \oplus \mathfrak{X}^h(TM).$$

The mapping

$$X \in \mathfrak{X}(M) \mapsto X^h := hX^c \in \mathfrak{X}^h(TM)$$

is called the *horizontal lifting* by h . We have the following basic relations among the vertical endomorphism J , a horizontal endomorphism h and the horizontal lifting:

$$(2.1a) \quad h \circ J = 0, \quad J \circ h = J;$$

$$(2.1b) \quad \forall X \in \mathfrak{X}(M) : JX^h = X^v;$$

$$(2.1c,d) \quad h[X^h, Y^h] = [X, Y]^h, \quad J[X^h, Y^h] = [X, Y]^v.$$

Immediate consequence of the decomposition $(*)$ the following

2nd local basis property. *If $(X_i)_{i=1}^n$ is a local basis for the module $\mathfrak{X}(M)$, then $(X_i^v, X_i^h)_{i=1}^n$ is a local basis for $\mathfrak{X}(TM)$.*

2.2 Definition. Suppose that h is a horizontal endomorphism on the manifold M . The vector forms

$$(2.2a) \quad H := [h, C] \in \Psi^1(TM),$$

$$(2.2b) \quad t := [J, h] \in \Psi^2(TM),$$

$$(2.2c) \quad R := -N_h := -\frac{1}{2}[h, h] \in \Psi^2(TM)$$

are called the *tension*, the *torsion* and the *curvature* of h , respectively. A horizontal endomorphism is said to be *homogeneous* if its tension vanishes.

2.3 Remark. H, t and R were introduced by GRIFONE [25]. It is easy to check that all of these tensors are semibasic, so they are completely determined by the following mappings:

$$(2.3a) \quad \eta : X \in \mathfrak{X}(M) \mapsto \eta(X) := H(X^c) = [X^h, C];$$

$$(2.3b) \quad \tau : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \tau(X, Y) := t(X^c, Y^c)$$

$$= [X^h, Y^v] - [Y^h, X^v] - [X, Y]^v;$$

$$(2.3c) \quad \varrho : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \varrho(X, Y) := R(X^c, Y^c) = -v[X^h, Y^h].$$

For details we refer to [50].

2.4 Lemma. *If h is a homogeneous horizontal endomorphism, then its torsion is 0-homogeneous and its curvature is 1-homogeneous.*

Proof. Routine calculation.

2.5 Definition. A *semispray* on the manifold M is a mapping

$$S : TM \rightarrow TTM, \quad v \mapsto S_v \in T_v TM$$

satisfying the following conditions:

- (i) S is smooth on TM ,
- (ii) $JS = C$.

A semispray S is called a *spray* if

- (iii) S is of class C^1 on TM

and

- (iv) $[C, S] = S$ (i.e., S is positive homogeneous of degree 2).

A manifold M endowed with a spray S will be mentioned as a *spray manifold*. A spray S is said to be *affine* (or *quadratic*) if it is C^2 on TM .

2.6. We recall (see [25]) that any horizontal endomorphism $h \in \text{End } \mathfrak{X}(TM)$ gives rise to a unique almost complex structure $F \in \text{End } \mathfrak{X}(TM)$, smooth over TM , characterized by the relations

$$(2.6a) \quad F \circ J = h, \quad F \circ h = -J.$$

F can explicitly be given by the formula

$$(2.6b) \quad F := h[S, h] - J,$$

where $S := h(S')$ (S' is an arbitrary semispray on M .) From these one can easily deduce the useful identities

$$(2.6c) \quad J \circ F = v, \quad F \circ v = h \circ F;$$

$$(2.6d) \quad v \circ F = F - F \circ v = F - h \circ F = -J.$$

2.7 Lemma. *If S is a semispray on the manifold M , then*

$$(2.7) \quad \forall Z \in \mathfrak{X}(TM) : J[JZ, S] = JZ.$$

Proof. See [25], p. 295. □

2.8. The fundamental relation between the horizontal endomorphisms and the semisprays was discovered, independently, by M. CRAMPIN and J. GRIFONE [15], [17], [25]. Their main result can be summarized as follows.

- (i) If $h \in \text{End } \mathfrak{X}(TM)$ is a horizontal endomorphism and S' is an arbitrary semispray on M , then $S := hS'$ is also a semispray on M . This semispray does

not depend on the choice of S' , it is horizontal with respect to h and satisfies the relation $h[C, S] = S$. S is called the *semispray associated to h* .

- (ii) Any semispray $S : TM \rightarrow TTM$ generates in a canonical way a horizontal endomorphism which can be given by the formula

$$(2.8) \quad h := \frac{1}{2} (1_{\mathfrak{X}(TM)} + [J, S]).$$

Then h is torsion free (i.e., $t = 0$) and the semispray associated to h is $\frac{1}{2}(S + [C, S])$. If, in addition, S is a spray, then h is homogeneous and its associated semispray is just the starting spray S .

- (iii) A horizontal endomorphism is generated by a semispray according to (2.8) if and only if it is torsion free.

2.9. In the sequel we shall call a horizontal endomorphism a *Berwald endomorphism* if it is generated by a *spray*. We emphasise that the Berwald endomorphisms are homogeneous and torsion free. It follows immediately that if h is a Berwald endomorphism with associated spray S , then the horizontal lifting with respect to h can be given by the formula

$$(2.9a) \quad X \in \mathfrak{X}(M) \mapsto X^h = \frac{1}{2}(X^c + [X^v, S]).$$

Conversely, it may be shown that if h is homogeneous horizontal endomorphism with vanishing torsion, then there is a spray which generates h according to (2.9a). For a proof we refer to [51]. Applying (2.8) and (2.9a) one can easily deduce the following relation:

$$(2.9b) \quad J[X^h, S] = X^h - X^c.$$

Indeed, $X^h = (2h - 1_{\mathfrak{X}(TM)})X^h = [J, S]X^h = [X^v, S] - J[X^h, S] = 2X^h - X^c - J[X^h, S]$, hence $J[X^h, S] = X^h - X^c$, as we claimed.

2.10 Semibasic trace. Now let us suppose that F is an almost complex structure on TM determined by an arbitrary horizontal endomorphism h according to 2.6. Consider a semibasic tensor $L \in \mathcal{T}_s^1(TM)$. We define the *semibasic trace* $\tilde{L}$ of L by recurrence as follows:

$$(2.10a) \quad \text{if } s = 1, \text{ then } \tilde{L} := \text{tr}(F \circ L);$$

$$(2.10b) \quad \text{if } s \geq 2, \text{ then } \forall X \in \mathfrak{X}(TM) : i_X \tilde{L} := \widetilde{i_X L}.$$

It can easily be seen that $\tilde{L}$ does not depend on the choice of the horizontal endomorphism h .

We shall need the following technical results.

2.11 Lemma. *Let $L \in \mathcal{T}_1^1(TM)$, $K \in \mathcal{T}_2^1(TM)$ be semibasic tensors. If $\omega \in \mathcal{T}_s^0(TM)$ is a symmetric semibasic tensor and $F \in \text{End } \mathfrak{X}(TM)$ is an almost complex structure determined by a horizontal endomorphism h , then*

$$(2.11a) \quad \widetilde{J} = n, \quad \widetilde{\omega \otimes J} = n\omega.$$

$$(2.11b) \quad \forall X \in \mathfrak{X}(TM) : \widetilde{\omega \otimes LX} = i_{FLX}\omega$$

$$(2.11c) \quad \widetilde{\omega \otimes L} = \widetilde{L}\omega,$$

$$(2.11d) \quad \forall X \in \mathfrak{X}(TM) : i_X \widetilde{L \otimes \omega} = i_{FL(X)}\omega,$$

$$(2.11e) \quad \widetilde{\omega \otimes K} = \omega \otimes \widetilde{K},$$

$$(2.11f) \quad \forall X, Y \in \mathfrak{X}(TM) : i_Y i_X \widetilde{K \otimes \omega} = i_{FK(X,Y)}\omega.$$

$$(2.11g) \quad \widetilde{\omega \odot L} = \widetilde{L}\omega + i_{FL}\omega$$

Using mathematical induction, the proofs of (2.11a-g) are quite straightforward; however we omit the lengthy calculations.

2.12 Remark. We recall the following analogous results concerning the semibasic trace of *vector forms* (see [64]). If ω is a semibasic scalar p -form on TM ($p \geq 1$) and L is a semibasic vector 1-form on TM , then

$$(2.12a) \quad \widetilde{\omega \otimes LX} = (-1)^{p-1} i_{FLX}\omega,$$

$$(2.12b) \quad \widetilde{\omega \wedge L} = \widetilde{L}\omega + (-1)^p i_{FL}\omega.$$

As particular cases, we have

$$(2.12c) \quad \widetilde{\omega \otimes C} = (-1)^{p-1} i_S\omega,$$

$$(2.12d) \quad \widetilde{\omega \wedge J} = (n-p)\omega.$$

where $X \in \mathfrak{X}(TM)$, S is a semispray on M and F is the almost complex structure determined by a horizontal endomorphism.

2.13 Lemma. *Suppose F is the almost complex structure determined by a horizontal endomorphism h and $A \in \mathcal{T}_s^1(TM)$ ($s \geq 1$) is a semibasic tensor homogeneous of degree r ($r \in \mathbb{Z}$). Then the tensor $F \circ A$ is $(r+1)$ -homogeneous.*

Proof. Notice first if A is r -homogeneous, then for any vector field $X_1, \dots, X_s$ on M we have

$$(r-1)A(X_1^c, \dots, X_s^c) = (\mathcal{L}_C A)(X_1^c, \dots, X_s^c) = [C, A(X_1^c, \dots, X_s^c)],$$

since $[C, X^c] \stackrel{(1.5d)}{=} 0$. Taking this into consideration,

$$\begin{aligned} J[C, FA(X_1^c, \dots, X_s^c)] &\stackrel{(1.3a)}{=} [J, C]FA(X_1^c, \dots, X_s^c) - [JFA(X_1^c, \dots, X_s^c), C] \\ &\stackrel{(1.5b)}{=} JFA(X_1^c, \dots, X_s^c) + [C, JFA(X_1^c, \dots, X_s^c)] \\ &\stackrel{(2.6c)}{=} A(X_1^c, \dots, X_s^c) + [C, A(X_1^c, \dots, X_s^c)] \\ &= rA(X_1^c, \dots, X_s^c), \end{aligned}$$

therefore

$$\begin{aligned} rFA(X_1^c, \dots, X_s^c) &= FJ[C, FA(X_1^c, \dots, X_s^c)] \stackrel{(2.6a)}{=} h[C, FA(X_1^c, \dots, X_s^c)] \\ &\stackrel{(1.3a)}{=} [h, C]FA(X_1^c, \dots, X_s^c) - [hFA(X_1^c, \dots, X_s^c), C] \\ &\stackrel{2.3}{=} [C, hFA(X_1^c, \dots, X_s^c)] \stackrel{(2.6c)}{=} [C, FA(X_1^c, \dots, X_s^c)] \\ &= (\mathcal{L}_C FA)(X_1^c, \dots, X_s^c) \end{aligned}$$

showing the $(r+1)$ -homogeneity of $F \circ A$. $\square$

2.14 Corollary. *If a semibasic $(1, s)$ tensor $(s \geq 1)$ on TM is r -homogeneous, then its semibasic trace is also r -homogeneous.*

Proof. This is obvious since tensor derivations commute with contractions. $\square$

2.15 Potential of semibasic forms. Suppose that $\alpha \in \Omega(TM)$ and $A \in \Psi(TM)$ are semibasic forms of degree not less than 1. If S is any semispray on M , then $\alpha^\circ := i_S \alpha$ and $A^\circ := i_S A$ are called the *potential* of α and A , respectively. Since the difference of two semisprays is a vertical vector field, the potentials are well-defined. It is also clear that α° and A° remain semibasic.

2.16 Lemma. *If $\alpha \in \Omega^{k+1}(TM)$ and $A \in \Psi^{k+1}(TM)$ are r -homogeneous semibasic forms, then their potentials are $(r+1)$ -homogeneous.*

Proof. We check only that α° is $(r+1)$ -homogeneous, the other case can be handled similarly. By 2.15, in forming α° we may take a spray S . Then, for any vector fields $X_1, \dots, X_k$ on M , we obtain:

$$\begin{aligned} &[\mathcal{L}_C (i_S \alpha)](X_1^c, \dots, X_k^c) \\ &= C[i_S \alpha(X_1^c, \dots, X_k^c)] - \sum_{i=1}^k i_S \alpha(X_1^c, \dots, [C, X_i^c], \dots, X_k^c) \\ &\stackrel{(1.5d)}{=} C[\alpha(S, X_1^c, \dots, X_k^c)] \\ &= (\mathcal{L}_C \alpha)(S, X_1^c, \dots, X_k^c) + \alpha([C, S], X_1^c, \dots, X_k^c) \\ &= (r\alpha)(S, X_1^c, \dots, X_k^c) + \alpha(S, X_1^c, \dots, X_k^c) \\ &= (r+1)(i_S \alpha)(X_1^c, \dots, X_k^c). \end{aligned}$$

This means by 1.7 that $\mathcal{L}_C \alpha^\circ = (r+1)\alpha^\circ$. □

2.17 Proposition. *Let $\alpha \in \Omega^k(TM)$ be an r -homogeneous semibasic form, and suppose that $k+r \neq 0$. Then*

$$(2.17a) \quad \alpha = \frac{1}{k+r} [(d_J \alpha)^\circ + d_J \alpha^\circ].$$

If, in particular, α is d_J -closed, i.e. $d_J \alpha = 0$, then

$$(2.17b) \quad \alpha = \frac{1}{k+r} d_J \alpha^\circ.$$

A proof can be found e.g. in [21].

2.18 Definition. Let S be a semispray over the manifold M . If h is the horizontal endomorphism generated by S and v is the vertical projector belonging to h , then the type $(1,1)$ tensor

$$(2.18a) \quad A := -v[h, S]$$

is said to be the *Jacobi endomorphism* determined by S . S is called *isotropic*, if

$$(2.18b) \quad A = \lambda J + \alpha \otimes C$$

where $\lambda \in C^\infty(TM)$ and α is a semibasic 1-form on TM . We say that the semispray S is *sectionally flat*, if

$$(2.18c) \quad A = \lambda J, \quad \lambda \in C^\infty(TM).$$

2.19 Remark. For any vector field X on M we have

$$\begin{aligned} A(X^v) &= -v([hX^v, S] - h[X^v, S]) = 0, \\ A(X^c) &= -v([hX^c, S] - h[X^c, S]) = -v[X^h, S], \end{aligned}$$

therefore the Jacobi endomorphism is *semibasic*.

2.20 Lemma. *The curvature of the horizontal endomorphism generated by a semispray S and the Jacobi endomorphism determined by S are related by the formula*

$$(2.20a) \quad R = -\frac{1}{3}[J, A].$$

If, in particular, S is a spray, then we have

$$(2.20b) \quad A = -R^\circ.$$

For a proof we refer to the book of J. Grifone and Z. Muzsnay [29].

2.21 Lemma. *The Jacobi endomorphism A determined by a spray is homogeneous of degree 2. If, in addition, A is isotropic, namely $A = \lambda J + \alpha \otimes C$, then λ is 2-homogeneous and α is 1-homogeneous.*

Proof. The 2-homogeneity of A is an immediate consequence of Lemma 2.4, (2.20b) and Lemma 2.16. If, in addition, $A = \lambda J + \alpha \otimes C$, then

$$\begin{aligned} \lambda J + \alpha \otimes C &= [C, \lambda J + \alpha \otimes C] = [C, \lambda J] + [C, \alpha \otimes C] \stackrel{(1.3k,l)}{=} \\ &= (C\lambda)J + \lambda[C, J] + \mathcal{L}_C \alpha \otimes C = (C\lambda - \lambda)J + \mathcal{L}_C \alpha \otimes C, \end{aligned}$$

from which it follows that $C\lambda = 2\lambda$, $\mathcal{L}_C \alpha = \alpha$. $\square$

2.22 Corollary. *If a spray is sectionally flat then it is flat in the sense that the horizontal endomorphism generated by the spray has vanishing curvature.*

Proof. Let S be a spray. Suppose that the Jacobi endomorphism determined by S is of the form $A = \lambda J$. Then, as we have just seen, λ is a 2-homogeneous function on TM . Using (2.20a), we get

$$R = -\frac{1}{3}[J, \lambda J] \stackrel{(1.3m)}{=} -\frac{1}{3}(d_J \lambda) \wedge J,$$

whence

$$\begin{aligned} A &\stackrel{(2.20b)}{=} \frac{1}{3}i_S((d_J \lambda) \wedge J) = \frac{1}{3}((C\lambda) - (d_J \lambda) \otimes C) \\ &\stackrel{(2.21)}{=} \frac{2}{3}\lambda J - \frac{1}{3}(d_J \lambda) \otimes C. \end{aligned}$$

Now the relation

$$\lambda J = \frac{2}{3}\lambda J - \frac{1}{3}(d_J \lambda) \otimes C$$

easily implies that $\lambda = 0$. Then $A = \lambda J = 0$, therefore $R \stackrel{(2.20a)}{=} -\frac{1}{3}[J, A]$ also vanishes. $\square$

2.23 Definition. A spray manifold (M, S) is said to be *isotropic* if the spray S is isotropic. If S is flat, then we also say that (M, S) is *flat*.

3. Finsler connections of Berwald type

3.1. Suppose that h is a horizontal endomorphism on the manifold M and let F be the almost complex structure determined by h . A pair (D, h) is said to be a *Finsler connection* on M , if D is a linear connection on the manifold TM or $\mathcal{TM}$ and the following conditions are satisfied:

$$(3.1a) \quad Dh = 0 \quad ("D \text{ is reducible}");$$

$$(3.1b) \quad DF = 0 \quad (D \text{ is almost complex}).$$

Then the mapping

$$h^*DC : X \in \mathfrak{X}(TM) \mapsto DC(hX) = D_{hX}C$$

is called the *h-deflection* of (D, h) .

An easy, but important consequence of (3.1b) is that D is completely determined by the restricted map $D \upharpoonright \mathfrak{X}(TM) \times \mathfrak{X}^v(TM)$. Explicitly, $\forall X, Y \in \mathfrak{X}(TM)$:

$$(3.1c) \quad D_{vX}hY = FD_{vX}JY,$$

$$(3.1d) \quad D_{hX}hY = FD_{hX}JY.$$

3.2. Suppose (D, h) is a Finsler connection on the manifold M . Denote by $\mathbb{T}$ and $\mathbb{K}$ the (classical) torsion and the curvature of D , respectively. In view of (3.1a) it is readily verified that the mappings

$$\mathbb{A} : (X, Y) \mapsto h\mathbb{T}(hX, hY) - \textit{h-horizontal torsion},$$

$$\mathbb{B} : (X, Y) \mapsto h\mathbb{T}(hX, JY) - \textit{h-mixed torsion},$$

$$\mathbb{R}^1 : (X, Y) \mapsto v\mathbb{T}(hX, hY) - \textit{v-horizontal torsion},$$

$$\mathbb{P}^1 : (X, Y) \mapsto v\mathbb{T}(hX, JY) - \textit{v-mixed torsion},$$

$$\mathbb{S}^1 : (X, Y) \mapsto v\mathbb{T}(JX, JY) - \textit{v-vertical torsion},$$

determine $\mathbb{T}$ completely. Similarly, $\mathbb{K}$ can be described by the following three mappings:

$$\mathbb{R} : (X, Y, Z) \mapsto \mathbb{K}(hX, hY)JZ - \textit{horizontal curvature},$$

$$\mathbb{P} : (X, Y, Z) \mapsto \mathbb{K}(hX, JY)JZ - \textit{mixed curvature},$$

$$\mathbb{Q} : (X, Y, Z) \mapsto \mathbb{K}(JX, JY)JZ - \textit{vertical curvature}.$$

The tensors $\mathbb{R}, \mathbb{P}, \mathbb{Q} \in \mathcal{T}_3^1(TM)$ are semibasic.

3.3 Definition. Let (D, h) be a Finsler connection on M . If $L \in \mathcal{T}_3^1(TM)$ is one of the horizontal, the mixed or the vertical curvatures of (D, h) , then the $(0, 2)$ -tensor

$$\tilde{L} : (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto \tilde{L}(X, Y) := \text{tr}[F \circ (Z \mapsto L(Y, Z)X)]$$

is said to be the *horizontal*, the *mixed*, or the *vertical Ricci tensor* of the Finsler connection, respectively. (F is an almost complex structure, e.g. the almost complex structure induced by h .) If L^* is defined by

$$L^*(X, Y, Z) := L(Y, Z)X,$$

then the Ricci tensor $\tilde{L}$ is the classical semibasic trace of L^* .

3.4 Berwald-type connections. Let us suppose that a horizontal endomorphism $h \in \Psi^1(TM)$ is given. We define the mapping

$$\overset{\circ}{D} : \mathfrak{X}(TM) \times \mathfrak{X}(TM) \rightarrow \mathfrak{X}(TM), \quad (X, Y) \mapsto \overset{\circ}{D}_X Y$$

by the following rules:

$$(3.4a) \quad \overset{\circ}{D}_{JX} JY := J[JX, Y],$$

$$(3.4b) \quad \overset{\circ}{D}_{hX} JY := v[hX, JY],$$

$$(3.4c) \quad \overset{\circ}{D}_{vX} hY := h[vX, Y],$$

$$(3.4d) \quad \overset{\circ}{D}_{hX} hY := hF[hX, JY]$$

and

$$\overset{\circ}{D}_X Y := \overset{\circ}{D}_{vX} vY + \overset{\circ}{D}_{hX} vY + \overset{\circ}{D}_{vX} hY + \overset{\circ}{D}_{hX} hY.$$

It is straightforward to prove that $(\overset{\circ}{D}, h)$ is a Finsler connection on M ; this Finsler connection is said to be the *Berwald-type Finsler connection induced by h* . If, in particular, h is a Berwald endomorphism, then we call $(\overset{\circ}{D}, h)$ a *Berwald connection*.

Replacing the vector fields in (3.4a)–(3.4d) by complete lifts X^c , Y^c , and taking into account (1.5c), 2.1 and (2.6c), we get the useful formulas

$$(3.4e,f) \quad \overset{\circ}{D}_{X^v} Y^v = 0, \quad \overset{\circ}{D}_{X^v} Y^h = 0.$$

$$(3.4g,h) \quad \overset{\circ}{D}_{X^h} Y^v = [X^h, Y^v], \quad \overset{\circ}{D}_{X^h} Y^h = F[X^h, Y^v].$$

$$(3.4i,j) \quad \overset{\circ}{D}_{X^v} FvY^c = [X, Y]^h - F[X,^v, Y^h], \quad \overset{\circ}{D}_{vX^c} Y^h = 0.$$

$$(3.4k) \quad \overset{\circ}{D}_J C = J.$$

If h is generated by the spray S , i.e., $(\overset{\circ}{D}, h)$ is the *Berwald connection* of (M, S)

$$(3.4l-n) \quad \overset{\circ}{D}_{X^v} S = X^h, \quad \overset{\circ}{D}_S X^v = X^c - X^h, \quad \overset{\circ}{D}_S X^h = FvX^c.$$

$$(3.4o,p) \quad \overset{\circ}{D}_{X^h} S = 0, \quad \overset{\circ}{D}_S S = 0.$$

3.5 Lemma. Suppose $(\overset{\circ}{D}, h)$ is a Berwald-type Finsler connection on M .

- (i) The h -deflection of $(\overset{\circ}{D}, h)$ coincides with the tension of h .
- (ii) The torsion $\overset{\circ}{\mathbb{T}}$ of $\overset{\circ}{D}$ can be represented in the form

$$(3.5) \quad \overset{\circ}{\mathbb{T}} = F \circ t + R,$$

where t and R are the torsion and the curvature of h , and F is the almost complex structure induced by h .

Proof.

- (i) Let S be an arbitrary semispray. For each vector field X on TM ,

$$\begin{aligned} H(hX) &:= [h, C]hX = [hX, C] - h[hX, C] = v[hX, C] \\ &= v[hX, JS] \stackrel{(3.4b)}{=} \overset{\circ}{D}_{hX} JS = h^*(\overset{\circ}{DC})(X) \\ &= h^*(\overset{\circ}{DC})(hX). \end{aligned}$$

This proves our assertion since H and $h^*(\overset{\circ}{DC})$ are semibasic tensors.

- (ii) Let $X, Y \in \mathfrak{X}(M)$. Then, using (3.4e)–(3.4h) we get:

$$\begin{aligned} \overset{\circ}{\mathbb{T}}(X^h, Y^v) &= 0, \quad \overset{\circ}{\mathbb{T}}(X^v, Y^v) = 0; \\ \overset{\circ}{\mathbb{T}}(X^h, Y^h) &= F([X^h, Y^v] - [Y^h, X^v]) - [X^h, Y^h] \\ &= F([X^h, Y^v] - [Y^h, X^v]) - h[X^h, Y^h] - v[X^h, Y^h] \\ &\stackrel{(2.6a), (2.3c)}{=} F([X^h, Y^v] - [Y^h, X^v] - [X, Y]^v) + R(X^h, Y^h) \\ &\stackrel{(2.3b)}{=} (F \circ t + R)(X^h, Y^h), \end{aligned}$$

thus (3.5) is also true. □

3.6 Corollary. If $(\overset{\circ}{D}, h)$ is a Berwald-type Finsler connection then

- (i) the h -deflection of $(\overset{\circ}{D}, h)$ vanishes if and only if h is homogeneous;
- (ii) the h -horizontal torsion $\overset{\circ}{\mathbb{A}}$ of $\overset{\circ}{D}$ is related to the torsion of h by the formula $\overset{\circ}{\mathbb{A}} = F \circ t$, therefore $\overset{\circ}{\mathbb{A}} = 0 \Leftrightarrow t = 0$, and then $\overset{\circ}{\mathbb{T}} = R$. □

3.7 Lemma. Suppose $(\overset{\circ}{D}, h)$ is a Berwald-type Finsler connection and $A \in \mathcal{T}_s^1(TM)$ is a semibasic tensor. Then

$$(3.7a) \quad \forall X \in \mathfrak{X}(M) : \overset{\circ}{D}_{X^v} A = \mathcal{L}_{X^v} A,$$

more precisely,

$$(3.7b) \quad \forall X_1, \dots, X_s \in \mathfrak{X}(M) : \left(\overset{\circ}{D}_{X^v} A \right) (X_1^c, \dots, X_s^c) = (\mathcal{L}_{X^v} A) (X_1^c, \dots, X_s^c) \\ = [X^v, A(X_1^c, \dots, X_s^c)].$$

Proof. Since $[X^v, X_i^h] \in \mathfrak{X}^v(TM)$ ($1 \leq i \leq s$) for all $X \in \mathfrak{X}(M)$ and A is semibasic, it follows that

$$(\mathcal{L}_{X^v} A) (X_1^h, \dots, X_s^h) = [X^v, A(X_1^h, \dots, X_s^h)].$$

Here

$$\begin{aligned} [X^v, A(X_1^h, \dots, X_s^h)] &\stackrel{(2.6c)}{=} [JFA(X_1^h, \dots, X_s^h), X^v] \\ &\stackrel{(1.3a)}{=} [J, X^v](FA(X_1^h, \dots, X_s^h)) - J[FA(X_1^h, \dots, X_s^h), X^v] \\ &\stackrel{(1.5c)}{=} J[JX^c, FA(X_1^h, \dots, X_s^h)] \\ &\stackrel{(3.4a)}{=} \overset{\circ}{D}_{X^v} A(X_1^h, \dots, X_s^h) \\ &\stackrel{(3.4f)}{=} \left(\overset{\circ}{D}_{X^v} A \right) (X_1^h, \dots, X_s^h), \end{aligned}$$

so we obtain the result. $\square$

4. The mixed curvature and the mixed Ricci tensor of a Berwald-type connection

4.1. If $(\overset{\circ}{D}, h)$ is a Finsler connection of Berwald-type, then we denote the horizontal, the mixed and the vertical curvature of $\overset{\circ}{D}$ by $\overset{\circ}{\mathbb{R}}$, $\overset{\circ}{\mathbb{P}}$ and $\overset{\circ}{\mathbb{Q}}$, respectively. In the centre of interest in this work is the mixed curvature, but for completeness we recall that $\overset{\circ}{\mathbb{Q}}$ *always vanishes*, while $\overset{\circ}{\mathbb{R}}$ can be given by the following formula:

$$(4.1) \quad \forall X, Y, Z \in \mathfrak{X}(TM) : \overset{\circ}{\mathbb{R}}(X, Y)Z = [J, R(X, Y)]Z.$$

For a good study of the surviving curvatures (in the Finslerian case) we refer to [55].

4.2. Suppose D is a linear connection on the manifold TM (or $\mathcal{T}M$). It will be convenient to introduce the operator

$$D_J, D_h : \mathcal{T}_s^r(TM) \rightarrow \mathcal{T}_{s+1}^r(TM)$$

by the rule

$$i_X D_J A := D_{JX} A \quad i_X D_h A := D_{hX} A \quad (X \in \mathfrak{X}(TM)).$$

Then, for example, (3.4a) can be written in the more compact form

$$\forall Y \in \mathfrak{X}(TM) : \overset{\circ}{D}_J JY = [J, JY].$$

4.3 Lemma (a J -Ricci formula). *Let $(\overset{\circ}{D}, h)$ be a Berwald-type connection on M . If $\alpha \in \Omega^1(TM)$ is a semibasic 1-form, then for any vector fields X, Y, Z on TM*

$$D_J D_J \alpha(X, Y, Z) = D_J D_J \alpha(Y, X, Z).$$

The proof is easy and is omitted.

4.4 Lemma. *Let h be a torsion-free horizontal endomorphism, $(\overset{\circ}{D}, h)$ the Berwald-type connection induced by h , and R the curvature of h . Then*

$$\mathfrak{S} [\overset{\circ}{D}_h R(X, Y, Z)] = 0,$$

where the symbol $\mathfrak{S}$ means cyclic permutation of the variables $X, Y, Z \in \mathfrak{X}(TM)$ and summation.

The proof is a straightforward calculation which we omit here, but see [63].

4.5 Corollary. *Under the preceding condition,*

$$(4.5) \quad \mathfrak{S} \overset{\circ}{\mathbb{R}}(X, Y)Z = 0 \quad (X, Y, Z \in \mathfrak{X}(TM)).$$

Proof. Apply the algebraic Bianchi identity to the curvature tensor field of D , and use 4.4. $\square$

4.6 Lemma (an h -Ricci formula). *Let h be a torsion-free horizontal endomorphism on M , and $(\overset{\circ}{D}, h)$ the corresponding Berwald-type connection. Then, for any 1-form α on TM and vector fields X, Y, Z on M , we have*

$$(4.6) \quad \begin{aligned} & \overset{\circ}{D}_h \overset{\circ}{D}_h \alpha(X^h, Y^h, Z^h) - \overset{\circ}{D}_h \overset{\circ}{D}_h \alpha(Y^h, X^h, Z^h) \\ &= -(\overset{\circ}{D}_J \alpha)(FR(X^h, Y^h), Z^h) - \alpha(F\overset{\circ}{\mathbb{R}}(X^h, Y^h)Z^h). \end{aligned}$$

Proof. Apply the classical Ricci formula (1.21) of [11]; split the curvature and the torsion of D according to 3.2, finally take into account that the vanishing of the weak torsion implies the coincidence of the torsion of D and the curvature of h (see Corollary 3.6). $\square$

4.7 Proposition. *Let h be a torsion-free horizontal endomorphism on M , and $(\overset{\circ}{D}, h)$ be Berwald-type connection induced by h . If α is a semibasic 1-form on TM , then $d_h\alpha$ is also semibasic, and we have the following relations:*

$$(4.7a) \quad d_h\alpha = h^*d\alpha;$$

$$(4.7b) \quad d_h\alpha(X^h, Y^h) = (\overset{\circ}{D}_h\alpha)(X^h, Y^h) - (\overset{\circ}{D}_h\alpha)(Y^h, X^h) \quad (X, Y \in \mathfrak{X}(M));$$

$$(4.7c,d) \quad \overset{\circ}{D}_J\alpha = \beta, \text{ where } \beta(X, Y) := d\alpha(JX, Y) \quad (X, Y \in \mathfrak{X}(TM)).$$

Proof. (1) Since α is semibasic, $d_h\alpha := i_h d\alpha - di_h\alpha = i_h d\alpha - d\alpha$. It can easily be checked that pairs of form (X^v, Y^v) , (X^v, Y^h) , (X^h, Y^v) ; $X, Y \in \mathfrak{X}(M)$ kill the 2-form $i_h d\alpha - d\alpha$, thus $d_h\alpha$ is semibasic.

(2) For any vector fields X, Y on M ,

$$\begin{aligned} (i_h d\alpha - d\alpha)(X^h, Y^h) &= i_h d\alpha(X^h, Y^h) - d\alpha(X^h, Y^h) \\ &= d\alpha(X^h, Y^h) = h^*d\alpha(X^h, Y^h); \end{aligned}$$

this and (1) imply (4.7a).

(3) Since on the one hand

$$\begin{aligned} d_h\alpha(X^h, Y^h) &\stackrel{(4.7a)}{=} d\alpha(X^h, Y^h) \\ &= X^h\alpha(Y^h) - Y^h\alpha(X^h) - \alpha([X^h, Y^h]) \\ &\stackrel{(2.1c)}{=} X^h\alpha(Y^h) - Y^h\alpha(X^h) - \alpha([X, Y]^h), \end{aligned}$$

on the other hand

$$\begin{aligned} \overset{\circ}{D}_h\alpha(X^h, Y^h) &= \left(\overset{\circ}{D}_{X^h}\alpha \right)(Y^h) = X^h\alpha(Y^h) - \alpha\left(\overset{\circ}{D}_{X^h}Y^h \right) \\ &\stackrel{(3.4h)}{=} X^h\alpha(Y^h) - \alpha(F[X^h, Y^v]), \end{aligned}$$

and h is torsion-free, we obtain:

$$\begin{aligned} &\overset{\circ}{D}_h\alpha(X^h, Y^h) - \overset{\circ}{D}_h\alpha(Y^h, X^h) \\ &= X^h\alpha(Y^h) - Y^h\alpha(X^h) - \alpha(F([X^h, Y^v] - [Y^h, X^v])) \\ &= X^h\alpha(Y^h) - Y^h\alpha(X^h) - \alpha(F[X, Y]^v) \\ &\stackrel{(2.6a)}{=} d_h\alpha(X^h, Y^h), \end{aligned}$$

whence (4.7b).

(4) From (4.7b) it is clear that the 2-tensor β is semibasic. So we have only to check that for any vector fields X, Y on M , $(\overset{\circ}{D}_J\alpha)(X^h, Y^h) = \beta(X^h, Y^h)$. But this is immediate:

$$\left(\overset{\circ}{D}_J\alpha\right)(X^h, Y^h) = \left(\overset{\circ}{D}_{X^v}\alpha\right)(Y^h) = X^v\alpha(Y^h) - \alpha\left(\overset{\circ}{D}_{X^v}Y^h\right) \stackrel{(3.4f)}{=} X^v\alpha(Y^h),$$

while $\beta(X^h, Y^h) = d\alpha(X^v, Y^h) = X^v\alpha(Y^h)$. $\square$

4.8 Proposition. *Keeping the hypothesis of 4.7, suppose that α is d_J -closed. Then we have*

$$(4.8) \quad \mathfrak{S}_{X,Y,Z \in \mathfrak{X}(M)} \left[\overset{\circ}{D}_h d_h \alpha(X^h, Y^h, Z^h) + \beta(X^h, FR(Y^h, Z^h)) \right] = 0.$$

Proof. (1) Under the condition $d_J\alpha = 0$ the tensor β is *symmetric*. Indeed, for any vector fields X, Y on M ,

$$\begin{aligned} & \beta(X^h, Y^h) - \beta(Y^h, X^h) \stackrel{(4.7c)}{=} \overset{\circ}{D}_J\alpha(X^h, Y^h) - \overset{\circ}{D}_J\alpha(Y^h, X^h) \\ & = X^v\alpha(Y^h) - Y^v\alpha(X^h) = d\alpha(X^v, Y^h) + d\alpha(X^h, Y^v) \\ & = i_J d\alpha(X^h, Y^h) = d_J\alpha(X^h, Y^h) = 0. \end{aligned}$$

(2) Now we calculate

$$\begin{aligned} & \mathfrak{S} \left[\overset{\circ}{D}_h d_h \alpha(X^h, Y^h, Z^h) \right] \\ & \stackrel{(4.7b)}{=} (\overset{\circ}{D}_h \overset{\circ}{D}_h \alpha)(X^h, Y^h, Z^h) - (\overset{\circ}{D}_h \overset{\circ}{D}_h \alpha)(X^h, Z^h, Y^h) + (\overset{\circ}{D}_h \overset{\circ}{D}_h \alpha)(Y^h, Z^h, X^h) \\ & \quad - (\overset{\circ}{D}_h \overset{\circ}{D}_h \alpha)(Y^h, X^h, Z^h) + (\overset{\circ}{D}_h \overset{\circ}{D}_h \alpha)(Z^h, X^h, Y^h) - (\overset{\circ}{D}_h \overset{\circ}{D}_h \alpha)(Z^h, Y^h, X^h) \stackrel{(4.6)}{=} \\ & \quad - \mathfrak{S} \left[(\overset{\circ}{D}_J\alpha)(FR(X^h, Y^h), Z^h) \right] - \mathfrak{S} \alpha(F\overset{\circ}{R}(X^h, Y^h)Z^h) \stackrel{(4.7c), (4.5)}{=} \\ & \quad - \mathfrak{S} \beta(X^h, FR(Y^h, Z^h)). \end{aligned} \quad \square$$

4.9 Lemma. *The mixed curvature $\overset{\circ}{\mathbb{P}}$ of a Berwald-type Finsler connection $(\overset{\circ}{D}, h)$ can be given by*

$$(4.9) \quad \forall X, Y, Z \in \mathfrak{X}(M) : \overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c = [[X^h, Y^v], Z^v].$$

$\overset{\circ}{\mathbb{P}}$ is symmetric in its second and third arguments. If, in addition, h is torsion free, then $\overset{\circ}{\mathbb{P}}$ is totally symmetric.

Proof.

$$\begin{aligned} \mathring{\mathbb{P}}(X^c, Y^c)Z^c &:= \mathbb{K}(hX^c, JY^c)JZ^c = \mathbb{K}(X^h, Y^v)Z^v := \mathring{D}_{X^h}\mathring{D}_{Y^v}Z^v \\ &\quad - \mathring{D}_{Y^v}\mathring{D}_{X^h}Z^v - \mathring{D}_{[X^h, Y^v]}Z^v \stackrel{(3.4e, g)}{=} -\mathring{D}_{Y^v}[X^h, Z^v] - \mathring{D}_{[X^h, Y^v]}Z^v. \end{aligned}$$

Since $[X^h, Y^v]$ is vertical, it can be combined from vertically lifted vector fields, so (3.4e) implies that $\mathring{D}_{[X^h, Y^v]}Z^v = 0$. The remainder term can be formed as follows:

$$\begin{aligned} -\mathring{D}_{Y^v}[X^h, Z^v] &= -\mathring{D}_{JY^c}JF[X^h, Z^v] \stackrel{(3.4a)}{=} -J[Y^v, F[X^h, Z^v]] \\ &= J[F[X^h, Z^v], Y^v] \stackrel{(1.3a)}{=} -[J, Y^v]F[X^h, Z^v] \\ &\quad + [JF[X^h, Z^v], Y^v] \stackrel{(1.5c), (2.6c)}{=} [[X^h, Z^v], Y^v] \\ &\stackrel{\text{Jacobi-identity}}{=} -[[Z^v, Y^v], X^h] - [[Y^v, X^h], Z^v] \\ &\stackrel{(1.4)}{=} [[X^h, Y^v], Z^v]. \end{aligned}$$

Thus (4.9) is verified, and, at the same time, we have got the relation

$$\mathring{\mathbb{P}}(X^c, Z^c)Y^c = [[X^h, Z^v], Y^v] = \mathring{\mathbb{P}}(X^c, Y^c)Z^c.$$

If h is torsion free and hence, in view of (2.3b),

$$[X^h, Y^v] - [Y^h, X^v] - [X, Y]^v = 0,$$

it follows that

$$\begin{aligned} [[X^h, Y^v], Z^v] &= [[Y^h, X^v], Z^v] + [[X, Y]^v, Z^v] \stackrel{(1.4)}{=} \\ &= [[Y^h, X^v], Z^v] = \mathring{\mathbb{P}}(Y^c, X^c)Z^c, \end{aligned}$$

thus $\mathring{\mathbb{P}}$ is totally symmetric. $\square$

4.10 Proposition. *If h is a homogeneous and torsion free horizontal endomorphism, then the mixed curvature of the Berwald-type connection $(\mathring{D}, h)$ has the following properties:*

$$(4.10a) \quad i_S \mathring{\mathbb{P}} = 0, \text{ for any semispray } S;$$

$$(4.10b) \quad \mathcal{L}_C \mathring{\mathbb{P}} = -2\mathring{\mathbb{P}}, \text{ i.e., } \mathring{\mathbb{P}} \text{ is homogeneous of degree } -1;$$

$$(4.10c) \quad \mathring{D}_J \mathring{\mathbb{P}} \text{ is totally symmetric.}$$

Proof.

