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The Holonomy of Spherically Symmetric Projective Finsler Metrics of Constant Curvature

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Abstract

In this paper, we investigate the holonomy group of n-dimensional projective Finsler metrics of constant curvature. We establish that in the spherically symmetric case, the holonomy group is maximal, and for a simply connected manifold it is isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$, the connected component of the identity of the group of smooth diffeomorphism on the $n-1$-dimensional sphere. In particular, the holonomy group of the n-dimensional standard Funk metric and the Bryant–Shen metrics are maximal and isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$. These results are the firsts describing explicitly the holonomy group of n-dimensional Finsler manifolds in the non-Berwaldian (that is when the canonical connection is non-linear) case.

Keywords Finsler geometry · Holonomy · Curvature · Diffeomorphism groups

Mathematics Subject Classification 53C29 · 53B40 · 22E65

1 Introduction

The holonomy group of a Riemannian or Finslerian manifold is a very natural algebraic object attached to the geometric structure: it is the group generated by parallel translations along loops with respect to the canonical connection. Riemannian holonomy groups have been extensively studied and their complete classification is now known.

On the Finslerian holonomy, relatively few results are known. Z.I. Szabó proved in [18] that in the case where the Finslerian parallel translation is linear, then there exist Riemannian metrics having the same holonomy group, and for Landsberg metrics L. Kozma showed in [8] that the holonomy groups are compact Lie groups consisting of isometries of the indicatrix with respect to an induced Riemannian metric. The first

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paper showing that the holonomy property of Finsler manifolds can be very different from the Riemannian is [9] by proving that the holonomy group of a Finsler manifold is not necessarily a compact Lie group. In [11] it was proven that there are 2-dimensional Finsler surfaces with infinite dimensional holonomy group, isomorphic to $\mathcal{D}iff_+(\mathbb{S}^1)$, the orientation preserving diffeomorphism group of the circle in the orientable case, and as a consequence, isomorphic to $\mathcal{D}iff(\mathbb{S}^1)$, the diffeomorphism group of the circle in the non-orientable case. We emphasize that explicit examples of Finsler holonomy groups for nonlinear connection existed only in the 2-dimensional cases.

In this article we are focusing on the holonomy structure of n-dimensional Finsler manifolds. We use the method developed in [7] for the investigation of holonomy properties of Finsler manifolds by constructing a tangent Lie algebra to the holonomy group, called the holonomy algebra, and its subalgebra: the infinitesimal holonomy algebra. After a brief introduction of the basic notation in Section 2, we give the most important element of this theory in Chapter 3. We give a sufficient condition on the holonomy group to be maximal in terms of the infinitesimal holonomy algebra in Theorem 3.3.

In Chapter 4 we investigate the holonomy structure of projective Finsler manifold (M, F) of constant curvature. In [17], Z. Shen proved that in the x-analytic case, those metrics are determined by their Finsler norm function and projective factor at a single point $x_o \in M$. We consider the *spherically symmetric* case, where there exists a point $x_o \in M$ where both the norm and the projective factor are nonzero multiples of the Euclidean norm. If the curvature vanishes, then the horizontal distribution associated with the canonical connection in the tangent bundle is integrable, and hence the holonomy group is trivial. This is why we are focusing on the non-flat case. In Theorem 4.5 we show that on a simply connected manifold, a spherically symmetric projective Finsler metrics of nonzero constant curvatures have maximal holonomy group. The holonomy groups in those cases are isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$, the connected component of the identity of the group of smooth diffeomorphism on the $n-1$-dimensional sphere. In particular, we obtain that the holonomy group of the n-dimensional standard Funk metric and the n-dimensional Bryant–Shen metrics are isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$.

2 Preliminaries

Throughout this article, M is a C^∞ smooth manifold, $\mathfrak{X}(M)$ is the vector space of smooth vector fields on M and $\mathcal{D}iff(M)$ is the group of all C^∞-diffeomorphism of M. The first and the second tangent bundles of M are denoted by (TM, π, M) and (TTM, τ, TM), respectively. Local coordinates (x^i) on M induce local coordinates (x^i, y^i) on TM.

2.1 Finsler Manifolds, Canonical Connection

A *Finsler manifold* is a pair $(M, \mathcal{F})$, where the norm function $F\colon TM \to \mathbb{R}_+$ is continuous, smooth on $\widehat{T}M := TM \backslash \{0\}$, its restriction $F_x = F|_{T_xM}$ is a positively homogeneous function of degree one and the symmetric bilinear form

$$g_{x,y}\colon (u,v) \;\mapsto\; g_{ij}(x,y)u^i v^j = \frac{1}{2}\frac{\partial^2 F_x^2(y+su+tv)}{\partial s\,\partial t}\bigg|_{t=s=0}$$

is positive definite at every $y \in \hat{T}_x M$. The hypersurface of $T_x M$ defined by

$$\mathcal{I}_x = \{\, y \in T_x M \mid F(x,y)=1 \,\}, \tag{1}$$

is called the *indicatrix* at $x \in M$. We note that at any point $x \in M$ the indicatrix is diffeomorphic to the $(n-1)$-dimensional sphere. *Geodesics* of (M, F) are determined by a system of 2nd order ordinary differential equation

$$\ddot{x}^i + 2G^i(x,\dot{x}) = 0, \quad i = 1,\dots,n, \tag{2}$$

in a local coordinate system (x^i, y^i) of TM, where $G^i(x,y)$ are given by

$$G^i(x,y) := \frac{1}{4}g^{il}(x,y)\Big(2\frac{\partial g_{jl}}{\partial x^k}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big)y^j y^k. \tag{3}$$

A vector field $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c(t)$ is said to be parallel with respect to the associated *homogeneous (nonlinear) connection* if it satisfies

$$D_{\dot c}X(t) := \Big(\frac{dX^i(t)}{dt} + G^i_j(c(t), X(t))\dot{c}^j(t)\Big)\frac{\partial}{\partial x^i} = 0, \tag{4}$$

where $G^i_j = \frac{\partial G^i}{\partial y^j}$.

