Metrics along the tangent bundle projection

Ph.D. thesis

Metrikák az érintönyaláb mentén

doktori (Ph.D.) értekezés tézisei

Tóth Viktória Zsuzsa

Debreceni Egyetem
Természettudományi Kar
Debrecen, 2003

Contents

Introduction

Finsler geometry is the main branch of mathematics where this dissertation falls down in. Since its first introduction, Finsler manifolds have been studied intensively and nowadays the theory has reached a huge level of development. Some of the most important breakthroughs in the history of Finsler geometry were obtained in the works [18, 22, 21, 76, 9, 4, 56, 31, 20, 86]. The reasons why present-day scientists remain attracted to Finsler geometry are not only motivated by its mathematical relevance or by its direct applicability in a lot of physical theories, such as mechanics [72], thermodynamics [6] or relativity and gauge theories [8, 16], but also by the fact that Finsler metrics seem to occur naturally in a lot of non-standard domains of applications, such as biology, ecology or paleontology [6, 7], control theory [50] and even financial sciences [7]. Standard textbooks on the mathematical theory of Finsler geometry are for example [61, 77], while we refer the modern reader to the books [15, 81, 91, 1, 82, 17] for recent surveys on Finsler geometry.

What is common to the bulk of the literature on Finsler geometry, is that the analysis is almost entirely based on computations in local coordinates. There is no doubt that classical tensor calculus still is a very important tool for discovering and proving intrinsic features in most fields of applied differential geometry. It is our believe, however, that it is of interest also to develop purely coordinate-free methods in such fields. Quite often, the more abstract approach reveals much better the geometric structures which are at work and thus paves the way to learning from these structures in related theories or applications. For example, coordinate-free intrinsic methods are indispensable tools to obtain classification theorems, see e.g. Szabó's results on the description of positive definite Berwald manifolds in [86].

More directly related to the results we wish to present in this dissertation, we can cite important work by Grifone [37, 38] and by Crampin [24, 25, 26], whose intrinsic methods for describing the geometry of a tangent bundle have had a great influence on many subsequent developments. The present work can be considered as a next step to realize the program initiated by J. Grifone and continued systematically by J. Szilasi and his students [90, 95, 97, 99, 100, 93, 92, 88, 89, 87]. Also our present analysis is essentially based on the techniques of tangent bundle geometry. But we add an additional feature to it: it has been observed in the past that for many important geometrical aspects, the vector and tensor fields of interest are vertical vector valued or, equivalently, can be identified with tensor fields along the tangent bundle projection $\tau : TM \to M$. So, working with sections of the pullback bundle τ^*TM rather than sections of $TTM \to TM$ one can avoid some unnecessary duplications of formulae and this is the line of approach we will follow here (see e.g. [28, 53, 54, 68, 51, 52, 55, 80, 29, 79, 30] for earlier work in this direction). For the above mentioned reasons, the basic philosophy of this dissertation can be formulated as follows:

> To investigate Finsler geometry with the aid of purely *intrinsic methods* only, by means of a calculus of *tensor fields along the tangent bundle projection.*

Of course, it is impossible to cover the whole range of work that has been done in the past in the field of Finsler geometry. Therefore, we have singled out two different subjects in which we intend to demonstrate the power of our approach. First of all we undertake a quite comprehensive survey of general theoretical elements of Finsler geometry: in Chapter 1 we present a detailed exposition of the conceptual and calculational background. In chapter 2 and 3 our strategy is to insert the theory of generalized Lagrange metrics and the foundations of Randers manifolds into a new approach to Finsler geometry, and investigate theoretically important particular problems within framework. Although the two subjects may seem to live on two different branches of Finsler geometry, we will show that they need not necessarily be completely unrelated with each other. For example, in section 3.6 we will apply some of the results that we obtained in the first part to the special situation of a Randers manifold.

Remark. In our next presentation, triplets of numbers printed in bold type, for example, **2.1.2**, refer to the corresponding theorem, definition etc. of the *Dissertation.*

Chapter 1

The background: Framework and calculus

(A) We work over an n-dimensional smooth manifold M and assume that its topology is Hausdorff, second countable and connected. $C^\infty(M)$ denotes the ring of real-valued smooth functions on M. $\mathcal{X}(M)$ and $\mathcal{A}^k(M)$ $(1 \leq k \leq n)$ stand for the $C^\infty(M)$-modules of vector fields and differential k-forms on M, respectively. $\mathcal{A}^0(M) := C^\infty(M)$; $\mathcal{A}(M) := \bigoplus_{k=0}^{n} \mathcal{A}^k(M)$ is the exterior algebra of M. The familiar wedge product $\wedge$ makes $\mathcal{A}(M)$ into a graded algebra over the ring $C^\infty(M)$. A *vector k-form* on M is a $C^\infty(M)$-multilinear skew-symmetric map $(\mathcal{X}(M))^k \to \mathcal{X}(M)$ $(1 \leq k \leq n)$. The $C^\infty(M)$-module of vector k-forms will be denoted by $\mathcal{B}^k(M)$. We agree that $\mathcal{B}^0(M) := \mathcal{X}(M)$ and we denote the direct sum $\bigoplus_{k=0}^{n} \mathcal{B}^k(M)$ by $\mathcal{B}(M)$. The symmetric product will be denoted by $\odot$.

(B) $\tau : TM \to M$ is the tangent bundle of M; $\overset{\circ}{T}M \subset TM$ is the (open) set of all nonzero tangent vectors. The natural projection $\overset{\circ}{T}M \to M$ is denoted by $\overset{\circ}{\tau}$. We shall remain in the smooth category, however, in Finsler geometry, the smoothness of some objects living on the tangent bundle will be guaranteed only over $\overset{\circ}{T}M$. The elements of the kernel of the tangent map $T\tau : TTM \to TM$ of the tangent bundle projection τ form the *vertical submanifold* VTM of TTM; VTM is the total manifold of the *vertical bundle* $V\tau : VTM \to TM$ to τ. The $C^\infty(TM)$-module $\mathcal{X}^v(TM)$ of the sections of the vertical bundle is called the module of *vertical vector fields* on TM. The *vertical lift* of a smooth function f on M is the function $f^v := f \circ \tau \in C^\infty(TM)$, the *complete lift* of f is the function $f^c : TM \to \mathbb{R}$, $v \mapsto f^c(v) := v(f)$. *Any vector field on TM is uniquely determined by its action on the complete lifts of smooth functions on M*, so, given a vector field X on M, there exist unique vector fields X^v and X^c on TM, such that $X^v f^c = (Xf)^v$ and $X^c f^c = (Xf)^c$ for all $X \in \mathcal{X}(M)$. X^v and X^c are said to be the *vertical* and the *complete lift of* X, respectively. *If* $(X_1, \dots, X_n)$ *is a local basis of vector fields on* M, *then* $(X_1^c, \dots, X_n^c, X_1^v, \dots, X_n^v)$ *is a local basis of vector fields on* TM (first local basis principle).

(C) The majority of our concepts lives on the pullback bundle $\tau^*\tau$ of the tangent bundle by its own projection. It is a vector bundle over TM with total manifold $\tau^*TM = TM \times_M TM := \{(v, w) \in TM \times TM \mid \tau(v) = \tau(w)\}$. The fibres of $\tau^*\tau$ are the n-dimensional real vector spaces $\{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M$, $v \in TM$. Any section of $\tau^*\tau$ is of the form

$$\tilde{X} : v \in TM \mapsto \tilde{X}(v) = (v, \underline{X}(v)) \in TM \times_M TM,$$

where $\underline{X} : TM \to TM$ is a smooth map such that $\tau \circ \underline{X} = \tau$. In particular, we have the specific section

$$\delta : v \in TM \mapsto \delta(v) := (v, v) \in TM \times_M TM$$

of $\tau^*\tau$, called the *canonical vector field along* τ. In the following we shall identify the sections of $\tau^*\tau$ with the smooth maps $\underline{X} : TM \to TM$ that satisfy the requirement $\tau \circ \underline{X} = \tau$. The $C^\infty(TM)$-module of such maps is denoted by $\mathcal{X}(\tau)$, and an element of this module is said to be a *vector field along the tangent bundle projection.* A special class of vector fields along the projection is formed by the sections of the form $\hat{X} := X \circ \tau$, where X is a vector field on M. For obvious reasons, $\hat{X}$ will be called the *lift of* X *into* $\mathcal{X}(\tau)$, or a *basic vector field* along τ. *If* $(X_1, \dots, X_n)$ *is a local basis of* $\mathcal{X}(M)$, *then* $(\hat{X}_1, \dots, \hat{X}_n)$ *is a local basis for* $\mathcal{X}(\tau)$ (second local basis principle).

(D) By a *one-form along* τ we mean an element of the dual module of $\mathcal{X}(\tau)$. As in the case of vector fields along τ, any one-form $\tilde{\alpha}$ along τ may be regarded as a smooth map of TM into $T^*M := \bigcup_{p \in M} (T_pM)^*$ that satisfies the condition $\tau^* \circ \tilde{\alpha} = \tau$, where τ^* is the natural projection $T^*M \to M$. We denote the $C^\infty(TM)$-module of these maps by $\mathcal{A}^1(\tau)$. For any one-form α on M, the map $\hat{\alpha} := \alpha \circ \tau$ is a one-form along τ, called the *lift of* α *into* $\mathcal{A}^1(\tau)$, or a *basic one-form* along τ. By a *k-fold contravariant, l-fold covariant tensor field,* briefly a *type* (k, l) *tensor field along* τ, we mean a $C^\infty(TM)$-multilinear map $\left(\mathcal{A}^1(\tau)\right)^k \times \left(\mathcal{X}(\tau)\right)^l \to C^\infty(TM)$. The $C^\infty(TM)$-module of these tensor fields will be denoted by $\mathcal{T}_l^k(\tau)$. We agree, as usual, that $\mathcal{T}_0^0(\tau) := C^\infty(TM)$. The elements of $\mathcal{T}_l^k(\tau)$ may indeed be regarded as 'fields' which smoothly assign to each element v of the base manifold TM a type (k, l) tensor on the fibre $\{v\} \times T_{\tau(v)}M \cong T_{\tau(v)}M$ over v. For example, if $g \in \mathcal{T}_2^0(\tau)$, then g may be interpreted as a smooth map $v \in TM \mapsto g_v$, where $g_v : T_{\tau(v)}M \times T_{\tau(v)}M \to \mathbb{R}$ is a bilinear form. Notice that any tensor field A on M induces a *basic tensor field* $\hat{A} := A \circ \tau$ along τ. Finally, by a $\tau^*\tau$-valued k-form on TM we mean a skew-symmetric $C^\infty(TM)$-multilinear map of $(\mathcal{X}(TM))^k$ into $\mathcal{X}(\tau)$.

(E) Most of our *canonical objects* may be identified from the short exact sequence

$$(*) \qquad 0 \to \tau^*TM \xrightarrow{\mathbf{i}} TTM \xrightarrow{\mathbf{j}} \tau^*TM \to 0$$

of vector bundles over TM. Here the map $\mathbf{j}$ is defined by $\mathbf{j}(z) := (v, T\tau(z))$, for all $v \in TM, z \in T_vTM$, while the simplest description of $\mathbf{i}$ uses local coordinates. Let $\left(U, (u)_{i=1}^n\right)$ be a chart on M, and let us consider the *induced chart*

$$(\tau^{-1}(U), (x^i)_{i=1}^n, (y^i)_{i=1}^n); \quad x^i := (u^i)^v, \; y^i := (u^i)^c \qquad (1 \leq i \leq n)$$

on TM. Then for any vectors $v, w \in T_{\tau(v)}M$,

$$\mathbf{i}(v,w) = \sum_{i=1}^n y^i(w)\left(\frac{\partial}{\partial y^i}\right)_v =: y^i(w)\left(\frac{\partial}{\partial y^i}\right)_v.$$

Note. In coordinate expressions the *Einstein summation convention* will sometimes be used: an index occurring twice in a product, once as a subscript and once as a superscript is to be summed from 1 to n (n is a fixed positive integer). Superscripts in a denominator act as subscripts.

The composite of $\mathbf{i}$ and δ yields another canonical object, the *Liouville vector field* $C := \mathbf{i} \circ \delta$ on TM. The short exact sequence $(*)$ gives rise to a short exact sequence

$$0 \to \mathcal{X}(\tau) \xrightarrow{\mathbf{i}} \mathcal{X}(TM) \xrightarrow{\mathbf{j}} \mathcal{X}(\tau) \to 0$$

of modules over $C^\infty(TM)$, where, for simplicity, we also denote by $\mathbf{i}$ and $\mathbf{j}$ the induced maps between the modules of sections. The map $\mathbf{i}$ is an isomorphism between $\mathcal{X}(\tau)$ and $\mathcal{X}^v(TM)$, so any vertical vector field on TM can be represented uniquely in the form $\mathbf{i}\tilde{X}$ ($\tilde{X} \in \mathcal{X}(\tau)$). The map $\mathbf{j}$ is surjective, therefore any vector field along τ is of the form $\mathbf{j}\xi$, $\xi \in \mathcal{X}(TM)$. $\mathbf{i}$ and $\mathbf{j}$ enable us to introduce our next canonical object, the *vertical endomorphism* $J := \mathbf{i} \circ \mathbf{j}$. J is a type $(1,1)$ tensor field on TM such that $ImJ = KerJ = \mathcal{X}^v(TM)$ and $J^2 = 0$.

