Dr. Kézi Csaba Gábor

Quantitative methods for Engineers

Part 1

Optimization and Basic Probability

Debreceni Egyetem Műszaki Kar
Műszaki Alaptárgyi Tanszék

DEBRECENI EGYETEM
MŰSZAKI KAR
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QUANTITATIVE METHODS FOR ENGINEERS PART 1

(Optimization and Basic Probability)

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Chapter 1

Optimization

1.1. Optimization of one-variabled functions

How we can solve optimization problems?

1. We have to read the problem. What is given? What is the unknown quantity to be optimized?
2. If we have a geometrically problem, we have to draw a figure.
3. We have to introduce variables. List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. We have to write an equation for the unknown quantity. If we can, express the unknown as a function of a single variable. Let this single variable function be f.
5. Compute the derivative function of f.
6. Find the critical points, that is, we have to solve the equation $f'(x) = 0$. Suppose that it is x_0 and $a \leq x_0 \leq b$.
7. Make a table of the values f at the endpoints of the domain of f and critical points.
8. Select the largest and smallest values of the function at the candidate points.

There is another way to give extremum point.

1. We have to read the problem. What is given? What is the unknown quantity to be optimized?
2. If we have a geometrically problem, we have to draw a figure.
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5. Compute the derivative function of f.
6. Find the critical points, that is, we have to solve the equation $f'(x) = 0$. Suppose that it is x_0 and $a \leq x_0 \leq b$.
7. Find the second derivative function of function f.
8. If $f''(x_0) > 0$ then x_0 is a local minimum. If $f''(x_0) < 0$ then x_0 is a local maximum.

1.2. Optimization of two-variabled functions

1.2.1. Definition. The function $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ has *local maximum* (other words *relative maximum*) at point $(x_0; y_0) \in D$ if there is disc centered at point $(x_0; y_0)$ such that $f(x; y) \leq f(x_0; y_0)$ for all point $(x; y)$ that lie inside the disc.

The function $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ has *local minimum* (other words *relative minimum*) at point $(x_0; y_0) \in D$ if there is disc centered at point $(x_0; y_0)$ such that $f(x; y) \geq f(x_0; y_0)$ for all point $(x; y)$ that lie inside the disc.

1.2.2. Theorem. If $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ is a totally differentiable function and has a local extremum at point $(x_0; y_0) \in D$ then the values of partial derivative functions at $(x_0; y_0)$ are zero, that is $f'_x(x_0; y_0) = 0$ and $f'_y(x_0; y_0) = 0$ equations are hold.

1.2.3. Definition. If $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ is a twice differentiable function then its *Hessian-matrix* at point $(x_0; y_0) \in D$ is as follows:

$$M(x_0; y_0) = \begin{pmatrix} f''_{xx}(x_0; y_0) & f''_{xy}(x_0; y_0) \\ f''_{yx}(x_0; y_0) & f''_{yy}(x_0; y_0) \end{pmatrix}$$

1.2.4. Theorem. If $f\colon D \subset \mathbb{R}^2 \to \mathbb{R}$ twice differentiable function and has a critical point at $(x_0; y_0) \in D$ and let the Hessian-matrix at $(x_0; y_0)$ is

$$M(x_0; y_0) = \begin{pmatrix} f''_{xx}(x_0; y_0) & f''_{xy}(x_0; y_0) \\ f''_{yx}(x_0; y_0) & f''_{yy}(x_0; y_0) \end{pmatrix}$$

Let denote $D_1 = f''_{xx}(x_0; y_0)$ and $D_2 = \det\big(M(x_0; y_0)\big)$.

If $D_1 > 0$ and $D_2 > 0$, then the function f has a local minimum at point $(x_0; y_0)$.

If $D_1 < 0$ and $D_2 > 0$, then the function f has a local maximum at point $(x_0; y_0)$.

If $D_2 < 0$, then the function has no local extremum at point $(x_0; y_0)$.

1.2.5. Remark. How we can find a local extremum?

1. We have to calculate the partial derivatives of function f.
2. We have to solve the system of equation

$$\left.\begin{aligned} f'_x(x; y) = 0 \\ f'_y(x; y) = 0 \end{aligned}\right\}.$$

The solutions of the system are the critical points.

3. We have to calculate the second ordered partial derivatives and we have to construct the Hessian-matrix:
$$M(x_0;y_0)=\begin{pmatrix} f''_{xx}(x_0;y_0) & f''_{xy}(x_0;y_0) \\ f''_{yx}(x_0;y_0) & f''_{yy}(x_0;y_0) \end{pmatrix}$$
4. We substitute the critical points to Hessian-matrix and we apply the theorem 1.2.4.

1.3. Optimization of three or more variabled functions

1.3.1. Definition. The function $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ has *local maximum* (*relative maximum*) at point $P \in D$ if there is disc centered at point P such that

$$f(x_1;\ldots;x_n) \le f(P)$$

for all point $(x_1;\ldots;x_n)$ that lie inside the disc.

The function $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ has *local minimum* (*relative minimum*) at point $P \in D$ if there is disc centered at point P such that

$$f(x_1;\ldots;x_n) \ge f(P)$$

for all point $(x_1;\ldots;x_n)$ that lie inside the disc.

1.3.2. Theorem. If $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ is a totally differentiable function and has a local extremum at point $P \in D$ then the values of partial derivative functions at P are zero.

1.3.3. Definition. If the function $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ is twice differentiable function, then its *Hessian-matrix* at point $P \in D$ is

$$M(P) = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \dots & x_n \end{array} \\ \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) & \dots & f''_{x_1x_n}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) & \dots & f''_{x_2x_n}(P) \\ \vdots & \vdots & \dots & \ddots \\ f''_{x_nx_1}(P) & f''_{x_nx_2}(P) & \dots & f''_{x_nx_n}(P) \end{pmatrix} \end{array}.$$

1.3.4. Theorem. Let $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ be twice differentiable function. Let the partial derivatives at point $P \in D$ are zero and the Hessian-matrix at point is

$$M(P) = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \dots & x_n \end{array} \\ \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) & \dots & f''_{x_1x_n}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) & \dots & f''_{x_2x_n}(P) \\ \vdots & \vdots & \dots & \ddots \\ f''_{x_nx_1}(P) & f''_{x_nx_2}(P) & \dots & f''_{x_nx_n}(P) \end{pmatrix} \end{array}.$$

Let

$$D_1 = f''_{x_1x_1}(P),$$

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$$D_2 = \det \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) \end{pmatrix},$$

and so on

$$D_i = \det \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) & \dots & f''_{x_1x_i}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) & \dots & f''_{x_2x_i}(P) \\ \vdots & \vdots & \ddots & \vdots \\ f''_{x_ix_1}(P) & f''_{x_ix_2}(P) & \dots & f''_{x_ix_i}(P) \end{pmatrix}.$$

If $D_i > 0$ for all $i = 1, 2, \dots, n$ then the function f has a local minimum at point P.

If $(-1)^i \cdot D_i > 0$ for all $i = 1, 2, \dots, n$ then the function f has a local maximum at point P.

1.3.5. Remark. How we can find a local extremum?

1. We have to calculate the partial derivatives of function f.
2. We have to solve the system of equations

$$\left.\begin{array}{r} f'_{x_1}(x_1; x_2; \dots; x_n) = 0 \\ f'_{x_2}(x_1; x_2; \dots; x_n) = 0 \\ \dots \\ f'_{x_n}(x_1; x_2; \dots; x_n) = 0 \end{array}\right\}.$$

   The solutions of the system are the critical points.

3. We have to calculate the second ordered partial derivatives and we have to construct the Hessian-matrix:

$$M(x_1; x_2; \dots; x_n) = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \dots & x_n \end{array} \\ \begin{pmatrix} f''_{x_1x_1} & f''_{x_1x_2} & \dots & f''_{x_1x_n} \\ f''_{x_2x_1} & f''_{x_2x_2} & \dots & f''_{x_2x_n} \\ \vdots & \vdots & \dots & \ddots \\ f''_{x_nx_1} & f''_{x_nx_2} & \dots & f''_{x_nx_n} \end{pmatrix} \end{array}.$$

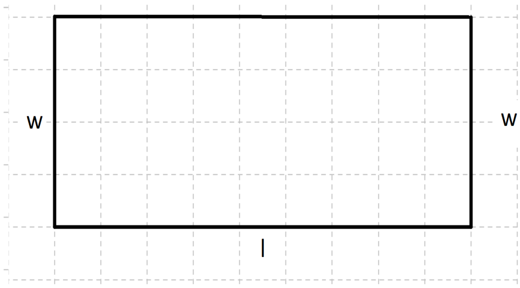
4. We substitute the critical points to Hessian-matrix and we apply the theorem 1.3.4.

1.4. Solved problems

1.4.1. Exercise. What is the smallest perimeter possible for a rectangle whose area is $25\,\mathrm{m}^2$, and what are its dimensions?

Solution:

Let w and l represent the length and width of the rectangle, respectively.



The area of the rectangle is 25, thus

$$25 = l \cdot w \qquad \Rightarrow \qquad w = \frac{25}{l}.$$

The perimeter of the rectangular is

$$P = 2l + 2w \quad \Rightarrow \quad P(l) = 2l + 2 \cdot \frac{25}{l} = 2l + \frac{50}{l} = 2l + 50l^{-1}.$$

The derivative function of the function P is

$$P'(l) = 2 - 50l^{-2} = 2 - \frac{50}{l^2}.$$

Solving the equation $P'(l) = 0$, we have

$$2 - \frac{50}{l^2} = 0 \quad \Rightarrow \quad 2l^2 - 50 = 0 \quad \Rightarrow \quad l^2 = 25.$$

Since $l > 0$ for the length of a rectangle, must be 5 and

$$w = \frac{25}{l} = \frac{25}{5} = 5.$$

Since

$$P''(l) = 100l^{-3} = \frac{100}{l^3} \qquad \Rightarrow \qquad P''(5) = \frac{100}{5^3} > 0,$$

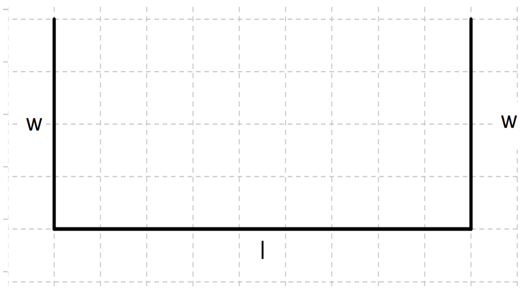
we have a minimum at $l = 5$. The minimum perimeter is

$$P = 2l + 2w = 2 \cdot 5 + 2 \cdot 5 = 20\,\mathrm{m}.$$

1.4.2. Exercise. A farmer has $200\,\mathrm{m}$ of fencing and wants to fence off a rectangular field that borders a straight river. If needs no fence along the river, what are the dimensions of the field that has the largest area?

Solution:

Let denote the sides of the rectangle by l and w.



The total length of the fence is $P = l + 2w$. Using the equation $P = 200$, we get that

$$l + 2w = 200 \qquad \Rightarrow \qquad l = 200 - 2w.$$

The area of the rectangle is

$$A = l \cdot w = (200 - 2w) \cdot w = 200w - 2w^2.$$

Thus the function we intend to maximize is $A(w) = 200w - 2w^2$. Note that $w \geq 0$ and $w \leq 100$, thus the domain of function A is the $[0; 100]$ interval. The derivative of function A is

$$A'(w) = 200 - 4w,$$

thus to find the extreme values we have to solve the equation $A'(w) = 0$, that is $200 - 4w = 0$. The solution of the equation above is $w = 50$.

The maximum value of A can be found at 50 or at the end points of the interval which is the domain of function A. Since

$$A(0) = 0;\ A(50) = 10\,000 - 5\,000 = 5\,000;\ A(100) = 0,$$

the maximum value as $A(50) = 5\,000$.

Alternatively, we could have observed that $A''(w) = -4 < 0$, thus A is always concave downward and local maximum at $w = 50$ must be an absolute maximum. Thus the dimensions of the rectangular field are 50 and 100 meters. The maximum area is

$$A = l \cdot w = 50 \cdot 100 = 5\,000\,\mathrm{m}^2.$$

1.4.3. Exercise. The quantity of water in a water storage is given as a function of time t is

$$V(t) = 4t^3 - 42t^2 + 120t \,\text{liters},$$

where t is in the interval $[0; 6]$ and t is the number of days since the beginning of the observation.

a) How much fluid was in the storage at the beginning and end of the observation?

b) Calculate the rate of change of the quantity of water.

c) What is the rate of change at $t = 1$?

d) Give the time interval when the function is increasing and the one when it is decreasing.

e) Find the global minimum and global maximum of function $V(t)$.

f) Sketch the graph of function $V(t)$.

Solution:

a) At the beginning of the observation there is

$$V(0) = 4 \cdot 0^3 - 42 \cdot 0^2 + 120 \cdot 0 = 0 \,\text{liters}.$$

At the end of the observation there is

$$V(6) = 4 \cdot 6^3 - 42 \cdot 6^2 + 120 \cdot 6 = 72 \,\text{liters}.$$

b) The rate of the change of function V is $V'(t) = 12t^2 - 84t + 120$.

c) The rate of change at $t = 1$ is

$$V'(1) = 12 \cdot 1^2 - 84 \cdot 1 + 120 = 48 \left[\frac{\text{liters}}{\text{day}}\right].$$

d) We have to solve the equation $12t^2 - 84t + 120 = 0$. Simplifying this equation, we get that $t^2 - 7t + 10 = 0$. Applying the quadratic formula, we get that

$$t_{1,2} = \frac{7 \pm \sqrt{49 - 40}}{2} = \frac{7 \pm 3}{2},$$

that is $t = 2\,\mathrm{day}$ or $t = 5\,\mathrm{day}$. The sign of the first derivative of function V is shown in the table below

	$0 < t < 2$	$t = 2$	$2 < t < 5$	$t = 5$	$5 < t < 6$
$V'(t)$	$+$	0	$-$	0	$+$
$V(t)$	$\nearrow$	local max.	$\searrow$	local min.	$\nearrow$

e) The values of function V at $t = 2$ and $t = 5$ are

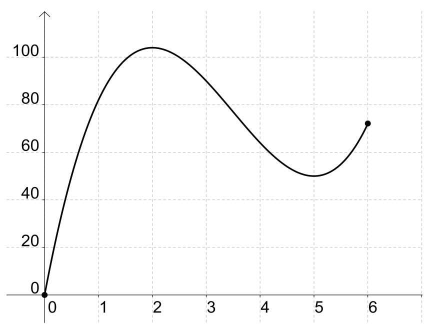
$$V(2) = 4 \cdot 2^3 - 42 \cdot 2^2 + 120 \cdot 2 = 104\,\mathrm{liters}$$
$$V(5) = 4 \cdot 5^3 - 42 \cdot 5^2 + 120 \cdot 5 = 50\,\mathrm{liters}$$

and

$$V(0) = 0 \quad \text{and} \quad V(6) = 72,$$

thus the minimum value is $0\,\mathrm{liters}$, the maximum value is $104\,\mathrm{liters}$.

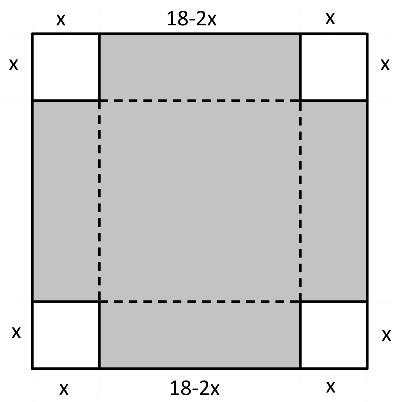
f) The graph of function V is



1.4.4. Exercise. An open box is made by cutting congruent squares from the corners of a $18\,\mathrm{cm}$ by $18\,\mathrm{cm}$ cardboard sheet. How large should the squares be so that the box has a maximum volume? What is the maximum capacity of the box?

Solution:

Let x be a side of the square in cm.



The volume of the box is

$$V(x) = (18-2x)^2 \cdot x = (324 - 72x + 4x^2) \cdot x = 4x^3 - 72x^2 + 324x.$$

The domain of function V is interval $[0; 9]$.
The derivative of function V is

$$V'(x) = 12x^2 - 144x + 324,$$

thus we have to solve the equation $12x^2 - 144x + 324 = 0$. If we simplify this equation, we get that

$$x^2 - 12x + 27 = 0.$$

Applying the quadratic formula

$$x_{1,2} = \frac{12 \pm \sqrt{144 - 4 \cdot 27}}{2} = \frac{12 \pm 6}{2},$$

that is $x_1 = 9$ or $x_2 = 3$. The second derivative of function V is

$$V''(x) = 24x - 144 \quad \Rightarrow \quad V''(9) = 72 > 0 \quad \text{and} \quad V''(3) = -72 < 0.$$

If $x = 3$ then $V''(3) < 0$ hence at $x = 3$ function V has a maximum. The maximum volume is

$$V = 4 \cdot 3^3 - 72 \cdot 3^2 + 324 \cdot 3 = 432\,\mathrm{cm}^3.$$

1.4.5. Exercise. A pencil cup with a capacity of $20\,\mathrm{dm}^3$ is to be constructed in the shape of a rectangular box with a square base and an open top. If the material for the sides costs 12 cents per inch squared and the material for the base costs 60 cents per inch squared, what should the dimensions of the cup be to minimize the construction cost?

<u>**Solution:**</u>

An open box with a square base can be defined as a square base prism without a top. In these premiums only two variables are defined and based on these two variables the surface area and volume can be expressed:

$$S = x^2 + 4xy \quad \text{and} \quad V = x^2 \cdot y,$$

where x is the length of the base y is the height of the box. Since

$$V = 20 \quad \Rightarrow \quad x^2 \cdot y = 20 \quad \Rightarrow \quad y = \frac{20}{x^2},$$

therefore

$$S(x) = x^2 + 4x \cdot \frac{20}{x^2} = x^2 + \frac{80}{x}.$$

We have to write the cost of the surface area as a function of one variable:

$$C(x) = 60x^2 + 12 \cdot \frac{80}{x} = 60x^2 + \frac{960}{x}.$$

Differentiate the objective function with respect to x, we get that

$$C'(x) = 120x - \frac{960}{x^2}.$$

We have to solve the equation

$$120x - \frac{960}{x^2} = 0 \quad \Rightarrow \quad 120x^3 - 960 = 0 \quad \Rightarrow \quad x^3 = 8,$$

thus $x = 2$. Since

$$C''(x) = 120 + \frac{1\,920}{x^3} > 0,$$

thus we have a minimum at $x = 2$. The height of the box is

$$y = \frac{20}{2^2} = 5\,\mathrm{dm}.$$

Therefore, the dimensions of the pencil cupthat minimize construction cost surface are:

$$x = 2\,\mathrm{dm} \quad \text{and} \quad y = 5\,\mathrm{dm}.$$

1.4.6. Exercise. The demand function of the company is

$$D(x) = 252 - 0.006x$$

and the cost function is

$$C(x) = 180x + 7\,200,$$

where x is the number of units demanded.

a) Give the marginal cost function.

b) Determine the revenue function.

c) Find the marginal revenue function.
d) Give the profit function.
e) Determine the marginal profit function.
f) Calculate the profit for 1 000 units.
g) Calculate the number of units, when the profit has maximum.

Solution:

a) Marginal cost function is

$$MC(x) = C'(x) = 180.$$

b) The revenue function is

$$R(x) = x \cdot D(x) = x \cdot (252 - 0.006x) = 252x - 0.006x^2.$$

c) The marginal revenue function is $MR(x) = R'(x) = 252 - 0.012x$.

d) The profit function is

$$\Pi(x) = R(x) - C(x) = 252x - 0.006x^2 - (180x + 7\,200) =$$
$$= -0.006x^2 + 72x - 7\,200.$$

e) The marginal profit function is $\Pi'(x) = -0.012x + 72$.

f) The profit for 1 000 units is

$$\Pi(1\,000) = -0.006 \cdot 1\,000^2 + 72 \cdot 1\,000 - 7\,200 = 58\,800.$$

g) The solution of equation $\Pi'(x) = 0$ is

$$-0.012x + 72 = 0 \qquad \Rightarrow \qquad x = 6\,000.$$

Since $\Pi''(x) = -0.012 < 0$, the profit function has maximum when the company produces 6 000 units. The maximum profit is

$$\Pi(6\,000) = -0.006 \cdot 6\,000^2 + 72 \cdot 6\,000 - 7\,200 = 208\,800.$$

1.4.7. Exercise. The estimated future profits of a small business are given by

$$\Pi(t) = 2t^2 - 12t + 118$$

thousand dollars, where t is the time in years from now.

a) What is the current annual profit?
b) Find the function $\Pi'(t)$.
c) What is the significance of $\Pi'(t)$?

d) When will the profit increase?
e) When will the profit decrease?
f) What is the minimum profit and when does it occur?
g) Find $\Pi'(t)$ at $t = 4$.