The *horizontal Berwald covariant derivative* $\nabla_X \xi$ of $\xi(x,y) = \xi^i(x,y)\frac{\partial}{\partial y^i}$ by the vector field $X(x) = X^i(x)\frac{\partial}{\partial x^i}$ is expressed locally by

$$\nabla_X \xi = \left(\frac{\partial \xi^i(x,y)}{\partial x^j} - G^k_j(x,y)\frac{\partial \xi^i(x,y)}{\partial y^k} + G^i_{jk}(x,y)\xi^k(x,y)\right)X^j\frac{\partial}{\partial y^i}, \tag{5}$$

where we denote $G^i_{jk}(x,y) := \frac{\partial G^i_j(x,y)}{\partial y^k}$. In the sequel we will use the simplified notation $\nabla_k \xi = \nabla_{\frac{\partial}{\partial x^k}}\xi$ for the horizontal Berwald covariant derivatives with respect to the coordinate directions.

The *Riemannian curvature tensor* field $R = R^i_{jk}(x,y)dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}$ has the expression

$$R^i_{jk}(x,y) = \frac{\partial G^i_j(x,y)}{\partial x^k} - \frac{\partial G^i_k(x,y)}{\partial x^j} + G^m_j(x,y)G^i_{km}(x,y) - G^m_k(x,y)G^i_{jm}(x,y).$$

2.2 Projective Finsler Manifold With Constant Curvature

A Finsler function F on an open subset $D \subset \mathbb{R}^n$ is said to be *projective* or projectively flat, if all geodesic curves are straight lines in D. A Finsler manifold is said to be *locally projective* or locally projectively flat, if at any point there is a local coordinate system (x^i) in which F is projective.

Let $(x^1, \dots, x^n)$ be a local coordinate system on M corresponding to the canonical coordinates of the Euclidean space which is projectively related to (M, F). Then the geodesic coefficients (3) and their derivatives have the form

$$G^i = \mathcal{P}(x,y)y^i, \qquad G^i_k = \frac{\partial \mathcal{P}}{\partial y^k} y^i + \mathcal{P}\delta^i_k, \qquad G^i_{kl} = \frac{\partial^2 \mathcal{P}}{\partial y^k \partial y^l} y^i + \frac{\partial \mathcal{P}}{\partial y^k}\delta^i_l + \frac{\partial \mathcal{P}}{\partial y^l}\delta^i_k, \tag{6}$$

where $\mathcal{P}$ is a 1-homogeneous function in y, called the projective factor of (M, F). According to [4, Lemma 8.2.1, p. 155] the projective factor can be computed using the formula

$$\mathcal{P}(x,y) = \frac{1}{2\mathcal{F}} \frac{\partial \mathcal{F}}{\partial x^i} y^i. \tag{7}$$

The Finsler manifold has *constant flag curvature* $\lambda \in \mathbb{R}$, if for any $x \in M$ the local expression of the Riemannian curvature is

$$R^i_{jk}(x,y) = \lambda\big(\delta^i_k g_{jm}(x,y)y^m - \delta^i_j g_{km}(x,y)y^m\big). \tag{8}$$

In this case the flag curvature of the Finsler manifold (cf. [4], Section 2.1) does not depend on the point, nor on the 2-flag.

Lemma 2.1 *([12]) The horizontal covariant derivative $\nabla_W R$ of the tensor field $R = R^i_{jk}(x,y)dx^j \wedge dx^k \frac{\partial}{\partial x^i}$ vanishes.*

Proof The Lemma is a consequence of the fact, that the horizontal covariant derivative of the Finsler function vanishes. Indeed, [16, Lemma 6.2.2, p. 85] yields

$$\nabla_w g_{(x,y)}(u,v) = -2L(u,v,w),$$

for any $u, v, w \in T_xM$, where L is the Landsberg curvature of the Finsler metric $\mathcal{F}$ [16, Chapter 6.2]. Moreover $\nabla_W y = 0$, $\nabla_W \mathrm{Id}_{TM} = 0$ for any vector field $W \in \mathfrak{X}^\infty(M)$, and $L_{(x,y)}(y,v,w) = 0$ (cf. equation 6.28, p. 85 in [16]). Hence we obtain $\nabla_W R = 0$. □

3 Holonomy

3.1 Parallel Translation and the Holonomy Group

Let $(M, \mathcal{F})$ be a Finsler manifold. The *parallel translation* $\tau_c \colon T_{c(0)}M \to T_{c(1)}M$ along a curve $c : [0,1] \to \mathbb{R}$ is defined by vector fields $X(t)$ along $c(t)$ which are solutions of the differential Eq. (4). Since $\tau_c : T_{c(0)}M \to T_{c(1)}M$ is a differentiable map between $\hat{T}_{c(0)}M$ and $\hat{T}_{c(1)}M$ preserving the value of the Finsler norm, it induces a map

$$\tau_c\colon \mathcal{I}_{c(0)} \longrightarrow \mathcal{I}_{c(1)}, \tag{9}$$

between the indicatrices.

The *holonomy group* $\mathcal{H}ol_x(M, \mathcal{F})$ of a Finsler manifold $(M, \mathcal{F})$ at a point $x \in M$ is the group generated by parallel translations along piece-wise differentiable closed curves starting and ending at x. Since the parallel translation (9) is 1-homogeneous and preserves the norm, one can consider it as a map on the indicatrices

$$\mathcal{T}_c : \mathcal{I}_x \to \mathcal{I}_x, \tag{10}$$

therefore, the holonomy group can be seen as a subgroup of the diffeomorphism group of the indicatrix:

$$\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}iff(\mathcal{I}_x). \tag{11}$$

3.2 Holonomy Algebra and Infinitesimal Holonomy Algebra

The tangent Lie algebra (see [7]) of the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ is called the *holonomy algebra* and is denoted as $\mathfrak{hol}_x(\mathcal{F})$. The holonomy algebra can give information about the holonomy property of the Finsler manifold.

When $\mathcal{H}ol_x(\mathcal{F})$ is a finite-dimensional Lie group, $\mathfrak{hol}_x(\mathcal{F})$ is its Lie algebra. In particular, for a Riemannian metric, $\mathcal{H}ol_x(\mathcal{F})$ is a Lie subgroup of the orthogonal group [1], and $\mathfrak{hol}_x(\mathcal{F})$ is its Lie algebra. In the Finslerian case, however, it may happen that $\mathcal{H}ol_x(\mathcal{F})$ is not a finite dimensional Lie group [10–12].

