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INDUCED AREAL SPACES AND REDUCTION OF THE CONNECTION IN AN AREAL SPACE TO CONNECTION IN A VECTOR BUNDLE

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Abstract

First we discuss the notion of area in a Finsler space, derive the fundamental function of an areal space induced by a Finsler space, construct concrete examples of areal spaces A_n^k (for any $n \geq 3$ and $n-1 \geq k \geq 2$) which are not induced by Finsler spaces, and give a scalar curvature which measures the power of the approximability of an A_n^k by areal spaces induced by Finsler spaces. The scene of the connection theory of an areal space is a fiber bundle η with the Grassmann cone as typical fiber. We consider a special vector bundle ξ and establish a strong bundle mapping $\eta \to \xi$ which makes correspond to areal connections H_η certain nonlinear connections H_ξ and conversely. This enables us to treat the areal connection as a vector bundle connection, which is more simple than the former one.

1 Finslerian area, areal fundamental function, areal space, induced areal space

A. Let $\phi \subset M$ be a submanifold of the manifold $M(\dim \phi = k < n = \dim M)$ given in local coordinates by $x^i = x^i(u^1, \ldots, u^k)$, $i = 1, \ldots, n$; u taken from a domain B of the parameter space E^k. Let σ be the tangent space of ϕ at $x(u) \in \phi : \sigma = T_{x(u)}\phi \subset T_{x(u)}M$, and $p(u) \subset \sigma$ the parallelotop in $T_x M$ spanned by the vectors $\frac{\partial x^i}{\partial u^\alpha}(u) du^\alpha$ $\alpha = 1, \ldots, k$ (no summation for α). $p(u)$ corresponds to a curvilinear parallelotop in ϕ having vertices with coordinates $x^i(u^\alpha + \varepsilon^\alpha du^\alpha)$, where ε^α is 0 or

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1. If there exists a metric structure in each T_xM and the area of $p(u)$ is denoted by $\|p(u)\|$, then the area of ϕ is defined by

$$\|\phi\| := \lim \sum \|p(u)\| du = \int_B \|p(u)\| du. \tag{1}$$

In case of a Riemannian space $V_n = (M, g)$ with metric tensor $g_{ij}(x)$ this yields $\int_B \sqrt{\tilde{g}} du$, where $\tilde{g} = \mathrm{Det}\, |\tilde{g}_{\alpha\beta}(u)|$, $\tilde{g}_{\alpha\beta} = g_{ik}(x(u)) = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\alpha}$ $\alpha, \beta = 1, \dots, k$.

We show this in a Finsler space $F_n = (M, L)$. Here $L : TM \to R^+$, $(x, y) \mapsto L(x, y)$ is the fundamental function which is supposed to be first order positively homogeneous in the second variable: $L(x, \lambda y) = |\lambda| L(x, y)$, $\lambda \in R^+$. Then each tangent space T_xM becomes a Minkowski space M_n (or an n-dimensional Banach space) with the norm $\|y\|_M = \|y\|_F = L(x, y)$, which is the Minkowskian and at the same time also the Finsler length of the vector $y \in T_xM$. The indicatrix $I(x_0)$ of F_n at x_0 is a star-shaped and symmetrical hypersurface in the tangent space $T_{x_0}M$ defined by $I(x_0) := \{y \mid L(x_0, y) = 1\}$. Hence the indicatrix plays the role of the euclidean unit sphere S^{n-1}, and makes the tangent space into a Minkowski space $(T_xM, I(x)) = M_n$. Then $I(x_0)$ and its interior $\mathcal{D}(x_0) = \{y \mid L(x_0, y) \leq 1\}$ play the role of the unit ball.