(a) Owing to Lemma 4.9, it is enough to check that

$$\forall X, Y \in \mathfrak{X}(M) : \mathring{\mathbb{P}}(X^c, S)Y^c = 0.$$

Calculating as before, we get:

$$\begin{aligned} \mathring{\mathbb{P}}(X^c, S)Y^c &= \mathring{D}_{X^h} \mathring{D}_{JS} Y^v - \mathring{D}_{JS} \mathring{D}_{X^h} Y^v - \mathring{D}_{[X^h, JS]} Y^c \\ &= \mathring{D}_{X^h} J[C, Y^c] - \mathring{D}_{JS} [X^h, Y^v] - \mathring{D}_{[X^h, C]} Y^c \\ &\stackrel{(1.5d), 2.2, (2.6c)}{=} -\mathring{D}_{JS} JF[X^h, Y^v] \stackrel{(3.4a)}{=} -J[C, F[X^h, Y^v]] \\ &= J[F[X^h, Y^v], C] \stackrel{(1.3a)}{=} -[J, C]F[X^h, Y^v] \\ &\quad + [JF[X^h, Y^v], C] \stackrel{(1.5b), (2.6c)}{=} -[X^h, Y^v] + [[X^h, Y^v], C] \\ &\stackrel{\text{Jacobi identity}}{=} -[X^h, Y^v] - [[Y^v, C], X^h] - [[C, X^h], Y^v] \\ &\stackrel{(1.5d), 2.2}{=} -[X^h, Y^v] - [Y^v, X^h] = 0. \end{aligned}$$

(b) Since the Lie brackets of C and the complete lifts vanish, for any vector fields $X, Y, Z \in \mathfrak{X}(M)$ we have

$$\left(\mathcal{L}_C \mathring{\mathbb{P}} \right) (X^c, Y^c, Z^c) = [C, \mathring{\mathbb{P}}(X^c, Y^c, Z^c)] \stackrel{(4.9)}{=} [C, [[X^h, Y^v], Z^v]].$$

Using the Jacobi identity repeatedly, in view of (1.5d) and the homogeneity of h it follows that

$$\begin{aligned} [C, [[X^h, Y^v], Z^v]] &= -[[X^h, Y^v], [Z^v, C]] - [Z^v, [C, [X^h, Y^v]]] \\ &= -[[X^h, Y^v], Z^v] - [Z^v, [C, [X^h, Y^v]]] = -[[X^h, Y^v], Z^v] \\ &\quad + [Z^v, [X^h, [Y^v, C]]] + [Z^v, [Y^v, [C, X^h]]] \\ &= -[[X^h, Y^v], Z^v] - [[X^h, Y^v], Z^v] \stackrel{(4.9)}{=} -2\mathring{\mathbb{P}}(X^c, Y^c)Z^c. \end{aligned}$$

Thus $\mathcal{L}_C \mathring{\mathbb{P}} = -2\mathring{\mathbb{P}}$.

(c) For any vector fields $X, Y, Z, U \in \mathfrak{X}(M)$,

$$\left(\mathring{D}_J \mathring{\mathbb{P}} \right) (X^c, Y^c, Z^c, U^c) \stackrel{4.2}{=} \left(\mathring{D}_{JX^c} \mathring{\mathbb{P}} \right) (Y^c, Z^c, U^c) = \mathring{D}_{JX^c} \mathring{\mathbb{P}}(Y^c, Z^c)U^c,$$

since $\mathring{\mathbb{P}}$ is semibasic, and, for example,

$$\begin{aligned} \mathring{D}_{JX^c} Y^c &= \mathring{D}_{JX^c} (vY^c + hY^c) = \mathring{D}_{JX^c} vY^c + \mathring{D}_{X^v} Y^h \\ &\stackrel{(3.4f)}{=} \mathring{D}_{JX^c} vY^c \in \mathfrak{X}^v(TM). \end{aligned}$$

Thus in view of Lemma 4.9 and the calculation in its proof, we get:

$$\begin{aligned}
\left(\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}\right)(X^c, Y^c, Z^c, U^c) &= \overset{\circ}{D}_{JX^c}[[Y^h, Z^v], U^v] = \overset{\circ}{D}_{JX^c}JF[[Y^h, Z^v], U^v] \\
&\stackrel{(3.4a)}{=} J[X^v, F[[Y^h, Z^v], U^v]] = [X^v, [[Y^h, Z^v], U^v]] \\
&= -[[Y^h, Z^v], [U^v, X^v]] - [U^v, [X^v, [Y^h, Z^v]]] \\
&= [U^v, [[Y^h, Z^v], X^v]] = [U^v, \overset{\circ}{\mathbb{P}}(Y^c, Z^c)X^c] \\
&\stackrel{(3.7b)}{=} \left(\overset{\circ}{D}_{U^v}\overset{\circ}{\mathbb{P}}\right)(Y^c, Z^c, X^c) = \left(\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}\right)(U^c, Y^c, Z^c, X^c).
\end{aligned}$$

This symmetry property together with the total symmetry of $\overset{\circ}{\mathbb{P}}$ implies the total symmetry of $\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}$. $\square$

4.11 Corollary. *Under the hypothesis of 4.10, the mixed Ricci tensor $\overset{\circ}{\mathbb{P}}$ of the Berwald-type connection $(\overset{\circ}{D}, h)$ has the following properties:*

$$(4.11a) \quad \mathcal{L}_C\overset{\circ}{\mathbb{P}} = \overset{\circ}{D}_C\overset{\circ}{\mathbb{P}} = -\overset{\circ}{\mathbb{P}} \text{ (i.e., } \overset{\circ}{\mathbb{P}} \text{ is } (-1)\text{-homogeneous);}$$

$$(4.11b) \quad \overset{\circ}{D}_J\overset{\circ}{\mathbb{P}} \text{ is totally symmetric.}$$

Proof. (4.11a) is a consequence of (4.10b) and Corollary 2.14. Since the semibasic trace operator clearly commutes with the tensor derivations, (4.11b) follows from (4.10c). $\square$

5. Yano-type connections

We start with a slight generalization of the Berwald-type connections.

5.1 Proposition. *Suppose h is a horizontal endomorphism on the manifold M with associated almost complex structure F . Let $\beta \in \mathcal{T}_2^0(TM)$ be a symmetric tensor, satisfying the condition*

$$(*) \quad \text{for any semispray } S, \quad i_S\beta = 0.$$

Let, finally, a vertical vector field $U \in \mathfrak{X}(TM)$ be given. Then there exists a unique Finsler connection (D, h) on M such that

- (i) *the v -mixed torsion of D is $\mathbb{P}^1 := \beta \otimes U$;*
- (ii) *the h -mixed torsion $\mathbb{B}$ of D vanishes.*

The table of rules of calculation of the covariant derivatives with respect to D is

$$(5.1a) \quad D_{JX}JY = J[JX, Y] = \overset{\circ}{D}_{JX}JY,$$

$$(5.1b) \quad D_{hX}JY = v[hX, JY] + \beta(X, Y)U = \overset{\circ}{D}_{hX}JY + \beta(X, Y)U,$$

$$(5.1c) \quad D_{vX}hY = h[vX, Y] = \overset{\circ}{D}_{vX}hY,$$

$$(5.1d) \quad D_{hX}hY = hF[hX, JY] + \beta(X, Y)FU = \overset{\circ}{D}_{hX}hY + \beta(X, Y)FU.$$

If, in addition,

(iii) the h -deflection of (D, h) vanishes,

(iv) the h -horizontal torsion of D vanishes

then the horizontal endomorphism h is homogeneous and torsion free.

Proof.

(a) *Unicity.* We show that (i) and (ii) imply (5.1a)–(5.1d).

$$(i) \Rightarrow (5.1b) \quad \forall X, Y \in \mathfrak{X}(TM) : \beta(X, Y)U \stackrel{(i)}{=} \mathbb{P}^1(X, Y)$$

$$\stackrel{3.2}{=} v(D_{hX}JY - D_{JY}hX - [hX, JY]) = D_{hX}JY - v[hX, JY], \text{ so (5.1b) is valid.}$$

$$(ii) \Rightarrow (5.1c) \quad \text{For any vector fields } X, Y \in \mathfrak{X}(TM),$$

$$0 \stackrel{(ii)}{=} \mathbb{B}(Y, X) \stackrel{3.2}{=} h(D_{hY}JX - D_{JX}hY - [hY, JX]) = -D_{JX}hY - h[hY, JX],$$

thus $D_{JX}hY = h[JX, hY]$. Replacing X by FX , we get

$$D_{vX}hY = D_{JFX}hY = h[vX, hY] = h[vX, Y] - h[vX, vY] = h[vX, Y].$$

This means that (5.1c) also holds. Now (5.1a) and (5.1d) follow from (5.1b) and (5.1c) by (3.1c) and (3.1d).

(b) *Existence.* Having the data h, β, U , we define a linear connection D on TM by the rules (5.1a)–(5.1d). A straightforward calculation shows that (D, h) is a Finsler connection satisfying (i) and (ii).

(c) The remainder is an immediate consequence of Lemma 3.5. □

5.2 Corollary. A Finsler connection $(\overset{\circ}{D}, h)$ on M is a Berwald-type connection if and only if the following axioms are satisfied:

(i) The v -mixed torsion of $\overset{\circ}{D}$ vanishes.

(ii) The h -mixed torsion of $\overset{\circ}{D}$ vanishes. □

5.3 Corollary and definition. Suppose h is a homogeneous and torsion free horizontal endomorphism on M . Let $\overset{\circ}{\mathbb{P}}$ be the mixed Ricci tensor of the Berwald-type connection $(\overset{\circ}{D}, h)$. There exists a unique Finsler connection (D, h) on M such that

- (i) the v -mixed torsion of D is $\mathbb{P}^1 := \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \otimes C$;
- (ii) the h -mixed torsion of D vanishes.

This Finsler connection is said to be the Yano-type connection induced by h . If, in particular, h is a Berwald endomorphism, then we speak of a Yano connection.

Proof. Recall that $\overset{\circ}{\mathbb{P}}$ is symmetric by Lemma 4.9, while the validity of the condition (*) in 5.1 is assured by (4.10a). $\square$

5.4 Remark. If $(\overset{\circ}{D}, h)$ and (D, h) are the Berwald-type and the Yano-type connections induced by h , then we see from (5.1a) and (5.1c) that $\overset{\circ}{D}_J = D_J$. In view of Lemma 3.7 and Proposition 4.10 this implies that

for any vector field $X \in \mathfrak{X}(M)$ and semibasic tensor $A \in \mathcal{T}_s^1(TM)$,

$$(5.4a) \quad D_{X^v} A = \mathcal{L}_{X^v} A;$$

$$(5.4b) \quad D_J \overset{\circ}{\mathbb{P}} \text{ is totally symmetric.}$$

5.5 Proposition. Suppose that h is a torsion free, homogeneous horizontal endomorphism. Let $(\overset{\circ}{D}, h)$ and (D, h) be the induced Berwald-type and Yano-type connections with mixed curvature and mixed Ricci tensors $\overset{\circ}{\mathbb{P}}, \mathbb{P}$ and $\overset{\circ}{\mathbb{P}}, \tilde{\mathbb{P}}$, respectively. Then we have the following relations:

$$(5.5a) \quad \begin{aligned} \mathbb{P}(X^c, Y^c)Z^c &= \left[\overset{\circ}{\mathbb{P}} - \frac{1}{n+1} \left(D_J \overset{\circ}{\mathbb{P}} \otimes C \right) \right] (X^c, Y^c, Z^c) \\ &\quad - \frac{1}{n+1} \left(\overset{\circ}{\mathbb{P}} \otimes J \right) (X^c, Z^c, Y^c) \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$;

$$(5.5b) \quad \tilde{\mathbb{P}} = \frac{2}{n+1} \overset{\circ}{\mathbb{P}}.$$

Proof.

(a) By the rules of calculation (5.1a)–(5.1c),

$$\begin{aligned}\mathbb{P}(X^c, Y^c)Z^c &= D_{X^h}D_{Y^v}Z^v - D_{Y^v}D_{X^h}Z^v - D_{[X^h, Y^v]}Z^v = -D_{Y^v}D_{X^h}Z^v \\ &= -D_{Y^v}([X^h, Z^v] + \frac{1}{n+1}\tilde{\mathbb{P}}(X^c, Z^c)C).\end{aligned}$$

Thus, taking into account (4.9) and 5.4,

$$\mathbb{P}(X^c, Y^c)Z^c = \mathring{\mathbb{P}}(X^c, Y^c)Z^c - \frac{1}{n+1}D_{Y^v}\left(\tilde{\mathbb{P}}(X^c, Z^c)C\right).$$

Now using the fact that $\left(D_J\tilde{\mathbb{P}}\right)(X^c, Y^c, Z^c) = D_{JX^c}\left[\tilde{\mathbb{P}}(Y^c, Z^c)\right]$ and – taking a semispray S –

$$D_JC(Y^c) = D_{JY^c}C = D_{JY^c}JS \stackrel{(5.1a)}{=} J[Y^v, S] \stackrel{(2.7)}{=} Y^v = J(Y^c),$$

it follows that

$$\begin{aligned}D_{Y^v}\left(\tilde{\mathbb{P}}(X^c, Z^c)C\right) &= Y^v\left(\tilde{\mathbb{P}}(X^c, Z^c)\right)C + \tilde{\mathbb{P}}(X^c, Z^c)D_{Y^v}C \\ &= \left(D_J\tilde{\mathbb{P}} \otimes C\right)(Y^c, X^c, Z^c) + \left(\tilde{\mathbb{P}} \otimes J\right)(X^c, Z^c, Y^c) \\ &\stackrel{(5.4b)}{=} \left(D_J\tilde{\mathbb{P}} \otimes C\right)(X^c, Y^c, Z^c) + \left(\tilde{\mathbb{P}} \otimes J\right)(X^c, Z^c, Y^c).\end{aligned}$$

This proves (5.5a).

(b) In view of Definition 3.3 and the symmetry of $\tilde{\mathbb{P}}$ and $D_J\tilde{\mathbb{P}}$,

$$\begin{aligned}\tilde{\mathbb{P}}(X^c, Y^c) &= tr[F \circ (Z^c \mapsto \mathbb{P}(Y^c, Z^c)X^c)] \\ &\stackrel{(5.5a)}{=} tr\left[F \circ (Z^c \mapsto \left(\mathring{\mathbb{P}}(Y^c, Z^c)X^c - \frac{1}{n+1}\left(D_J\tilde{\mathbb{P}} \otimes C\right)(Y^c, Z^c, X^c) \right. \right. \\ &\quad \left. \left. - \frac{1}{n+1}\left(\tilde{\mathbb{P}} \otimes J\right)(X^c, Y^c, Z^c)\right)\right] \\ &= \tilde{\mathbb{P}}(X^c, Y^c) - \frac{1}{n+1}\widetilde{D_J\tilde{\mathbb{P}} \otimes C(X^c, Y^c)} - \frac{1}{n+1}\widetilde{\tilde{\mathbb{P}} \otimes J(X^c, Y^c)}.\end{aligned}$$

Since $\widetilde{\mathbb{P}} \otimes J = n\widetilde{\mathbb{P}}$, and (taking a semispray S)

$$\begin{aligned} \widetilde{D_J \mathbb{P} \otimes C} &= \widetilde{D_J \mathbb{P} \otimes JS} \stackrel{(2.11b)}{=} i_{FJS} D_J \widetilde{\mathbb{P}} = i_{hS} D_J \widetilde{\mathbb{P}} \\ &= D_{JhS} \widetilde{\mathbb{P}} = D_C \widetilde{\mathbb{P}} = \mathcal{L}_C \widetilde{\mathbb{P}} \stackrel{(4.11a)}{=} -\widetilde{\mathbb{P}}, \end{aligned}$$

we have:

$$\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}} - \frac{n}{n+1} \widetilde{\mathbb{P}} + \frac{1}{n+1} \widetilde{\mathbb{P}} = \frac{2}{n+1} \widetilde{\mathbb{P}}. \quad \square$$

5.6 Remark. Using the symmetric product (1.1), (5.5a) can also be written in the form

$$(5.6) \quad \mathbb{P} = \overset{\circ}{\mathbb{P}} - \frac{1}{n+1} \left[\left(D_J \overset{\circ}{\mathbb{P}} \right) \otimes C + \overset{\circ}{\mathbb{P}} \odot J - \overset{\circ}{\mathbb{P}} \otimes J - J \otimes \overset{\circ}{\mathbb{P}} \right].$$

6. The Douglas tensor of a Berwald endomorphism

6.1. Two sprays S and $\overline{S}$ over a manifold M are said to be *projectively equivalent* if there is a function $\lambda : TM \rightarrow \mathbb{R}$ satisfying the conditions

- (i) λ is smooth on TM , and C^1 on TM ;
- (ii) $\overline{S} = S + \lambda C$.

Then λ is automatically 1-homogeneous (i.e., $C\lambda = \lambda$). Conversely, if a spray S and a 1-homogeneous function λ , satisfying (i), are given, then $\overline{S} = S + \lambda C$ is also a spray. In this case we speak of a *projective change* of the spray, and we say that the spray manifolds (M, S) and $(M, \overline{S})$ are *projectively equivalent*.

6.2 Lemma. Suppose that h is a Berwald endomorphism with associated spray S . Consider a projective change $S \rightarrow \overline{S} := S + \lambda C$ of S . Then h changes by the formula

$$(6.2a) \quad h \rightarrow \overline{h} = h + \frac{1}{2} \lambda J + \frac{1}{2} d_J \lambda \otimes C.$$

$\overline{h}$ remains homogeneous and torsion free. The horizontal lifting with respect to $\overline{h}$ is the mapping

$$(6.2b) \quad X \in \mathfrak{X}(M) \mapsto X^{\overline{h}} = X^h + \frac{1}{2} \lambda X^v + \frac{1}{2} (X^v \lambda) C.$$

The proof is an easy calculation so we omit it.

6.3 Definition. Suppose h is a Berwald endomorphism on the manifold M . If (D, h) is the Yano connection induced by h and $\mathbb{P}$ is the mixed curvature of D , then the tensor

$$(6.3) \quad \mathbb{D} := \mathbb{P} - \frac{1}{2} \left(\tilde{\mathbb{P}} \otimes J + J \otimes \tilde{\mathbb{P}} \right)$$

is said to be the *Douglas tensor* of the Berwald endomorphism.

6.4 Remark.

- (a) It is clear that $\mathbb{D}$ is semibasic and symmetric. By the definition, for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$(6.4a) \quad \mathbb{D}(X^c, Y^c)Z^c = \mathbb{P}(X^c, Y^c)Z^c - \frac{1}{2} \left(\tilde{\mathbb{P}}(X^c, Y^c)Z^v + \tilde{\mathbb{P}}(Y^c, Z^c)X^v \right).$$

- (b) It is easy to express $\mathbb{D}$ in terms of the Berwald connection $(\overset{\circ}{D}, h)$: in view of (6.4a), (5.5a) and (5.5b) we have

$$\begin{aligned} \mathbb{D}(X^c, Y^c)Z^c &= \overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c - \frac{1}{n+1} \left(\overset{\circ}{D}_J \tilde{\mathbb{P}} \otimes C \right) (X^c, Y^c, Z^c) \\ &\quad - \frac{1}{n+1} \tilde{\mathbb{P}}(X^c, Z^c)Y^v - \frac{1}{1+n} \tilde{\mathbb{P}}(X^c, Y^c)Z^v - \frac{1}{n+1} \overset{\circ}{\mathbb{P}}(Y^c, Z^c)X^v \\ &= \left[\overset{\circ}{\mathbb{P}} - \frac{1}{n+1} \left(\overset{\circ}{D}_J \tilde{\mathbb{P}} \otimes C + \tilde{\mathbb{P}} \odot J \right) \right] (X^c, Y^c, Z^c). \end{aligned}$$

In a more compact form:

$$(6.4b) \quad \mathbb{D} = \overset{\circ}{\mathbb{P}} - \frac{1}{n+1} \left(\overset{\circ}{D}_J \tilde{\mathbb{P}} \otimes C + \tilde{\mathbb{P}} \odot J \right).$$

6.5 Proposition. Let $\mathbb{D}$ be the Douglas tensor of a Berwald endomorphism. Then

$$(6.5a) \quad \text{for any semispray } S, \quad i_S \mathbb{D} = 0;$$

$$(6.5b) \quad \tilde{\mathbb{D}} = 0 \text{ (i.e., the semibasic trace of } \mathbb{D} \text{ vanishes).}$$

Proof.

- (a) Let us first observe that $i_S \tilde{\mathbb{P}} := \widetilde{i_S \overset{\circ}{\mathbb{P}}} \stackrel{(4.10a)}{=} 0$, so for any vector fields $Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned} (i_S \mathbb{P})(Y^c, Z^c) &\stackrel{(5.5a), (4.10a)}{=} -\frac{1}{n+1} \left[\left(D_C \tilde{\mathbb{P}} \right) (Y^c, Z^c) \otimes C + \left[i_S \tilde{\mathbb{P}}(Z^c) \right] Y^v \right] \\ &= -\frac{1}{n+1} \left(D_C \tilde{\mathbb{P}} \right) (Y^c, Z^c) \otimes C \stackrel{(4.11a)}{=} \frac{1}{n+1} \tilde{\mathbb{P}}(Y^c, Z^c) C. \end{aligned}$$

Thus

$$\begin{aligned}
(i_S \mathbb{D})(Y^c, Z^c) &= \mathbb{D}(S, Y^c, Z^c) = (i_S \mathbb{P})(Y^c, Z^c) - \frac{1}{2} [\tilde{\mathbb{P}}(S, Y^c)Z^v + \tilde{\mathbb{P}}(Y^c, Z^c)C] \\
&= \frac{1}{n+1} \tilde{\mathbb{P}}(Y^c, Z^c)C - \frac{1}{2} (i_S \tilde{\mathbb{P}}(Y^c)Z^v + \tilde{\mathbb{P}}(Y^c, Z^c)C) \\
&\stackrel{(5.5b)}{=} \left(\frac{1}{n+1} \tilde{\mathbb{P}} \otimes C - \frac{1}{n+1} i_S \tilde{\mathbb{P}} \otimes J - \frac{1}{n+1} \tilde{\mathbb{P}} \otimes C \right) (Y^c, Z^c) \\
&\stackrel{(4.10a)}{=} 0.
\end{aligned}$$

(b) Since

$$\overset{\circ}{D}_J \tilde{\mathbb{P}} \otimes C = \overset{\circ}{D}_J \tilde{\mathbb{P}} \otimes JS \stackrel{(2.11b)}{=} i_{FJS} \overset{\circ}{D}_J \tilde{\mathbb{P}} = i_{hS} \overset{\circ}{D}_J \tilde{\mathbb{P}} = \overset{\circ}{D}_{JhS} \tilde{\mathbb{P}} = \overset{\circ}{D}_C \tilde{\mathbb{P}} \stackrel{(4.11a)}{=} -\tilde{\mathbb{P}},$$

and

$$\tilde{\mathbb{P}} \odot J \stackrel{(2.11g)}{=} \tilde{J} \tilde{\mathbb{P}} + i_{FJ} \tilde{\mathbb{P}} \stackrel{(2.11a)}{=} n \tilde{\mathbb{P}} + i_h \tilde{\mathbb{P}} \stackrel{1,2}{=} (n+2) \tilde{\mathbb{P}};$$

using (6.4b) we get:

$$\tilde{\mathbb{D}} = \tilde{\mathbb{P}} - \frac{1}{n+1} \left(-\tilde{\mathbb{P}} + (n+2) \tilde{\mathbb{P}} \right) = \tilde{\mathbb{P}} - \tilde{\mathbb{P}} = 0. \quad \square$$

6.6 Theorem. *The Douglas tensor of a Berwald endomorphism is invariant under any projective change of the associated spray.*

Proof. Suppose h is a Berwald endomorphism on M with associated spray S . Denote by $\mathbb{D}$ the Douglas tensor of h . Consider a projective change

$$S \rightarrow \bar{S} := S + \lambda C, \quad C\lambda = \lambda.$$

Then $\bar{S}$ generates a Berwald endomorphism $\bar{h}$ (6.2); let $\bar{\mathbb{D}}$ be the Douglas tensor of $\bar{h}$. First we express the mixed curvature $\bar{\mathbb{P}}$ of the Berwald connection $(\bar{D}, \bar{h})$ in terms of the mixed curvature $\overset{\circ}{\mathbb{P}}$ of the Berwald connection $(\overset{\circ}{D}, h)$.

For any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}
\bar{\mathbb{P}}(X^c, Y^c)Z^c &\stackrel{(4.9)}{=} \left[[X^{\bar{h}}, Y^v], Z^v \right] \stackrel{(6.2b)}{=} \left[\left[X^h + \frac{1}{2} \lambda X^v + \frac{1}{2} (X^v \lambda) C, Y^v \right], Z^v \right] \\
&= [[X^h, Y^v], Z^v] + \frac{1}{2} [[\lambda X^v, Y^v], Z^v] + \frac{1}{2} [[(X^v \lambda) C, Y^v], Z^v] \\
&= \overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c + \frac{1}{2} [Z^v (Y^v \lambda)] X^v + \frac{1}{2} [Z^v [Y^v (X^v \lambda)]] C \\
&\quad + \frac{1}{2} [Z^v (X^v \lambda)] Y^v + \frac{1}{2} [Y^v (X^v \lambda)] Z^v.
\end{aligned}$$

Consider the tensor

$$\alpha : (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto \alpha(X, Y) := dd_J \lambda(JX, Y).$$

α is obviously symmetric and semibasic, and it makes possible to abbreviate the above expression as follows:

$$(6.6a) \quad \overset{\circ}{\mathbb{P}} = \overset{\circ}{\mathbb{P}} + \frac{1}{2} \alpha \odot J + \frac{1}{2} (D_J \alpha) \otimes C.$$

In the next step we derive the relation between the semibasic traces of $\overset{\circ}{\mathbb{P}}$ and $\overset{\circ}{\mathbb{P}}$. By (2.11a), (2.11b) and (2.11g) we obtain from (6.6a):

$$\begin{aligned} \overset{\circ}{\mathbb{P}} &= \overset{\circ}{\mathbb{P}} + \frac{1}{2} n \alpha + \frac{1}{2} i_{FJ} \alpha + \frac{1}{2} i_{FJS} D_J \alpha \\ &= \overset{\circ}{\mathbb{P}} + \frac{1}{2} n \alpha + \alpha + \frac{1}{2} D_C \alpha. \end{aligned}$$

Now we calculate the covariant derivative $D_C \alpha$. For any vector fields $X, Y \in \mathfrak{X}(M)$:

$$\begin{aligned} (D_C \alpha)(X^c, Y^c) &= (\mathcal{L}_C \alpha)(X^c, Y^c) = C \alpha(X^c, Y^c) = C[X^v(Y^v \lambda)] \\ &= [C, X^v](Y^v \lambda) + X^v[C(Y^v \lambda)] \stackrel{(1.5d)}{=} -X^v(Y^v \lambda) \\ &\quad + X^v([C, Y^v] \lambda + Y^v(C \lambda)) \stackrel{(1.5d)}{=} -X^v(Y^v \lambda) \\ &\quad - X^v(Y^v \lambda) + X^v(Y^v \lambda) = -X^v(Y^v \lambda) = -\alpha(X^c, Y^c). \end{aligned}$$

(using the 1-homogeneity of λ). Thus $D_C \alpha = -\alpha$, therefore

$$(6.6b) \quad \overset{\circ}{\mathbb{P}} = \overset{\circ}{\mathbb{P}} + \frac{1}{2} (n+1) \alpha$$

and, in view of (5.5b),

$$(6.6c) \quad \widetilde{\mathbb{P}} = \widetilde{\mathbb{P}} + \alpha.$$

On the other hand, by (5.6),

$$\begin{aligned} \widetilde{\mathbb{P}} &= \overset{\circ}{\mathbb{P}} - \frac{1}{n+1} \left[\left(D_J \overset{\circ}{\mathbb{P}} \right) \otimes C + \overset{\circ}{\mathbb{P}} \odot J - \overset{\circ}{\mathbb{P}} \otimes J - J \otimes \overset{\circ}{\mathbb{P}} \right] \\ &\stackrel{(6.6a), (6.6b)}{=} \overset{\circ}{\mathbb{P}} + \frac{1}{2} \alpha \odot J + \frac{1}{2} (D_J \alpha) \otimes C - \frac{1}{n+1} \left(D_J \overset{\circ}{\mathbb{P}} \right) \otimes C - \frac{1}{2} (D_J \alpha) \otimes C \\ &\quad - \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \odot J - \frac{1}{2} \alpha \odot J + \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \otimes J + \frac{1}{2} \alpha \otimes J \\ &\quad + \frac{1}{n+1} J \otimes \overset{\circ}{\mathbb{P}} + \frac{1}{2} J \otimes \alpha \stackrel{(5.6)}{=} \overset{\circ}{\mathbb{P}} + \frac{1}{2} (\alpha \otimes J + J \otimes \alpha) \\ &= \overset{\circ}{\mathbb{P}} + \frac{1}{2} \left[\left(\widetilde{\mathbb{P}} - \overset{\circ}{\mathbb{P}} \right) \otimes J + J \otimes \left(\widetilde{\mathbb{P}} - \overset{\circ}{\mathbb{P}} \right) \right], \end{aligned}$$

thus

$$\bar{\mathbb{P}} - \frac{1}{2} (\tilde{\mathbb{P}} \otimes J + J \otimes \tilde{\mathbb{P}}) = \mathbb{P} - \frac{1}{2} (\tilde{\mathbb{P}} \otimes J + J \otimes \tilde{\mathbb{P}}),$$

i.e., $\bar{\mathbb{D}} = \mathbb{D}$, as was to be shown. $\square$

6.7. Now we are going roughly to clarify the meaning of the Douglas tensor. Suppose (M, ∇) is an affinely connected manifold. Then ∇ determines a horizontal endomorphism h (smooth on the whole tangent manifold TM !) which is related to h by

$$(6.7) \quad [X^h, Y^v] = (\nabla_X Y)^v \quad (X, Y \in \mathfrak{X}(M)).$$

Obviously, h is a Berwald endomorphism, and the Berwald connection $(\overset{\circ}{D}, h)$ is just (∇^h, h) , where ∇^h is the horizontal lift ([21]) of ∇ . Then for any vector fields $X, Y, Z \in \mathfrak{X}(M)$ we have

$$\overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c \stackrel{4.9}{=} [[X^h, Y^v], Z^v] = [(\nabla_X Y)^v, Z^v] \stackrel{(1.4)}{=} 0,$$

thus the mixed curvature of $\overset{\circ}{D}$ vanishes, hence *the Douglas tensor of h also vanishes*. As for the converse and the whole story, we have the following important result.

6.8 Theorem (J. DOUGLAS - Z. SHEN). *Suppose M is an orientable manifold and let h be a Berwald endomorphism on M . The associated spray of h is projectively equivalent to the spray determined by a linear connection on M if and only if the Douglas tensor of h vanishes.*

Proof. The “if” part is an immediate consequence of the preceding Remark and Theorem 6.6. Now we prove the “only if” part. Following SHEN’s idea, we consider an arbitrary volume form μ on M and extend it to a volume form $\tilde{\mu}$ on TM as follows:

$$\tilde{\mu} := \mu_1 \wedge \mu_2; \quad \mu_1 := \pi^* \mu, \quad \mu_2 : v \in TM \mapsto (\mu_2)_v \in \wedge^n T_v^v TM,$$

$$\forall z_1, \dots, z_n \in T_v^v TM : (\mu_2)_v(z_1, \dots, z_n) := \mu_{\pi(v)}(j_v(z_1), \dots, j_v(z_n)),$$

where $j_v : T_v^v TM \rightarrow T_{\pi(v)}M$ is the well-known canonical isomorphism. Having this volume form, one can speak of the *divergence* of any vector field $X \in \mathfrak{X}(TM)$ in the usual manner: $\text{div}_{\tilde{\mu}} X$ is defined by the relation

$$\mathcal{L}_X \tilde{\mu} = (\text{div}_{\tilde{\mu}} X) \tilde{\mu}.$$

It is easy to check that, in particular,

$$\text{div}_{\tilde{\mu}} C = n.$$

Now we turn to the associated spray S of h . Observe that $\operatorname{div}_{\tilde{\mu}} S$ is *1-homogeneous*, i.e.

$$\mathcal{L}_C \operatorname{div}_{\tilde{\mu}} S = \operatorname{div}_{\tilde{\mu}} S.$$

Indeed,

$$\begin{aligned} \mathcal{L}_C \mathcal{L}_S \tilde{\mu} &= \mathcal{L}_C [(\operatorname{div}_{\tilde{\mu}} S) \tilde{\mu}] = [\mathcal{L}_C (\operatorname{div}_{\tilde{\mu}} S)] \tilde{\mu} + (\operatorname{div}_{\tilde{\mu}} S) \mathcal{L}_C \tilde{\mu} \\ &= [\mathcal{L}_C (\operatorname{div}_{\tilde{\mu}} S)] \tilde{\mu} + (n \operatorname{div}_{\tilde{\mu}} S) \tilde{\mu}. \end{aligned}$$

The left hand side can also be written as follows:

$$\begin{aligned} \mathcal{L}_C \mathcal{L}_S \tilde{\mu} &= \mathcal{L}_{[C, S]} \tilde{\mu} + \mathcal{L}_S \mathcal{L}_C \tilde{\mu} = \mathcal{L}_S \tilde{\mu} + n \mathcal{L}_S \tilde{\mu} \\ &= (\operatorname{div}_{\tilde{\mu}} S) \tilde{\mu} + (n \operatorname{div}_{\tilde{\mu}} S) \tilde{\mu}. \end{aligned}$$

Comparing the right hand sides, we get the results.

Now let

$$\lambda := -\frac{1}{n+1} \operatorname{div}_{\tilde{\mu}} S, \quad \bar{S} := S + \lambda C = S - \frac{1}{n+1} (\operatorname{div}_{\tilde{\mu}} S) C.$$

Then $\bar{S}$ is divergence-free:

$$\begin{aligned} \operatorname{div}_{\tilde{\mu}} \bar{S} &= \operatorname{div}_{\tilde{\mu}} S - \frac{1}{n+1} (\operatorname{div}_{\tilde{\mu}} S \operatorname{div}_{\tilde{\mu}} C + \mathcal{L}_C \operatorname{div}_{\tilde{\mu}} S) \\ &= \operatorname{div}_{\tilde{\mu}} S - \frac{1}{n+1} (n \operatorname{div}_{\tilde{\mu}} S + \operatorname{div}_{\tilde{\mu}} S) = 0. \end{aligned}$$

After some calculation, for the tensor α constructed in the proof of 6.6 now we obtain the relation

$$\alpha = -\frac{2}{n+1} \overset{\circ}{\mathbb{P}}.$$

Substituting this into 6.6(a), we get:

$$(6.8a) \quad \overset{\circ}{\mathbb{P}} = \overset{\circ}{\mathbb{P}} - \frac{1}{n+1} \left(D_J \overset{\circ}{\mathbb{P}} \otimes C + \overset{\circ}{\mathbb{P}} \odot J \right) \stackrel{(6.4b)}{=} \mathbb{D}.$$

From this follows (c.f. 6.7) that the vanishing of $\mathbb{D}$ implies the desired projective equivalence. $\square$

6.9 Remark. The local version of 6.8 was proved by J. DOUGLAS [23], the global result is due to Z. SHEN [48]. Our proof is a more conceptual and coordinate-free realization of SHEN's ingenious thought. Pay attention to the remarkable relation (6.8a): the mixed curvature of the Berwald endomorphism induced by the projectively deformed (divergence-free) spray is just the Douglas tensor!

THE WEYL TENSOR OF A SPRAY MANIFOLD

THE FUNDAMENTAL PROJECTIVE INVARIANTS ...

6.7. Now we are going roughly to clarify the meaning of the Douglas tensor. Suppose (M, ∇) is an affinely connected manifold. Then ∇ determines a horizontal endomorphism h (smooth on the whole tangent manifold TM !) which is related to h by

$$(6.7) \quad [X^h, Y^v] = (\nabla_X Y)^v \quad (X, Y \in \mathfrak{X}(M)).$$

Obviously, h is a Berwald endomorphism, and the Berwald connection $(\overset{\circ}{D}, h)$ is just (∇^h, h) , where ∇^h is the horizontal lift ([21]) of ∇ . Then for any vector fields $X, Y, Z \in \mathfrak{X}(M)$ we have

$$\overset{\circ}{\mathbb{P}}(X^c, Y^c)Z^c \stackrel{4.9}{=} [[X^h, Y^v], Z^v] = [(\nabla_X Y)^v, Z^v] \stackrel{(1.4)}{=} 0,$$

thus the mixed curvature of $\overset{\circ}{D}$ vanishes, hence *the Douglas tensor of h also vanishes*. As for the converse and the whole story, we have the following important result.

7. The Weyl tensor of a spray manifold

For the sake of completeness, we collect here some basic facts about the Weyl tensor, which is the other fundamental projective invariant of a spray manifold. More or less, our brief exposition follows del Castillo's note [20].

7.1 Theorem and definition. *Let h be a Berwald endomorphism generated by the spray S . If R is the curvature of h , $\tilde{R}$ is the semibasic trace of R , $\tilde{R}^\circ$ is the potential of $\tilde{R}$ and*

$$(7.1a) \quad r := \frac{1}{n-1}\tilde{R}^\circ, \quad \sigma = \frac{1}{n+1}(\tilde{R} + d_J r),$$

then the tensor

$$(7.1b) \quad W := R - \sigma \wedge J + d_J \sigma \otimes C$$

is invariant under the projective changes of S . W is called the Weyl tensor of h , or of the spray manifold (M, S) .

For a proof we refer to del Castillo's paper [20].

7.2 Proposition. *The potential of the Weyl tensor (7.1b) is given by*

$$(7.2a) \quad W^\circ = R^\circ - rJ + \frac{3\tilde{R} + (2-n)d_J r}{n+1} \otimes C,$$

and we have the relation

$$(7.2b) \quad W = \frac{1}{3}[J, W^\circ].$$

The semibasic trace of W , as well as of W° , vanishes:

$$(7.2c) \quad \widetilde{W} = \widetilde{W}^\circ = 0.$$

Proof of $\widetilde{W} = 0$. $W := R - \sigma \wedge J + d_J \sigma \otimes C \stackrel{(7.1a)}{=} R - \frac{1}{n+1}\tilde{R} \wedge J - \frac{1}{n+1}d_J r \wedge J + \frac{1}{n+1}d_J \tilde{R} \otimes C$, so

$$\begin{aligned} \widetilde{W} &\stackrel{(2.12a,b)}{=} \tilde{R} - \frac{n-1}{n+1}\tilde{R} - \frac{n-1}{n+1}d_J r - \frac{1}{n+1}(d_J \tilde{R})^\circ \\ &= \frac{2}{n+1}\tilde{R} - \frac{1}{n+1}d_J \tilde{R}^\circ - \frac{1}{n+1}i_S d_J \tilde{R}. \end{aligned}$$

Since

$$i_S d_J \tilde{R} \stackrel{(1.3g)}{=} -d_J i_S \tilde{R} + \mathcal{L}_J S \tilde{R} + i_{[J,S]} \tilde{R} \stackrel{(2.4)}{=} -d_J \tilde{R}^\circ + \tilde{R} + i_{2h-\text{id}} \tilde{R} = -d_J \tilde{R}^\circ + 2\tilde{R},$$

we get

$$\widetilde{W} = \frac{2}{n+1}\tilde{R} - \frac{1}{n+1}d_J \tilde{R}^\circ + \frac{1}{n+1}d_J \tilde{R}^\circ - \frac{2}{n+1}\tilde{R} = 0. \quad \square$$

7.3 Proposition. *A spray manifold (M, S) is isotropic, if and only if, its Weyl tensor vanishes.*

Proof. Consider a spray manifold (M, S) . Then by (2.20b), the Jacobi endomorphism determined by S is $A = -R^\circ$, where R is the curvature of the horizontal endomorphism $h = \frac{1}{2}(\text{id} + [J, S])$.