Considering the tangent spaces of both sides in (11) we obtain

$$\mathfrak{hol}_x(\mathcal{F}) \subset \mathfrak{X}(\mathcal{I}_x). \tag{12}$$

The most important properties of the holonomy algebra are given by the following

Proposition 3.1 ([7]) *Let $(M, \mathcal{F})$ be a Finsler manifold. Then the holonomy algebra $\mathfrak{hol}_x(\mathcal{F})$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{I}_x)$, and its exponential image is in the topological closure of the holonomy group, that is*

$$\exp\big(\mathfrak{hol}_x(\mathcal{F})\big) \subset \overline{\mathcal{H}ol_x(\mathcal{F})}, \tag{13}$$

where the overline denotes the topological closure of the holonomy group with respect to the C^∞–topology of $\mathcal{D}iff(\mathcal{I}_x)$.

From Proposition 3.1 one can obtain, that a Lie subalgebra of $\mathfrak{X}(\mathcal{I}_x)$ generated by any subset of $\mathfrak{hol}_x(\mathcal{F})$ is also a Lie subalgebra of $\mathfrak{hol}_x(\mathcal{F})$, and in particular, its elements have the tangent property to the holonomy group $\mathcal{H}ol_x(\mathcal{F})$.

One can show that for any point $x \in M$ and any tangent vectors $X_1, X_2 \in T_x M$, the curvature vector field $R(X_1, X_2) \in \mathfrak{X}(\mathcal{I}_x)$ considered as

$$y \to R_{(x,y)}(X_1, X_2), \tag{14}$$

and its successive convariant derivatives are tangent to the holonomy group [7]. It follows that the Lie subalgebra of vector fields on the indicatrix $\mathcal{I}_x$ generated by the curvature vector fields (14) and their successive covariant derivatives

$$\mathfrak{hol}_x^*(\mathcal{F}) := \big\langle \nabla_{X_k} \dots \nabla_{X_3} R(X_1, X_2) \;\big|\; X_1, \dots, X_k \in \mathfrak{X}(M) \big\rangle_{Lie}, \tag{15}$$

is a Lie subalgebra of $\mathfrak{hol}_x(\mathcal{F})$. From (13) we get

$$\exp\big(\mathfrak{hol}_x^*(\mathcal{F})\big) \subset \overline{\mathcal{H}ol_x(\mathcal{F})}. \tag{16}$$

Definition 3.2 The Lie algebra $\mathfrak{hol}_x^*(\mathcal{F})$ defined in (15) is called the *infinitesimal holonomy algebra* of the Finsler space $(M, \mathcal{F})$ at $x \in M$.

One has the inclusion of Lie algebras:

$$\mathfrak{hol}_x^*(\mathcal{F}) \subset \mathfrak{hol}_x(\mathcal{F}) \subset \mathfrak{X}(\mathcal{I}_x), \tag{17}$$

therefore, at the level of groups, we get

$$\exp\big(\mathfrak{hol}_x^*(\mathcal{F})\big) \subset \exp\big(\mathfrak{hol}_x(\mathcal{F})\big) \subset \overline{\mathcal{H}ol_x(\mathcal{F})} \subset \mathcal{D}iff(\mathcal{I}_x). \tag{18}$$

We give a sufficient condition on the holonomy group to be maximal in terms of the infinitesimal holonomy algebra in the following

Theorem 3.3 *Let $(M, \mathcal{F})$ be an n-dimensional simply connected Finsler manifold. If the infinitesimal holonomy algebra $\mathfrak{hol}_x^*(\mathcal{F})$ at some point $x \in M$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_x)$ of the vector fields on the indicatrix $\mathcal{I}_x$, then the holonomy group is maxmial:*

$$\overline{\mathcal{H}ol_x(\mathcal{F})} \cong \mathcal{D}iff_o(\mathbb{S}^{n-1}), \tag{19}$$

that is its closure is isomorphic to the connected component of the identity in the diffeomorphism group of the $n-1$-dimensional sphere.

Proof Let M be a simply connected n-dimensional manifold. At any point $x \in M$ we have (11). Since M is simply connected, any closed curve can be shrunk to a point, therefore

$$\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}iff_o(\mathcal{I}_x). \tag{20}$$

On the other hand, from the hypotheses, the infinitesimal holonomy algebra $\mathfrak{hol}^*_x(\mathcal{F})$ is dense in $\mathfrak{X}(\mathcal{I}_x)$, therfore its closure satisfies

$$\overline{\mathfrak{hol}^*_x(\mathcal{F})} = \mathfrak{X}(\mathcal{I}_x). \tag{21}$$

Since the exponential mapping is continuous (c.f. Lemma 4.1 in [15], p. 79), we have

$$\exp\left(\overline{\mathfrak{hol}^*_x(\mathcal{F})}\right) \subset \overline{\exp(\mathfrak{hol}^*_x(\mathcal{F}))}, \tag{22}$$

hence taking into account (21), (22), (16), and (20) respectively, we have

$$\exp(\mathfrak{X}(\mathcal{I}_x)) = \exp\left(\overline{\mathfrak{hol}^*_x(\mathcal{F})}\right) \subset \overline{\exp(\mathfrak{hol}^*_x(\mathcal{F}))} \subset \overline{\mathcal{H}ol_x(\mathcal{F})} \subset \mathcal{D}iff_o(\mathcal{I}_x), \tag{23}$$

which gives for the generated groups the following relations

$$\big\langle \exp(\mathfrak{X}(\mathcal{I}_x)) \big\rangle_{\text{group}} \subset \overline{\mathcal{H}ol_x(\mathcal{F})} \subset \mathcal{D}iff_o(\mathcal{I}_x). \tag{24}$$

Moreover, the conjugation map $Ad : \mathcal{D}iff_o(\mathcal{I}_x) \times \mathfrak{X}(\mathcal{I}_x) \longrightarrow \mathfrak{X}(\mathcal{I}_x)$ satisfies the relation

$$h(\exp s\xi)\, h^{-1} = \exp s\, Ad_h \xi,$$

for every $h \in \mathcal{D}iff_o(\mathcal{I}_x)$ and $\xi \in \mathfrak{X}(\mathcal{I}_x)$.