In order to obtain the Finsler area $\|\phi\|_F$ of ϕ let us consider a euclidean metric $\| \ \|_E$ and two domains D_1, D_2 in $\sigma \subset T_{x(u)}\phi \subset T_xM = M_n$. Since in a Minkowski space the measures of domains in a linear subspace relate as their euclidean measures, we have ([9], [8], [1])

$$\|D_1\|_M : \|D_2\|_M = \|D_1\|_E : \|D_2\|_E. \tag{2}$$

The value of the fraction on the right hand side is independent of the applied euclidean measure. Thus (2) determines the Minkowski norm of a domain up to a factor (up to the unit). Now choose $D_1 = p(u)$ and $D_2 = \sigma \cap \mathcal{D}$. Then

$$\|p(u)\|_M = \frac{\|p(u)\|_E}{\|\sigma \cap \mathcal{D}\|_E} \cdot \|\sigma \cap \mathcal{D}\|_M. \tag{3}$$

Since $\mathcal{D}$ plays the role of the unit ball of the Minkowski space it is natural to define $\|\sigma \cap \mathcal{D}\|_M$ as the area of the k-dimensional ($k = \dim \sigma$) euclidean unit ball denoted by $\omega^{(k)}$. Furthermore we define $\lambda p(u)$, $\lambda \in R^+$ as the image of $p(u)$ in the homothety with the factor $\sqrt[k]{\lambda}$: $\lambda p(u) := (\mathrm{Hom}\, \sqrt[k]{\lambda}) p(u)$. For $\lambda \in R^-$ after the homothety Hom $\sqrt[k]{|\lambda|}$ a change of the orientation is to be performed. Hence $\|\lambda p(u)\|_E = \lambda \|p(u)\|_E$ and this yields

$$\|\lambda p(u)\|_F = \lambda \|p(u)\|_F, \quad \lambda \in R$$

i.e. $\|p(u)\|_F$ is first order homogeneous. For a $p(u) \subset \sigma$ we define $p_1(u) := \frac{1}{\|p(u)\|_E} p(u)$. Hence $\|p_1(u)\|_E = 1$ and

$$\|p(u)\|_F = \|p(u)\|_E \, \|p_1(\dot{u})\|_F .$$

Furthermore we obtain from (3)

$$\|p_1(u)\|_F = \|p_1(u)\|_M = \frac{\omega^{(k)}}{\|T_{x(u)}\phi \cap \mathcal{D}(x(u))\|_E}. \tag{4}$$

Thus $\|\phi\|_F = \int_B \|p(u)\|_F du$ is determined; and a calculation (see [1] or [2]) results in

$$\|\phi\|_F = k\omega^{(k)} \int_B \left[\int_{S^{k-1}(\varphi)} I^2(x(u),\varphi) \sqrt{g_S(\varphi)} d\varphi \right]^{-1} \sqrt{g(u)} du, \tag{5}$$

where S^{k-1} is the unit sphere in $T_{x(u)}\phi = T_x M \cap \sigma$, φ means spherical coordinates on it, g_S is the determinant of the metric tensor on S^{k-1}, $g(u)$ is the determinant of $g_{\alpha\beta}(u) = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^i}{\partial u^\beta}$ and $I(x(u),\varphi)$ means the euclidean distance of the point of $I(x(u)) \cap \sigma$ in the direction determined by φ from the origin 0.

B. For a fixed F_n, $\|p(u)\|_F$ can be calculated for any ϕ, and thus for any x and p. So L defines a function $\bar{A}(x,p) = \|p(x)\|_F$. By (2) two $p_1, p_2 \in T_x M$ lying in the same σ and having the same euclidean area and orientation have also the same Minkowskian area, and thus $\bar{A}(x,p_1) = \bar{A}(x,p_2)$. Such $p - s$ can be identified. They form an equivalence class Π in the ensemble $\mathcal{P}$ of the parallelotops p. A Π at a point x is determined by its plane position σ and a real, which means its euclidean area. The orientation can be taken into consideration as the sign of the area (of the real). Hence these Π-s can be identified also with the simple k-vectors, or with the exterior products $\overset{1}{v} \wedge \cdots \wedge \overset{k}{v}$ of k vectors $\overset{\alpha}{v} \in T_x M$, $\alpha = 1, \ldots, k$. Thus the Π-s coincide with the elements of a Grassmann cone $GK_{n,k}$ which equals $G_{n,k} \times R$, $G_{n,k}$ being the Grassmann manifold of the k dimensional linear subspaces of the vector space $T_x M$. Hence