(a) *Sufficiency.* If the Weyl tensor of (M, S) vanishes then (7.1b) leads to the relation $R = \sigma \wedge J - d_J \sigma \otimes C$, where

$$\sigma = \frac{1}{n+1}(\tilde{R} + d_J r), \quad r := \frac{1}{n-1}\tilde{R}^\circ.$$

For the potential of R we obtain

$$R^\circ = i_S(\sigma \wedge J) - i_S(d_J \sigma \otimes C) = \sigma^\circ J - \sigma \otimes C - (d_J \sigma)^\circ \otimes C,$$

therefore

$$A = -R^\circ = -\sigma^\circ J + (\sigma + (d_J \sigma)^\circ) \otimes C.$$

This means that S is isotropic.

(b) *Necessity.* Suppose that

$$A = -R^\circ = \lambda J + \alpha \otimes C,$$

where $\lambda \in C^\infty(TM)$ and α is a semibasic 1-form on TM . In 2.21 we have already learnt that λ is 2-homogeneous, α is 1-homogeneous. Applying the relation $R = -\frac{1}{3}[J, A]$, R can also be expressed as follows:

$$R = -[J, \lambda J + \alpha \otimes C] \stackrel{(1.3m,n)}{=} -\frac{1}{3}(d_J \lambda \wedge J + d_J \alpha \otimes C - \alpha \wedge J).$$

From this expression we get

$$\tilde{R} \stackrel{(2.12c,d)}{=} -\frac{1}{3}((n-1)d_J \lambda - (d_J \lambda)^\circ - (n-1)\alpha),$$

and hence

$$\tilde{R}^\circ = -\frac{1}{3}(2(n-1)\lambda - (n-1)\alpha^\circ).$$

On the other hand,

$$\tilde{R}^\circ = \widetilde{R^\circ} = -\widetilde{\lambda J} - \widetilde{\alpha \otimes C} = -n\lambda - \alpha^\circ.$$

Comparing the last two relations, it follows that $\lambda = -\alpha^\circ$. Thus

$$\tilde{R}^\circ = -\frac{1}{3}(-3(n-1)\alpha^\circ), \quad r := \frac{1}{n-1}\tilde{R}^\circ = -\frac{1}{3}(-3\alpha^\circ).$$

Now we are in a position to express the 1-form $\sigma := \frac{1}{n+1}(\tilde{R} + d_J r)$ in terms of α and λ . By the result of our previous calculations,

$$\sigma = -\frac{1}{3} \frac{1}{n+1} \left((n-1)d_J \lambda - (d_J \alpha)^\circ - (n-1)\alpha - 3d_J \alpha^\circ \right).$$

Since

$$\begin{aligned} (d_J \alpha)^\circ &:= i_S d_J \alpha = -d_J i_S \alpha + \mathcal{L}_C \alpha + i_{[J, S]} \alpha \\ &= -d_J \alpha^\circ + \alpha + i_{2h-\text{id}} \alpha = -d_J \alpha^\circ + 2\alpha, \end{aligned}$$

we have

$$\sigma = -\frac{1}{3} \left(\frac{n-1}{n+1} d_J \lambda - \frac{2}{n+1} d_J \alpha^\circ - \alpha \right) = -\frac{1}{3} (d_J \lambda - \alpha);$$

hence

$$d_J \sigma = -\frac{1}{3} (-d_J \alpha), \quad \sigma \wedge J = -\frac{1}{3} (d_J \lambda \wedge J - \alpha \wedge J).$$

Putting all these together,

$$\begin{aligned} W &:= R - \sigma \wedge J + d_J \sigma \otimes C \\ &= -\frac{1}{3} (d_J \lambda \wedge J + d_J \alpha \otimes C - \alpha \wedge J - d_J \lambda \wedge J + \alpha \wedge J - d_J \alpha \otimes C) = 0. \end{aligned}$$

This concludes the proof. $\square$

7.4 Lemma. *Let (M, S) be a spray manifold. If a semibasic tensor $T \in \mathcal{T}_s^1(TM)$ is invariant under the projective changes of the spray, then its vertical covariant differential with respect the Berwald – and therefore the Yano connection – is also invariant.*

Proof. Since the horizontal lifts with respect to the Berwald endomorphisms of two projectively related sprays differ only in a vertical vector field (see (6.2b)), and the semibasic tensors kill the vertical vector fields, the implication is immediate.

7.5 Corollary and definition. *Let W be the Weyl tensor of a spray manifold (M, S) . Then the tensor W^* defined by*

$$(7.5) \quad \overset{\circ}{D}_J W(Z, X, Y) = W^*(X, Y, Z) \quad (X, Y, Z \in \mathfrak{X}(TM)).$$

is invariant under the projective changes of the spray S . W^* is mentioned as the Weyl tensor of the Berwald connection of $(\overset{\circ}{D}, h)$. It can explicitly be given by the formula

$$\begin{aligned} W^*(X, Y, Z) &:= \overset{\circ}{\mathbb{R}}(X, Y)Z - \left(\overset{\circ}{D}_J A \otimes J - \overset{\circ}{D}_J B \otimes C \right) (Z, X, Y) \\ &\quad + \left(\overset{\circ}{D}_J A \otimes J \right) (Z, Y, X) + (B \otimes J)(X, Y, Z), \end{aligned}$$

where

$$A(X) := \frac{1}{n^2 - 1} \left(n i_{S_\circ} \tilde{\mathbb{R}}(X) + \tilde{\mathbb{R}}(X, S_\circ) \right),$$

$$B(X, Y) := \frac{1}{n + 1} \left(\tilde{\mathbb{R}}(X, Y) - \tilde{\mathbb{R}}(Y, X) \right)$$

($S_\circ$ is an arbitrary semispray over M).

The *proof* is a very long and tedious computation, so we have to omit it.

Now we are in a position to derive an essential result of the paper [5], without any calculation.

7.6 Proposition. *Suppose h is a Berwald endomorphism with vanishing Douglas tensor. Then the Weyl tensor of the Berwald connection $(\overset{\circ}{D}, h)$ is a vertical lift, i.e., roughly speaking, its components depend only on the position.*

Proof. In view of 6.8 the associated spray of h is projectively equivalent with the associated spray of a linear connection ∇ . Since the Weyl tensor W^* is a projective invariant by (7.5), it follows that W^* is the Weyl tensor of ∇ . More precisely, W^* is the Weyl tensor of the Berwald connection $(\nabla^{\bar{h}}, \bar{h})$, where $\bar{h}$ is the Berwald endomorphism determined by ∇ (c.f. 6.7). $\square$

To complete this section we present without proof, a description of the Weyl tensor W^* in terms of the Yano connection.

7.7 Proposition. *let (M, S) be a spray manifold with associated Berwald endomorphism h . Suppose $\tilde{\mathbb{R}}$ is the h -Ricci tensor of the Yano connection (D, h) and define a tensor field K by the formula*

$$K(X, Y) := \frac{1}{n^2 - 1} \left(n \tilde{\mathbb{R}}(X, Y) + \tilde{\mathbb{R}}(Y, X) \right).$$

Then the Weyl tensor W^* of (D, h) can be expressed in the form

$$W^*(X, Y, Z) = \mathbb{R}(X, Y)Z - K \otimes J(Z, X, Y) + K \otimes J(Z, Y, X) \\ + K \otimes J(X, Y, Z) - K \otimes J(Y, X, Z),$$

$X, Y, Z \in \mathfrak{X}(TM)$. More compactly,

$$W^* = \mathbb{R} - \text{Sym}K \odot J - \text{Alt}K \wedge J + 3\text{Alt}K \otimes J + \text{Sym}K \otimes J + 2J \otimes \text{Sym}K.$$

II. FINSLER-METRIZABILITIES OF SPRAY MANIFOLDS

8. Hilbert 1-forms and Lagrangians

8.1 Hilbert 1-forms. (1) A 1-form λ on TM is said to be a *Hilbert 1-form* (other terms: *Poincaré* or *Poincaré-Cartan 1-form*), if it is semibasic and d_J -closed, i.e., if $i_J\lambda = 0$ and $d_J\lambda = 0$.

(2) It follows immediately from 2.17 that any 0-homogeneous Hilbert 1-form λ is “ d_J -exact”, namely $\lambda = d_J\lambda^\circ$.

(3) The vertical lift of a 1-form is a trivial example of a Hilbert 1-form.

8.2 Lemma. *If λ is a 0-homogeneous Hilbert 1-form, then the 1-form $(d\lambda)^\circ$ is semibasic.*

Proof. Take a spray S on M and consider the Berwald endomorphism h generated by S . — For any vector field X on M ,

$$\begin{aligned} (d\lambda)^\circ(X^v) &= (i_S d\lambda)(X^v) = d\lambda(S, X^v) = -X^v\lambda(S) - \lambda([S, X^v]) \\ &= -X^v\lambda^\circ + \lambda(h[X^v, S]) \stackrel{(2.9a)}{=} -X^v\lambda^\circ + \lambda(X^h) \\ &\stackrel{8.1(2)}{=} -X^v\lambda^\circ + d_J\lambda^\circ(X^h) = -X^v\lambda^\circ + X^v\lambda^\circ = 0. \quad \square \end{aligned}$$

8.3 Proposition. *Suppose that (M, S) is a spray manifold, and let $(\overset{\circ}{D}, h)$ be the Berwald connection determined by S . If λ is a 0-homogeneous Hilbert 1-form on TM , then*

$$(8.3) \quad d_h\lambda = \overset{\circ}{D}_J i_S d\lambda - \overset{\circ}{D}_S \overset{\circ}{D}_J \lambda.$$

Proof. (1) $d_h\lambda$ is semibasic by 4.7, and an easy calculation shows that the 2-forms $\overset{\circ}{D}_J i_S d\lambda$, $\overset{\circ}{D}_S \overset{\circ}{D}_J \lambda$ are also semibasic. Thus to verify (8.3) it is enough to check that the 2-form $\overset{\circ}{D}_S \overset{\circ}{D}_J \lambda - \overset{\circ}{D}_J i_S d\lambda + d_h\lambda$ kills the pairs of form (X^h, Y^h) , where X, Y are vector fields on M .

(2) Using the representation $\lambda = d_J\lambda^\circ$, it follows that $\lambda(X^h) = X^v\lambda(S)$.

$$\begin{aligned} (3) \quad &\overset{\circ}{D}_S \overset{\circ}{D}_J \lambda(X^h, Y^h) \stackrel{(3.4n)}{=} S[\overset{\circ}{D}_J \lambda(X^h, Y^h)] - \overset{\circ}{D}_J \lambda(FvX^c, Y^h) - \\ &\overset{\circ}{D}_J \lambda(X^h, FvY^c) \stackrel{(3.4e,f)}{=} S(X^v\lambda(Y^h)) - vX^c\lambda(Y^h) + \lambda(\overset{\circ}{D}_{vX^c} Y^h) - X^v\lambda(FvY^c) \\ &+ \lambda(\overset{\circ}{D}_{X^v} FvY^c) \stackrel{(3.4i,j)}{=} S(X^v\lambda(Y^h)) - X^c\lambda(Y^h) + X^h\lambda(Y^h) - X^v\lambda(FvY^c) \\ &+ \lambda([X, Y]^h) - \lambda(F[X^v, Y^h]). \end{aligned}$$

(4)

$$\begin{aligned} \mathring{D}_J i_S d\lambda(X^h, Y^h) &= (\mathring{D}_{X^v} i_S d\lambda)(Y^h) \stackrel{(3.4f)}{=} X^v[d\lambda(S, Y^h)] \\ &\stackrel{(2)}{=} X^v(S\lambda(Y^h)) - [X^v, Y^h]\lambda(S) - Y^h\lambda(X^h) - X^v\lambda(FvY^c). \end{aligned}$$

(5) In view of (3), (4) and (3) from the proof of 4.7 we obtain:

$$\begin{aligned} (\mathring{D}_S \mathring{D}_J \lambda - \mathring{D}_J i_S d\lambda + d_h \lambda)(X^h, Y^h) &= [S, X^v]\lambda(Y^h) - X^c\lambda(Y^h) + 2X^h\lambda(Y^h) + \\ JF[X^v, Y^h]\lambda(S) - \lambda(F[X^v, Y^h]) &\stackrel{(2.9a)}{=} JF[X^v, Y^h]\lambda(S) - \lambda(F[X^v, Y^h]) = \\ d_J i_S \lambda(F[X^v, Y^h]) - \lambda(F[X^v, Y^h]) &\stackrel{8.1(2)}{=} 0. \end{aligned} \quad \square$$

8.4 Definitions. (1) By a *Lagrange function*, briefly *Lagrangian*, we mean a continuous function $L : TM \rightarrow \mathbb{R}$ which is smooth on TM and satisfies the condition $L(0) = 0$.

(2) A Lagrangian $L : TM \rightarrow \mathbb{R}$ is said to be a *Finsler-Lagrangian*, or a *Finsler structure* on M , if L is 1-homogeneous, and the *fundamental 2-form* $\omega := dd_J E := \frac{1}{2} dd_J L^2$ is symplectic. $E := \frac{1}{2} L^2$ is mentioned as the *energy function* (briefly the *energy*) of the Finsler-Lagrangian L . A Finsler-Lagrangian L is called a *positive Finsler-Lagrangian* if $L(v) = 0$ implies that $v = 0$.

(3) A manifold endowed with a Finsler structure is called a *Finsler manifold*. A horizontal endomorphism h on a Finsler manifold with energy E is *conservative* if $d_h E = 0$.

8.5 Remark. If L is a Lagrangian, then $d_J L$ is obviously a Hilbert 1-form.

8.6 Some basic facts. Let (M, L) be a Finsler manifold with the energy $E := \frac{1}{2} L^2$ and the fundamental 2-form $\omega := dd_J E$.

(1) *The Hilbert 1-form $d_J L$ is 0-homogeneous.* — The verification is immediate:

$$\mathcal{L}_C d_J L \stackrel{(1.3h)}{=} d_J \mathcal{L}_C L - d_{[J, C]} L \stackrel{(1.5b)}{=} d_J L - d_J L = 0.$$

(2) $i_C dd_J L = 0$. — Indeed, $i_C dd_J L = i_C dd_J L + d[d_J L(C)] = i_C dd_J L + di_C d_J L = \mathcal{L}_C(d_J L) \stackrel{(1)}{=} 0$. This means that C belongs to the null-space of $dd_J L$, therefore $dd_J L$ is not symplectic (c.f. [10], 1.41).

(3) $i_C dd_J E = d_J E$. — This is an easy consequence of (2).

(4) Nondegeneracy of the fundamental 2-form guarantees that for any 1-form $\alpha \in \Omega^1(TM)$ there exists a unique vector field $\alpha^\#$ (read: α sharp) on TM such that $i_{\alpha^\#} \omega = \alpha$. Thus we obtain an isomorphism from $\Omega^1(TM)$ onto $\mathfrak{X}(TM)$, called the *sharp operator* with respect to ω .

(5) *Any Finsler manifold is a spray manifold* in a natural manner. — Indeed, if $S := -(dE)^\#$ over TM and $S(0) = 0$, then S is a spray on M , called the *canonical spray* of (M, L) .

(6) *On any Finsler manifold there exists a unique conservative Berwald endomorphism* generated by the canonical spray according to (2.8). — This is the *fundamental lemma of Finsler geometry* due to J. GRIFONE [25]. The horizontal endomorphism in question we call the *Barthel endomorphism* of the Finsler manifold.

8.7 Lemma. *If (M, L) is a Finsler manifold and S is its canonical spray, then $i_S dd_J L = 0$.*

Proof. Take a vector field X on M , and let X^h be its horizontal lift by the Barthel endomorphism h . — On the one hand,

$$\begin{aligned} i_S dd_J L(X^v) &= dd_J L(S, X^v) = -X^v d_J L(S) - d_J L([S, X^v]) \\ &= -X^v L - J[S, X^v]L \stackrel{(2.7)}{=} -X^v L + X^v L = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} i_S dd_J L(X^h) &= SX^v L - X^h L - J[S, X^h]L \stackrel{(2.9b)}{=} SX^v L - X^c L \\ &= [S, X^v]L + X^v SL - X^c L \stackrel{(2.9a)}{=} -2X^h L + X^v SL \\ &= -2d_h L(X^h) + X^v [d_h L(S)] = 0, \end{aligned}$$

since h is conservative. □

8.8 Proposition. *Suppose $L : TM \rightarrow \mathbb{R}$ is a 1-homogeneous, positive Lagrangian. L is a positive Finsler-Lagrangian if and only if $\text{rank } dd_J L = 2(n-1)$.*

Proof. Let us define the tensor fields g and $\bar{\mu}$ by

$$(8.8a, b) \quad g(JX, JY) := dd_J E(JX, Y), \quad \bar{\mu}(JX, JY) := dd_J L(JX, Y),$$

for all $X, Y \in \mathfrak{X}(TM)$. First we show that $dd_J E$ is non-degenerate if and only if $\text{rank } g = n$. Take a local basis $(X_i)_{i=1}^n$ of $\mathfrak{X}(M)$ and consider the local basis $(X_i^v, X_i^c)_{i=1}^n$ of $\mathfrak{X}(TM)$. Using the following notations $g_{ij} := g(X_i^v, X_j^v) = dd_J E(X_i^v, X_j^c)$ and $\alpha_{ij} := dd_J E(X_i^c, X_j^c)$ the matrix of $dd_J E$ has the form

$$\begin{pmatrix} 0 & \dots & 0 & & & \\ \vdots & \ddots & \vdots & & (g_{ij}) & \\ 0 & \dots & 0 & & & \\ & & & 0 & & \\ & (-g_{ij}) & & (-\alpha_{kl}) & \ddots & (\alpha_{kl}) \\ & & & & & 0 \end{pmatrix}.$$

From this the implication is immediate. Next we prove that $\text{rank } dd_J L = 2(n-1)$ if and only if $\text{rank } \bar{\mu} = n-1$. Take the Liouville vector field C and a semispray

$S_\circ$ and extend them to a basis of $\mathfrak{X}(TM)$. Note first that $i_C dd_J L \stackrel{8.6(2)}{=} 0$ and for any $X \in \mathfrak{X}(TM)$ we have $i_{S_\circ} dd_J L(JX) = -JX(L) - J[S_\circ, JX]L \stackrel{(2.7)}{=} 0$. Suppose $(C, JX_1, \dots, JX_{n-1}, S_\circ, X_1, \dots, X_{n-1})$ is our considered basis. Let us introduce $\bar{\mu}_{ij} := \bar{\mu}(JX_i, JX_j) = dd_J L(JX_i, X_j)$ and $\beta_{ij} := dd_J L(X_i, X_j)$ ($1 \leq i, j \leq n-1$). The matrix of $dd_J L$ is written in the following form:

$$\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & (\bar{\mu}_{ij}) & \\ 0 & \dots & 0 & 0 & & \\ 0 & \dots & 0 & 0 & & \\ \vdots & (-\bar{\mu}_{ij}) & & (-\beta_{kl}) & \ddots & (\beta_{kl}) \\ 0 & & & & & 0 \end{pmatrix}.$$

From this one can easily deduce our statement. Finally we show that g is of rank n if and only if $\text{rank } \bar{\mu} = n-1$. Since

$$dd_J E = dd_J \left(\frac{1}{2} L^2 \right) = d(Ld_J L) = dL \wedge d_J L + Ldd_J L,$$

so for any vector field X, Y on (TM)

$$dd_J E(JX, Y) = dL(JX) \otimes dL(JY) + Ldd_J L(JX, Y).$$

By the definition of g and $\bar{\mu}$ we obtain that

$$(*) \quad g(JX, JY) = dL(JX) \otimes dL(JY) + L\bar{\mu}(JX, JY).$$

dL has a $n-1$ -dimensional kernel on the vertical bundle which does not contain the Liouville vector field C . Now choose $n-1$ vector fields from the kernel and extend with C to a basis of the vertical bundle. In this case the equation $(*)$ has the form

$$\begin{pmatrix} (g_{ij}) \end{pmatrix} = \begin{pmatrix} L^2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & (\bar{\mu}_{ij}) & & \\ 0 & & & \end{pmatrix},$$

which proves 8.8. □

8.9 Proposition. *Let $\bar{L}$ be a 1-homogeneous Lagrangian on the manifold M and $\bar{\mu}$ as in 8.8. Suppose that for any $v \in TM$ and $X \neq \lambda C \in \mathfrak{X}(TM)$ ($\lambda \in C^\infty(TM)$) $\bar{\mu}(X, X)(v) > 0$. Then there exists – locally – a positive Finsler–Lagrangian L such that the tensor μ arising from L according to (8.8b) coincides with $\bar{\mu}$.*

Proof. (C.f. [13] page 243.)

Suppose $\bar{L}$ is a 1-homogeneous Lagrangian on M . Fix a $v \in TM$ and choose a $w \in TM$ such that $\pi(v) = \pi(w)$ and $w \neq \xi v$ ($\xi \geq 0$). Taking a chart $(U, (u)_{i=1}^n)$ on M and introducing the induced chart $(\pi^{-1}(U), x^i, y^i)_{i=1}^n$ ($x^i := u^i \circ \pi$, $y^i := du^i$) consider the Taylor expansion of $\bar{L}$

$$\bar{L}(w) = \bar{L}(v) + (w^i - v^i) \frac{\partial \bar{L}}{\partial y^i}(v) + \frac{1}{2} (w^i - v^i)(w^j - v^j) \frac{\partial^2 \bar{L}}{\partial y^i \partial y^j}(z),$$

where $z = v + \theta(w - v)$ ($0 < \theta < 1$), $w = w^i \left(\frac{\partial}{\partial u^i} \right)_{\pi(w)}$, $v = v^i \left(\frac{\partial}{\partial u^i} \right)_{\pi(v)}$. This can be written as follows

$$\bar{L}(w) = \bar{L}(v) - C(v)\bar{L} + w_v^\uparrow \bar{L} + \frac{1}{2} (w - v)_z^\uparrow \left((w - v)_z^\uparrow \bar{L} \right),$$

where $w_v^\uparrow$ is the vertical lift of w to v , that is $w_v^\uparrow = w^i \left(\frac{\partial}{\partial y^i} \right)_v$. By the definition $\bar{\mu}$ this yields

$$\bar{L}(w) = \bar{L}(v) - C(v)\bar{L} + w_v^\uparrow \bar{L} + \frac{1}{2} \bar{\mu}_z \left((w - v)_z^\uparrow, (w - v)_z^\uparrow \right).$$

Since $\bar{L}$ is 1-homogeneous we obtain that

$$\bar{L}(w) - w_v^\uparrow \bar{L} = \frac{1}{2} \bar{\mu}_z \left((w - v)_z^\uparrow, (w - v)_z^\uparrow \right).$$

By our assumption v and w have not the same direction, therefore $(w - v)_z^\uparrow \neq \lambda(v)C(v)$ ($\lambda \in C^\infty(TM)$), but then $\bar{\mu}_z$ is strictly positive, whence

$$(*) \quad \bar{L}(w) - w_v^\uparrow \bar{L} = \bar{L}(w) - w^i \frac{\partial \bar{L}}{\partial y^i}(v) > 0.$$

Now, take the half of the Euclidean unit ball of $T_{\pi(v)}M$ which is in opposite direction to v , that is the set B defined by

$$(y^1)^2 + (y^2)^2 + \dots + (y^n)^2 = 1, \quad y^i v^i \leq 0.$$

This is a compact set. Let k denote the minimum of the function $\bar{L}(w) - w_v^\uparrow \bar{L}$ on B and define c_i by

$$c_i = \frac{k}{2} \frac{v^i}{\sqrt{(v^1)^2 + (v^2)^2 + \dots + (v^n)^2}} - \frac{\partial \bar{L}}{\partial y^i}(v).$$

Set $L(w) := \bar{L}(w) + c_i y^i(w) = \bar{L}(w) - w^i \frac{\partial \bar{L}}{\partial y^i}(v) + c_i y^i(w)$, ($w \in T_{\pi(v)}M \setminus \{0\}$). We show that $L(w) > 0$ on the Euclidean unit ball of $T_{\pi(v)}M$. If $v^i w^i > 0$, then

$L(w) > 0$ by (*). Suppose that $v^i w^i \leq 0$. In this case $\bar{L}(w) - \frac{\partial \bar{L}}{\partial y^i}(v)$ is decreased by the absolute value of

$$\begin{aligned} & \frac{k}{2} \frac{v^i w^i}{\sqrt{(v^1)^2 + \dots + (v^n)^2}} = \frac{k}{2} \frac{v^i w^i}{\sqrt{(v^1)^2 + \dots + (v^n)^2} \sqrt{(w^1)^2 + \dots + (w^n)^2}} \\ & = \frac{\langle v, w \rangle}{\|v\| \|w\|} \geq \frac{k}{2}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $\| \cdot \|$ is the corresponding Euclidean norm. Therefore we obtain that $L(w) \geq \frac{k}{2} > 0$. By the homogeneity of L this yields $L(w) > 0$, $w \in T_{\pi(v)}M \setminus \{0\}$. Since $c_i y^i = (c_i u^i)^c$, so L and $\bar{L}$ differs in only a complete lift of a function. We obtain by a routine calculation that $\forall X, Y \in \mathfrak{X}(TM) : dd_J f^c(JX, Y) = 0$, whence $\bar{\mu} = \mu$. $\square$

8.10 Proposition (the fundamental relation). *Let (M, S) be a spray manifold and $\bar{L}$ a Finsler-structure on M . Then, over TM , the canonical spray $\bar{S}$ arising from $\bar{L}$ can be represented in the form*

$$(8.10) \quad \bar{S} = S - \frac{S\bar{L}}{\bar{L}}C - \bar{L}(i_S dd_J \bar{L})^\sharp,$$

where the sharp operator is taken with respect to the fundamental 2-form of $(M, \bar{L})$.

Proof. Consider the energy $\bar{E} := \frac{1}{2}\bar{L}^2$ of $(M, \bar{L})$. Then

$$\begin{aligned} i_S dd_J \bar{E} &= i_S d(\bar{L} d_J \bar{L}) = i_S (d\bar{L} \wedge d_J \bar{L} + \bar{L} dd_J \bar{L}) = (S\bar{L})d_J \bar{L} - d\bar{L}(C\bar{L}) \\ &+ \bar{L} i_S dd_J \bar{L} = \frac{S\bar{L}}{\bar{L}} d_J \bar{E} - d\bar{E} + \bar{L} i_S dd_J \bar{L} \stackrel{8.6(3),(4),(5)}{=} \frac{S\bar{L}}{\bar{L}} i_C dd_J \bar{E} \\ &+ i_{\bar{S}} dd_J \bar{E} + \bar{L} i_{(i_S dd_J \bar{L})^\sharp} dd_J \bar{E} = i_{\bar{S} + \frac{S\bar{L}}{\bar{L}}C + \bar{L}(i_S dd_J \bar{L})^\sharp} dd_J \bar{E}, \end{aligned}$$

whence (8.10). $\square$

9. Projectively Finsler spray manifolds

9.1 Definitions. (1) Two *Finsler manifolds* (with common base manifold) are said to be *projectively equivalent* if they are projectively equivalent as spray manifolds.

(2) A spray manifold (M, S) is projectively equivalent to a Finsler manifold $(M, \bar{L})$ if S is projectively equivalent to the canonical spray of $(M, \bar{L})$. In this case (M, S) is called *Finsler-metrizable in a broad sense* or *projectively Finsler*.

(3) A spray manifold (M, S) is said to be *Finsler-metrizable in a natural sense* if there exists a Finsler-Lagrangian on M , whose canonical spray is the given spray S . Then we also say that S is *Finsler-variational*.

9.2 Theorem. *Let (M, S) be a spray manifold endowed with the Berwald connection $(\overset{\circ}{D}, h)$ induced by S . If $\overline{L}$ is a Finsler-Lagrangian on M , then the following conditions are equivalent:*

- (1) (M, S) is projectively equivalent to $(M, \overline{L})$.
- (2) $i_S dd_J \overline{L} = 0$.
- (3) $d_h d_J \overline{L} = 0$.
- (4) $\overset{\circ}{D}_h d_J \overline{L}(X, Y) = \overset{\circ}{D}_h d_J \overline{L}(Y, X) \quad (X, Y \in \mathfrak{X}(\mathcal{T}M))$.

Proof. Let $\overline{S}$ be the canonical spray of $(M, \overline{L})$.

(1) $\implies$ (2). In view of the assumption $\overline{S}$ can be written in the form $\overline{S} = S + fC$ ($f \in C^\infty(\mathcal{T}M)$). Thus $i_S dd_J \overline{L} = i_{\overline{S}} dd_J \overline{L} - f i_C dd_J \overline{L} \stackrel{8.7, 8.6(2)}{=} 0$.

(2) $\implies$ (1). This is clear by (8.10).

(2) $\implies$ (3). We first notice that $d_h \circ \mathcal{L}_C - \mathcal{L}_C \circ d_h = [d_h, \mathcal{L}_C] \stackrel{(1.3h)}{=} d_{[h, C]} = 0$. Thus $\mathcal{L}_C d_h d_J \overline{L} = d_h \mathcal{L}_C d_J \overline{L} \stackrel{8.6}{=} 0$, which means that $d_h d_J \overline{L}$ is a 0-homogeneous 2-form. Hence, applying (2.17a), it can be represented in the form

$$(*) \quad d_h d_J \overline{L} = \frac{1}{2} [(d_J d_h d_J \overline{L})^\circ + d_J (d_h d_J \overline{L})^\circ].$$

By the vanishing of the weak torsion of h

$$d_J \circ d_h + d_h \circ d_J = [d_J, d_h] =: d_{[J, h]} = 0,$$

therefore

$$d_J d_h d_J \overline{L} = -d_h d_J^2 \overline{L} \stackrel{(1.5b)}{=} 0.$$

In view of 8.6(1) and 8.2, $i_S dd_J \overline{L}$ is a semibasic 1-form. Hence

$$(d_h d_J \overline{L})^\circ = i_S d_h d_J \overline{L} \stackrel{(4.7a)}{=} i_S h^* dd_J \overline{L} = i_S dd_J \overline{L} \stackrel{(2)}{=} 0.$$

We find that both terms on the right hand side of $(*)$ vanish. Thus $d_h d_J \overline{L} = 0$.

$$(3) \implies (2). \quad i_S dd_J \overline{L} \stackrel{8.2, 8.6(1)}{=} i_S h^* dd_J \overline{L} \stackrel{(4.7a)}{=} i_S d_h d_J \overline{L} \stackrel{(3)}{=} 0.$$

$$(3) \iff (4). \quad \text{Immediate from (4.7b).} \quad \square$$

9.3 Remark. We call the equivalent relations (2)–(4) in 9.2 the *fundamental* or the *Rapcsák equations of projective equivalence*.

9.4 Corollary. *Let (M, S) be a spray manifold and h the Berwald endomorphism generated by S . (M, S) is projectively Finsler if and only if there exists a 0-homogeneous, d_h -closed Hilbert 1-form λ on TM such that the 2-form $dd_J(\lambda^\circ)^2$ is symplectic.*

Proof. The *necessity* of the condition is obvious: if a Finsler manifold $(M, \bar{L})$ is projectively equivalent to (M, S) , then by the Rapcsák equation 9.2(3) the Hilbert 1-form $d_J \bar{L}$ has all the required properties. — To verify the *sufficiency*, let $\bar{L} := \lambda^\circ$. Then $\bar{L}$ is a Finsler-Lagrangian, because it is 1-homogeneous by 2.16 and $dd_J \bar{L}^2$ is symplectic by assumption. It remains to check that $\bar{L}$ satisfies 9.2(3). Using our previous results, this is quite immediate:

$$i_S dd_J \bar{L} = i_S dd_J \lambda^\circ \stackrel{(2.17b)}{=} i_S d\lambda \stackrel{8.2}{=} h^* i_S d\lambda = i_S h^* d\lambda \stackrel{(4.7a)}{=} i_S d_h \lambda = 0,$$

since λ is d_h -closed. $\square$

9.5 Corollary. *Let (M, S) be a spray manifold with the Berwald connection $(\overset{\circ}{D}, h)$ induced by S . S is Finsler-variational if and only if there exists a 0-homogeneous Hilbert 1-form λ on TM satisfying the conditions*

$$(9.5a-b) \quad \overset{\circ}{D}_h \lambda = 0, \quad dd_J(\lambda^\circ)^2 \text{ is symplectic.}$$

Proof. (1) Assume that $(M, \bar{L})$ is a Finsler manifold with canonical spray S . Then the Barthel endomorphism $\bar{h}$ arising from $\bar{L}$ is just the Berwald endomorphism h . Since $\lambda := d_J \bar{L}$ is a 0-homogeneous Hilbert 1-form satisfying (9.5b), our only task is to check (9.5a). — For any vector fields X, Y on M ,

$$\begin{aligned} (\overset{\circ}{D}_h \lambda)(X^h, Y^h) &= (\overset{\circ}{D}_h d_J \bar{L})(X^h, Y^h) = (\overset{\circ}{D}_{X^h} d_J \bar{L})(Y^h) = X^h(Y^v \bar{L}) \\ &\quad - d_J \bar{L}(F[X^h, Y^v]) = X^h(Y^v \bar{L}) - [X^h, Y^v] \bar{L} = Y^v(X^h \bar{L}) \\ &= Y^v(d_h \bar{L}(X^h)) = Y^v(d_h \bar{L}(X^h)) = 0 \end{aligned}$$

since $\bar{h}$ is conservative. Thus (9.5a) holds.

(2) Conversely, let λ be a 0-homogeneous Hilbert 1-form on TM , satisfying (9.5a-b). Then, as we have already learnt, $\bar{L} := \lambda^\circ$ is a Finsler-Lagrangian. We show that the canonical spray of the Finsler manifold $(M, \bar{L})$ is just the given spray S , i.e. $i_S dd_J \bar{L}^2 = -d\bar{L}^2$, or equivalently,

$$(*) \quad (i_S d\bar{L})\lambda + \bar{L} i_S d\lambda = 0.$$

Here $i_S d\bar{L} = i_S d(i_S \lambda) = d(i_S \lambda)(S) = S(\lambda S) \stackrel{(3.4p)}{=} S(\lambda S) - \lambda(\overset{\circ}{D}_S S) = (\overset{\circ}{D}_S \lambda)(S) = \overset{\circ}{D}_h \lambda(S, S) \stackrel{(9.5a)}{=} 0$. The 1-form $i_S d\lambda$ in $(*)$ is semibasic by 8.3, and $i_S d\lambda = i_S d_h \lambda$ from the proof of 9.4. Since for any vector fields X, Y on M ,

$$d_h \lambda(X^h, Y^h) \stackrel{(4.7b)}{=} \overset{\circ}{D}_h \lambda(X^h, Y^h) - \overset{\circ}{D}_h \lambda(Y^h, X^h) \stackrel{(9.5a)}{=} 0,$$

the term $\bar{L} i_S d\lambda$ also vanishes in $(*)$. This concludes the proof. $\square$

9.6 Lemma. *Let α be a 1-form on M . If $\tilde{\alpha} := (\alpha^v)^\circ$, then $d_J \tilde{\alpha} = \alpha^v$.*

Proof. Let $(\overset{\circ}{D}, h)$ be a Berwald connection on M induced by the Spray S . For any vector fields X, Y on M ,

$$\begin{aligned} (\overset{\circ}{D}_J \alpha^v)(X^h, Y^h) &= (\overset{\circ}{D}_{X^v} \alpha^v)(Y^h) = X^v \alpha^v(Y^h) - \alpha^v(D_{X^v} Y^h) \\ &\stackrel{(3.4f)}{=} X^v \alpha^v(Y^h) = X^v \alpha^v(Y^c) = X^v[\alpha(Y)]^v = 0. \end{aligned}$$

Thus we get

$$\begin{aligned} 0 &= (\overset{\circ}{D}_J \alpha^v)(X^h, S) = (\overset{\circ}{D}_{X^v} \alpha^v)(S) = X^v \alpha^v(S) - \alpha^v(D_{X^v} S) \\ &\stackrel{(3.4l)}{=} X^v \tilde{\alpha} - \alpha^v(X^h) = X^v \tilde{\alpha} - \alpha^v(X^c) = d_J \tilde{\alpha}(X^c) - \alpha^v(X^c), \end{aligned}$$

as claimed. $\square$

9.7 An application. Let (M, L) be a Finsler manifold, α a 1-form on M , and let us consider the function $\tilde{\alpha}$ defined in 9.6. If the function $\overline{L} := L + \tilde{\alpha}$ is also a Finsler-Lagrangian, then we speak of a *Randers-change* of the Finsler structure L . In view of 9.2(2), $(M, \overline{L})$ is projectively equivalent to (M, L) if and only if

$$i_S(d\alpha)^v = i_S d\alpha^v \stackrel{9.6}{=} i_S dd_J \tilde{\alpha} \stackrel{8.7}{=} i_S dd_J(L + \tilde{\alpha}) = i_S dd_J \overline{L} = 0.$$

An immediate calculation shows that $(d\alpha)^v$ is a 0-homogeneous, d_J -closed semibasic 2-form on TM . Hence, by (2.17b), $(d\alpha)^v = \frac{1}{2}d_J[i_S(d\alpha)^v]$. — From these observations we infer the following result: *a Randers-change $\overline{L} := L + \tilde{\alpha}$ yields a Finsler manifold projectively equivalent to (M, L) if and only if α is a closed 1-form on M .* By means of the classical tensor calculus, this was originally proved by M. HASHIGUCHI and Y. ICHIJYŌ [30].

10. Helmholtz-type condition

10.1 The theorem of M. Crampin. *Let S be a semispray over the manifold M , and let h be the horizontal endomorphism generated by S according to (2.8). If a 2-form ω on TM satisfies the conditions*

$$(10.1a) \quad \mathcal{L}_S \omega = 0,$$

$$(10.1b) \quad \omega(JX, JY) = 0 \quad (X, Y \in \mathfrak{X}(TM)),$$

$$(10.1c) \quad d\omega(hX, JY, JZ) = 0 \quad (X, Y, Z \in \mathfrak{X}(TM)),$$

then there is a smooth function K defined on an open subset of TM such that $\omega = dd_J K$.

For a proof we refer to Crampin's paper [16] or the book of de León and Rodríguez [21].

10.2 Remark. Crampin's theorem takes its origin in the inverse problem of the calculus of variations (or of the Lagrangian mechanics). In modern terms this problem can be formulated as follows. *Given a semispray S over a manifold M , find the conditions for the existence of a regular Lagrangian $L : TM \rightarrow \mathring{\mathbb{R}}$ such that*

$$(10.2a) \quad i_S dd_J L = d(CL - L).$$

The semispray S is called a *Lagrangian semispray* if there exists a regular Lagrangian L which satisfies (10.2a).

Now choose a chart $(U, (u^i)_{i=1}^n)$ for M , and let us consider the induced chart $(\pi^{-1}(U), (x^i)_{i=1}^n, (y^i)_{i=1}^n)$ on TM . Let $S \upharpoonright \pi^{-1}(U) = y^i \frac{\partial}{\partial x^i} + f^i \frac{\partial}{\partial y^i}$. It has been known for a long time that S is Lagrangian if there exists a family $(g_{ij})_{1 \leq i, j \leq n}$ of smooth function over $\pi^{-1}(U) \cap TM$ such that

$$(10.2b) \quad \det(g_{ij}) \neq 0,$$

$$(10.2c) \quad g_{ij} = g_{ji} \quad (1 \leq i, j \leq n),$$

$$(10.2d) \quad S(g_{ij}) = \frac{1}{2} \left(g_{kj} \frac{\partial f^k}{\partial y^i} + g_{ik} \frac{\partial f^k}{\partial y^j} \right),$$

$$(10.2e) \quad \begin{aligned} & g_{ik} \left(\frac{\partial f^k}{\partial x^j} + \frac{1}{4} \frac{\partial f^k}{\partial y^l} \frac{\partial f^l}{\partial y^j} - \frac{1}{2} S \left(\frac{\partial f^k}{\partial y^j} \right) \right) \\ &= g_{jk} \left(\frac{\partial f^k}{\partial x^i} + \frac{1}{4} \frac{\partial f^k}{\partial y^l} \frac{\partial f^l}{\partial y^i} - \frac{1}{2} S \left(\frac{\partial f^k}{\partial y^i} \right) \right). \end{aligned}$$

Relations (10.2b-e) are the so-called *Helmholtz conditions*. Crampin's theorem provides an elegant coordinate-free, 'geometric' formulation of these famous conditions. With regard to this fact, we shall also refer to (10.1a-c) as Helmholtz conditions. Another equivalent geometric reformulation of (10.2b-e) is due to J. F. Cariñena and E. Martínez [14].