Solution:

a) Since
$$\Pi(0) = 2 \cdot 0^2 - 12 \cdot 0 + 118 = 118,$$
the current annual profit is $118\,000$\$.

b) The function $\Pi'(t)$ is
$$\Pi'(t) = 4t - 12.$$

c) Function $\Pi'(t)$ is the rate of change in profit with time.

d) We have to solve the inequality $\Pi'(t) \geq 0$, that is
$$4t - 12 \geq 0 \qquad \Rightarrow \qquad t \geq 3.$$

e) We have to solve the inequality $\Pi'(t) \leq 0$, that is
$$4t - 12 \leq 0 \qquad \Rightarrow \qquad t \leq 3.$$

f) We have to solve the equation $\Pi'(t) = 0$, that is, $t = 3$.
Since $\Pi''(t) = 4 > 0$, thus the profit function has a minimum at $t = 3$.

g) The value of function $\Pi'(t)$ at $t = 4$ is
$$\Pi'(4) = 4 \cdot 4 - 12 = 4.$$

1.4.8. Exercise. The revenue per month earned by the Couture clothing chain at time t is $R(t) = N(t) \cdot S(t)$, where $N(t)$ is the number of stores and $S(t)$ is average revenue per store per month. Couture embarks on a two-part campaign: (A) to build new stores at a rate of 5 stores per month, and (B) to use advertising to increase average revenue per store at a rate of \$10 000 per month. Assume that $N(0) = 50$ and $S(0) = \$150\,000$.

a) Show that total revenue will increase at the rate
$$R'(t) = 5 \cdot S(t) + 10\,000 \cdot N(t).$$
Note that the two terms in the Product Rule correspond to the separate effects of increasing the number of stores on the one hand, and the average revenue per store on the other.

b) Calculate the value $R'(0)$.

c) If Couture can implement only one leg (A or B) of its expansion at $t = 0$, which choice will grow revenue most rapidly?

Solution:

a) Given $R(t) = N(t) \cdot S(t)$, it follows that

$$R'(t) = N'(t) \cdot S(t) + N(t) \cdot S'(t).$$

We are told that $N'(t) = 5$ stores per month and $S'(t) = 10\,000$ dollars per month. Therefore,

$$R'(t) = 5 \cdot S(t) + 10\,000 \cdot N(t).$$

b) Using the previous part and the given values of $N(0)$ and $S(0)$, we can find

$$R'(0) = 5 \cdot S(0) + 10\,000 \cdot N(0) = 5 \cdot 150\,000 + 10\,000 \cdot 50 = 1\,250\,000.$$

c) From the previous part, we see that of the two terms contributing to total revenue growth, the term $5S(0)$ is larger than the term $10\,000N(0)$. Thus, if only one leg of the campaign can be implemented, it should be part A: increase the number of stores by 5 per month.

1.4.9. Exercise. A company can produce LCD digital alarm clocks at a cost of \$6 each while fixed costs are \$16. Therefore, the company's cost function is $C(x) = 6x + 16$.

a) Find the average cost function.

b) Find the marginal average cost function.

c) Evaluate the marginal cost at $x = 20$. Round the answer to the nearest cent.

Solution:

a) The average cost function is

$$AC(x) = \frac{C(x)}{x} = 6 + \frac{16}{x}.$$

b) The marginal average cost function is a differentiate function of the average cost function, that is

$$MAC(x) = -\frac{16}{x^2}.$$

c) The value of marcginal cost function at $x = 20$ is

$$MAC(20) = -\frac{16}{400} = -0.04,$$

that is -4 cents.

1.4.10. Exercise. Find all critical points of the function

$$f(x;y) = x^2 - 2x + 2y^2 - 4y.$$

Solution:

The first ordered partial derivatives of function f are

$$f'_x(x;y) = 2x - 2$$
$$f'_y(x;y) = 4y - 4.$$

We have to solve the system of equations

$$\left.\begin{aligned} 2x - 2 &= 0 \\ 4y - 4 &= 0 \end{aligned}\right\} .$$

The solution of the system is $(1;1)$. The critical point of the function f is $P = (1;1)$.

1.4.11. Exercise. Find all critical points of the function

$$f(x;y) = x^3 - 2y^2 - xy + 2.$$

Solution:

The first ordered partial derivatives of function f are

$$f'_x(x;y) = 3x^2 - y$$
$$f'_y(x;y) = -4y - x.$$

We have to solve the system of equations

$$\left.\begin{aligned} 3x^2 - y &= 0 \\ -4y - x &= 0 \end{aligned}\right\} .$$

From the second equation, we get that $x = -4y$. If we substitute this to the first equation, we get that

$$48y^2 - y = 0 \qquad \Rightarrow \qquad y \cdot (48y - 1) = 0,$$

thus $y_1 = 0$ and $y_2 = \frac{1}{48}$, hence $x_1 = 0$ and $x_2 = -\frac{1}{12}$.
The critical points of the function f are $P_1 = (0;0)$ and $P_2 = \left(\frac{1}{12}; -\frac{1}{48}\right)$.

1.4.12. Exercise. The critical point for $f(x;y) = x^2 - 4x + y^2 - 8y + 2$ is $P = (2;4)$. Determine if the critical point is a local (relative) maximum or minimum.

Solution:

The first ordered partial derivatives of function f are

$$f'_x(x;y) = 2x - 4$$
$$f'_y(x;y) = 2y - 8.$$

The second ordered partial derivatives are

$$f''_{xx}(x;y) = 2 \qquad f''_{xy}(x;y) = 0$$
$$f''_{yx}(x;y) = 0 \qquad f''_{yy}(x;y) = 2,$$

thus the Hessian-matrix is as follows:

$$M(x_0;y_0) = \begin{pmatrix} f''_{xx}(x_0;y_0) & f''_{xy}(x_0;y_0) \\ f''_{yx}(x_0;y_0) & f''_{yy}(x_0;y_0) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

If we substitute the coorinates of point P to the Hessian-matrix, we get the same matrix:

$$M(P_1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $D_1 = 2$ and

$$D_2 = \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4,$$

therefore D_1 and D_2 are positive, thus at P has a local minimum.

1.4.13. Exercise. Calculate the local extremum of the function

$$f(x;y) = 2x^3 + 16y^3 - 24xy.$$

Solution:

The first ordered partial derivatives of function f are

$$f'_x(x;y) = 6x^2 - 24y$$
$$f'_y(x;y) = 48y^2 - 24x.$$

We have to solve the system of equations

$$\left.\begin{aligned} 6x^2 - 24y = 0 \\ 48y^2 - 24x = 0 \end{aligned}\right\}.$$

If we simplify the equations, we get that

$$\left.\begin{aligned} x^2 - 4y = 0 \\ 2y^2 - x = 0 \end{aligned}\right\}.$$

From the second equation, we get that $x = 2y^2$. If we substitute this to the first equation, we get that

$$4y^4 - 4y = 0 \qquad \Rightarrow \qquad y^4 - y = 0,$$

thus

$$y \cdot (y^3 - 1) = 0.$$

The solutions of this equation are $y_1 = 0$ and $y_2 = 1$, hence $x_1 = 0$ and $x_2 = 2$. The critical points of the function f are $P_1 = (0; 0)$ and $P_2 = (2; 1)$.

The general form of Hessian-matrix for two variables function is as follows:

$$M(x; y) = \begin{pmatrix} f''_{xx}(x; y) & f''_{xy}(x; y) \\ f''_{yx}(x; y) & f''_{yy}(x; y) \end{pmatrix}.$$

In this exercise the second ordered partial derivatives are

$$f''_{xx}(x; y) = 12x \qquad\qquad f''_{xy}(x; y) = -24$$
$$f''_{yx}(x; y) = -24 \qquad\qquad f''_{yy}(x; y) = 96y,$$

thus the Hessian-matrix is as follows:

$$M(x; y) = \begin{pmatrix} 12x & -24 \\ -24 & 96y \end{pmatrix}.$$

If we substitute the point P_1 to the Hessian-matrix, we get that

$$M(P_1) = \begin{pmatrix} 0 & -24 \\ -24 & 0 \end{pmatrix}.$$

Since $D_1 = 0$ and

$$D_2 = \det \begin{pmatrix} 0 & -24 \\ -24 & 0 \end{pmatrix} = 0 - 576 = -576.$$

therefore at point P_1 has no local extremum.

The Hessian-matrix at point P_2 is as follows:

$$M(P_2) = \begin{pmatrix} 24 & -24 \\ -24 & 96 \end{pmatrix}.$$

Since $D_1 = 24$ and

$$D_2 = \det \begin{pmatrix} 24 & -24 \\ -24 & 96 \end{pmatrix} = 24 \cdot 96 - (-24)^2 = 1\,728,$$

therefore D_1 and D_2 are positive, thus at P_2 has a local minimum. Its value is:

$$f(2;1) = 2\cdot 2^3 + 16\cdot 1^3 - 24\cdot 2\cdot 1 = -16.$$

1.4.14. Exercise. Find the local extremum of function

$$f(x;y) = x^3 - 75x + 2y^2 - 4y.$$

<u>Solution:</u>

The partial derivatives f are

$$f'_x(x;y) = 3x^2 - 75$$
$$f'_y(x;y) = 4y - 4.$$

The solutions of system of equation are

$$\left.\begin{aligned} 3x^2 - 75 &= 0 \\ 4y - 4 &= 0 \end{aligned}\right\}$$

$P_1 = (5;1)$ and $P_2 = (-5;1)$.

The second ordered partial derivatives of function f are

$$f''_{xx}(x;y) = 6x \qquad f''_{xy}(x;y) = 0$$
$$f''_{yx}(x;y) = 0 \qquad f''_{yy}(x;y) = 4.$$

The Hessian-matrix is

$$M(x;y) = \begin{pmatrix} f''_{xx}(x;y) & f''_{xy}(x;y) \\ f''_{yx}(x;y) & f''_{yy}(x;y) \end{pmatrix} = \begin{pmatrix} 6x & 0 \\ 0 & 4 \end{pmatrix}.$$

The Hessian-matrix at point P_1 is

$$M(P_1) = \begin{pmatrix} 30 & 0 \\ 0 & 4 \end{pmatrix}.$$

Since $D_1 = 30$ and

$$D_2 = \det \begin{pmatrix} 30 & 0 \\ 0 & 4 \end{pmatrix} = 120 - 0 = 120,$$

therefore at point P_1 has a local minimum of function f. The value of function f is

$$f(5;1) = 5^3 - 75\cdot 5 + 2\cdot 1^2 - 4\cdot 1 = -252.$$

The value of matrix at point P_2 is

$$M(P_2) = \begin{pmatrix} -30 & 0 \\ 0 & 4 \end{pmatrix}.$$

Since $D_1 = -30$ and

$$D_2 = \det \begin{pmatrix} -30 & 0 \\ 0 & 4 \end{pmatrix} = -120 - 0 = -120,$$

therefore $D_2 < 0$ thus at point P_2 there is no local extremum.

1.4.15. Exercise. Let $f(x;y) = \ln(x^2+y^2+1)+5$ be a two variabled function. Calculate the local extremum of the function.

Solution:

The partial derivatives of f are

$$f'_x(x;y) = \frac{2x}{x^2+y^2+1}$$

$$f'_y(x;y) = \frac{2y}{x^2+y^2+1}.$$

The solution of the equation

$$\left.\begin{aligned} \frac{2x}{x^2+y^2+1} &= 0 \\ \frac{2y}{x^2+y^2+1} &= 0 \end{aligned}\right\}$$

is $P = (0;0)$.

The second ordered partial derivatives are

$$f''_{xx}(x;y) = \frac{2\cdot(x^2+y^2+1) - 4x^2}{(x^2+y^2+1)^2} = \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}$$

and

$$f''_{yy}(x;y) = \frac{2\cdot(x^2+y^2+1) - 4y^2}{(x^2+y^2+1)^2} = \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2}$$

and

$$f''_{yx}(x;y) = f''_{xy}(x;y) = \frac{-4xy}{(x^2+y^2+1)^2}.$$

The Hessian-matrix is

$$M(x;y) = \begin{pmatrix} \dfrac{-2x^2+2y^2+2}{(x^2+y^2+1)^2} & \dfrac{-4xy}{(x^2+y^2+1)^2} \\ \dfrac{-4xy}{(x^2+y^2+1)^2} & \dfrac{2x^2-2y^2+2}{(x^2+y^2+1)^2} \end{pmatrix}.$$

The Hessian-matrix at point P is

$$M(P) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The value of D_1 is 1, and

$$D_2 = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 - 0 = 1.$$

Since D_1 and D_2 are positive real numbers, we get that at point P there is a local minimum of function f. The value of function f at point P is:

$$f(0;0) = \ln(0^2 + 0^2 + 1) + 5 = 5.$$

1.4.16. Exercise. Find three real numbers whose sum is 150 and the sum of whose squares is as small as possible.

Solution:

Let the numbers are x, y and z. The sum of three numbers is

$$x + y + z = 150.$$

If we express variable z from the previous equation, we get that

$$z = 150 - x - y,$$

thus we have to find the local minimum of the function

$$f(x;y) = x^2 + y^2 + (150 - x - y)^2.$$

The partial derivatives of function f are

$$f'_x(x;y) = 2x + 2 \cdot (150 - x - y) \cdot (-1) = 4x + 2y - 300$$

$$f'_y(x;y) = 2y + 2 \cdot (150 - x - y) \cdot (-1) = 2x + 4y - 300.$$

The solution of the system

$$\left.\begin{aligned} 4x + 2y - 300 = 0 \\ 2x + 4y - 300 = 0 \end{aligned}\right\}$$

is $P = (50; 50)$.

The second ordered partial derivatives are

$$f''_{xx}(x;y) = 4 \qquad f''_{xy}(x;y) = 2$$
$$f''_{yx}(x;y) = 2 \qquad f''_{yy}(x;y) = 4,$$

thus the Hessian-matrix is

$$M(x;y) = \begin{pmatrix} f''_{xx}(x;y) & f''_{xy}(x;y) \\ f''_{yx}(x;y) & f''_{yy}(x;y) \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

The Hessian-matrix at point is the same, that is

$$M(P) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

Since $D_1 = 4$ and

$$D_2 = \det \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = 16 - 4 = 12,$$

therefore D_1 and D_2 are positive real numbers, thus there is a minimum at point P. The value of z is

$$z = 150 - 50 - 50 = 50.$$

The sum of squares is

$$50^2 + 50^2 + 50^2 = 7\,500.$$

1.4.17. Exercise. A small company produces speakers and subwoofers for computers that they sell through a website. After extensive research, the company has developed a revenue function,

$$R(x;y) = 120x - 4.5x^2 + 160y - 2y^2$$

thousand dollars, where x is the number of subwoofers produced and sold in thousands and y is the number of speakers produced and sold in thousands. The corresponding cost function is

$$C(x;y) = 3x^2 + 3y^2 + 5xy - 5y + 50$$

thousand dollars. Find the production levels that maximize the profit.

Solution:

By subtracting the cost from the revenue, we get the profit function:

$$\begin{aligned}\Pi(x;y) &= R(x;y) - C(x;y) = \\ &= 120x - 4.5x^2 + 160y - 2y^2 - (3x^2 + 3y^2 + 5xy - 5y + 50) = \\ &= 120x - 7.5x^2 + 165y - 5y^2 - 5xy - 50.\end{aligned}$$

The first ordered partial derivatives are

$$f'_x(x;y) = 120 - 15x - 5y$$

$$f'_y(x;y) = 165 - 10y - 5x.$$

We have to solve the system

$$\left.\begin{aligned}120 - 15x - 5y = 0\\ 165 - 10y - 5x = 0\end{aligned}\right\}.$$

If we multiply the second equation by -3, we get that

$$\left.\begin{aligned}120 - 15x - 5y = 0\\ -495 + 30y + 15x = 0\end{aligned}\right\}.$$

If we add the second equation to the first equation, we get that

$$375 = 25y \qquad \Rightarrow \qquad y = 15,$$

thus $x = 3$. The critical point is $P = (3;15)$.

The second ordered partial derivatives are

$$f''_{xx}(x;y) = -15 \qquad\qquad f''_{xy}(x;y) = -5$$
$$f''_{yx}(x;y) = -5 \qquad\qquad f''_{yy}(x;y) = -10,$$

thus the Hessian-matrix is

$$M(x;y) = \begin{pmatrix} f''_{xx}(x;y) & f''_{xy}(x;y) \\ f''_{yx}(x;y) & f''_{yy}(x;y) \end{pmatrix} = \begin{pmatrix} -15 & -5 \\ -5 & -10 \end{pmatrix}.$$

The Hessian-matrix at point is the same, that is

$$M(P) = \begin{pmatrix} -15 & -5 \\ -5 & -10 \end{pmatrix}.$$

Since $D_1 = -15$ and

$$D_2 = \det \begin{pmatrix} -15 & -5 \\ -5 & -10 \end{pmatrix} = 150 - 25 = 125,$$

therefore D_1 is negative and D_2 is positive real numbers, thus there is a maximum at point P. The maximum value of profit function Π is

$$\Pi(3;15) = 120 \cdot 3 - 7.5 \cdot 3^2 + 165 \cdot 15 - 5 \cdot 15^2 - 5 \cdot 3 \cdot 15 - 50 = 1\,367.5.$$

At a production level of 3 thousand subwoofers and 15 thousand speakers, the company will win 1 367.5 thousand dollars.

1.4.18. Exercise. Find the point on the plane given by $x + y - z = 1$ that is closest to the point $P = (0; -3; 2)$ and calculate their distance.

Solution:

Let the arbitrary point of the plane $x + y - z = 1$ is $Q = (x; y; z)$. The distance of point P and Q is

$$d_{PQ} = \sqrt{x^2 + (y+3)^2 + (z-2)^2}.$$

Since the point Q lies on the plane, therefore

$$x + y - z = 1 \qquad \Rightarrow \qquad z = x + y - 1,$$

thus

$$d_{PQ} = \sqrt{x^2 + (y+3)^2 + (z-2)^2} = \sqrt{x^2 + (y+3)^2 + (x+y-3)^2}.$$

We will minimize the distance squared:

$$f(x;y) = x^2 + (y+3)^2 + (x+y-3)^2$$

The partial derivatives of function f are

$$f'_x(x;y) = 2x + 2 \cdot (x + y - 3) = 4x + 2y - 6$$
$$f'_y(x;y) = 2 \cdot (y+3) + 2 \cdot (x + y - 3) = 2x + 4y.$$

We have to solve the system

$$\left.\begin{aligned} 4x + 2y &= 6 \\ 2x + 4y &= 0 \end{aligned}\right\}.$$

The solution of the system is $S = (2; -1)$.

The second ordered partial derivatives are

$$f''_{xx}(x;y) = 4 \qquad f''_{xy}(x;y) = 2$$
$$f''_{yx}(x;y) = 2 \qquad f''_{yy}(x;y) = 4,$$

hence the Hessian-matrix is

$$M(x;y) = \begin{pmatrix} f''_{xx}(x;y) & f''_{xy}(x;y) \\ f''_{yx}(x;y) & f''_{yy}(x;y) \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

The Hessian-matrix at point S is

$$M(S) = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}.$$

Since $D_1 = 4$ and

$$D_2 = \det \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} = 16 - 4 = 12,$$

we get that there is a local minimum at point S. We then calculate the distance from $Q = (2; -1; 0)$ to P is $\sqrt{12}$.

1.4.19. Exercise. Find all critical points of the function

$$f(x;y;z) = x^2 - 2x + 2y^2 - 4y + z^2 - 2z + 1.$$

Solution:

The first ordered partial derivatives of function f are

$$f'_x(x;y;z) = 2x - 2$$
$$f'_y(x;y;z) = 4y - 4$$
$$f'_z(x;y;z) = 2z - 2.$$

We have to solve the system of equations

$$\left.\begin{aligned} 2x - 2 &= 0 \\ 4y - 4 &= 0 \\ 2z - 2 &= 0 \end{aligned}\right\}.$$

The solution of the system is $(1;1;1)$. The critical point of the function f is $P = (1;1;1)$.

1.4.20. Exercise. Find the local extremum of function

$$f(x;y;z) = x^2 - xy + y^2 + z^2 - 2z + 10.$$

Solution:

The partial derivatives of function f are

$$\begin{aligned} f'_x(x;y;z) &= 2x - y \\ f'_y(x;y;z) &= -x + 2y \\ f'_z(x;y;z) &= 2z - 2. \end{aligned}$$

We have to solve the system

$$\left.\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 0 \\ 2z - 2 &= 0 \end{aligned}\right\}.$$

The solution of the system is $P = (0;0;1)$.

The second ordered partial derivatives of f are

$$\begin{array}{lll} f''_{xx} = 2 & f''_{xy} = -1 & f''_{xz} = 0 \\ f''_{yx} = -1 & f''_{yy} = 2 & f''_{yz} = 0 \\ f''_{zx} = 0 & f''_{zy} = 0 & f''_{zz} = 2 \end{array}$$

thus the Hessian-matrix is

$$M(x;y;z) = \begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

If we substitute the point P to the Hessian-matrix, we get that

$$M(P) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $D_1 = 2$ and

$$D_2 = \det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 3,$$

moreover

$$D_3 = \det \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 6,$$

we get that $D_i > 0$ for all $i = 1, 2, 3$ are positive real numbers, thus there is a local minimum at point P. The value of function at point P is

$$f(0; 0; 1) = 0^2 - 0 \cdot 0 + 0^2 + 1^2 - 2 \cdot 1 + 10 = 9.$$

1.4.21. Exercise. Find the local extremum of function

$$f(x; y; z) = x^3 - 9xy + y^3 + z^2 - 2z + 10.$$

Solution:

The partial derivatives of function f are 0

$$\begin{aligned} f'_x(x; y; z) &= 3x^2 - 9y \\ f'_y(x; y; z) &= 3y^2 - 9x \\ f'_z(x; y; z) &= 2z - 2. \end{aligned}$$

We have to solve the system

$$\left.\begin{aligned} 3x^2 - 9y &= 0 \\ 3y^2 - 9x &= 0 \\ 2z - 2 &= 0 \end{aligned}\right\} .$$

The solutions of the system are $P_1 = (0; 0; 1)$ and $P_2 = (3; 3; 1)$.