$$\begin{aligned} \bar{A} :& M \times GK_{n,k} \to R \\ & x, \Pi \mapsto \bar{A}(\alpha, \Pi) \end{aligned} \tag{6}$$

$\bar{A} \in C^\infty(M \times GK_{n,k})$ and 1 order homogeneous in Π.

Let $\rho : \mathcal{P} \to GK_{n,k}$ be a mapping which orders to a parallelotop p its class $\Pi \in GK_{n,k} : \rho(p) = \Pi$. Then by

$$\hat{A}(x,p) := \bar{A}(x,\rho(p)) \quad \hat{A} : M \times \mathcal{P} \to R$$

we can extend $\bar{A}$ to $\hat{A}$ in such a way that $\hat{A}$ is constant on each equivalence class Π. Conversely, if $\hat{A}$ is constant on each equivalence class Π, then it determines $\bar{A}(x,\Pi)$. Hence they are equivalent in this sense. ρ is univalent, but its inverse $\rho^{-1} : GK_{n,k} \to \mathcal{P}$ is multivalent $(1,\infty)$. $\rho^{-1}(\Pi_0)$ consists of all those parallellotops $p \subset \sigma \subset T_xM$ spanned by $\overset{1}{v},\dots,\overset{k}{v}$; $\overset{1}{w},\dots,\overset{k}{w}$; … which have the same euclidean area with sign, or which form the same exterior products $\overset{1}{v} \wedge \cdots \wedge \overset{k}{v} = \overset{1}{w} \wedge \cdots \wedge \overset{k}{w} = \dots$.

C. Now an areal space A_n^k is a couple of a manifold M and a smooth and in its second variable first order homogeneous fundamental function

$$\begin{aligned} A^*: \quad & M \times GK_{n,k} \to R \\ & x,\Pi \mapsto A^*(x,\Pi). \end{aligned}$$

This fundamental function can be replaced by its extended function

$$\begin{aligned} A: \quad & M \times \mathcal{P} \to R \\ & x,p \mapsto A(x,p) := A^*(x,\rho(p)). \end{aligned}$$

Then the area of a surface element p in this areal space $A_n^k = (M,A)$ is defined by

$$\|p\|_A := A(x,p) \tag{7}$$

and the surface of $\phi \subset M$ is

$$\|\phi\|_A := \int_B A(x(u),p(x(u)))du.$$

If $k=1$ then ϕ is a curve $x^i = x^i(u)$, $\sigma = T_{x(u)}\phi \subset T_{x(u)}M$ is a line, $p(u) \subset \sigma$ is an oriented segment of this line which can be considered as a vector $y \in T_xM$ and $A(x,p) = A(x,y)$ becomes the fundamental function $L(x,y)$ of a Finsler space F_n. Thus $A_n^1 = F_n$. For $k = n-1$ we obtain a special Hamilton space, the so called Cartan space ([4], [5]).

What we have shown is that a Finsler space F_n induces an areal space A_n^k, or the fundamental function L of an F_n determines by (4) the fundamental function $A(x,p) = \|p\|_F = \|p\|_E \|p_1\|_F$ of an $A_n^k : L \Rightarrow A$. Such an A_n^k will be denoted by $A_n^k(F_n)$.

D. Now the following question naturally arises: Is every A_n^k induced by an F_n? Our answer is negative for every n $(n \geq 3)$ and k $(n-1 \geq k \geq 2)$.