10.3 Proposition. *Let (M, S) be a spray manifold and suppose that a 2-form ω satisfies the conditions (10.1a-c). If, in addition, ω is 1-homogeneous and has maximal rank then there exists - locally - a Finsler energy E such that S is the canonical spray of the Finsler manifold (M, E) .*

Proof. According to 10.1 there exists a function K such that $\omega = dd_J K$. First we show that the condition $\mathcal{L}_C \omega = \omega$ guarantees the existence of a 2-homogeneous function E such that $\omega = dd_J E$. Since

$$\begin{aligned} dd_J K &= \mathcal{L}_C dd_J K = i_C ddd_J K + di_C dd_J K = di_C dd_J K \\ &= d\mathcal{L}_C d_J K - ddi_C d_J K = d\mathcal{L}_C d_J K \stackrel{(1.5b)}{=} dd_J \mathcal{L}_C K - dd_J K, \end{aligned}$$

it follows that $d(d_J 2K - d_J \mathcal{L}_C K) = 0$. Thus by the Poincaré lemma, there exists a function F such that

$$d_J(2K - \mathcal{L}_C K) = dF.$$

The 1-form on the left-hand side of this relation vanishes on the vertical vector fields, so the function F has to be a vertical lift. Suppose that $F = f^v$. Then

$$d_J(2K - \mathcal{L}_C K) = df^v$$

and, consequently,

$$CK - 2K = f^c + h^v,$$

where h is another smooth function on M . Now let E

$$E := K + \frac{1}{2}h^v + f^c.$$

Then E is 2-homogeneous since

$$CE = C(K + \frac{1}{2}h^v + f^c) = CK + f^c = 2K + h^v + f^c + f^c = 2E.$$

Obviously, we also have $dd_J K = dd_J E$. Since the 2-form $dd_J E$ is of maximal rank, E is a Finsler energy on the manifold M . Consider the canonical spray S_E of (M, E) . Now, on the one hand

$$(*) \quad i_{S_E} dd_J E = -dE,$$

on the other hand

$$0 \stackrel{(10.1a)}{=} \mathcal{L}_S dd_J E = i_S ddd_J E + di_S dd_J E = di_S dd_J E,$$

so there exists a function G defined on an open subset of TM such that

$$(**) \quad i_S dd_J E = -dG.$$

Taking the difference of $(*)$ and $(**)$ we obtain

$$i_{S_E - S} dd_J E = d(G - E).$$

$S_E - S$ is a vertical vector field therefore the 1-form on the left-hand side vanishes on the vertical vector fields by (10.1b). This implies as before, that $G - E$ is a vertical lift. If $G - E = k^v$, then $i_{S_E - S} dd_J E = dk^v := i_{(dk^v)^\#} dd_J E$, therefore

$$S_E = S + (dk^v)^\#.$$

(The sharp operator $\#$ is taken with respect to ω .) The sprays S and S_E are 2-homogeneous. To conclude the proof, we show that $(dh^v)^\#$ is 0-homogeneous and hence it has to vanish.

$$\begin{aligned} i_{[C, (dk^v)^\#]} \omega &= \mathcal{L}_C i_{(dk^v)^\#} \omega - i_{(dk^v)^\#} \mathcal{L}_C \omega = \mathcal{L}_C dk^v - i_{(dk^v)^\#} \omega \\ &= d\mathcal{L}_C k^v - i_{(dk^v)^\#} \omega = -i_{(dk^v)^\#} \omega. \end{aligned}$$

□

10.4 Proposition. *Let S be a spray over the manifold M and ω be a 2-form satisfying the conditions (10.1a-c). Define the tensor μ by the relation $\mu(JX, JY) := \omega(JX, Y)$ ($X, Y \in \mathfrak{X}(TM)$). If, in addition, $i_C\omega = 0$ and for any $v \in TM$ and $X \neq \lambda C \in \mathfrak{X}(TM)$ ($\lambda \in C^\infty(TM)$) $\bar{\mu}(X, X)(v) > 0$, then there exists – locally – a positive Finsler-Lagrangian L such that its canonical spray is projectively equivalent to the given one.*

Proof. First we show that there exists a 1-homogeneous Lagrangian $\bar{L}$ such that $\omega = dd_J\bar{L}$. By Crampin's theorem $\omega = dd_JK$, where K is a function on a region of TM . Since

$$0 = i_C dd_JK = \mathcal{L}_C d_JK + di_C d_JK = \mathcal{L}_C d_JK \stackrel{(1.5b)}{=} d_J \mathcal{L}_C K - d_J K,$$

therefore

$$d_J(\mathcal{L}_C K - K) = 0.$$

Obviously, the function $\mathcal{L}_C K - K$ vanishes on the vertical vector fields, so it has to be a vertical lift of a function on M . Suppose $\mathcal{L}_C K - K = f^v$. Define $\bar{L}$ by

$$\bar{L} := \mathcal{L}_C K.$$

Since $\mathcal{L}_C \bar{L} := \mathcal{L}_C \mathcal{L}_C K = \mathcal{L}_C K = \bar{L}$, $\bar{L}$ is 1-homogeneous and, obviously $\omega = dd_J\bar{L}$. According to 8.9 there exists a positive Finsler-Lagrangian L such that $dd_J L = \omega$. We show that $i_S dd_J L = 0$. Since

$$0 \stackrel{(10.1a)}{=} \mathcal{L}_S dd_J L = i_S ddd_J L + di_S dd_J L = di_S dd_J L,$$

therefore $i_S dd_J L = dF$. The 1-form $i_S dd_J L$ vanishes on the vertical vector fields, this, as again, implies that F is a vertical lift of a function on M . Suppose that $F = h^v$. We claim that $i_S dd_J L$ is a 1-homogeneous semibasic 1-form. Indeed,

$$\begin{aligned} \mathcal{L}_C i_S dd_J L &= \mathcal{L}_C (\mathcal{L}_S d_J L - di_S d_J L) = \mathcal{L}_{[C, S]} d_J L + \mathcal{L}_S \mathcal{L}_C d_J L - \mathcal{L}_C di_S d_J L \\ &= \mathcal{L}_S d_J L - \mathcal{L}_C di_S d_J L = i_S dd_J L + \mathcal{L}_C di_S d_J L - \mathcal{L}_C di_S d_J L = i_S dd_J L. \end{aligned}$$

Since $\mathcal{L}_C dh^v(X^c) = C(dh^v(X^c)) = C(X^c(h^v)) = C(Xh)^v = 0$ it follows that dh^v is 0-homogeneous and therefore it has to vanish. According to 9.2(2) this means that S is projective equivalent to the canonical spray of (M, L) . $\square$

10.5 Lemma. *Suppose that g is a non-degenerate, symmetric $(0, 2)$ -tensor on the vertical subbundle and let $\mathcal{C}_\flat := \overset{\circ}{D}_J J^* g$, where $\overset{\circ}{D}$ is the Berwald connection determined an arbitrarily chosen horizontal endomorphism h . If*

- (i) $\mathcal{C}_\flat$ is totally symmetric,
- (ii) $\mathcal{C}_\flat$ has no potential,

then the $(0, 2)$ tensor g_E defined by $g_E(JX, JY) := dd_J E(JX, Y)$, where $E := \frac{1}{2}g(C, C)$; $X, Y \in \mathfrak{X}(TM)$, equals to the given tensor g .

Proof. For any vector field X on M and semispray S over M we get

$$\begin{aligned} 0 &\stackrel{(ii)}{=} \mathcal{C}_\flat(X^h, S, S) = \overset{\circ}{D}_J J^* g(X^h, S, S) \stackrel{(i)}{=} X^v g(C, C) - 2g(J\overset{\circ}{D}_{X^v} S, C) \\ &\stackrel{(3.4l)}{=} X^v g(C, C) - 2g(X^v, C), \end{aligned}$$

whence

$$(*) \quad X^v g(C, C) = 2g(X^v, C).$$

Similarly, for any vector fields X, Y on M we have

$$\begin{aligned} 0 &= \mathcal{C}_\flat(X^h, Y^h, S) = \overset{\circ}{D}_J J^* g(X^h, Y^h, S) \stackrel{(3.4e)}{=} X^v g(Y^v, C) - g(Y^v, J\overset{\circ}{D}_{X^v} S) \\ &\stackrel{(3.4l)}{=} X^v g(Y^v, C) - g(X^v, Y^v), \end{aligned}$$

therefore

$$(**) \quad X^v g(Y^v, C) = g(X^v, Y^v).$$

Now, by relations $(*)$ and $(**)$ it follows that

$$X^v(Y^v g(C, C)) \stackrel{(*)}{=} X^v(2g(Y^v, C)) \stackrel{(**)}{=} 2g(X^v, Y^v).$$

On the other hand,

$$\begin{aligned} 2g_E(X^v, Y^v) &= 2g_E(JX^h, JY^h) = 2dd_J E(X^v, Y^h) = 2X^v d_J E(Y^h) - 2Y^h d_J E(X^v) \\ &\quad - 2d_J E([X^v, Y^h]) = 2X^v Y^v E = X^v Y^v g(C, C), \end{aligned}$$

which concludes the proof. $\square$

10.6 Proposition. Let (M, S) be a spray manifold and let $(\overset{\circ}{D}, h)$ denote the Berwald connection determined by the Berwald endomorphism h arising from S . Suppose that g is a non-degenerate, symmetric $(0, 2)$ -tensor in the vertical bundle, and let $\mathcal{C}_\flat := \overset{\circ}{D}_J J^* g$. If

$$(10.6a) \quad \mathcal{C}_\flat \text{ is totally symmetric,}$$

$$(10.6b) \quad i_S \mathcal{C}_\flat = 0,$$

$$(10.6c) \quad \overset{\circ}{D}_S J^* g = 0,$$

then S is the canonical spray of the Finsler energy $E := \frac{1}{2}g(C, C)$.

Proof. First we show that the function $E := \frac{1}{2}g(C, C)$ is 2-homogeneous. According to (10.6b)

$$\begin{aligned} 0 &= \mathcal{C}_b(S, S, S) := \overset{\circ}{D}_J J^* g(S, S, S) := (\overset{\circ}{D}_C J^* g)(S, S) \\ &= Cg(C, C) - 2g(J\overset{\circ}{D}_C C, C) \stackrel{(3.4k)}{=} Cg(C, C) - 2g(C, C) = 2CE - 4E, \end{aligned}$$

therefore $CE = 2E$, as we claimed. Since g is non-degenerate, therefore the 2-form $dd_J E$ is symplectic. All these mean that E is a Finsler energy on M . It remains only to check that E is conservative. Applying our conditions,

$$0 \stackrel{(10.6c)}{=} \overset{\circ}{D}_S J^* g(S, S) = Sg(C, C) - 2g(J\overset{\circ}{D}_S S, C) \stackrel{(3.4p)}{=} Sg(C, C) = 2SE,$$

therefor $SE = 0$. Choose a vector field X on M . Using this observation and (10.6c) again we get

$$\begin{aligned} 0 &\stackrel{(10.6c)}{=} \overset{\circ}{D}_S J^* g(X^h, S) = Sg(X^v, C) - g(J\overset{\circ}{D}_S X^h, C) - g(X^v, J\overset{\circ}{D}_S S) \\ &\stackrel{(3.4n,p)}{=} Sg(X^v, C) - g(vX^c, C) \stackrel{10.5}{=} SX^v E - vX^c E \\ &= [S, X^v]E + X^v SE - vX^c E \stackrel{(2.9a)}{=} X^c E - 2X^h E - vX^c E \\ &= -X^h E = -d_h E(X^c), \end{aligned}$$

whence h is indeed a conservative Berwald endomorphism on M . In view of 8.6 it has to coincide with the Barthel endomorphism of (M, E) or, equivalently, S has to be the canonical spray from which h is arising. $\square$

10.8 Lemma. Suppose that S is a spray over the manifold M and $(\overset{\circ}{D}, h)$ is the Berwald connection generated by S . Let R and A be the curvature and the Jacobi endomorphism of h , respectively. If $\bar{\mu}$ is a symmetric $(0, 2)$ -tensor on the vertical bundle satisfying the relations

$$(10.8a,b) \quad i_C \bar{\mu} = 0, \quad \overset{\circ}{D}_J J^* \bar{\mu} \text{ is totally symmetric},$$

then the following assertions are equivalent:

- (1) $\sum_{X, Y, Z \in \mathfrak{X}(TM)} \bar{\mu}(JX, R(Y, Z)) = 0,$
- (2) $\bar{\mu}(JX, A(Y)) = \bar{\mu}(A(X), JY),$

where X, Y are arbitrary vector fields on TM .

Proof. (1) $\implies$ (2). For any semispray $S_\circ$ over M we have

$$\begin{aligned}
0 &= \sum_{X,Y,S_\circ \in \mathfrak{X}(TM)} \bar{\mu}(JX, R(Y, S_\circ)) \\
&= \bar{\mu}(JX, R(Y, S_\circ)) + \bar{\mu}(JY, R(S_\circ, X)) + \bar{\mu}(JS_\circ, R(X, Y)) \\
&\stackrel{2.5}{=} \bar{\mu}(JX, R(Y, S_\circ)) + \bar{\mu}(JY, R(S_\circ, X)) + \bar{\mu}(C, R(X, Y)) \\
&\stackrel{(10.8a)}{=} \bar{\mu}(JX, R(Y, S_\circ)) + \bar{\mu}(JY, R(S_\circ, X)) \\
&\stackrel{2.2}{=} -\bar{\mu}(JX, R(S_\circ, Y)) + \bar{\mu}(JY, R(S_\circ, X)) \\
&\stackrel{(2.20b)}{=} \bar{\mu}(JX, A(Y)) - \bar{\mu}(JY, A(X)).
\end{aligned}$$

(2) $\implies$ (1). For any vector fields X, Y on M we have

$$\begin{aligned}
&-3R(X^h, Y^h) \stackrel{(2.20a)}{=} [J, A](X^h, Y^h) \\
&\stackrel{(1.3b)}{=} [X^v, A(Y^h)] + [A(X^h), Y^v] + J \circ A([X^h, Y^h]) + A \circ J([X^h, Y^h]) \\
&\quad - A([X^v, Y^h]) - A([X^h, Y^v]) - J[A(X^h), Y^h] - J[X^h, A(Y^h)] \\
&\stackrel{2.19}{=} [X^v, A(Y^h)] + [A(X^h), Y^v]
\end{aligned}$$

therefore

$$(*) \quad -3R(X^h, Y^h) = [X^v, A(Y^h)] + [A(X^h), Y^v].$$

By our assumption $\bar{\mu}(X^v, A(Y^h)) - \bar{\mu}(A(X^h), Y^v) = 0$, whence for any vector fields X, Y, Z on M we have

$$0 = Z^v \bar{\mu}(X^v, A(Y^h)) - Z^v \bar{\mu}(A(X^h), Y^v).$$

Similarly

$$\begin{aligned}
0 &= -X^v \bar{\mu}(Z^v, A(Y^h)) + X^v \bar{\mu}(A(Z^h), Y^v), \\
0 &= Y^v \bar{\mu}(Z^v, A(X^h)) - Y^v \bar{\mu}(A(Z^h), X^v).
\end{aligned}$$

Adding the last three equations we obtain

$$\begin{aligned}
0 &= Z^v \bar{\mu}(X^v, A(Y^h)) - Z^v \bar{\mu}(A(X^h), Y^v) \\
&\quad - X^v \bar{\mu}(Z^v, A(Y^h)) + X^v \bar{\mu}(A(Z^h), Y^v) \\
&\quad + Y^v \bar{\mu}(Z^v, A(X^h)) - Y^v \bar{\mu}(A(Z^h), X^v) \\
&= (\overset{\circ}{D}_J J^* \bar{\mu})(Z^h, X^h, F A(Y^h)) + \bar{\mu}(X^v, [Z^v, A(Y^h)])
\end{aligned}$$

$$\begin{aligned}
& - (\overset{\circ}{D}_J J^* \bar{\mu})(Z^h, Y^h, FA(X^h)) - \bar{\mu}(Y^v, [Z^v, A(X^h)]) \\
& - (\overset{\circ}{D}_J J^* \bar{\mu})(X^h, Z^h, FA(Y^h)) - \bar{\mu}(Z^v, [X^v, A(Y^h)]) \\
& + (\overset{\circ}{D}_J J^* \bar{\mu})(X^h, Y^h, FA(Z^h)) + \bar{\mu}(Y^v, [X^v, A(Z^h)]) \\
& + (\overset{\circ}{D}_J J^* \bar{\mu})(Y^h, Z^h, FA(X^h)) + \bar{\mu}(Z^v, [Y^v, A(X^h)]) \\
& - (\overset{\circ}{D}_J J^* \bar{\mu})(Y^h, X^h, FA(Z^h)) + \bar{\mu}(X^v, [Y^v, A(Z^h)]).
\end{aligned}$$

Due to the symmetry of $\overset{\circ}{D}_J J^* \bar{\mu}$ the right-hand side of the last equality reduces to

$$\begin{aligned}
0 &= \bar{\mu}(X^v, [Z^v, A(Y^h)] + [A(Z^h), Y^v]) \\
&+ \bar{\mu}(Y^v, [X^v, A(Z^h)] + [A(X^h), Z^v]) \\
&+ \bar{\mu}(Z^v, [Y^v, A(X^h)] + [A(Y^h), X^v]).
\end{aligned}$$

Taking into account (*), this yields the desired relation

$$\bigotimes_{X,Y,Z \in \mathfrak{X}(M)} \bar{\mu}(JX^h, R(Y^h, Z^h)) = 0. \quad \square$$

10.9 Theorem (the alternative theorem). (i) *Let (M, S) be a spray manifold with Berwald connection induced by S . If (M, S) is projectively equivalent to a Finsler manifold $(M, \bar{L})$, then the tensor field*

$$(10.9a) \quad \bar{\mu} : (JX, JY) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto \bar{\mu}(JX, JY) := dd_J \bar{L}(JX, Y)$$

has the properties

$$(10.9b,c) \quad \overset{\circ}{D}_S J^* \bar{\mu} = 0; \quad \bigotimes_{X,Y,Z \in \mathfrak{X}(TM)} \bar{\mu}(JX, R(Y, Z)) = 0.$$

(ii) *Conversely, let (M, L^*) be a Finsler manifold, and suppose that the tensor field μ^* formed from L^* by the rule of (10.9a) satisfies (10.9b,c). Then*

- (I) (M, S) is projectively equivalent to (M, L^*) , or
- (II) there exists, locally in general, a 1-form η on M such that — also locally — (M, S) is projectively equivalent to the Finsler manifold $(M, L^* + \tilde{\eta})$, $\tilde{\eta} := (\eta^v)^\circ$.

Proof. (i) Since $d_J \bar{L}$ is a 0-homogeneous Hilbert 1-form, $J^* \bar{\mu} = \overset{\circ}{D}_J d_J \bar{L}$ by (4.7c), while (8.3) yields the relation $d_h d_J \bar{L} = \overset{\circ}{D}_J i_S dd_J \bar{L} - \overset{\circ}{D}_S J^* \bar{\mu}$. By the Rapcsák equations 9.2(2),(3) this implies $\overset{\circ}{D}_S J^* \bar{\mu} = 0$. Similarly, (10.9c) is an immediate consequence of (4.7c), (4.8) and 9.2(3).

(ii) If $i_S dd_J L^* = 0$, then we can stop by 9.2, so we have only to consider the case $i_S dd_J L^* \neq 0$. To make our reasoning more transparent, we divide it into several steps.

Step 1. We show that the 2-form $\overset{\circ}{D}_J i_S dd_J L^*$ is a vertical lift, namely: there exists a 2-form Λ on M , such that

$$(1) \quad \overset{\circ}{D}_J i_S dd_J L^* = \Lambda^\vee.$$

Notice first that (8.3) and (10.9b) yield

$$(2) \quad \overset{\circ}{D}_J i_S dd_J L^* = d_h d_J L^*.$$

This implies that $\overset{\circ}{D}_J i_S dd_J L^*$ is *skew-symmetric*. Thus, taking into account 4.3, we infer

$$(3) \quad \begin{aligned} \overset{\circ}{D}_J \overset{\circ}{D}_J i_S dd_J L^* &\text{ is symmetric in its first two variables} \\ &\text{ and skew-symmetric in its second two variables.} \end{aligned}$$

Now we calculate. For any vector fields X, Y, Z on M ,

$$\begin{aligned} \overset{\circ}{D}_J \overset{\circ}{D}_J i_S dd_J L^*(X^h, Y^h, Z^h) &\stackrel{(2)}{=} \overset{\circ}{D}_J d_h \overset{\circ}{D}_J L^*(X^h, Y^h, Z^h) \\ &\stackrel{(3.4f)}{=} X^v [(d_h d_J L^*)(Y^h, Z^h)] \stackrel{(4.7a)}{=} X^v [(dd_J L^*)(Y^h, Z^h)] \\ &= X^v (Y^h(Z^v L^*) - Z^h(Y^v L^*) - J[Y^h, Z^h]L^*) \\ &\stackrel{(2.1d)}{=} X^v ([Y^h, Z^v]L^* + Z^v(Y^h L^*) - [Z^h, Y^v]L^* - Y^v(Z^h L^*) - [Y, Z]^v L^*) \\ &\stackrel{t=0}{=} X^v (Z^v(Y^h L^*)) - X^v(Y^v(Z^h L^*)). \end{aligned}$$

Using this result we obtain

$$(4) \quad \bigoplus_{X, Y, Z \in \mathfrak{X}(M)} \overset{\circ}{D}_J \overset{\circ}{D}_J i_S dd_J L^*(X^h, Y^h, Z^h) = 0.$$

(3) and (4) imply that $\overset{\circ}{D}_J(\overset{\circ}{D}_J i_S dd_J L^*) = 0$, from which (1) immediately follows.

Step 2. We claim that

$$(5) \quad i_S \Lambda^\vee = i_S dd_J L^*.$$

Starting with the obvious relation $\overset{\circ}{D}_J \Lambda^\vee = 0$, we have, for any vector fields X, Y on M ,

$$\begin{aligned} 0 &= \overset{\circ}{D}_J \Lambda^\vee(X^h, S, Y^h) = (\overset{\circ}{D}_{X^v} \Lambda^\vee)(S, Y^h) \stackrel{(3.4l)}{=} X^v \Lambda^\vee(S, Y^h) - \Lambda^\vee(X^h, Y^h) \\ &= (\overset{\circ}{D}_J i_S \Lambda^\vee - \Lambda^\vee)(X^h, Y^h). \end{aligned}$$

Thus $\overset{\circ}{D}_J i_S \Lambda^v = \overset{\circ}{D}_J i_S dd_J L^*$, and hence

$$i_S dd_J L^* = i_S \Lambda^v + \alpha^v, \quad \alpha \in \Omega^1(M).$$

But $i_S dd_J L^*$ and $i_S \Lambda^v$ are both 1-homogeneous (see 2.16) while α^v is 0-homogeneous, so α^v must vanish.

Step 3. We show that the 2-form Λ^v is closed. — By (4.8) and (10.9c),

$$\begin{aligned} 0 &= \underset{X,Y,Z \in \mathfrak{X}(M)}{\mathfrak{S}} \overset{\circ}{D}_h d_h dd_J L^*(X^h, Y^h, Z^h) \stackrel{(1),(2)}{=} \underset{X,Y,Z \in \mathfrak{X}(M)}{\mathfrak{S}} (\overset{\circ}{D}_h \Lambda^v)(X^h, Y^h, Z^h) \\ &\stackrel{(3.4h)}{=} X^h \Lambda^v(Y^h, Z^h) - Y^h \Lambda^v(X^h, Z^h) + Z^h \Lambda^v(X^h, Y^h) \\ &\quad - \Lambda^v(F([X^h, Y^v] - [Y^h, X^v]), Z^h) + \Lambda^v(F([X^h, Z^v] - [Z^h, X^v]), Y^h) \\ &\quad - \Lambda^v(F([Y^h, Z^v] - [Z^h, Y^v]), X^h) \stackrel{t=0, (2.6a)}{=} X^h \Lambda^v(Y^h, Z^h) \\ &\quad - Y^h \Lambda^v(X^h, Z^h) + Z^h \Lambda^v(X^h, Y^h) - \Lambda^v([X, Y]^h, Z^h) \\ &\quad + \Lambda^v([X, Z]^h, Y^h) - \Lambda^v([Y, Z]^h, X^h). \end{aligned}$$

Thus, since $[X, Y]^h = [X^h, Y^h] - v[X^h, Y^h]$, etc., and Λ^v is semibasic, we obtain

$$\begin{aligned} 0 &= X^h \Lambda^v(Y^h, Z^h) - Y^h \Lambda^v(X^h, Z^h) + Z^h \Lambda^v(X^h, Y^h) - \Lambda^v([X^h, Y^h], Z^h) \\ &\quad + \Lambda^v([X^h, Z^h], Y^h) - \Lambda^v([Y^h, Z^h], X^h) = (d\Lambda^v)(X^h, Y^h, Z^h) \end{aligned}$$

whence our claim.

Step 4. Since Λ^v is closed and $d\Lambda^v = (d\Lambda)^v$, Λ is also closed. Thus there exists, at least locally, a 1-form η on M such that $d\eta = \Lambda$. Consider the function $\bar{L} := L^* - \tilde{\eta}$, $\tilde{\eta} := (\eta^v)^\circ$, defined on a region of TM . We show that $\bar{L}$ satisfies 9.2(2). Notice first that for any vector fields X, Y on M ,

$$\begin{aligned} dd_J \tilde{\eta}(X^c, Y^c) &= X^c(Y^v \tilde{\eta}) - Y^c(X^v \tilde{\eta}) - J[X^c, Y^c] \tilde{\eta} \stackrel{9.4, (1.4), (1.5c)}{=} X^c(\eta(Y))^v \\ &\quad - Y^c(\eta(X))^v - (\eta([X, Y]))^v = [(d\eta)(X, Y)]^v = [\Lambda(X, Y)]^v = \Lambda^v(X^c, Y^c). \end{aligned}$$

Since $dd_J \tilde{\eta}$ is clearly semibasic, this means that $dd_J \tilde{\eta} = \Lambda^v$. Thus

$$i_S dd_J \bar{L} = i_S dd_J L^* - i_S dd_J \tilde{\eta} = i_S dd_J L^* - i_S \Lambda^v \stackrel{(5)}{=} i_S \Lambda^v - i_S \Lambda^v = 0,$$

concluding the proof of (II). $\square$

10.10 Theorem. *Let (M, S) be a spray manifold and suppose that there exists a symmetric $(0, 2)$ -tensor μ on the vertical subbundle satisfying the following conditions:*

$$(1) \quad i_C \mu = 0,$$

- (2) $\overset{\circ}{D}_S J^* \mu = 0$,
- (3) $\underset{X,Y,Z \in \mathfrak{X}(TM)}{\mathfrak{S}} \mu(JX, R(Y, Z)) = 0$,
- (4) $\overset{\circ}{D}_J J^* \mu$ is totally symmetric ,
- (5) If $X \in \mathfrak{X}^v(TM)$ is not proportional to the Liouville vector field C , then $\mu(X, X)$ is a positive-valued function on TM .

Then there exists a positive Finsler-Lagrangian L such that its canonical spray is projectively equivalent to the given spray S .

Proof. Let h , v and F be the induced horizontal endomorphism, the vertical projector and the almost complex structure induced by S , respectively. Consider the $(0, 2)$ tensor $\bar{\mu}$ given by

$$\bar{\mu}(X, Y) := \mu(vX, vY) + \mu(JX, JY), \quad \text{for all } X, Y \in \mathfrak{X}(TM).$$

Let us also introduce a 2-form ω by the rule:

$$\omega(X, FY) := \bar{\mu}(X, Y), \quad \text{for all } X, Y \in \mathfrak{X}(TM).$$

One can easily deduce that

$$(*) \quad \omega(JX, JY) = \omega(hX, hY) = 0,$$

whence (10.1b) is satisfied. Next we show that (10.1a) is also valid. We check that $\mathcal{L}_S \omega$ vanishes on the following pairs of vector fields: (X^h, Y^h) , (X^v, Y^h) , (X^v, Y^v) .

$$\begin{aligned}
 \text{Case 1.} \quad \mathcal{L}_S \omega(X^h, Y^h) &= S\omega(X^h, Y^h) - \omega([S, X^h], Y^h) - \omega(X^h, [S, Y^h]) \\
 &\stackrel{(*)}{=} \omega([S, X^h], Y^h) + \omega(X^h, [S, Y^h]) \\
 &\stackrel{(2.6a)}{=} \omega([S, X^h], FY^v) - \omega([S, Y^h], FY^v) \\
 &= \mu([S, X^h], Y^v) - \mu([S, Y^h], X^v) \\
 &= \mu(v[S, X^h], Y^v) - \mu(v[S, Y^h], X^v) \\
 &\stackrel{2.19}{=} \mu(A(X^h), Y^v) - \mu(A(Y^h), X^v).
 \end{aligned}$$

By 10.8, the last equation is equivalent to (3).

$$\begin{aligned}
\text{Case 2.} \quad \mathcal{L}_S \omega(X^v, Y^h) &= S\omega(X^v, Y^h) - \omega([S, X^v], Y^h) - \omega(X^v, [S, Y^h]) \\
&\stackrel{(*)}{=} S\omega(X^v, FY^v) - \omega([S, X^v], FY^v) - \omega(X^v, h[S, Y^h]) \\
&\stackrel{(2.6a)}{=} S\mu(X^v, Y^v) - \mu([S, X^v], Y^v) - \mu(X^v, J[S, Y^h]) \\
&\stackrel{(2.9b)}{=} S\mu(X^v, Y^v) - \mu([S, X^v], Y^v) - \mu(X^v, vY^c) \\
&\stackrel{(2.9a)}{=} S\mu(X^v, Y^v) - \mu(vX^c, Y^v) - \mu(X^v, vY^c) \\
&\stackrel{(3.4m)}{=} \overset{\circ}{D}_S J^* \mu(X^h, Y^h) \stackrel{(2)}{=} 0.
\end{aligned}$$

$$\begin{aligned}
\text{Case 3.} \quad \mathcal{L}_S \omega(X^v, Y^v) &= S\omega(X^v, Y^v) - \omega([S, X^v], Y^v) - \omega(X^v, [S, Y^v]) \\
&\stackrel{(*)}{=} \omega(Y^v, h[S, X^v]) - \omega(X^v, h[S, Y^v]) \\
&\stackrel{(2.6a)}{=} -\mu(Y^v, J[S, X^v]) + \mu(X^v, J[S, Y^v]) \\
&\stackrel{(2.7)}{=} -\mu(Y^v, X^v) + \mu(X^v, Y^v) = 0,
\end{aligned}$$

due to the symmetry of μ .

Finally, we show that (4) implies (10.1c).

$$\begin{aligned}
d\omega(X^h, Y^v, Z^v) &= X^h \omega(Y^v, Z^v) - Y^v \omega(X^h, Z^v) + Z^v \omega(X^h, Y^v) \\
&\quad - \omega([X^h, Y^v], Z^v) + \omega([X^h, Z^v], Y^v) - \omega([Y^v, Z^v], X^h) \\
&= -Y^v \omega(X^h, Z^v) + Z^v \omega(X^h, Y^v),
\end{aligned}$$

since ω vanishes on pairs of vertical vector fields, the bracket of vertical lifts is zero and the bracket of a vertically and a horizontally lifted vector field is vertical. Using the definition μ we obtain

$$\begin{aligned}
d\omega(X^h, Y^v, Z^v) &= -Y^v \omega(X^h, Z^v) + Z^v \omega(X^h, Y^v) \\
&= Y^v \omega(Z^v, X^h) - Z^v \omega(Y^v, X^h) \\
&= -Y^v \mu(Z^v, X^v) + Z^v \mu(Y^v, X^v) \\
&\stackrel{(3.4f)}{=} \overset{\circ}{D}_J J^*(X^h, Y^h, Z^h) + \overset{\circ}{D}_J J^*(Z^h, Y^h, X^h) \stackrel{(4)}{=} 0.
\end{aligned}$$

Now our statement is an immediate consequence of Proposition 10.4. $\square$

III. PROJECTION ONTO THE INDICATRIX BUNDLE OF A FINSLER MANIFOLD

11. Finslerian Berwald and Cartan connection

11.1 The Cartan tensors. We agree that in the whole Chapter III (M, L) stands for a Finsler manifold endowed with a *positive* Finsler-Lagrangian L . We denote by F the almost complex structure associated to the Barthel endomorphism of (M, L) (2.6), (8.6(6)). Then the mapping

$$(11.1a) \quad g : \mathfrak{X}(TM) \times \mathfrak{X}(TM) \rightarrow C^\infty(TM), \quad (X, Y) \mapsto g(X, Y) := \omega(X, FY)$$

is a well-defined pseudo-Riemannian metric on TM . We introduce the *first Cartan tensors* $\mathcal{C}$ and $\mathcal{C}_b$ by the relations

$$(11.1b) \quad g(\mathcal{C}(X, Y), JZ) := \frac{1}{2} \mathcal{L}_{JX}(J^*g)(Y, Z) \quad \text{and} \quad J \circ \mathcal{C} = 0,$$

$$(11.1c) \quad \mathcal{C}_b(X, Y, Z) := g(\mathcal{C}(X, Y), JZ); \quad X, Y, Z \in \mathfrak{X}(TM).$$

(cf. 10.6). The *second Cartan tensors* $\mathcal{C}'$, $\mathcal{C}'_b$ are defined as follows: for any vector fields X, Y, Z on TM ,

$$(11.1d) \quad g(\mathcal{C}'(X, Y), JZ) := \frac{1}{2} (\mathcal{L}_{hX}g)(JY, JZ) \quad \text{and} \quad J \circ \mathcal{C}' = 0,$$

$$(11.1e) \quad \mathcal{C}'_b(X, Y, Z) := g(\mathcal{C}'(X, Y), JZ).$$

Now we summarize the basic properties of the Cartan tensors.

$$(11.1f) \quad \mathcal{C} \text{ and } \mathcal{C}' \text{ are semibasic.}$$

$$(11.1g) \quad \mathcal{C}_b \text{ and } \mathcal{C}'_b \text{ are totally symmetric.}$$

For any semispray S_0 ,

$$(11.1h) \quad i_{S_0}\mathcal{C} = i_{S_0}\mathcal{C}_b = 0, \quad i_{S_0}\mathcal{C}' = i_{S_0}\mathcal{C}'_b = 0.$$

We have the following computational formulas:

$$(11.1i) \quad \forall X, Y, Z \in \mathfrak{X}(M) : 2\mathcal{C}_b(X^c, Y^c, Z^c) = X^v[Y^v(Z^v E)];$$

$$(11.1j) \quad \forall X, Y, Z \in \mathfrak{X}(M) : 2\mathcal{C}'_b(X^c, Y^c, Z^c) = [Y^v, [X^h, Z^v]] E.$$

For details and a somewhat more general approach we refer to [55].

11.2 The Finslerian Berwald and Cartan connection. As it is well-known, there exist a number of 'nice' Finsler connections on a Finsler manifold [50]. Since every Finsler manifold is a spray manifold (8.6(5)) the Berwald connection $\overset{\circ}{D}$ arising from the canonical spray of a Finsler manifold is naturally available. In our subsequent considerations we shall also need the *Cartan connection* D . The simplest way to define it is to prescribe the rules for calculation, with the help of the Berwald derivatives. we present these rules in the following tabular form:

Berwald connection	Cartan connection
$\overset{\circ}{D}_{JX}JY = J[JX, Y]$	$D_{JX}JY = J[JX, Y] + \mathcal{C}(X, Y)$
$\overset{\circ}{D}_{hX}JY = v[hX, JY]$	$D_{hX}JY = v[hX, JY] + \mathcal{C}'(X, Y)$
$\overset{\circ}{D}_{JX}hY = h[JX, Y]$	$D_{JX}hY = h[JX, Y] + FC(X, Y)$
$\overset{\circ}{D}_{hX}hY = hF[hX, JY]$	$D_{hX}hY = hF[hX, JY] + FC'(X, Y)$

($X, Y \in \mathfrak{X}(TM)$).

Using vertically and horizontally lifted vector fields, we get more convenient expressions:

Berwald connection	Cartan connection
$\overset{\circ}{D}_{X^v}Y^v = 0$	$D_{X^v}Y^v = \mathcal{C}(X^h, Y^h)$
$\overset{\circ}{D}_{X^h}Y^v = [X^h, Y^v]$	$D_{X^h}Y^v = [X^h, Y^v] + \mathcal{C}'(X^h, Y^h)$
$\overset{\circ}{D}_{X^v}Y^h = 0$	$D_{X^v}Y^h = FC(X^h, Y^h)$
$\overset{\circ}{D}_{X^h}Y^h = F[X^h, Y^v]$	$D_{X^h}Y^h = F[X^h, Y^v] + FC'(X^h, Y^h)$

($X, Y \in \mathfrak{X}(M)$ are arbitrary vector fields).

11.3 Torsion and curvature. As we have already mentioned in Section 3, the classical torsion and curvature tensor of a Finsler connection can be decomposed into five "partial" torsions and three partial curvatures. As for the Berwald and the Cartan connection, we have the following result ([26], [50]):

	Berwald connection	Cartan connection
h -horizontal torsion	$\overset{\circ}{\mathbb{A}} = 0$	$\mathbb{A} = 0$
h -mixed torsion	$\overset{\circ}{\mathbb{B}} = 0$	$\mathbb{B} = -F \circ \mathcal{C}$
v -horizontal torsion	$\overset{\circ}{\mathbb{R}}^1 = -\frac{1}{2}[h, h]$	$\mathbb{R}^1 = -\frac{1}{2}[h, h]$
v -mixed torsion	$\overset{\circ}{\mathbb{P}}^1 = 0$	$\mathbb{P}^1 = \mathcal{C}'$
v -vertical torsion	$\overset{\circ}{\mathbb{S}}^1 = 0$	$\mathbb{S}^1 = 0$
horizontal curvature	$\overset{\circ}{\mathbb{R}} = \overset{\circ}{D}_J \overset{\circ}{\mathbb{R}}^1$	$\mathbb{R}$
mixed curvature	$\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h = [[X^h, Y^v], Z^v]$	$\mathbb{P}$
vertical curvature	$\overset{\circ}{\mathbb{Q}} = 0$	$\mathbb{Q}(X^h, Y^h)Z^h$ $= \mathcal{C}(Y^h, FC(X^h, Z^h))$ $- \mathcal{C}(X^h, FC(Y^h, Z^h))$

$(X, Y, Z \in \mathfrak{X}(M))$. $\mathbb{R}$ and $\mathbb{P}$ can be explicitly given with the help of $\overset{\circ}{\mathbb{R}}$ and $\overset{\circ}{\mathbb{P}}$ respectively, but we shall not need these quite complicated expressions.

11.4 Lemma (the fifth Bianchi identity). *Let (D, h) be the Cartan connection on a Finsler manifold. For any vector fields $X, Y, Z \in \mathfrak{X}(TM)$,*

$$\begin{aligned} & (D_h X \mathbb{Q})(Y, Z) - (D_{JY} \mathbb{P})(X, Z) + (D_{JZ} \mathbb{P})(X, Y) \\ &= \mathbb{P}(FC(X, Y), Z) - \mathbb{P}(FC(Z, X), Y) - \mathbb{Q}(FC'(X, Y), Z) + \mathbb{Q}(FC'(Z, X), Y). \end{aligned}$$

Proof. See [26], A.26. □

11.5 Lemma. *For any vector fields $X, Y, Z, U \in \mathfrak{X}(M)$,*

$$\begin{aligned} (D_h \mathbb{Q})(X^h, Y^h, Z^h, U^h) &= (D_{X^h} \mathcal{C})(Z^h, FC(Y^h, U^h)) - (D_{X^h} \mathcal{C})(Y^h, FC(Z^h, U^h)) \\ &+ \mathcal{C}(Z^h, F(D_{X^h} \mathcal{C})(Y^h, U^h)) - \mathcal{C}(Y^h, F(D_{X^h} \mathcal{C})(Z^h, U^h)). \end{aligned}$$

The proof is a routine but lengthy calculation so we omit it.