The second ordered partial derivatives of f are

$$\begin{aligned} f''_{xx} &= 6x & f''_{xy} &= -9 & f''_{xz} &= 0 \\ f''_{yx} &= -9 & f''_{yy} &= 6y & f''_{yz} &= 0 \\ f''_{zx} &= 0 & f''_{zy} &= 0 & f''_{zz} &= 2, \end{aligned}$$

thus the Hessian-matrix is

$$M(x; y; z) = \begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{yx} & f''_{yy} & f''_{yz} \\ f''_{zx} & f''_{zy} & f''_{zz} \end{pmatrix} = \begin{pmatrix} 6x & -9 & 0 \\ -9 & 6y & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

If we substitute the coordinates point P_1 to the Hessian-matrix, we get that

$$M(P_1) = \begin{pmatrix} 0 & -9 & 0 \\ -9 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $D_1 = 0$ and

$$D_2 = \det \begin{pmatrix} 0 & -9 \\ -9 & 0 \end{pmatrix} = -81,$$

thus at P_1 there is no extremum point.
If we substitute the coordinates point P_2 to the Hessian-matrix, we get that

$$M(P_2) = \begin{pmatrix} 18 & -9 & 0 \\ -9 & 18 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since $D_1 = 18$ and

$$D_2 = \det \begin{pmatrix} 18 & -9 \\ -9 & 18 \end{pmatrix} = 243,$$

and

$$D_3 = \det \begin{pmatrix} 18 & -9 & 0 \\ -9 & 18 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 486$$

thus at P_2 there is a local minimum.
The value of function at point P_2 is

$$f(3;3;1) = 3^3 - 9 \cdot 3 \cdot 3 + 3^3 + 1^2 - 2 \cdot 1 + 10 = -18.$$

1.4.22. Exercise. In a certain office, Computers A, B and C are utilized for a, b and c hours, respectively. If the daily output f is a function of a, b and c, namely

$$f(a;b;c) = 23a + 29b - 2a^2 - 4b^2 - ab - c^2 + 2c,$$

find the values of a, b and c that maximize f.

Solution:

The partial derivatives of function f are

$$f'_a(a;b;c) = 23 - 4a - b$$
$$f'_b(a;b;c) = 29 - 8b - a$$
$$f'_c(a;b;c) = -2c + 2.$$

We have to solve the system

$$\left.\begin{aligned} 23 - 4a - b &= 0 \\ 29 - 8b - a &= 0 \\ -2c + 2 &= 0 \end{aligned}\right\}.$$

The solution of system is $P = (5;3;1)$.

The second ordered partial derivatives are

$$\begin{array}{lll} f''_{aa}(a;b;c) = -4 & f''_{ab}(a;b;c) = -1 & f''_{ac}(a;b;c) = 0 \\ f''_{ba}(a;b;c) = -1 & f''_{bb}(a;b;c) = -8 & f''_{bc}(a;b;c) = 0 \\ f''_{ca}(a;b;c) = 0 & f''_{cb}(a;b;c) = 0 & f''_{cc}(a;b;c) = -2 \end{array}$$

hence the Hessian-matrix is

$$M(a;b;c) = \begin{pmatrix} f''_{aa} & f''_{ab} & f''_{ac} \\ f''_{ba} & f''_{bb} & f''_{bc} \\ f''_{ca} & f''_{cb} & f''_{cc} \end{pmatrix} = \begin{pmatrix} -4 & -4 & 0 \\ -1 & -8 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The Hessian-matrix at point P is

$$M(P) = \begin{pmatrix} -4 & -1 & 0 \\ -1 & -8 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Since $D_1 = -4$ and

$$D_2 = \det \begin{pmatrix} -4 & -1 \\ -1 & -8 \end{pmatrix} = 32 - 1 = 31,$$

and

$$D_3 = \det \begin{pmatrix} -4 & -1 & 0 \\ -1 & -8 & 0 \\ 0 & 0 & -2 \end{pmatrix} = -2 \cdot 31 = -62,$$

thus the function f has a maximum.

The value of function at point P is

$$f(5;3;1) = 23 \cdot 5 + 29 \cdot 3 - 2 \cdot 5^2 - 4 \cdot 3^2 - 15 - 1^2 + 2 \cdot 1 = 102.$$

1.4.23. Exercise. Find the local extremum of function

$$f(x;y;z;u) = x^3 + y^2 - 6xy + 2z^2 - 4z + u^2 - 6u + 3.$$

Solution:

The partial derivatives are

$$f'_x(x;y;z;u) = 3x^2 - 6y$$
$$f'_y(x;y;z;u) = 2y - 6x$$
$$f'_z(x;y;z;u) = 4z - 4$$
$$f'_u(x;y;z;u) = 2u - 6.$$

We have to solve the system of equations

$$\left.\begin{aligned} 3x^2 - 6y &= 0 \\ 2y - 6x &= 0 \\ 4z - 4 &= 0 \\ 2u - 6 &= 0 \end{aligned}\right\}.$$

The solutions are

$$P_1 = (0;0;1;3), \qquad P_2 = (6;18;1;3).$$

The Hessian matrices is

$$M(x;y;z;u) = \begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} & f''_{xu} \\ f''_{yx} & f''_{yy} & f''_{yz} & f''_{yu} \\ f''_{zx} & f''_{zy} & f''_{zz} & f''_{zu} \\ f''_{ux} & f''_{uy} & f''_{uz} & f''_{uu} \end{pmatrix}.$$

Since

$$\begin{array}{llll} f''_{xx} = 6x & f''_{xy} = -6 & f''_{xz} = 0 & f''_{xu} = 0 \\ f''_{yx} = -6 & f''_{yy} = 2 & f''_{yz} = 0 & f''_{yu} = 0 \\ f''_{zx} = 0 & f''_{zy} = 0 & f''_{zz} = 4 & f''_{zu} = 0 \\ f''_{ux} = 0 & f''_{uy} = 0 & f''_{uz} = 0 & f''_{uu} = 2, \end{array}$$

we have

$$M(x;y;z;u) = \begin{pmatrix} 6x & -6 & 0 & 0 \\ -6 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

The value of matrices M at point P_1 is

$$M(P_1) = \begin{pmatrix} 0 & -6 & 0 & 0 \\ -6 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

In this case the first element of the matrices is $D_1 = 0$.
Let denote

$$D_2 = \det \begin{pmatrix} 0 & -6 \\ -6 & 2 \end{pmatrix} = -36.$$

Since D_2 is negative then at point P_1 there is no extremum point of function f.
The value of matrices M at point P_2 is

$$M(P_2) = \begin{pmatrix} 36 & -6 & 0 & 0 \\ -6 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The first element of matrices is $D_1 = 36$.
Let denote

$$D_2 = \det \begin{pmatrix} 36 & -6 \\ -6 & 2 \end{pmatrix} = 36.$$

Let

$$D_3 = \det \begin{pmatrix} 36 & -6 & 0 \\ -6 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} = 4 \cdot 36 = 144.$$

The determinant of matrices M is

$$D_4 = \det \begin{pmatrix} 36 & -6 & 0 & 0 \\ -6 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2 \cdot 144 = 288.$$

All determinant are positive thus at point P_2 there is a local minimum.

1.5. Unsolved problems

1.5.1. Exercise. A company sells a product at a price $p(q) = 100 - 2q$, where q is the quantity sold. Find the value of q that maximizes the company's revenue.

1.5.2. Exercise. A firm's total cost for producing x units of a good is given by

$$C(x) = x^3 - 12x^2 + 45x + 100.$$

Determine the level of production x that minimizes the total cost.

1.5.3. Exercise. A monopolist's total profit from selling x units of a product is represented by the profit function

$$\Pi(x) = -x^2 + 14x - 24.$$

Calculate the number of units x that the firm should produce and sell to maximize its profit.

1.5.4. Exercise. A firm's production function is given by

$$Q(L; K) = L^{0.75} \cdot K^{0.25},$$

where L is labor and K is capital. The wage rate (cost of labor) is \$20 per unit, and the cost of capital is \$40 per unit. The firm sells its output at a price of \$10 per unit. Determine the combination of labor and capital that maximizes the firm's profit.

1.5.5. Exercise. A company needs to produce 200 units of a good. The production function is

$$Q(L; K) = 4L^{0.5} \cdot K^{0.5}.$$

The price of labor is \$15 per unit, and the price of capital is \$30 per unit. Find the combination of labor and capital that minimizes the company's total cost.

1.5.6. Exercise. An individual seeks to maximize their utility, which is represented by the utility function

$$U(X, Y) = X^{0.4} \cdot Y^{0.6},$$

where X and Y are quantities of two goods. The price of X is \$2, and the price of Y is \$3. The individual has a budget of \$120. Determine the optimal consumption bundle of goods X and Y.

1.5.7. Exercise. A firm can sell its product in two different markets. The revenue functions for these markets are given by

$$R_1(Q_1) = 60Q_1 - 0.5Q_1^2$$

and

$$R_2(Q_2) = 40Q_2 - 0.2Q_2^2,$$

where Q_1 and Q_2 are the quantities sold in each market. The firm has a total of 100 units of the product to distribute between the two markets. Determine how many units should be sold in each market to maximize total revenue.

1.5.8. Exercise. A consumer derives satisfaction from consuming goods A and B, with the utility function

$$U(A, B) = 2A^{0.5} + 3B^{0.5}.$$

The price per unit of good A is \$4, and the price per unit of good B is \$6. If the consumer has \$240 to spend, find the quantities of goods A and B that maximize the consumer's surplus.

1.5.9. Exercise. A firm's production function is given by

$$Q(L; K; M) = 2L^{0.3} \cdot K^{0.5} \cdot M^{0.2},$$

where L is labor, K is capital, and M is raw materials. The prices per unit are \$30 for labor, \$50 for capital, and \$20 for raw materials. If the firm aims to produce goods with a budget of \$1000, find the combination of labor, capital, and raw materials that maximizes output.

Chapter 2

Introduction to probability

2.1. Sample space and events

2.1.1. Remark. In the study of statistics, we are concerned basically with the presentation and interpretation of chance outcomes that occur in a planned study or scientific investigation. For example, we might classify items coming off an assembly line as ”defective” or ”nondefective”.
The statistician is often dealing with either numerical data, representing counts or measurements, or categorical data, which can be classified according to some criterion.
Statisticians use the word experiment to describe any process that generates a set of data. A simple example of a statistical experiment is the tossing of a coin. In this experiment, there are only two possible outcomes, heads or tails. When a coin is tossed repeatedly, we cannot be certain that a given toss will result in a head. However, we know the entire set of possibilities for each toss.

2.1.2. Definition. An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a *random experiment*.

2.1.3. Definition. The set of all possible outcomes of a random experiment is called the *sample space* of the experiment. The sample space is denoted as S. Each outcome in a sample space is called an element or a member of the sample space, or simply a *sample point*.

2.1.4. Example. Consider the experiment of tossing a die. If we are interested in the number that shows on the top face, the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

2.1.5. Definition. A sample space is *discrete* if it consists of a finite or countable infinite set of outcomes. A sample space is *continuous* if it contains an interval (either finite or infinite) of real numbers.

2.1.6. Example. Find the sample space for the experiment of tossing a coin once.

In this situation there are two possible outcomes, heads or tails. Thus the sample space is

$$S = \{H, T\},$$

where H and T represent head and tail, respectively. The sample space is discrete.

2.1.7. Example. Find the sample space for the experiment of measuring (in hours) the lifetime of a transistor.

Clearly all possible outcomes are all nonnegative real numbers. That is,

$$S = \{t \mid t \geq 0\} = [0; \infty[,$$

where t represents the life of a transistor in hours. This sample space is continuous.

2.1.8. Definition. An event is a subset of a sample space.

2.1.9. Example. Given the sample space

$$S = \{t \in \mathbb{R} \mid t \geq 0\} = [0; \infty[,$$

where t is the life in years of a certain electronic component. In this case the event A that the component fails before the end of the fifth year is the subset

$$A = \{t \in \mathbb{R} \mid 0 \leq t < 5\} = [0; 5[.$$

2.1.10. Example. In the tossing of a die we might let A be the event that an even number occurs and B the event that a number greater than 3 shows. Then the subsets $A = \{2, 4, 6\}$ and $B = \{4, 5, 6\}$ are subsets of the same sample space $S = \{1, 2, 3, 4, 5, 6\}$.

2.1.11. Definition. The *complement* of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol $\overline{A}$.

2.1.12. Example. Let R be the event that a red card is selected from an ordinary deck of 52 playing cards, and let S be the entire deck. Then $\overline{R}$ is the event that the card selected from the deck is not a red card but a black card.

2.1.13. Definition. The *intersection* of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

2.1.14. Example. Let E be the event that a person selected at random in a classroom is majoring in engineering, and let F be the event that the person is female. Then $E \cap F$ is the event of all female engineering students in the classroom.

2.1.15. Definition. Two events A and B are *mutually exclusive*, or disjoint, if $A \cap B = \emptyset$, that is, if A and B have no elements in common.

2.1.16. Definition. The collection of events

$$A_1, A_2, \ldots, A_n$$

are *mutually exclusive* if

$$A_i \cap A_j = \emptyset \quad (i \neq j).$$

2.1.17. Example. Let $A = \{a, e, i, o, u\}$ and $B = \{l, r, s, t\}$. In this case $A \cap B = \emptyset$. That is, A and B have no elements in common and, therefore, cannot both simultaneously occur.

2.1.18. Example. If we roll a fair die and event A means that the number on the top of the die is even and let denote B when the number on the top of the die is odd. In this case A and B are mutually exclusive event, hence $A \cap B = \emptyset$.

2.1.19. Definition. The *union* of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

2.1.20. Example. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$. In this case

$$A \cup B = \{1, 2, 3, 4, 5\}.$$

2.2. Probability of events

2.2.1. Remark. In this section, we introduce the notion of probability for discrete sample spaces. (Remember it means that a finite or countable infinite set of outcomes).

The restriction to these sample spaces enables us to simplify the concepts and the presentation without excessive mathematics. Probability is used to quantify the likelihood, or chance, that an outcome of a random experiment will occur. ”The chance of rain today is 30%” is a statement that quantifies our feeling about the possibility of rain. The likelihood of an outcome is quantified by assigning a number from the interval $[0;1]$ to the outcome (or a percentage from 0 to 100%). Higher numbers indicate that the outcome is more likely than lower numbers. A 0 indicates an outcome will not occur. A probability of 1 indicates that an outcome will occur with certainty.

2.2.2. Definition. Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $\frac{1}{N}$.

2.2.3. Definition. For a discrete sample space, the probability of an event E, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.

2.2.4. Example. A random experiment can result in one of the outcomes

$$\{a, b, c, d\}$$

with probabilities 0.1, 0.3, 0.5, and 0.1, respectively. Let A denote the event a, b, B the event b, c, d, and C the event d. Then,

$$P(A) = 0.1 + 0.3 = 0.4$$
$$P(B) = 0.3 + 0.5 + 0.1 = 0.9$$
$$P(C) = 0.1.$$

On the other hand

$$\overline{A} = \{c, d\}, \quad \overline{B} = \{a\}, \quad \overline{C} = \{a, b, c\}$$

thus

$$P(\overline{A}) = 0.5 + 0.1 = 0.6$$
$$P(\overline{B}) = 0.1$$
$$P(\overline{C}) = 0.3 + 0.5 + 0.1 = 0.9.$$

Since $A \cap B = \{b\}$ therefore $P(A \cap B) = 0.3$.

Since $A \cup B = \{a, b, c, d\}$ hence $P(A \cup B) = 0.1 + 0.3 + 0.5 + 0.1 = 1$.

2.2.5. Definition. *Probability* is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties: If S is the sample space and A is any event in a random experiment,

- $P(S) = 1$;
- $0 \leq P(A) \leq 1$;
- For two events A_1 and A_2 with $A_1 \cap A_2 = \emptyset$ it is fulfilled that

$$P(A_1 \cup A_2) = P(A_1) + P(A_2).$$

2.2.6. Theorem. If A and B are events such that $A \subset B$ then $P(A) \leq P(B)$.

Proof: Let A and B events such that $A \subset B$. Since

$$B = A \cup (B \setminus A),$$

we have

$$P(B) = P\big(A \cup (B \setminus A)\big) = P(A) + P(B \setminus A) = P(A) + P(B \setminus A).$$

We know that $P(B \setminus A) \geq 0$ thus $P(A) \leq P(B)$. ■

2.2.7. Theorem. If A is an arbitrary event then $P(\overline{A}) = 1 - P(A)$.

Proof: It is well-known fact $S = A \cup \overline{A}$ and $A \cap \overline{A} = \emptyset$ thus

$$1 = P(S) = P(A \cup \overline{A}) = P(A) + P(\overline{A}),$$

hence we get that $P(\overline{A}) = 1 - P(A)$. ■

2.2.8. Example. If we toss a fair die once, what is the probability that the score on the top of die at least 2? To the solution in first step we introduce the notation $A = \{1\}$. In this situation

$$P(\overline{A}) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}.$$

2.2.9. Theorem. The probability of emptyset is zero.

Proof: Let denote the sample space S. In this case

$$P(\emptyset) = P(\overline{S}) = 1 - P(S) = 1 - 1 = 0$$

which is the statement of the theorem. ■

2.2.10. Theorem. If A and B are events then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Proof: Since $A \cup B = A \cup (B \setminus A)$ we have

$$P(A \cup B) = P(A \cup (B \setminus A)).$$

We know that $B \setminus A = B \setminus (A \cap B)$ thus

$$P(A \cup B) = P(A \cup (B \setminus (A \cap B))) = P(A) + P(B) - P(A \cap B)$$

which is the equality to be proved. ■

2.2.11. Example. If $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cap B) = 0.1$ then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.1 = 0.6.$$

2.2.12. Corollary. If A and B are mutually exclusive events then

$$P(A \cup B) = P(A) + P(B).$$

Proof: If A and B are mutually exclusive events then

$$A \cap B = \emptyset$$

hence we get that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) =$$
$$= P(A) + P(B) - P(\emptyset) = P(A) + P(B)$$

which is the statement of the theorem. ■

2.2.13. Example. If $P(A) = 0.2$, $P(B) = 0.3$ and $A \cap B = \emptyset$ that is A and B are mutually exclusive events then

$$P(A \cup B) = P(A) + P(B) = 0.2 + 0.3 = 0.5.$$

2.2.14. Theorem. If A, B and C are events then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) -$$
$$- P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

2.2.15. Theorem. If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is $P(A) = \frac{n}{N} = \frac{n(A)}{n(S)}$, where $n(A)$ and $n(S)$ mean the number of the element of set A and S, respectively.

2.2.16. Example. We toss a coin twice. What is the probability that we have exactly one head? In this example the sample space is

$$S = \{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$$

Let denote set A the outcomes when we have exactly one head, that is

$$A = \{(H, T), (T, H)\}.$$

The probability of event A is $P(A) = \frac{n(A)}{n(S)} = \frac{2}{4} = \frac{1}{2}$.

2.3. Conditional probability and multiplication rule

2.3.1. Remark. A digital communication channel has an error rate of one bit per every thousand transmitted. Errors are rare, but when they occur, they tend to occur in bursts that affect many consecutive bits. If a single bit is transmitted, we might model the probability of an error as $1/100$. However, if the previous bit was in error, because of the bursts, we might believe that the probability that the next bit is in error is greater than $1/100$.

In a thin film manufacturing process, the proportion of parts that are not acceptable is 1%. However, the process is sensitive to contamination problems that can increase the rate of parts that are not acceptable. If we knew that during a particular shift there were problems with the filters used to control contamination, we would assess the probability of a part being unacceptable as higher than 2%. These examples illustrate that probabilities need to be reevaluated as additional information becomes available.

2.3.2. Definition. Let A and B are arbitrary events and assume that the probability of B is not zero. The *conditional probability* of an event A given an event B, denoted as $P(A|B)$, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

2.3.3. Example. A day's production of 900 manufactured parts contains 100 parts that do not meet customer requirements. Two parts are selected randomly without replacement from the batch. What is the probability that the second part is defective given that the first part is defective?

Let A denote the event that the first part selected is defective, and let B denote the event that the second part selected is defective. The probability needed can be expressed as $P(B|A)$. If the first part is defective, prior to selecting the second part, the batch contains 899 parts, of which 49 are defective, therefore

$$P(B|A) = \frac{49}{899}.$$

2.3.4. Theorem. If A and B are arbitrary events and assume that the probability of B is not zero then

$$P(A \cap B) = P(A|B) \cdot P(B).$$

2.3.5. Example. The probability that an automobile battery subject to high engine compartment temperature suffers low charging current is 0.8. The probability that a battery is subject to high engine compartment temperature is 0.08.