First we give negative examples among the A_3^2. Let T_xM be endowed with a euclidean metric, Σ a fixed plane through the origin 0 of $T_xM = E_3$ and η an arbitrarily small positive real. Denote by $\Delta \subset G_{3,2}$ those 2-planes σ through

0 in E_3 whose normals make an angle not greater than η with Σ (Δ is also the set of those σ which make an angle not greater than η with the normal of Σ.) Now consider an $A_3^2 = (M, A)$ such that

$$A(x, p_1) \begin{cases} = 1 & \text{if } \sigma(p_1) \in \Delta \\ \neq 1 & \text{if } \sigma(p_1) \notin \Delta \end{cases} \quad (\|p_1\|_E = 1). \tag{8}$$

If A_3^2 is determined by an F_3, then according to (4)

$$A(x, p_1) = \frac{\pi}{\|\sigma \cap \mathcal{D}(x)\|_E} \quad \forall x, p_1. \tag{9}$$

Then by (8)

$$\begin{matrix} a) \\ b) \end{matrix} \quad \|\sigma \cap \mathcal{D}(x)\|_E \begin{cases} = \pi & \sigma \in \Delta \\ \neq \pi & \sigma \notin \Delta, \end{cases} \tag{10}$$

where $\mathcal{D}(x)$ is the indicatrix body of F_3. The unit ball S^2 as a $\mathcal{D}(x)$ clearly satisfies (10,a). However, according to [3] S^2 is the only symmetrical star-shaped body with smooth boundary satisfying (10,a). But $\mathcal{D}(x) = S^2$ does not satisfy (10,b). Thus for an A_3^2, where $A(x, p)$ is given by (8), there exists no $\mathcal{D}(x)$ satisfying (10), and hence this $A_3^2 = (M, A)$ is no $A_3^2(F_3)$.

Now we can give examples for A_n^2 being no $A_n^2(F_n)$. Introduce again in T_xM a euclidean metric $\| \ \|_E$, and let E_3 be a 3-dimensional linear subspace of T_xM, Σ a fix plane in E_3 and η and Δ as in the previous paragraph (i.e. Δ denotes those 2-planes σ in E_3 through 0, whose normals in E_3 make an angle not greater than η with Σ). Consider an $A_n^2 = (M, A)$ such that $A(x, p)$ is given by (8). If A_n^2 is determined by an F_n with indicatrix body $\mathcal{D}(x)$, then

$$A(x, p_1) = \frac{\pi}{\|\sigma \cap \mathcal{D}_3(x)\|_E} \quad (\|p_1\|_E = 1), \ \forall \sigma \subset E_3$$

where $\mathcal{D}_3(x) := E_3 \cap \mathcal{D}(x)$ which is a symmetrical star-shaped body with smooth boundary. Hence we obtain again

$$\begin{matrix} a) \\ b) \end{matrix} \quad \|\sigma \cap \mathcal{D}_3(x)\|_E \begin{cases} = \pi & \sigma \in \Delta \\ \neq \pi & \sigma \notin \Delta, \end{cases} \quad (\sigma \subset E_3). \tag{11}$$

Then by (11,a) and [3] $\mathcal{D}_3(x)$ should be an S^2, but because of (11,b) it cannot be so. Thus (11) is satisfied by no star-shaped symmetrical body. Therefore an $A_n^2 = (M, A)$ given by (8) is no $A_n^2(F_n)$.

The proof of the statement that also for $k > 2$ there are A_n^k which cannot be induced by an F_n can be led on the same way, provided that the theorem of

[3] used in the previous two paragraphs can be extended to higher dimensions. This extension says the following: Denote by $\tilde{\Delta}$ the set of those hyperplanes through the midpoint of a symmetrical body $\mathcal{D}_{k+1}$ of the euclidean $k+1$-space whose normals make an angle not greater than η with a fixed hyperplane $\tilde{\Sigma}$, where η is an arbitrarily small positive constant. We claim:

The only symmetrical star-shaped body with smooth boundary which is intersected by elements of $\tilde{\Delta}$ in figures of area $\omega^{(k)}$ is the unit ball S^{k-1}.