(F) A *horizontal map for* τ is a (right) splitting $\mathcal{H} : \tau^*TM \to TTM$ of the short exact sequence $(*)$, i.e. a strong bundle map such that $\mathbf{j} \circ \mathcal{H} = 1_{\tau^*TM}$. The existence of a horizontal map is guaranteed by the second countability of the topology of the base manifold. Let $\mathcal{H}_v := \mathcal{H} \upharpoonright \{v\} \times T_{\tau(v)}M$ $(v \in TM)$, $HTM := \bigcup_{v \in TM} Im\mathcal{H}_v$, and let $H\tau$ be the natural projection of HTM onto TM. There is a unique smooth manifold structure on HTM which makes $H\tau : HTM \to TM$ into a vector bundle. This vector bundle is said to be the *horizontal bundle* induced by $\mathcal{H}$ and denoted by $H\tau$. Then
$TTM = HTM \oplus VTM$; fibrewise $T_vTM = Im\mathcal{H}_v \oplus V_vTM$ ($V_vTM := KerT_v\tau$) for all $v \in TM$. The sections of $H\tau$ are called *($\mathcal{H}$-)horizontal vector fields* on TM. For the $C^\infty(TM)$-module of horizontal vector fields we use the notation $\mathcal{X}^h(TM)$, then $\mathcal{X}(TM) = \mathcal{X}^h(TM) \oplus \mathcal{X}^v(TM)$. Any horizontal map $\mathcal{H}$ can be used to define a lifting process of vector fields on M to vector fields on TM. The *horizontal lift* of $X \in \mathcal{X}(M)$ is the horizontal vector field X^h given by $X^h(v) = \mathcal{H}(v, X(\tau(v)))$ for all $v \in TM$). Equivalently, $X^h = \mathcal{H} \circ \hat{X}$. Any right splitting $\mathcal{H}$ of $(*)$ induces a left splitting $\mathcal{V} : TTM \to \tau^*TM$ of $(*)$ such that $\mathcal{V} \circ \mathbf{i} = 1_{\tau^*TM}$; $Ker\mathcal{V} = Im\mathcal{H}$ and hence $\mathcal{V} \circ \mathcal{H} = 0$. Thus, specifying a horizontal map for τ, we arrive at the fundamental 'double exact' sequence

$$0 \rightleftarrows \tau^*TM \underset{\mathcal{V}}{\overset{\mathbf{i}}{\rightleftarrows}} TTM \underset{\mathcal{H}}{\overset{\mathbf{j}}{\rightleftarrows}} \tau^*TM \rightleftarrows 0.$$

$\mathcal{V}$ is called the *vertical map* belonging to $\mathcal{H}$. The maps $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$ and $\mathbf{v} = 1_{TTM} - \mathbf{h}$ are said to be (respectively) the *horizontal* and the *vertical projectors* determined by $\mathcal{H}$. $\mathbf{h}$ and $\mathbf{v}$ are obviously $(1,1)$ tensor fields on TM, or equivalently, vector one-forms on TM, i.e. $\mathbf{h}, \mathbf{v} \in \mathcal{B}^1(TM)$. We have the relations $\mathbf{h}^2 = \mathbf{h}$, $Im\mathbf{h} = HTM$, $Ker\mathbf{h} = VTM$; $\mathbf{v} = \mathbf{i} \circ \mathcal{V}$, $\mathbf{v}^2 = \mathbf{v}$, $Im\mathbf{v} = VTM$, $Ker\mathbf{v} = HTM$. Concerning these technical tools, we collect here some useful formulae:

$$J \circ \mathbf{h} = J, \qquad \mathbf{h} \circ J = 0, \qquad J \circ \mathbf{v} = 0, \qquad \mathbf{v} \circ J = J,$$
$$JX^h = X^v, \quad J[X^h, Y^h] = [X, Y]^v, \quad \mathbf{h}X^c = X^h, \quad \mathbf{h}[X^h, Y^h] = [X, Y]^h.$$

In general there is no canonical way to specify a horizontal map. However, in the presence of some additional structure, a horizontal map may be given canonically. We recall here a well-known and very important construction. Suppose that ξ is a *semispray* on M, i.e. $\xi : TM \to TTM$ is a C^1 vector field which is smooth on $\overset{\circ}{T}M$ and has the property $J\xi = C$. Then the map

$$X \in \mathcal{X}(M) \mapsto X^h := \frac{1}{2}(X^c + [X^v, \xi]) \in \mathcal{X}(\overset{\circ}{T}M)$$

defines a horizontal lifting and, as a consequence, a horizontal map $\mathcal{H}$ for τ. $\mathcal{H}$ will be mentioned as the *horizontal map generated by the semispray* ξ. If ξ is a *spray*, i.e. $[C,\xi]=\xi$, then the horizontal map $\mathcal{H}$ is *homogeneous* in the sense that $[X^h,C]=0$ for all $X\in\mathcal{X}(M)$.

(G) Let r be an integer. By a *graded derivation* of degree r of the exterior algebra $\mathcal{A}(M)$ we mean an $\mathbb{R}$-linear map $\mathcal{D}:\mathcal{A}(M)\to\mathcal{A}(M)$ such that $\mathcal{D}\big(\mathcal{A}^k(M)\big)\subset\mathcal{A}^{k+r}(M)$ for all $k\in\{0,\dots,n\}$ and $\mathcal{D}(\alpha\wedge\beta)=(\mathcal{D}\alpha)\wedge\beta+(-1)^{rk}\alpha\wedge\mathcal{D}\beta$ if $\alpha\in\mathcal{A}^k(M)$, $\beta\in\mathcal{A}(M)$. The *graded commutator* of two graded derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ is given by $[\mathcal{D}_1,\mathcal{D}_2]:=\mathcal{D}_1\circ\mathcal{D}_2-(-1)^{r_1r_2}\mathcal{D}_2\circ\mathcal{D}_1$ where r_1 and r_2 are the degrees of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. $[\mathcal{D}_1,\mathcal{D}_2]$ is also a graded derivation whose degree is r_1+r_2. The classical graded derivations of $\mathcal{A}(M)$ are the *substitution operator* i_X (induced by $X\in\mathcal{X}(M)$), the Lie derivative d_X (with respect to $X\in\mathcal{X}(M)$), and the *exterior derivative* d; their degrees are $-1,0$, and 1, respectively. i_X,d_X and d are related by H. Cartan's 'magic' formula

$$d_X=i_X\circ d+d\circ i_X=[i_X,d].$$

In the *Frölicher-Nijenhuis theory* of derivations two graded derivations of $\mathcal{A}(M)$ are associated to any vector k-form $K\in\mathcal{B}^k(M)$: the derivation i_K of degree $k-1$ defined by $i_K\restriction C^\infty(M):=0$ and $i_K\alpha:=\alpha\circ K$ for $\alpha\in\mathcal{A}^1(M)$, and the derivation d_K of degree k defined as the graded commutator $d_K:=[i_K,d]=i_K\circ d-(-1)^{k-1}d\circ i_K$. As an immediate consequence, we obtain:

if $f\in C^\infty(M)$ and $K\in\mathcal{B}^k(M)$, then $d_Kf=i_Kdf=df\circ K$.

A characteristic property of d_K is expressed by

$$[d,d_K]:=d\circ d_K-(-1)^kd_K\circ d=0.$$

For any vector k-form K and vector l-form L on M there is a unique vector $(k+l)$-form on M, denoted by $[K,L]$, such that $d_{[K,L]}=[d_K,d_L]$. $[K,L]$ is said to be the *Frölicher–Nijenhuis bracket* of K and L. If K and L are 0-forms, i.e. vector fields on M, then $[K,L]$ is the usual bracket of vector fields. If $L:=Y\in\mathcal{X}(M)=\mathcal{B}^0(M)$ and $K\in\mathcal{B}^1(M)$, then

$$[K,Y]X=[KX,Y]-K[X,Y]\qquad\text{for all } X\in\mathcal{X}(M).$$

If K and L are both vector one-forms, or in other words type $(1,1)$ tensor fields on M, then for any vector fields X,Y in $\mathcal{X}(M)$ we have

$$\begin{aligned}&[K,L](X,Y)=[KX,LY]+[LX,KY]+(K\circ L+L\circ K)[X,Y]\\&-K[LX,Y]-K[X,LY]-L[KX,Y]-L[X,KY].\end{aligned}$$

Notice that in our calculations over TM the operators i_J and $d_J = i_J \circ d - d \circ i_J$ will play a distinguished role.

A more or less analogous theory of derivations of $\mathcal{A}(\tau)$ was elaborated by E. Martínez, J.F. Cariñena and W. Sarlet [53, 54], see also [90]. We shall borrow only one ingredient from this theory, the *v-exterior derivative* d^v defined by

$$(d^v F)(\tilde{X}) := dF(\mathbf{i}\tilde{X}) = (\mathbf{i}\tilde{X})F, \; d^v\hat{\alpha} := 0 \text{ for all } F \in C^\infty(TM) \text{ and } \alpha \in \mathcal{A}^1(M).$$

We can easily deduce the important relation $d^v \circ d^v = 0$.

(H) Consider for a given vector field $\tilde{X}$ along τ the map $\nabla^v_{\tilde{X}}$ whose action on functions, vector fields and one-forms along the projection is given by

$$\nabla^v_{\tilde{X}} F := (d^v F)(\tilde{X}), \quad \nabla^v_{\tilde{X}}\tilde{Y} := \mathbf{j}[\mathbf{i}\tilde{X}, \mathcal{H}\tilde{Y}], \; (\nabla^v_{\tilde{X}}\tilde{\alpha})(\tilde{Y}) := \nabla^v_{\tilde{X}}\big(\tilde{\alpha}(\tilde{Y})\big) - \tilde{\alpha}(\nabla^v_{\tilde{X}}\tilde{Y})$$
$$(F \in C^\infty(TM), \tilde{Y} \in \mathcal{X}(\tau), \tilde{\alpha} \in \mathcal{A}^1(\tau)).$$

The formula

$$\begin{aligned}(\nabla^v_{\tilde{X}}\tilde{A})(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \tilde{X}_l) &:= (\mathbf{i}\tilde{X})\tilde{A}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \tilde{X}_l)\\ &-\sum_{i=1}^{k}\tilde{A}(\tilde{\alpha}_1, \dots, \nabla^v_{\tilde{X}}\tilde{\alpha}_i, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \tilde{X}_l)\\ &-\sum_{j=1}^{l}\tilde{A}(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \nabla^v_{\tilde{X}}\tilde{X}_j, \dots, \tilde{X}_l)\end{aligned}$$

extends the action of $\nabla^v_{\tilde{X}}$ to a general type (k, l) tensor field along the projection. The *canonical v-covariant differential* is the operator ∇^v that maps a (k, l) tensor field $\tilde{A}$ along τ onto the $(k, l+1)$ tensor field $\nabla^v\tilde{A}$ along τ by the rule

$$(\nabla^v\tilde{A})(\tilde{X}, \tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \tilde{X}_l) := (\nabla^v_{\tilde{X}}\tilde{A})(\tilde{\alpha}_1, \dots, \tilde{\alpha}_k, \tilde{X}_1, \dots, \tilde{X}_l).$$

In the same way, specifying a horizontal map $\mathcal{H}$ for τ, and starting from

$$\nabla^h_{\tilde{X}} F := (\mathcal{H}\tilde{X})F, \quad \nabla^h_{\tilde{X}}\tilde{Y} := \mathcal{V}[\mathcal{H}\tilde{X}, \mathbf{i}\tilde{Y}], \quad (\nabla^h_{\tilde{X}}\tilde{\alpha})(\tilde{Y}) := \nabla^h_{\tilde{X}}\big(\tilde{\alpha}(\tilde{Y})\big) - \tilde{\alpha}(\nabla^h_{\tilde{X}}\tilde{Y})$$

$(F \in C^\infty(TM)$, $\tilde{Y} \in \mathcal{X}(\tau)$, $\tilde{\alpha} \in \mathcal{A}^1(\tau))$ we can introduce the *h-covariant differential* ∇^h. Having these partial differentials, the map

$$\nabla : (\xi, \tilde{Y}) \in \mathcal{X}(TM) \times \mathcal{X}(\tau) \mapsto \nabla_\xi\tilde{Y} := \nabla^v_{\mathcal{V}\xi}\tilde{Y} + \nabla^h_{\mathbf{j}\xi}\tilde{Y} \in \mathcal{X}(\tau)$$

is a *covariant derivative operator* in the vector bundle $\tau^*\tau$ in the sense that ∇ is an $\mathbb{R}$-bilinear map satisfying the conditions $\nabla_{F\xi}\tilde{Y} = F\nabla_\xi \tilde{Y}$ and $\nabla_\xi F\tilde{Y} = (\xi F)\tilde{Y} + F\nabla_\xi \tilde{Y}$ concerning the multiplication with a smooth function F on TM. The covariant derivative operator ∇ is said to be the *Berwald derivative in* $\tau^*\tau$ induced by $\mathcal{H}$. Explicitly,

$$\nabla_\xi \tilde{Y} = \mathbf{j}[\mathbf{v}\xi, \mathcal{H}\tilde{Y}] + \mathcal{V}[\mathbf{h}\xi, \mathbf{i}\tilde{Y}] \qquad \text{for all } \xi \in \mathcal{X}(TM) \text{ and } \tilde{Y} \in \mathcal{X}(\tau).$$

The canonical v-covariant differential ∇^v is intimately related to the v-exterior derivative d^v: we have

$$d^v\tilde{\alpha} = (k+1) Alt \nabla^v \tilde{\alpha} \qquad \text{for all } \tilde{\alpha} \in \mathcal{A}^k(\tau),$$

where Alt is the alternator in $\mathcal{A}^k(\tau)$. We say that a one-form $\tilde{\alpha}$ along τ is ∇^v-*exact* if there exists a function $F \in C^\infty(TM)$ such that $\nabla^v F = \tilde{\alpha}$.

Lemma. *A one-form $\tilde{\alpha}$ along τ is ∇^v-exact if, and only if, $\nabla^v\tilde{\alpha}$ is symmetric, i.e.*

$$(\nabla^v\tilde{\alpha})(\tilde{X}, \tilde{Y}) = (\nabla^v\tilde{\alpha})(\tilde{Y}, \tilde{X}) \qquad \textit{for all } \tilde{X}, \tilde{Y} \in \mathcal{X}(\tau).$$

(I) Finally, we point out that the Frölicher-Nijenhuis formalism provides a concise and extremely elegant way to define the basic geometric data of a horizontal map. Namely, let $\mathcal{H}$ be a horizontal map over M, and let $\mathbf{h}$ be the horizontal projector determined by $\mathcal{H}$. Then the vector forms

$$\mathbf{t} := [\mathbf{h}, C] \in \mathcal{B}^1(TM), \quad \mathbf{T} := [J, \mathbf{h}] \in \mathcal{B}^2(TM), \quad \Omega := -\frac{1}{2}[\mathbf{h}, \mathbf{h}] \in \mathcal{B}^2(TM)$$

are said to be the *tension*, the *torsion* and the *curvature* of $\mathcal{H}$, respectively. It is known that a *horizontal map is generated by a semispray if, and only if, its torsion vanishes* (a theorem of M. Crampin).

Chapter 2

Generalized Lagrange manifolds

In this chapter we treat Finsler metrics as a subclass in a more general metric geometry. The concept of a Finsler manifold is as old as the concept of a Riemannian manifold, since it was Riemann himself who suggested the investigation of the more general Finsler metric in his *Habilitationsvortrag* of 1854 (see e.g. [23, 75]). However, Finsler geometers usually do not refer to a metric-like structure as the cornerstone of their theory: traditionally Finsler geometry is cast in terms of a 1-homogeneous function, called the *fundamental function*, or a 2-homogeneous function, the so-called *energy*, and only secondary is the Finsler metric introduced as the Hessian of the energy. In contrast with this point of view, in the second chapter of this dissertation, **we intend to treat the Finsler metric as being prior to the energy**. Speaking in coordinate terms, the most striking difference between a Riemannian metric and a Finsler metric is that the local components of the latter can depend on the fibre coordinates of the tangent manifold. In the past, a lot of models have been proposed to describe Finsler geometry. In our experience it turns out to be convenient to think about Finsler metrics as a special subset in the class of symmetric and non-degenerate (0,2)-tensor fields of the pullback bundle $\tau^*TM \to TM$. In this dissertation, we will refer to all such tensor fields as *metrics*.