Let C denote the event that a battery suffers low charging current, and let T denote the event that a battery is subject to high engine compartment temperature. The probability that a battery is subject to low charging current and high engine compartment temperature is

$$P(C \cap T) = P(C|T) \cdot P(T) = 0.8 \cdot 0.08 = 0.064.$$

2.3.6. Definition. Let S be a sample space. A collection of events

$$A_1, A_2, \ldots, A_n,$$

such that

$$A_1 \cup A_2 \cup \ldots \cup A_n = S$$

is said to be *exhaustive*.

2.3.7. Theorem. Assume that

$$A_1, A_2, \ldots, A_n$$

are mutually exclusive and exhaustive events and B is an arbitrary event. Then

$$P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n).$$

2.3.8. Corollary. Assume that

$$A_1, A_2, \ldots, A_n$$

are mutually exclusive and exhaustive events and B is an arbitrary event. Then

$$P(B) = P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2) + \ldots + P(B|A_n) \cdot P(A_n).$$

The name of this formula is *Total Probability Rule*.

2.3.9. Example. In a particular production run, 10% of the chips are subjected to high levels of contamination, 30% to medium levels of contamination, and 60% to low levels of contamination. The probability of failure 0.1 at high levels, 0.01 at medium levels and 0.001 at low levels. We will calculate, what is the probability that a product using one of these chips fails?

Let F denote that the chips fails.

Let H denote the event that a chip is exposed to high levels of contamination.

Let M denote the event that a chip is exposed to medium levels of contamination.

Let L denote the event that a chip is exposed to low levels of contamination.

Then

$$\begin{aligned} P(F) &= P(F|H) \cdot P(H) + P(F|M) \cdot P(M) + P(F|L) \cdot P(L) = \\ &= 0.1 \cdot 0.1 + 0.3 \cdot 0.01 + 0.6 \cdot 0.001 = 0.0136. \end{aligned}$$

2.4. Independence

2.4.1. Definition. Let A and B are two events such that $P(B) \neq 0$. The events A and B are *independent* if $P(A|B) = P(B)$.

2.4.2. Theorem. Let A and B are two events such that $P(B) \neq 0$. The events A and B are independent if and only if

$$P(A \cap B) = P(A) \cdot P(B).$$

Proof: If A and B are independent then

$$P(A|B) = P(A).$$

By definition of the independence, we get that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Using the equations, we get that

$$P(A) = \frac{P(A \cap B)}{P(B)} \qquad \Rightarrow \qquad P(A \cap B) = P(A) \cdot P(B),$$

which is the proven identity. ■

2.4.3. Definition. The events are $A_1, A_2, \ldots, A_n$ are *independent* if for any subset of these events

$$A_{i_1}, A_{i_2}, \ldots, A_{i_k},$$

we have

$$P\big(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}\big) = P\big(A_{i_1}\big) \cdot P\big(A_{i_2}\big) \cdot \ldots \cdot P\big(A_{i_k}\big).$$

2.5. Bayes's theorem

2.5.1. Theorem. Assume that A and B are arbitrary events, such that $P(B) > 0$. Then

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}.$$

Proof: By the definition of conditional probability, we have

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \qquad \Rightarrow \qquad P(A \cap B) = P(A|B) \cdot P(B).$$

Using the definition of conditional probability, we get that

$$P(B|A) = \frac{P(B \cap A)}{P(A)} \qquad \Rightarrow \qquad P(A \cap B) = P(B|A) \cdot P(A).$$

Therefore, we get that

$$P(A|B) \cdot P(B) = P(B|A) \cdot P(A).$$

Using the condition $P(B) \neq 0$, the

$$P(A|B) \cdot = \frac{P(B|A) \cdot P(A)}{P(B)}.$$

equation holds. ∎

2.5.2. Theorem. Assume that

$$A_1, A_2, \ldots, A_n$$

are mutually exclusive and exhaustive events and B is an arbitrary event, such that $P(B) > 0$ and $i \in \{1, 2, , \ldots, n\}$. Then

$$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n)}.$$

2.5.3. Corollary. Assume that

$$A_1, A_2, \ldots, A_n$$

are mutually exclusive and exhaustive events and B is an arbitrary event, such that $P(B) > 0$ and $i \in \{1, 2, , \ldots, n\}$. Then

$$P(A_i|B) = \frac{P(B|A_i) \cdot P(A_i)}{P(B)}.$$

2.6. Counting techniques

2.6.1. Theorem. Assume that an operation can be described as a sequence of k steps and the number of ways of completing step 1 is n_1, the number of ways of completing step 2 is n_2 for each way of completing step 1 and the number of ways of completing step 3 is n_3 for each way of completing step 2 and so forth. The total number of ways of completing the operation is

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k.$$

2.6.2. Example. An automobile manufacturer provides vehicles equipped with selected options. Each vehicle is ordered with or without an automatic transmission, with or without air conditioning, with one of four choices of a stereo system and with one of five exterior colors (red, blue, white, black, grey).

In this situation there are 2 options for transmission, two options for air conditioning, four options for stereo system and five options for colours thus the sample space contains

$$2 \cdot 2 \cdot 4 \cdot 5 = 80$$

outcomes.

2.6.3. Definition. Another useful calculation is the number of ordered sequences of the elements of a set. A *permutation* of the elements is an ordered sequence of the elements.

2.6.4. Example. Consider the set $S = \{a, b, c\}$.

In this case abc, acb, bac, bca, cab, and cba are all of the permutations of the elements of S.

2.6.5. Theorem. The number of permutations of n different elements is $n!$, where

$$n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n.$$

2.6.6. Theorem. The number of permutations of objects of

$$n = k_1 + k_2 + \ldots + k_r$$

which k_1 are of one type, k_2 are of a second type, ..., and k_r are of an rth type is

$$\frac{n!}{k_1! \cdot k_2! \cdot \ldots \cdot k_r!}.$$

2.6.7. Example. There are 5 balls in a box, 3 of them are red and 2 are white.

These balls can be arranged in

$$\frac{5!}{3! \cdot 2!} = \frac{4 \cdot 5}{2} = 10$$

different ways.

2.6.8. Definition. Another counting problem of interest is the number of subsets of r elements that can be selected from a set of n elements. Here, order is not important. These are called *combinations*.

2.6.9. Theorem. The number of combinations, subsets of size r that can be selected from a set of n elements, is denoted as C_n^r, where

$$C_n^r = \binom{n}{r} = \frac{n!}{r! \cdot (n-r)!}.$$

2.6.10. Example. A bin of 30 manufactured parts contains three defective parts and 27 nondefective parts. A sample of three parts is selected from the 30 parts without replacement. That is, each part can only be selected once and the sample is a subset of the 30 parts. How many different samples are there of size three that contain exactly two defective parts?

A subset containing exactly two defective parts can be formed by first choosing the two defective parts from the three defective parts. Using the previous theorem, we get that

$$\binom{3}{2} = \frac{3!}{2! \cdot 1!} = 3.$$

Then, the second step is to select the remaining one part from the 27 acceptable parts in the bin. The second step can be completed in

$$\binom{27}{1} = \frac{27!}{1! \cdot 26!} = 27.$$

Therefore, from the multiplication rule, the number of subsets of size three that contain exactly two defective items is

$$3 \cdot 27 = 81.$$

2.7. Solved problems

2.7.1. Exercise. Consider a random experiment of tossing a coin twice. Find the sample space S, if we wish to observe the exact sequences of heads and tails obtained.

Solution:

The sample space S, is given by

$$S = \{HH, HT, TH, TT\}$$

where, for example, HT indicates a head on the first throw and a tail on the second throw. We remark that there are four sample points in S.

2.7.2. Exercise. Consider a random experiment of tossing a coin three times. Find the sample space S, if we wish to observe the exact sequences of heads and tails obtained.

Solution:

The sample space S, is given by

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

where, for example, HTH indicates a head on the first and third throws and a tail on the second throw. We remark that there are eight sample points in S.

2.7.3. Exercise. Consider an experiment of drawing two cards at random from a bag containing four cards marked with the integers 1 through 4. Find the sample space S, of the experiment if the first card is replaced before the second is drawn.

Solution:

The sample space S, contains 16 ordered pairs:

$$\{(i, j) \,|\, 1 \leq i \leq 4 \text{ and } 1 \leq j \leq 4\}$$

where the first number indicates the first number drawn. Thus the sample space is

$$S = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4),$$
$$(3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}.$$

2.7.4. Exercise. A car repair is performed either on time or late and either satisfactorily or unsatisfactorily. What is the sample space for a car repair?

Solution:

The sample space is

$$S = \{(T,S),(T,U),(L,S),(L,U)\}$$

where T means on time, L means late, S means satisfactorly and U means unsatisfactorly.

2.7.5. Exercise. An automobile manufacturer provides vehicles equipped with selected options. Each vehicle is ordered with or without an automatic transmission, with or without air conditioning, with one of three choices of a stereo system and with one of four exterior colors (red, blue, white, black). If the sample space consists of the set of all possible vehicle types, what is the number of outcomes in the sample space?

Solution:

There are 2 options for transmission, two options for air conditioning, three options for stereo system and four options for colours thus the sample space contains

$$2 \cdot 2 \cdot 3 \cdot 4 = 48$$

outcomes.

2.7.6. Exercise. An experiment consists of rolling a die until a 6 is obtained. Find the sample space S, if we are interested in all possibilities.

Solution:

The sample space is

$$\begin{aligned} S = \{&6,\\ &(1,6),(2,6),(3,6),(4,6),(5,6),(4,2),(4,3),(4,4),\\ &(1,1,6),(1,2,6),(1,3,6),(1,4,6),(1,5,6),(2,1,6),\dots,(5,5,6)\} \end{aligned}$$

where the first line indicates that a 6 is obtained in one throw, the second line indicates that a 6 is obtained in two throws, and so forth.

2.7.7. Exercise. Find the sample space for the experiment consisting of measurement of the voltage output v from a transducer, the maximum and minimum of which are $+5$ and -5 volts, respectively.

Solution:

The sample space is

$$S = \{v \in \mathbb{R} \mid -5 \leq v \leq 5\} = [-5; 5].$$

2.7.8. Exercise. An experiment has five outcomes, I, II, III, IV, and V such that

$$P(I) = 0.08, \quad P(II) = 0.20, \quad P(III) = 0.33.$$

what are the possible values for the probability of outcome V if outcomes IV and V are equally likely?

Solution:

The sample space is

$$S = \{I, II, III, IV, V\}.$$

The sum of the probability of all outcomes is 1 thus

$$1 = P(I) + P(II) + P(III) + P(IV) + P(V).$$

Substituting data and supposing that $P(IV) = P(V) = x$ we get that

$$1 = P(I) + P(II) + P(III) + P(IV) + P(V) = 0.08 + 0.2 + 0.33 + 2x.$$

Solving the equation we get that $x = 0.195$.

2.7.9. Exercise. A company's advertising expenditure is either low with probability 0.3, average with probability 0.5, or high with probability p. What is the value of p?

Solution:

The sample space is

$$S = \{\text{low}, \text{average}, \text{high}\}.$$

Since $0.3 + 0.5 + p = 1$, we have $p = 0.2$.

2.7.10. Exercise. Two fair dice are thrown, one red and one blue. What is the probability that the red die has a score that is strictly greater than the score of the blue die?

Solution:

The sample space is

$$\begin{aligned} S = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \end{aligned}$$

where (i,j) $(i,j = 1,2,3,4,5,6)$ means that the score on the top of red die is i and the score on the top of blue die is j. The total outcomes is $n(S) = 6^2 = 36$. Let denote A when the red die has a score stricly than the score of the blue die, that is

$$\begin{aligned} A = \{(2,1),(3,1),\ &(3,2),(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(5,4),\\ &(6,1),(6,2),(6,3),(6,4),(6,5)\}\,. \end{aligned}$$

The probability of event A is

$$P(A) = \frac{n(A)}{n(S)} = \frac{15}{36} = \frac{5}{12}.$$

2.7.11. Exercise. Let A be the event that a person is female, let B be the event that a person has black hair, and let C be the event that a person has brown eyes. Describe the kinds of people in the following events:

a) $A \cap B$;

b) $A \cup \overline{C}$;

c) $\overline{A} \cap B \cap C$;

d) $A \cap (B \cup C)$.

<u>Solution:</u>

Let denote $A = \{\text{female}\}$, $B = \{\text{black hair}\}$, $C = \{\text{brown eyes}\}$.

a) $A \cap B$: females with black hair;

b) $A \cup \overline{C}$: all females and any male who does not brown eyes;

c) $\overline{A} \cap B \cap C$: males with black hair and brown eyes;

d) $A \cap (B \cup C)$: females with either black hair or brown eyes or both.

2.7.12. Exercise. If $P(A) = 0.5$, $P(A \cap B) = 0.1$ and $P(A \cup B) = 0.8$ find the probability of event B.

Solution:

Using the equation

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

we get that

$$0.8 = 0.5 + P(B) - 0.1$$

thus $P(B) = 0.4$.

2.7.13. Exercise. A car repair can be performed either on time or late and either satisfactorily or unsatisfactorily. The probability of a repair being on time and satisfactory is 0.2. The probability of a repair being on time is 0.7. The probability of a repair being satisfactory is 0.4. What is the probability of a repair being late and unsatisfactory?

Solution:

Let denote event on time T and event satisfactorily S. Substitute the data, we get that

$$P(T \cap S) = 0.2, \quad P(T) = 0.7, \quad P(S) = 0.4.$$

By the de-Morgan identity we have

$$\overline{T} \cap \overline{S} = \overline{T \cup S}$$

hence

$$P(\overline{T} \cap \overline{S}) = P()\overline{T \cup S}) = 1 - P(T \cup S).$$

Using the equation

$$P(T \cup S) = P(T) + P(S) - P(T \cap S).$$

Substituting the data, we get that

$$P(T \cup S) = P(T) + P(S) - P(T \cap S) = 0.7 + 0.4 - 0.2 = 0.9$$

thus

$$P(\overline{T} \cap \overline{S}) = P(\overline{T \cup S}) = 1 - P(T \cup S) = 1 - 0.9 = 0.1.$$

2.7.14. Exercise. A bag contains 200 balls that are either red or blue and either dull or shiny. There are 55 shiny red balls, 91 shiny balls, and 79 red balls. A ball is chosen at random.

a) What is the probability that it is either a shiny ball or a red ball?

b) What is the probability that it is a dull blue ball?

Solution:

Let denote the event A that a red ball has been chosen and let denote B the event that a shiny ball has been chosen.

a) The probability of event A is $P(A) = \frac{79}{200}$.
The probability of event B is $P(B) = \frac{91}{200}$.
The probability of event $A \cap B$ is $P(A \cap B) = \frac{55}{200}$.
Using the equation

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Substituting the data, we get that

$$P(A \cup B) = \frac{79}{200} + \frac{91}{200} - \frac{55}{200} = \frac{115}{200}.$$

b) The probability that we chose blue ball

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - \frac{115}{200} = \frac{85}{200}.$$

2.7.15. Exercise. A day's production of 900 manufactured parts contains 100 parts that do not meet customer requirements. Three parts are selected randomly without replacement from the batch. what is the probability that the first two are defective and the third is not defective?

Solution:

The probability is

$$\frac{99}{900} \cdot \frac{98}{899} \cdot \frac{800}{898}.$$

2.7.16. Exercise. A car repair is either on time or late and either satisfactory or unsatisfactory. If a repair is made on time, then there is a probability of 0.8 that it is satisfactory. There is a probability of 0.7 that a repair will be made on time. What is the probability that a repair is made on time and is satisfactory?

Solution:

Let denote B the event that the car repaired has been performed on time.

Let denote A the event that the car repaired has been performed satisfactorily.

Using the notations, we get that

$$P(B) = 0.7 \qquad P(A|B) = 0.8.$$

We have to find the probability of event $A \cap B$. Applying the definition of conditional probability we get that

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Substituting data, we have

$$0.8 = \frac{P(A \cap B)}{0.7} \qquad \Rightarrow \qquad P(A \cap B) = 0.56.$$

2.7.17. Exercise. There is a 60 per cent chance that the event A will occur. If event A does not occur, then there is a 10 per cent chance that B will occur. What is the probability that at least one of the events A or B occurs?

Solution:

By the text we know that

$$P(A) = 0.6 \qquad \text{and} \qquad P\big(B|\overline{A}\big) = 0.1.$$

We have to find the probability of event $A \cup B$.
By the definition of the conditional probability we get that

$$0.1 = P\big(B|\overline{A}\big) = \frac{P\big(B \cap \overline{A}\big)}{P\big(\overline{A}\big)}.$$

Substitute the data, we get that

$$0.1 = \frac{P\big(B \cap \overline{A}\big)}{0.4},$$

that is $P\big(B \cap \overline{A}\big) = 0.04$.
Since $B \setminus (B \cap A) = B \cap \overline{A}$ and $B \subset (B \cap A)$, we have

$$P\big(B \cap \overline{A}\big) = P\big(B \setminus (B \cap A)\cap\big) = P(B) - P(B \cap A).$$

Using the addition rule we get that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.04 = 0.64,$$

which is the searched probability.

2.7.18. Exercise. A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected randomly without replacement from the batch. What is the probability that the second part is defective given that the first part is defective?

Solution:

Let B denote the event that the first part selected is defective, and let A denote the event that the second part selected is defective. The probability needed can be expressed as $P(A|B)$. If the first part is defective, prior to selecting the second part, the batch contains 849 parts, of which 49 are defective. Therefore

$$P(A|B) = \frac{49}{849}.$$

2.8. Problems without solution

2.8.1. Exercise. An experiment has three outcomes, I, II, and III. If outcome I is twice as likely as outcome III, and outcome III is three times as likely as outcome II, what are the probability values of the three outcomes?

2.8.2. Exercise. The rise time of a reactor is measured in minutes (and fractions of minutes). Let the sample space for the rise time of each batch be positive, real numbers. Consider the rise times of two batches. Let A denote the event that the rise time of batch 1 is less than 72 minutes, and let B denote the event that the rise time of batch 2 is greater than 50 minutes. Describe the sample space for the rise time of two batches graphically and show each of the following events on a two-dimensional plot: $A\cap$ and $A \cup B$.

2.8.3. Exercise. When a company introduces initiatives to reduce its carbon footprint, its costs will either increase, stay the same, or decrease. Suppose that the probability that the costs increase is 0.05, and the probability that the costs stay the same is 0.2.

a) What is the probability that costs will decrease?

b) What is the probability that costs will not increase?

2.8.4. Exercise. A bag contains balls that are either red or blue and either dull or shiny. What is the sample space when a ball is chosen from the bag?

2.8.5. Exercise. A manager supervises the operation of three power plants, plant X, plant Y, and plant Z. At any given time, each of the three plants can be classified as either generating electricity (1) or being idle (0). What is the sample space?

2.8.6. Exercise. A fair coin is tossed three times. What is the probability that exatly two heads will be obtained?

2.8.7. Exercise. Find the sample space for the experiment consisting of measurement of the voltage output v from a transducer, the maximum and minimum of which are -10 and $+10$ volts, respectively. Is this sample space discrete or continuous?

2.8.8. Exercise. A sample space contains 10 equally likely outcomes. If the probability of event A is 0.3, how many outcomes are in event A?

2.8.9. Exercise. A ball is chosen at random from a bag containing 100 balls that are either red or blue and either dull or shiny. There are 30 red shiny balls and 50 blue balls.

a) What is the probability of the chosen ball being shiny conditional on it being red?
b) What is the probability of the chosen ball being dull conditional on it being red?

2.8.10. Exercise. A gene can be either type A or type B, and it can be either dominant or recessive. If the gene is type B, then there is a probability of 0.3 that it is dominant. There is also a probability of 0.2 that a gene is type B and it is dominant. What is the probability that a gene is of type A?

2.8.11. Exercise. There is a 5% probability that the plane used for a commercial flight has technical problems, and this causes a delay in the flight. If there are no technical problems with the plane, then there is still a 30% probability that the flight is delayed due to all other reasons. What is the probability that the flight is delayed?

2.8.12. Exercise. In a process that manufactures aluminum cans, the probability that a can has a flaw on its side is 0.03, the probability that a can has a flaw on the top is 0.05, and the probability that a can has a flaw on both the side and the top is 0.01. What is the probability that a randomly chosen can has a flaw? What is the probability that it has no flaw?

2.8.13. Exercise. The probability that a bearing fails during the first month of use is 0.08. What is the probability that it does not fail during the first month?

2.8.14. Exercise. A batch of 300 containers for frozen orange juice contains five that are defective. Two are selected, at random, without replacement from the batch.

a) What is the probability that the second one selected is defective given that the first one was defective?
b) What is the probability that both are defective?
c) What is the probability that both are acceptable?

Chapter 3

Discrete random variables

3.1. Some basic notions

3.1.1. Definition. A random variable is a function that associates a real number with each element in the sample space. We shall use a capital letter, say X, to denote a random variable and its corresponding small letter, x in this case, for one of its values.

3.1.2. Example. Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. Let random variable X be the number of red balls. The possible outcomes, that is the sample space is

$$S = \{\text{RR}, \text{RB}, \text{BR}, \text{BB}\}.$$

The values of the random variable

$$X = \{0, 1, 2\}.$$

3.1.3. Definition. A random variable is called a *discrete random variable* if the range is finite or countable infinite.

3.1.4. Example. Discrete random variables: number of scratches on a surface, proportion of defective parts among 100 tested, number of transmitted bits received in error, number of chairs in the classroom.