The proof of this extended statement which will be published elsewhere uses the same ideas as the original proof [3].

These show that $A_n^k(F_n)$ is a special case only of an A_n^k. The difference between them is greater than between a euclidean and a Riemannian space, because the Riemannian metric in a tangent space is euclidean, but for the fundamental function $A(x,p)$ of a general $A_n^k = (M, A)$ there does not exist, even at a point $\overset{x}{0}$, an indicatrix body $\mathcal{D}(x)$ such that

$$A(\overset{x}{0}, p_1) = \frac{\omega^{(k)}}{\|\sigma \cap \mathcal{D}(\overset{x}{0})\|_E} \quad \forall \sigma \in G_{n,k}. \tag{12}$$

Denoting by $\mathbb{D} := \{\mathcal{D}\}$ the set of the symmetrical star-shaped bodies $\mathcal{D}$ with smooth boundary of the E_n centered at the origin 0 of E_n we can define

$$K(x) := \inf_{\mathcal{D} \in \mathbb{D}} \int_{G_{n,k}} \left| \|\sigma \cap \mathcal{D}(x)\|_E A(x, p_1) - \omega^{(k)} \right| d\sigma,$$
$$\sigma \in G_{n,k} \ (\|p_1\|_E = 1) \tag{13}$$

as the curvature of $A_n^k = (M, A)$ at $x \in M$. Then $K(x) = 0$ means that A_n^k is induced by an F_n, and $K(x)$ measures the deviation from this situation in a certain sense.

If $\mathcal{D}(x)$ is a field of bodies establishing the infimum in (13), then the Finsler space determined by this $\mathcal{D}(x)$ is a best osculation of $A_n^k = (M, A)$ among the Finsler spaces. The infimum may occur for several different $\mathcal{D}(x)$'s. Restricting the domain of integration to a submanifold $\Delta \subset G_{n,k}$ we obtain another infimum: $K_\Delta(x) \leq K(x)$, and the Finsler space determined by the $\mathcal{D}_\Delta(x)$ producing this restricted infimum yields a weaker osculation of the A_n^k.

We remark that both factors $\|\sigma \cap \mathcal{D}(x)\|_E$ and $A(x, p_1)$ in the integrand of (13) depend on the choice of the euclidean space E and the euclidean metric $\| \ \|_E$, but their product does not depend on it. Indeed, if we make T_xM into another euclidean space $\overline{E}$, and for a line segment AB we have $\|AB\|_{\overline{E}} = \lambda\|AB\|_E$, then $\|p_1\|_{\overline{E}} = \lambda^k \|p_1\|_E = \lambda^k$ $\ p_1 \subset \sigma$ and from this $\left\|\frac{1}{\mu} p_1\right\|_{\overline{E}} = 1$ $(\mu = \lambda^k)$. In σ there exists a $\bar{p}_1$ homothetic to p_1 and having area 1 in

$\overline{E}: \|\bar{p}_1\|_{\overline{E}} = 1$. From these $\bar{p}_1 = \frac{1}{\mu}p_1$. This yields $A(x,\bar{p}_1) = A\left(x, \frac{1}{\mu}p_1\right) = \frac{1}{\mu}A(x,p_1)$. On the other hand $\|\sigma \cap \mathcal{D}(x)\|_{\overline{E}} = \mu\|\sigma \cap \mathcal{D}(x)\|_E$. These show the invariance of $\|\sigma \cap \mathcal{D}(x)\|_E A(x,p_1)$ with respect to the euclidean metric chosen.