In [39], M. Hashiguchi gave a necessary and sufficient condition for a metric to be the Hessian of some Finsler energy and used for the first time the adjective *normal* to distinguish Finsler metrics from all others. As we mentioned in the *Introduction*, a large number of scientific areas make use of a normal metric. However, also the study of less restricted classes of metrics is necessary since there remain a lot of theories that use a (not normal) metric (see e.g. [72] Ch. XI, Ch. XII and the references therein). In the first part of this dissertation we will study metrics in a broader context, meaning that they need not necessarily be the Hessian of some energy. Thus, we will investigate subclasses of metrics that satisfy each a certain aspect of the very

restricted condition of 'normality'.
There exists a long history of attempts to generalize Finsler geometry, mainly written in the language of classical tensor calculus. Here, we will mention only two papers which have a direct link with our dissertation. In [98] and [71], the two authors considered a subclass of metrics that is more general than the class of normal metrics. These two subclasses are different from each other: J.R. Vanstone, building upon earlier work of A. Moór [73], studied certain aspects of homogeneous metrics, while the metrics of R. Miron satisfy a weaker condition than M. Hashiguchi's. We will come back to the precise characterization of these two classes in chapter two. For now, it is important to note that a lot of applications in metric geometry involve the use of a metric derivative. Both above mentioned papers have in common that the authors were able to provide a local coordinate formulation for a 'canonical' metric derivative in their subclass. The main reason why they could find such a formulation is related to the ability of their subclass to generate a *regular Lagrangian*. The regularity of this Lagrangian implies the existence of a canonical semispray, which in turn leads to an associated horizontal distribution. In general, such a horizontal distribution is not 'canonically' available for an arbitrary metric. Its presence makes it possible to simplify the problem of metric derivatives to the search for two appropriate tensor fields on the pullback bundle. Later, in [72], R. Miron and M. Anastasiei recognized this idea in a theorem that gives an explicit coordinate formulation for all 'metrical connections' when the availability of a horizontal distribution is assumed (which is a priori not related to the metric). Therefore, the main goal of chapter two can be formulated as follows: **we will look for metrics that provide us in a natural way a regular Lagrangian**. Now we describe the contents of Chapter 2 of our dissertation in detail.

⋆ ⋆ ⋆

In **section 1***, firstly we discuss some basic properties of the* **Hessian** *of a function. We deal with the* **regularity** *of a* **Lagrangian** *and the existence of a* **Lagrange vector field***. Its integral curves are the solutions of the well-known* **Euler–Lagrange equations***.*

Lemma (2.1.2, 2.1.3). *Let $F : TM \longrightarrow \mathbb{R}$ be a smooth function.*

(a) The Hessian $g_F := H^{\nabla} F := \nabla^v(\nabla^v F)$ is a symmetric type $(0,2)$ tensor field along τ.

(b) The v-covariant differential $\nabla^v g_F$ *of* g_F *is a totally symmetric type* $(0,3)$ *tensor field along* τ.

(c) For the Hessian of a smooth function $F \in C^\infty(TM)$ *we have*
$g_F(\hat{X}, \hat{Y}) := H^\nabla F(\hat{X}, \hat{Y}) = X^v(Y^v F)$ *for all* $X, Y \in \mathcal{X}(M)$.

Definition. Let $L : TM \longrightarrow \mathbb{R}$ be a smooth function, called *Lagrangian*.

(a) The function $E_L := CL - L$ is said to be the *energy function* associated to L.

(b) The two-form $\omega_L := dd_J L$ is called the *Lagrange two-form*.

Proposition (2.1.6). *Let a Lagrangian* L *on* TM *be given. The Lagrange two-form* ω_L *is non-degenerate if, and only if, the Hessian* $g_L := H^\nabla L$ *is non-degenerate.*

Definition. A Lagrangian $L : TM \longrightarrow \mathbb{R}$ is said to be *regular*, if the Lagrange two-form ω_L is non-degenerate. A manifold M endowed with a regular Lagrangian, formally a pair (M, L), where L is a regular Lagrangian, is called a *Lagrange manifold*.

⋆ ⋆ ⋆

In **section 2** *we introduce* **(generalized Lagrange) metrics***, their* **(absolute) energy** *and their associated* **Lagrange one- and two-forms** *and we also define variational metrics. We introduce the* **first Cartan tensor** *and the* **lowered first Cartan tensor** *of a metric and show that the symmetry of these tensors characterizes the very important subclass of* **variational** *metrics. Also, we deduce some elementary properties of the first Cartan tensors.*

Definition (2.2.1). (a) By a *generalized Lagrange metric* (or briefly a *metric*) we mean a symmetric and non-degenerate type $(0,2)$ tensor field along τ. A manifold M endowed with a metric, formally a pair (M, g), where g is a metric, is called a *generalized Lagrange manifold*.

(b) Let (M, g) be a generalized Lagrange manifold.

(1) The smooth function $E_g := \frac{1}{2} g(\delta, \delta)$ is called the *(absolute) energy* of (M, g).

(2) The one-form $\Theta_g \in \mathcal{A}^1(TM)$ defined by $\Theta_g(\xi) := g(\mathbf{j}\xi, \delta)$ for all $\xi \in \mathcal{X}(TM)$ and the two-form $\omega_g := d\Theta_g$ on TM are said to be the *Lagrange one-form* and the *Lagrange two-form* associated to g, respectively.

(c) A generalized Lagrange metric g is called *variational* if there is a Lagrangian $L : TM \longrightarrow \mathbb{R}$ such that $g = H^{\nabla}L := \nabla^v(\nabla^v L) =: g_L$.

Definition (2.2.3). Let (M, g) be a generalized Lagrange manifold. The *canonical v-covariant derivative*

$$\mathcal{C}_\flat := \nabla^v g \in \mathcal{T}_3^0(\tau)$$

of g is said to be the *lowered first Cartan tensor* of g. The type $(1,2)$ tensor field $\mathcal{C}$ along τ determined by

$$g\big(\mathcal{C}(\tilde{X}, \tilde{Y}), \tilde{Z}\big) = (\nabla^v g)(\tilde{X}, \tilde{Y}, \tilde{Z}) =: \mathcal{C}_\flat(\tilde{X}, \tilde{Y}, \tilde{Z}) \quad (\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{X}(\tau))$$

is called the *first Cartan tensor* of g.

Proposition (2.2.4.6). *A generalized Lagrange metric g along τ is variational if, and only if, it satisfies the integrability condition*

$$\nabla^v g(\tilde{X}, \tilde{Y}, \tilde{Z}) = \nabla^v g(\tilde{Y}, \tilde{X}, \tilde{Z}) \quad (\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathcal{X}(\tau)),$$

i.e. the lowered first Cartan tensor of g is symmetric in its first two variables.

⋆⋆⋆

In **section 3** *we investigate two different ways to characterize that the energy is regular (and thus gives rise to a semispray). This subclass will be called the class of* E**-regular metrics***. Next we show when the Lagrange two-form of a metric is non-degenerate and give a necessary and sufficient condition for it to be the Lagrange two-form of a regular Lagrangian (leading again to a semispray). These two properties are called respectively* **Miron-regular** *and* **weakly variational***. A metric that is weakly variational with respect to its own energy is a* **weakly normal** *metric. We indicate that the subclass of metrics are, in our terminology, weakly normal Miron-regular metrics. Further we pay some attention to the* **semi-Finsler** *case, when the metric is variational and Miron-regular, and-among others- we characterize the difference between the two possible associated semisprays.*

Definition (2.3.2). A generalized Lagrange metric g is said to be E*-regular*, if the absolute energy $E := \frac{1}{2}g(\delta, \delta) : TM \longrightarrow \mathbb{R}$ is a regular Lagrangian.

Lemma (2.3.3). *A metric tensor g along τ is E-regular if, and only if, the type $(1,1)$ tensor field $\tilde{A} : \mathcal{X}(\tau) \longrightarrow \mathcal{X}(\tau), \tilde{X} \longmapsto \tilde{A}(\tilde{X})$ along τ defined by $g\big(\tilde{A}(\tilde{X}), \tilde{Y}\big) = g_E(\tilde{X}, \tilde{Y})$ is injective for all $\tilde{Y} \in \mathcal{X}(\tau)$.*

Proposition (2.3.6). *If g is a variational metric, then the $(1,1)$ tensor field $\tilde{A}$ acts by the rule $\tilde{A}(\hat{X}) = \hat{X} + 2\mathcal{C}(\hat{X}, \delta) + \frac{1}{2}(\nabla^v_{\hat{X}}\mathcal{C})(\delta, \delta) + \frac{1}{2}\mathcal{C}\big(\mathcal{C}(\delta, \delta), \hat{X}\big)$.*

Proposition (2.3.9.3). *The Lagrange two-form ω_g of the generalized Lagrange manifold (M, g) is non-degenerate if, and only if, the tensor*

$$\tilde{B} : \tilde{X} \in \mathcal{X}(\tau) \longmapsto \tilde{B}(\tilde{X}) := \tilde{X} + \mathcal{C}(\tilde{X}, \delta)$$

is injective.

Definition (2.3.9.4). A generalized Lagrange metric g along τ is said to be *regular in Miron's sense*, briefly *Miron-regular*, if the map

$$\tilde{B} : \tilde{X} \in \mathcal{X}(\tau) \longmapsto \tilde{X} + \mathcal{C}(\tilde{X}, \delta) \in \mathcal{X}(\tau)$$

is injective.

Definition (2.3.10.1). A generalized Lagrange manifold (M, g) is said to be a *semi-Finsler manifold*, if g is variational and Miron-regular. If $g = H^{\nabla}L$, then the function $\tilde{E_L} := CE_L - E_L$ is called the *principal energy* of the semi-Finsler manifold.

Proposition (2.3.10.4). *Let $(M, g) = (M, H^{\nabla}L)$ be a semi-Finsler manifold. Let ξ_g be the canonical spray of (M, g), ξ_L the Lagrange vector field for L. The difference $\xi := \xi_g - \xi_L$ is the unique (necessarily vertical) vector field on TM such that $i_\xi \omega_g = i_{[C,\xi_L]-\xi_L}\omega_L$, where ω_g and ω_L are the Lagrange two-forms associated to g and L, respectively.*

Definition (2.3.11.1). A generalized Lagrange metric g is said to be *weakly variational* if the first Cartan tensor of g has the symmetry property $g\big(\mathcal{C}(\tilde{X}, \tilde{Y}), \delta\big) = g\big(\mathcal{C}(\tilde{Y}, \tilde{X}), \delta\big)$, or, equivalently, if $\mathcal{C}_\flat(\tilde{X}, \tilde{Y}, \delta) = \mathcal{C}_\flat(\tilde{Y}, \tilde{X}, \delta)$ for all $\tilde{X}, \tilde{Y} \in \mathcal{X}(\tau)$.

Proposition (2.3.11.6). *For a generalized Lagrange metric g along τ the following conditions are equivalent:*

- *(i) The type $(0,2)$ tensor field γ_g given by $\gamma_g(\tilde{X}, \tilde{Y}) := g\big(\tilde{B}(\tilde{X}), \tilde{Y}\big)$ for all $\tilde{X}, \tilde{Y} \in \mathcal{X}(\tau)$ is a metric along τ.*
- *(ii) g is Miron-regular and weakly variational.*
- *(iii) γ_g is the Hessian of a regular Lagrangian.*

Corollary (2.3.11.7). *If g is a weakly variational metric, then the tensor $\tilde{A}$ acts by the rule* $\tilde{A}(\tilde{X}) = \tilde{X} + 2\mathcal{C}(\tilde{X}, \delta) + \frac{1}{2}(\nabla^v_{\tilde{X}}\mathcal{C}) + \frac{1}{2}\mathcal{C}\big(\tilde{X}, \mathcal{C}(\delta, \delta)\big)$ $(\tilde{X} \in \mathcal{X}(\tau))$.

Definition (2.3.12.2). A generalized Lagrange metric is said to be *weakly normal* if its lowered first Cartan tensor has the property $\mathcal{C}_\flat(., \delta, \delta) = 0$.

Proposition (2.3.12.3). *If g is a weakly normal metric, then*

(i) $\gamma_g = g_E$, *therefore* $\tilde{A} = \tilde{B}$;

(ii) $g_E(\tilde{X}, \delta) = g(\tilde{X}, \delta) = \tilde{\Theta}_g(\tilde{X})$ *for all* $\tilde{X} \in \mathcal{X}(\tau)$;

(iii) the absolute energy of g is homogeneous of degree two, i.e. $CE = 2E$.

Corollary (2.3.12.5). *If the metric g is both Miron-regular and weakly normal, then the semispray ξ_E defined by $i_{\xi_E}\omega_E = -dE$ is actually a spray.*

⋆ ⋆ ⋆

For a variational metric, the **Helmholtz conditions** *are the necessary and sufficient conditions for a semispray to be the Lagrange vector field of the regular Lagrangian (whose Hessian is exactly the metric). In* **section 4** *we have shown that also E-regular metrics and Miron-regular weakly variational metrics give rise to a regular Lagrangian. We prove conditions for a semispray to be the Lagrange vector field of these Lagrangians.*

Definition (2.4.1). Let a semispray ξ be given, and let $\mathcal{H}$ be the horizontal map generated by ξ. If $\mathcal{V}$ is the vertical map belonging to $\mathcal{H}$, then the type (1,1) tensor field $\tilde{\Phi}$ along τ given by

$$\tilde{\Phi}(\tilde{X}) := \mathcal{V}[\xi, \mathcal{H}\tilde{X}] \quad \text{for all } \tilde{X} \in \mathcal{X}(\tau)$$

is said to be the *Jacobi endomorphism* determined by ξ.

Horizontal Lie derivatives (2.4.2). Assume that a horizontal map $\mathcal{H}$ is given for τ. For all vector fields ξ on TM, we define an operator $\mathcal{L}^h_\xi$ by the rules $\mathcal{L}^h_\xi F := \xi F$, if $F \in C^\infty(TM)$; $\mathcal{L}^h_\xi \tilde{Y} := \mathbf{j}[\xi, \mathcal{H}\tilde{Y}]$, if $\tilde{Y} \in \mathcal{X}(\tau)$, and extend it into a tensor derivation of the full tensor algebra of the tensor fields along τ. If, in particular, ξ is a semispray and $\mathcal{H}$ is generated by ξ, then we call $\mathcal{L}^h_\xi$ the *dynamical derivative* with respect to ξ.

Proposition (2.4.5). *Let g be an E-regular metric, and let a semispray ξ be given. ξ is the Lagrangian vector field for E if, and only if, for all vector fields $\tilde{X}, \tilde{Y}$ along τ we have* *(i)* $g\Big(\tilde{A}\big(\tilde{\Phi}(\tilde{X})\big), \tilde{Y}\Big) = g\Big(\tilde{X}, \tilde{A}\big(\tilde{\Phi}(\tilde{Y})\big)\Big)$;

(ii) $(\mathcal{L}^h_\xi g)\big(\tilde{A}(\tilde{X}), \tilde{Y}\big) = -g\big((\mathcal{L}^h_\xi \tilde{A})(\tilde{X}), \tilde{Y}\big)$.

Proposition (2.4.6). *Assume that the metric g is both Miron-regular and weakly variational and let L_g be the Lagrangian arising from g. A semispray ξ is the Lagrangian vector field for L_g if, and only if, for all vector fields $\tilde{X}, \tilde{Y}$ along τ we have*

(i) $g\Big(\tilde{B}\big(\tilde{\Phi}(\tilde{X})\big), \tilde{Y}\Big) = g\Big(\tilde{X}, \tilde{B}\big(\tilde{\Phi}(\tilde{Y})\big)\Big)$;

(ii) $(\mathcal{L}^h_\xi g)\big(\tilde{B}(\tilde{X}), \tilde{Y}\big) = -g\big((\mathcal{L}^h_\xi \tilde{B})(\tilde{X}), \tilde{Y}\big)$.