3.1.5. Example. A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used. Let the random variable X denote the number of lines in use. Then X can assume any of the integer values 0 through 48. When the system is observed, if 10 lines are in use, $x = 10$.

3.1.6. Remark. Many physical systems can be modeled by random variables. The distribution of the random variables can be used in different applications and examples. In this chapter, we present the analysis of several random experiments and discrete random variables that frequently arise in applications.

3.2. Probability mass function and cumulative distribution function

3.2.1. Definition. The *probability distribution* of a random variable X is a description of the probabilities associated with the possible values of X.

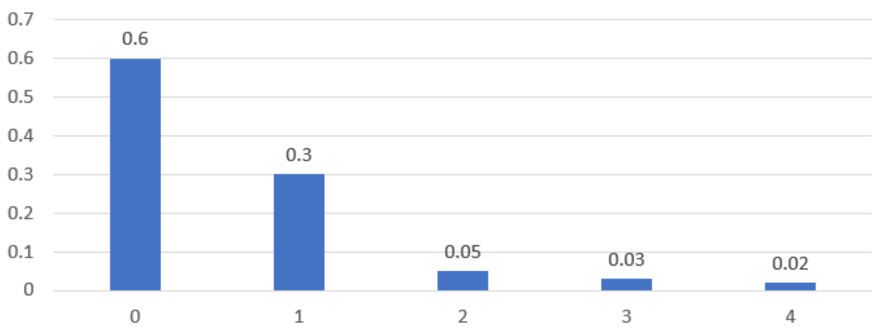
3.2.2. Remark. For a discrete random variable, the distribution is often specified by just a list of the possible values along with the probability of each. In some cases, it is convenient to express the probability in terms of a formula.

3.2.3. Example. There is a chance that a bit transmitted through a digital transmission channel is received in error. Let X equal the number of bits in error in the next four bits transmitted. The possible values for X are $\{0, 1, 2, 3, 4\}$. Based on a model for the errors that is presented in the following section, probabilities for these values will be determined. Suppose that the probabilities are

$$P(X = 0) = 0.6; \quad P(X = 1) = 0.3; \quad P(X = 2) = 0.05;$$

$$P(X = 3) = 0.03; \quad P(X = 4) = 0.02.$$

The probability distribution of X is specified by the possible values along with the probability of each. A graphical description of the probability distribution of X is shown in figure below



3.2.4. Definition. For a discrete random variable X with possible values

$$x_1, x_2, \ldots, x_n$$

a *probability mass function* (PMF) is a function f such that

- $f(x_i) \geq 0$ for all $i = 1, 2, \ldots .n$;
- $\sum\limits_{i=1}^{n} f(x_i) = 1$;
- $f(x_i) = P(X = x_i)$ for all $i = 1, 2, \ldots, n$.

3.2.5. Remark. The probability mass function is one way to describe the distribution of a discrete random variable. As we will see later on, probability mass function cannot be defined for continuous random variables but there is another important function, namely the cumulative distribution function (CDF) of a random variable. The advantage of the cumulative distribution function is that it can be defined for any kind of random variable (discrete, continuous, and mixed).

3.2.6. Definition. The *cumulative distribution function* F of random variable X is defined as $F_X(x) = P(X < x)$, for all $x \in \mathbb{R}$.

3.3. Expected value and variance

3.3.1. Definition. The *expected value* of the random variable X is

$$E(X) = \sum_{x=1}^{\infty} x \cdot f(x),$$

where f is the probability mass function of the random variable.

3.3.2. Remark. The expected value (also called expectation, expectancy, expectation operator, mathematical expectation, mean, average or first moment) is a generalization of the weighted average.

Informally, the expected value is the arithmetic mean of a large number of independently selected outcomes of a random variable.

3.3.3. Example. Let X represent the outcome of a roll of a fair six-sided die. More specifically, X will be the number of pips showing on the top face of the die after the toss. The possible values for X are $1, 2, 3, 4, 5$, and 6, all of which are equally likely with a probability of $1/6$. The expected value is

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = 3.5.$$

3.3.4. Example. The roulette game consists of a small ball and a wheel with 38 numbered pockets around the edge. As the wheel is spun, the ball bounces around randomly until it settles down in one of the pockets. Suppose random variable X represents the (monetary) outcome of a $\$1$ bet on a single number ("straight up" bet). If the bet wins (which happens with probability $1/38$ in American roulette), the payoff is $\$35$; otherwise the player loses the bet. The expected profit from such a bet will be

$$E(\text{ gain from } \$1 \text{ bet }) = -1 \cdot \frac{37}{38} + 35 \cdot \frac{1}{38} = -\frac{1}{19}.$$

That is, the expected value to be won from a $\$1$ bet is $-1/19$. Therefore, in 190 bets, the net loss will probably be about $\$10$.

3.3.5. Theorem. If X and Y are discrete random variables then

$$E(X + Y) = E(X) + E(Y).$$

3.3.6. Theorem. If X is a discrete random variable and $a \in \mathbb{R}$ then

$$E(a \cdot X) = a \cdot E(X).$$

3.3.7. Theorem. If X and Y are discrete random variables such that $X \leq Y$ then

$$E(X) \leq E(Y).$$

3.3.8. Theorem. If X is a discrete random variable with probability mass function h then

$$E\big(h(X)\big) = \sum_{x=1}^{\infty} g(x) \cdot f(x),$$

where f is the probability mass function of the random variable.

3.3.9. Definition. The *variance* of the random variable X is

$$Var(X) = E\big(X - E(X)\big)^2.$$

3.3.10. Theorem. The variance of the random variable X is

$$Var(X) = E(X)^2 - \big(E(X)\big).$$

Proof: By definition

$$Var(X) = E\big(X - E(X)\big)^2 = E\big(X^2 - 2X \cdot E(X) + E^2(X)\big).$$

Using the properties of the expected value we get that

$$Var(X) = E(X^2) - 2 \cdot E^2(X) + E^2(X) = E(X^2) - E^2(X)$$

and we have the formula to be proved. ■

3.3.11. Example. Two new product designs are to be compared on the basis of revenue potential. Marketing feels that the revenue from design A can be predicted quite accurately to be \$4 million. The revenue potential of design B is more difficult to assess. Marketing concludes that there is a probability of 0.2 that the revenue from design B will be \$7 million, but there is a 0.8 probability that the revenue will be only \$4 million. Which design do you prefer?

Let X denote the revenue from design A. Because there is no uncertainty in the revenue from design A, we can model the distribution of the random variable X as \$3 million with probability 1. Therefore, $E(X) = 4$ million.

Let Y denote the revenue from design B. The expected value of Y in millions of dollars is

$$E(Y) = 7 \cdot 0.2 + 4 \cdot 0.8 = 4.6.$$

Because $E(Y)$ exceeds $E(X)$, we might prefer design B.

However, the variability of the result from design B is larger. Since

$$E(Y^2) = 7^2 \cdot 0.2 + 4^2 \cdot 0.8 = 49 \cdot 0.2 + 16 \cdot 0.8 = 22.6,$$

thus

$$Var(Y) = 22.5 - 4.6^2 = 1.34.$$

The standard deviation is

$$STD(Y) = \sqrt{1.34} \approx 1.1576.$$

3.4. Solved problems

3.4.1. Exercise. Consider the experiment of throwing a fair die. Let X be the random variable which assigns 1 if the number that appears is even and 0 if the number that appears is odd.

a) What is the range of X?

b) Find $P(X = 1)$ and $P(X = 0)$.

c) Determine the expected value of X.

d) Calculate the variance of X.

e) Find the standard deviation of X.

<u>**Solution:**</u>

a) The sample space S on which X is defined consists of 6 points which are equally likely:
$$S = \{1, 2, 3, 4, 5, 6)\}.$$
The range of random variable X is $\{0, 1\}$.

b) The probabilities:
$$P(X = 0) = \frac{3}{6} = \frac{1}{2}, \quad P(X = 1) = \frac{3}{6} = \frac{1}{2}.$$

c) The expected value of X is
$$E(X) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}.$$

d) Since $E(X^2) = 0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2} = \frac{1}{2}$, we have
$$Var(X) = E(X^2) - \big(E(X)\big)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

e) The standard deviation of X is
$$STD(X) = \sqrt{Var(X)} = \sqrt{\frac{1}{4}} = \frac{1}{2}.$$

3.4.2. Exercise. Consider the function
$$f(k) = \frac{2k+1}{25}, \quad k = 0, 1, 2, 3, 4.$$

a) Verify that f is a probability mass function.

b) Calculate the expected value.

c) Determine the variance.
d) Find the standard deviation.

Solution:

a) The values of the function f are

$$f(0) = \frac{1}{25} = 0.04$$

$$f(1) = \frac{2 \cdot 1 + 1}{25} = \frac{3}{25} = 0.12$$

$$f(2) = \frac{2 \cdot 2 + 1}{25} = \frac{5}{25} = 0.2$$

$$f(3) = \frac{2 \cdot 3 + 1}{25} = \frac{7}{25} = 0.28$$

$$f(4) = \frac{2 \cdot 4 + 1}{25} = \frac{9}{25} = 0.36.$$

Since all probabilites are non-negative values and

$$f(0)+f(1) + f(2) + f(3) + f(4) =$$
$$= 0.04 + 0.12 + 0.2 + 0.28 + 0.36 = 1$$

thus the function f is a probability mass function.

b) The expected value is

$$E(X) = 1 \cdot 0.12 + 2 \cdot 0.2 + 3 \cdot 0.28 + 4 \cdot 0.36 = 2.8.$$

c) Since

$$E(X^2) = 1^2 \cdot 0.12 + 2^2 \cdot 0.2 + 3^2 \cdot 0.28 + 4^2 \cdot 0.36 = 9.2,$$

we get that

$$Var(X) = 9.2 - 2.8^2 = 1.36.$$

d) The standard deviation is

$$STD(X) = \sqrt{1.36} \approx 1.166.$$

3.4.3. Exercise. Consider the probability mass function

$$f(k) = A \cdot \sqrt{k}, \quad k = 1, 4, 9.$$

Determine the value of A.

Solution:

The values of the function f are

$$f(1) = A \cdot \sqrt{1} = A$$

$$f(4) = A \cdot \sqrt{4} = 2A$$

$$f(9) = A \cdot \sqrt{9} = 3A.$$

Since

$$f(1) + f(4) + f(9) = A + 2A + 3A = 6A$$

we have $6A = 1$ thus $A = \frac{1}{6}$.

3.4.4. Exercise. We flip a fair coin twice. Let X be a random variable denoted by the number of heads that appear.

a) Describe the distribution of the random variable.
b) Determine the probability mass function.
c) Plot the graph of probability mass function.
d) Find the cumulative distribution function.
e) Plot the graph of cumulative distribution function.
f) Calculate the expected value.
g) Determine the variance.
h) What is the value of standard deviation.

Solution:

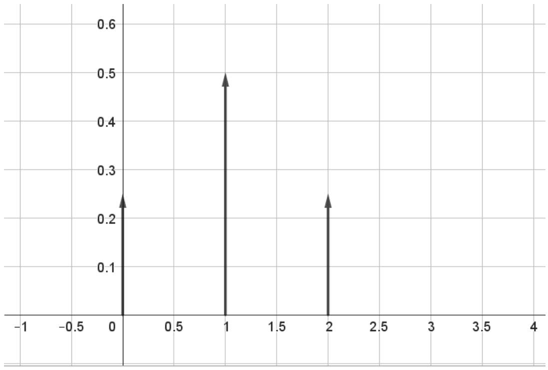
a) The values of the random variable are $0; 1; 2$. The probability of the values:

$$P(X = 0) = \frac{1}{4}, \quad P(X = 1) = \frac{1}{2}, \quad P(X = 2) = \frac{1}{4}.$$

b) The probability mass function is

$$f(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ \frac{1}{4} & \text{if} \quad 0 < x \leq 1 \\ \frac{1}{2} & \text{if} \quad 1 < x \leq 2 \\ \frac{1}{4} & \text{if} \quad x > 2. \end{cases}$$

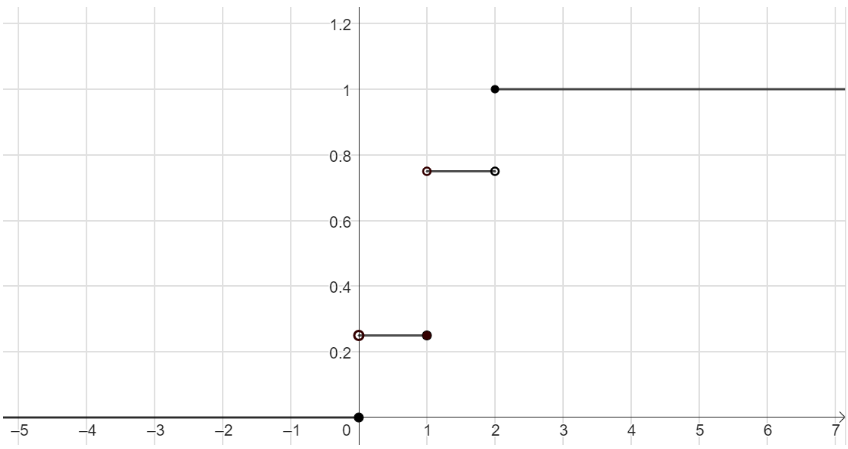
c) The graph of probability mass function:



d) Find the cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ \frac{1}{4} & \text{if} \quad 0 < x \leq 1 \\ \frac{1}{4} + \frac{1}{2} = \frac{3}{4} & \text{if} \quad 1 < x \leq 2 \\ \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 & \text{if} \quad x > 2. \end{cases}$$

e) The graph of cumulative distribution function is



f) The expected value is

$$E(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1.$$

g) To the value of variance in first step we have to calculate $E(X^2)$. Since

$$E(X^2) = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4}$$

thus the variance is

$$Var(X) = E(X^2) - E^2(X) = 1.5 - 1^2 = 0.5.$$

h) Standard deviation is the root of the variance, that is

$$STD(X) = \sqrt{0.5} \approx 0.7071.$$

3.4.5. Exercise. We toss a fair coin three times. Let denote variable X the number of heads!

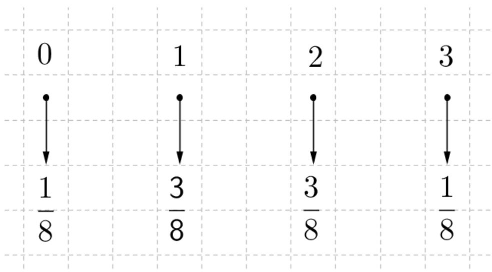
a) Determine the distribution of random variable, that is find the values of all probabilities.
b) Calculate the expected value.
c) Find the variance and standard deviation.
d) Determine the cumulative distribution function.
e) Draw the graph of cumulative distribution function.

<u>**Solution:**</u>

a) If we toss a coin three times then the sample space is

$$S = \{HHH, HTH, HHT, HTT, THH, TTH, THT, TTT\}.$$

The number of heads are 0, 1, 2 or 3. The probabilities of values:



b) The expected values is

$$E(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3}{2} = 1.5.$$

This means that if our experiment is that if a fair coin is flipped three times we toss it up and perform the experiment several times and write everything down the number of heads in the experiment then the average of the values obtained in this way is for 1.5 is approaching.

c) Since

$$E(X^2) = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} = 3,$$

we have

$$Var(X) = E(X^2) - \big(E(X)\big)^2 = 3 - \left(\frac{3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}.$$

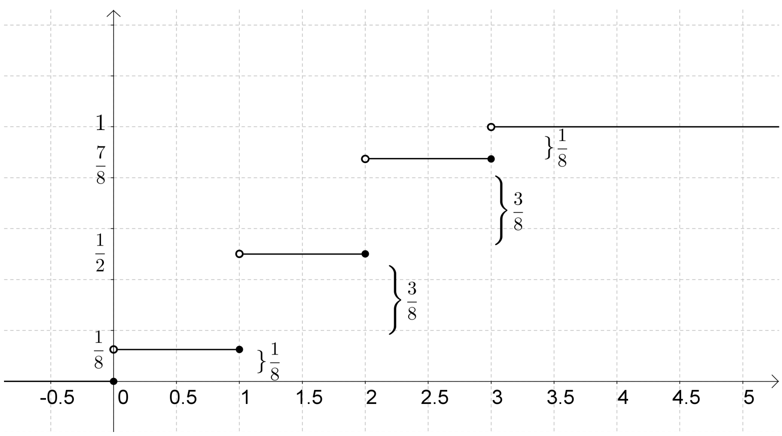
The standard deviation is

$$STD(X) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \approx 0.866.$$

d) The cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \frac{1}{8}, & \text{if } 0 < x \leq 1 \\ \frac{1}{2}, & \text{if } 1 < x \leq 2 \\ \frac{7}{8}, & \text{if } 2 < x \leq 3 \\ 1, & \text{if } x > 3. \end{cases}$$

e) The graph of the distribution function is a step function, where the limit at minus infinity is 0, the limit at infinity is 1, and the height of each step is the value of each probability:



3.4.6. Exercise. We roll a fair die. Let denote random variable X the number on top.

a) Determine the distribution of random variable, that is find the values of all probabilities.

b) Calculate the expected value.

c) Find the variance and standard deviation.

d) Determine the cumulative distribution function.

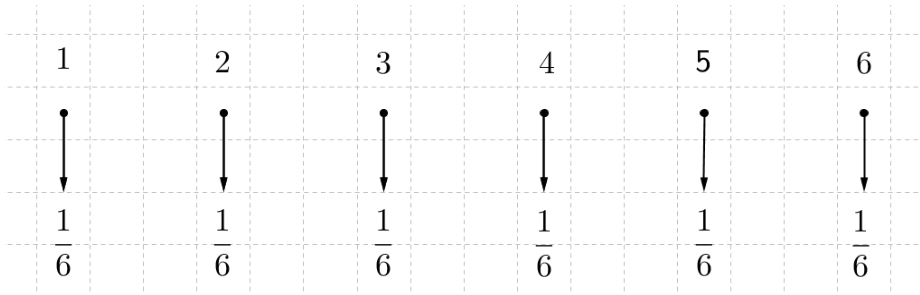
e) Draw the graph of cumulative distribution function.

Solution:

a) The sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

All value occur with same probability since the die is fair thus the probabilities for each value are $\frac{1}{6}$:



b) The expected value is

$$E(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{21}{6} = 3.5.$$

This means that if our experiment is to toss a regular die once and perform the experiment several times and write down the number rolled for each attempt, then the average of the values obtained in this way is closed to 3.5.

c) Since

$$E(X^2) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6} = \frac{91}{6},$$

we have

$$Var(X) = E(X^2) - (E(X))^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

The standard deviation is

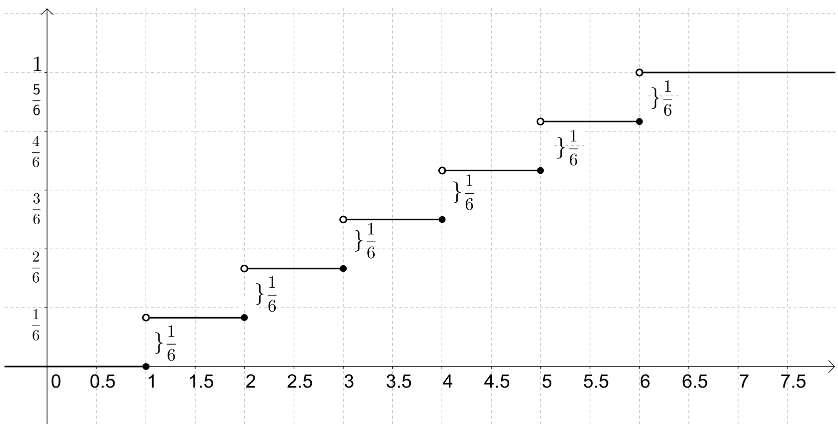
$$Var(X) = \sqrt{\frac{35}{12}} \approx 1.63.$$

.

d) The cumulative distribution function is as follow

$$F(x) = \begin{cases} 0, & \text{if } x \leq 1 \\ 1/6, & \text{if } 1 < x \leq 2 \\ 2/6, & \text{if } 2 < x \leq 3 \\ 3/6, & \text{if } 3 < x \leq 4 \\ 4/6, & \text{if } 4 < x \leq 5 \\ 5/6, & \text{if } 5 < x \leq 6 \\ 1, & \text{if } x > 6. \end{cases}$$

e) The graph of the cumulative distribution function:



3.4.7. Exercise. When examining bicycles, it was determined that 5 out of 100 bicycles need to be repaired after 1 year, 20 after 2 years, 60 after 3 years, and the others after 4 years. Let denote random variable X the time elapsed until the first reparation.

a) Determine the distribution of random variable.

b) Calculate the expected value.

c) Find the variance.

d) Compute the standard deviation.

e) Determine the cumulative distribution function.

f) Draw the graph of cumulative distribution function.