E. A more difficult problem is to decide about the fundamental function $A(x,p)$ of an A_n^k whether it is induced by an F_n or not. By (4) and (7) the following relation must hold in an A_n^k:

$$A(x,p_1) = \frac{\omega^{(k)}}{\|\sigma \cap \mathcal{D}(x)\|_E}$$

where σ is the k-plane of p_1, or p_1 is a parallelotop in σ such that $\|p_1\|_E = 1$. In our case for a fixed $\overset{x}{0}$ $A(\overset{x}{0},p)$ is given and $\mathcal{D}(\overset{x}{0})$ is the unknown. The question is the existence of such a $\mathcal{D}$ at any $\overset{x}{0} \in M$.

For an A_3^2 this means the following. Let $\mathfrak{i},\mathfrak{j},\mathfrak{k}$ be an orthonormed basis in the T_xM endowed with a euclidean metric. Denote by $\mathfrak{a}$ the unit vector of the intersection line a of the plane σ through the origin 0 and the plane $[\mathfrak{i},\mathfrak{j}]$. Let $P \in I(x) \cap \sigma$, $\alpha = \sphericalangle(\mathfrak{a},\mathfrak{i})$, $\beta = \sphericalangle(\sigma,[\mathfrak{i},\mathfrak{j}])$, $t = \sphericalangle(\mathfrak{a},\vec{OP})$, $\mathfrak{b} \in \sigma$ be such that $\mathfrak{b} \perp \mathfrak{a}$, $\mathfrak{b}^2 = 1$, $\mathfrak{b}$ unit vector orthogonal to $\mathfrak{a}$ in $[\mathfrak{i},\mathfrak{j}]$, $\mathfrak{c}$ the normalized of $\vec{OP}$ with endpoint P^*, and (φ,ν) the spherical coordinates of P^*. Then $\vec{OP}^* = \cos t\mathfrak{a} + \sin t\mathfrak{b} = \sin\nu(\cos\varphi\mathfrak{i} + \sin\varphi\mathfrak{j}) + \cos\nu\mathfrak{k}$, $\mathfrak{a} = \cos\alpha\mathfrak{i} + \sin\alpha\mathfrak{j}$, where $\mathfrak{b} = \cos\beta\mathfrak{b}' + \sin\beta\mathfrak{k}$ and $\mathfrak{b}' = -\sin\alpha\mathfrak{i} + \cos\alpha\mathfrak{j}$. From these

$$\begin{aligned}\varphi &= \operatorname{arctg}\frac{\sin\alpha + \operatorname{tg}t\cos\beta}{\cos\alpha - \operatorname{tg}t\cos\beta)} = \varphi(\alpha,\beta;t)\\ \nu &= \arccos(\sin t\sin\beta) = \nu(\beta;t).\end{aligned} \tag{14}$$

α and β can be considered as coordinates (parameters) of σ. Then for fixed α and β $I(\varphi(\alpha,\beta;t),\ \nu(\beta;t))$ gives the curve $I \cap \sigma$, and

$$\begin{aligned}\int_0^\pi I^2(\varphi(\alpha,\beta;t),\nu(\beta;t))dt &= T(\alpha,\beta)\\ T(\alpha,\beta) := \|\sigma(\alpha,\beta) \cap \mathcal{D}\|_E &= \frac{\pi}{A(\overset{x}{0},p)}\end{aligned} \tag{15}$$

is the euclidean surface area of the intersection $\sigma \cap \mathcal{D}$. If $A(x,p)$ is given, then (15) is an integral equation for $I(\varphi,\nu) \Leftrightarrow \mathcal{D}$ (applying (14)). A_3^2 is induced by an F_3 iff (15) has a solution for $I(\varphi,\nu)$. I do not know a concrete and simple condition for this. In case of an A_n^k, (15) is more complicated.