⋆⋆⋆

In **section 5** *we discuss* **Moór-Vanstone metrics** *which are homogeneous and E-regular at the same time. We point out that the intersection of all above mentioned subclasses of metrics is constituted by the* **normal metrics**. *We show how the* **classical point of view** *in Finsler geometry (by means of the energy) is incorporated in our approach.*

Definition (2.5.1, 2.5.2). A generalized Lagrange metric g is said to be *homogeneous* if $\nabla^v_\delta g = 0$, and *normal*, if its first Cartan tensor has the property $\mathcal{C}(., \delta) = 0$.

Proposition (2.5.4). *A metric is variational with respect to a homogeneous function of degree 2 if, and only if, it is normal.*

Definition A. Let a function $E : TM \longrightarrow \mathbb{R}$ be given. Assume:

(E1) E is of class C^1 on TM, smooth on $\overset{\circ}{T}M$.

(E2) E is positive-homogeneous of degree 2, i.e.$CE = 2E$.

(E3) The Lagrangian two-form $\omega_E := dd_J E$ is non-degenerate, i.e. E is a regular Lagrangian.

(E4) $\forall v \in TM$: $E(v) \geq 0$, $E(0) = 0$.

Then (M, E) is said to be a *Finsler manifold with energy E*; $g_E := \nabla^v \nabla^v E$ is mentioned as a *Finsler metric* or an *energy metric*.

Definition B. Let a function $L : TM \longrightarrow \mathbb{R}$ be given. Assume:

(L1) L is continuous on TM, smooth on $\overset{\circ}{T}M$.

(L2) L is positive-homogeneous of degree 1.

(L3) If $E := \frac{1}{2}L^2$, then the two-form $\omega_E := dd_J E$ is non-degenerate.

(L4) $\forall v \in TM$: $L(v) \geq 0$, $L(0) = 0$.

Then (M, L) is said to be a *Finsler manifold with fundamental function* L; $E := \frac{1}{2}L^2$ is called the *energy* associated to L.

⋆ ⋆ ⋆

In **section 6** *we present a brief survey on the interrelations among the various metrics that have been discussed. (Some of the information is summarized in two tabulars at the end of this section (p. 20, 21)). In* **section 7** *we enter the discussion about* **metric derivatives**. *For any given torsion-free horizontal map (meaning that it is generated by a semispray), we are able to give a* **canonical metric derivative**. *We also show how the difference between an arbitrary metric derivative and this canonical one can be characterized by means of two tensors. In the special case that the metric is normal, the canonical metric derivative is nothing but the well-known* **Cartan derivative** *of Finsler geometry. In this way we have arrived again in the field of Finsler geometry and so we are ready to start our study on Randers manifolds.*

Definition (2.7.1). The *lowered second Cartan tensor* of a generalized Lagrange metric g with respect to a horizontal map $\mathcal{H}$ is the type $(0,3)$ tensor field $\mathcal{C}^h_\flat := \nabla^h g$, where ∇^h is the h-covariant derivative induced by the Berwald derivative ∇ arising from $\mathcal{H}$. The type $(1,2)$ tensor field $\mathcal{C}^h$ along τ given by

$$g\big(\mathcal{C}^h(\tilde{X},\tilde{Y}),\tilde{Z}\big) := \mathcal{C}^h_\flat(\tilde{X},\tilde{Y},\tilde{Z}) \quad \text{for all } \tilde{X},\tilde{Y},\tilde{Z} \in \mathcal{X}(\tau)$$

is called the *second Cartan tensor* of g with respect to $\mathcal{H}$.

Definition (2.7.3). Let a metric g and a horizontal map $\mathcal{H}$ be given, and let us consider the first and the second Cartan tensor $\mathcal{C}$ and $\mathcal{C}^h$ of g, respectively. If $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ are defined by

$$g(\overset{\circ}{\mathcal{C}}(\tilde{X},\tilde{Y}),\tilde{Z}) := \mathcal{C}_\flat(\tilde{X},\tilde{Y},\tilde{Z}) + \mathcal{C}_\flat(\tilde{Y},\tilde{Z},\tilde{X}) - \mathcal{C}_\flat(\tilde{Z},\tilde{X},\tilde{Y})$$

and

$$g(\overset{\circ}{\mathcal{C}}{}^h(\tilde{X},\tilde{Y}),\tilde{Z}) := \mathcal{C}^h_\flat(\tilde{X},\tilde{Y},\tilde{Z}) + \mathcal{C}^h_\flat(\tilde{Y},\tilde{Z},\tilde{X}) - \mathcal{C}^h_\flat(\tilde{Z},\tilde{X},\tilde{Y})$$

$\big(\tilde{X},\tilde{Y},\tilde{Z} \in \mathcal{X}(\tau)\big)$, then $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ are well-defined symmetric type $(1,2)$ tensor fields along τ.

Definition. Let a horizontal map $\mathcal{H}$ be specified for τ, and let D be a covariant derivative operator in $\tau^*\tau$.

(1) The $\tau^*\tau$-valued two-forms $T^h(D) := d^D\mathbf{j}$ and $T^v(D) := d^D\mathcal{V}$ are said to be the *horizontal* and *vertical torsion* of D, respectively. (d^D is the covariant exterior derivative with respect to D.)

(2) The maps $\mathcal{T}$ and $\mathcal{S}$ given by

$$\mathcal{T}(\tilde{X},\tilde{Y}) := T^h(D)(\mathcal{H}\tilde{X},\mathcal{H}\tilde{Y}) \text{ and } \mathcal{S}(\tilde{X},\tilde{Y}) := T^h(D)(\mathcal{H}\tilde{X},\mathbf{i}\tilde{Y}) \ (\tilde{X},\tilde{Y} \in \mathcal{X}(\tau))$$

are said to be the *h-horizontal* and *h-mixed torsion* of D, respectively. D is called *symmetric* if $\mathcal{T} = 0$ and $\mathcal{S} = 0$.

(3) The maps $\mathbf{R^1}$, $\mathbf{P^1}$, and $\mathbf{Q^1}$ defined by the formulae

$$\mathbf{R^1}(\tilde{X},\tilde{Y}) := T^v(D)(\mathcal{H}\tilde{X},\mathcal{H}\tilde{Y}),\ \mathbf{P^1}(\tilde{X},\tilde{Y}) := T^v(D)(\mathcal{H}\tilde{X},\mathbf{i}\tilde{Y})$$

and

$$\mathbf{Q^1}(\tilde{X},\tilde{Y}) := T^v(D)(\mathbf{i}\tilde{X},\mathbf{i}\tilde{Y}) \quad \text{for all } \tilde{X},\tilde{Y} \in \mathcal{X}(\tau)$$

are called the *v-horizontal, v-mixed* and *v-vertical torsion* of D, respectively. We shall denote by $(\mathbf{R^1})_0$ the *semibasic* tensor field given by

$$(\mathbf{R^1})_0(\xi,\eta) := \mathbf{i}\mathbf{R^1}(\mathbf{j}\xi,\mathbf{j}\eta) \quad \text{for all } \xi,\eta \in \mathcal{X}(M).$$

Theorem (2.7.8). *Let a metric g along τ be given. Suppose that $\mathcal{H}$ is a horizontal map with vanishing torsion, and let ∇ be the Berwald derivative induced by $\mathcal{H}$ in $\tau^*\tau$. If $\overset{\circ}{\mathcal{C}}$ and $\overset{\circ}{\mathcal{C}}{}^h$ are the tensors introduced above, then the rules*

$$D_{\mathbf{i}\tilde{X}}\tilde{Y} := \nabla_{\mathbf{i}\tilde{X}}\tilde{Y} + \frac{1}{2}\overset{\circ}{\mathcal{C}}(\tilde{X},\tilde{Y}),\ D_{\mathcal{H}\tilde{X}}\tilde{Y} := \nabla_{\mathcal{H}\tilde{X}} + \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h(\tilde{X},\tilde{Y})$$

define a symmetric, metric derivative in $\tau^*\tau$. For the torsions of D we have:

$$\mathcal{T} = 0,\ \mathcal{S} = \frac{1}{2}\overset{\circ}{\mathcal{C}},\ (\mathbf{R^1})_0 = \Omega,\ \mathbf{P^1} = \frac{1}{2}\overset{\circ}{\mathcal{C}}{}^h,\ \mathbf{Q^1} = 0.$$

Corollary (2.7.9). *Suppose that g is a variational metric, namely $g = \nabla^v \nabla^v L$, and let $\mathcal{H}$ be the horizontal map generated by the Lagrangian vector field ξ_L. Then, with the notation of* **2.7.8**, *the rules*

$$D_{\mathbf{i}\tilde{X}}\tilde{Y} := \nabla_{\mathbf{i}\tilde{X}}\tilde{Y} + \frac{1}{2}\mathcal{C}(\tilde{X},\tilde{Y}), \quad D_{\mathcal{H}\tilde{X}}\tilde{Y} := \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\mathcal{C}^h(\tilde{X},\tilde{Y})$$

define a symmetric metric derivative in $\tau^*\tau$. The torsions of D are

$$\mathcal{T} = 0,\ \mathcal{S} = -\frac{1}{2}\mathcal{C},\ (\mathbf{R^1})_0 = \Omega,\ \mathbf{P^1} = \frac{1}{2}\mathcal{C}^h,\ \mathbf{Q^1} = 0.$$

Corollary and definition (2.7.10). *Let g be a normal metric in $\tau^*\tau$, and let $\mathcal{H}$ be the canonical horizontal map of the Finsler manifold $(M, E) = \big(M, \frac{1}{2}g(\delta,\delta)\big)$. There is a* **unique** *covariant derivative operator D in $\tau^*\tau$ satisfying the following conditions:*

C.covd.1. *D is metrical, i.e. $Dg = 0$.*

C.covd.2. *The h-horizontal torsion $\mathcal{T}$ of D vanishes.*

C.covd.3. *The v-vertical torsion $\mathbf{Q^1}$ of D vanishes.*

This covariant derivative operator is said to be the **Cartan derivative** *in (M, E). The rules for calculations with respect to D are the following:*

$$\boxed{\begin{array}{c} D_{\mathbf{i}\tilde{X}}\tilde{Y} = \nabla_{\mathbf{i}\tilde{X}}\tilde{Y} + \frac{1}{2}\mathcal{C}(\tilde{X},\tilde{Y}) = \mathbf{j}[\mathbf{i}\tilde{X},\mathcal{H}\tilde{Y}] + \frac{1}{2}\mathcal{C}(\tilde{X},\tilde{Y}) \\ D_{\mathcal{H}\tilde{X}}\tilde{Y} = \nabla_{\mathcal{H}\tilde{X}}\tilde{Y} + \frac{1}{2}\mathcal{C}^h(\tilde{X},\tilde{Y}) = \mathcal{V}[\mathcal{H}\tilde{X},\mathbf{i}\tilde{Y}] + \frac{1}{2}\mathcal{C}^h(\tilde{X},\tilde{Y}). \end{array}}$$

(∇ is the Berwald derivative induced by $\mathcal{H}$, $\tilde{X},\tilde{Y} \in \mathcal{X}(\tau)$). In particular, for any basic vector fields $\hat{X},\hat{Y}$ along $\overset{\circ}{\tau}$ we have

$$\boxed{D_{X^v}\hat{Y} = \frac{1}{2}\mathcal{C}(\hat{X},\hat{Y}), \quad D_{X^h}\hat{Y} = \mathcal{V}[X^h, Y^v] + \frac{1}{2}\mathcal{C}^h(\hat{X},\hat{Y}).}$$

D is **associated** *to $\mathcal{H}$, in the sense that $D\delta = \mathcal{V}$.*

A description of all metric derivatives (2.7.11). Let a metric tensor g in $\tau^*\tau$ and a horizontal map $\mathcal{H}$ for τ be given. The *difference* of two covariant derivative operators D^1 and D^2 can be characterized by means of a $\mathcal{C}^\infty(TM)$-bilinear map $\varrho : \mathcal{X}(TM) \times \mathcal{X}(\tau) \longrightarrow \mathcal{X}(\tau)$ such that

$$D^1_\xi \tilde{Y} - D^2_\xi \tilde{Y} = \varrho(\xi, \tilde{Y}) \quad \text{for all } \xi \in \mathcal{X}(TM) \text{ and } \tilde{Y} \in \mathcal{X}(\tau).$$

ϱ can be decomposed into a v-part ϱ^v and an h-part ϱ^h given by

$$\varrho^v(\tilde{X},\tilde{Y}) := \varrho(\mathbf{i}\tilde{X},\tilde{Y}) \quad \text{and} \quad \varrho^h(\tilde{X},\tilde{Y}) := \varrho(\mathcal{H}\tilde{X},\tilde{Y}) \quad \text{for all } \tilde{X},\tilde{Y} \in \mathcal{X}(\tau).$$

Then we have $\varrho(\xi,\tilde{Y}) = \varrho^v(\mathcal{V}\xi,\tilde{Y}) + \varrho^h(\mathbf{j}\xi,\tilde{Y})$ for all $\xi \in \mathcal{X}(TM),\ \tilde{Y} \in \mathcal{X}(\tau)$.

Theorem (2.7.11). *Let D be the metric derivative constructed above. If Φ and Ψ are arbitrary type (1,2) tensor fields along τ, and the tensor fields ϱ^v and ϱ^h are determined by the relations*

(1) $g\big(\varrho^v(\tilde{X},\tilde{Y}),\tilde{Z}\big) = \frac{1}{2}\Big(g\big(\Phi(\tilde{X},\tilde{Y}),\tilde{Z}\big) - g\big(\Phi(\tilde{X},\tilde{Z}),\tilde{Y}\big)\Big)$,

(2) $g\big(\varrho^h(\tilde{X},\tilde{Y}),\tilde{Z}\big) = \frac{1}{2}\Big(g\big(\Psi(\tilde{X},\tilde{Y}),\tilde{Z}\big) - g\big(\Psi(\tilde{X},\tilde{Z}),\tilde{Y}\big)\Big)$,

then the covariant derivative operator $\overline{D}$ defined by

(3) $\overline{D}_\xi\tilde{Y} := D_\xi\tilde{Y} + \varrho^v(\mathcal{V}\xi,\tilde{Y}) + \varrho^h(\mathbf{j}\xi,\tilde{Y})$ *for all* $\xi \in \mathcal{X}(TM),\ \tilde{Y} \in \mathcal{X}(\tau)$

is a metric derivative in $\tau^*\tau$. Conversely, any metric derivative in $\tau^*\tau$ can be represented in this form.

Note. A coordinate version of this theorem can also be found in [72]. In fact, Miron and Anastasiei make use the so called *Obata operators*. In our context, the first Obata operator can be best viewed as a map $Ob : \mathcal{T}_2^1(\tau) \longrightarrow \mathcal{T}_2^1(\tau)$, $\tilde{A} \longmapsto Ob_{\tilde{A}}$ given by

$$g\big(Ob_{\tilde{A}}(\tilde{X},\tilde{Y}),\tilde{Z}\big) := \frac{1}{2}\Big(g\big(\tilde{A}(\tilde{X},\tilde{Y}),\tilde{Z}\big) - g\big(\tilde{A}(\tilde{X},\tilde{Z}),\tilde{Y}\big)\Big).$$

The second Obata operator $Ob^* : \mathcal{T}_2^1(\tau) \longrightarrow \mathcal{T}_2^1(\tau)$ has an analogous definition: for all $\tilde{B} \in \mathcal{T}_2^1(\tau)$ and $\tilde{X},\tilde{Y},\tilde{Z} \in \mathcal{X}(\tau)$,

$$g\big(Ob^*_{\tilde{B}}(\tilde{X},\tilde{Y}),\tilde{Z}\big) := \frac{1}{2}\Big(g\big(\tilde{B}(\tilde{X},\tilde{Y}),\tilde{Z}\big) + g\big(\tilde{B}(\tilde{X},\tilde{Z}),\tilde{Y}\big)\Big).$$

Therefore, (1) and (2) in our above Theorem can be restated putting $\varrho^v := Ob_\Phi$ and $\varrho^h := Ob_\Psi$, respectively.