Solution:

a) The sample space is

$$S = \{1, 2, 3, 4\}.$$

The probabilities of the values:

1	2	3	4
$\frac{5}{100}$	$\frac{20}{100}$	$\frac{60}{100}$	$\frac{15}{100}$

b) The expected value is

$$E(X) = 1 \cdot \frac{5}{100} + 2 \cdot \frac{20}{100} + 3 \cdot \frac{60}{100} + 4 \cdot \frac{15}{100} = 2.85.$$

Thus, on average, a bicycle breaks down for the first time after 2.85 years.

c) Since

$$E(X^2) = 1^2 \cdot \frac{5}{100} + 2^2 \cdot \frac{20}{100} + 3^2 \cdot \frac{60}{100} + 4^2 \cdot \frac{15}{100} = 8.65,$$

we have

$$Var(X) = E(X^2) - \big(E(X)\big)^2 = 8.65 - 2.85^2 = 0.5275.$$

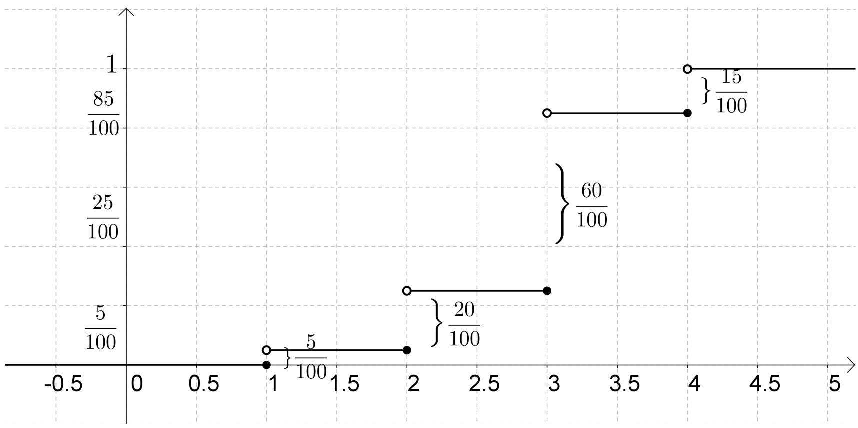
d) The standard deviation is

$$STD(X) = \sqrt{0.5275} \approx 0.7263.$$

e) The cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x \leq 1 \\ \frac{5}{100}, & \text{if } 1 < x \leq 2 \\ \frac{25}{100}, & \text{if } 2 < x \leq 3 \\ \frac{85}{100}, & \text{if } 3 < x \leq 4 \\ 1, & \text{if } x > 4. \end{cases}$$

f) The graph of cumulative distribution function is as follows



3.4.8. Exercise. There are three good parts and two bad parts in a box. We will choose three of them at random, without repetition. Let the value of the random variable X be the number of selected good parts.

a) Write down the distribution of the probability variable.

b) Determine the expected value.

c) Compute the variance of the random variable.

d) Calculate the standard deviation of the random variable.

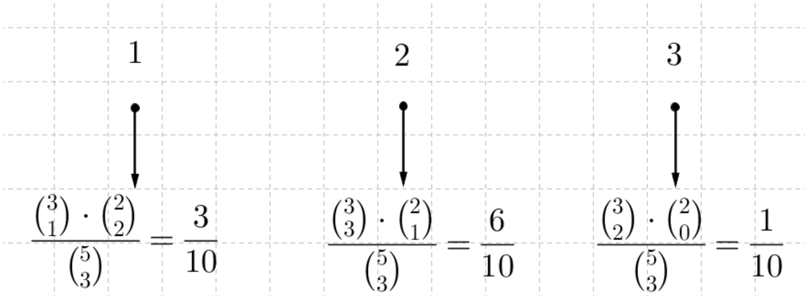
e) What is the probability that there are at least 2 good parts among selected?

<u>**Solution:**</u>

a) The values of the random variable are:

$$1, 2, 3.$$

The probability of the values are:



b) The expected value is

$$E(X) = 1 \cdot \frac{3}{10} + 2 \cdot \frac{6}{10} + 3 \cdot \frac{1}{10} = 1.8.$$

c) Since

$$E(X^2) = 1^2 \cdot \frac{3}{10} + 2^2 \cdot \frac{6}{10} + 3^2 \cdot \frac{1}{10} = 3.6,$$

we have

$$Var(X) = E(X^2) - \big(E(X)\big)^2 = 3.6 - 1.8^2 = 0.36.$$

d) The standrad deviation is

$$STD(X) = \sqrt{0.36} = 0.6.$$

e) The probability that there are at least 2 good parts among selected is

$$0.6 + 0.1 = 0.7.$$

3.4.9. Exercise. Do they form a probability distribution

$$p^2;\ 2pq;\ q^2$$

if $0 \leq p \leq 1$ and $q = 1 - p$?

Solution:

Since $p^2 \geq 0$, $p \cdot q \geq 0$, $q^2 \geq 0$ and

$$p^2 + 2pq + q^2 = (p+q)^2 = (p+1-p)^2 = 1$$

we get that $p^2, 2pq, q^2$ probability distribution.

3.4.10. Exercise. Do they form a probability distribution

$$p^3;\ 3p^2q;\ 3pq^2;\ q^3$$

if $0 \leq p \leq 1$ and $q = 1 - p$?

Solution:

Since $p^3 \geq 0$, $3p^2 \cdot q \geq 0$, $3pq^2 \geq 0$, $q^3 \geq 0$ and

$$p^3 + 3p^2q + 3pq^2 + q^3 = (p+q)^3 = (p+1-p)^3 = 1$$

we get that $p^3, 3p^2q, 3pq^2, q^3$ probability distribution.

3.4.11. Exercise. The values of the random variable X are $0, 1, 2, 3$. It is known that

$$P(X=0) = 0.1,\ P(X=1) = 0.3,\ P(X=2) = 0.2.$$

a) What is the probability of value 3?

b) Determine the expected value.

Solution:

a) Let denote x the probability of value 3. In this situation

$$0.1 + 0.3 + 0.2 + x = 1 \qquad \Rightarrow \qquad x = 0.4.$$

b) The expected value is

$$E(X) = 0 \cdot 0.1 + 1 \cdot 0.3 + 2 \cdot 0.2 + 3 \cdot 0.4 = 1.9.$$

3.4.12. Exercise. Let X be the number of software bugs detected in a code review session. If X follows a distribution with

$$P(X=k) = \frac{1}{(k+1)\cdot(k+2)} \quad \text{for } k = 0, 1, 2, \ldots,$$

find the range of k where $P(X=k)$ is maximized.

Solution:

Given the probability mass function

$$P(X=k) = \frac{1}{(k+1)\cdot(k+2)}$$

for a discrete random variable X, which represents the number of software bugs detected. We need to find the value of k for which this probability mass function is maximized. In first step we have to Analyze the probability mass function. It is a decreasing function of k because as k increases, both $k+1$ and $k+2$ in the denominator increase, making the overall fraction smaller.

To find the maximum value of the probability mass function, we observe its behavior at the smallest value of k. Since k can be as small as 0 (as k is a non-negative integer), we evaluate the probability mass function at $k=0$.

Evaluating $P(X = k)$ at $k = 0$:

$$P(X = 0) = \frac{1}{(0+1)\cdot(0+2)} = \frac{1}{2}.$$

Since the probability mass function is decreasing for increasing k, the maximum value of $P(X = k)$ is at $k = 0$.

3.4.13. Exercise. In a communication system, let X be the number of packets lost per day. If X follows a distribution with

$$P(X = k) = \frac{2}{3^k},$$

calculate the probability that more than 2 packets are lost on a given day.

Solution:

Given the probability mass function for the number of packets lost per day X as

$$P(X = k) = \frac{2}{3^k} \text{ for } k = 1, 2, 3, \dots,$$

we are to calculate the probability that more than 2 packets are lost, that is, $P(X > 2)$.

Using the complement rule, the probability of more than 2 packets being lost is the complement of the probability of 2 or fewer packets being lost, hence

$$P(X > 2) = 1 - P(X \leq 2).$$

Calculating the probability of losing exactly 1 or 2 packets, we get that

$$P(X = 1) = \frac{2}{3^1} = \frac{2}{3}$$
$$P(X = 2) = \frac{2}{3^2} = \frac{2}{9}.$$

Sum these probabilities we get that

$$P(X \leq 2) = P(X = 1) + P(X = 2) = \frac{2}{3} + \frac{2}{9}.$$

Using the previous results, we get that

$$P(X > 2) = 1 - P(Z \leq 2) = 1 - \left(\frac{2}{3} + \frac{2}{9}\right) = 1 - \frac{6}{9} - \frac{2}{9}$$
$$= 1 - \frac{8}{9} = \frac{1}{9}.$$

The probability that more than 2 packets are lost on a given day is $\frac{1}{9}$.

3.5. Unsolved problems

3.5.1. Exercise. Consider a discrete random variable Y representing the number of defective parts in a batch, where Y follows the distribution

$$P(Y = k) = \frac{1}{2^k} \text{ for } k = 1, 2, 3, \ldots.$$

Determine the value of k for which $P(Y = k)$ first falls below 0.05.

3.5.2. Exercise. Let X be a discrete random variable representing the total load (in tons) on a bridge, with $P(X = k) = \frac{5}{6^k}$ for $k = 1, 2, 3, \ldots$. Find the probability that the load exceeds 4 tons.

3.5.3. Exercise. Let X be a discrete random variable representing the number of heads in 3 coin flips, where each coin flip has a probability of 0.5 of landing heads. Calculate the expected value and the standard deviation of X.

3.5.4. Exercise. A discrete random variable X has a probability mass function defined as follows

$$P(X = k) = \frac{k+1}{10}$$

for $k = 0, 1, 2, 3$. Verify that this is a valid probability mass function, and then calculate the expected value and variance of X.

3.5.5. Exercise. Suppose a discrete random variable X can take on values -1, 0, and 1 with probabilities 0.2, 0.5, and 0.3 respectively. Find the expected value, variance, and compute the probability that $|X|$ is less than 1.

3.5.6. Exercise. Consider a random variable X that takes on values from 1 to 5 with equal probability. Calculate the expected value, variance, and determine and plot the cumulative distribution function of X.

Chapter 4

Special discrete random variables

4.1. Discrete uniform distribution

4.1.1. Remark. The simplest discrete random variable is one that assumes only a finite number of possible values, each with equal probability. A random variable X that assumes each of the values $x_1, x_2, \dots, x_n$, with equal probability $\frac{1}{n}$, is frequently of interest.

4.1.2. Definition. A random variable X has a discrete uniform distribution if each of the n values in its range, say, $x_1, x_2, \dots, x_n$, has equal probability. Then,

$$f(x_i) = \frac{1}{n}.$$

4.1.3. Theorem. Suppose X is a discrete uniform random variable on the consecutive integers $a, a+1, a+2, \dots, b$ for $a \leq b$. The expected value of X is

$$E(X) = \frac{a+b}{2}.$$

Proof: Suppose the range of the discrete random variable X is the consecutive integers $a, a+1, \dots, b$ for $a \leq b$. The range of X contains $b-a+1$ values each with probability $1/(b-a+1)$. Now,

$$E(X) = \sum_{k=a}^{b} k \cdot \frac{1}{b-a+1}$$

Using the algebraic identity

$$\sum_{k=a}^{b} k = \frac{2a + (b-a)\cdot 1}{2} \cdot (b-a+1) = \frac{(b+a)\cdot(b-a+1)}{2} =$$

$$= \frac{b^2 - ab + b + ab - a^2 + a}{2} = \frac{b^2 - a^2 + b + a}{2} =$$

$$= \frac{(b-a)\cdot(b+a) + b + a}{2} = \frac{(b+a)\cdot(b-a+1)}{2}$$

we get that

$$E(X) = \sum_{k=a}^{b} k \cdot \frac{1}{b-a+1} = \frac{(b+a)\cdot(b-a+1)}{2} \cdot \frac{1}{b-a+1} = \frac{a+b}{2},$$

which is the formula to prove. ∎

4.1.4. Theorem. Suppose X is a discrete uniform random variable on the consecutive integers $a, a+1, a+2, \ldots, b$ for $a \leq b$. The variance of X is

$$Var(X) = \frac{(b-a+1)^2 - 1}{12}.$$

4.1.5. Example. Let the random variable X have a discrete uniform distribution on the integers $0 \leq x \leq 100$. In this situation the expected value is

$$E(X) = \frac{0+100}{2} = 50.$$

The variance is

$$Var(X) = \frac{(100-0+1)^2 - 1}{12} = \frac{101^2 - 1}{12} = 850.$$

The standard deviation is

$$STD(X) = \sqrt{850} \approx 29.15.$$

4.2. Binomial distribution

4.2.1. Definition. The **binomial distribution** is a discrete probability distribution that models the number of successes in a fixed number of independent Bernoulli trials. A Bernoulli trial is an experiment with exactly two possible outcomes, typically termed "success" and "failure".

The binomial distribution is characterized by two parameters:

- n: Number of trials.
- p: Probability of success on an individual trial.

The probability of observing exactly k successes in n trials is given by the probability mass function

$$P(X = k) = \binom{n}{k} \cdot p^k \cdot (1 - p)^{n-k},$$

where X is the random variable representing the number of successes, and $\binom{n}{k}$ is the binomial coefficient.

4.2.2. Theorem. The expected value and variance of a binomial distribution are:

- $E(X) = n \cdot p$
- $Var(X) = n \cdot p \cdot (1 - p)$.

4.2.3. Remark. The binomial distribution has extensive applications in various fields, including statistics, biology, finance, and quality control.

4.2.4. Example. Consider the following random experiments and random variables: 1. Flip a coin 100 times. Let X number of heads obtained.

2. A worn machine tool produces 2% defective parts. Let X number of defective parts in the next 25 parts produced.

3. Of all bits transmitted through a digital transmission channel, 5% are received in error. Let X the number of bits in error in the next six bits transmitted.

4. A multiple choice test contains 5 questions, each with four choices, and you guess at each question. Let X the number of questions answered correctly.

These examples illustrate that a general probability model that includes these experiments as particular cases would be very useful.

Each of these random experiments can be thought of as consisting of a series of repeated, random trials: 8 flips of the coin in experiment 1, the production of 20 parts in experiment 3, and so forth.

The random variable in each case is a count of the number of trials that meet a specified criterion. The outcome from each trial either meets the criterion that X counts or it does not; consequently, each trial can be summarized as resulting in either a success or a failure.

For example, in the multiple choice experiment, for each question, only the choice that is correct is considered a success. Choosing any one of the three incorrect choices results in the trial being summarized as a failure.

4.3. Poisson distribution

4.3.1. Definition. The Poisson distribution is a discrete probability distribution that describes the probability of a given number of events occurring in a fixed interval of time or space. It is applicable in situations where events occur independently and the average rate at which they occur is constant.

The probability of observing k events in an interval is given by the formula

$$P(\lambda = k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!},$$

where

- k is the number of occurrences of an event (the function's argument);
- λ is the event rate, also known as the rate parameter (the average number of events in an interval);
- e is Euler's number ($e \approx 2.718$).

4.3.2. Theorem. For a Poisson distribution, the mean and the variance are both equal to λ, that is

$$\text{Mean} = \text{Variance} = \lambda.$$

4.3.3. Theorem. The Poisson distribution has no memory. The probability of an event occurring in the future is independent of how long it has been since the last event.

4.3.4. Example. The Poisson distribution is used in various fields such as physics, finance, medicine, and traffic engineering to model random events in time or space.

4.4. Solved problems

4.4.1. Exercise. We roll a fair die. Let denote random variable X the number on top.

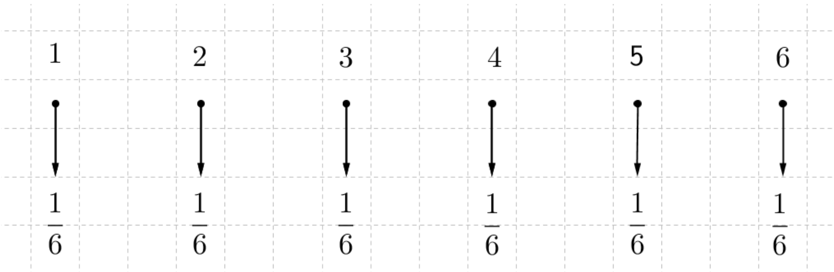
a) Determine the distribution of random variable, that is find the values of all probabilities.
b) Calculate the expected value.
c) Find the variance.
d) Calculate the standard deviation.
e) Determine the cumulative distribution function.
f) Draw the graph of cumulative distribution function.

Solution:

a) The sample space is

$$S = \{1, 2, 3, 4, 5, 6\}.$$

All value occur with same probability since the die is fair thus the probabilities for each value are $\frac{1}{6}$, therefore we have a discrete uniform distribution:



b) The expected value is

$$E(X) = \frac{a+b}{2} = \frac{1+6}{2} = 3.5.$$

c) The variance is

$$Var(X) = \frac{(b-a+1)^2-1}{12} = \frac{(6-1+1)^2-1}{12} = \frac{35}{12}.$$

d) The standard deviation is

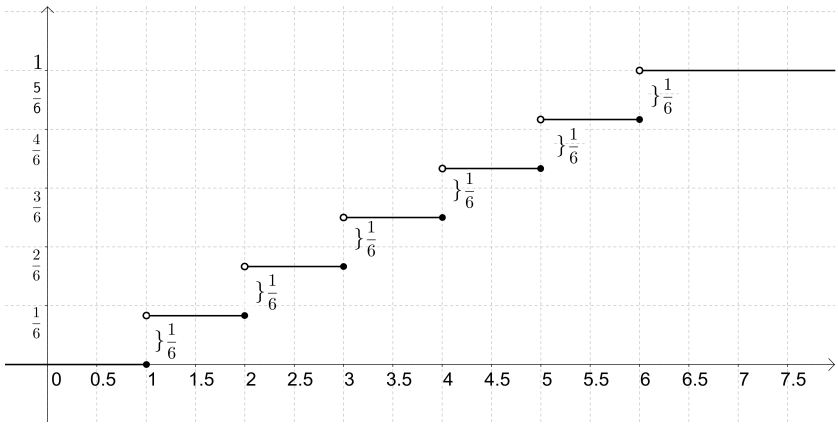
$$Var(X) = \sqrt{\frac{35}{12}} \approx 1.63.$$

.

e) The cumulative distribution function is as follow

$$F(x) = \begin{cases} 0, & \text{if } x \le 1 \\ 1/6, & \text{if } 1 < x \le 2 \\ 2/6, & \text{if } 2 < x \le 3 \\ 3/6, & \text{if } 3 < x \le 4 \\ 4/6, & \text{if } 4 < x \le 5 \\ 5/6, & \text{if } 5 < x \le 6 \\ 1, & \text{if } x > 6. \end{cases}$$

f) The graph of the cumulative distribution function:



4.4.2. Exercise. We have a discrete uniform distribution X such that

$$E(X) = 4 \qquad Var(X) = 2.$$

Determine the parameters a and b.

Solution:

Since the expected value is 4,

$$\frac{a+b}{2} = 4,$$

we have $a + b = 8$, that is $b = 8 - a$.

On the other hand the variance is 2, that is

$$\frac{(b-a+1)^2-1}{12}=2 \quad \Rightarrow \quad (b-a+1)^2=25,$$

thus $b-a+1=5$ or $b-a+1=-5$. Consider $b>a$. In this situation $b-a+1=5$. Using the formula $b=8-a$, we get that

$$8-a-a+1=5 \qquad \Rightarrow \qquad a=2$$

and $b=8-a=8-2=6$.

4.4.3. Exercise. A manufacturing process produces widgets with a 2% defect rate. If a quality control engineer selects 50 widgets at random for inspection:

a) What is the probability that exactly 3 widgets are defective?

b) What is the probability that at least 1 widget is defective?

Solution:

The defect rate is $p=0.02$, the number of widgets is $n=50$.

a) The probability of exactly 3 defective widgets is

$$P(X=3)=\binom{50}{3}\cdot 0.02^3\cdot 0.98^{47}\approx 0.18.$$

b) The probability of at least 1 defective widget is

$$P(X\geq 1)=1-P(X=0)=1-\binom{50}{0}\cdot 0.02^0\cdot 0.98^{50}\approx 0.64.$$

4.4.4. Exercise. Let X be a binomial distribution with parameters $n=7$ and $p=\frac{1}{2}$. Calculate the value of $P(X=3)$.

Solution:

Using the definition of binomial distribution, we get that

$$P(\xi=3)=\binom{7}{3}\cdot\left(\frac{1}{2}\right)^3\cdot\left(1-\frac{1}{2}\right)^{7-3}=0.2734.$$

4.4.5. Exercise. Let X be a binomial distribution with parameters $n=8$ and $p=\frac{2}{3}$.

a) Calculate the probability $P(X=2)$.

b) Find the expected value.

c) Determine the variance.

d) Calculate the standard deviation.

Solution:

a) The probability is

$$P(X=2)=\binom{8}{2}\cdot\left(\frac{2}{3}\right)^2\cdot\left(1-\frac{2}{3}\right)^{8-2}=0.017.$$

b) The expected value is

$$E(X)=n\cdot p=8\cdot\frac{2}{3}=\frac{16}{3},$$

c) The variance is

$$Var(X)=n\cdot p\cdot(1-p)=8\cdot\frac{2}{3}\cdot\frac{1}{3}=\frac{16}{9},$$

d) The standard deviation is

$$STD(X)=\sqrt{n\cdot p\cdot(1-p)}=\sqrt{8\cdot\frac{2}{3}\cdot\frac{1}{3}}=\sqrt{\frac{16}{9}}=\frac{4}{3}.$$

4.4.6. Exercise. We roll a die twelve times. Let denote X the number of 3.

a) Find the expected value.

b) Calculate the variance.

c) Determine the standard deviation.

d) Calculate the probability $P(X<2)$.