12. General remarks on Landsberg and Berwald manifolds

12.1 Proposition and definition (J.-G. DIAZ). *Keeping the notation introduced in Section 11, let (M, L) be a Finsler manifold. The following assertions are equivalent:*

- (i) *The mixed curvature $\mathbb{P}$ of the Cartan connection vanishes.*
- (ii) *The second Cartan tensor $\mathcal{C}'$ vanishes.*
- (iii) *The tensor $D_h \mathcal{C}$ is totally symmetric.*
- (iv) *$\overset{\circ}{\mathbb{P}} = -D_h \mathcal{C}$.*

If one, and therefore all, of the conditions (i)–(iv) are satisfied, then (M, L) is called a Landsberg manifold.

An intrinsic *proof* of this important result (presented in the language of Dazord's formalism) can be found in DIAZ's thesis [22]. As for the traditional treatment, we refer to [38].

12.2 Further characterizations of a Landsberg manifold

12.2.1. *(M, L) is a Landsberg manifold if and only if $\overset{\circ}{D}_h = D_h$.*

Proof. It can be seen immediately from the table in 11.2 that the properties $\mathcal{C}' = 0$ and $\overset{\circ}{D}_h = D_h$ are equivalent. □

12.2.2 Since $\mathbb{P}^1 = \mathcal{C}'$ (11.3), *the vanishing of the v-mixed torsion of the Cartan connection characterizes the Landsberg manifolds.*

12.2.3. *A Finsler manifold is a Landsberg manifold if and only if the Berwald connection is h-metrical, i.e., $\overset{\circ}{D}_h g = 0$.*

Proof. For any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned} \mathcal{C}'_b(X^h, Y^h, Z^h) &\stackrel{(11.1e)}{:=} g(\mathcal{C}'(X^h, Y^h), Z^v) \stackrel{(11.1d)}{:=} \frac{1}{2} (\mathcal{L}_{X^h} g)(JY^h, JZ^h) \\ &= \frac{1}{2} (\mathcal{L}_{X^h} g)(Y^v, Z^v) = \frac{1}{2} [X^h g(Y^v, Z^v) - g([X^h, Y^v], Z^v) - g(Y^v, [X^h, Z^v])] \\ &\stackrel{11.2}{=} \frac{1}{2} [X^h g(Y^v, Z^v) - g(\overset{\circ}{D}_{X^h} Y^v, Z^v) - g(Y^v, \overset{\circ}{D}_{X^h} Z^v)] = \frac{1}{2} (\overset{\circ}{D}_{X^h} g)(Y^v, Z^v). \end{aligned}$$

Hence

$$(12.2.3) \quad 2\mathcal{C}'_b = \overset{\circ}{D}_h g,$$

which proves our assertion. $\square$

Remark that in MATSUMOTO's monograph [38] Landsberg manifolds are defined by the property $\overset{\circ}{D}_h g = 0$.

12.3 Corollary. *In a Landsberg manifold we have*

$$(12.3) \quad \overset{\circ}{D}_h \mathbb{Q} = D_h \mathbb{Q} = 0.$$

Proof. This follows from the fifth Bianchi identity (11.4), since – as we have just seen – in Landsberg manifolds $\mathbb{P} = 0$, $\mathcal{C}' = 0$ and $\overset{\circ}{D}_h = D_h$. $\square$

12.4 Corollary. *The mixed curvature of the Berwald connection in a Landsberg manifold satisfies the identity*

$$\begin{aligned} (12.4) \quad &\overset{\circ}{\mathbb{P}}(X^h, Y^h) F\mathcal{C}(Z^h, U^h) + \mathcal{C}(Y^h, F(\overset{\circ}{\mathbb{P}}(X^h, Z^h)U^h)) \\ &= \overset{\circ}{\mathbb{P}}(Z^h, X^h) F\mathcal{C}(Y^h, U^h) + \mathcal{C}(Z^h, F(\overset{\circ}{\mathbb{P}}(X^h, Y^h)U^h)) \\ &\quad (X, Y, Z, U \in \mathfrak{X}(M)). \end{aligned}$$

Proof. This is an immediate consequence of (11.5), since in a Landsberg manifold $D_h \mathbb{Q} = 0$ and $D_h \mathcal{C} = -\overset{\circ}{\mathbb{P}}$. $\square$

12.5 Proposition. *The lowered mixed curvature tensor $\overset{\circ}{\mathbb{P}}_{\flat}$ of the Berwald connection defined by*

$$(12.5) \quad \overset{\circ}{\mathbb{P}}_{\flat}(X, Y, Z, U) := g\left(\overset{\circ}{\mathbb{P}}(X, Y)Z, JU\right) \quad (X, Y, Z, U \in \mathfrak{X}(TM))$$

is totally symmetric in a Landsberg manifold.

Proof. For any vector fields X, Y, Z, U on M we have

$$\begin{aligned} \overset{\circ}{\mathbb{P}}_{\flat}(X^h, Y^h, Z^h, U^h) &= g\left(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, JU^h\right) \stackrel{(11.1a), (2.6a)}{=} \omega\left(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, U^h\right) \\ &= d(d_J E)\left(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, U^h\right) = \overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(d_J E)(U^h) \\ &\quad - U^h(d_J E)\left(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h\right) - d_J E\left(\left[\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, U^h\right]\right). \end{aligned}$$

Since $\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h \in \mathfrak{X}^v(TM)$, the second and the third term of the last expression vanish by (1.5a) and (1.2). The first term can be formed as follows:

$$\begin{aligned} \overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(d_J E)(U^h) &= \overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(U^v E) \\ &= \left[\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, U^v\right]E - U^v\left(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(E)\right). \end{aligned}$$

Applying Lemma 3.7, it follows that here the first term is

$[\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h, U^h)]E$, while

$$U^v\left(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(E)\right) \stackrel{11.3}{=} U^v([X^h, Y^v], Z^v)E \stackrel{(11.1j)}{=} -2U^v(C'_b(X^h, Y^h, Z^h)) = 0,$$

since the underlying manifold is Landsberg. Thus we obtain the relation

$$\overset{\circ}{\mathbb{P}}_{\flat}(X^h, Y^h, Z^h, U^h) = \left[\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h, U^h)\right]E.$$

In view of (4.10c) this shows that $\overset{\circ}{\mathbb{P}}_{\flat}$ is totally symmetric. $\square$

12.6 Definition [55]. *A Finsler manifold (M, L) is said to be a Berwald manifold if there is a linear connection ∇ on M such that for any vector fields $X, Y \in \mathfrak{X}(M)$, $(\nabla_X Y)^v = [X^h, Y^v]$.*

12.7 Remark. The linear connection ∇ in the previous definition is clearly unique, so it can be mentioned as *the* linear connection of the Berwald manifold. It is readily seen that the horizontal endomorphism induced by ∇ is just the Barthel endomorphism, therefore *the canonical spray of a Berwald manifold is smooth on the whole tangent manifold TM* . The converse of the last statement is also true. Let us note finally that the Berwald connection of a Berwald manifold is just the horizontal lift ([21], p. 134) of the linear connection ∇ .

12.8 Lemma. *A Finsler manifold (M, L) is a Berwald manifold if and only if the condition*

$$(12.8) \quad \text{for all } X, Y \text{ on } M \text{ } [X^h, Y^v] \text{ is a vertical lift}$$

is fulfilled.

Proof. The necessity of (12.8) is evident. To see the sufficiency, let us consider the mapping

$$\nabla : (X, Y) \in \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \nabla_X Y \in \mathfrak{X}(M), \quad (\nabla_X Y)^v = [X^h, Y^v],$$

where X^h is the horizontal lift of X by the Barthel endomorphism. Then an easy calculation shows that ∇ is a well-defined linear connection on M . $\square$

12.9 Corollary. *A Finsler manifold is a Berwald manifold if and only if the mixed curvature of the Berwald connection vanishes.*

Proof. If (M, L) is a Berwald manifold, then for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h \stackrel{11.3}{=} [[X^h, Y^v], Z^v] = [(\nabla_X Y)^v, Z^v] = 0.$$

Conversely, if $\overset{\circ}{\mathbb{P}}$ vanishes, then for any X, Y, Z on M we have $0 = [[X^h, Y^v], Z^v] \Rightarrow \forall X, Y \in \mathfrak{X}(M) : [X^h, Y^v] \text{ is a vertical lift}$, and it follows by (12.8) that (M, L) is a Berwald manifold. $\square$

12.10 Remark. This is a well-known observation, of course. As for the usual “coordinate-proof”, we also refer to [38].

13. Projection onto the indicatrix bundle

13.1 By the *indicatrix bundle* of the Finsler manifold (M, L) we mean the $(2n-1)$ -dimensional submanifold

$$U(M) := \{v \in TM \mid L(v) = 1\}.$$

of TM .

13.2 Lemma [27]. *Let us denote by $C_0^\infty(TM)$ the ring of those smooth functions on TM which are homogeneous of degree 0. The module $\mathfrak{X}[U(M)]$ is canonically isomorphic to the $C_0^\infty(TM)$ -module*

$$(13.2) \quad \mathfrak{X}_0(TM) := \{X \in \mathfrak{X}(TM) \mid [C, X] = -X \text{ and } XE = 0\}. \quad \square$$

13.3 Lemma. *With respect to the metric g given by (11.1a), the Liouville vector field is everywhere orthogonal to the indicatrix bundle, i.e., $g(C, X) = 0$ for any vector fields $X \in \mathfrak{X}[U(M)]$. The vector field $\frac{1}{L}C$ is a unit normal vector field of the indicatrix bundle.*

Proof. In view of Lemma 13.2 it is enough to show that for any vector field $X \in \mathfrak{X}_0(TM)$, we have $g(C, X) = 0$. This can be seen by an easy calculation:

$$\begin{aligned} g(C, X) &\stackrel{(11.1a)}{=} \omega(C, FX) = i_C \omega(FX) \stackrel{(8.6(3))}{=} d_J E(FX) \\ &= dE(J \circ F(X)) = dE(X - hX) \stackrel{(8.6(6))}{=} dE(X) = XE \stackrel{(13.2)}{=} 0. \quad \square \end{aligned}$$

13.4 Lemma. *Suppose that (M, L) is a positive Finsler manifold. The mapping*

$$\tau := I - \frac{1}{2E} dE \otimes C$$

is a projector. For any vector field $X \in \mathfrak{X}(TM)$, $\tau(X) \upharpoonright U(M)$ is a tangent vector field to the indicatrix bundle.

Proof.

$$\begin{aligned} \forall X \in \mathfrak{X}(TM) : \tau(X) &= X - \frac{1}{2E} dE(X)C \stackrel{(8.6(5))}{=} X + \frac{1}{2E} i_S \omega(X)C = \\ &= X - \frac{1}{2E} \omega(X, S)C = X - \frac{1}{2E} \omega(X, hS)C \stackrel{(2.6a)}{=} X - \frac{1}{2E} \omega(X, FC)C \\ &\stackrel{(11.1a)}{=} X - \frac{1}{2E} g(X, C)C = X - \frac{g(X, C)}{g(C, C)}C. \end{aligned}$$

Thus we have

$$(13.4) \quad \forall X \in \mathfrak{X}(TM) : \tau(X) = X - \frac{g(X, C)}{g(C, C)}C.$$

In particular, for any vector field $X \in \mathfrak{X}(TM)$,

$$\tau(\tau(X)) = \tau(X) - \frac{g(\tau(X), C)}{g(C, C)}C.$$

Since $g(\tau(X), C) = g(X, C) - \frac{g(X, C)}{g(C, C)}g(C, C) = 0$, $\tau(X) \upharpoonright U(M)$ is tangent to $U(M)$ and $\tau(\tau(X)) = \tau(X)$. $\square$

13.5 Proposition. *Using the above notation, let $\kappa := J^* \circ \tau$. Then*

$$(13.5a) \quad \kappa = J - \frac{1}{2E} d_J E \otimes C.$$

The semibasic trace of κ is

$$(13.5b) \quad \tilde{\kappa} = n - 1.$$

The lowered tensor κ_b of κ defined by

$$(13.5c) \quad \kappa_b(X, Y) := g(\kappa X, JY) \quad (X, Y \in \mathfrak{X}(TM))$$

is symmetric.

Proof. The relation (13.5a) is evident by the definition. It is also clear that κ is semibasic. Since

$$F \circ \kappa \stackrel{(2.6a)}{=} h - \frac{1}{2E} F \circ (d_J E \otimes C),$$

in view of definition 2.10a, we have

$$\tilde{\kappa} := tr(F \circ \kappa) = tr(h) - \frac{1}{2E} tr[F \circ (d_J E \otimes C)] = tr(h) - \frac{1}{2E} \widetilde{d_J E \otimes C}.$$

An easy calculation shows that $tr(h) = n$, while, using formula (2.11b),

$$\widetilde{d_J E \otimes C} = i_S d_J E = dE(C) = CE = 2E,$$

thus (13.5b) is proved. Finally, for any vector fields $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned} \kappa_b(X^h, Y^h) &:= g(\kappa X^h, JY^h) \stackrel{(13.5a)}{=} g(X^v - \frac{1}{2E}(X^v E)C, Y^v) \\ &= g(X^v, Y^v) - \frac{1}{2E}(X^v E)\omega(Y^v, S) \\ &= g(X^v, Y^v) + \frac{1}{2E}(X^v E)i_S \omega(Y^v) \\ &\stackrel{(8.6(5))}{=} g(X^v, Y^v) - \frac{1}{2E}(X^v E)dE(Y^v) \\ &= g(X^v, Y^v) - \frac{1}{2E}(X^v E)(Y^v E), \end{aligned}$$

showing the symmetry of κ_b . □

13.6 Lemma. Consider the operator κ given by (13.5a). For any vector fields $X, Y \in \mathfrak{X}(M)$, we have

$$(13.6) \quad \kappa F\mathcal{C}(X^h, Y^h) = \mathcal{C}(X^h, Y^h) = \mathcal{C}(X^h, F\kappa Y^h).$$

Proof.

$$\begin{aligned} \kappa F\mathcal{C}(X^h, Y^h) &= \mathcal{C}(X^h, Y^h) - \frac{1}{2E} d_J E(F\mathcal{C}(X^h, Y^h))C \stackrel{(8.6(3))}{=} \mathcal{C}(X^h, Y^h) \\ &\quad - \frac{1}{2E} i_C \omega(F\mathcal{C}(X^h, Y^h))C = \mathcal{C}(X^h, Y^h) - \frac{1}{2E} g(C, \mathcal{C}(X^h, Y^h))C \stackrel{(11.1c)}{=} \\ &\quad \mathcal{C}(X^h, Y^h) - \frac{1}{2E} \mathcal{C}_b(X^h, Y^h, S)C \stackrel{(11.1h)}{=} \mathcal{C}(X^h, Y^h); \end{aligned}$$

$$\begin{aligned} \mathcal{C}(X^h, F\kappa Y^h) &= \mathcal{C}\left(X^h, F\left(Y^v - \frac{1}{2E}(Y^v E)C\right)\right) \\ &= \mathcal{C}(X^h, Y^h) - \frac{1}{2E}(Y^v E)\mathcal{C}(X^h, S) = \mathcal{C}(X^h, Y^h) \end{aligned}$$

□

13.7 Corollary. *For any vector fields $X, Y, Z \in \mathfrak{X}(M)$, we have*

$$(13.7) \quad \kappa_b(X^h, FC(Y^h, Z^h)) = \mathcal{C}_b(X^h, Y^h, Z^h).$$

Proof. $\kappa_b(X^h, FC(Y^h, Z^h)) \stackrel{(13.5)}{=} \kappa_b(FC(Y^h, Z^h), X^h) \stackrel{(13.5c)}{:=} g(\kappa FC(Y^h, Z^h), JX^h)$
 $\stackrel{(13.6)}{=} g(\mathcal{C}(Y^h, Z^h), JX^h) \stackrel{(11.1c)}{=} \mathcal{C}_b(Y^h, Z^h, X^h) \stackrel{(11.1g)}{=} \mathcal{C}_b(X^h, Y^h, Z^h). \quad \square$

13.8 Definition. *Let us suppose that (M, L) is a positive Finsler manifold and $A \in \mathcal{T}_r^0(TM)$, $K \in \mathcal{T}_r^1(TM)$ ($r \in \mathbb{N}^+$). Consider the operator $\tau = I - \frac{1}{2E}dE \otimes C$. The tensors A^* and K^* , given by*

$$A^*(X_1, \dots, X_r) := A(\tau(X_1), \dots, \tau(X_r))$$

and

$$K^*(X_1, \dots, X_r) := \tau[K(\tau(X_1), \dots, \tau(X_r))]$$

($X_i \in \mathfrak{X}(TM)$, $1 \leq i \leq r$) are called the projected tensors of A and K , respectively.

13.9 Remark. Since for any vector field $X \in \mathfrak{X}(M)$

$$\tau(X^h) = X^h - \frac{1}{2E}(X^h E)C \stackrel{(8.6(6))}{=} X^h,$$

it follows that in case of *semibasic* tensors $A \in \mathcal{T}_r^0(TM)$ and $K \in \mathcal{T}_r^1(TM)$

$$A^* = A, \quad K^* = \tau \circ K.$$

13.10 Proposition. *Consider the Douglas tensor*

$$(13.10a) \quad \mathbb{D} = \overset{\circ}{\mathbb{P}} - \frac{1}{n+1} \left(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}} \otimes C + \overset{\circ}{\mathbb{P}} \odot J \right) \in \mathcal{T}_3^1(TM)$$

of the Finsler manifold (M, L) (6.4b). The projected tensor of $\mathbb{D}$ is

$$(13.10b) \quad \mathbb{D}^* = \overset{\circ}{\mathbb{P}} - \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \odot \kappa + \frac{1}{E} \mathcal{C}'_b \otimes C.$$

Proof. Since $\mathbb{D}$ is semibasic, by the preceding remark $\mathbb{D}^* = \tau \circ \mathbb{D}$. So, for any vector fields $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned} \mathbb{D}^*(X^h, Y^h, Z^h) &= \mathbb{D}(X^h, Y^h, Z^h) - \frac{1}{2E} \mathbb{D}(X^h, Y^h, Z^h)(E)C \\ &\stackrel{(13.10a)}{=} \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h) - \frac{1}{n+1} \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h)C \\ &\quad - \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \odot J(X^h, Y^h, Z^h) - \frac{1}{2E} \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h)(E)C \\ &\quad + \frac{1}{n+1} \frac{1}{2E} \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h)(CE)C \\ &\quad + \frac{1}{n+1} \frac{1}{2E} \left[\overset{\circ}{\mathbb{P}} \odot J(X^h, Y^h, Z^h) \right](E)C. \end{aligned}$$

The second and the fifth term on the left hand side cancel. Applying (11.1j) and **11.3** we obtain that

$$\begin{aligned} \left[\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h \right] E &= [[X^h, Y^h], Z^h] E = -[Z^h, [X^h, Y^h]] E \\ &= -2\mathcal{C}'_b(X^h, Z^h, Y^h) = -2\mathcal{C}'_b(X^h, Y^h, Z^h), \end{aligned}$$

hence the fourth term is

$$\frac{1}{E} \mathcal{C}'_b \otimes C(X^h, Y^h, Z^h).$$

It is also easy to check that the sixth term can be written in the form

$$\frac{1}{n+1} \frac{1}{2E} \left(\overset{\circ}{\mathbb{P}} \odot d_J E \right) \otimes C(X^h, Y^h, Z^h),$$

which, together with the third term, yields the second term of (13.10b). $\square$

13.11 Corollary. *The Douglas tensor and its projected tensor are related as follows:*

$$(13.11) \quad \mathbb{D}^* = \mathbb{D} + \left[\frac{1}{E} \mathcal{C}'_b + \frac{1}{n+1} \left(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}} + \frac{1}{2E} \overset{\circ}{\mathbb{P}} \odot d_J E \right) \right] \otimes C. \quad \square$$

13.12 Theorem. *If (M, L) is a positive Finsler manifold of dimension $n > 2$, then the vanishing of the projected Douglas tensor is equivalent to the vanishing of the Douglas tensor.*

Proof. Let us consider the Douglas tensor $\mathbb{D}$ and the projected Douglas tensor $\mathbb{D}^*$ of (M, E) . Since $\mathbb{D}^* = \tau \circ \mathbb{D}$, if $\mathbb{D} = 0$, then $\mathbb{D}^*$ also vanishes.

Conversely, assume that $\mathbb{D}^* = 0$. Then, from (13.10b) we obtain

$$\overset{\circ}{\mathbb{P}} = \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \odot \kappa - \frac{1}{E} \mathcal{C}'_b \otimes C,$$

and hence

$$(13.12) \quad \left(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}} \right)^* = \frac{1}{n+1} \left[\overset{\circ}{D}_J \left(\overset{\circ}{\mathbb{P}} \odot \kappa \right) \right]^* - \left[\overset{\circ}{D}_J \left(\frac{1}{E} \mathcal{C}'_b \otimes C \right) \right]^*.$$

Step 1. We evaluate the left hand side of (13.12) term by term. Let $X, Y, Z, U \in \mathfrak{X}(M)$ be arbitrary vector fields.

$$(a) \quad \left[\overset{\circ}{D}_J \left(\overset{\circ}{\mathbb{P}} \odot \kappa \right) \right]^* (X^h, Y^h, Z^h, U^h) = \left[\overset{\circ}{D}_J \left(\overset{\circ}{\mathbb{P}} \odot \kappa \right) \right] (X^h, Y^h, Z^h, U^h) - \frac{1}{2E} \left[\left(\overset{\circ}{D}_J \left(\overset{\circ}{\mathbb{P}} \odot \kappa \right) \right) (X^h, Y^h, Z^h, U^h) \right] (E) C = \left[\overset{\circ}{D}_{X^v} \left(\overset{\circ}{\mathbb{P}} \odot \kappa \right) \right] (Y^h, Z^h, U^h) - \frac{1}{2E} \left[\left(\overset{\circ}{D}_{X^v} \left(\overset{\circ}{\mathbb{P}} \odot \kappa \right) \right) (Y^h, Z^h, U^h) \right] (E) C.$$

Using the product rule and taking into account the facts that

$$(i) \quad \overset{\circ}{D}_{X^v} \kappa Y^h \stackrel{(13.5a)}{=} \overset{\circ}{D}_{X^v} (Y^v - \frac{1}{2E} (Y^v E) C) \stackrel{(11.2)}{=} -X^v \left(\frac{1}{2E} Y^v E \right) C -$$

$$\frac{1}{2E} (Y^v E) \overset{\circ}{D}_{X^v} C = -X^v \left(\frac{1}{2E} Y^v E \right) C - \frac{1}{2E} (Y^v E) X^v$$

$$(\text{since } \overset{\circ}{D}_{X^v} C = \overset{\circ}{D}_{JX^h} JS \stackrel{(3.4a)}{=} J[JX^h, S] \stackrel{(2.7)}{=} JX^h = X^v);$$

$$(ii) \quad (\kappa Y^h) E = Y^v E - \frac{1}{2E} (Y^v E) (CE) = Y^v E - Y^v E = 0;$$

$$(iii) \quad \left(\overset{\circ}{D}_{X^v} \kappa Y^h \right) E \stackrel{(i)}{=} -2E \cdot X^v \left(\frac{1}{2E} Y^v E \right) - \frac{1}{2E} (X^v E) (Y^v E),$$

after a somewhat lengthy but quite straightforward calculation we obtain

$$(1) \quad \left[\overset{\circ}{D}_J \left(\overset{\circ}{\mathbb{P}} \odot \kappa \right) \right]^* (X^h, Y^h, Z^h, U^h) = \left[\left(\overset{\circ}{D}_{X^v} \overset{\circ}{\mathbb{P}} \right) \odot \kappa - \frac{1}{2E} \left(\overset{\circ}{\mathbb{P}} \odot d_J E \right) \otimes \kappa X^h \right] (Y^h, Z^h, U^h).$$

$$(b) \quad \left[\overset{\circ}{D}_J \left(\frac{1}{E} \mathcal{C}'_b \otimes C \right) \right]^* (X^h, Y^h, Z^h, U^h) = \left[\overset{\circ}{D}_J \left(\frac{1}{E} \mathcal{C}'_b \otimes C \right) \right] (X^h, Y^h, Z^h, U^h) - \frac{1}{2E} \left[\left(\overset{\circ}{D}_J \left(\frac{1}{E} \mathcal{C}'_b \otimes C \right) \right) (X^h, Y^h, Z^h, U^h) \right] (E) C = \left[\overset{\circ}{D}_{X^v} \left(\frac{1}{E} \mathcal{C}'_b \otimes C \right) \right] (Y^h, Z^h, U^h) - \frac{1}{2E} \left[\left(\overset{\circ}{D}_{X^v} \left(\frac{1}{E} \mathcal{C}'_b \otimes C \right) \right) (Y^h, Z^h, U^h) \right] (E) C \stackrel{(11.2)}{=} X^v \left[\frac{1}{E} \mathcal{C}'_b (Y^h, Z^h, U^h) \right] C + \frac{1}{E} \mathcal{C}'_b (Y^h, Z^h, U^h) \overset{\circ}{D}_{X^v} C - \frac{1}{2E} X^v \left[\frac{1}{E} \mathcal{C}'_b (Y^h, Z^h, U^h) \right] (CE) C - \frac{1}{2E^2} \mathcal{C}'_b (Y^h, Z^h, U^h) \left(\overset{\circ}{D}_{X^v} C \right) (E) C \stackrel{(i)}{=} \frac{1}{E} \mathcal{C}'_b (Y^h, Z^h, U^h) X^v - \frac{1}{2E} \frac{1}{E} \mathcal{C}'_b (Y^h, Z^h, U^h) (X^v E) C = \frac{1}{E} \mathcal{C}'_b (Y^h, Z^h, U^h) \kappa (X^h).$$

This result and (1) yield the formula

$$(2) \quad \left(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}} \right)^* (X^h, Y^h, Z^h, U^h) = \frac{1}{n+1} \left[\left(\overset{\circ}{D}_{X^v} \overset{\circ}{\mathbb{P}} \right) \odot \kappa \right] (Y^h, Z^h, U^h) - \kappa (X^h) \left(\frac{1}{2E} \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \odot d_J E + \frac{1}{E} \mathcal{C}' \right) (Y^h, Z^h, U^h).$$

Step 2. Since the tensor $\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}$ is totally symmetric (4.10c) and $(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}})^* = \tau \circ \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}$, the projected tensor $(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}})^*$ is also totally symmetric. Hence the semibasic (1,4) tensor A defined by

$$A(X^h, Y^h, Z^h, U^h) := \left(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}} \right)^* (X^h, Y^h, Z^h, U^h) - \left(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}} \right)^* (U^h, Y^h, Z^h, X^h)$$

vanishes identically. On the other hand, in view of (2), for any vector fields $X, Y, Z, U \in \mathfrak{X}(M)$ we have

$$\begin{aligned} A(X^h, Y^h, Z^h, U^h) &= \frac{1}{n+1} \left[\left(\overset{\circ}{D}_{X^v} \overset{\circ}{\mathbb{P}} \right) \odot \kappa \right] (Y^h, Z^h, U^h) \\ &\quad - \kappa(X^h) \left(\frac{1}{E} \mathcal{C}'_b + \frac{1}{2E} \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \odot d_J E \right) (Y^h, Z^h, U^h) \\ &\quad - \frac{1}{n+1} \left(\overset{\circ}{D}_{U^v} \overset{\circ}{\mathbb{P}} \odot \kappa \right) (Y^h, Z^h, X^h) \\ &\quad + \kappa(U^h) \left(\frac{1}{E} \mathcal{C}'_b + \frac{1}{2E} \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \odot d_J E \right) (Y^h, Z^h, X^h). \end{aligned}$$

Introducing the symmetric semibasic (0,3) tensor

$$(3) \quad \alpha := \frac{1}{E} \mathcal{C}'_b + \frac{1}{2E} \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \odot d_J E + \frac{1}{n+1} \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}},$$

and taking into account that $\overset{\circ}{D}_{X^v} \overset{\circ}{\mathbb{P}} = i_{X^h} \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}$ ($X \in \mathfrak{X}(M)$), where $\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}$ is also totally symmetric by Corollary 4.11, we finally obtain that

$$A = \alpha \otimes \kappa - \kappa \otimes \alpha.$$

Since the semibasic trace of A clearly vanishes, it follows that

$$\widetilde{\alpha \otimes \kappa} - \widetilde{\kappa \otimes \alpha} = 0.$$

Here

$$\begin{aligned} \widetilde{\alpha \otimes \kappa} &\stackrel{(2.11c)}{=} \widetilde{\kappa \alpha} \stackrel{(13.5b)}{=} (n-1)\alpha, \\ i_X \widetilde{\kappa \otimes \alpha} &\stackrel{(2.11d)}{=} i_{F\kappa(X)} \alpha \quad (X \in \mathfrak{X}(TM)), \end{aligned}$$

and therefore for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$(4) \quad (n-1)\alpha(X^h, Y^h, Z^h) - \alpha(F\kappa(X^h), Y^h, Z^h) = 0.$$

To complete the proof, we show that

$$(5) \quad \alpha(F\kappa(X^h), Y^h, Z^h) = \alpha(X^h, Y^h, Z^h).$$

We evaluate the right side of (3) on the triplet $(F\kappa(X^h), Y^h, Z^h)$ term by term.

$$\begin{aligned} (i) \quad & \frac{1}{E} C'_b(F\kappa(X^h), Y^h, Z^h) = \frac{1}{E} C'_b(X^h, Y^h, Z^h) - \frac{1}{E} C'_b\left(\frac{1}{2E}(X^v E)S, Y^h, Z^h\right) \\ & \stackrel{(11.1h)}{=} \frac{1}{E} C'_b(X^h, Y^h, Z^h); \\ (ii) \quad & \frac{1}{n+1} \frac{1}{2E} \left(\overset{\circ}{\mathbb{P}} \odot d_J E \right) (F\kappa(X^h), Y^h, Z^h) = \frac{1}{n+1} \frac{1}{2E} \left(\overset{\circ}{\mathbb{P}} \odot d_J E \right) (X^h, Y^h, Z^h) - \\ & \frac{1}{n+1} \frac{1}{4E^2} (X^v E) \left(\overset{\circ}{\mathbb{P}} \odot d_J E \right) (S, Y^h, Z^h) = \frac{1}{n+1} \frac{1}{2E} \left(\overset{\circ}{\mathbb{P}} \odot d_J E \right) (X^h, Y^h, Z^h) - \\ & \frac{1}{n+1} \frac{1}{4E^2} (X^v E) \overset{\circ}{\mathbb{P}}(Y^h, Z^h) C(E) = \frac{1}{n+1} \frac{1}{2E} \left(\overset{\circ}{\mathbb{P}} \odot d_J E \right) (X^h, Y^h, Z^h) - \\ & \frac{1}{n+1} \frac{1}{2E} (X^v E) \overset{\circ}{\mathbb{P}}(Y^h, Z^h), \text{ using the fact that } i_S \overset{\circ}{\mathbb{P}} := \widetilde{i_S \overset{\circ}{\mathbb{P}}} \stackrel{(4.10a)}{=} 0; \\ (iii) \quad & \frac{1}{n+1} \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}(F\kappa(X^h), Y^h, Z^h) = \frac{1}{n+1} \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h) - \\ & \frac{1}{n+1} \frac{1}{2E} (X^v E) \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}(S, Y^h, Z^h) = \frac{1}{n+1} \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h) - \\ & \frac{1}{n+1} \frac{1}{2E} (X^v E) \overset{\circ}{D}_C \overset{\circ}{\mathbb{P}}(Y^h, Z^h) = \frac{1}{n+1} \overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h) + \\ & \frac{1}{n+1} \frac{1}{2E} (X^v E) \overset{\circ}{\mathbb{P}}(Y^h, Z^h), \text{ since } \overset{\circ}{D}_C \overset{\circ}{\mathbb{P}} = -\overset{\circ}{\mathbb{P}} \text{ by (4.11a).} \end{aligned}$$

Adding the corresponding sides of (i)–(iii) we obtain the desired relation (5). (4) and (5) imply that

$$(n-2)\alpha = 0,$$

from which we see, by the assumption $n > 2$, that $\alpha = 0$. Owing to (13.11) and (3) this means that $\mathbb{D} = 0$, concluding the proof. $\square$

14. Landsberg manifolds with vanishing Douglas tensor

14.1 Lemma. *Suppose that (M, L) is a Landsberg manifold whose Douglas tensor vanishes. Then the mixed curvature of the Berwald connection can be represented in the form*

$$(14.1) \quad \overset{\circ}{\mathbb{P}} = \frac{1}{n+1} \overset{\circ}{\mathbb{P}} \odot \kappa.$$

Proof. This is an immediate consequence of (13.10b), the trivial implication $\mathbb{D} = 0 \Rightarrow \mathbb{D}^* = 0$, and the Proposition 12.1. $\square$

14.2 Proposition. *Let (M, L) be a Landsberg manifold with vanishing Douglas tensor. Then the semibasic trace of the mixed curvature of the Berwald connection can be expressed in the form*

$$(14.2) \quad \tilde{\mathbb{P}} = \frac{1}{n-1} \tilde{\mathbb{P}}^{\#} \kappa_b,$$

where $\tilde{\mathbb{P}}^{\#}$ is the $(1,1)$ tensor metrically equivalent to the $(0,2)$ tensor $\tilde{\mathbb{P}}$: for any vector fields $X, Y \in \mathfrak{X}(TM)$,

$$g(\tilde{\mathbb{P}}^{\#}(X), JY) = \tilde{\mathbb{P}}(X, Y).$$

Proof. Let $X, Y, Z, U \in \mathfrak{X}(M)$ be arbitrary vector fields. We set out from the fact that the tensor $\tilde{\mathbb{P}}_b$ defined by (12.5) is totally symmetric, whence

$$\begin{aligned} 0 &= \tilde{\mathbb{P}}_b(X^h, Y^h, Z^h, U^h) - \tilde{\mathbb{P}}_b(X^h, Y^h, U^h, Z^h) \stackrel{(12.5)}{=} g(\tilde{\mathbb{P}}(X^h, Y^h)Z^h, JU^h) - \\ &g(\tilde{\mathbb{P}}(X^h, Y^h)U^h, JZ^h) \stackrel{(14.1)}{=} \frac{1}{n+1} \left[g(\tilde{\mathbb{P}}(X^h, Y^h)\kappa_b(Z^h), U^v) + g(\tilde{\mathbb{P}}(Y^h, Z^h)\kappa_b(X^h), U^v) \right. \\ &+ g(\tilde{\mathbb{P}}(Z^h, X^h)\kappa_b(Y^h), U^v) - g(\tilde{\mathbb{P}}(X^h, Y^h)\kappa_b(U^h), Z^v) - g(\tilde{\mathbb{P}}(Y^h, U^h)\kappa_b(X^h), Z^v) - \\ &\left. g(\tilde{\mathbb{P}}(U^h, X^h)\kappa_b(Y^h), Z^v) \right] \stackrel{(13.5c)}{=} \frac{1}{n+1} \left[\tilde{\mathbb{P}}(X^h, Y^h)\kappa_b(Z^h, U^h) + \tilde{\mathbb{P}}(Y^h, Z^h)\kappa_b(X^h, U^h) \right. \\ &+ \tilde{\mathbb{P}}(Z^h, X^h)\kappa_b(Y^h, U^h) - \tilde{\mathbb{P}}(X^h, Y^h)\kappa_b(U^h, Z^h) - \tilde{\mathbb{P}}(Y^h, U^h)\kappa_b(X^h, Z^h) - \\ &\left. \tilde{\mathbb{P}}(U^h, X^h)\kappa_b(Y^h, Z^h) \right] = \frac{1}{n+1} \left[\tilde{\mathbb{P}}(Y^h, Z^h)\kappa_b(X^h, U^h) + \tilde{\mathbb{P}}(Z^h, X^h)\kappa_b(Y^h, U^h) - \right. \\ &\left. \tilde{\mathbb{P}}(Y^h, U^h)\kappa_b(X^h, Z^h) - \tilde{\mathbb{P}}(U^h, X^h)\kappa_b(Y^h, Z^h) \right], \text{ using the symmetry of } \kappa_b \text{ in the} \end{aligned}$$

last step. In view of the definition of the tensors $\tilde{\mathbb{P}}^{\#}$ and κ_b , we finally obtain the relation

$$\begin{aligned} &g(\tilde{\mathbb{P}}^{\#}(Z^h), Y^v)\kappa_b(X^h, U^h) + \tilde{\mathbb{P}}(Z^h, X^h)g(\kappa_b(U^h), Y^v) \\ &- g(\tilde{\mathbb{P}}^{\#}(U^h), Y^v)\kappa_b(X^h, Z^h) - \tilde{\mathbb{P}}(U^h, X^h)g(\kappa_b(Z^h), Y^v) = 0. \end{aligned}$$

Since the vector field $Y \in \mathfrak{X}(M)$ is arbitrary and the tensor g is nondegenerate, this implies

$$(1) \quad \begin{aligned} &\tilde{\mathbb{P}}^{\#}(Z^h)\kappa_b(X^h, U^h) + \tilde{\mathbb{P}}(Z^h, X^h)\kappa_b(U^h) - \tilde{\mathbb{P}}^{\#}(U^h)\kappa_b(X^h, Z^h) \\ &- \tilde{\mathbb{P}}(U^h, X^h)\kappa_b(Z^h) = 0. \end{aligned}$$

Now let us define the semibasic (1,3) tensors $\overline{B}$ and B as follows: for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}\overline{B}(X^h, Y^h, Z^h) &:= \overset{\circ}{\mathbb{P}}^{\#}(Y^h)\kappa_b(X^h, Z^h) + \overset{\circ}{\mathbb{P}}(Y^h, X^h)\kappa(Z^h) \\ &\quad - \overset{\circ}{\mathbb{P}}^{\#}(Z^h)\kappa_b(X^h, Y^h) - \overset{\circ}{\mathbb{P}}(Z^h, X^h)\kappa(Y^h); \\ B(X^h, Y^h, Z^h) &:= \overline{B}(Y^h, X^h, Z^h).\end{aligned}$$

From (1) it is obvious that $\overline{B} = B = 0$, whence $\widetilde{\overline{B}} = \widetilde{B} = 0$. Let us observe that

$$B = \overset{\circ}{\mathbb{P}}^{\#} \otimes \kappa_b + \overset{\circ}{\mathbb{P}} \otimes \kappa - \kappa_b \otimes \overset{\circ}{\mathbb{P}}^{\#} - \kappa \otimes \overset{\circ}{\mathbb{P}}.$$

Thus, applying (2.11c), (2.11d) and (13.5b), it follows that for any vector fields X, Y on M ,

$$\begin{aligned}(2) \quad \widetilde{B}(X^h, Y^h) &= \kappa_b \left(F \overset{\circ}{\mathbb{P}}^{\#}(X^h), Y^h \right) + (n-1) \overset{\circ}{\mathbb{P}}(X^h, Y^h) \\ &\quad - \widetilde{\overset{\circ}{\mathbb{P}}^{\#} \kappa_b}(X^h, Y^h) - \overset{\circ}{\mathbb{P}}(F\kappa(X^h), Y^h) = 0.\end{aligned}$$

In this formula

$$\begin{aligned}(i) \quad \kappa_b \left(F \overset{\circ}{\mathbb{P}}^{\#}(X^h), Y^h \right) &= g \left(\kappa F \overset{\circ}{\mathbb{P}}^{\#}(X^h), Y^v \right) \\ &= g \left(\overset{\circ}{\mathbb{P}}^{\#}(X^h), Y^v \right) - \frac{1}{2E} d_J E \left(F \overset{\circ}{\mathbb{P}}^{\#}(X^h) \right) g(C, Y^v) \\ &= \overset{\circ}{\mathbb{P}}(X^h, Y^h) - \frac{1}{2E} g \left(C, \overset{\circ}{\mathbb{P}}^{\#}(X^h) \right) g(C, Y^v) \\ &= \overset{\circ}{\mathbb{P}}(X^h, Y^h) - \frac{1}{2E} \overset{\circ}{\mathbb{P}}(X^h, S) g(C, Y^v) \stackrel{(4.10a)}{=} \overset{\circ}{\mathbb{P}}(X^h, Y^h);\end{aligned}$$

$$(ii) \quad \overset{\circ}{\mathbb{P}}(F\kappa(X^h), Y^h) = \overset{\circ}{\mathbb{P}}(X^h, Y^h) - \frac{1}{2E} (X^v E) \overset{\circ}{\mathbb{P}}(S, Y^h) = \overset{\circ}{\mathbb{P}}(X^h, Y^h),$$

whence

$$\overset{\circ}{\mathbb{P}}(X^h, Y^h) + (n-1) \overset{\circ}{\mathbb{P}}(X^h, Y^h) - \widetilde{\overset{\circ}{\mathbb{P}}^{\#} \kappa_b}(X^h, Y^h) - \overset{\circ}{\mathbb{P}}(X^h, Y^h) = 0,$$

which proves the relation (14.2). $\square$

14.3 Corollary. *If (M, L) is a Landsberg manifold with vanishing Douglas tensor, then the mixed curvature of the Berwald connection can be expressed as follows:*

$$(14.3) \quad \overset{\circ}{\mathbb{P}} = \frac{1}{n^2 - 1} \overset{\sim}{\overset{\circ}{\mathbb{P}}}^{\#} \kappa_b \odot \kappa.$$