Since the Lie algebra $\mathfrak{X}(\mathcal{I}_x)$ is invariant under conjugation, therefore the group $\big\langle \exp(\mathfrak{X}(\mathcal{I}_x)) \big\rangle_{\text{group}}$ is also invariant under conjugation and consequently, it is a non-trivial normal subgroup of $\mathcal{D}iff_o(\mathcal{I}_x)$. From [19, Theorem 1.] we know that $\mathcal{D}iff_o(\mathcal{I}_x)$ is a simple group, its only non-trivial normal subgroup is itself, we get that

$$\big\langle \exp \mathfrak{X}(\mathcal{I}_x) \big\rangle_{\text{group}} = \mathcal{D}iff_o(\mathcal{I}_x),$$

and (24) reads as

$$\mathcal{D}iff_o(\mathcal{I}_x) \subset \overline{\mathcal{H}ol_x(\mathcal{F})} \subset \mathcal{D}iff_o(\mathcal{I}_x), \tag{25}$$

that is

$$\overline{\mathcal{H}ol_x(\mathcal{F})} = \mathcal{D}iff_o(\mathcal{I}_x). \tag{26}$$

Since $\mathcal{I}_x$ and $\mathbb{S}^{n-1}$ are diffeomorphic, their diffeomorphism groups and the connected component of their diffeomorphism groups are isomorphic, therefore from (26) we obtain

$$\overline{\mathcal{H}ol_x(\mathcal{F})} \cong \mathcal{D}iff_o(\mathbb{S}^{n-1}). \tag{27}$$

□

4 The Holonomy of Spherically Symmetric Projective Finsler Metrics With Constant Flag Curvature

Z. Shen in his paper [17] investigated projective Finsler metrics with constant flag curvature, and give a complete classification in the x-analytical case. In particular, he showed that a projective Finsler metrics $\mathcal{F}$ with constant flag curvature is completely determined – using an x_o centered coordinate system – by $\mathcal{F}(0, y)$ and $\mathcal{P}(0, y)$, where $\mathcal{P}$ denotes the projective factor. In this chapter we investigate the case when those data are spherically symmetrical, that is at some particular point $x_o \in M$, the Finsler function and the projective factor are both a multiple of the Euclidean norm

$$\mathcal{F}(x_o, y) = c_1\,|y|\,, \quad \mathcal{P}(x_o, y) = c_2\,|y|\,, \tag{28}$$

with $c_1, c_2 \neq 0$.

Remark 4.1 Without loss of generality, the constant c_1 may be considered equal to 1, and instead of (28) to have at $x_o \in M$ the relations:

$$\mathcal{F}(x_o, y) = |y|\,, \qquad \mathcal{P}(x_o, y) = c \cdot |y|\,, \tag{29}$$

where $c \neq 0$. Indeed, there is no loss of generality, since replacing the Finsler norm function $\mathcal{F}$ by a positive constant multiple of it does not change the geodesic equation (2), the parallelism (4), therefore it does not change the holonomy sturcture.

We note that if (29) is satisfied, then the indicatrix at x_o is

$$\mathcal{I}_{x_o} = \big\{ y \in T_{x_o}M \;\big|\; |y| = 1 \big\} = \mathbb{S}^{n-1} \quad \subset \mathbb{R}^n, \tag{30}$$

the $n-1$ dimensional Euclidean sphere. Moreover, for the geodesic coefficients (6) at x_o we have

$$\begin{aligned} G^i(x_o, y) &= c\,|y|\,y^i, \\ G^i_j(x_o, y) &= c\left(\frac{y^i y^j}{|y|} + |y|\,\delta^i_j\right), \\ G^i_{jk}(x_o, y) &= c\left(\frac{y^i}{|y|}\delta^j_k + \frac{y^j}{|y|}\delta^i_k + \frac{y^k}{|y|}\delta^i_j - \frac{y^i y^j y^k}{|y|^3}\right). \end{aligned} \tag{31}$$

Using [4, Lemma 8.2.1] we get for a projective metric:

$$\frac{\partial \mathcal{P}}{\partial x^m} = \mathcal{P}\frac{\partial \mathcal{P}}{\partial y^m} - \lambda \mathcal{F}\frac{\partial \mathcal{F}}{\partial y^m} = \tfrac{1}{2}\frac{\partial(\mathcal{P}^2 - \lambda\mathcal{F}^2)}{\partial y^m}, \tag{32}$$

therefore at x_o where we have (29) we get

$$\frac{\partial \mathcal{P}}{\partial x^m}(x_o, y) = (c^2 - \lambda)y^m, \tag{33}$$

and at x_o we have

$$\frac{\partial G^i_k}{\partial x^m}(x_o, y) = (c^2 - \lambda)(y^i \delta^m_k + y^m \delta^i_k). \tag{34}$$

From (8) we get that at x_o the coefficients of the curvature tensor are

$$R^l_{ij}(x_o, y) = \lambda\big(\delta^l_j \delta^i_m y^m - \delta^l_i \delta^j_m y^m\big) = \lambda\big(\delta^l_j y^i - \delta^l_i y^j\big), \tag{35}$$

therefore the curvature vector fields are

$$\xi_{ij} = R^s_{ij} \tfrac{\partial}{\partial y^s} = \lambda \left(y^i \frac{\partial}{\partial y^j} - y^j \frac{\partial}{\partial y^i} \right), \tag{36}$$

the infinitesimal generators of rotations.
Let us introduce the multiindex notation: for $\mathbf{m} := (m_1, \dots m_n)$ its length is $\ell(\mathbf{m}) = m_1 + \cdots + m_n$, and $\mathbf{y}^{\mathbf{m}} = \prod_{i=1}^n (y^i)^{m_i} = (y^1)^{m_1} \dots (y^n)^{m_n}$. We define

$$\mathcal{A}_p = Span_{\mathbb{R}} \left\{ \frac{\mathbf{y}^{\mathbf{m}}}{|\mathbf{y}|^{\ell(\mathbf{m})}} \, \xi_{ij} \Big|_{\widehat{T}_{x_o} M} \right\}_{1 \le i, j \le n, \, \ell(\mathbf{m}) = p}, \tag{37}$$

and introduce the Lie algebra

$$\mathcal{A} := \oplus_{p=0}^{\infty} \mathcal{A}_p. \tag{38}$$

Remark 4.2 The elements of $\mathcal{A}$ can be seen as 1-homogeneous vector fields on $\widehat{T}_{x_o} M$, or equivalently, as vector fields on the indicatrix $\mathcal{I}_{x_o} \simeq \mathbb{S}^{n-1}$ with polynomial coefficients. Indeed, in (37) the denominators of the coefficients of the curvature vector fields ξ_{ij} is $|y|^p$ which is identically 1 on the indicatrix at the particular point $x_o \in M$.