Chapter 3

Randers manifolds

This part of the dissertation is devoted to one extremely important particular class of Finsler manifolds. Prominent examples of Finsler manifolds (besides Riemannian manifolds) where an explicit expression of the fundamental function is given, are named after Kropina [48, 84, 64], Matsumoto [62, 3], Antonelli [5]. In this chapter we will investigate *Randers manifolds*. The main novelty of our contribution lies in the purely intrinsic character of the methods and results. We elaborate an efficient machinery of forms and vector fields that are abundantly used in the coordinate-free description of the basic geometry data of Randers manifolds.
As their name suggests, Randers metrics were introduced by G. Randers in 1941 [74], and named after him for the first time by R.S. Ingarden. The original interest for Randers manifolds came from physics: in optics, Randers metrics were found to describe the motion of a relativistic electron, but also in other physical areas (see e.g. the remarks in [6, 15]) many applications followed. Not only physicists, but also pure geometers started to show interest in the subject, because Randers manifolds supply one of the most basic examples of Finsler manifolds: by adding a one-form, their fundamental function perturbs the fundamental function of a Riemannian manifold. A lot of invariants in Finsler geometry were explicitly calculated for the first time for Randers manifolds. Randers metrics were seen in a more general class of metrics which emerged in the study of what are now called (α, β)-metrics. For a general survey of results and applications of Randers manifolds we refer to [15] and [66]. Among the many papers on the subject we cite only a few [14, 41, 42, 43, 44, 45, 46, 58, 59, 63, 65, 67, 83, 85, 78] which have a direct bearing on the subject matter of this dissertation.

One of the most important features that distinguishes Riemannian geometry from Finsler geometry is the existence of a unique torsion-free and metrical linear connection (i.e. the Levi-Civita connection) on the base manifold M.

Finsler manifolds lack this property and all four famous linear connections (Berwald, Hashiguchi, Chern-Rund and Cartan) can be regarded as several attempts to partially recover this property, either by relaxing the torsion condition, or by softening the metrical condition. Therefore, it is an interesting question to investigate under what condition a Finsler manifold gives rise to a linear connection on M. For example, at a local coordinate level, the (horizontal) connection coefficients of the Berwald connection would form a linear connection on M if they would not explicitly depend on the coordinates of the (tangent) fibre. Equivalently: if we would derive these connection coefficients with respect to the fibre coordinates, we should find zero. Of course, this is nothing but looking for vanishing (mixed) curvature. A Finsler manifold whose Berwald derivative satisfies this property is called a *Berwald manifold*. We present a completely new proof of a famous criterion for a Randers manifold to be of Berwald type. Randers manifolds suffer an inevitable obstacle. The bigger part of the outcome can only be given in implicit terms: due to the tremendous complexity of the involved calculations it is almost impossible to find direct and compact expressions for most of the basic data. In particular, it is almost impossible to write down an explicit formula for the Cartan derivative of a Randers manifold, since it would involve, among other, the calculation of the second Cartan tensor. The definition of this tensor involves both the Riemann-Finsler metric and the Berwald connection associated to the Barthel endomorphism of the Randers manifold. A direct formulation of both the Barthel endomorphism and its Berwald connection is presented and, to say the least, they are already very complicated! Obviously, the computation of the second Cartan tensor will lead to a direct expression of a tensor which is too complex to be of use. Even without an explicit formula, the Cartan connection is frequently used, also in the study of Randers manifolds. The reason for this can easily be found in the one major advantage that the Cartan derivative has above all other famous derivatives in the Finsler literature: the Cartan derivative is fully metrical. But, it remains possible to find a number of connections which share this property with respect to a given metric: every, arbitrarily chosen, horizontal map can be shown to produce such a derivative. In relation to Randers manifolds, it is obvious now that this horizontal map should not be the Barthel endomorphism, since we have a more simple horizontal map at our disposal: we will use the Barthel endomorphism of the underlying Riemannian manifold instead of the Barthel endomorphism of the whole Randers manifold. Mainly due to the easy relation between the Berwald derivative associated to this horizontal lift (which is at the same time also the Cartan derivative) and the Levi-Civita connection associated

to the Riemannian metric, the calculation of the second Cartan tensor of the Randers metric *with respect to this horizontal lift* will be much less complicated then the original second Cartan tensor of the Randers manifold.
Two Finsler manifolds are said to be conformally equivalent if their energy metrics are proportional to each other. In this case we also speak of a conformal change of the metric. By Knebelman's observation the proportionality factor is a vertical lift of a function on the base manifold (see e.g. [94]). According to this definition, we present a new framework for the conformal theory of Randers metrics.We prove in a coordinate-free manner that under a conformal change of a Randers metric the underlying Riemannian manifolds are also conformally equivalent. In the course of the proof, we obtain an intimate relation between the Levi-Civita derivatives. This leads us to build up a conformally invariant symmetric covariant derivative operator on the base manifold. The corresponding spray and horizontal endomorphism can be relatively easily described. Finally, we have a look at the projective equivalence of two sprays. Roughly speaking, two sprays over a manifold are said to be projectively equivalent, if their geodesics are the same as point sets. Using an appropriate formulation of the definition, we give a necessary and sufficient condition for the projective equivalence of the sprays arising from the conformally invariant symmetric covariant derivative operator and from the Levi-Civita derivative. Now we present a detailed view on the contents of chapter 3.

⋆⋆⋆

In **section 1** *we deal with again the necessary machinery of forms and vector fields and derive some useful technical result.*

Note. Consider a Riemannian metric α on M. This leads to a one-form $\bar{\alpha}$ along τ given by $\bar{\alpha}(\tilde{X}) := \hat{\alpha}(\tilde{X}, \delta)$ for all $\tilde{X} \in \mathcal{X}(\tau)$. In particular, we write L_α^2 instead of $\bar{\alpha}(\delta)$.

Lemma (3.1.1). *Let β be a 1-form on M. For any vector field X on M, $X^v \bar{\beta} = \beta(X) \circ \tau = [\beta(X)]^v = \hat{\beta}(\hat{X})$.*

Lemma (3.1.2). *We have the relations*

$$X^v L_\alpha^2 = 2\bar{\alpha}(\hat{X}) = 2\hat{\alpha}(\delta, \hat{X}), \qquad Y^v(X^v L_\alpha^2) = 2\hat{\alpha}(\hat{X}, \hat{Y}) = 2[\alpha(X,Y)]^v,$$
$$X^v L_\alpha = \frac{1}{L_\alpha}\bar{\alpha}(\hat{X}), \qquad Y^v(X^v L_\alpha) = \frac{1}{L_\alpha}\hat{\alpha}(\hat{X}, \hat{Y}) - \frac{1}{L_\alpha^3}\bar{\alpha}(\hat{X})\bar{\alpha}(\hat{Y}).$$

for all $X, Y \in \mathcal{X}(M)$.

Notation. Let α be a Riemannian metric on M. For any $\tilde{\theta} \in \mathcal{X}^*(\tau)$, $\tilde{\theta}^{\sharp_{\hat{\alpha}}}$ is the unique vector field along τ such that $\tilde{\theta}(\tilde{X}) := \hat{\alpha}(\tilde{\theta}^{\sharp_{\hat{\alpha}}}, \tilde{X})$ for all $\tilde{X} \in \mathcal{X}(\tau)$, where $\sharp_{\hat{\alpha}}$ is a musical endomorphism with respect to the metric tensor $\hat{\alpha}$ along the projection.

Definition. The *norm* of a basic one-form $\hat{\beta}$ with respect to the metric tensor $\hat{\alpha}$ is $||\,\hat{\beta}\,||_{\hat{\alpha}} := \sqrt{\hat{\beta}(\hat{\beta}^{\sharp_{\hat{\alpha}}})} = \sqrt{\hat{\alpha}(\hat{\beta}^{\sharp_{\hat{\alpha}}}, \hat{\beta}^{\sharp_{\hat{\alpha}}})}$.

Lemma (3.1.7). *Consider the Levi-Civita connection ∇ of the Riemannian metric α. Its action on the 1-form β gives rise to a (0,2)-tensor $\nabla\beta$ on M, with the associated (0,2)-tensor $\widehat{\nabla\beta}$ along the projection such that*

$$\overset{\alpha}{\nabla}_{\mathbf{i}\tilde{X}}\hat{\beta} = 0, \quad \text{and} \quad (\overset{\alpha}{\nabla}_{\mathcal{H}_\alpha}\hat{\beta})(\tilde{Y}) = \widehat{\nabla\beta}(\tilde{X}, \tilde{Y}),$$

where $\mathcal{H}_\alpha$ is the canonical horizontal map of the Riemannian manifold (M, α).

Notation. (a) In the special case when the first or the second argument of $\widehat{\nabla\beta}$ is δ, we obtain two 1-forms, denoted by $\overline{\nabla\beta}$ and $\overline{\overline{\nabla\beta}}$, along the projection such that

$$\overline{\nabla\beta}(\tilde{Y}) := \widehat{\nabla\beta}(\delta, \tilde{Y}), \quad \overline{\overline{\nabla\beta}}(\tilde{Y}) := \widehat{\nabla\beta}(\tilde{Y}, \delta) \quad \text{for all } \tilde{Y} \in \mathcal{X}(\tau).$$

In particular, we shall use the notation $\overline{\nabla\beta}(\delta)$ for both $\overline{\nabla\beta}(\delta)$ and $\overline{\overline{\nabla\beta}}(\delta)$.
(b) The symmetric and skew-symmetric extension of $\widehat{\nabla\beta}$ will be denoted respectively by $Sym\widehat{\nabla\beta}$ and $Alt\widehat{\nabla\beta}$:

$$Sym\widehat{\nabla\beta}(\tilde{X}, \tilde{Y}) := \tfrac{1}{2}\Big(\widehat{\nabla\beta}(\tilde{X}, \tilde{Y}) + \widehat{\nabla\beta}(\tilde{Y}, \tilde{X})\Big),$$

$$Alt\widehat{\nabla\beta}(\tilde{X}, \tilde{Y}) := \tfrac{1}{2}\Big(\widehat{\nabla\beta}(\tilde{X}, \tilde{Y}) - \widehat{\nabla\beta}(\tilde{Y}, \tilde{X})\Big).$$

In particular, we shall write

$$\overline{Sym\nabla\beta}(\tilde{Y}) := Sym\widehat{\nabla\beta}(\delta, \tilde{Y}), \quad \overline{Alt\nabla\beta}(\tilde{Y}) := Alt\widehat{\nabla\beta}(\delta, \tilde{Y}).$$

Lemma (3.1.12). *$S_\alpha\bar{\beta} = \overline{\nabla\beta}(\delta)$, where S_α is the canonical spray of the Riemannian manifold (M, α).*

⋆ ⋆ ⋆

In **section 2** *we define the Randers manifolds and prove a sufficient condition for a Randers manifold to be a (positive-definite) Finsler manifold. We present a direct expression for the* **metric** *and the* **angular metric***.*

Definition (3.2.1). Let (M, α) be a Riemannian manifold, β a one-form on M, and let the functions L_α and $\bar{\beta}$ be given by

$$L_\alpha(v) := \sqrt{\alpha_{\tau(v)}(v, v)}, \quad \bar{\beta}(v) := \beta_{\tau(v)}(v).$$

Then $(M, L_\alpha + \bar{\beta}) =: (M, L)$ is said to be the *Randers manifold* constructed from the Riemannian manifold by perturbation with $\bar{\beta}$ (or with β).

Proposition (3.2.3). *The fundamental function of a Randers manifold* $(M, L_\alpha + \bar{\beta})$ *is strictly positive if and only if* $|| \hat{\beta} ||_{\hat{\alpha}} < 1$.

Proposition (3.2.4). *The energy metric* g *of the Randers manifold* (M, L) *can be represented as follows:*

$$g = \frac{L}{L_\alpha}\hat{\alpha} - \frac{\bar{\beta}}{L_\alpha^3}\bar{\alpha} \otimes \bar{\alpha} + \frac{1}{L_\alpha}\bar{\alpha} \odot \hat{\beta} + \hat{\beta} \otimes \hat{\beta}.$$

Proposition (3.2.5). *If* $|| \hat{\beta} ||_{\hat{\alpha}} < 1$*, then the energy metric of the Randers manifold* $(M, L_\alpha + \bar{\beta})$ *is positive definite.*

Corollary (3.2.6). *If* $|| \hat{\beta} ||_{\hat{\alpha}} < 1$*, then the Randers manifold arising from the Riemannian manifold* (M, L_α) *by perturbation with* $\bar{\beta}$ *is a Finsler manifold.*

Proposition (3.2.9). *In a Randers manifold the angular metric takes the form* $k = \frac{L}{L_\alpha}\hat{\alpha} - \frac{L}{L^3{}_\alpha}\bar{\alpha} \otimes \bar{\alpha}$*. Its relation to the energy metric* g *is given by* $k = g - (\frac{1}{L_\alpha}\bar{\alpha} + \hat{\beta}) \otimes (\frac{1}{L_\alpha}\bar{\alpha} + \hat{\beta})$.

⋆⋆⋆

In **section 3** *we give an explicit expression for the* **canonical spray** *and an intrinsic formula for the difference between the* **horizontal lift** *generated by the canonical spray and the horizontal lift of the underlying Riemannian structure.*

Corollary (3.3.7). *The canonical spray* S *of a Randers manifold* $(M, L) = (M, L_\alpha + \bar{\beta})$ *can be expressed as follows:*

$$S = S_\alpha + \frac{1}{L}\Big(2L_\alpha \overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_{\hat{\alpha}}}) - \overline{\nabla\beta}(\delta)\Big)C - 2L_\alpha \mathbf{i}(\overline{Alt\nabla\beta})^{\sharp_{\hat{\alpha}}}.$$

Proposition (3.3.9). *The horizontal lift of a vector field X on M is given by the formula*

$$\begin{aligned} X^h &= X^{h_\alpha} + \frac{\mu}{2L}X^v - \frac{\mu}{2L^2}\tilde{P}(\hat{X})C + \frac{1}{L}\tilde{Q}(\hat{X})C - \frac{1}{L_\alpha}\bar{\alpha}(\hat{X})\mathbf{i}(\overline{Alt\nabla\beta})^{\sharp_{\hat{\alpha}}} \\ &- L_\alpha\big[X^v, \mathbf{i}(\overline{Alt\nabla\beta})^{\sharp_{\hat{\alpha}}}\big], \end{aligned}$$

where $\mu := 2L_\alpha\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_{\hat{\alpha}}}) - \overline{\nabla\beta}(\delta)$*, while* $\tilde{P}$ *and* $\tilde{Q}$ *are one-forms along* τ *defined by* $\tilde{P}(\hat{X}) := \frac{1}{L_\alpha}\bar{\alpha}(\hat{X}) + \hat{\beta}(\hat{X})$ *and* $\tilde{Q}(\hat{X}) := \frac{1}{L_\alpha}\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_{\hat{\alpha}}})\bar{\alpha}(\hat{X}) + L_\alpha Alt\widehat{\nabla\beta}(\hat{X}, \hat{\beta}^{\sharp_{\hat{\alpha}}}) - \overline{Sym\nabla\beta}(\hat{X})$*, for all* $X \in \mathcal{X}(M)$.