Proof. Immediate from (14.1) and (14.2). $\square$

14.4 Corollary. *With the hypothesis as above, for any vector fields $X, Y, Z, U \in \mathfrak{X}(M)$, we have*

$$(14.4) \quad \overset{\sim}{\overset{\circ}{\mathbb{P}}}^{\#} [\mathcal{C}_b(X^h, Z^h, U^h)\kappa(Y^h) + \kappa_b(Z^h, U^h)\mathcal{C}(X^h, Y^h) - \mathcal{C}_b(X^h, Y^h, U^h)\kappa(Z^h) - \kappa_b(Y^h, U^h)\mathcal{C}(X^h, Z^h)] = 0.$$

Proof. Easy calculation, applying (12.4) and (14.3). $\square$

14.5 Theorem. *Suppose that (M, L) is an $n > 2$ dimensional Landsberg manifold. If the Douglas tensor of (M, L) vanishes, then (M, L) is a Berwald manifold.*

Proof. Let us introduce the semibasic $(1, 4)$ tensors $\overline{\beta}$ and β defined by

$$\begin{aligned} \overline{\beta}(X^h, Y^h, Z^h, U^h) &:= \text{left hand side of (14.4),} \\ \beta(X^h, Y^h, Z^h, U^h) &:= \overline{\beta}(Y^h, X^h, U^h, Z^h) \quad (X, Y, Z, U \in \mathfrak{X}(M)). \end{aligned}$$

Then $\overline{\beta}$, β and their semibasic traces vanish identically. It is immediate that for any vector fields $X, Y, Z, U \in \mathfrak{X}(M)$,

$$\begin{aligned} \beta(X^h, Y^h, Z^h, U^h) &= \overset{\sim}{\overset{\circ}{\mathbb{P}}}^{\#} (\kappa \otimes \mathcal{C}_b + \mathcal{C} \otimes \kappa_b - \mathcal{C}_b \otimes \kappa)(X^h, Y^h, Z^h, U^h) \\ &\quad - \overset{\sim}{\overset{\circ}{\mathbb{P}}}^{\#} (\kappa_b \otimes \mathcal{C})(X^h, Z^h, Y^h, U^h). \end{aligned}$$

Applying (2.11c–f) and (13.5b), the relation $\tilde{\beta} = 0$ yields

$$(1) \quad \overset{\sim}{\overset{\circ}{\mathbb{P}}}^{\#} [\mathcal{C}_b(F\kappa(X^h), Y^h, Z^h) + \kappa_b(F\mathcal{C}(X^h, Y^h), Z^h) - (n-1)\mathcal{C}_b(X^h, Y^h, Z^h) - \kappa_b(X^h, Z^h)\tilde{\mathcal{C}}(Y^h)] = 0.$$

In this formula

$$\begin{aligned} (i) \quad \mathcal{C}_b(F\kappa(X^h), Y^h, Z^h) &= \mathcal{C}_b(X^h, Y^h, Z^h) - \frac{1}{2E}(X^v E)\mathcal{C}_b(S, Y^h, Z^h) \\ &= \mathcal{C}_b(X^h, Y^h, Z^h), \end{aligned}$$

$$(ii) \quad \kappa_b(F\mathcal{C}(X^h, Y^h), Z^h) \stackrel{(13.7)}{=} \mathcal{C}_b(X^h, Y^h, Z^h),$$

whence

$$\widetilde{\overset{\circ}{\mathbb{P}}^\#} \left[(n-3)\mathcal{C}_b(X^h, Y^h, Z^h) + \kappa_b(X^h, Z^h)\tilde{\mathcal{C}}(Y^h) \right] = 0.$$

This can also be written in the form

$$\widetilde{\overset{\circ}{\mathbb{P}}^\#} \left[(n-3)g(\mathcal{C}(X^h, Y^h), Z^v) + g(\tilde{\mathcal{C}}(Y^h)\kappa(X^h), Z^v) \right] = 0.$$

This implies

$$(2) \quad \widetilde{\overset{\circ}{\mathbb{P}}^\#} \left[(n-3)\mathcal{C}(X^h, Y^h) + \kappa(X^h)\tilde{\mathcal{C}}(Y^h) \right] = 0.$$

Finally let us consider the semibasic $(1, 2)$ tensor

$$\gamma := \widetilde{\overset{\circ}{\mathbb{P}}^\#} \left[(n-3)\mathcal{C} + \kappa \otimes \tilde{\mathcal{C}} \right].$$

In view of (2) γ and, at the same time, $\tilde{\gamma}$ vanish identically. Applying (2.11d) it follows that for any vector field $X \in \mathfrak{X}(M)$,

$$0 = \tilde{\gamma}(X^h) = \widetilde{\overset{\circ}{\mathbb{P}}^\#} \left[(n-3)\tilde{\mathcal{C}}(X^h) + \tilde{\mathcal{C}}(F\kappa(X^h)) \right].$$

In this formula

$$\begin{aligned} \tilde{\mathcal{C}}(F\kappa(X^h)) &= \tilde{\mathcal{C}}(X^h) - \frac{1}{2E}(X^v E)\tilde{\mathcal{C}}(S) \\ &= \tilde{\mathcal{C}}(X^h) - \frac{1}{2E}(X^v E)\widetilde{i_S \mathcal{C}} \stackrel{(11.1h)}{=} \tilde{\mathcal{C}}(X^h), \end{aligned}$$

so we obtain the relation

$$(n-2)\widetilde{\overset{\circ}{\mathbb{P}}^\#} \tilde{\mathcal{C}} = 0.$$

Since $n > 2$, from this it follows that $\widetilde{\overset{\circ}{\mathbb{P}}^\#} = 0$ or $\tilde{\mathcal{C}} = 0$. In the first case (14.3) implies that $\overset{\circ}{\mathbb{P}} = 0$. Then, according to Corollary 12.9, (M, L) is a Berwald manifold. In the second case taking the semibasic trace of both sides of the relation $\overset{\circ}{\mathbb{P}} = -D_h \mathcal{C}$ (12.1 (iv)) we obtain that $0 = D_h \tilde{\mathcal{C}} = -\overset{\circ}{\mathbb{P}}$, and then (14.1) yields the desired conclusion $\overset{\circ}{\mathbb{P}} = 0$. $\square$

IV. APPLICATIONS TO TWO-DIMENSIONAL FINSLER MANIFOLDS

15. Basic constructions

15.1 The Berwald frame. Throughout this chapter, let (M, L) be a two-dimensional Finsler manifold and suppose the Riemann-Finsler metric g arising from L is positive definite. Starting from the Liouville vector field C and the canonical spray S of (M, L) , let us first consider the unit vector fields

$$C_0 := \frac{1}{L}C \quad \text{and} \quad S_0 := \frac{1}{L}S.$$

Next, using the Gram-Schmidt process we construct, at least locally, a g -orthonormal basis (C_0, X_0) of $\mathfrak{X}^v(TM)$. Applying the almost complex structure F associated to the Barthel endomorphism of (M, L) , we obtain a local g -orthonormal basis (FX_0, S_0) of $\mathfrak{X}^h(TM)$. The quadruple

$$(C_0, X_0, FX_0, S_0)$$

constructed in this way is a (local) orthonormal basis of $\mathfrak{X}(TM)$; it is called the *Berwald frame* of the Finsler manifold (M, L) .

Note that

$$(15.1) \quad X_0 E = 0.$$

This can be seen by a straightforward calculation:

$$\begin{aligned} 0 &= g(C, X_0) \stackrel{(11.1a)}{=} \omega(C, FX_0) \stackrel{(8.6(3))}{=} d_J E(FX_0) \\ &= dE[JF(X_0)] \stackrel{(2.6c)}{=} dE[v(X_0)] = dE(X_0) = X_0 E. \end{aligned}$$

15.2 Proposition. *The members of the Berwald frame have the following homogeneity properties:*

$$(15.2a) \quad [C, C_0] = -C_0,$$

$$(15.2b) \quad [C, S_0] = 0,$$

$$(15.2c) \quad [C, X_0] = -X_0,$$

$$(15.2d) \quad [C, FX_0] = 0.$$

Proof. Taking into account the fact that the function $\frac{1}{L}$ is homogeneous of degree -1 , i.e., that

$$(1) \quad C \left(\frac{1}{L} \right) = -\frac{1}{L},$$

we obtain

$$[C, C_0] = [C, \frac{1}{L}C] = C \left(\frac{1}{L} \right) C = -\frac{1}{L}C = -C_0,$$

which shows (15.2a). Similarly, since S is a spray and hence $[C, S] = S$,

$$[C, S_0] = \frac{1}{L}[C, S] + C \left(\frac{1}{L} \right) S = \frac{1}{L}S + C \left(\frac{1}{L} \right) S \stackrel{(1)}{=} 0,$$

so (15.2b) is also true.

In view of the definition (11.1b) and the property (11.1h) of the first Cartan tensor $\mathcal{C}$,

$$\begin{aligned} 0 &= 2g(\mathcal{C}(S, FX_0), X_0) = \mathcal{L}_C(J^*g)(FX_0, FX_0) \\ &= Cg(X_0, X_0) - 2g(J[C, FX_0], X_0) \stackrel{15.1}{=} -2g(J[C, FX_0], X_0). \end{aligned}$$

In the same way we find that

$$\begin{aligned} 0 &= 2g(\mathcal{C}(S, FX_0), C) = Cg(X_0, C) - g(J[C, FX_0], C) - g(X_0, J[C, S]) \\ &\stackrel{15.1}{=} -g(J[C, FX_0], C) - g(X_0, C) \stackrel{15.1}{=} -g(J[C, FX_0], C). \end{aligned}$$

It follows from the last two relations that $J[C, FX_0] = 0$, and so $[C, FX_0]$ is vertical. Thus,

$$X_0 = vX_0 \stackrel{(2.6c)}{=} JF(X_0) \stackrel{(1.5b)}{=} [J, C](FX_0) = [X_0, C] - J[FX_0, C] = -[C, X_0]$$

which proves the relation (15.2c).

Taking into account again the fact that the vector field $[FX_0, C]$ is vertical, from the homogeneity of the Barthel endomorphism (8.6(6)) we obtain

$$\begin{aligned} 0 &= [h, C](FX_0) = [h(FX_0), C] - h[FX_0, C] \\ &= [h(FX_0), C] \stackrel{(2.6c)}{=} [F(vX_0), C] = [FX_0, C] \end{aligned}$$

whence (15.2d). □

15.3 Proposition. *With the hypothesis above, the following relations hold:*

$$(15.3a) \quad C'(FX_0, FX_0) = v[X_0, FX_0] \quad (v := 1 - h),$$

$$(15.3b) \quad v[X_0, S] = 0,$$

$$(15.3c) \quad \text{for all } Y \in \mathfrak{X}(TM) : D_{hY}X_0 = 0,$$

where D is the Cartan connection of (M, L) .

Proof. According to the definition of the second Cartan tensor C' (11.1d),

$$\begin{aligned}
2g(\mathcal{C}'(FX_0, FX_0), X_0) &= (\mathcal{L}_{h(FX_0)}g)(J(FX_0), J(FX_0)) \stackrel{(2.6c)}{=} (\mathcal{L}_{FX_0}g)(X_0, X_0) = \\
&= FX_0g(X_0, X_0) - 2g([FX_0, X_0], X_0) \stackrel{15.1}{=} 2g([X_0, FX_0], X_0) = 2g(v[X_0, FX_0], X_0) + \\
&+ 2g(h[X_0, FX_0], X_0) = 2g(v[X_0, FX_0], X_0),
\end{aligned}$$

using the g -orthogonality of the vertical and horizontal subbundle in the last step. From this we obtain the relation

$$(1) \quad g(\mathcal{C}'(FX_0, FX_0) - v[X_0, FX_0], X_0) = 0.$$

Furthermore, by the properties (11.1g), (11.1h) of the second Cartan tensor, we can write

$$\begin{aligned}
0 &= 2g(\mathcal{C}'(FX_0, FX_0), C) = FX_0g(X_0, C) - g([FX_0, X_0], C) - g(X_0, [FX_0, C]) \\
&\stackrel{15.1, (15.2d)}{=} g([X_0, FX_0], C) = g(v[X_0, FX_0], C) + g(h[X_0, FX_0], C) \\
&= g(v[X_0, FX_0], C).
\end{aligned}$$

This means that the equality

$$(2) \quad g(\mathcal{C}'(FX_0, FX_0) - v[X_0, FX_0], C) = 0$$

is valid automatically. The relations (1) and (2) show that the vertical vector field $\mathcal{C}'(FX_0, FX_0) - v[X_0, FX_0]$ is orthogonal to two nowhere vanishing vertical vector fields. Hence it must be the zero vector field, which proves (15.3a).

Similarly, on the one hand

$$\begin{aligned}
0 &= 2g(\mathcal{C}'(S, FX_0), X_0) = Sg(X_0, X_0) - 2g([S, X_0], X_0) = 2g([X_0, S], X_0) \\
&= 2g(v[X_0, S], X_0) + 2g(h[X_0, S], X_0) = 2g(v[X_0, S], X_0)
\end{aligned}$$

on the other hand

$$\begin{aligned}
0 &= 2g(\mathcal{C}'(S, FX_0), C) = Sg(X_0, C) - g([S, X_0], C) - g(X_0, [S, C]) = g([X_0, S], C) \\
&+ g(X_0, S) = g([X_0, S], C) = g(v[X_0, S], C) + g(h[X_0, S], C) = g(v[X_0, S], C),
\end{aligned}$$

which imply the relation (15.3b).

To prove the formula (15.3c) it is sufficient to check that $D_{FX_0}X_0$ and D_SX_0 vanish. But this is immediate:

$$\begin{aligned}
D_{FX_0}X_0 &\stackrel{(11.2)}{=} v[FX_0, X_0] + \mathcal{C}'(FX_0, FX_0) \stackrel{(15.3a)}{=} 0, \\
D_SX_0 &\stackrel{(11.2)}{=} v[S, X_0] \stackrel{(15.3b)}{=} 0.
\end{aligned}$$

□

15.4 Definition and remark. *The function*

$$(15.4) \quad \lambda := g(\mathcal{C}(FX_0, FX_0), X_0)$$

is said to be the main scalar of (M, L) with respect to the Berwald frame (C_0, X_0, FX_0, S_0) . – λ depends only on the choice of X_0 and specifying X_0 , it is uniquely determined up to sign.

15.5 Lemma. *With the help of the main scalar and the vector field X_0 , the first Cartan tensors can be represented in the form*

$$(15.5) \quad \mathcal{C} = \lambda i_{X_0} \omega \otimes i_{X_0} \omega \otimes X_0 \quad \text{and} \quad \mathcal{C}_b = \lambda i_{X_0} \omega \otimes i_{X_0} \omega \otimes i_{X_0} \omega,$$

where ω is the fundamental two-form.

Proof. The vertical vector field $\mathcal{C}(FX_0, FX_0)$ can be uniquely represented as a linear combination

$$\mathcal{C}(FX_0, FX_0) = \lambda_1 X_0 + \lambda_2 C_0, \quad \lambda_1, \lambda_2 \in C^\infty(TM).$$

Since on the one hand

$$g(\mathcal{C}(FX_0, FX_0), C_0) = \frac{1}{L} \mathcal{C}_b(FX_0, FX_0, S) \stackrel{(11.1h)}{=} 0,$$

on the other hand

$$g(\mathcal{C}(FX_0, FX_0), C_0) = g(\lambda_1 X_0 + \lambda_2 C_0, C_0) = \lambda_2,$$

it follows that $\lambda_2 = 0$. So

$$\lambda := g(\mathcal{C}(FX_0, FX_0), X_0) = g(\lambda_1 X_0, X_0) = \lambda_1,$$

hence $\mathcal{C}(FX_0, FX_0) = \lambda X_0$, where λ is the main scalar. From this observation we infer immediately (15.5). $\square$

15.6 Corollary. *A positive definite two-dimensional Finsler manifold is a Riemann-manifold if and only if its main scalar vanishes.*

Proof. This is an immediate consequence of (15.5), because (M, L) is a Riemannian manifold if and only if $\mathcal{C} = 0$. $\square$

15.7 Proposition. *A positive definite two-dimensional Finsler manifold is a Berwald manifold if and only if the horizontal vector fields annihilate the main scalar, i.e., $d_h \lambda = 0$.*

Proof. We first recall that a Finsler manifold is a Berwald manifold if and only if the h -covariant derivative, with respect to the Cartan connection, of the first Cartan tensor vanishes. (An intrinsic proof of this well-known fact is available in [55].) Thus we can argue as follows:

$$\begin{aligned} (M, L) \text{ is a Berwald manifold} &\Leftrightarrow \forall Y \in \mathfrak{X}(TM) : D_{hY} \mathcal{C} = 0 \\ &\Leftrightarrow \forall Y \in \mathfrak{X}(TM) : 0 = (D_{hY} \mathcal{C})(FX_0, FX_0) = D_{hY}[\mathcal{C}(FX_0, FX_0)] \\ &\quad - 2\mathcal{C}(D_{hY} FX_0, FX_0) \stackrel{(\text{FINS2})/[50]}{=} D_{hY}[\mathcal{C}(FX_0, FX_0)] - 2\mathcal{C}(FD_{hY} X_0, FX_0) \stackrel{(15.3c)}{=} \\ &\quad D_{hY}[\mathcal{C}(FX_0, FX_0)] \stackrel{(15.5)}{=} D_{hY}(\lambda X_0) = [(hY)\lambda]X_0 + \lambda D_{hY} X_0 \stackrel{(15.3c)}{=} [(hY)\lambda]X_0 \\ &= [d_h \lambda(Y)]X_0 \Leftrightarrow d_h \lambda = 0. \end{aligned} \quad \square$$

15.8 Corollary. *The second Cartan tensor of (M, L) is completely determined by the formula*

$$(15.8) \quad \mathcal{C}'(FX_0, FX_0) = -S(\lambda)X_0.$$

Proof. Since $D_S \mathcal{C} = -\mathcal{C}'$ (see [26], Prop. (A.12)), taking into account the previous proof we obtain

$$\begin{aligned} \mathcal{C}'(FX_0, FX_0) &= -(D_S \mathcal{C})(FX_0, FX_0) = -[d_h \lambda(S)]X_0 \\ &= -[d\lambda(hS)]X_0 = [-(d\lambda)S]X_0 = -S(\lambda)X_0. \end{aligned} \quad \square$$

15.9 Proposition and definition. *Let us consider the curvature tensor $R = -\frac{1}{2}[h, h]$ of (M, L) . Then*

$$(15.9a) \quad R(FX_0, S_0) = g(R(FX_0, S_0), X_0)X_0,$$

and R is uniquely determined by this formula on the domain of the Berwald frame constructed in 15.1. The function

$$(15.9b) \quad \kappa := g(R(FX_0, S_0), X_0)$$

is said to be the Gauss curvature of (M, L) .

Proof. Since R is a semibasic tensor of type $(1, 2)$, it is uniquely determined by its value on the local basis (FX_0, S_0) of $\mathfrak{X}^h(TM)$. So our only task is to verify the equality (15.9a). Starting with the definition of the Nijenhuis torsion, we obtain

$$\begin{aligned} R(FX_0, S_0) &= -([hFX_0, hS_0] + h^2[FX_0, S_0] - h[FX_0, hS_0] - h[hFX_0, S_0]) = \\ &= -[FX_0, S_0] - h[FX_0, S_0] + 2h[FX_0, S_0] = -v[FX_0, S_0]. \end{aligned}$$

Now, if

$$R(FX_0, S_0) = f_1X_0 + f_2C_0 \quad (f_1, f_2 \in C^\infty(TM)),$$

then on the one hand

$$g(R(FX_0, S_0), C) = g(f_1X_0 + f_2C_0, C) = f_2g(C_0, C) = \frac{1}{L}f_2g(C, C) = Lf_2$$

on the other hand, using repeatedly the fact that $d_h E = 0$,

$$\begin{aligned} g(R(FX_0, S_0), C) &= -g(v[FX_0, S_0], C) = -\omega(v[FX_0, S_0], FC) = -\omega(v[FX_0, S_0], S) \\ &= i_S \omega(v[FX_0, S_0]) \stackrel{(8.6(5))}{=} -v[FX_0, S_0](E) \\ &= h[FX_0, S_0](E) - [FX_0, S_0](E) = [S_0, FX_0](E) \\ &= S_0[FX_0(E)] - FX_0[S_0(E)] = 0. \end{aligned}$$

These imply that $f_2 = 0$, $R(FX_0, S_0) = f_1X_0$, and

$$f_1 = g(f_1X_0, X_0) = g(R(FX_0, S_0), X_0)$$

whence (15.9a). $\square$

15.10 Theorem (E. CARTANS's "permutation formulas"). *For the Lie brackets of the members of the Berwald frame we have*

$$(15.10a-c) \quad \boxed{\begin{aligned} [X_0, FX_0] &= -\frac{1}{L}S_0 - \lambda FX_0 - S(\lambda)X_0 \\ [S_0, X_0] &= -\frac{1}{L}FX_0 \\ [FX_0, S_0] &= -\kappa X_0 \end{aligned}},$$

where λ is the main scalar, κ is the Gauss curvature of (M, L) .

Proof. Since

$$[X_0, FX_0] = v[X_0, FX_0] + h[X_0, FX_0] \stackrel{(15.3a), (15.8)}{=} -S(\lambda)X_0 + h[X_0, FX_0],$$

to prove the relation (15.10a) it remains to be shown that

$$(1) \quad h[X_0, FX_0] = -\frac{1}{2E}S_0 - \lambda FX_0.$$

First we observe that

$$\begin{aligned} 2\lambda &\stackrel{(15.4)}{=} 2g(\mathcal{C}(FX_0, FX_0), X_0) = X_0g(X_0, X_0) \\ &\quad - 2g(J[X_0, FX_0], X_0) = -2g(J[X_0, FX_0], X_0) \end{aligned}$$

whence

$$(2) \quad g(J[X_0, FX_0], X_0) = -\lambda.$$

Similarly,

$$\begin{aligned} 0 &= 2g(\mathcal{C}(FX_0, FX_0), C) = X_0g(X_0, C) - g(J[X_0, FX_0], C) - g(X_0, J[X_0, S]) \\ &= -g(J[X_0, FX_0], C) - g(X_0, J[X_0, S]) \stackrel{(2.7)}{=} -g(J[X_0, FX_0], C) - g(X_0, X_0). \end{aligned}$$

Hence it follows that

$$(3) \quad g(J[X_0, FX_0], C_0) = -\frac{1}{L}.$$

Now we use orthonormal expansion to express $J[X_0, FX_0]$ in terms of the local basis (X_0, C_0) of $\mathfrak{X}^v(TM)$:

$$\begin{aligned} J[X_0, FX_0] &= g(J[X_0, FX_0], X_0)X_0 + g(J[X_0, FX_0], C_0)C_0 \\ &\stackrel{(2), (3)}{=} -\lambda X_0 - \frac{1}{L}C_0. \end{aligned}$$

In view of the identity $F \circ J = h$, from this we obtain (1) and hence (15.10a).

For the second formula (15.10b) we have

$$\begin{aligned} [S_0, X_0] &= h[S_0, X_0] + v[S_0, X_0] \stackrel{(15.3b)}{=} h[S_0, X_0] = -FJ[X_0, \frac{1}{L}S] \\ &\stackrel{(15.1)}{=} -\frac{1}{L}FJ[X_0, S_0] \stackrel{(2.7)}{=} -\frac{1}{L}FX_0. \end{aligned}$$

To prove (15.10c), first it will be shown that the vector field $[FX_0, S_0]$ is vertical, i.e.,

$$(4) \quad h[FX_0, S_0] = 0.$$

The vanishing of the h -horizontal torsion of the Cartan connection (11.3) yields

$$h[FX_0, S_0] = D_{FX_0}S_0 - D_{S_0}FX_0 = D_{FX_0}S_0 - FD_{S_0}X_0 \stackrel{(15.3c)}{=} D_{FX_0}S_0.$$

Using the fact that $d_h E = 0$ and the vanishing of the h -deflection of the Cartan connection ([50], (M4)) we obtain

$$\begin{aligned} D_{FX_0}S_0 &= D_{FX_0} \left(\frac{1}{L}S \right) = FX_0 \left(\frac{1}{L} \right) S + \frac{1}{L} D_{FX_0}S \\ &= \frac{1}{L} D_{FX_0}FC = \frac{1}{L} F D_{FX_0}C = 0, \end{aligned}$$

thus (4) is true. By this observation and taking into account (15.9) it follows that

$$[FX_0, S_0] = v[FX_0, S_0] = -R(FX_0, S_0) = -\kappa X_0,$$

completing the proof. $\square$

15.11 Proposition (“Bianchi identity”).

$$(15.11) \quad \boxed{\lambda\kappa + X_0(\kappa) + S_0(S\lambda) = 0}.$$

Proof. Starting with the Jacobi identity, taking into account that both FX_0 and S_0 are horizontal vector fields and $d_h E = 0$, finally using (15.10a-c) we obtain

$$\begin{aligned} 0 &= [X_0, [S_0, FX_0]] + [FX_0, [X_0, S_0]] + [S_0, [FX_0, X_0]] \stackrel{(15.10a-c)}{=} [X_0, \kappa X_0] + \\ &[FX_0, \frac{1}{L}FX_0] + [S_0, \frac{1}{L}S_0 + \lambda FX_0 + S(\lambda)X_0] = (X_0\kappa)X_0 + \lambda[S_0, FX_0] + \\ &S_0(\lambda)FX_0 + S(\lambda)[S_0, X_0] + S_0(S\lambda)X_0 \stackrel{(15.10b-c)}{=} (X_0\kappa)X_0 + \lambda\kappa X_0 + S_0(\lambda)FX_0 - \\ &\frac{1}{L}S(\lambda)FX_0 + S_0(S\lambda)X_0 = [X_0(\kappa) + \lambda\kappa + S_0(S\lambda)]X_0 \end{aligned}$$

whence (15.11). $\square$

16. Two-dimensional Landsberg manifolds with vanishing Douglas tensor

16.1 Proposition. *A positive definite Finsler manifold of dimension two is a Landsberg manifold if and only if the main scalar is a first integral of the canonical spray, i.e.,*

$$(16.1) \quad S(\lambda) = 0.$$

Proof. This is an immediate consequence of (15.8) and 12.1. $\square$

16.2 Lemma. *Suppose that (M, L) is a (positive definite, two-dimensional) Landsberg manifold. Then the mixed curvature and the mixed Ricci tensor of the Berwald connection are completely determined by the formulas*

$$(16.2a) \quad \overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0 = -FX_0(\lambda)X_0,$$

$$(16.2b) \quad \overset{\circ}{\mathbb{P}}(FX_0, FX_0) = -FX_0(\lambda),$$

where λ is the main scalar of (M, L) .

Proof. (16.2b) is a trivial consequence of (16.2a). To prove (16.2a), we first recall that the relation

$$\overset{\circ}{\mathbb{P}}(X, Y)Z = -(D_{hX}\mathcal{C})(Y, Z) \quad (X, Y, Z \in \mathfrak{X}(\mathcal{T}M))$$

holds in any Landsberg manifold (see e.g. (12.1)/(iv)). Thus, in our case

$$\begin{aligned} \overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0 &= -(D_{FX_0}\mathcal{C})(FX_0, FX_0) = -D_{FX_0}\mathcal{C}(FX_0, FX_0) \\ &\quad + 2\mathcal{C}(D_{FX_0}FX_0, FX_0) \stackrel{(15.5)}{=} -D_{FX_0}\lambda X_0 \\ &\quad + 2\mathcal{C}(FD_{FX_0}X_0, FX_0) \stackrel{(15.3c)}{=} -FX_0(\lambda)X_0 \end{aligned}$$

whence (16.2a). $\square$

16.3 Proposition. *The Douglas tensor of a positive definite two-dimensional Landsberg manifold is completely determined by*

$$(16.3) \quad \boxed{\mathbb{D}(FX_0, FX_0)FX_0 = \frac{1}{3}[X_0(FX_0(\lambda) + 2\lambda FX_0(\lambda))C]}.$$

Proof. As we know from (6.4) and (6.5), $\mathbb{D}$ is semibasic, symmetric, and for any semispray S_0 , $i_{S_0}\mathbb{D} = 0$. Thus, in two dimensions $\mathbb{D}$ is completely determined by its value on the triplet (FX_0, FX_0, FX_0) . According to 6.4/(b),

$$\begin{aligned}
\mathbb{D}(FX_0, FX_0)FX_0 &= \overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0 - \frac{1}{3} \left(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}} \right) (FX_0, FX_0, FX_0)C - \\
&\overset{\circ}{\mathbb{P}}(FX_0, FX_0)X_0 \stackrel{(16.2a)}{=} \stackrel{(16.2b)}{=} -\frac{1}{3} \left(\overset{\circ}{D}_{X_0} \overset{\circ}{\mathbb{P}} \right) (FX_0, FX_0)C = \\
&-\frac{1}{3} \left[X_0 \left(\overset{\circ}{\mathbb{P}}(FX_0, FX_0) \right) - 2 \overset{\circ}{\mathbb{P}} \left(\overset{\circ}{D}_{X_0} FX_0, FX_0 \right) \right] C \stackrel{(16.2b)}{=} \\
&\frac{1}{3} \left[X_0(FX_0(\lambda)) + 2 \overset{\circ}{\mathbb{P}} \left(\overset{\circ}{D}_{X_0} FX_0, FX_0 \right) \right] C.
\end{aligned}$$

There remains only to calculate the second member of the right-hand side. Applying the rules of calculation of the Berwald connection (3.4) we obtain

$$\begin{aligned}
\overset{\circ}{D}_{X_0} FX_0 &= F \overset{\circ}{D}_{JFX_0} JFX_0 = FJ[X_0, FX_0] = h[X_0, FX_0] \\
&\stackrel{(15.10a)}{=} -\frac{1}{L} S_0 - \lambda(h \circ F)X_0 - S(\lambda)hX_0 = -\frac{1}{L} S_0 - \lambda FX_0.
\end{aligned}$$

Hence

$$\begin{aligned}
\overset{\circ}{\mathbb{P}} \left(\overset{\circ}{D}_{X_0} FX_0, FX_0 \right) &= -\frac{1}{2E} \overset{\circ}{\mathbb{P}}(S, FX_0) - \lambda \overset{\circ}{\mathbb{P}}(FX_0, FX_0) \\
&\stackrel{(4.10a)}{=} -\lambda \overset{\circ}{\mathbb{P}}(FX_0, FX_0) \stackrel{(16.2b)}{=} \lambda FX_0(\lambda),
\end{aligned}$$

and the result follows. $\square$

16.4 Remark. Using the same technique, it may be proved that the effect of the tensors $\overset{\circ}{\mathbb{P}}$, $\overset{\circ}{\mathbb{P}}$ and $\mathbb{D}$ can be described analogously in *any* (positive definite, two-dimensional) Finsler manifold. More precisely, the following relations are fulfilled in the general case:

$$(16.4a) \quad \overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0 = -[FX_0(\lambda) + X_0(S\lambda)]X_0 + 2\frac{S(\lambda)}{L}C_0,$$

$$(16.4b) \quad \overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0 = -FX_0(\lambda) - X_0(S\lambda),$$

$$\begin{aligned}
(16.4c) \quad \mathbb{D}(FX_0, FX_0)FX_0 &= \frac{1}{3} \left[X_0(FX_0(\lambda)) + X_0(X_0(S\lambda)) \right. \\
&\quad \left. + 2\lambda FX_0(\lambda) + 2\lambda X_0(S\lambda) + 3\frac{S(\lambda)}{E} \right] C.
\end{aligned}$$

16.5 Lemma. *Suppose that (M, L) is a positive definite two-dimensional Landsberg manifold with vanishing Douglas tensor. The iterated Lie derivatives of the main scalar with respect to the vector fields X_0, FX_0, S_0 (up to fifth order) can be expressed as follows:*

$$(16.5a) \quad \mathcal{L}_{FX_0} \mathcal{L}_{X_0} \lambda = -\lambda \mathcal{L}_{FX_0} \lambda,$$

$$(16.5b) \quad \mathcal{L}_{S_0} \mathcal{L}_{X_0} \lambda = -\frac{1}{L} \mathcal{L}_{FX_0} \lambda,$$

$$(16.5c) \quad \mathcal{L}_{FX_0} \mathcal{L}_{X_0}^2 \lambda = \left(\lambda^2 - \mathcal{L}_{X_0} \lambda - \frac{1}{2E} \right) \mathcal{L}_{FX_0} \lambda,$$

$$(16.5d) \quad \mathcal{L}_{S_0} \mathcal{L}_{X_0}^2 \lambda = \frac{3}{L} \lambda \mathcal{L}_{FX_0} \lambda,$$

$$(16.5e) \quad \mathcal{L}_{FX_0} \mathcal{L}_{X_0}^3 \lambda = \left(-\lambda^3 + 3\lambda \mathcal{L}_{X_0} \lambda - \mathcal{L}_{X_0}^2 \lambda + \frac{2}{E} \lambda \right) \mathcal{L}_{FX_0} \lambda,$$

$$(16.5f) \quad \mathcal{L}_{S_0} \mathcal{L}_{X_0}^3 \lambda = \frac{1}{L} \left(4\mathcal{L}_{X_0} \lambda - 7\lambda^2 + \frac{1}{2E} \right) \mathcal{L}_{FX_0} \lambda,$$

$$(16.5g) \quad \mathcal{L}_{FX_0} \mathcal{L}_{X_0}^4 \lambda = \left(-\mathcal{L}_{X_0}^3 \lambda + 4\lambda \mathcal{L}_{X_0}^2 \lambda + 3(\mathcal{L}_{X_0} \lambda)^2 - 6\lambda^2 \mathcal{L}_{X_0} \lambda \right. \\ \left. + \frac{4}{E} \mathcal{L}_{X_0} \lambda - \frac{11}{2E} \lambda^2 + \lambda^4 + \frac{1}{(2E)^2} \right) \mathcal{L}_{FX_0} \lambda$$

$$(\mathcal{L}_{X_0}^n := \mathcal{L}_{X_0} \circ \dots \circ \mathcal{L}_{X_0} \text{ (n times)}).$$

Proof. We shall verify only the first three formulas, the remaining ones can be handled in the same way. First we observe that the vanishing of the Douglas tensor implies by (16.3) the relation

$$(16.5i) \quad \boxed{X_0(FX_0(\lambda)) = -2\lambda FX_0(\lambda)}.$$

From now on we calculate.

$$(a) \quad \mathcal{L}_{FX_0} \mathcal{L}_{X_0} \lambda = [FX_0, X_0] \lambda + \mathcal{L}_{X_0} \mathcal{L}_{FX_0} \lambda \stackrel{(16.5i)}{=} [FX_0, X_0] \lambda - 2\lambda FX_0(\lambda) \\ \stackrel{(15.10a), (16.1)}{=} \frac{1}{L} (S_0 \lambda) + \lambda (FX_0) \lambda - 2\lambda (FX_0) \lambda \stackrel{(16.1)}{=} -\lambda (FX_0) \lambda = -\lambda \mathcal{L}_{FX_0} \lambda,$$

thus (16.5a) is proved.

$$(b) \quad \mathcal{L}_{S_0} \mathcal{L}_{X_0} \lambda = [S_0, X_0] \lambda + X_0(S_0 \lambda) \stackrel{(15.10b), (16.1)}{=} -\frac{1}{L} (FX_0) \lambda = -\frac{1}{L} \mathcal{L}_{FX_0} \lambda,$$

so we have obtained (15.5b).

$$(c) \quad \mathcal{L}_{FX_0} \mathcal{L}_{X_0}^2 \lambda = \mathcal{L}_{FX_0} \mathcal{L}_{X_0} (X_0 \lambda) = [FX_0, X_0] (X_0 \lambda) + X_0 [FX_0 (X_0 \lambda)] \\ \stackrel{(15.10a), (16.1)}{=} \frac{1}{L} S_0 (X_0 \lambda) + \lambda (FX_0) (X_0 \lambda) + X_0 [FX_0 (X_0 \lambda)] \stackrel{(16.5b), (16.5a)}{=} \\ -\frac{1}{2E} \mathcal{L}_{FX_0} \lambda - \lambda^2 \mathcal{L}_{FX_0} \lambda + X_0 [FX_0 (X_0 \lambda)] \stackrel{(16.5a)}{=} \left(-\frac{1}{2E} - \lambda^2 \right) \mathcal{L}_{FX_0} \lambda \\ + X_0 (-\lambda (FX_0) \lambda) = \left(-\frac{1}{2E} - \lambda^2 \right) \mathcal{L}_{FX_0} \lambda - (X_0 \lambda) \mathcal{L}_{FX_0} \lambda - \lambda \mathcal{L}_{X_0} \mathcal{L}_{FX_0} \lambda \stackrel{(16.5i)}{=} \\ \left(-\frac{1}{2E} - \lambda^2 - \mathcal{L}_{X_0} \lambda \right) \mathcal{L}_{FX_0} \lambda + 2\lambda^2 \mathcal{L}_{FX_0} \lambda = \left(\lambda^2 - \mathcal{L}_{X_0} \lambda - \frac{1}{2E} \right) \mathcal{L}_{FX_0} \lambda,$$

showing that (16.5c) is also valid. $\square$

16.6 Theorem. *If a positive definite two-dimensional Landsberg manifold has a vanishing Douglas tensor, then it is a Berwald manifold.*

Proof.

(A) In the next, quite tedious calculations our aim is to show that

$$(1) \quad \mathcal{L}_{FX_0}\lambda = 0.$$

Then on the one hand

$$0 = (FX_0)\lambda = [(F \circ v)X_0]\lambda = [h(FX_0)]\lambda = (d_h\lambda)(FX_0)$$

on the other hand

$$(d_h\lambda)(S) = (d\lambda)(hS) = (d\lambda)S = S(\lambda) \stackrel{(16.1)}{=} 0,$$

so it follows that $d_h\lambda = 0$ and, in view of Proposition 15.7, (M, L) is a Berwald manifold.

Notice that our subsequent reasoning relies heavily on the fact that

$$(2) \quad \lambda\kappa = -X_0(\kappa).$$

This relation is an immediate consequence of the Bianchi identity (15.11) and the property (16.1).