Solution:

a) The expected value is

$$E(X)=n\cdot p=12\cdot\frac{1}{6}=2,$$

which means that out of 12 rolls, we expect (on average) to get a 3 twice.

b) The variance is

$$Var(X)=n\cdot p\cdot(1-p)=12\cdot\frac{1}{6}\cdot\frac{5}{6}=\frac{5}{3}.$$

c) The standard deviation is

$$STD(X)=\sqrt{n\cdot p\cdot(1-p)}=\sqrt{12\cdot\frac{1}{6}\cdot\frac{5}{6}}=\sqrt{\frac{5}{3}}.$$

d) The probability is

$$P(X < 2) = P(X = 0) + P(X = 1) = \left(\frac{5}{6}\right)^{12} + 12 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{11} =$$
$$= 0.3813.$$

4.4.7. Exercise. The expected value and standard deviation of a binomially distributed random variable are given by 6 and 2, respectively. Determine the parameters of distribution.

Solution:

Using the formula of the expected value, we get that $n \cdot p = 6$. Applying the formula of the standard deviation, we have $\sqrt{n \cdot p \cdot (1 - p)} = 2$. We have to solve the system of equations below

$$\left.\begin{aligned} n \cdot p &= 6 \\ n \cdot p \cdot (1 - p) &= 4 \end{aligned}\right\} .$$

If we substitute the value of $n{\cdot}p$ to the second equation we get that $6{\cdot}(1{-}p) = 4$, that is $p = \frac{1}{3}$ thus $n = 18$.

4.4.8. Exercise. A customer service center receives an average of 10 calls per hour. What is the probability that they will receive exactly 15 calls in the next hour?

Solution:

Given, $\lambda = 10$ and $k = 15$. The probability of receiving exactly 15 calls is given by the Poisson formula:

$$P(X = k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}.$$

Substituting the given values we get that

$$P(X = 15) = \frac{10^{15} \cdot e^{-10}}{15!} \approx 0.0347.$$

Thus, the probability is approximately 3.47%.

4.4.9. Exercise. A bookstore sells an average of 3 rare books per week. What is the probability that they sell exactly one rare book in a week?

Solution:

Here, $\lambda = 3$ and $k = 1$ thus

$$P(X = 1) = \frac{3^1 \cdot e^{-3}}{1!} = P(X = 1) = 3e^{-3} \approx 0.1494.$$

Therefore, the probability is approximately 14.94

4.5. Unsolved problems

4.5.1. Exercise. Thickness measurements of a coating process are made to the nearest hundredth of a millimeter. The thickness measurements are uniformly distributed with values

$$0.15, 0.16, 0.17, 0.18, 0.19.$$

Determine the mean and variance of the coating thickness for this process.

4.5.2. Exercise. Product codes of 2, 3, or 4 letters are equally likely. What is the mean and standard deviation of the number of letters in 100 codes?

4.5.3. Exercise. In a communication network, each link has a 5% chance of failure. For a particular route consisting of 20 independent links:

a) Calculate the probability that exactly 2 links will fail.

b) Determine the probability that the route will have no failed links.

4.5.4. Exercise. A software engineer is testing a new program. Each test has an 8% chance of finding a bug. If the engineer runs 100 independent tests:

a) What is the probability of finding exactly 10 bugs?

b) What is the expected number of tests to be run before a bug is found?

4.5.5. Exercise. In a digital signal processing application, each bit has a 0.1% chance of being corrupted during transmission. If a message of 10,000 bits is sent:

a) Find the probability that there are no corrupted bits in the message.

b) Calculate the probability of having more than 15 corrupted bits.

4.5.6. Exercise. A structural engineer assesses the failure probability of individual components in a bridge structure. Each component has a 0.5% chance of failure under a specific load. If the bridge consists of 200 such components:

a) Compute the probability that exactly 1 component fails.

b) What is the probability that at least 2 components fail?

Chapter 5

Continuous random variables

5.1. Probability density function and cumulative distribution function

5.1.1. Remark. A continuous random variable is a type of random variable that can take an infinite number of possible values.

In mathematical terms, continuous random variables are defined over a continuous range of values within the real numbers.

Unlike discrete random variables, which have countable outcomes, continuous random variables can assume any value within a specified interval or intervals.

In this case the cumulative distribution function is continuous.

5.1.2. Definition. The *probability density function* (PDF) of a continuous random variable X is a function f that defines the probability distribution of X. The probability density function is the derivative function of the cumulative distribution function.

5.1.3. Theorem. For any two points a and b where $a \leq b$, the probability that X falls within the interval $[a; b]$ is given by the integral of f over $[a; b]$, that is

$$P(a \leq X \leq b) = \int_a^b f(x)\,\mathrm{d}x.$$

5.1.4. Theorem. The probability density function f must satisfy the following conditions:

(1) $f(x) \geq 0$ for all x.

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

5.1.5. Theorem. The cumulative distribution function (CDF), denoted as F, of a continuous random variable X the probability that X will take a value less than or equal to x, that is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt.$$

5.1.6. Remark. We remind you that the cumulative distribution function F has the following properties:

(1) $0 \leq F(x) \leq 1$ for all x.

(2) $F(x)$ is a non-decreasing function.

(3) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$.

5.2. Expected value and variance

5.2.1. Definition. The *expected value* (or *mean*) of a continuous random variable X with a PDF f is given by

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,\mathrm{d}x.$$

It represents the average or central value of X.

5.2.2. Definition. The *variance* of a continuous random variable X defined as

$$Var(X) = E[(X - E(X))^2] = \int_{-\infty}^{\infty} \big(x - E(X)\big)^2 \cdot f(x)\,\mathrm{d}x.$$

The standard deviation, σ, is the square root of the variance, providing a measure of dispersion in the same units as X.

5.2.3. Theorem. The variance of a continuous random variable X can be calculated by the formula

$$Var(X) = E(X^2) - \big(E(X)\big)^2,$$

where

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,\mathrm{d}x.$$

5.3. Solved problems

5.3.1. Exercise. Let X be a continuous random variable with the probability density function (PDF) given by:

$$f_X(x) = \begin{cases} \frac{3}{8}x^2 & \text{for } 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

a) Verify if $f_X(x)$ is a valid PDF.

b) Calculate the cumulative distribution function (CDF) of X.

c) Find the expected value.

d) Determine the variance of X.

Solution:

a) To verify if $f_X(x)$ is a valid PDF, we need to ensure that the integral over its entire range equals 1. Since

$$\int_0^2 \frac{3}{8} \cdot x^2 dx = \left[\frac{3}{8} \cdot \frac{x^3}{3}\right]_0^2 = \left[\frac{x^3}{8}\right]_0^2 = \frac{8}{8} = 1$$

thus $f_X(x)$ is a valid PDF.

b) The CDF of X, $F_X(x)$, is found by integrating $f_X(x)$, that is

$$F_X(x) = \int_0^x \frac{3}{8} \cdot t^2 dt = \left[\frac{3}{8} \cdot \frac{t^3}{3}\right]_0^x = \left[\frac{t^3}{8}\right]_0^x = \frac{x^3}{8}, \text{ for } 0 \leq x \leq 2.$$

Using the previous results, we get that the cumulative distribution function is

$$F_X(x) = \begin{cases} 0 & \text{for} x < 0, \\ \frac{x^3}{8} & \text{for } 0 \leq x \leq 2, \\ 1 & \text{for} x < 0. \end{cases}$$

c) The expected value $E[X]$ is calculated as:

$$E(X) = \int_0^2 x \cdot \frac{3}{8} \cdot x^2 dx = \int_0^2 \frac{3}{8} \cdot x^3 dx = \left[\frac{3}{8} \cdot \frac{x^4}{4}\right]_0^2 = \frac{3}{32} \cdot 16 = \frac{3}{2}.$$

The variance requires $E(X^2)$:

$$E(X^2) = \int_0^2 x^2 \cdot \frac{3}{8} \cdot x^2 dx = \int_0^2 \frac{3}{8} \cdot x^4 dx =$$
$$= \left[\frac{3}{8} \cdot \frac{x^5}{5}\right]_0^2 = \frac{3}{40} \cdot 32 = \frac{12}{5}.$$

Thus,

$$Var(X) = E(X^2) - \big(E(X)\big)^2 = \frac{12}{5} - \left(\frac{3}{2}\right)^2 = \frac{12}{5} - \frac{9}{4} = \frac{3}{20}.$$

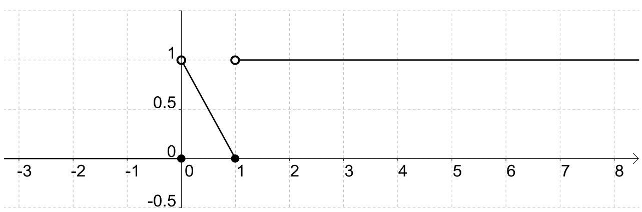
5.3.2. Exercise. Can the function $F \colon \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = \begin{cases} 0, & \text{ha } x \leq 0 \\ 1 - x, & \text{ha } 0 < x \leq 1 \\ 1, & \text{ha } x > 1 \end{cases}$$

be the distribution function of some random variable?

Solution:

The graph of function can be seen below:



This function is monotone decreasing on interval $[0;1]$ thus it cannot be distribution function.

5.3.3. Exercise. Consider the function $F \colon \mathbb{R} \to \mathbb{R}$,

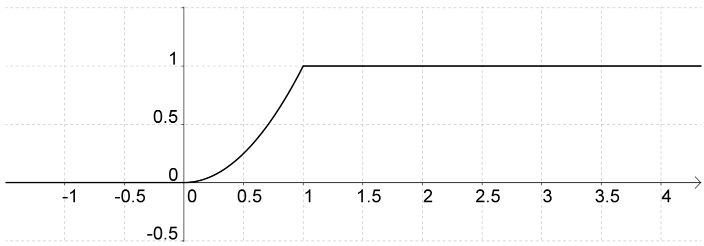
$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x^2, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}.$$

a) Sketch the graph of function F.
b) Prove that function F is cumulative distribution function.
c) Calculate the probability $P(X < 0.5)$.

d) Calculate the probability $P(X > 2)$.
e) Determine the probability density function.
f) Find the expected value.
g) Calculate the variance.
h) Determine the standard deviation.

Solution:

a) The graph of function F is as follows:



b) The function F is
- monotone increasing;
- continuous;
- limit of function F at $-\infty$ is zero;
- limit of function F at ∞ is ∞,

thus the function F is cumulative distribution function.

c) The value of probability that $X < 0.5$ is

$$P(X < 0.5) = F(0.5) = 0.5^2 = 0.25.$$

d) The value of probability that $X > 0.5$ is

$$P(X > 2) = 1 - F(2) = 1 - 1 = 0.$$

e) The probability density function is the derivative function of cumulative distribution function, that is

$$f(x) = F'(x) = \begin{cases} 2x, & \text{if } 0 < x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

f) The expected value is

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,\mathrm{d}x = \int_{0}^{1} x \cdot 2x\,\mathrm{d}x = \int_{0}^{1} 2x^2\,\mathrm{d}x = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}.$$

g) To the variance we have to calculate $E(X^2)$. Since

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,\mathrm{d}x = \int_{0}^{1} x^2 \cdot 2x\,\mathrm{d}x = \int_{0}^{1} 2x^3\,\mathrm{d}x = \left[\frac{x^4}{2}\right]_0^1 = \frac{1}{2},$$

we have

$$Var(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{18}.$$

h) The standard deviation is the square root of the variance, that is

$$STD(X) = \sqrt{\frac{1}{18}} \approx 0.2357.$$

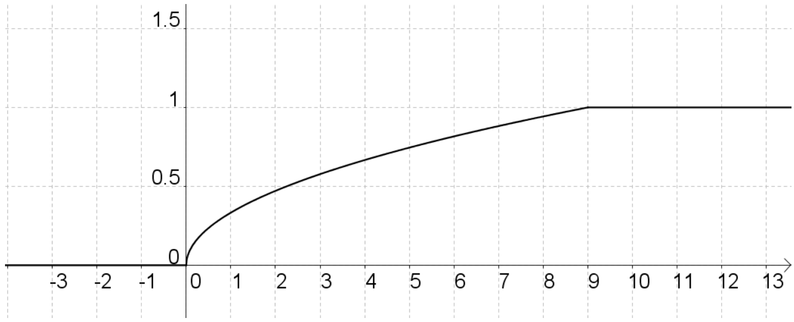
5.3.4. Exercise. Consider the function $F\colon \mathbb{R} \to \mathbb{R}$,

$$F(x) = \begin{cases} 0, & \text{ha } x \leq 0 \\ \frac{1}{3} \cdot \sqrt{x}, & \text{ha } 0 < x \leq 9 \\ 1, & \text{ha } x > 9 \end{cases} \cdot$$

a) Sketch the graph of function F.
b) Prove that function F is cumulative distribution function.
c) Calculate the probability $P(X < 1)$.
d) Calculate the probability $P(1 < X < 2)$.
e) Calculate the probability $P(X > 4)$.
f) Determine the probability density function.
g) Find the expected value.
h) Calculate the variance.
i) Determine the standard deviation.

<u>Solution:</u>

a) The graph of function F is as follows



b) The function F is
- monotone increasing;
- continuous;
- limit of function F at $-\infty$ is zero;
- limit of function F at ∞ is ∞,

thus the function F is cumulative distribution function.

c) The probability $P(X < 1)$ is

$$P(X < 1) = F(1) = \frac{1}{3} \cdot \sqrt{1} = \frac{1}{3}.$$

d) The probability $P(1 < X < 2)$ is

$$P(1 < X < 2) = F(2) - F(1) = \frac{1}{3} \cdot \sqrt{2} - \frac{1}{3} \approx 0.1381.$$

e) The probability $P(X > 4)$ is

$$P(X > 4) = 1 - F(4) = 1 - \frac{1}{3} \cdot \sqrt{4} = 1 - \frac{2}{3} = \frac{1}{3}.$$

f) The probability density function is the derivative function of cumulative distribution function, that is

$$f(x) = F'(x) = \begin{cases} \frac{1}{3} \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}}, & \text{if } 0 < x \leq 9 \\ 0, & \text{otherwise,} \end{cases}$$

thus

$$f(x) = \begin{cases} \frac{1}{6 \cdot \sqrt{x}}, & \text{if } 0 < x \leq 9 \\ 0, & \text{otherwise} \end{cases}.$$

g) The expected value is

$$E(X) = \int\limits_{-\infty}^{\infty} x \cdot f(x)\,\mathrm{d}x = \int\limits_{0}^{9} x \cdot \frac{1}{6 \cdot \sqrt{x}}\,\mathrm{d}x = \int\limits_{0}^{9} \frac{1}{6} \cdot \sqrt{x}\,\mathrm{d}x =$$

$$= \int\limits_{0}^{9} \frac{1}{6} \cdot x^{\frac{1}{2}}\,\mathrm{d}x = \left[\frac{1}{6} \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9 = \frac{1}{6} \cdot 9^{\frac{3}{2}} \cdot \frac{2}{3} = 3.$$

h) Since

$$E(X^2) = \int\limits_{-\infty}^{\infty} x^2 \cdot f(x)\,\mathrm{d}x = \int\limits_{0}^{9} x^2 \cdot \frac{1}{6 \cdot \sqrt{x}}\,\mathrm{d}x = \int\limits_{0}^{9} \frac{1}{6} \cdot \sqrt{x^3}\,\mathrm{d}x =$$

$$= \int\limits_{0}^{9} \frac{1}{6} \cdot x^{\frac{3}{2}}\,\mathrm{d}x = \left[\frac{1}{6} \cdot \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^9 = \frac{1}{6} \cdot 9^{\frac{5}{2}} \cdot \frac{2}{5} = \frac{1}{6} \cdot 243 \cdot \frac{2}{5} = \frac{81}{5},$$

thus

$$Var(X) = \frac{81}{5} - 9 = \frac{36}{5}.$$

i) The standard deviation is

$$STD(X) = \sqrt{\frac{36}{5}} = \frac{6}{\sqrt{5}}.$$

5.3.5. Exercise. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} A \cdot x, & \text{if } 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

a) Determine the value of A such that f can be a probability density function of some random variable X.
b) Calculate the expected value!
c) Find the variance.
d) What is the value of standrad deviation?

Solution:

a) Since

$$\int\limits_{0}^{2} A \cdot x\,\mathrm{d}x = \left[A \cdot \frac{x^2}{2}\right]_0^2 = 2A,$$

we have $2A = 1$, thus $A = \frac{1}{2}$.

b) The expected value is

$$E(X) = \int_0^2 \frac{1}{2} \cdot x \cdot x \,\mathrm{d}x = \int_0^2 \frac{1}{2} \cdot x^2 \,\mathrm{d}x \left[\frac{1}{2} \cdot \frac{x^3}{3}\right]_0^2 = \frac{4}{3}.$$

c) Since

$$E(X^2) = \int_0^2 \frac{1}{2} \cdot x \cdot x^2 \,\mathrm{d}x = \int_0^2 \frac{1}{2} \cdot x^3 \,\mathrm{d}x \left[\frac{1}{2} \cdot \frac{x^4}{4}\right]_0^2 = 2,$$

thus the variance is

$$Var(X) = 2 - \left(\frac{4}{3}\right)^2 = 2 - \frac{16}{9} = \frac{2}{9}.$$

d) The standard deviation is the squre root of the variance, that is

$$STD(X) = \frac{\sqrt{2}}{3}.$$

5.3.6. Exercise. Let A is a real number and f be a probability density function, such that

$$f(x) = \begin{cases} A \cdot \cos x, & \text{if } 0 < x < \frac{\pi}{2} \\ 0, & \text{otherwise} \end{cases}.$$

a) Determine the value of A.

b) Calculate the expected value.

Solution:

a) Since

$$\int_0^{\frac{\pi}{2}} A \cdot \cos x \,\mathrm{d}x = \left[A \cdot \sin x\right]_0^{\frac{\pi}{2}} = A \cdot \sin \frac{\pi}{2} = A,$$

thus $A = 1$.

b) The expected value is

$$E(X) = \int_0^{\frac{\pi}{2}} x \cdot \cos x \,\mathrm{d}x.$$

Using integral by parts, we have

$$\int x \cdot \cos x \, \mathrm{d}x = x \cdot \sin x - \int \sin x \, \mathrm{d}x = x \cdot \sin x + \cos x + c,$$

therefore

$$E(X) = \int_0^{\frac{\pi}{2}} x \cdot \cos x \, \mathrm{d}x = \Big[x \cdot \sin x + \cos x \Big]_0^{\frac{\pi}{2}} =$$

$$= \frac{\pi}{2} \cdot \sin \frac{\pi}{2} + \cos \frac{\pi}{2} - 1 = \frac{\pi}{2} - 1 \approx 0.57.$$

5.3.7. Exercise. Shots are fired at a target with a radius of one unit, which is circular in shape. Assume that every shot hits the target and that the location of the hit is uniformly distributed across the target. Let r denote the distance of the hit location from the center of the target. Determine the cumulative distribution function of the random variable!

Solution:

To solve this problem, we need to determine the cumulative distribution function, of the distance r from the center of the target to the point where a shot lands, given that the shots are uniformly distributed over the area of the target.

To find the distribution function of r, we recognize that the probability of a shot landing within a distance r from the center corresponds to the ratio of the area of a circle of radius r to the area of the target. The area of a circle with radius r is $r^2 \cdot \pi$, and the area of the target circle is π, because the radius of the target is 1 unit.

The cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \dfrac{x^2 \cdot \pi}{\pi}, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x > 1, \end{cases}$$

that is

$$F_x(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x^2, & \text{if } 0 < x \leq 1 \; . \\ 1, & \text{if } x > 1. \end{cases}$$

5.3.8. Exercise. The contaminants on the surface of a given object can be considered spherical, with their diameter well modeled by the probability density function

$$f(x) = \begin{cases} \dfrac{a}{x^3}, & \text{if } x > 2 \\ 0, & \text{otherwise} \end{cases}.$$

a) Determine the value of a.

b) Calculate the expected value.

Solution:

a) Since f is a probability density function, then

$$1 = \int_2^\infty \frac{a}{x^3}\,dx = a \cdot \lim_{c\to\infty} \int_2^c x^{-3}\,dx = a \cdot \lim_{c\to\infty} \left[\frac{x^{-2}}{-2}\right]_2^c =$$

$$= a \lim_{c\to\infty} \left[\frac{-1}{2x^2}\right]_2^c = \frac{a}{8},$$

thus $a = 8$.

b) The expected value is

$$E(X) = \int_2^\infty x \cdot \frac{8}{x^3}\,dx = \int_2^\infty \frac{8}{x^2}\,dx = \int_2^\infty 8 \cdot x^{-2}\,dx =$$

$$= \lim_{c\to\infty} \left[8 \cdot \frac{x^{-1}}{-1}\right]_2^c = \lim_{c\to\infty} \left[-\frac{8}{x}\right]_2^c = \lim_{c\to\infty} \left(-\frac{8}{c} + 4\right) = 4.$$

5.4. Unsolved problems

5.4.1. Exercise. Let X be a continuous random variable with the probability density function (PDF) given by:

$$f(x) = \begin{cases} \frac{2}{9}x & \text{for } 0 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$

a) Verify if $f(x)$ is a valid PDF.

b) Calculate the cumulative distribution function (CDF) of X.

c) Find the expected value.

d) Determine the variance of X.