⋆ ⋆ ⋆

In **section 4** *we derive the rules for calculation for the* **Berwald derivative** *of the Randers manifold, together with those of the* **first Cartan tensors***. Further we compute an intrinsic formula for the* **Cartan vector field***.*

Proposition (3.4.1). *Let* $X, Y \in \mathcal{X}(M)$*. The Berwald derivative* $(\overset{\circ}{D}, \mathcal{H})$ *for a Randers manifold* $(M, L) = (M, L_\alpha + \bar{\beta})$ *is given by*

$$\begin{aligned} (1)\ \overset{\circ}{D}_{X^v}\hat{Y} &= 0, \\ (2)\ \overset{\circ}{D}_{X^h}\hat{Y} &= \overset{\alpha}{\nabla}_{X^{h_\alpha}}\hat{Y} + \Big(\frac{\mu}{2L^2}P(\hat{X}) - \frac{1}{L}Q(\hat{X})\Big)\hat{Y} + \Big(\frac{\mu}{2L^2}P(\hat{Y}) - \frac{1}{L}Q(\hat{Y})\Big)\hat{X} \\ &+\Big(\frac{1}{L^2}P(\hat{X})Q(\hat{Y}) + \frac{1}{L^2}Q(\hat{X})P(\hat{Y}) - \frac{\mu}{L^3}P(\hat{X})P(\hat{Y})\Big)\delta \\ &-\frac{1}{LL_\alpha}\Big(\bar{\alpha}(\hat{X})Alt\widehat{\nabla\beta}(\hat{Y}, \hat{\beta}^{\sharp_{\hat{\alpha}}}) + \bar{\alpha}(\hat{Y})Alt\widehat{\nabla\beta}(\hat{X}, \hat{\beta}^{\sharp_{\hat{\alpha}}})\Big)\delta \\ &+\frac{1}{L}Sym\widehat{\nabla\beta}(\hat{X}, \hat{Y})\delta + \frac{\mu}{2L^3}k(\hat{X}, \hat{Y})\delta - \frac{1}{L^2}k(\hat{X}, \hat{Y})\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_{\hat{\alpha}}})\delta \\ &+\frac{1}{L}k(\hat{X}, \hat{Y})(\overline{Alt\nabla\beta})^{\sharp_{\hat{\alpha}}} + \frac{1}{L_\alpha}\bar{\alpha}(\hat{X})\mathcal{V}_\alpha\big[Y^v, \mathbf{i}(\overline{Alt\nabla\beta})^{\sharp_{\hat{\alpha}}}\big] \\ &+\frac{1}{L_\alpha}\bar{\alpha}(\hat{Y})\mathcal{V}_\alpha\big[X^v, \mathbf{i}(\overline{Alt\nabla\beta})^{\sharp_{\hat{\alpha}}}\big] + L_\alpha\mathcal{V}_\alpha\big[X^v, \big[Y^v, \mathbf{i}(\overline{Alt\nabla\beta})^{\sharp_{\hat{\alpha}}}\big]\big]. \end{aligned}$$

Proposition (3.4.2). *The first lowered Cartan tensor* $\mathcal{C}_\flat$ *and the first*

Cartan tensor $\mathcal{C}$ of a Randers manifold (M, L) can be expressed as follows:

$$\mathcal{C}_\flat = \frac{1}{L_\alpha}\Big(\hat{\beta}\odot\hat{\alpha} - \frac{\bar{\beta}}{L_\alpha^2}\bar{\alpha}\odot\hat{\alpha} - \frac{1}{L_\alpha^2}\hat{\beta}\odot\bar{\alpha}\otimes\bar{\alpha} + \frac{\bar{\beta}}{L_\alpha^4}\bar{\alpha}\odot\bar{\alpha}\otimes\bar{\alpha}\Big)$$
$$= \frac{1}{L}\Big(\hat{\beta} - \frac{\bar{\beta}}{L_\alpha^2}\bar{\alpha}\Big)\odot k,$$
$$\mathcal{C} = \frac{1}{L}\hat{\beta}\odot id - \frac{\bar{\beta}}{LL_\alpha^2}\bar{\alpha}\odot id - \frac{1}{L^2}\hat{\beta}\odot P\otimes\delta + \frac{\bar{\beta}}{L^2L_\alpha^2}\bar{\alpha}\odot P\otimes\delta$$
$$+\frac{L_\alpha}{L^2}k\otimes\hat{\beta}^{\sharp_{\hat{\alpha}}} - \frac{\bar{\beta} + L_\alpha||\,\hat{\beta}\,||_{\hat{\alpha}}^2}{L^3}k\otimes\delta.$$

The trace of the first Cartan tensor is given by

$$tr\mathcal{C} = (n+1)\frac{1}{L}\Big(\hat{\beta} - \frac{\bar{\beta}}{L_\alpha^2}\bar{\alpha}\Big).$$

Corollary (3.4.3). *The lowered Cartan tensor is related to the angular metric and the trace of the first Cartan tensor by $\mathcal{C}_\flat = \frac{1}{n+1}(k\odot tr\mathcal{C})$.*

Definition (3.4.5). The Cartan vector field $\mathcal{C}^*$ of a Finsler manifold is the unique vector field along the projection such that $g(\mathcal{C}^*, \tilde{X}) = (tr\mathcal{C})(\tilde{X})$ for any vector field $\tilde{X}\in\mathcal{X}(\tau)$.

Proposition (3.4.11). *The Cartan vector field of a Randers manifold $(M, L) = (M, L_\alpha + \bar{\beta})$ can be decomposed as follows:*

$$\mathcal{C}^* = -(n+1)\frac{1}{L^3}(\bar{\beta} + L_\alpha||\,\hat{\beta}\,||_{\hat{\alpha}}^2)\delta + (n+1)\frac{L_\alpha}{L^2}\hat{\beta}^{\sharp_{\hat{\alpha}}}.$$

⋆⋆⋆

Section 5 *is devoted to a completely new proof for the following famous result.* **Theorem.** *A Randers manifold $(M, L_\alpha+\bar{\beta})$ is a Berwald manifold if and only if $\nabla\beta = 0$, where ∇ is the Levi-Civita connection of the Riemanian manifold (M, α).*

⋆⋆⋆

In **section 6** *we relate our results on generalized Lagrange metric with the theory of Randers manifolds, by means of the choice of an appropriate* **met-**

ric derivative. *We also show how this metric derivative can be used to characterize the Randers manifolds of Berwald type.*

Proposition (3.6.2). *The second Cartan tensors of g with respect to $\mathcal{H}_\alpha$ are given by*

$$i_{\hat{X}}\mathcal{C}_\flat^{h_\alpha} = \frac{1}{L}\overline{\overline{\overline{\nabla\beta}}}(\hat{X})k + (\overset{\alpha}{\nabla}_{X^{h_\alpha}}\hat{\beta}) \odot P,$$

and

$$\overset{\circ}{\mathcal{C}}{}^{h_\alpha} = \frac{1}{L}\overline{\overline{\overline{\nabla\beta}}} \odot id - \frac{1}{L}\overline{\overline{\overline{\nabla\beta}}} \odot P \otimes \delta + \frac{L_\alpha}{L}\overline{B} \odot P - \frac{L_\alpha}{L}\overline{\overline{B}} \odot P$$
$$+2\frac{1}{L}Sym\widehat{\nabla\beta} \otimes \delta + \frac{L_\alpha}{L^2}k \otimes (\overline{\overline{\overline{\nabla\beta}}})^{\sharp_{\hat{\alpha}}} - \frac{L_\alpha}{L^3}\overline{\overline{\overline{\nabla\beta}}}(\hat{\beta}^{\sharp_{\hat{\alpha}}})k \otimes \delta$$
$$-\frac{2L_\alpha}{L^2}\nabla^v(\overline{Alt\nabla\beta}(\hat{\beta}^{\sharp_{\hat{\alpha}}})) \odot P \otimes \delta - \frac{2}{n+1}\overline{Alt\nabla\beta} \odot P \otimes \mathcal{C}^*$$
$$-\frac{1}{(n+1)L}\overline{\nabla\beta}(\delta) \otimes k \otimes \mathcal{C}^*.$$

($\overline{B}$ and $\overline{\overline{B}}$ are defined by $\hat{\alpha}(\overline{B}(\tilde{X}), \tilde{Y}) := \widehat{\nabla\beta}(\tilde{X}, \tilde{Y})$, $\hat{\alpha}(\overline{\overline{B}}(\tilde{X}), \tilde{Y}) := \widehat{\nabla\beta}(\tilde{Y}, \tilde{X})$; $X, Y \in \mathcal{X}(M)$).

Proposition (3.6.3). *Let $\mathcal{H}_\alpha$ be the canonical horizontal map for the Riemannian manifold (M, α), and let us consider the Berwald derivative $\overset{\alpha}{\nabla}$ induced by $\mathcal{H}_\alpha$. If*

(1) $\overset{\alpha}{D}_{X^v}\hat{Y} := \mathcal{C}(\hat{X}, \hat{Y})$,

(2) $\overset{\alpha}{D}_{X^{h_\alpha}}\hat{Y} := \overset{\alpha}{\nabla}_{X^{h_\alpha}}\hat{Y} + \overset{\circ}{\mathcal{C}}{}^{h_\alpha}(\hat{X}, \hat{Y})$,

then $\overset{\alpha}{\nabla}$ is a metric derivative for the Randers manifold $(M, L_\alpha + \bar{\beta})$.

Corollary (3.6.4). *For a Randers manifold $(M, L_\alpha + \bar{\beta})$, the following properties are equivalent*

(1) The manifold is a Berwald manifold.

(2) $\nabla\beta = 0$.

(3) $\overset{\alpha}{D}_{\mathcal{H}_\alpha\tilde{X}}\hat{\beta} = 0$.

(4) $\overset{\alpha}{D}_{X^{h_\alpha}}\hat{Y} = \widehat{\nabla_X Y}$, *for any vector fields X, Y on the base manifold.*

⋆ ⋆ ⋆

In **section 7** *we turn to the* **conformal equivalence** *of two Randers manifolds. Given a Randers manifold, we present an explicit, coordinate-free description of a covariant derivative that is invariant under any conformal change of the fundamental function* $L = L_\alpha + \bar{\beta}$.

Definition. Two Finsler manifolds with common base manifold M are said to be *conformally equivalent* if their energy metrics g and g_c are related by $g_c = \phi g$, where $\phi \in C^\infty(TM)$ and $\phi > 0$.In this case we also speak of a *conformal change* of the metric.

By Knebelman's observation ϕ is vertical lift, so it can be written in the form $e^{2\sigma^v}$, where $\sigma \in C^\infty(M)$. A conformal change $g \longrightarrow e^{2\sigma^v} g$ of the energy metric is equivalent to a 'conformal change' $L \longrightarrow e^{\sigma^v} L$ of the fundamental function.

Proposition (3.7.7). *Given a Randers manifold* $(M, L) = (M, L_\alpha + \bar{\beta})$, *let*

$$\begin{aligned} \nu &:= \frac{1}{\|\beta\|_\alpha^2}\Big(\nabla_{\beta^{\sharp_\alpha}}\beta - \frac{1}{n-1}(div\beta^{\sharp_\alpha})\beta\Big), \\ \nu^{\sharp_\alpha} &:= \frac{1}{\|\beta\|_\alpha^2}\Big(\nabla_{\beta^{\sharp_\alpha}}\beta^{\sharp_\alpha} - \frac{1}{n-1}(div\beta^{\sharp_\alpha})\beta^{\sharp_\alpha}\Big). \end{aligned}$$

Then the formula $\overset{\star}{\nabla} = \nabla + \nu \odot id - \alpha \otimes \nu^{\sharp_\alpha}$ *gives a symmetric linear connection on* M*, which is invariant under the conformal changes of* L.

⋆ ⋆ ⋆

In **section 8** *we describe the difference between the horizontal map generated by the conformally invariant covariant derivative and the horizontal map generated by the Riemannaian metric. Next, we define when two sprays are said to be* **projectively equivalent**. *We show: if the spray generated by the conformally invariant derivative is projectively equivalent to the spray of the Randers manifold then the former must be equal to the spray generated by the underlying Riemannian structure.*

Proposition (3.8.2). *The horizontal endomorphism arising from* $\overset{\star}{\nabla}$ *can be expressed as follows:* $h_{\overset{\star}{\nabla}} = h_\alpha + \hat{\nu} \otimes C + \bar{\nu} id - \bar{\alpha} \otimes \hat{\nu}^{\sharp_{\hat{\alpha}}}$. *The spray* $\overset{\star}{S}$ *belonging to* $h_{\overset{\star}{\nabla}}$ *is given by* $\overset{*}{S} = S_\alpha + 2\bar{\nu} C - L_\alpha^2 \hat{\nu}^{\sharp_{\hat{\alpha}}}$.

Definition. Two sprays, S_1 and S_2, over a manifold M are said to be *projectively equivalent* if there exists a function $\lambda : TM \longrightarrow \mathbb{R}$ such that

(1) λ is a class of C^1 over $\overset{\circ}{T}M$ and smooth over $\mathcal{T}M$,

(2) $S_1 = S_2 + \lambda C$.

Proposition (3.8.4). *The canonical sprays S and S_α are projectively equivalent if, and only if, the deforming one-form β is closed.*

Proposition (3.8.5). *If S is projectively equivalent with $\overset{\star}{S}$, then $\overset{\star}{S}$ is just the canonical spray of the underlying Riemannian manifold (M, α).*

Magyar nyelvű összefoglaló

Tárgyát tekintve a disszertáció két szorosan összekapcsolódó részre tagolható, melyeknek közös gyökerei a Finsler-geometriában találhatók. Mivel a Finsler-geometriában mind a mai napig nem alakult ki egy zavaró különbségektől mentes, egyöntetűnek tekinthető terminológia, valamint konszenzus a jelölésrendszert illetően, mondandónk világos kifejtése érdekében szükségesnek éreztük a Finsler-geometria általános elméleti alapjainak egy meglehetősen részletes áttekintését, rögzítve ezáltal egyben a jelölés - és szóhasználatot is. Így a dolgozat teljes és meglehetősen terjedelmes **első fejezet**ét ilyen általánosságoknak szenteltük, elsősorban a [91] munkára támaszkodva. Munkánk fő célja az általánosított Lagrange-metrikákkal és a Randers-sokaságokkal kapcsolatos néhány alapvető probléma koordinátamentes eszközökkel történő vizsgálata, a klasszikus tenzorkalkulus egy továbbfejlesztett változatának keretei között. Tárgyalásunkban alapvető szerepet játszik a nemlineáris konnexiók Grifone-féle elmélete [37], [38], szoros összefüggésben a vektorértékű differenciálformák A. Frölicher és A. Nijenhuis [35] által kidolgozott kalkulusával, s ez utóbbinak a pull-back nyaláb keretei közé történt adaptálásával (Martinez-Cariñena-Sarlet elmélet, [53]-[54], s ld. [91]-et is).