Lemma 4.3 *The Lie algebra $\mathcal{A}$ satisfies $\mathcal{A} \subset \mathfrak{hol}^*_{x_o}(\mathcal{F})$.*

Proof The infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is generated by the curvature vector fields, their successive horizontal covariant derivatives, and contains their Lie brackets. We will show, using mathematical induction, that the generating elements of $\mathcal{A}_p$ $(p \ge 0)$ can be express as linear combination of elements of $\mathfrak{hol}^*_{x_o}(\mathcal{F})$.

• $p = 0$. The elements of $\mathcal{A}_0$ are just the linear combination of curvature vector fields ξ_{ij} with constant coefficients, and by definitions they are elements of $\mathfrak{hol}^*_{x_o}(\mathcal{F})$.

• $p = 1$. From Lemma 2.1 we know, that the horizontal covariant derivative of the curvature tensor vanishes. It follows that for the curvature vector fields $\xi_{ij} = R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, we get

$$\nabla_k \xi_{ij} = G^s_{ki} \xi_{sj} + G^s_{kj} \xi_{is}. \tag{39}$$

At the particular point $x_o \in M$, using the formula (31) and (36), one can obtain:

$$\nabla_k \xi_{ij} = \tfrac{c}{|y|} \left(2y^k \xi_{ij} + \delta^k_i y^s \xi_{sj} - \delta^k_j y^s \xi_{si} \right). \tag{40}$$

As a consequence, for any pairwise different indeces i, j, k we get

$$\tfrac{y^k}{|\mathbf{y}|}\xi_{ij} = \tfrac{1}{2c}\nabla_k\xi_{ij} \quad \in \mathfrak{hol}^*_{x_o}(\mathcal{F}) \qquad i \neq j,\ i \neq k,\ j \neq k, \tag{41}$$

showing that (41) are in $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. Moreover, for $k = i$ from (40), we get the linear system:

$$\begin{pmatrix} \nabla_1\xi_{1j} \\ \vdots \\ [\nabla_j\xi_{jj}] \\ \vdots \\ \nabla_n\xi_{nj} \end{pmatrix} = c \begin{pmatrix} 3 & 1 & \dots & 1 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 3 \end{pmatrix} \begin{pmatrix} \frac{y^1}{|y|}\xi_{1j} \\ \vdots \\ \left[\frac{y^j}{|y|}\xi_{jj}\right] \\ \vdots \\ \frac{y^n}{|y|}\xi_{nj} \end{pmatrix}, \qquad j = 1, \dots, n, \tag{42}$$

where in the column matrices the (trivially zero) terms written in square brackets are missing. Since the quadratic matrix in (42) is invertible, we get that $\frac{y^i}{|y|}\xi_{ij}$ can be expressed as a linear combination of covariant derivatives of curvature vector fields, therefore they are also elements of $\mathfrak{hol}^*_{x_o}(M)$. Therefore we obtained that the generating elements of $\mathcal{A}_1$ are in $\mathfrak{hol}^*_{x_o}(M)$, therefore $\mathcal{A}_1 \subset \mathfrak{hol}^*_{x_o}(M)$.

- $p = 2$. The second covariant derivatives of the curvature vector fields are elements of $\mathfrak{hol}^*_{x_o}(M, \mathcal{F})$. Calculating their expression we get

$$\begin{aligned} \nabla_m\nabla_k\xi_{ij} =& (\lambda + c^2)(\delta^m_j\xi_{ki} - \delta^m_i\xi_{kj}) + c^2(\delta^k_j\xi_{mi} - \delta^k_i\xi_{mj}) + 2(c^2 - \lambda)\delta^m_k\xi_{ij} \\ &+ 4\tfrac{c^2}{|y|^2}\left(y^m y^k\xi_{ij} + \delta^i_k y^m y^s\xi_{sj} - \delta^j_k y^m y^s\xi_{si} - \delta^j_m y^k y^s\xi_{si} + \delta^i_m y^k y^s\xi_{sj}\right). \end{aligned} \tag{43}$$

As special cases we can get (different letters denote different indeces, repeated indeces does not mean summation in the formula below):

$$\nabla_m\nabla_k\xi_{ij} = +4c^2\tfrac{y^m y^k}{|y|^2}\xi_{ij} \tag{44a}$$

$$\nabla_k\nabla_k\xi_{ij} = 2(c^2 - \lambda)\xi_{ij} + 4c^2\tfrac{(y^k)^2}{|y|^2}\xi_{ij}, \tag{44b}$$

$$\nabla_i\nabla_i\xi_{ij} = -3\lambda\xi_{ij} + 4c^2\tfrac{(y^i)^2}{|y|^2}\xi_{ij}, +8c^2\sum_{s=1}^{n}\tfrac{y^i y^s}{|y|^2}\xi_{sj} \tag{44c}$$

$$\nabla_i\nabla_k\xi_{ij} = -(\lambda + c^2)\xi_{kj} + 4c^2\tfrac{y^i y^k}{|y|^2}\xi_{ij} + 4c^2\sum_{s=1}^{n}\tfrac{y^k y^s}{|y|^2}\xi_{sj} \tag{44d}$$

From the Eq. (44a) we obtain that $\frac{y^k y^m}{|y|^2}\xi_{ij}$ can be expressed as a constant multiple of the second derivatives of curvature vector fields, therefore it is in $\mathfrak{hol}^*_{x_o}(M, \mathcal{F})$. Similarly, from Eq. (44b) we get that $\frac{(y^k)^2}{|y|^2}\xi_{ij}$ can be expressed with the combination of the curvature vector fields and teir second covariant derivatives, therefore they are elements of $\mathfrak{hol}^*_{x_o}(M, \mathcal{F})$. Equation (44d) can be considered as a linear system on $\frac{y^k y^i}{|y|^2}\xi_{ij}$