(B) We start with the “permutation formula” (15.10c) and apply both its sides to the main scalar. Taking into account (16.1), we obtain

$$(3) \quad \mathcal{L}_{S_0}\mathcal{L}_{FX_0}\lambda = \kappa X_0(\lambda).$$

Now we evaluate the vector field $[S_0, X_0]$ on the function $FX_0(\lambda)$.

$$\begin{aligned} [S_0, X_0](FX_0(\lambda)) &= S_0[X_0(FX_0(\lambda))] - X_0[S_0(FX_0(\lambda))] \stackrel{(16.5i), (3)}{=} \\ &-2S_0[\lambda(FX_0)(\lambda)] - X_0(\kappa X_0(\lambda)) \stackrel{(16.1)}{=} -2\lambda S_0[(FX_0)\lambda] - X_0(\kappa)X_0(\lambda) \\ &- \kappa X_0(X_0\lambda) \stackrel{(3), (2)}{=} -2\lambda\kappa X_0(\lambda) + \lambda\kappa X_0(\lambda) - \kappa X_0(X_0\lambda) = \\ &- \kappa(\lambda X_0(\lambda) + X_0(X_0\lambda)). \end{aligned}$$

Since, on the other side, $[S_0, X_0] \stackrel{(15.10b)}{=} -\frac{1}{L}FX_0$, it follows that

$$(4) \quad \frac{1}{L}\mathcal{L}_{FX_0}^2\lambda = \kappa(\lambda X_0(\lambda) + X_0(X_0\lambda)).$$

Applying the vector field X_0 to both sides of (4), owing to (15.1) we obtain

$$\begin{aligned} \frac{1}{L}\mathcal{L}_{X_0}\mathcal{L}_{FX_0}^2\lambda &= X_0(\kappa)(\lambda X_0(\lambda) + X_0(X_0\lambda)) \\ &+ \kappa \left[(\mathcal{L}_{X_0}\lambda)^2 + \lambda\mathcal{L}_{X_0}^2\lambda + \mathcal{L}_{X_0}^3\lambda \right] \stackrel{(2)}{=} \kappa(-\lambda^2\mathcal{L}_{X_0}\lambda + (\mathcal{L}_{X_0}\lambda)^2 + \mathcal{L}_{X_0}^3\lambda), \end{aligned}$$

i.e.,

$$(5) \quad \frac{1}{L}\mathcal{L}_{X_0}\mathcal{L}_{FX_0}^2\lambda = \kappa(-\lambda^2\mathcal{L}_{X_0}\lambda + (\mathcal{L}_{X_0}\lambda)^2 + \mathcal{L}_{X_0}^3\lambda).$$

The Lie derivatives of the two sides of (16.5i) with respect to FX_0 yield

$$(6) \quad \frac{1}{L} \mathcal{L}_{FX_0} \mathcal{L}_{X_0} \mathcal{L}_{FX_0} \lambda = \frac{1}{L} [-2(\mathcal{L}_{FX_0} \lambda)^2 - 2\lambda \mathcal{L}_{FX_0}^2 \lambda].$$

Taking the difference of (5) and (6), and then substituting the term $\frac{1}{L} \mathcal{L}_{FX_0}^2 \lambda$ from the right-hand side of (4) we obtain

$$\begin{aligned} \frac{1}{L} [X_0, FX_0](FX_0(\lambda)) &= \kappa(\lambda^2 \mathcal{L}_{X_0} \lambda + (\mathcal{L}_{X_0} \lambda)^2 + \mathcal{L}_{X_0}^3 \lambda + 2\lambda \mathcal{L}_{X_0}^2 \lambda) \\ &\quad + \frac{2}{L} (\mathcal{L}_{FX_0} \lambda)^2. \end{aligned}$$

The left-hand side of this equality can also be written in the form

$$\begin{aligned} \frac{1}{L} [X_0, FX_0](FX_0(\lambda)) &\stackrel{(15.10a), (16.1)}{=} -\frac{1}{2E} S_0[FX_0(\lambda)] - \frac{1}{L} \lambda(FX_0)(FX_0(\lambda)) \\ &\stackrel{(3), (4)}{=} -\kappa \left[\frac{1}{2E} X_0(\lambda) + \lambda^2 X_0(\lambda) + \lambda X_0(X_0 \lambda) \right], \end{aligned}$$

so it follows that

$$(7) \quad \frac{2}{L} (\mathcal{L}_{FX_0} \lambda)^2 + \kappa(\mathcal{L}_{X_0}^3 \lambda + 3\lambda \mathcal{L}_{X_0}^2 \lambda + (\mathcal{L}_{X_0} \lambda)^2 + 2\lambda^2 \mathcal{L}_{X_0} \lambda + \frac{1}{2E} \mathcal{L}_{X_0} \lambda) = 0.$$

Now we apply the vector field X_0 to (7). Taking into account that

$$\begin{aligned} X_0 \left[\frac{2}{L} (\mathcal{L}_{FX_0} \lambda)^2 \right] &\stackrel{(15.1)}{=} \frac{2}{L} \mathcal{L}_{X_0} (\mathcal{L}_{FX_0} \lambda)^2 = \frac{4}{L} (\mathcal{L}_{FX_0} \lambda) \mathcal{L}_{X_0} \mathcal{L}_{FX_0} \lambda \\ &\stackrel{(16.5i)}{=} -\frac{8\lambda}{L} (\mathcal{L}_{FX_0} \lambda)^2 \stackrel{(7)}{=} 4\kappa \lambda \left(\mathcal{L}_{X_0}^3 \lambda + 3\lambda \mathcal{L}_{X_0}^2 \lambda + (\mathcal{L}_{X_0} \lambda)^2 + 2\lambda^2 \mathcal{L}_{X_0} \lambda + \frac{1}{2E} \mathcal{L}_{X_0} \lambda \right), \end{aligned}$$

we obtain the relation

$$\begin{aligned} (X_0 \kappa) &\left(\mathcal{L}_{X_0}^3 \lambda + 3\lambda \mathcal{L}_{X_0}^2 \lambda + (\mathcal{L}_{X_0} \lambda)^2 + 2\lambda^2 \mathcal{L}_{X_0} \lambda + \frac{1}{2E} \mathcal{L}_{X_0} \lambda \right) \\ &\quad + \kappa \left(\mathcal{L}_{X_0}^4 \lambda + 5(\mathcal{L}_{X_0} \lambda) \mathcal{L}_{X_0}^2 \lambda + 7\lambda \mathcal{L}_{X_0}^3 \lambda + 8\lambda (\mathcal{L}_{X_0} \lambda)^2 \right. \\ &\quad \left. + 14\lambda^2 \mathcal{L}_{X_0}^2 \lambda + \frac{1}{2E} \mathcal{L}_{X_0}^2 \lambda + 8\lambda^3 \mathcal{L}_{X_0} \lambda + \frac{2\lambda}{E} \mathcal{L}_{X_0} \lambda \right) = 0. \end{aligned}$$

Using (2), this takes the form

$$\begin{aligned} (8) \quad \kappa &\left[\mathcal{L}_{X_0}^4 \lambda + 6\lambda \mathcal{L}_{X_0}^3 \lambda + \left(5\mathcal{L}_{X_0} \lambda + 11\lambda^2 + \frac{1}{2E} \right) \mathcal{L}_{X_0}^2 \lambda \right. \\ &\quad \left. + \left(7\mathcal{L}_{X_0} \lambda + 6\lambda^2 + \frac{3}{2E} \right) \lambda \mathcal{L}_{X_0} \lambda \right] = 0. \end{aligned}$$

(C) To conclude the proof, we finally discuss the relation (8).

- (a) If $\kappa = 0$, then we see from (7) that $\mathcal{L}_{FX_0}\lambda = 0$. This means by (A) that (M, L) is a Berwald manifold.
- (b) In the case $\kappa \neq 0$ the second factor has to vanish on the left hand side of (8). Then we take the Lie derivative of this factor with respect to the vector field FX_0 . Applying the relations (16.5a–g), after a somewhat lengthy but quite straightforward calculation we obtain

$$(9) \quad \left[\mathcal{L}_{X_0}^3 \lambda + 3\lambda \mathcal{L}_{X_0}^2 \lambda + (\mathcal{L}_{X_0} \lambda)^2 + 2\lambda^2 \mathcal{L}_{X_0} \lambda + \frac{1}{2E} \mathcal{L}_{X_0} \lambda \right] \mathcal{L}_{FX_0} \lambda = 0.$$

If $\mathcal{L}_{FX_0} \lambda = 0$, then the process ends. Otherwise the first factor on the left hand side of (9) is zero, but, owing to (7), this also yields the desired relation $\mathcal{L}_{FX_0} \lambda = 0$. $\square$

17. Two-dimensional Finsler manifolds admitting a special conformal change

17.1 Two Finsler structures $\tilde{L}$ and L over the same manifold M are said to be *conformally equivalent* if there exists a positive smooth function $\varphi : TM \rightarrow \mathbb{R}$ such that $\tilde{g} = \varphi g$, where $\tilde{g}$ and g the Riemann-Finsler metrics of $\tilde{L}$ and L respectively. In this case we also speak of the conformal change of the Riemann-Finsler metric g . φ always can be written in the form $\varphi = \exp \circ \alpha^v$, $\alpha^v = \alpha \circ \pi$, $\alpha \in C^\infty(M)$ [46]. The changed metric will be denoted by g_α , so $g_\alpha := \varphi g$.

17.2 Problem. *Determine all Finsler manifolds of dimension 2 which admit a conformal change such that the mixed curvature tensor of the Finslerian Berwald connection remains invariant.*

We remark that in case of the vanishing of the mixed curvature tensor of the Berwald connection this leads to the two-dimensional Matsumoto's problem posed in [39], namely: *Are there conformally equivalent Berwald manifolds?* (For a solution see [59].) As we shall see the Berwald's list of two-dimensional Finsler manifolds with constant main scalar can be also interpreted as particular solutions of our problem. It will be proved that the mixed curvature tensor of these manifolds is invariant under any conformal change. We shall describe a slightly modified example in detail.

17.3 Suppose that $\mathring{\mathbb{P}}_\alpha = \mathring{\mathbb{P}}$ under the change $E_\alpha = \varphi E$ and denote by h_α and h the corresponding Barthel endomorphisms, respectively. Then for any vector field $X, Y, Z \in \mathfrak{X}(M)$ we have

$$\begin{aligned} 0 &= \mathring{\mathbb{P}}_\alpha(X^c, Y^c)Z^c - \mathring{\mathbb{P}}(X^c, Y^c)Z^c \stackrel{(4.9)}{=} [[X^{h_\alpha}, Y^v], Z^v] - [[X^h, Y^v], Z^v] \\ &= [[X^{h_\alpha}, Y^v] - [X^h, Y^v], Z^v], \end{aligned}$$

which means that the vector field $[X^{h_\alpha}, Y^v] - [X^h, Y^v]$ is a vertical lift (see e.g. [55]). Therefore, as an easy local calculation shows, the components of the difference tensor $h_\alpha - h$ are linear on the tangent spaces and, consequently, the difference of the associated sprays S_α and S is a quadratic vector field. As it is known [54],

$$S_\alpha = S - \alpha^c C + E(d\alpha^v)^\#,$$

where α^c is the complete lift of α ($\alpha^c := S\alpha^v$) and $i_{(d\alpha^v)^\#}\omega = d\alpha^v$. Evaluating both sides on the function α^c we have

$$Eg((d\alpha^v)^\#, (d\alpha^v)^\#) = (S_\alpha - S)\alpha^c + (\alpha^c)^2,$$

where the function of the right-hand side is quadratic. This means that it can be considered as a (local) Riemannian energy function E_R under the regularity condition $d_p\alpha \neq 0$ ($p \in M$). Let $grad_R\alpha$ and $grad_R^v\alpha$ be the Riemannian gradient of the function α and its vertical lift, respectively. Since the manifold is of dimension 2, the vector fields

$$JX := (d\alpha^v)^\#, \quad Y^v := grad_R^v\alpha \quad \text{and} \quad C$$

are linearly dependent. Let $c(t) = v + tY(p)$, where $p \in M$ and $v \in T_pM \cap Ker\alpha^c$. Then c is an integral curve of $grad_R^v\alpha$. Evaluating their Gram-determinant with respect to g along c , we have

$$\begin{aligned} & \left(2Eg(JX, JX)g(Y^v, Y^v) + 2g(JX, Y^v)g(Y^v, C)g(JX, C) - g^2(JX, C)g(Y^v, Y^v) \right. \\ & \quad \left. - g^2(Y^v, C)g(JX, JX) - 2Eg^2(JX, Y^v) \right) \circ c = 0. \end{aligned}$$

Since

$$\begin{aligned} Eg(JX, JX) &= E_R, \quad g(JX, Y^v) = (Y\alpha)_p, \quad g(Y^v, C) = Y^v E, \\ g(JX, C) &= \alpha^c, \quad g(Y^v, Y^v) = Y^v(Y^v E) \end{aligned}$$

we arrive at the following differential equation:

$$2(a^2 + t^2b^2(1 - b^2))y(t)y''(t) - (a^2 + t^2b^2)(y'(t))^2 + 4tb^4y(t)y'(t) - 4b^4(y(t))^2 = 0,$$

where $y := E \circ c$, $a^2 = 2E_R(v)$, $b^2 = 2E_R(Y(p))$ and $0 < b^2 \leq 1$ by the Cauchy-Schwarz inequality. Introducing $z = \frac{y'}{y}$ we obtain the following Riccati-type equation:

$$(a^2 + t^2b^2(1 - b^2))z'(t) + \frac{1}{2}(a^2 + t^2b^2(1 - 2b^2))(z(t))^2 + 2tb^4z(t) - 2b^4 = 0.$$

If the constants a and b are arbitrary, to find the solutions seems to be very complicated, so we discuss only the case $b^2 = 1$. If $b^2 = 1$, then the solution with the initial condition $z(0) = \frac{K}{a}$ has the form

$$z(t) = 2 \frac{2t + Ka}{2t^2 + tKa + 2a^2} = \frac{2t + Ka}{\left(t + \frac{Ka}{4}\right)^2 + a^2\left(1 - \frac{K^2}{16}\right)}.$$

We have the following cases:

(A) $K^2 < 16$. Then

$$E \circ c(t) = 4K^* \left(E_R \circ c(t) + KL_R(v) \frac{t}{4} \right) \exp \frac{2K}{\sqrt{16 - K^2}} \left(\arctan \frac{4t + KL_R(v)}{L_R(v)\sqrt{16 - K^2}} - \arctan \frac{2K}{\sqrt{16 - K^2}} \right).$$

(B) $K^2 = 16$. Then

$$E \circ c(t) = K^* \left(t + \frac{KL_R(v)}{4} \right)^2 \exp \frac{-2KL_R(v)}{4t + KL_R(v)}.$$

(C) $K^2 > 16$. Then

$$E \circ c(t) = K^* \left(t + \frac{L_R(v)}{4} (K - \sqrt{K^2 - 16}) \right)^{\left(1 + \frac{K}{\sqrt{K^2 - 16}}\right)} \left(t + \frac{L_R(v)}{4} (K + \sqrt{K^2 - 16}) \right)^{\left(1 - \frac{K}{\sqrt{K^2 - 16}}\right)}.$$

We are going to discuss the first case in detail under further simplification. We suppose that the constants K and K^* are depend only on the position. Note that in general they are just homogeneous of degree zero, i.e. $K_{tv} = K_v$ for any positive real number t and, similarly, $K_{tv}^* = K_v^*$.

17.4 Let M be a two-dimensional Finsler manifold with the energy given by the first case. Fix a point $p \in M$ and choose a coordinate system (u, v) on M such that $\left(\left(\frac{\partial}{\partial u} \right)_p, \left(\frac{\partial}{\partial v} \right)_p \right)$ form an orthonormal basis for $T_p M$ with respect to the

Riemann metric g_R . Denote the induced coordinates on TU by $((u, v), (x, y))$. Then the Finsler energy has the form

$$E((u, v), (x, y)) = \left(\frac{1}{2}(x^2 + y^2) + \frac{K(u, v)}{4}|x|y \right) \exp A,$$

where

$$A := \frac{2K(u, v)}{\sqrt{16 - K^2(u, v)}} \left(\arctan \frac{4y + K(u, v)|x|}{|x|\sqrt{16 - K^2(u, v)}} - \arctan \frac{2K(u, v)}{\sqrt{16 - K^2(u, v)}} \right),$$

and $-4 < K(u, v) < 4$.

Routine calculation yields the matrix (g_{ij}) of the Riemann-Finsler metric g . It is of the form

$$(g_{ij}) = \frac{\exp A}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}|x|y} \begin{pmatrix} \frac{1}{2}(x^2 + y^2) & \frac{K|x|x}{4} \\ \frac{K|x|x}{4} & \frac{1}{2}(x^2 + y^2) + \frac{K|x|y}{2} + \frac{K^2x^2}{8} \end{pmatrix},$$

where $K := K(u, v)$.

17.5 Remark. The energy function in this case is singular: the matrix of the Riemann-Finsler metric becomes 0 along the y -axis (i.e, if $x = 0$).

Next we determine the second member of the Berwald frame (C_0, X_0, FX_0, S_0) .

17.6 Proposition.

$$X_0 = \frac{1}{L} \begin{cases} \left(-\frac{Kx}{2} - y \right) \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, & \text{if } x > 0, \\ \left(\frac{Kx}{2} - y \right) \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}, & \text{if } x < 0. \end{cases}$$

Proof. We deal with only the case $x > 0$, the other case can be treated similarly. We show that $g(X_0, C) = 0$ and $g(X_0, X_0) = 1$.

$$\begin{aligned} g(X_0, X_0) &= \frac{\exp A}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy} \left(\left(-\frac{Kx}{2L} - \frac{y}{L} \right)^2 \frac{1}{2}(x^2 + y^2) \right. \\ &\quad \left. + 2 \left(-\frac{Kx}{2L} - \frac{y}{L} \right) \frac{Kx^3}{4L} + \frac{x^2}{L^2} \left(\frac{1}{2}(x^2 + y^2) + \frac{Kxy}{2} + \frac{K^2x^2}{8} \right) \right) \\ &= \frac{\exp A}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy} \left(\frac{K^2}{8L^2} x^2(x^2 + y^2) + \frac{Kxy}{2L^2}(x^2 + y^2) \right. \\ &\quad \left. + \frac{y^2}{2L^2}(x^2 + y^2) - \frac{K^2x^4}{4L^2} - \frac{Kx^3y}{2L^2} + \frac{x^2}{2L^2}(x^2 + y^2) + \frac{Kx^3y}{2L^2} + \frac{K^2x^4}{8L^2} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{\exp A}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy} \frac{2}{L^2} \left(\frac{K^2 x^4}{16} + \frac{K^2 x^2 y^2}{16} + \frac{K x^3 y}{4} + \frac{K x y^3}{4} + \frac{x^2 y^2}{4} \right. \\
&\quad \left. + \frac{y^4}{4} - \frac{K^2 x^4}{8} - \frac{K x^3 y}{4} + \frac{x^4}{4} + \frac{x^2 y^2}{4} + \frac{K x^3 y}{4} + \frac{K^2 x^4}{16} \right) \\
&= \frac{\exp A}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy} \frac{1}{E} \left(\frac{1}{4}(x^4 + 2x^2 y^2 + y^4) + \frac{K}{4}xy(x^2 + y^2) + \frac{K^2}{16}x^2 y^2 \right) \\
&= \frac{\exp A}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy} \frac{1}{E} \left(\frac{1}{2}(x^2 + y^2) + \frac{Kxy}{4} \right)^2 \\
&= \left(\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy \right) \frac{1}{E} \exp A = 1.
\end{aligned}$$

The Liouville vector field C locally has the form

$$C = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$$

therefore

$$\begin{aligned}
g(X_0, C) &= \frac{\exp A}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy} \left(\left(-\frac{Kx^2}{2L} - \frac{xy}{L} \right) \frac{1}{2}(x^2 + y^2) \right. \\
&\quad \left. + \frac{Kx^4}{4L} + \frac{Kx^2 y}{4} \left(-\frac{Kx^2}{2L} - \frac{xy}{L} \right) + \frac{xy}{L} \left(\frac{1}{2}(x^2 + y^2) + \frac{Kxy}{2} + \frac{K^2 x^2}{8} \right) \right) \\
&= \frac{\exp A}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy} \left(-\frac{Kx^2}{4L}(x^2 + y^2) - \frac{xy}{2L}(x^2 + y^2) + \frac{K^2 x^4}{4L} \right. \\
&\quad \left. - \frac{K^2 x^3 y}{8L} - \frac{Kx^2 y^2}{4L} + \frac{xy}{2L}(x^2 + y^2) + \frac{Kx^2 y^2}{2L} + \frac{K^2 x^3 y}{8L} \right) = 0. \quad \square
\end{aligned}$$

17.7 Proposition. *The main scalar λ is given by*

$$\lambda = \begin{cases} \frac{K}{2L}, & \text{if } x > 0, \\ -\frac{K}{2L}, & \text{if } x < 0. \end{cases}$$

Proof. The main scalar (15.4) of a two-dimensional Finsler manifold is defined by

$$\lambda := g(\mathcal{C}(FX_0, FX_0), X_0) = \mathcal{C}_b(FX_0, FX_0, FX_0).$$

Since the Cartan tensors have no potential (11.1h)

$$\mathcal{C}_b(FX_0, FX_0, FX_0) = \tilde{\mathcal{C}}(FX_0),$$

therefore

$$\lambda = X_0^1 \tilde{\mathcal{C}}_1 + X_0^2 \tilde{\mathcal{C}}_2.$$

It is known (see e.g. [38]) that

$$\tilde{\mathcal{C}}_1 = \frac{\partial}{\partial x} \ln \sqrt{\det g_{ij}}, \quad \tilde{\mathcal{C}}_2 = \frac{\partial}{\partial y} \ln \sqrt{\det g_{ij}}.$$

By straightforward calculation we have $\det (g_{ij}) = \exp 2A$, whence

$$\tilde{\mathcal{C}}_1 = \frac{\partial A}{\partial x} = \begin{cases} -\frac{Ky}{4} \frac{1}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy}, & \text{if } x > 0, \\ \frac{Ky}{4} \frac{1}{\frac{1}{2}(x^2 + y^2) - \frac{K}{4}xy}, & \text{if } x < 0, \end{cases}$$

$$\tilde{\mathcal{C}}_2 = \frac{\partial A}{\partial y} = \begin{cases} -\frac{Kx}{4} \frac{1}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy}, & \text{if } x > 0, \\ \frac{Kx}{4} \frac{1}{\frac{1}{2}(x^2 + y^2) - \frac{K}{4}xy}, & \text{if } x < 0. \end{cases}$$

In case of $x > 0$,

$$\begin{aligned} \lambda &= \frac{1}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy} \left(\left(-\frac{Kx}{2L} - \frac{y}{L} \right) \left(-\frac{Ky}{4} \right) + \frac{Kx^2}{4L} \right) \\ &= \frac{1}{\frac{1}{2}(x^2 + y^2) + \frac{K}{4}xy} \frac{K}{2L} \left(\frac{Kxy}{4} + \frac{y^2}{2} + \frac{x^2}{2} \right) = \frac{K}{2L}. \end{aligned}$$

If $x < 0$, then

$$\begin{aligned} \lambda &= \frac{1}{\frac{1}{2}(x^2 + y^2) - \frac{K}{4}xy} \left(\left(\frac{Kx}{2L} - \frac{y}{L} \right) \frac{Ky}{4} - \frac{Kx^2}{4L} \right) \\ &= \frac{1}{\frac{1}{2}(x^2 + y^2) - \frac{K}{4}xy} \left(-\frac{K}{2L} \right) \left(-\frac{Kxy}{4} + \frac{y^2}{2} + \frac{x^2}{2} \right) = -\frac{K}{2L}. \end{aligned}$$

□

17.8 Theorem. *For a two-dimensional Finsler manifold M with the energy given by (A) the following are equivalent:*

- (1) *The manifold is Douglas.*
- (2) *The manifold is Landsberg.*
- (3) *The manifold is Berwald.*

Proof. (i) (1) $\implies$ (3).

Suppose that the Douglas tensor vanishes. Then by (16.4c)

$$(*) \quad X_0(FX_0(\lambda)) + X_0(X_0(S\lambda)) + 2\lambda FX_0(\lambda) + 2\lambda X_0(S\lambda) + 3\frac{S(\lambda)}{E} = 0.$$

Since the main scalar has the form $\lambda = \frac{K}{2L}$ (in case of $x > 0$), where K is constant on each fibre, it follows from (15.1) that

$$X_0\lambda = 0.$$

Using the "permutation formulas" we have

$$\begin{aligned} X_0(FX_0(\lambda)) &= [X_0, FX_0]\lambda \stackrel{(15.10a)}{=} -\frac{1}{L}S_0\lambda - \lambda FX_0\lambda \\ X_0(S\lambda) &= X_0(LS_0\lambda) \stackrel{(15.1)}{=} LX_0(S_0\lambda) = L[X_0, S_0]\lambda \stackrel{(15.10b)}{=} FX_0\lambda, \end{aligned}$$

therefore

$$X_0(X_0(S\lambda)) = X_0(FX_0(\lambda)) = -\frac{1}{L}S_0\lambda - \lambda FX_0\lambda.$$

Substituting these into (*) we obtain that vanishing of the Douglas tensor is equivalent to the following:

$$(**) \quad \frac{2S_0\lambda}{L} + \lambda(FX_0\lambda) = 0.$$

The Lie derivative of (**) with respect to X_0 yields

$$\frac{2}{L}X_0(S_0\lambda) + \lambda(X_0(FX_0\lambda)) = 0.$$

As above, also by the "permutation formulas" we obtain that

$$\frac{2}{L^2}FX_0\lambda - \frac{\lambda(S_0\lambda)}{L} - \lambda^2(FX_0\lambda) = 0.$$

Here $-\frac{\lambda(S_0\lambda)}{L} = \frac{\lambda^2}{2}(FX_0\lambda)$ by (**) and using this we get

$$(FX_0\lambda)\left(\frac{2}{L^2} - \frac{\lambda^2}{2}\right) = 0.$$

Suppose $\frac{2}{L^2} - \frac{\lambda^2}{2} = 0$. Since $\lambda = \frac{K}{2L}$ this yields $K^2 = 16$, which contradicts the assumption $-4 < K < 4$, whence $FX_0\lambda = 0$. But this means by (**) that $S_0\lambda$ also vanishes, that is $d_h\lambda = 0$ and this implies by (15.7) that the manifold is Berwald.

(ii) (2) $\implies$ (1).

Suppose the manifold is Landsberg. Then $S\lambda = 0$ by (15.1). Since $\lambda = \frac{K}{2L}$, $0 = S(K \circ \pi) = K^c$ and this implies that K is constant (not only on each fibre, but on the whole TU) and, evidently $d_h\lambda = 0$ in this case. $\square$

17.9 Theorem. *If a two-dimensional Finsler manifold has constant main scalar in Berwald's sense, that is $L\lambda$ is constant on each fibre, then the mixed curvature of the Finslerian Berwald connection is invariant under any conformal change.*

Proof. Suppose that $L\lambda = K \circ \pi$ and consider a conformal change $g_\alpha = \varphi g$. Then

$$g_\alpha\left(\frac{X_0}{\sqrt{\varphi}}, \frac{X_0}{\sqrt{\varphi}}\right) = \frac{\varphi}{\varphi} g(X_0, X_0) = 1,$$

therefore

$$X_{0\alpha} = \frac{X_0}{\sqrt{\varphi}} \quad \text{and, similarly} \quad C_{0\alpha} = \frac{C_0}{\sqrt{\varphi}}.$$

This implies that

$$\lambda_\alpha = \frac{\lambda}{\sqrt{\varphi}},$$

since

$$\begin{aligned} \lambda_\alpha &= g_\alpha(\mathcal{C}_\alpha(F_\alpha X_{0\alpha}, F_\alpha X_{0\alpha}), X_{0\alpha}) = g_\alpha(\mathcal{C}(FX_{0\alpha}, FX_{0\alpha}), X_{0\alpha}) \\ &= \varphi g\left(\mathcal{C}\left(\frac{FX_0}{\sqrt{\varphi}}, \frac{FX_0}{\sqrt{\varphi}}\right), \frac{X_0}{\sqrt{\varphi}}\right) = \frac{\lambda}{\sqrt{\varphi}}, \end{aligned}$$

using that $F_\alpha X_{0\alpha} - FX_{0\alpha}$ is vertical and $\mathcal{C}$ is invariant under the conformal change (see e.g. [54]). On the one hand

$$\begin{aligned} \overset{\circ}{\mathbb{P}}(F_\alpha X_{0\alpha}, F_\alpha X_{0\alpha})F_\alpha X_{0\alpha} &= \overset{\circ}{\mathbb{P}}(FX_{0\alpha}, FX_{0\alpha}), FX_{0\alpha} \\ &= \overset{\circ}{\mathbb{P}}\left(\frac{FX_0}{\sqrt{\varphi}}, \frac{FX_0}{\sqrt{\varphi}}\right) \frac{FX_0}{\sqrt{\varphi}} = \frac{1}{\sqrt{\varphi}^3} \overset{\circ}{\mathbb{P}}(FX_0, FX_0)FX_0, \end{aligned}$$

on the other hand

$$\begin{aligned}
& \overset{\circ}{\mathbb{P}}_{\alpha}(F_{\alpha}X_{0\alpha}, F_{\alpha}X_{0\alpha})F_{\alpha}X_{0\alpha} \\
& \stackrel{(16.4a)}{=} -[F_{\alpha}X_{0\alpha}(\lambda_{\alpha}) + X_{0\alpha}(S_{\alpha}\lambda_{\alpha})]X_{0\alpha} + 2\frac{S_{\alpha}\lambda_{\alpha}}{L_{\alpha}}C_{0\alpha} \\
& = -\left[\frac{F_{\alpha}X_0}{\sqrt{\varphi}}\left(\frac{\lambda}{\sqrt{\varphi}}\right) + \frac{X_0}{\sqrt{\varphi}}\left(S_{\alpha}\left(\frac{K \circ \pi}{L_{\alpha}}\right)\right)\right]\frac{X_0}{\sqrt{\varphi}} + \frac{2}{L_{\alpha}}S_{\alpha}\left(\frac{K \circ \pi}{L_{\alpha}}\right)\frac{C_0}{\sqrt{\varphi}} \\
& = -\left[\frac{F_{\alpha}X_0}{\sqrt{\varphi}}\left(\frac{K \circ \pi}{L\sqrt{\varphi}}\right) + \frac{X_0}{\sqrt{\varphi}}\left(S_{\alpha}\left(\frac{K \circ \pi}{L_{\alpha}}\right)\right)\right]\frac{X_0}{\sqrt{\varphi}} + \frac{2}{L_{\alpha}}S_{\alpha}\left(\frac{K \circ \pi}{L_{\alpha}}\right)\frac{C_0}{\sqrt{\varphi}} \\
& = -\left[\frac{F_{\alpha}X_0}{\sqrt{\varphi}}\left(\frac{K \circ \pi}{L_{\alpha}}\right) + \frac{X_0}{\sqrt{\varphi}}\left(S_{\alpha}\left(\frac{K \circ \pi}{L_{\alpha}}\right)\right)\right]\frac{X_0}{\sqrt{\varphi}} + \frac{2}{L_{\alpha}}S_{\alpha}\left(\frac{K \circ \pi}{L_{\alpha}}\right)\frac{C_0}{\sqrt{\varphi}}
\end{aligned}$$

Since the Lie derivative of L_{α} with respect to $F_{\alpha}X_0$, S_{α} is zero and $F_{\alpha}X_{0\alpha} - F X_{0\alpha}$ is vertical, we obtain

$$\begin{aligned}
& \overset{\circ}{\mathbb{P}}_{\alpha}(F_{\alpha}X_{0\alpha}, F_{\alpha}X_{0\alpha})F_{\alpha}X_{0\alpha} \\
& = -\left[\frac{1}{L_{\alpha}\sqrt{\varphi}}F_{\alpha}X_0(K \circ \pi) + \frac{X_0}{\sqrt{\varphi}}\left(\frac{K^c}{L_{\alpha}}\right)\right]\frac{X_0}{\sqrt{\varphi}} + \frac{2K^c}{L_{\alpha}^2}\frac{C_0}{\sqrt{\varphi}} \\
& = -\left[\frac{1}{L\varphi}F X_0(K \circ \pi) + K^c X_{0\alpha}\left(\frac{1}{L_{\alpha}}\right) + \frac{1}{\sqrt{\varphi}L_{\alpha}}X_0K^c\right]\frac{X_0}{\sqrt{\varphi}} + \frac{2K^c}{\varphi L^2\sqrt{\varphi}}C_0 \\
& \stackrel{(15.1)}{=} -\left[\frac{1}{\varphi}F X_0\left(\frac{K \circ \pi}{L}\right) + \frac{1}{\varphi L}(X_0K^c)\right]\frac{X_0}{\sqrt{\varphi}} + \frac{2K^c}{L^2\sqrt{\varphi^3}}C_0 \\
& \stackrel{(15.1)}{=} -\frac{1}{\sqrt{\varphi^3}}\left[F X_0\lambda + X_0\left(\frac{K^c}{L}\right)\right]X_0 + \frac{2K^c}{L^2\sqrt{\varphi^3}}C_0 \\
& = -\frac{1}{\sqrt{\varphi^3}}\left[F X_0\lambda + X_0\left(\frac{S(K \circ \pi)}{L}\right)\right]X_0 + \frac{2S(K \circ \pi)}{L^2\sqrt{\varphi^3}}C_0 \\
& \stackrel{d_h L=0}{=} -\frac{1}{\sqrt{\varphi^3}}\left[F X_0\lambda + X_0\left(S\left(\frac{K \circ \pi}{L}\right)\right)\right]X_0 + \frac{2}{L\sqrt{\varphi^3}}S\left(\frac{K \circ \pi}{L}\right)C_0 \\
& = \frac{1}{\sqrt{\varphi^3}}\left(-[F X_0\lambda + X_0(S\lambda)]X_0 + \frac{2S\lambda}{L}C_0\right) \\
& = \frac{1}{\sqrt{\varphi^3}}\overset{\circ}{\mathbb{P}}(F X_0, F X_0)F X_0 = \overset{\circ}{\mathbb{P}}(F_{\alpha}X_{0\alpha}, F_{\alpha}X_{0\alpha})F_{\alpha}X_{0\alpha}. \quad \square
\end{aligned}$$

A COORDINATE FORMULARY

Here we summarize the coordinate expressions of the most important formulas to make our thesis easy to understand.

1. Taking a chart $(U, (u)_{i=1}^n)$ of our n -dimensional manifold M in the whole section we work on the induced map $(\pi^{-1}(U), x^i, y^i)_{i=1}^n$, where π is the canonical projection and $x^i := u^i \circ \pi$, $y^i := du^i$ and Latin indices run from 1 to n . We also use the Einstein convention: if an index in a formula appears twice, once on the top, the other time on the bottom, then it means summation from 1 to n .

The complete and the vertical lifts of the coordinate vector fields $\left(\frac{\partial}{\partial u^i}\right)$, namely $\left(\frac{\partial}{\partial u^i}\right)^c = \left(\frac{\partial}{\partial x^i}\right)$, $\left(\frac{\partial}{\partial u^i}\right)^v = \left(\frac{\partial}{\partial y^i}\right)$, form a local basis of $\mathfrak{X}(TM)$. The *vertical endomorphism* and the *Liouville field* (1.5) has the following local form:

$$J = dx^i \otimes \frac{\partial}{\partial y^i}, \quad C = y^i \frac{\partial}{\partial y^i}.$$

2. If T is a semibasic tensor (1.6) of type $(1, s)$, then it has the form

$$T = T_{i_1 \dots i_s}^j dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial y^j}.$$

The *semibasic trace* (2.10) of T can be written

$$\tilde{T} = T_{i_1 \dots i_{s-1} j}^j dx^{i_1} \otimes \dots \otimes dx^{i_{s-1}}.$$

The *potential* (2.15) of T is usually noted by

$$i_S T = y^i T_{ii_2 \dots i_s}^j dx^{i_2} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial y^j} = T_{0i_2 \dots i_s}^j dx^{i_2} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial y^j}.$$

3. On the induced chart a *spray* (2.5) S can be written as

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}.$$

The homogeneity condition 2.5(iv) leads to

$$y^k \frac{\partial G^i}{\partial y^k} = 2G^i.$$

The *horizontal endomorphism* h generated by the spray S (2.8) has the coordinate expression

$$h = dx^i \otimes \frac{\partial}{\partial x^i} - \Gamma_j^i dx^j \otimes \frac{\partial}{\partial y^i},$$

where

$$\frac{\partial G^i}{\partial y^k} = \Gamma_k^i,$$

and therefore the *horizontal lift* (2.1) with respect to h can locally be given by

$$\left(\frac{\partial}{\partial u^i} \right)^h = \frac{\partial}{\partial x^i} - \Gamma_i^k \frac{\partial}{\partial y^k}.$$

4. The *Berwald connection* (3.4) induced by the Berwald endomorphism h is completely determined by the rules for calculation

$$\overset{\circ}{D} \left(\frac{\partial}{\partial u^i} \right)^v \left(\frac{\partial}{\partial u^j} \right)^v = 0,$$

$$\overset{\circ}{D} \left(\frac{\partial}{\partial u^i} \right)^h \left(\frac{\partial}{\partial u^j} \right)^v = G_{ij}^k \left(\frac{\partial}{\partial u^k} \right)^v,$$

where

$$G_{ij}^k = \frac{\partial \Gamma_i^k}{\partial y^j} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}.$$

Suppose T is a semibasic tensor of type (1,1) (of type (1,s) is analogous), so it is locally given as

$$T = T_i^k dx^i \otimes \frac{\partial}{\partial y^k},$$

the usual notation for the *vertical* and *horizontal differential* (4.2) (with respect to the Berwald connection) in the classical tensor calculus has the following form:

$$\overset{\circ}{D}_J T = T_{i,j}^k dx^j \otimes dx^i \otimes \frac{\partial}{\partial y^k},$$

and

$$\overset{\circ}{D}_h T = T_{i;j}^k dx^j \otimes dx^i \otimes \frac{\partial}{\partial y^k}.$$

We have the following non-vanishing data (3.2):
the *v-horizontal torsion* $\mathbb{R}^1$:

$$\begin{aligned} \mathbb{R}^1 &= \left(\left(\frac{\partial}{\partial u^i} \right)^h \left(\Gamma_j^k \right) - \left(\frac{\partial}{\partial u^j} \right)^h \left(\Gamma_i^k \right) \right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^k} \\ &= \left(\frac{\partial \Gamma_j^k}{\partial x^i} - \frac{\partial \Gamma_i^k}{\partial x^j} + \Gamma_j^s \frac{\partial \Gamma_i^k}{\partial y^s} - \Gamma_i^s \frac{\partial \Gamma_j^k}{\partial y^s} \right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^k} \\ &:= R_{ij}^k dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^k}, \end{aligned}$$

the *horizontal curvature* $\overset{\circ}{\mathbb{R}}$:

$$\overset{\circ}{\mathbb{R}} := R_{ijt}^k dx^i \otimes dx^j \otimes dx^t \otimes \frac{\partial}{\partial y^k} = R_{ij,t}^k dx^i \otimes dx^j \otimes dx^t \otimes \frac{\partial}{\partial y^k},$$

explicitly

$$R_{ijt}^k = \frac{\partial G_{jt}^k}{\partial x^i} - \frac{\partial G_{it}^k}{\partial x^j} + G_{jt}^s G_{is}^k + \Gamma_j^s \frac{\partial G_{is}^k}{\partial y^t} - G_{it}^s G_{js}^k - \Gamma_i^s \frac{\partial G_{js}^k}{\partial y^t},$$

the *mixed curvature* $\overset{\circ}{\mathbb{P}}$:

$$\overset{\circ}{\mathbb{P}} = -\frac{\partial G_{ij}^t}{\partial y^k} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^t} := -G_{ijk}^t dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^t},$$

the *horizontal Ricci tensor* $\overset{\circ}{\mathbb{R}}$:

$$\begin{aligned} \overset{\circ}{\mathbb{R}} &= R_{ikt}^k dx^t \otimes dx^i \\ &= \left(\frac{\partial G_{kt}^k}{\partial x^i} - \frac{\partial G_{it}^k}{\partial x^k} + G_{kt}^s G_{is}^k + \Gamma_k^s \frac{\partial G_{is}^k}{\partial y^t} - G_{it}^s G_{ks}^k - \Gamma_i^s \frac{\partial G_{ks}^k}{\partial y^t} \right) dx^t \otimes dx^i, \end{aligned}$$

the *mixed Ricci tensor* $\overset{\circ}{\mathbb{P}}$:

$$\overset{\circ}{\mathbb{P}} = -G_{ikt}^k dx^t \otimes dx^i := -G_{it} dx^i \otimes dx^t.$$

5. The *Yano connection* (5.3) generated by the spray S is determined by:

$$D \left(\frac{\partial}{\partial u^i} \right)^v \left(\frac{\partial}{\partial u^j} \right)^v = 0,$$

$$D \left(\frac{\partial}{\partial u^t} \right)^h \left(\frac{\partial}{\partial u^j} \right)^v = \left(G_{ij}^k - \frac{1}{n+1} G_{ij} y^k \right) \left(\frac{\partial}{\partial u^k} \right)^v.$$

The *mixed curvature* (5.5a) is given by

$$\begin{aligned} \mathbb{P} &= \left(-G_{ijk}^t + \frac{1}{n+1} \frac{\partial G_{ik}}{\partial y^j} y^t + \frac{1}{n+1} G_{ik} \delta_j^t \right) dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^t} \\ &:= P_{ijk}^t dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^t} \end{aligned}$$

and the *mixed Ricci tensor* (5.5b) can be formulated as

$$\tilde{\mathbb{P}} = \frac{2}{n+1} G_{ij} dx^i \otimes dx^j := P_{ij} dx^i \otimes dx^j.$$

6. The local expression of the *Douglas tensor* (6.3) in terms of the Yano connection is the following:

$$\begin{aligned} \mathbb{D} &= \left(P_{ijk}^t - \frac{1}{2} (P_{ij} \delta_k^t + P_{jk} \delta_i^t) \right) dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^t} \\ &:= D_{ijk}^t dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^t}, \end{aligned}$$

and with the data of the Berwald connection (6.4) it can be written as

$$D_{ijk}^t = -G_{ijk}^t + \frac{1}{n+1} \left(\frac{\partial G_{ik}}{\partial y^j} y^t + G_{ik} \delta_j^t + G_{kj} \delta_i^t + G_{ji} \delta_k^t \right).$$

7. The *projective change* of the spray S (6.1) locally can be formed by

$$\bar{G}^i = G^i - \frac{1}{2} \lambda y^i,$$

and the relation of the horizontal lifts (6.2) generated by two projectively equivalent sprays is given by

$$\left(\frac{\partial}{\partial u^i} \right)^{\bar{h}} = \left(\frac{\partial}{\partial u^i} \right)^h + \frac{1}{2} \lambda \frac{\partial}{\partial y^i} + \frac{1}{2} \frac{\partial \lambda}{\partial y^i} y^k \frac{\partial}{\partial y^k},$$

so

$$\bar{\Gamma}_k^i = \Gamma_k^i - \frac{1}{2} \frac{\partial \lambda}{\partial y^k} y^i - \frac{1}{2} \lambda \delta_k^i.$$

8. Let us suppose that L is a *Finsler-Lagrangian* (8.4) on M . The *fundamental 2-form* (8.4(2)) can be written as

$$\omega = \frac{\partial^2 E}{\partial x^k \partial y^i} dx^k \wedge dx^i + \frac{\partial^2 E}{\partial y^k \partial y^i} dy^k \wedge dy^i,$$

and therefore the *Riemann Finsler metric* on the vertical subbundle has the form

$$g = \frac{\partial^2 E}{\partial y^k \partial y^i} dy^k \otimes dy^i := g_{ki} dy^k \otimes dy^i,$$

and in the natural basis consisting of $\left(\frac{\partial}{\partial u^i} \right)^v, \left(\frac{\partial}{\partial u^i} \right)^h$ the *Riemann Finsler metric* (11.1a) defined on the whole $\mathcal{T}M$ is given by the matrix

$$g = \begin{pmatrix} (g_{ij}) & 0 \\ 0 & (g_{ij}) \end{pmatrix}.$$

The coordinate functions of the *canonical spray* (8.6(5)) are determined by

$$G^i = \frac{1}{2}g^{js} \left(y^k \frac{\partial^2 E}{\partial x^k \partial y^j} - \frac{\partial E}{\partial x^j} \right),$$

where g^{ij} denotes the inverse of g_{ij} .