A dolgozat **második fejezet**ének középpontjában a Lagrange- és Finsler-sokaságok egy közös általánosítása áll. A Finsler-sokaságok elvben egyidősek a Riemann-sokaságokkal, hiszen Riemann maga vetette föl az 1854 június 12-én, Göttingen-ben tartott híres habilitációs előadásában (*Über die Hypothesen welche der Geometrie zu Grunde liegen*; ld. [23], [75]) az általánosabb, mai szóhasználattal élve Finsler-metrikák vizsgálatának gondolatát. Hagyományosan a Finsler-geometria egy elsőfokú pozitív homogén függvényre, az ún. *alapfüggvényre*, vagy pedig egy másodfokú pozitív homogén függvényre, az ún. *energiára* épül, ezek birtokában - az energia Hesse-tenzoraként - csak második lépésben kerül sor metrikus tenzor megadására. E megközelítéstől eltérően, disszertációnkban közvetlenül egy "irányfüggő" metrikus tenzorból indulunk ki, s ebből származtatjuk az energiafüggvényt. Tapasztalataink azt mutatják, hogy kifejezetten célszerű a Finsler-metrikákat úgy tekinteni, mint a *pull-back nyaláb* szimmetrikus, nemelfajuló (0,2)-tenzormezőinek egy speciális típusát. M. Hashiguchi egy, a hagyományos tenzorkalkulus nyelvén megírt, de nagyon modern szemléletű és világos dolgozatában ([39]) már 1958-ban szükséges és elegendő feltételt adott arra vonatkozóan, hogy milyen feltétel mellett állítható elő egy metrika valamilyen Finsler-energia Hesse-tenzoraként, s ugyanitt ő használta először a "*normális*" jelzőt a Finsler-metrikák más metrikáktól való megkülönböztetésére. A normális metrikáknak számos alkalmazása van a fizikában, a biológiában, a közgazdaságtudományban, de az általánosabb metrikákban is sok alkalmazási lehetőség rejlik (ld. például [72], Ch. XI, Ch. XII). Mindezek alapján, a dolgozat első felében olyan metrikákat vizsgálunk, amelyek nem feltétlenül állíthatók elő egy energiafüggvény Hesse-tenzoraként, de bizonyos értelemben eleget tesznek a "normalitás" feltételének.

Két dolgozatot, a [71] és a [98] munkát külön is meg kell említenünk, mint e fejezet közvetlen előzményeit. Ezekben a szerzők olyan, egymástól különböző metrikákat tekintenek, amelyek a normális metrikáknál jóval általánosabbak. Moór Arthur korábbi kutatásaira (ld. [73]) alapozva, a [98] dolgozatban J. R. Vanstone homogén metrikákat vizsgál különböző aspektusokból, míg R. Miron a Hashiguchi által adott kritériumnál gyengébb feltételnek megfelelő metrikákat tanulmányoz [71]-ben. Mindezen metrikáknak pontos jellemzésével szolgálunk a disszertáció második fejezetében. Fontos megjegyeznünk, hogy a metrikus geometriában számos alkalommal szükség van valamilyen *metrikus derivált* alkalmazására. Ez a helyzet a fent említett dolgozatok esetében is, ahol az egyes metrikák tulajdonságainak felderítése egy kanonikus metrikus derivált bevezetését igényli. Ennek egzisztenciáját az adott metrikákból származó reguláris Lagrange-függvény biztosítja, amely kanonikus módon szemisprayt, és ezáltal horizontális struktúrát generál. (Egy tetszőleges metrikából általában nem nyerhető "kanonikus módon" horizontális struktúra!) Horizontális struktúra birtokában már bevezethető a Berwald-derivált, melynek egy metrikus deriválttal képzett különbsége tenzor a pull-back nyalábon. Ez az észrevétel lehetővé teszi, hogy a metrikus deriváltak meghatározásával kapcsolatos problémákat a pull-back nyalábon konstruált, alkalmas tenzormező kiválasztására redukáljuk. R. Miron és M. Anastasiei ezt felismerve - adott horizontális struktúra birtokában - lokálisan leírta az összes metrikus deriváltakat [72]. Mi a metrikus deriváltak leírását egy áttekinthető, koordinátamentes (s ilyen értelemben "intrinsic") formában valósítottuk meg.

A disszertáció második fejezetének legfőbb eredménye az összes olyan metrikák meghatározása, amelyek természetes módon reguláris Lagrange-függvényt származtatnak. Az ehhez vezető út vázlatosan a következő:

(1) Az alkalmazott jelölések és fogalmak bevezetése után egy függvény

*Hesse-tenzor*ának néhány alapvető tulajdonságát származtatjuk, majd a Lagrange-függvények *regularitás*át és a *Lagrange-vektormező*ket tár-gyaljuk. (Az utób-biak integrálgörbéi a jól ismert *Euler-Lagrange egyenletek* megoldásai.)

(2) Definiáljuk az *(általánosított Lagrange-) metrikák*at, azok *(abszolút) ener-giáját*, és a metrikából származó *Lagrange 1*-és *2-formát*. Bevezetjük az *első Cartan-tenzor*t, igazoljuk néhány egyszerűbb tulajdonságát, és megmutatjuk, hogy ennek szimmetriája az ún. *variációs metrikák*at jellemzi.

(3) A továbbiakban a metrika regularitását két különböző úton tárgyaljuk. Egyfelől egy metrikát *E-reguláris*nak nevezünk, ha a metrikából származó abszolút energia reguláris, míg *Miron-reguláris metrikáról* szólunk, ha a metrikából származó Lagrange 2-forma nemelfajuló. Reguláris metrikából szemispray konstruálható. Ezek után definiáljuk a *gyengén variációs metrikák*at, melyeknek egy speciális osztályát alkotják a *gyengén normális metrikák*. Ebben a megközelítésben a R. Miron által vizsgált metrikák az egyidejűleg gyengén normális és Miron-reguláris metrikáknak felelnek meg. Ezt követően külön figyelmet szentelünk a szemi-Finsler esetnek. Egy metrikát *szemi-Finsler-metrikának* nevezünk, ha variációs és Miron-reguláris. Leírjuk egy szemi-Finsler-sokaságon adott Lagrange-vektormező és a kanonikus spray közötti relációt.

(4) Tekintve egy olyan reguláris Lagrange-függvényt, amelynek Hesse-tenzora variációs metrika, a nevezetes *Helmholtz-kritériumok* szükséges és elegendő feltételeket szolgáltatnak arra vonatkozóan, hogy mikor lesz egy szemispray az illető reguláris Lagrange-függvényhez tartozó Lagrange-vektormező. Megmutatjuk, hogy egy E-reguláris metrika, valamint egy egyidejűleg Miron-reguláris és gyengén variációs metrika reguláris Lagrange-függvényt származtat. Igazoljuk ezek után, hogy egy adott szemispray, bizonyos Helmholtz-típusú feltételek teljesítése mellett, egy ilyen Lagrange-függvényhez tartozó Lagrange-vektormezővel egyezik meg.

(5) Bevezetjük a *Moór-Vanstone-metrikákat*, mint homogén és E-reguláris metrikákat. Rámutatunk arra, hogy az eddig tárgyalt összes metrikák halmazának metszetét a *normális metrikák* alkotják. Ezt követően megmutatjuk, hogy megközelítésünk miként vezet el a Finsler-sokaságok hagyományos értelmezéséhez.

(6) Végezetül rátérünk a metrikus deriváltak tanulmányozására. Tetszőleges torziómentes horizontális leképezéshez (amely a torziómentesség folytán szemisprayből származik) megadható egy kanonikus metrikus derivált. E kanonikus derivált és egy tetszőlegesen választott metrikus derivált közötti relációt két tenzor segítségével fejezzük ki. Ha a metrika normális, akkor a kanonikus derivált a klasszikus *Cartan-derivált*ra redukálódik. Így ismét a Finsler-geometria felségterületére jutunk vissza, s immár belevághatunk a Randers-sokaságok vizsgálatába.

A disszertáció **harmadik fejezet**ében tehát a *Randers-sokaságok*kal foglalkozunk. Amint azt az elnevezés is sejteti, a Randers-metrika fogalmát elsőként egy ilyen nevű kutató, mégpedig G. Randers vezette be 1941-ben [74], magát az elnevezést R. S. Ingarden alkalmazta először. A Randers-metrikákhoz eredetileg fizikai meggondolások vezettek el, s későbbi alkalmazásaikban is a fizikáé a főszerep. Ugyanakkor a geométerek érdeklődését is hamarosan felkeltették ezek a metrikák, mivel igen természetes példáit nyújtják speciális, de már nem-Riemann Finsler-metrikáknak. Az alapfüggvényük úgy származik, hogy egy Riemann-sokaság alapfüggvényéhez egy 1-formából származó (az érintősokaságon ható) függvényt hozzáadunk. Jegyezzük rögtön meg, hogy a Randers-sokaságok elmélete ma már egy sokkal általánosabb elméletnek, az (α, β)*-metrikák* elméletének is része [15], [66]. Ekkor az alapfüggvény egy Finsler-sokaság alapfüggvényének és egy 1-formából származó függvénynek az összege.
A Randers-sokaságok általunk követett koordinátamentes tárgyalása során, mintegy a kiépített fogalmi és kalkulatív apparátus hatékonyságának "teszteléseként", egyik célunk annak a problémának az újragondolása volt, hogy milyen feltétel mellett válik egy Randers-sokaság Berwald-sokasággá. A történet talán onnan indul, hogy míg egy Riemann-sokaságon egyértelműen létezik torziómentes metrikus derivált, a *Levi-Civita derivált*, addig egy Finsler-sokaságon több nevezetes kovariáns derivált is megadható (így például *a Berwald-, a Cartan-, a Hashiguchi- és a Chern-Rund*-féle), melyeket vagy a metrikára, és/vagy a torziók-ra vonatkozó feltételek finomításával nyerünk. Az említett kovariáns deriváltakat *nevezetes Finsler-konnexiók*nak is hívjuk. Kézenfekvően vetődik fel az a kérdés, hogy egy nevezetes Finsler-konnexió milyen esetben és miként számaztat kovariáns deriváltat az alapsokaságon. A Berwald-derivált esetén ez akkor teljesül, ha a horizontális rész konnexióparaméterei csak a "helytől függenek", illetve, ezzel ekvivalens módon, ha a Berwald-derivált vegyes görbületi tenzora eltűnik. Azt a Finsler-sokaságot, amelyben a Berwald-derivált eleget tesz ennek a feltételnek, *Berwald-*

*sokaság*nak nevezzük. Természetes módon vetődik fel a probléma: mi a szükséges és elegendő feltétele annak, hogy egy Randers-sokaság Berwald-sokasággá váljon? Noha a kérdést megválaszoló tétel már-már klasszikus, az itt közölt bizonyítás eredeti, mind alapgondolatát, mind technikáját tekintve.
Külön figyelmet szentelünk az *első Cartan-tenzorok*nak, amelyek - mint az a 2. fejezetben is kiderült - már a legalapvetőbb összefüggések és geometriai konstrukciók feltárásához, illetve megvalósításához nélkülönözhetetlenek. Ugyanakkor az ún. *második Cartan-tenzorok* explicit módon történő megadása a Riemann-Finsler-metrika és a Barthel-endomorfizmus bonyolultsága miatt gyakorlatilag lehetetlennek látszik, s ezért egy Randers-sokaságon a Cartan-derivált koordinátamentes leírása is meglehetősen reménytelen feladat. Lehetőséget találtunk azonban arra, hogy egy, a Cartan-deriválttal szorosan rokon *metrikus* deriváltat alkalmazzunk a Randers-sokaságok elméletének kiépítésében. A Barthel-endomorfizmus említett komplikáltsága miatt, az alapsokaság Riemann-struktúrája által indukált Barthel-endomorfizmusból származó második Cartan-tenzorok segítségével képeztünk egy olyan metrikus deriváltat, amely jóval egyszerűbb a Finsler-struktúra által indukált Barthel-endomorfizmushoz tartozó társánál.
Végezetül, egy, az alapsokaságon konform-invariánsnak bizonyuló kovariáns deriváltat konstruálunk, és megmutatjuk, hogy az ebből származó spray abban és csak abban az esetben projektíven ekvivalens a Randers-sokaságon adott kanonikus sprayvel, ha éppen a Riemann-struktúrából származó kanonikus sprayről van szó.

Részletezve:

(1) A fejezet ismét a tárgyalás során alkalmazásra kerülő, speciális technikai eszközök bevezetésével indul.

(2) A Randers-sokaságok definíciója után elegendő feltételt adunk arra, hogy egy Randers-sokaság (pozitív-definit) Finsler-sokasággá váljon. Kiszámítjuk a sokaság Riemann-Finsler metrikáját.

(3) Meghatározzuk a sokaságon a kanonikus sprayt, s leírjuk a Randers- és a Riemann-struktúra Barthel-endomorfizmusai, valamint a belőlük származó horizontális liftek között fennálló relációt.

(4) Koordinátamentes formában előállítjuk a Berwald-deriváltat, és ezzel együtt az első Cartan-tenzorokat, továbbá az ún. a Cartan-vektormezőt is.

(5) Új, geometriai meggondolásokon alapuló bizonyítást adunk arra a klasszikus tételre, amely szükséges és elegendő feltételt állapít meg arra vonatkozólag, hogy egy Randers-sokaság Berwald-sokaságra redukálódjon.

(6) Tekintjük az alapsokaság Riemann-struktúrája által indukált Barthel-endomorfizmust és meghatározzuk a hozzátartozó második Cartan-tenzorokat. Ezek segítségével egy olyan kovariáns deriváltat vezetünk be, amely a Randers-sokaság Riemann-Finsler-metrikájára nézve metrikus. Ennek alkalmazásával egy új kritériumot adunk arra, hogy egy Randers-sokaság Berwald-sokasággá váljon.

(7) Megmutatjuk, hogy egy Randers-metrika konform változtatása során az alapsokaság metrikus tenzorai is konform-ekvivalensek egymással, és felírjuk a Levi-Civita deriváltak közötti összefüggést. Ezek alapján az alapsokaságon felépítünk egy konform-invariánsnak bizonyuló kovariáns deriváltat, és kifejezzük a hozzátartozó, valamint a Riemann-struktúrából származó horizontális endomorfizmusok és sprayk közötti kapcsolatot. A sprayk közötti összefüggésből adódik, hogy a konform-invariáns kovariáns deriválthoz tartozó spray abban az esetben lesz projektíven ekvivalens a Randers- sokaság kanonikus sprayjével, ha egybeesik a Riemann-struktúrából származó társával.

Bibliography

[1] M. Abate and G. Patrizio, *Finsler Metrics - A Global Approach*, Springer-Verlag, Berlin (1994).

[2] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Second Edition, Benjamin-Cummings, Reading, Mass. (1978).

[3] T. Aikou, M. Hashiguchi and K. Yamauchi, *On Matsumoto's Finsler space with time measure*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. & Chem.) **23** (1990), 1–12.

[4] H. Akbar-Zadeh, *Les espaces de Finsler et certaines de leurs généralisations*, Ann. Sci. Ecole Norm. Sup. **80** no. 3 (1963), 1–79.

[5] P.L. Antonelli and H. Shimada, *On 1-form Finsler connections with constant coefficients*, Tensor N.S. **50** no. 3 (1991), 263–275.