$$\begin{pmatrix} \nabla_1\nabla_k\xi_{1j} + (c^2+\lambda)\xi_{kj} \\ \vdots \\ [\nabla_j\nabla_k\xi_{jj} + (c^2+\lambda)\xi_{kj}] \\ \vdots \\ \nabla_n\nabla_k\xi_{nj} + (c^2+\lambda)\xi_{kj} \end{pmatrix} = 4c^2 \begin{pmatrix} 2 & 1\dots & 1 \\ 1 & \ddots & \vdots \\ \vdots & \ddots & 1 \\ 1\dots & 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{y^k y^1}{|y|^2}\xi_{1j} \\ \vdots \\ \left[\frac{y^k y^j}{|y|^2}\xi_{jj}\right] \\ \vdots \\ \frac{y^k y^n}{|y|^2}\xi_{nj} \end{pmatrix} \tag{45}$$

where the elements on the left hand side are elements of the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. The $(n-1) \times (n-1)$ matrix appearing in (45) is regular, therefore the terms $\frac{y^k y^i}{|y|^2}\xi_{ij}$ can be expressed as linear combination of elements of the left hand side which are elements of $\mathfrak{hol}^*_{x_o}(\mathcal{F})$. It follows that $\frac{y^k y^i}{|y|^2}\xi_{ij} \in \mathfrak{hol}^*_{x_o}(\mathcal{F})$. Finally, using (44c), one can express $\frac{y^i y^i}{|y|^2}\xi_{ij}$ with the help of the other terms appearing in those equations. Since these are all in $\mathfrak{hol}^*_{x_o}(\mathcal{F})$, we can get that $\frac{y^i y^i}{|y|^2}\xi_{ij} \in \mathfrak{hol}^*_{x_o}(\mathcal{F})$. One can conclude that the elements generating $\mathcal{A}_2$ can be expressed as a linear combination of elements in the infinitesimal holonomy algebra, therefore $\mathcal{A}_2 \subset \mathfrak{hol}^*_{x_o}(\mathcal{F})$.

- Let us assume that $A_\ell \in \mathfrak{hol}^*_{x_o}(M, \mathcal{F})$ for $1 \leq \ell \leq p$ and we will show that $A_{p+1} \in \mathfrak{hol}^*_{x_o}(M, \mathcal{F})$. First we observe that

$$\sum_{s=1}^{n} \tfrac{y^s}{|y|}\xi_{ks} = \frac{1}{\lambda}\left(\sum_{s=1}^{n} \tfrac{y^s y^k}{|y|}\tfrac{\partial}{\partial y^s} - |y|\tfrac{\partial}{\partial y^k}\right) = \frac{1}{\lambda}\left(\tfrac{y^k}{|y|}C - |y|\tfrac{\partial}{\partial y^k}\right) \in \mathcal{A}_1, \tag{46}$$

where C denotes the canonica Liouville (or radial) vector field. Let us consider for the multiindex $\mathbf{m} := (m_1, \dots m_n)$ where $\ell(\mathbf{m}) = m_1 + \cdots + m_n = p$ the vector field

$$\frac{\mathbf{y}^{\mathbf{m}}}{|\mathbf{y}|^{\ell(\mathbf{m})}}\xi_{ij} = \frac{y_1^{m_1} y_2^{m_2} \dots y_n^{m_n}}{|y|^p}\xi_{ij} \qquad \in \mathcal{A}_p. \tag{47}$$

By the induction hypothesis $\mathcal{A}_1 \subset \mathfrak{hol}^*_{x_o}(M)$ and $\mathcal{A}_p \subset \mathfrak{hol}^*_{x_o}(M)$, therefore – using the fact that $\mathfrak{hol}^*_{x_o}(M)$ is a Lie algebra – we get that

$$[\mathcal{A}_1, \mathcal{A}_p] \subset \mathfrak{hol}^*_{x_o}(M). \tag{48}$$

In particular, the Lie bracket of the elements (46) and (47) is an element of $\mathfrak{hol}^*_{x_o}(M)$. If i, j, k are pairwise different indeces, then one can obtain for the multiindex $\mathbf{m}$ of length $\ell(\mathbf{m}) = p$:

$$\begin{aligned}\left[\frac{\mathbf{y}^{\mathbf{m}}}{|\mathbf{y}|^{\ell(\mathbf{m})}}\,\xi_{ij},\sum_{s=1}^{n}\frac{y^s}{|y|}\xi_{ks}\right] &= \left[\frac{\mathbf{y}^{\mathbf{m}}}{|\mathbf{y}|^{p}}\,\xi_{ij},\,\tfrac{1}{\lambda}\left(\tfrac{y^k}{|y|}C-|y|\,\tfrac{\partial}{\partial y_k}\right)\right]\\ &= m_k\frac{\mathbf{y}^{\mathbf{m}-\mathbf{1}_\mathbf{k}}}{|\mathbf{y}|^{p-1}}\,\xi_{ij}-p\frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_\mathbf{k}}}{|\mathbf{y}|^{p+1}}\,\xi_{ij},\end{aligned} \tag{49}$$

where $\mathbf{1}_\mathbf{k}=(0,\ldots,1,\ldots0)$ denotes the multiindex having 1 at the kth position. From (48) the left hand side of (49) is in the infinitesimal holonomy, and from the induction hypothesys $\frac{\mathbf{y}^{\mathbf{m}-\mathbf{1}_\mathbf{k}}}{|\mathbf{y}|^{p-1}}\,\xi_{ij}\in\mathcal{A}_{p-1}$, is also an element of $\mathfrak{hol}^*_{x_o}(M)$. It follows that

$$\frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_\mathbf{k}}}{|\mathbf{y}|^{p+1}}\,\xi_{ij}\in\mathfrak{hol}^*_{x_o}(\mathcal{F}),\qquad i\neq j,\ k\neq i,\ k\neq j. \tag{50}$$

Similarly,

$$\left[\frac{\mathbf{y}^{\mathbf{m}}}{|\mathbf{y}|^{\ell(\mathbf{m})}}\,\xi_{ij},\sum_{s=1}^{n}\frac{y^s}{|y|}\xi_{is}\right]=m_i\frac{\mathbf{y}^{\mathbf{m}-\mathbf{1}_i}}{|\mathbf{y}|^{p-1}}\,\xi_{ij}+(1-p)\frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_i}}{|\mathbf{y}|^{p+1}}\,\xi_{ij}+\sum_{s=1,s\neq i}^{n}\frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_s}}{|\mathbf{y}|^{p+1}}\,\xi_{sj}. \tag{51}$$

Using (50) we obtain that

$$\frac{\mathbf{y}^{\mathbf{m}+\mathbf{1}_i}}{|\mathbf{y}|^{p+1}}\,\xi_{ij}\in\mathfrak{hol}^*_{x_o}(\mathcal{F}),\qquad i\neq j. \tag{52}$$