9. *The fundamental relation* (8.10). Suppose S is a spray on the manifold M with coordinate functions G^i and $\bar{L}$ is a Finsler Lagrangian on M , then the coordinate function $\bar{G}^i$ of the canonical spray arising from $\bar{L}$ is written in the form

$$\bar{G}^i = G^i + \frac{y^i}{2\bar{L}} \left(y^s \frac{\partial \bar{L}}{\partial x^s} - 2G^s \frac{\partial \bar{L}}{\partial y^s} \right) + \frac{1}{2}\bar{L}g^{ik} \left(y^s \frac{\partial^2 \bar{L}}{\partial x^s \partial y^k} - 2G^s \frac{\partial^2 \bar{L}}{\partial y^s \partial y^k} - \frac{\partial \bar{L}}{\partial x^k} \right)$$

10. The condition for a spray S to be *projectively Finsler* is given by the *Rapcsák equations* (9.2) which can locally be formulated as

$$(*) \quad y^s \frac{\partial^2 \bar{L}}{\partial x^s \partial y^k} - 2G^s \frac{\partial^2 \bar{L}}{\partial y^s \partial y^k} - \frac{\partial \bar{L}}{\partial x^k} = 0,$$

or with the simpler notation

$$(*') \quad \bar{L}_{;i} - y^s \bar{L}_{;s;i} = 0,$$

and

$$(**) \quad \frac{\partial^2 \bar{L}}{\partial x^k \partial y^i} - \Gamma_k^s \frac{\partial^2 \bar{L}}{\partial y^s \partial y^i} - \frac{\partial^2 \bar{L}}{\partial x^i \partial y^k} - \Gamma_i^s \frac{\partial^2 \bar{L}}{\partial y^s \partial y^k} = 0,$$

or equivalently

$$(**') \quad \bar{L}_{;i;j} - \bar{L}_{;j;i} = 0.$$

The condition $(*)$ is the local version of (9.2(2)) and $(**)$ is of (9.2(3)) and (9.2(4)), since the last two ones have the same coordinate form.

11. *The alternative theorem* (10.9):

(i) Suppose S is a spray with coordinate function G^i and $\bar{L}$ is projectively equivalent to S , then the tensor locally given by

$$(1) \quad \mu = \frac{\partial^2 \bar{L}}{\partial y^i \partial y^k} dx^i \otimes dx^k := \mu_{ik} dx^i \otimes dx^k$$

has the properties

$$(2) \quad y^s \frac{\partial \mu_{ik}}{\partial x^s} - 2G^s \frac{\partial \mu_{ik}}{\partial x^s} - \mu_{sk} \Gamma_i^s - \mu_{si} \Gamma_k^s = 0,$$

or with simplified notation

$$(2') \quad \mu_{ik;0} = 0,$$

and

$$(3) \quad \mu_{ir}R_{jk}^r + \mu_{jr}R_{ki}^r + \mu_{kr}R_{ij}^r = 0,$$

where R_{ij}^r are the components of $\mathbb{R}^1$.

(ii) Conversely, suppose that L^* is a Finsler structure and the tensor μ^* formed by (1) satisfies (2) and (3). Then

- (I) (M, S) is projectively equivalent to (M, L^*) , or
- (II) there exists, locally in general, a 1-form $\eta = \eta_i du^i$ on M such that — also locally — (M, S) is projectively equivalent to the Finsler manifold $(M, L^* + y^i(\eta_i \circ \pi))$.

12. The next theorem (10.10) reduces the condition for a spray being projectively Finsler to a first order PDE with an algebraic condition.

Suppose there exist a symmetric μ on vertical bundle satisfying (2) and (3) and has the following properties:

- (i) $\mu_{0i}(= y^j \mu_{ij}) = 0$,
- (ii) $\frac{\partial \mu_{ik}}{\partial y^s} (= \mu_{ik.s})$ is totally symmetric,
- (iii) if $v \neq 0$ and $w \neq \lambda v$, then $\mu_{ik}(v)w^i w^k$ is strictly positive,

then there exist a positive Finsler Lagrangian L which is projective equivalent to the given spray S .

13. The *first Cartan tensor* (11.1) of a Finsler manifold is locally written as

$$\mathcal{C}_b = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k} dx^i \otimes dx^j \otimes dx^k = \frac{1}{2} g_{ij.k} := \mathcal{C}_{ijk} dx^i \otimes dx^j \otimes dx^k,$$

and the metrically equivalent (1,2) tensor is given by

$$\mathcal{C} = \mathcal{C}_{ijk} g^{ks} dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^s} := \mathcal{C}_{ij}^s dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^s}.$$

The *second Cartan tensors* (11.1) have the form

$$\mathcal{C}'_b = \frac{1}{2} g_{ij;k} := \mathcal{C}'_{ijk} dx^i \otimes dx^j \otimes dx^k,$$

and

$$\mathcal{C}' = \mathcal{C}'_{ijk} g^{ks} dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^s} := \mathcal{C}'_{ij}{}^s dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^s}.$$

14. The *Cartan connection* (11.2) of a Finsler manifold is determined by the following two rules:

$$D \left(\frac{\partial}{\partial u^i} \right)^v \left(\frac{\partial}{\partial u^j} \right)^v = \mathcal{C}_{ij}^k \left(\frac{\partial}{\partial u^k} \right)^v,$$

and

$$D \left(\frac{\partial}{\partial u^i} \right)^h \left(\frac{\partial}{\partial u^j} \right)^v = \left(G_{ij}^k + C_{ij}'^k \right) \left(\frac{\partial}{\partial u^k} \right)^v.$$

The surviving torsion tensors (11.3) are:
the *h-mixed torsion*:

$$\mathbb{B} = -C_{ij}^k dx^i \otimes dx^j \otimes \left(\frac{\partial}{\partial u^k} \right)^h,$$

the *v-horizontal torsion*:

$$\mathbb{R}^1 = \overset{\circ}{\mathbb{R}}^1 = R_{ij}^k dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^k},$$

the *v-mixed torsion*:

$$\mathbb{P}^1 = C_{ij}'^k dx^i \otimes dx^j \otimes \frac{\partial}{\partial y^k}.$$

15. The matrix of the *projector onto the indicatrix bundle* (13.4) of a Finsler manifold in the natural local basis has the following form:

$$\tau = \begin{pmatrix} \delta_j^i - \frac{1}{2E} y^i \frac{\partial E}{\partial y^j} & 0 \\ 0 & \delta_j^i \end{pmatrix},$$

and therefore the semibasic tensor κ derived from τ by (13.5) is given by

$$\kappa = \left(\delta_j^i - \frac{1}{2E} y^i \frac{\partial E}{\partial y^j} \right) dx^j \otimes \frac{\partial}{\partial y^i} := \kappa_j^i dx^j \otimes \frac{\partial}{\partial y^i}.$$

Suppose T is a semibasic tensors of type (1,s), so it is locally written as

$$T = T_{i_1 \dots i_s}^j dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial y^j},$$

and τ is the projector onto the indicatrix bundle. The *projected tensor* of T (13.9) denoted by T^* is locally defined by

$$T^* = \left(\delta_j^k - \frac{1}{2E} y^k \frac{\partial E}{\partial y^j} \right) T_{i_1 \dots i_s}^j dx^{i_1} \otimes \dots \otimes dx^{i_s} \otimes \frac{\partial}{\partial y^k}.$$

The consequence of the definition (13.8) we have: if ω is a semibasic tensor of type (0,s), then its projected tensor ω^* coincides with the given one ω .

16. The coordinate expression of the *projected Douglas tensor* (13.10) can be written as

$$\mathbb{D}^* := D_{ijk}^{*s} dx^i \otimes dx^j \otimes dx^k \otimes \frac{\partial}{\partial y^s},$$

where

$$D_{ijk}^{*s} = -G_{ijk}^s + \frac{1}{n+1}(G_{ij}\kappa_k^s + G_{jk}\kappa_i^s + G_{ki}\kappa_j^s) + \frac{1}{E}y^s\mathcal{C}'_{ijk}$$

so the Douglas tensor and its projected tensor are related as follows:

$$\begin{aligned} D_{ijk}^{*s} &= D_{ijk}^s + y^s\alpha_{ijk}, \\ \alpha_{ijk} &= \frac{1}{E}\mathcal{C}'_{ijk} - \frac{1}{n+1}\left(G_{ij.k} + \frac{1}{2E}G_{ij}\frac{\partial E}{\partial y^k} + \frac{1}{2E}G_{jk}\frac{\partial E}{\partial y^i} + \frac{1}{2E}G_{ki}\frac{\partial E}{\partial y^j}\right) \end{aligned}$$

The proof of the theorem (13.12) stating the vanishing of D_{ijk}^{*s} implies the vanishing of D_{ijk}^s (the converse is trivial) if the dimension of the manifold $n > 2$ is based on the following facts:

(i) the mixed curvature of the Berwald connection has the form:

$$G_{ijk}^s = \frac{1}{n+1}(G_{ij}\kappa_k^s + G_{jk}\kappa_i^s + G_{ki}\kappa_j^s) + \frac{1}{E}y^s\mathcal{C}'_{ijk}$$

(ii) the vertical covariant differential (with respect to the Berwald connection) is symmetric and therefore its projected tensor is totally symmetric too, so

$$0 = (G_{ijk.l}^s)^* - (G_{ljk.i}^s)^* = \alpha_{ijk}\kappa_l^s - \alpha_{jkl}\kappa_i^s$$

(iii) taking the semibasic trace of the last equation (contracting by $i = s$) we obtain

$$(n-2)\alpha_{ijk} = 0.$$

16. The *Landsberg manifolds* are characterized - among others - by the vanishing of the v-mixed torsion of the Cartan connection that is $\mathcal{C}_{ij}^k = 0$ and therefore $\mathcal{C}'_{ijk} = 0$. That means the mixed curvature of the Berwald connection in a Landsberg manifold with vanishing Douglas tensor (14.1) has a simpler form:

$$(4) \quad G_{ijk}^s = \frac{1}{n+1}(G_{ij}\kappa_k^s + G_{jk}\kappa_i^s + G_{ki}\kappa_j^s)$$

and its semibasic trace (14.2) is also special:

$$G_{ij} = \frac{1}{n-1}(G_{is}g^{sj})(\kappa_i^s g_{sj}) := \frac{1}{n-1}G\kappa_{ij},$$

therefore

$$(5) \quad G_{ijk}^s = \frac{1}{n^2-1}G(\kappa_{ij}\kappa_k^s + \kappa_{jk}\kappa_i^s + \kappa_{ki}\kappa_j^s).$$

The theorem (14.5) states if in a Landsberg manifold (the dimension of the manifold $n > 2$) the Douglas tensor vanishes then it is a Berwald manifold (which is characterized by $G_{ijk}^s = 0$). The proof is based on the previous form of G_{ijk}^s and the special form of the *fifth Bianchi identity* of the Cartan connection (11.4) on Landsberg manifolds. Using these one can obtain

$$(n-2)G\mathcal{C}_{ik}^k := (n-2)G\mathcal{C}_i = 0,$$

which implies that $G = 0$ or $\mathcal{C}_i = 0$. In the first case $G_{ijk}^s = 0$ by (5), and therefore the manifold is Berwald. In the second case use the equation (12.1(iv)) which locally can be formed as $G_{ijk}^s = \mathcal{C}_{ij;k}^s$, take the semibasic trace of it $G_{ij} = \mathcal{C}_{i;j}$, so the assumption $\mathcal{C}_i = 0$ yields to $G_{ij} = 0$ and from (4) we conclude again $G_{ijk}^s = 0$.

SUMMARY

Our PhD dissertation treats some current problems, as well as some old problems from a new point of view of spray and Finsler geometry in the framework of the theory initiated by JOSEPH GRIFONE in the early seventies of the last century ([25],[26]). We apply mainly the calculus of vector-valued differential forms elaborated by A. FRÖLICHER and A. NIJENHUIS [24] combining it with (and simplifying at the same time) a quite systematic use of a moving frame field consisting of vertically and completely (or vertically and horizontally) lifted vector fields.

Spray geometry grew out the theory of 'affinely connected manifolds', i.e. the theory of manifolds endowed with a torsion-free linear connection. HERMANN WEYL was the first to point out the importance of this structure, independently of the metric. The differential equations for the geodesics on such a manifold are quadratic in the velocities, in modern terms: the geodesic spray of a linear connection is 2-homogeneous and of class C^2 . If we weaken the differentiability hypothesis and require the 1-times continuous differentiability, we arrive at the concept of (general) **sprays**. This seemingly mild generalization leads us to the territory of the *general geometry of paths*. The local study of sprays is a classical field in differential geometry, whose first golden age was in the twenties-thirties of the last century. Several outstanding mathematicians worked in this area, e.g. L. BERWALD, E. CARTAN, J. DOUGLAS, M. S. KNEBELMANN, T. Y. THOMAS, O. VEULEN and others. L. P. EISENHART's excellent book 'Non-Riemannian Geometry' (Amer. Math. Soc. Colloq. Publ. **8**, 1927) gives a good picture of the spirit and technique of these investigations.

A renaissance of spray geometry began in the 1970's, with a recognition of the fundamental role of sprays in the geometrical background of Lagrangian mechanics [16, 18], and, in particular, in the foundation of Finsler geometry [25, 26]. Indeed, in differential-geometric terms, the dynamics of a time-independent Lagrangian dynamical system is determined by a spray acting on the tangent manifold of the configuration space of the system. The 'canonical spray' of a Finsler manifold arises from the energy determined by a suitable Lagrangian (see 8.4 and 8.6). J. KLEIN, J. GRIFONE and M. CRAMPIN did pioneering work in this field, and our Thesis follows the path which was opened by them.

One of the main topics of our dissertation belongs to the territory of the *projective geometry of sprays*. Roughly speaking, two sprays over the same manifold are said to be projectively equivalent if they have the same geodesics as point sets, i.e., if they have common *pregeodesics*. (Recall: a curve is called pregeodesic of a spray if it has a reparametrization as a geodesic). A diffeomorphism of a manifold endowed with a spray is said to be a *projective transformation* if it sends geodesics

pregeodesics. Roughly speaking again, in the projective geometry of sprays we are interested in those properties which are invariant under projective transformations. The main ingredient of Part I of our thesis is a projectively invariant tensor, the so-called *Douglas tensor*. In the context of affine sprays, a fundamental projectively invariant tensor, the *Weyl projective curvature tensor* (briefly Weyl tensor) has already been constructed [61]. It is well-known that an affine spray is projectively flat, i.e. projectively equivalent to a spray of a torsion-free linear connection with zero curvature if and only if the projective Weyl tensor vanishes, provided that $n \geq 3$. (In the two-dimensional case the Weyl tensor is identically 0.) The counterpart of the Weyl curvature tensor in the more general context on non-affine (i.e. not necessarily affine) sprays was intrinsically constructed by L. del Castillo [20] in 1976, using the Frölicher–Nijenhuis formalism; see also [34].

In order to build up the Douglas tensor, we represent the nonlinear connections by *horizontal endomorphisms*, i.e. by projectors whose kernel is the vertical subbundle (differentiability on the zero section is not required!). Any horizontal endomorphism gives rise to a special Finsler connection, called a *Berwald-type connection*. A Berwald-type connection is said to be a *Berwald connection* if the horizontal endomorphism is generated by a spray. In this case the horizontal endomorphism will be mentioned as a *Berwald endomorphism*. Any Berwald-type connection has two surviving “partial curvatures”, the horizontal and the mixed curvature. A careful analysis of the behavior of the mixed curvature of a Berwald connection under a projective change of the associated spray yields the Douglas tensor. However, in order to identify the Douglas tensor we shall follow a slightly different path. As a generalization of Berwald-type connections, we introduce the so-called *Yano-type connections*. If, in particular, we start from a Berwald endomorphism, then the construction results in a *Yano connection*, called also - unfortunately - a projective connection. In our present approach the definition, as well as the proof of the projective invariance of the Douglas tensor will be given in terms of a Yano connection. Although the central object of this study is the Douglas tensor, a very brief section will also be devoted to the projective Weyl tensors. This includes a simple proof of the following, quite recently discovered (see [5]), important fact: the vanishing of the Douglas tensor implies that one of the Weyl tensors is a vertical lift, i.e., its components depend only on the position.

In *part II* we start with the definition of *Hilbert 1-forms* and introduce the concept of *Finsler manifolds*. With the help of the so called *Finsler-Lagrangian* L we can introduce the *canonical spray* S of a Finsler manifold and the canonical horizontal endomorphism generated by S which is called *Barthel endomorphism*. Since every Finsler manifold is a spray manifold, we can also speak of the projective equivalence of a spray manifold and a Finsler manifold, and of that of two Finsler manifolds (which have, of course, a common carrier manifold). A spray manifold (M, S) is said to be *Finsler-metrizable in the broad sense* or — following SHEN’s terminology [48] — *projectively Finsler*, if there exists a Finsler-Lagrangian $L : TM \rightarrow \mathbb{R}$ such that the Finsler manifold (M, L) is projectively equivalent to (M, S) . If, in particular, the canonical spray of (M, L) coincides with the given spray S , then

we say that (M, S) is *Finsler-metrizable in the natural sense* or that S is a *Finsler-variational spray*. The latter concept is a faithful analogue of the variability of a spray (or a semispray) used in the classical inverse problem of the calculus of variations ([19], [28], [29], [33]). Our problem is *to find criteria for the Finsler-metrizability of a spray both in the broad sense and in the natural sense*. The key ingredients will be *the fundamental equations of projective equivalence*. These provide equivalent (and, in our presentation, intrinsically formulated) second order partial differential equations for the Finsler-Lagrangian to be determined. Their coordinate version was discovered by A. RAPCSÁK in the early sixties (see [42], [43], [44], [45] and, for a recent account, [48]), hence we call them *Rapcsák's equations*.

In his paper [16] M. CRAMPIN presented a stimulating intrinsic reformulation of the Helmholtz conditions from the classical inverse problem of the calculus of variations through the existence of a 2-form on the tangent manifold. Prescribing for this 2-form some extra conditions of homogeneity and regularity we derive necessary and sufficient conditions for the metrization of a spray in both senses. Next we investigate the question of metrization from another point of view. We present equivalent conditions for a spray to be Finsler-variational assuming the existence of a symmetric, non-degenerate $(0,2)$ tensor field on the vertical bundle with some further properties. Similarly, our *alternative theorem* gives a characterization of projectively Finsler spray manifolds through the existence of a $(0,2)$ tensor field on the tangent manifold, satisfying some, partly quite complicated, algebraic conditions. The significance of our alternative theorem lies in the fact that it reduces the problem of Finsler-metrizability in the broad sense to a *first order* partial differential equation. Finally we give its integrability condition and supporting it by some regularity conditions we derive through the Rapcsák equation necessary and sufficient conditions for a spray to be projectively Finsler.

In *part III* we present an intrinsic description of the *projection onto the indicatrix bundle* of a Finsler manifold and give a new, coordinate-free proof for the theorem: *a Landsberg manifold with vanishing Douglas tensor is a Berwald manifold*. In the two-dimensional case this was shown by L. BERWALD in the third part [9] of his famous series on Finsler and Cartan geometries. More than 40 years later it was announced by H. IZUMI [32] that this is also true in higher dimensions. IZUMI's statement was proved by the group S. BÁCSÓ, F. ILOSVAY and B. KIS in [3], using the machinery of classical tensor calculus and utilizing an important observation of T. SAKAGUCHI [47] on the projected Douglas tensor. Finsler manifolds with vanishing Douglas tensor were baptized *Douglas manifolds* by M. MATSUMOTO and S. BÁCSÓ [4]. Thus, briefly speaking, *any Landsberg-Douglas manifold is a Berwald manifold*.

The method of the projection onto the indicatrix bundle is based on the following fact: one of the canonical geometrical objects on the tangent bundle, the Liouville vector field, is a normal vector field to the “unit sphere bundle” or “indicatrix bundle” $U(M)$ of any Finsler manifold (M, L) . Keeping this in mind, the construction of the orthogonal projector into $U(M)$ is straightforward. As a next step, we define the *projected tensor* of a tensor given on the tangent bundle. On

this conceptual basis, we compute the projected Douglas tensor, which makes it possible to elaborate a new, coordinate-free proof of SAKAGUCHI's important observation that the projected Douglas tensor and the Douglas tensor vanish together. We note that in local coordinates our formulas, including the coordinate expression of the projected Douglas tensor, do not coincide with those of SAKAGUCHI. The reason for this is the difference in the underlying bundle structures, which implies that the notions of "Finsler tensor fields" are also different. Most of the "essential" tensors are semibasic in our framework, while the vector field variables of MATSUMOTO's theory [38] used by SAKAGUCHI correspond to the vertical vector fields in our approach.

Finally the reader will find the deduction of the main result of this part. Due to the special form of the mixed curvature of the Berwald connection in Landsberg-Douglas manifolds we derive an important relation (14.3) from which the theorem mentioned above may be concluded.

Unfortunately, our elegant method fails in two dimensions. Then the Douglas tensor has surviving components only along the Liouville vector field. Since the Liouville vector field is orthogonal to the unit sphere bundle, it follows immediately that *the projected Douglas tensor of a two-dimensional Finsler manifold vanishes identically*. In the next part we complete the story interpreting the previously cited paper of BERWALD on the two-dimensional Finsler manifolds from our present day standpoint. In BERWALD's own words: "*The Landsberg spaces, the extremals of which form a quasigeodesic system of curves, are identical with the two-dimensional affinely connected Finsler spaces.*" ([9], p. 110.) In our language, the condition on the extremals states that the geodesics of a Landsberg manifold coincide with those of a linear connection on the underlying manifold. Since by the Douglas-Shen theorem (6.8) this property is equivalent to the vanishing of the Douglas tensor, our formulation is indeed a translation of BERWALD's theorem.

Our approach is based on BERWALD's original ideas, and it may be considered as an intrinsic version of them. In this section we restrict our investigation to the *positive definite two-dimensional Finsler manifolds*. (It is possible to generalize the theorem to two-dimensional Finsler manifolds with nondegenerate Riemann-Finsler metric. This needs a little modification of Berwald's method as it turns out from the paper [7], where the machinery of classical tensor calculus is applied.) In this part we sketch a general coordinate-free theory of the two-dimensional (positive definite) Finsler manifolds, introducing the *Berwald frame*, the *main scalar* and the *Gaussian curvature* and deriving E. CARTAN's "*permutation formulas*" as well as the "*Bianchi identity*". Next we turn to the two-dimensional Landsberg-Douglas manifolds proving the statement mentioned above.

Finally we investigate the following problem: *what kind of (two-dimensional) Finsler manifolds admit a conformal change leaving the mixed curvature of the Berwald connection invariant?* If the mixed curvature of the Berwald connection remains invariant, then the manifold admits an associated Riemannian metric [58] and we can form the Riemannian gradient $grad_R \alpha$ of the scale function α . Fix a nonzero tangent vector v such that $v\alpha = 0$ and consider the integral curve

$c(t) := v + t \operatorname{grad}_R \alpha(p)$ of the vertical lift of $\operatorname{grad}_R \alpha$. We set up a differential equation for the energy along this integral curve. To find the general solution seems to be very complicated, so we restrict ourselves here to a simplified case (fixing a parameter). Then the solutions essentially are the singular Finsler metrics with constant main scalar determined by Berwald [9]. We show that on these manifolds the mixed curvature of the Berwald connection remains invariant under *any* conformal change. We determine the vertical members of the Berwald frame and compute the main scalar. Finally we show that if the Douglas tensor vanishes, then we get a Berwald manifold, also in this case.

A brief concluding chapter is devoted to the coordinate expressions of the geometric objects and basic relations (identities) we were dealing with in an invariant form.

ÖSSZEFOGLALÓ

Disszertációnk új nézőpontból és modern eszközökkel tárgyal problémákat a pályaterek (spray sokaságok) és a Finsler-sokaságok geometriájából. Megközelítésünk jelentős mértékben JOSEPH GRIFONE hetvenes években kidolgozott elméletét [25], [26] követi. Munkánk során technikai eszközként és különböző geometriai objektumok megkonstuálásakor elsősorban a vektorértékű differenciálformák Frölicher–Nijenhuis kalkulusát alkalmazzuk, amelyet a számolások egyszerűbbé tétele érdekében kombinálunk egy alkalmas, az alapsokaság lokális bázisát képező vektormezők vertikális és teljes (vagy vertikális és horizontális) liftjeiből álló frame–mező használatával.

A spray sokaságok geometriája az affinösszefüggő sokaságok – torziómentes lineáris konnexióval ellátott sokaságok – elméletéből fejlődött ki. Egy ilyen sokaság geodetikusainak differenciálegyenlete kvadratikus, mai szóhasználatnál élve: a geodetikus spray 2-homogén és C^2 -osztályú. Gyenítve a differenciálhatósági feltételt, és csupán C^1 -osztályúságot követelve meg jutunk az (általános) spray fogalmához. Ezen sprayk lokális tanulmányozása, a 'pályaterek' geometriájának vizsgálata a múlt század huszas éveiben kezdődött és elsősorban L. BERWALD, J. DOUGLAS, M. S. KNEBELMAN, T. Y. THOMAS, O. VEBLEN nevéhez fűződik a megalapozása. A magyar kutatók közül RAPCSÁK ANDRÁS akadémikus vitte tovább lényeges mértékben ezt az elméletet az 1960-as években; disszertációnk sokban adós az ő tudományos örökségének.

A sprayk vizsgálata az 1970-es években került ismét előtérbe, amikor felismerték, hogy a Lagrange–mechanika geometriai megalapozásában alapvető szerepet játszanak. Egy időfüggetlen Lagrange–rendszer dinamikáját a rendszer konfigurációs terén ható spray határozza meg. A Finsler-sokaságok rendelkeznek egy 'kanonikus spray'-vel, melyet egy alkalmas Lagrange–függvény által meghatározott energiafüggvényből származtatunk (lásd 8.4 és 8.6). J. KLEIN, J. GRIFONE és M. CRAMPIN végzett úttörő munkát ezen a területen, disszertációnk az általuk kijelölt irányban lép tovább.

Munkánk egyik központi témája a sprayk projektív geometriájának vizsgálata. Ugyanazon sokaságon adott két sprayt projektívan ekvivalens, ha a geodetikusaik, mint ponthalmazok megegyeznek. Adott spray esetén egy vele projektívan ekvivalens sprayre való áttéréstől azt mondjuk, hogy *projektív változtatást* hajtottunk végre. A sprayk projektív geometriájának tanulmányozása során a projektív változtatással szemben invariáns tulajdonságokat keresünk. Disszertációnk első fejezetében a projektívan invariáns Douglas–tenzor koordinátamentes ("intrinsic") előállítását konstruáljuk meg. Affinösszefüggő esetben projektívan invariáns a

Weyl-tenzor [60]. Ennek nemlineáris általánosítása L. DEL CASTILLO nevéhez fűződik [20] (lásd még [34]).

A koordinátás formájában jól ismert Douglas-tenzor "beazonosításához" vezető út vázlatosan a következő. Először is értelmezzük a nemlineáris (homogén) konnexió fogalmát, amely megközelítésünkben egy ún. horizontális endomorfizmust jelent: az érintősokaságon ható olyan $(1,1)$ -tenzort, mely projektor és eltűnik a vertikális nyalábon. Minden horizontális endomorfizmus származtat egy Finsler-konnexiót, melyet *Berwald-típusúnak*, ill., ha a horizontális endomorfizmus spray-ből nyerhető, akkor *Berwald-konnexiónak* hívunk. Ha megvizsgáljuk, mi történik projektív változtatás során a Berwald-konnexió vegyes görbületi tenzorával, eljuthatunk az invariánsan maradó Douglas-tenzorhoz. Munkánk során mégsem teljesen ezt az utat követjük: a Berwald-típusú, ill. a Berwald-konnexió általánosításaként bevezetjük a Yano-típusú, ill. a Yano-konnexiót, s ezen konnexió adataival építjük fel a Douglas-tenzort és bizonyítjuk be projektív invarianciáját. (Természetesen megadjuk a Douglas-tenzornak a Berwald konnexió adataival való kifejezését is.) A teljesség kedvéért egy rövid részt szánunk a másik projektíven invariáns görbületi tenzorra is: a del Castillo által intrinsic módon felépített Weyl-tenzornak [20] a Berwald- (vagy a Yano-)konnexió szerinti vertikális kovariáns differenciáljaként bevezetjük a Berwald-konnexió Weyl-tenzorát, majd egyszerű bizonyítást adunk arra a Bácsó Sándor és M. Matsumoto által fölfedezett tényre [5], hogy a Douglas-tenzor eltűnése azt vonja maga után, hogy *a Berwald-konnexió Weyl-tenzora vertikális lift* (a komponensei 'csak a helytől függenek').

A második fejezetben rögzítjük a *Finsler-sokaságok* definícióját, bevezetjük a *kanonikus sprayt*, mely a kanonikus nemlineáris konnexiót (az ún. *Barthel-endomorfizmust*) származtatja. Egy sokaságon adott sprayt *tágabb értelemben metrizálható*nak vagy *projektíven Finslernek* mondunk, ha létezik olyan Finsler-alapfüggvény, melyhez tartozó kanonikus spray projektíven ekvivalens az eredetivel. Ha a spray megegyezik egy Finsler-alapfüggvényhez tartozó kanonikus sprayvel, akkor *természetes értelemben metrizálható* vagy *Finsler-variációs sprayről* beszélünk (v.ö. [42]). A második eset vizsgálata éppen a variációszámítás inverz problémája homogén esetben. Kitűzött célunk, hogy adott spray esetén a mindkét értelemben vett metrizálhatóságra szükséges és elegendő feltételrendszert adjunk. Projektíven Finsler-alapfüggvény megtalálására RAPCSÁK ANDRÁS állított föl másodrendű parciális differenciálegyenleteket [42], [43], [44], [45] a hatvanas években. Ezen egyenletek, ill. az általunk megtalált koordinátamentes formájuk képezik a kiindulópontot vizsgálataink számára.

Egy igen nagyhatásúnak bizonyult dolgozatában [16] M. CRAMPIN a variációszámítás inverz problémájának *Helmholtz-feltételeire* adott ekvivalens, koordinátamentes formában megfogalmazott feltételrendszert, megkövetelve egy, az érintősokaságon ható, alkalmas 2-forma létezését. Erre a 2-formára további homogenitási és regularitási tulajdosságokat előírva, disszertációnkban *szükséges és elegendő feltételeket vezetünk le egy spray mindkét értelemben vett metrizálhatóságára*. Ezek után a metrizálhatóságot más oldalról is megközelítjük. Egy spray Finsler-variációsságára adunk ekvivalens feltételeket úgy, hogy egy szimmetrikus, nemelfajuló $(0,2)$ -tenzor létezését kívánjuk meg (melynek további feltételeket is ki kell elégítenie). Ha-

sonlóan, az *alternatíva-tétel*nek keresztelt eredményünk egy spray tágabb értelemben vett metrizálhatóságára ad ekvivalens feltételeket úgy, hogy olyan $(0, 2)$ -tenzor létezését követeli meg az érintőnyalábon, mely még további algebrai feltételeket is teljesít. A tétel jelentősége az, hogy a *tágabb értelemben vett metrizálhatóságra vonatkozó egyenleteket elsőrendű parciális differenciálegyenletekre redukálja*. Megadjuk ezen egyenletek integrabilitási feltételeit és - kiegészítve ezeket homogenitási és regularitási követelményekkel - a Rapcsák-egyenletek felhasználásával szükséges és elegendő feltételeket vezetünk le arra, hogy egy spray projektívan Finsler legyen.

A harmadik fejezetben az induktrix-nyalábra történő vetítés intrinszc konstrukcióját írjuk le, valamint új, koordinátamentes bizonyítást adunk a következő állításra: *ha egy Landsberg-sokaságon a Douglas-tenzor eltűnik, akkor a sokaság Berwald-sokaság*. 2-dimenzióban ezt Berwald bizonyította be 1941-ben megjelent posztumusz dolgozatában [9]. Több mint negyven évvel később H. IZUMI mutatott rá arra [32], hogy Berwald tétele igaz magasabb dimenzióban is. Izumi állítását S. BÁCSÓ, F. ILOSVAY és B. KIS igazolta [3], felhasználva T. SAKAGUCHI azon észrevételét [47], miszerint a Douglas-tenzor és az induktrix-nyalábra való vetítettje egyszerre tűnik el.

Az induktrix-nyalábra való vetítés azon az egyszerű tényen alapul, hogy a Liouville-vektormező (a kanonikus vertikális vektormező) az induktrix-nyalábra merőleges vektormező, így a vetítés operátora a lineáris algebrából ismert módon megadható. (Ezt az operátort elsőként J. GRIFONE vezette be [27].) Ezután egy, az érintősokaságon adott tenzor levetítettjét definiáljuk. Kiszámítjuk a vetített Douglas-tenzort – és ezzel minden készen áll ahhoz, hogy igazoljuk Sakaguchi észrevételét.

A fejezet fő eredményéhez a következő megfontolások figyelembevételével jutunk el: Landsberg-sokaságokon a Berwald-konnexió vegyes görbületi tenzora speciális alakú, a Douglas-tenzor (ill. a levetített Douglas-tenzor) eltűnése ezt a formát tovább specializálja. Ebből az alakból vezetünk le egy olyan önmagában is érdekes formulát, melyből nyomban adódik a következtetés, hogy az ilyen sokaságok Berwald-sokaságok.

Ez az elegáns eljárás a 2-dimenziós esetben nem működik, ugyanis ilyenkor a Douglas-tenzornak csak a Liouville vektormező irányában van nemzérus komponense, s így a vetített Douglas-tenzor automatikusan eltűnik. A következő fejezetben, más módszerrel, bebizonyítjuk az állítást a kétdimenziós esetben is. Berwald eredeti gondolatmenetét követjük; a fejezet első két része úgy tekinthető, mint annak egy intrinszc verziója. Az első részben bevezetjük a *Berwald-framet*, a *főskalárt*, a *Gauss-görbületet*, levezetjük E. CARTAN *permutációs formuláit* és a *Bianchi-identitást*. A második részben bebizonyítjuk az említett állítást.

Az utolsó részben a következő problémát vizsgáljuk: milyen (2-dimenziós) Finsler-sokaságokban lehetséges olyan konform változtatás, melynek során a Berwald-konnexió vegyes görbületi tenzora invariáns marad. Ismeretes [58], hogy ebben az esetben a sokaság egy (lokális) Riemann-metrikával bír, így képezhető az α skálázó függvény ezen Riemann-metrika szerinti gradiense. Rögzítve egy v érintővektort úgy, hogy $v\alpha = 0$, a $c(t) := v + t\text{grad}_R\alpha(p)$ görbe éppen az α skálázó függvény Riemann-gradiense vertikális liftjének a v -ből induló integrálgörbéje. Levezetünk

egy differenciálegyenletet, melyet a Finsler-energiafüggvénynek teljesítenie kell ezen görbe mentén. Az egyenlet megoldása általános esetben nagyon bonyolult, ezért disszertációnkban csak egy egyszerűbb esetet vizsgálunk (az egyik paramétert rögzítjük). Ekkor a megoldások megegyeznek a BERWALD által meghatározott, konstans főskalárral rendelkező, szinguláris Finsler-energiafüggvényekkel. Megmutatjuk, hogy konstans főskalár esetén minden konform változtatás invariánsan hagyja a Berwald-konnexió vegyes görbületi tenzorát. Azt is bebizonyítjuk, hogy ekkor a Douglas-tenzor eltűnéséből következik a sokaság Berwald-volta.

Megemlítjük, hogy dolgozatunk rövid záró fejezetében összegyűjtöttük a legfontosabb formulák koordinátakifejezését a lokális számolásokat inkább preferáló olvasók kedvéért.

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- [58]SZ. VATTAMÁNY and CS. VINCZE, *On a new geometrical derivation of two-dimensional Finsler manifolds with constant main scalar*, (submitted).
- [59]CS. VINCZE, *On conformal equivalence of Berwald manifolds all of whose indicatrices have positive curvature*, (submitted).
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APPENDIX

List of Publications

- [1] J. SZILASI and SZ. VATTAMÁNY, *On the projective geometry of sprays*, Differ. Geom. Appl **12** (2000), 183–208.
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R. L. LOVAS, *Affine and projective vector fields on spray manifolds*, Periodica Mathematica Hungarica, (to appear) [1], [4]

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CS. VINCZE, *On geometric vector fields of Minkowski spaces and their applications*, (to appear) [4]

Participation in Conferences

The 11th National Seminar on Finsler and Lagrange Geometry, 17–20 February, 2000, Bacau, Romania

Colloquium on Differential Geometry and its Applications, 25–30 July, 2000, Debrecen, Hungary

Workshop on Finsler Geometry and its Applications, 11–15 August, 2003, Debrecen, Hungary

**ON THE PROJECTIVE GEOMETRY AND METRIZABILITY
OF SPRAY MANIFOLDS**

értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a matematika tudományágban

Írta: Vattamány Szabolcs okleveles matematika-fizika szakos tanár

Készült a Debreceni Egyetem Matematika doktori iskolája
(Differenciálgeometria programja) keretében

Témavezető: Dr. Szilasi József egyetemi docens

A doktori szigorlati bizottság:

elnök:	Dr.	Dr.
tagok:	Dr.	Dr.
	Dr.	Dr.

.....

A doktori szigorlat időpontja: 200... ..

Az értekezés bírálói:

Dr.	Dr.
Dr.	Dr.
Dr.	Dr.

A bírálóbizottság:

elnök:	Dr.	Dr.
tagok:	Dr.	Dr.
	Dr.	Dr.
	Dr.	Dr.
	Dr.	Dr.

Az értekezés védésének időpontja: 200... ..