[6] P.L. Antonelli, R.S. Ingarden and M. Matsumoto, *The theory of sprays and Finsler spaces with applications in physics and biology*, Fundamental theories of physics **58**, Kluwer Academic Publishers (1993).

[7] P.L. Antonelli and T.J. Zastawniak, *Fundamentals of Finslerian diffusion with applications*, Fundamental Theories of Physics **101**, Kluwer Academic Publishers Group (1999).

[8] G.S. Asanov, *Finsler geometry, relativity and gauge theories*, Fundamental theories of physics, Kluwer Academic Publishers (1985).

[9] L. Auslander, *On curvature in Finsler Geometry*, Trans. Amer. Math. Soc. **79** (1955), 378–388.

[10] E. K. Ayassou, *Cohomologies définies pour une 1-forme vectorielle plate*, Thèse de Doctorat, Université de Grenoble, (1985).

[11] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type. A generalization of the notion of Berwald space*, Publ. Math. Debrecen **51** (1997), 385–406.

[12] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type II. Projectively flat spaces*, Publ. Math. Debrecen **53** (1998), 423–438.

[13] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas type III.*, in: P.L. Antonelli, ed. *Finslerian Geometries*, Kluwer, Dordercht (2000), 89–94.

[14] S. Bácsó and M. Matsumoto, *Randers spaces with the h-curvature tensor H dependent on position alone*, Publ. Math. Debrecen **57** (2000), 185–192.

[15] D. Bao, S.-S. Chern and Z. Shen, *An Introduction to Riemann-Finsler Geometry*, Graduate Texts in Mathematics 200, Springer-Verlag, New York (2000).

[16] A. Bejancu, *Finsler geometry and applications*, Ellis Horwood Series: Mathematics and its Applications (1990).

[17] A. Bejancu and H.R. Farran, *Geometry of Pseudo-Finsler Submanifolds*, Kluwer Academic Publishers, Dordrecht (2001).

[18] L. Berwald, *Untersuching der Krümmung allgemeiner metrische Räume auf Grund des ihnen herrschenden Parallelism*, Mathematische Zeitschrift **25** (1926), 40–73.

[19] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin (1987).

[20] F. Brickell, *A theorem on homogeneous functions*, J. London Math. Soc. **42** (1967), 325–329.

[21] H. Busemann, *The geometry of Finsler spaces*, Bull. Amer. Math. Soc. **56** (1950), 5–16.

[22] E. Cartan, *Les espaces de Finsler*, Actualités **79**, Paris (1934).

[23] S.S. Chern, *Finsler geometry is just Riemannian geometry without the quadratic restriction*, Notices Amer. Math. Soc. **43** no. 9, 959–963.

[24] M. Crampin, *On horizontal distributions on the tangent bundle of a differentiable manifold*, J. London Math. Soc. **3** no. 2 (1971), 178–182.

[25] M. Crampin, *Generalized Bianchi identities for horizontal distributions*, Math. Proc. Camb. Phil. Soc. **94** (1983), 125–132.

[26] M. Crampin, *Tangent bundle geometry for Lagrangian dynamics*, J. Phys. A **16** no. 16 (1983), 3755–3772.

[27] M. Crampin, *Jet bundle techniques in analytical mechanics*, Quaderni del consiglio nazionale delle ricerche, gruppo nazionale di fisica matematica **47** (1995).

[28] M. Crampin, *Connections of Berwald type*, Publ. Math. Debrecen, **57**, no. 3-4 (2000), 455–473.

[29] M. Crampin, G.E. Prince, W. Sarlet and G. Thompson, *The inverse problem of the calculus of variations: separable systems*, Acta Appl. Math. **57** no.3 (1999), 239–254.

[30] M. Crampin, W. Sarlet, E. Martínez, G.B. Byrnes and G.E. Prince, *Towards a geometrical understanding of Douglas's solution of the inverse problem of the calculus of variations*, Inverse Problems **10** (1994), 245–260.

[31] A. Deicke, *Über die Finsler-Räume mit* $A_i = 0$, Arch. Math. **4** (1953), 45–51.

[32] M. de Leon and P. Rodrigues, *Methods of differential geometry in analytical mechanics*, North-Holland Mathematics studies **158**, North-Holland (1989).

[33] J.G. Diaz, *Etude des tenseurs de courbure en géométrie Finslerienne*, Thèse IIIème cycle, *Publ. Dép. Math. Lyon* (1972).

[34] J.G. Diaz and G. Grangier, *Courbure et holonomie des variétés Finslèriennes*, Tensor, N.S. **30** (1976), 95–109.

[35] A. Frölicher and A. Nijenhuis, *Theory of vector-valued differential forms*, Proc. Kon. Ned. Akad. A **59** (1956), 338–359.

[36] W. Greub, S. Halperin and R. Vanstone, *Connections, Curvature, and Cohomology*, Vol II, Pure and Applied Mathematics **47**/II, Academic Press, New York (1972).

[37] J. Grifone, *Structure presque tangente et connexions I*, Ann. Inst. Fourier, Grenoble **22** no.1 (1972), 287–334.

[38] J. Grifone, *Structure presque tangente et connexions II*, Ann. Inst. Fourier, Grenoble **22** no.3 (1972), 291–338.

[39] M. Hashiguchi, *On parallel displacements in Finsler spaces*, J. Math. Soc. Japan **10** no. 4 (1958), 365–379.

[40] M. Hashiguchi, *How to get examples of Finsler spaces*, Conf. Sem. Mat. Univ. Bari **225**, Bari (1987)

[41] M. Hashiguchi, S. Hōjō, M. Matsumoto, *Landsberg spaces of dimension two with (α, β)-metric*, *Tensor, N.S.* **57** (1996), 145–153.

[42] M. Hashiguchi and Y. Ichijyō, *On some special (α, β)-metric*, J. Korean Math. Soc. **10** (1973), 17–26.

[43] M. Hashiguchi and Y. Ichijyō, *Randers spaces with rectilinear geodesics*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. & Chem.) **13** (1980), 33–40.

[44] Y. Ichijyō and M. Hashiguchi, *On the condition that a Randers space be conformally flat*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. & Chem.) **22** (1989), 7–14.

[45] Y. Ichijyō and M. Hashiguchi, *On locally flat generalized (α, β)-metrics and conformally flat generalized Randers metrics*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. & Chem.) **27** (1994), 17–25.

[46] S. Kikuchi, *On the condition that a space with (α, β)-metric be locally Minkowskian*, Tensor, N.S. **33** (1979), 242–246.

[47] I. Kolář, P.W. Michor, J. Slovák, *Natural operations in differential geometry*, Springer-Verlag Berlin Heidelberg (1993).

[48] V.V. Kropina, *Projective two-dimensional Finsler spaces with special metric* (in Russian), Trudy Sem. Vektor. Tenzor. Anal. **11** (1961), 277–292.

[49] O. Krupková, *Variational metrics on $\mathbb{R} \times TM$ and the geometry of nonconservative mechanics*, Math. Slovaca **44** no. 3 (1994), 315–335.

[50] C. López and E. Martínez, *Sub-Finslerian metric associated to an optimal control system*, SIAM J. Control Optim. **39** (2000), 798–811.

[51] R.L. Lovas, *Lie derivatives and Killing vector fields in Finsler geometry*, Proc. International Conference on Non-Euclidean Geometry in Modern Physics (2002),35-50.

[52] E. Martínez and J.F. Cariñena, *Geometric characterization of linearisable second-order differential equations*, Math. Proc. Cambridge Phil. Soc. **119** no. 2 (1996), 373–381.

[53] E. Martínez, J.F. Cariñena and W. Sarlet, *Derivations of differential forms along the tangent bundle projection*, Diff. Geometry and its Applications **2** (1992) 17–43.

[54] E. Martínez, J.F. Cariñena and W. Sarlet, *Derivations of differential forms along the tangent bundle projection II*, Diff. Geometry and its Applications **3** (1993) 1–29.

[55] E. Martínez, J.F. Cariñena and W. Sarlet, *Geometric characterization of separable second-order differential equations*, Math. Proc. Cambridge Phil. Soc. **113** no. 1 (1993), 205–224.

[56] M. Matsumoto, *Affine transformations of Finsler spaces*, J. Math. Kyoto Univ. **3** (1963), 1–35.

[57] M. Matsumoto, *Conformal changes and conformally Minkowski spaces*, Symp. Finsler Geom., Shiobara, Japan (1988).

[58] M. Matsumoto, *On C-reducible Finsler spaces*, Tensor, N.S. **24** (1972), 29–37.

[59] M. Matsumoto, *On Finsler spaces with Randers' metric and special forms of important tensors*, J. Math. Kyoto. Univ. **14** (1974), 477–498.

[60] M. Matsumoto, *History of Finsler Geometry* (manuscript), 30 pages, Debrecen (1979).

[61] M. Matsumoto, *Foundation of Finsler geometry and special Finsler manifolds*, Kaiseisha Press, Saikawa, Otsu, Japan (1986).

[62] M. Matsumoto, *A slope of a mountain in a Finsler surface with respect to time measure*, J. Math. Kyoto Univ. **29** (1989), 17–25.

[63] M. Matsumoto, *Randers spaces of constant curvature*, Rep. on Math. Phys. **28** (1989), 249–261.

[64] M. Matsumoto, *Finsler spaces of constant curvature with Kropina metric*, Tensor, N.S. **50** (1991), 194–201.

[65] M. Matsumoto, *The Berwald connection of a Finsler space with an (α, β)-metric*, Tensor, N.S. **50** (1991), 18–21.

[66] M. Matsumoto, *Theory of Finsler spaces with (α, β)-metric*, Rep. Math. Phys. **31** no.1 (1992), 43–83.

[67] M. Matsumoto and S. Hōjō, *A conclusive theorem on C-reducible Finsler spaces*, Tensor, N.S. **32** (1978), 225–229.

[68] T. Mestdag and W. Sarlet, *The Berwald-type connection associated to time-dependent second-order differential equations*, Houston J. of Math., **27** (4), 763–797 (2001).

[69] T. Mestdag and V. Tóth, *On the geometry of Randers manifolds*, Rep. Math. Phys. **43** no. 11 (2002), 5654–5674.

[70] T. Mestdag, J. Szilasi and V. Tóth, *On the geometry of generalized metrics*, Publ. Math. (Debrecen) **62/3-4** (2003), 511-545.

[71] R. Miron, *Metrical Finsler structures and metrical Finsler connections*, J. Math. Kyoto. Univ. **23** (1983), 219–224.

[72] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications*, Fundamental Theories of Physics **59**, Kluwer Academic Publishers (1994).

[73] A. Moór, *Entwicklung einer Geometrie der allgemeinen metrischen Linienelementräume*, Acta Sci. Math. Szeged **17** (1956), 85–120.

[74] G. Randers, *On a asymmetrical metric in the four-space of general relativity*, Phys. Rev. **59** no. 2 (1941), 159–199.

[75] B. Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, edited by H. Weyl, Springer (1921). See also M. Spivak, *A comprehensive introduction to differential geometry.*, Vol. II., Brandeis Univ., Waltham, Mass. (1970).

[76] H. Rund, *Über die Parallelverschiebung in Finslerschen Raümen*, Mathematische Zeitschrift **54** (1951), 115–128.

[77] H. Rund, *The Differential Geometry of Finsler Spaces*, Grundlehren der Mathematischen Wissenschaffen, **101**, Springer-Verlag, Berlin (1959).

[78] V.S. Sabau and H. Shimada, *Classes of Finsler spaces with (α,β)-metrics*, Rep. Math. Phys. **47** no.1 (2001), 31–48.

[79] W. Sarlet, M. Crampin and E. Martínez, *The integrability conditions in the inverse problem of the calculus of variations for second-order ordinary differential equations*, Acta Appl. Math. **54** no.3 (1998), 233–273.

[80] W. Sarlet, G. Thompson and G.E. Prince, *The inverse problem of the calculus of variations: The use of geometrical calculus in Douglas' analysis*, Trans. Amer. Math. Soc. **354** no.7 (2002), 2897–2919.

[81] Z. Shen, *Lectures on Finsler geometry*, World Scientific Publishing Co., Singapore (2001).

[82] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, Dordrecht (2001).

[83] Z. Shen, *Finsler metrics with* $\mathbf{K} = 0$ and $\mathbf{S} = 0$, preprint, *arXiv:math.DG/0109060.*

[84] C. Shibata, *On Finsler spaces with Kropina metric*, Rep. on Math. Phys. **13** (1978), 117–128.

[85] C. Shibata, H. Shimada, M. Azuma amd H. Yasuda, *On Finsler spaces with Randers' metric*, Tensor, N.S. **31** (1977), 219–226.

[86] Z.I. Szabó, *Positive definite Berwald spaces, Structure theorems on Berwald spaces*, Tensor, N.S. **35** (1981), 25–39.

[87] Sz. Szakál, *On the conformal theory of Ichijyō manifolds*, Rend. Circ. Mat. Palermo, Serie II, Suppl. **69** (2002), 245–254.

[88] Sz. Szakál and J. Szilasi, *A new approach to generalized Berwald manifolds I*, SUT Journal of Math. **37** (2001), no. 1, 19–41.

[89] Sz. Szakál and J. Szilasi, *A new approach to generalized Berwald manifolds II*, Publ. Math. Debrecen **60** (2002), 429–453.

[90] J. Szilasi, *Notable Finsler connections on a Finsler manifold*, Lecturas Matematicas **19** (1998), 7–34.

[91] J. Szilasi, *A setting for Spray and Finsler Geometry*, in: Handbook of Finsler Geometry (ed. P.L. Antonelli), to be published by Kluwer Academic Publishers, Dordrecht.

[92] J. Szilasi and Sz. Vattamány, *Erratum to: "On the projective geometry of sprays"*[Diff. Geom. Appl. **12** (2000),185–206],Diff. Geom. Appl. **13** (2000),97–118.

[93] J. Szilasi and Sz. Vattamány, *On the Finsler-metrizabilities of spray manifolds*, Per. Math. Hun. **44** no. 1 (2002), 81–100.

[94] J. Szilasi and Cs. Vincze, *On conformal equivalence of Riemann-Finsler metrics*, Publ. Math. Debrecen **52** (1998), 167-185.

[95] J. Szilasi and Cs. Vincze, *A new look at Finsler connections and special Finsler manifolds*, Acta Math. Acad. Paed. Nyiregyháriensis **16** (2000), 33–63.

[96] L. Tamássy and M. Matsumoto, *Direct method to characterize conformally Minkowski Finsler spaces*, Tensor, N.S. **33** (1979), 380-384.

[97] V. Tóth, *On Randers manifolds*, Studii si Cercetări Stiinifice (Ser. Mat.) **10** (2000), 289–296.

[98] J.R. Vanstone, *A generalization of Finsler geometry*, Canad. J. Math. **14** (1962), 87–112.

[99] Sz. Vattamány, *Projection onto the indicatrix bundle of a Finsler manifold*, Publ. Math. Debrecen, **58** (2001), 193–221.

[100] Sz. Vattamány and Cs. Vincze, *Two-dimensional Landsberg manifolds with vanishing Douglas tensor*, Annales Univ. Sci. Budapest **44** (2001), 11-26.

[101] H. Yasuda and H. Shimada, *On Randers spaces of scalar curvature*, Rep. on Math. Phys **11** (1977), 347–360.

[102] N.L. Youssef, *Semi-projective changes*, Tensor, N.S. **55** (1994), 131–141.