From (50) and (52) we get that $\mathcal{A}_{p+1}\subset\mathfrak{hol}^*_{x_o}(\mathcal{F})$ which completes the proof of Lemma 4.3. ☐

Proposition 4.4 *Let $(M,\mathcal{F})$ be a projectively flat spherically symmetric Finsler manifold of constant curvature $\lambda\neq 0$ and x_o a point where (29) is satisfied. Then the infinitesimal holonomy algebra $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is dense in $\mathfrak{X}(\mathcal{I}_{x_o})$.*

Proof Let $(M,\mathcal{F})$ be a projectively flat spherically symmetric Finsler manifold of constant curvature $\lambda\neq 0$ and x_o a point where (29) is satisfied. According to Remark 4.2, the elements of $\mathcal{A}$ can be considered as vector fileds on the indicatrix $\mathcal{I}_{x_o}\simeq\mathbb{S}^{n-1}$ with polynomial coefficients that is

$$\mathcal{A}=\left\{\,Q^{ij}\xi_{ij}\mid Q^{ij}\in\mathcal{P}ol(\mathbb{S}^{n-1})\,\right\}, \tag{53}$$

where

$$\mathcal{P}ol(\mathbb{S}^{n-1})=\mathbb{R}[y^1,\ldots,y^n]\big|_{\mathbb{S}^{n-1}}=\left\{p\,\big|_{\mathbb{S}^{n-1}}\;\middle|\;p\in\mathbb{R}[y^1,\ldots,y^n]\right\}, \tag{54}$$

is the algebra of polynomial functions on $\mathbb{S}^{n-1}\subset\mathbb{R}^n$.

Nachbin's theorem [13, 14] gives an analog for Stone–Weierstrass theorem for algebras of real values C^k functions on a C^k manifold, $k=1,\ldots,\infty$. It states that if $\mathcal{S}$ is a subalgebra of the algebra of C^k smooth functions on a finite dimensional C^k smooth manifold M, and $\mathcal{S}$ separates the points of M and also separates the tangent vectors of M in the sense that for each point $x \in M$ and tangent vector $v \in T_xM$, there is an $f \in \mathcal{S}$ such that $df_x(v) \neq 0$, then $\mathcal{S}$ is dense in $C^k(M)$. Clearly, $\mathcal{P}ol(\mathbb{S}^{n-1})$, the algebra of polynomial functions on $\mathbb{S}^{n-1}$ satisfies Nachbin's conditions, therefore $\mathcal{P}ol(\mathbb{S}^{n-1})$ is dense in $C^\infty(\mathbb{S}^{n-1})$ with respect to the C^∞ topology:

$$\overline{\mathcal{P}ol(\mathbb{S}^{n-1})} = C^\infty(\mathbb{S}^{n-1}). \tag{55}$$

On the other hand, any vector field $X \in \mathfrak{X}(\mathbb{S}^{n-1})$ can be written as $X = X^{ij}\xi_{ij}$, where X^{ij} are smooth function on $\mathbb{S}^{n-1}$, and ξ_{ij} are the infinitesimal generator of rotations which are the curvature vector fields (36), that is

$$\mathfrak{X}(\mathbb{S}^{n-1}) = \left\{ X^{ij}\xi_{ij} \mid X^{ij} \in C^\infty(\mathbb{S}^{n-1}) \right\}, \tag{56}$$

and from (55) we get that (53) is a dense subset in (56) with respect to the C^k topology, that is

$$\overline{\mathcal{A}} = \mathfrak{X}(\mathbb{S}^{n-1}). \tag{57}$$

From Lemma 4.3 we get

$$\mathcal{A} \subset \mathfrak{hol}^*_{x_o}(\mathcal{F}) \subset \mathfrak{X}(\mathbb{S}^{n-1}), \tag{58}$$

it follows

$$\overline{\mathcal{A}} \subset \overline{\mathfrak{hol}^*_{x_o}(\mathcal{F})} \subset \mathfrak{X}(\mathbb{S}^{n-1}), \tag{59}$$

and from (57) one can obtain

$$\overline{\mathfrak{hol}^*_{x_o}(\mathcal{F})} = \mathfrak{X}(\mathbb{S}^{n-1}) \tag{60}$$

that is $\mathfrak{hol}^*_{x_o}(\mathcal{F})$ is dense in $\mathfrak{X}(\mathbb{S}^{n-1})$ with respect to the C^∞ topology. □

Theorem 4.5 *Let $(M, \mathcal{F})$ be a simply connected, projectively flat spherically symmetric Finsler manifold of constant curvature $\lambda \neq 0$ and x_o a point where (29) is satisfied. Then the holonomy group $\mathcal{H}ol_{x_o}(\mathcal{F})$ is maximal, that is its closure is isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$, the connected component of the identity of the group of smooth diffeomorphism on the $n-1$-dimensional sphere.*

Proof The proof is a direct consequence of Theorem 3.3 and Proposition 4.4. Indeed, let $(M, \mathcal{F})$ be a projectively flat spherically symmetric Finsler manifold of constant curvature $\lambda \neq 0$ and x_o a point where (29) is satisfied. From Proposition 4.4 we get that the infinitesimal holonomy algebra $hol^*_{x_o}(\mathcal{F})$ is dense in $\mathfrak{X}(\mathcal{I}_{x_o})$, and from Theorem 3.3 we get that in that case the holonomy group is maxmial

$$\overline{\mathcal{H}ol_x(\mathcal{F})} \cong \mathcal{D}iff_o(\mathbb{S}^{n-1}), \tag{61}$$

that is its closure is isomorphic to the connected component of the identity in the diffeomorphism group of the $n-1$-dimensional sphere. □

Example 1 (P. Funk, [5, 6]) The *standard Funk manifold* $(\mathbb{D}^n, \mathcal{F})$ defined by the metric function

$$\mathcal{F}(x,y)=\frac{\sqrt{|y|^2-\big(|x|^2|y|^2-\langle x,y\rangle^2\big)}}{1-|x|^2}\pm\frac{\langle x,y\rangle}{1-|x|^2} \tag{62}$$

on the unit open disk $\mathbb{D}^n \subset \mathbb{R}^n$ is projectively flat with constant flag curvature $\lambda=-\frac{1}{4}$. Its projective factor can be computed using formula (7):

$$\mathcal{P}(x,y)=\frac{1}{2}\,\frac{\pm\sqrt{|y|^2-\big(|x|^2|y|^2-\langle x,y\rangle^2\big)}+\langle x,y\rangle}{1-|x|^2}. \tag{63}$$

At $x_o=(0,\dots,0)\in\mathbb{D}^n$ we have $\mathcal{F}(x_o,y)=|y|$ and $\mathcal{P}(x_0,y)=\pm\frac{1}{2}\,|y|$. Thus, at x_o the standard Funk metric satisfies the condition of Theorem 4.5, therefore its holonomy group $\mathcal{H}ol_{x_o}(\mathcal{F})$ is maximal and isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$.