5.4.2. Exercise. Can the function $F\colon \mathbb{R} \to \mathbb{R}$ defined as

$$F(x) = \begin{cases} 0, & \text{ha } x \leq 0 \\ -x^2, & \text{ha } 0 < x \leq 1 \\ 1, & \text{ha } x > 1 \end{cases}$$

be the distribution function of some random variable?

5.4.3. Exercise. Consider the function $F\colon \mathbb{R} \to \mathbb{R}$,

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x^3, & \text{if } 0 < x \leq 1 \\ 1, & \text{if } x > 1 \end{cases}.$$

a) Sketch the graph of function F.

b) Prove that function F is cumulative distribution function.

c) Calculate the probability $P(X < 0.5)$.

d) Calculate the probability $P(X > 2)$.

e) Determine the probability density function.

f) Find the expected value.

g) Calculate the variance.

h) Determine the standard deviation.

5.4.4. Exercise. Consider the function $F\colon \mathbb{R} \to \mathbb{R}$,

$$F(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x^4, & \text{if } 0 < x \le 1 \\ 1, & \text{if } x > 1 \end{cases}.$$

a) Sketch the graph of function F.
b) Prove that function F is cumulative distribution function.
c) Calculate the probability $P(X < 3)$.
d) Calculate the probability $P(X > 1)$.
e) Determine the probability density function.
f) Find the expected value.
g) Calculate the variance.
h) Determine the standard deviation.

5.4.5. Exercise. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} A \cdot x^2, & \text{if } 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}.$$

a) Determine the value of A such that f can be a probability density function of some random variable X.
b) Calculate the expected value!
c) Find the variance.
d) What is the value of standrad deviation?

5.4.6. Exercise. Consider the function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} \dfrac{A}{x^2}, & \text{if } 1 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

a) Determine the value of A such that f can be a probability density function of some random variable X.
b) Calculate the expected value!
c) Find the variance.
d) What is the value of standrad deviation?

Chapter 6

Special continuous random variables

6.1. Uniform distribution

6.1.1. Remark. A continuous *uniform distribution* is one of the simplest probability distributions used in probability. It is defined over a continuous range $[a, b]$, where every interval of equal length within $[a; b]$ has an equal probability of containing a random variable X.

6.1.2. Definition. The probability density function of a continuous uniform distribution on interval $[a; b]$ is given by

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

6.1.3. Remark. The probability density function is constant between a and b, indicating that the random variable is equally likely to fall anywhere within the interval.

6.1.4. Theorem. The cumulative distribution function for the continuous uniform distribution is defined as:

$$F(x) = \begin{cases} 0 & \text{for } x < a, \\ \frac{x-a}{b-a} & \text{for } a \leq x < b, \\ 1 & \text{for } x \geq b. \end{cases}$$

6.1.5. Theorem. The expected value of a continuous uniform distribution is

$$E(X) = \frac{a+b}{2}.$$

6.1.6. Remark. It is the midpoint of the interval $[a; b]$, reflecting the symmetry and uniformity of the distribution.

6.1.7. Theorem. The variance of a continuous uniform distribution is given by

$$Var(X) = \frac{(b-a)^2}{12}.$$

6.2. Exponential distribution

6.2.1. Remark. The *exponential distribution* is a continuous probability distribution commonly used to model the time between events in a Poisson process. It is characterized by its rate parameter λ, which represents the average number of events occurring in a unit time interval.

6.2.2. Definition. The probability density function of the exponential distribution for a random variable X is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

and $f(x) = 0$ when $x < 0$.

6.2.3. Theorem. The cumulative distribution function of the exponential distribution is defined as

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

and $F(x) = 0$ when $x < 0$.

6.2.4. Theorem. The mean an exponentially distributed random variable are directly related to its rate parameter λ:

$$E(X) = \frac{1}{\lambda}.$$

6.2.5. Theorem. The variance of the exponential distribution is

$$Var(x) = \frac{1}{\lambda^2}.$$

6.2.6. Theorem. A unique property of the exponential distribution is its memorylessness, which means the distribution of the time until the next event does not depend on how much time has already elapsed. Mathematically, this is described as:

$$P(X > s + t | X > s) = P(X > t)$$

for any $s, t \geq 0$.

6.2.7. Remark. The exponential distribution has wide applications across various fields, such as:

- Reliability engineering for modeling time until failure,
- Queueing theory for the time between arrivals of events,
- Telecommunications for modeling message transmission times,
- Nuclear physics for radioactive decay processes.

6.3. Normally distribution

6.3.1. Definition. The *normal distribution*, also known as the *Gaussian distribution*, is a continuous probability distribution that is symmetric around its mean, showing that data near the mean are more frequent in occurrence than data far from the mean.

6.3.2. Definition. Let m is an arbitrary real number and $\sigma > 0$. The probability density function of the normal distribution for a random variable X is given by:

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

6.3.3. Theorem. The mean of the normal distribution is m, the standard deviation of the normal distribution is σ.

6.3.4. Theorem. The normal distribution has several key properties:

(1) It is symmetric around the mean, μ.
(2) The mean, median, and mode of the distribution are equal.
(3) The total area under the curve of the distribution is equal to 1.
(4) Approximately 68% of the data within one standard deviation of the mean, 95% within two standard deviations, and 99.7% within three standard deviations (the 68-95-99.7 rule).

6.3.5. Definition. The *standard normal distribution* is a special case of the normal distribution with a mean of 0 and a standard deviation of 1. The probability density function of the standard normal distribution is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}.$$

6.3.6. Theorem. Let X be the normal distribution with expected value m and standard deviation σ. Let denote F is the cumulative distribution function of X. Let denote φ the cumulative distribution function if standard normal distribution. In this case

$$F(x) = \varphi\left(\frac{x-m}{\sigma}\right).$$

6.3.7. Remark. The normal distribution is widely used across various fields such as statistics, economics, engineering, and natural and social sciences. It is used to model errors, human characteristics, and natural phenomena that tend to cluster around a single mean value.

6.4. Solved problems

6.4.1. Exercise. Let X be a continuous random variable that follows a uniform distribution over the interval $[2; 8]$. Calculate the probability that X is less than 5.

Solution:

The probability density function of a uniform distribution is $f(x) = \frac{1}{b-a}$ for x in $[a; b]$. For X in $[2; 8]$, $f(x) = \frac{1}{8-2} = \frac{1}{6}$. The probability that $X < 5$ is given by

$$P(X < 5) = \int_2^5 \frac{1}{6}\,\mathrm{d}x = \frac{5-2}{6} = \frac{1}{2}.$$

6.4.2. Exercise. Find the expected value and variance of a uniform distribution on the interval $[0; 10]$.

Solution:

For a uniform distribution over $[a; b]$, the expected value $E(X)$ and variance $Var(X)$ are given by

$$E(X) = \frac{a+b}{2}, \quad Var(X) = \frac{(b-a)^2}{12}.$$

Thus, for $[0; 10]$, we have

$$E(X) = \frac{0+10}{2} = 5, \quad Var(X) = \frac{(10-0)^2}{12} = \frac{100}{12} \approx 8.33.$$

6.4.3. Exercise. A continuous random variable X follows a uniform distribution over $[-3; 3]$. Calculate the cumulative distribution function for X.

Solution:

The cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{for } x < -3, \\ \dfrac{x-(-3)}{3-(-3)} & \text{for } -3 \le x < 3, \\ 1 & \text{for } x \ge 3. \end{cases}$$

Simplifying, we get that

$$F(x) = \begin{cases} 0 & \text{for } x < -3, \\ \dfrac{x+3}{6} & \text{for } -3 \le x < 3, \\ 1 & \text{for } x \ge 3. \end{cases}$$

6.4.4. Exercise. An economist models the price X (in dollars) of a certain commodity within a market as a continuous random variable following a uniform distribution over the interval $[50; 150]$. Determine the probability that the price will be less than \$100.

Solution:

The probability density function is

$$f(x) = \frac{1}{150-50} = \frac{1}{100},$$

when $x \in [50; 150]$. The probability that $X < 100$ is

$$P(X < 100) = \int_{50}^{100} \frac{1}{100}\,\mathrm{d}x = \frac{100-50}{100} = \frac{1}{2} = 0.5.$$

6.4.5. Exercise. An engineer models the lifespan of a certain type of mechanical bearing as uniformly distributed between 2000 and 5000 hours. Calculate the expected lifespan and the standard deviation.

Solution:

Let X denote the random variable. The expected value is

$$E(X) = \frac{2000+5000}{2} = 3500 \text{ hours.}$$

The variance is

$$Var(X) = \frac{(5000-2000)^2}{12} = \frac{9,000,000}{12} = 750,000.$$

The standard deviation is the square root of the variance, that is

$$STD(X) = \sqrt{Var(X)} = \sqrt{750,000} \approx 866.03 \text{ hours.}$$

6.4.6. Exercise. Let X be a uniform distribution on interval $[1; 3]$.

a) Write down the cumulative distribution function of X.

b) Sketch the graph of cumulative distribution function.

c) Determine the probability density function.

d) Sketch the graph of probability density function.

e) Calculate the expected vaule.

f) Determine the standard deviation.

g) What is the probability that $X < 1.5$?

h) What is tha probability that $X > 2$?

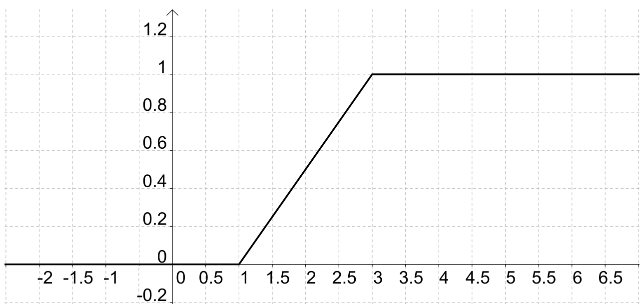
i) Calculate the probability that X between 1.2 and 1.3.

Solution:

a) The cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x < 1 \\ \dfrac{x-1}{2}, & \text{if } 1 \leq x \leq 3 \\ 1, & \text{if } x > 3. \end{cases}$$

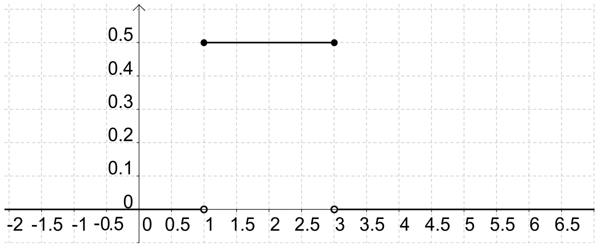
b) The graph of the cumulative distribution function is as follows



c) The probability density function is

$$f(x) = \begin{cases} \dfrac{1}{2}, & \text{if } 1 \leq x \leq 3 \\ 0, & \text{otherwise.} \end{cases}$$

d) The graph of probability density function is as follows



e) The expected value is

$$E(X) = \frac{1+3}{2} = 2.$$

f) The variance is

$$Var(X) = \frac{(3-1)^2}{12} = \frac{1}{3}.$$

The standard deviation is

$$STD(X) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

g) The probability of $X < 1.5$ is

$$P(X < 1.5) = F(1.5) = \frac{1.5-1}{2} = 0.25.$$

h) The probability of $X > 2$ is

$$P(X > 2) = 1 - F(2) = 1 - \frac{2-1}{2} = 0.5.$$

i) The probability of X between 1.2 and 1.3 is

$$P(1.2 < \xi < 1.3) = F(1.3) - F(1.2) = \frac{1.3-1}{2} - \frac{1.2-1}{2} = 0.05.$$

6.4.7. Exercise. Trams arrive at a tram stop every 15 minutes. Upon arriving at the tram stop, we see that no tram will come within 1 minute. Let X be the length of the waiting time.

a) Determine the cumulative distribution function.
b) Find the probability density function.
c) Calculate the expected value.
d) Give the variance.
e) Calculate the probability that we have to wait less than 5 minutes.

f) What is the probability that we have to wait at least 7 minutes?

Solution:

a) The cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x < 1 \\ \dfrac{x-1}{14}, & \text{if } 1 \leq x \leq 15 \\ 1, & \text{if } x > 15. \end{cases}$$

b) The probability density function is as follows

$$f(x) = \begin{cases} \dfrac{1}{14}, & \text{if } 1 \leq x \leq 15 \\ 0, & \text{otherwise.} \end{cases}$$

c) The expected value is

$$E(X) = \frac{1+15}{2} = 8.$$

d) The variance is

$$Var(X) = \frac{(15-1)^2}{12} = \frac{196}{12} = \frac{49}{3}.$$

e) The probability of $X < 5$ is

$$P(X < 5) = F(5) = \frac{2}{7}.$$

f) The probability of $X > 7$ is

$$P(X > 7) = 1 - P(X < 7) = 1 - F(7) = 1 - \frac{3}{7} = \frac{4}{7}.$$

6.4.8. Exercise. Let X be a uniform distribution on interval $[a; b]$. Its expected value is 4 and variance is $\frac{4}{3}$.

a) Calculate the value of a and b.

b) Determine the cumulative distribution function.

c) Find the probability density function.

Solution:

a) Since

$$E(X) = \frac{a+b}{2} \quad \text{and} \quad Var(X) = \frac{(b-a)^2}{12},$$

thus

$$\left.\begin{aligned} \frac{a+b}{2} &= 4 \\ \frac{(b-a)^2}{12} &= \frac{4}{3} \end{aligned}\right\},$$

that is

$$\left.\begin{aligned} a+b &= 8 \\ (b-a)^2 &= 16 \end{aligned}\right\}.$$

From the first equation, we get that $b = 8 - a$. Substitute to the second equation, we get that

$$(8-a-a)^2 = 16 \qquad \Rightarrow \qquad (8-2a)^2 = 16,$$

thus $8-2a = 4$ or $8-2a = -4$. We get that $a_1 = 6$ and $a_2 = 2$ thus $b_1 = 2$ and $b_2 = 6$. Since $a < b$, we have just $a = 2$ and $b = 6$.

b) The cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x < 2 \\ \frac{x-2}{4}, & \text{if } 2 \leq x \leq 6 \\ 1, & \text{if } x > 6. \end{cases}$$

c) The probability density function os

$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } 2 \leq x \leq 6 \\ 0, & \text{otherwise.} \end{cases}$$

6.4.9. Exercise. Bettina usually wakes up at 8 a.m., and a given standard deviation $\frac{2}{\sqrt{3}}$ according to a uniform distribution. What is the probability that Bettina woke up before 7 today?

Solution:

Since X is a uniform distribution, we have

$$E(X) = \frac{a+b}{2} \quad \text{and} \quad Var(X) = \frac{(b-a)^2}{12},$$

that is we have to solve the system of equation

$$\left.\begin{aligned} \frac{a+b}{2} &= 8 \\ \frac{(b-a)^2}{12} &= \frac{4}{3} \end{aligned}\right\}.$$

Using algebraic identity, we get that

$$\left.\begin{aligned} a+b &= 16 \\ (b-a)^2 &= 16 \end{aligned}\right\}.$$

From the first equation it follows that $b = 16 - a$. Substituting this to the second equation, we get that

$$(16 - a - a)^2 = 16 \qquad \Rightarrow \qquad (16 - 2a)^2 = 16,$$

thus $16 - 2a = 4$ and $16 - 2a = -4$, that is $a_1 = 6$, and $a_2 = 10$. From these we get that $b_1 = 10$ and $b_2 = 6$. Using the relation $a < b$, we have $a = 6$ and $b = 10$.

The cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x < 6 \\ \dfrac{x-6}{4}, & \text{if } 6 \le x \le 10 \\ 1, & \text{if } x > 10. \end{cases}$$

The probability is

$$P(X < 7) = F(7) = \frac{7-6}{4} = 0.25.$$

6.4.10. Exercise. Let X be an exponential distribution with parameter $\lambda = 5$.

a) Give the expected value.
b) Determine the variance.
c) Find the standard deviation.
d) Determine the cumulative distribution function.
e) Determine the probability density function.
f) Calculate the probability $P(\xi = 3)$.
g) Determine the probability $P(-2 < \xi < 3)$.

Solution:

a) The expected value is

$$E(X) = \frac{1}{\lambda} = \frac{1}{5}.$$

b) The variance is

$$Var(X) = \frac{1}{\lambda} = \frac{1}{5}.$$

c) The standard deviation is

$$STD(X) = \sqrt{\frac{1}{\lambda}} = \frac{1}{\sqrt{5}}.$$

d) The cumulative distribution function is

$$F(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1 - \mathrm{e}^{-5x}, & \text{if } x > 0 \end{cases}.$$

e) The probability density function is

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 5\mathrm{e}^{-5x}, & \text{if } x > 0 \end{cases}.$$

f) If X is a continuous random variable then $P(X = k) = 0$ for all real number k, thus $P(X = 3) = 0$.

g) The probability that X between -2 and 3 is

$$P(-2 < X < 3) = F(3) - F(-2) = 1 - e^{-15} - 0 = 1 - \frac{1}{e^{15}}.$$

6.4.11. Exercise. The probability that one has to wait more than 6 minutes at a gas station is 0.1. The length of the waiting time follows an exponential distribution. What is the probability that upon arriving at the gas station, one will be served within 3 minutes?

Solution:

Since

$$0.1 = P(X > 6) = 1 - P(X \leq 6) = 1 - F(6) = 1 - \left(1 - e^{-6\lambda}\right) = e^{-6\lambda},$$

we have

$$\lambda = \frac{\ln 0.1}{-6}.$$

Using the value of λ, we get that

$$P(X \leq 3) = F(3) = 1 - e^{-3\lambda} = 1 - e^{-3\frac{\ln(0.1)}{-6}} = 1 - \sqrt{0.1} = 0.684.$$

6.4.12. Exercise. The average height of a group of people is 175 cm, with a standard deviation of 3 cm. The size of the height can be considered to follow a normal distribution.

a) What is the probability that the height of a randomly selected person deviates less than 3 cm from the average?
b) What is the probability that a selected person's height is at least 173 cm?
c) What is the probability that the height of a randomly selected person falls between 173 cm and 177 cm?

<u>Solution:</u>

a) The probability is

$$\begin{aligned} P(172 < X < 178) &= F(178) - F(172) = \\ &= \Phi\left(\frac{178-175}{3}\right) - \Phi\left(\frac{172-175}{3}\right) = \\ &= \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 = \\ &= 2 \cdot 0.8413 - 1 = 0.6826. \end{aligned}$$

b) The probability is

$$P(X > 173) = 1 - F(173) = 1 - \Phi\left(\frac{173-175}{3}\right) = 1 - \Phi\left(-\frac{2}{3}\right) =$$

$$= \Phi\left(\frac{2}{3}\right) = 0.7486.$$

c) The probability is

$$\begin{aligned} P(173 < X < 177) &= F(177) - F(173) = \\ &= \Phi\left(\frac{177-175}{3}\right) - \Phi\left(\frac{173-175}{3}\right) = \\ &= \Phi\left(\frac{2}{3}\right) - \Phi\left(-\frac{2}{3}\right) = 2\Phi\left(\frac{2}{3}\right) - 1 = \\ &= 2 \cdot 0.7486 - 1 = 0.5028. \end{aligned}$$

6.4.13. Exercise. On a given day, the closing price of a stock in forints is a normally distributed random variable with an expected value of 100 forints and a standard deviation of 5 forints. If we bought the stock at 100 forints at the opening, what is the probability that we will have at least a 10% profit at the close?

Solution:

The expected value is $m = 100$ and the standard deviation is $\sigma = 4$. The probability is

$$P(X \geq 110) = 1 - F(110) = 1 - \Phi\left(\frac{110 - 100}{5}\right) =$$
$$= 1 - \Phi\left(2\right) = 1 - 0.9772 = 0.0228.$$

6.5. Unsolved problems

6.5.1. Exercise. Let X be a uniform distribution on interval $[-1; 5]$.

a) Write down the cumulative distribution function of X.
b) Sketch the graph of cumulative distribution function.
c) Determine the probability density function.
d) Sketch the graph of probability density function.
e) Calculate the expected vaule.
f) Determine the standard deviation.
g) What is the probability that $X < 3$?
h) What is tha probability that $X > 2$?
i) Calculate the probability that X between 1 and 2.

6.5.2. Exercise. Let X be a uniform distribution on interval $[a; b]$. Its expected value is 4 and variance is 3.

a) Calculate the value of a and b.
b) Determine the cumulative distribution function.
c) Find the probability density function.

6.5.3. Exercise. Let X be an exponential distribution with parameter $\lambda = \frac{1}{2}$.

a) Give the expected value.
b) Determine the variance.
c) Find the standard deviation.
d) Determine the cumulative distribution function.
e) Determine the probability density function.
f) Calculate the probability $P(\xi = 3)$.
g) Determine the probability $P(-2 < \xi < 3)$.

6.5.4. Exercise. The average height of a group of people is 170 cm, with a standard deviation of 5 cm. The size of the height can be considered to follow a normal distribution.

a) What is the probability that the height of a randomly selected person deviates less than 5 cm from the average?
b) What is the probability that a selected person's height is at least 170 cm?
c) What is the probability that the height of a randomly selected person falls between 168 cm and 171 cm?

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