Example 2 The *Bryant–Shen metric* $\mathcal{F}_\alpha$, $(|\alpha|<\frac{\pi}{2})$, are the elements of a 1-parameter family of projective Finsler metric with constant flag curvature $\lambda=1$:

$$\mathcal{F}_\alpha=\sqrt{\frac{\sqrt{A}+B}{2D}+\Big(\frac{C}{D}\Big)^2}+\frac{C}{D},\quad \mathcal{P}_\alpha=-\sqrt{\frac{\sqrt{A}-B}{2D}-\Big(\frac{C}{D}\Big)^2}-\frac{\widetilde{C}}{D}, \tag{64}$$

where

$$\begin{aligned}
A&=\big(\cos(2\alpha)\,|y|^2+|x|^2|y|^2-\langle x,y\rangle^2\big)^2+\big(\sin(2\alpha)\,|y|^2\big)^2,\\
B&=\cos(2\alpha)\,|y|^2+|x|^2|y|^2-\langle x,y\rangle^2,\\
C&=\sin(2\alpha)\langle x,y\rangle,\\
\widetilde{C}&=\big(\cos(2\alpha)+|x|^2\big)\langle x,y\rangle,\\
D&=|x|^4+2\cos(2\alpha)\,|x|^2+1.
\end{aligned} \tag{65}$$

The norm function and the projective factor at $0\in\mathbb{R}^n$ have the form

$$\mathcal{F}_\alpha(0,y)=|y|\,\cos\alpha,\qquad \mathcal{P}_\alpha(0,y)=|y|\,\sin\alpha,\qquad |\alpha|<\frac{\pi}{2}, \tag{66}$$

in a local coordinate system corresponding to the Euclidean canonical coordinates, centered at $0\in\mathbb{R}^n$. R. Bryant in [2, 3] introduced and studied this class of Finsler metrics on $\mathbb{S}^2$ where great circles are geodesics. Z. Shen generalized its construction and obtained the expression (64) in [17, Example 7.1.]. Since the condition of Theorem 4.5, therefore the holonomy group $\mathcal{H}ol_{x_o}(\mathcal{F}_\alpha)$ of the Bryant–Shen metric is maximal and isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$.

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References

1. Borel, A., Lichnerowicz, A.: Groupes d'holonomie des variétés riemanniennes. C. R. Acad. Sci. Paris **234**, 1835–1837 (1952)
2. Bryant, R.L., Finsler structures on the 2-sphere satisfying $K = 1$. In Finsler geometry (Seattle, WA,: vol. 196 of Contemp. Math. Amer. Math. Soc. Providence, RI **1996**, 27–41 (1995)
3. Bryant, R.L.: Projectively flat Finsler 2-spheres of constant curvature. Selecta Math. (N.S.) **2**, 161–203 (1997)
4. Chern, S.-S., Shen, Z.: *Riemann-Finsler geometry*, vol. 6 of *Nankai Tracts in Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, (2005)
5. Funk, P.: Über Geometrien, bei denen die Geraden die Kürzesten sind. Math. Ann. **101**(1), 226–237 (1929)
6. Funk, P.: Eine Kennzeichnung der zweidemensionalen elliptischen Geometrie. Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. **II**(172), 251–269 (1963)
7. Hubicska, B., Muzsnay, Z.: Tangent lie algebra of a diffeomorphism group and application to holonomy theory. J. Geom. Anal. **30**, 107–123 (2020)
8. Kozma, L., On.: Landsberg spaces and holonomy of Finsler manifolds. In Finsler geometry (Seattle, WA,: vol. 196 of Contemp. Math. Amer. Math. Soc. Providence, RI **1996**, 177–186 (1995)
9. Muzsnay, Z., Nagy, P.T.: Finsler manifolds with non-Riemannian holonomy. Houston J. Math. **38**(1), 77–92 (2012)
10. Muzsnay, Z., Nagy, P.T.: Characterization of projective finsler manifolds of constant curvature having infinite dimensional holonomy group. Publ. Math. Debrecen **84**(1–2), 17–28 (2014)
11. Muzsnay, Z., Nagy, P.T.: Finsler 2-manifolds with maximal holonomy group of infinite dimension. Differential Geom. Appl. **39**, 1–9 (2015)
12. Muzsnay, Z., Nagy, P.T.: Projectively flat finsler manifolds with infinite dimensional holonomy. Forum Math. **27**(2), 767–786 (2015)
13. Nachbin, L.: Sur les algèbres denses de fonctions différentiables sur une variété. C. R. Acad. Sci. Paris **228**, 1549–1551 (1949)
14. Nachbin, L.A.: look at approximation theory. In Approximation theory and functional analysis (Proc. Internat. Sympos. Approximation Theory, Univ. Estadual de Campinas, Campinas,: vol. 35 of Notas de Matemática, 66; North-Holland Math. Stud. North-Holland, Amsterdam-New York **1979**, 309–331 (1977)
15. Omori, H.: *Infinite-dimensional Lie groups*, vol. 158 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1997. Translated from the 1979 Japanese original and revised by the author
16. Shen, Z.: Differential geometry of spray and Finsler spaces. Kluwer Academic Publishers, Dordrecht (2001)
17. Shen, Z.: Projectively flat Finsler metrics of constant flag curvature. Trans. Amer. Math. Soc. **355**(4), 1713–1728 (2003)
18. Szabó, Z.I.: Positive definite Berwald spaces. Structure theorems on Berwald spaces. Tensor **1**, 25–39 (1981)
19. Thurston, W.: Foliations and groups of diffeomorphisms. Bull. Amer. Math. Soc. **80**, 304–307 (1974)