2 Connections

A In a Finsler space we measure the length of a vector $v \in V^n \simeq T_xM$, in a areal space the area of a parallelotop (or a class of parallelotops) $p \in GK_{n,k}$ is measured. Therefore the scene of the connection theory of a Finsler space is a vector bundle with typical fiber V^n, and the scene of the connection theory of an areal space is a fiber bundle $\eta = (\overline{E}, \overline{\pi}, M, GK_{n,k})$ with total space $\overline{E}$, projection operator $\overline{\pi}$, base manifold M and typical fiber $GK_{n,k}$, which is much more complicated than a vector bundle. We want to show that connections H_η in η can be reduced to certain special nonlinear connections H_ξ in a vector bundle $\xi = (E, \pi, M, V^{kn})$ $(n = \dim M)$.

The total space of ξ is a Whitney sum $E = \overset{k}{\oplus} TM$. Connections in ξ were already studied by R. Miron, M. Anastasiei, M. Kirkovits [6] and others. Let $U \subset M$ be a coordinate neighbourhood in M with local coordinates x^i, and $y \in V^{kn}$ with components y^a $a, b, \dots = 1, \dots, kn$. A nonlinear connection H_ξ in ξ is given by the splitting $T_zE = V_zE \oplus H_zE$ $(z = (x,y) \in \pi^{-1}(U) \subset E)$. H_zE is spanned locally by the vectors $\delta_i(z) = \frac{\partial}{\partial x^i}(z) - N_i^a(x,y)\frac{\partial}{\partial y^a}(z)$, where $N_i^a(z)$ are the coefficients of H_ξ. H_ξ is homogeneous (abbreviated by hom.) if for any homothety $\mu_t : \pi^{-1}(x) \to \pi^{-1}(x)$, $z = (x,y) \mapsto \hat{z} = (x, ty)$, $t \in R$ satisfies the relation $d\mu_t \delta_i(x,y) = \delta_i(x,ty)$. This is equivalent with the homogeneity of $N_i^a : N_i^a(x,ty) = tN_i^a(x,y)$.

Any integer a $(1 \leq a \leq kn)$ can uniquely be represented in the form $a = (\alpha - 1)n + j$ (greek indices run from 1 to k) with certain α and j. Let V^n be a vector space and $\overset{\alpha}{v} \in V^n$ with components $(\overset{\alpha}{v})^j$. Hence $y^a = (\overset{\alpha}{v})^j$ establishes a $1:1$ correspondence

$$V^{kn} \ni y \longleftrightarrow (\overset{1}{v}, \dots, \overset{k}{v}) = p \in \mathcal{P}$$

between the elements of V^{kn} and the parallelotops p. Thus the $1:\infty$ mapping $\rho^{-1} : GK_{n,k} \to \mathcal{P}$ of section 1.B is equivalent with $\overline{\pi}^{-1}(x) \simeq GK_{n,k} \to V^{kn} \simeq \pi^{-1}(x)$ denoted similarly by $\rho^{-1}(x)$. Then $\rho(x) : \pi^{-1}(x) \simeq V^{kn} \to GK_{n,k} \simeq \overline{\pi}^{-1}(x)$, and $\pi\rho^{-1}\Pi = x$ and $\overline{\pi}\rho z = x$. ρ^{-1} establishes a classification of V^{kn} into equivalence classes $\rho^{-1}(\Pi) = \{y_0, y_1, \dots, y\}$. y_0 and y are equivalent: $y_0 = (\overset{1}{v}_0, \dots, \overset{k}{v}_0) \sim (\overset{1}{v}, \dots, \overset{k}{v}) = y$ if $\overset{\alpha}{v}_0$ and $\overset{\alpha}{v}$ span the same k-plane $\sigma : \sigma_{y_0} = \sigma_y$ and the parallelotops p_0 and p spanned by them have the same euclidean area: $\|p_0\|_E = \|p\|_E$. Hence the condition of $y_0 \sim y$ is the existence of a determinant $\det |s^\alpha_\beta| = 1$ such that $\overset{\alpha}{v} = s^\alpha_\beta \overset{\beta}{v}_0$. Then in V^{kn} $y = sly_0$, where sl is a special unimodular linear transformation in V^{nk} with a quadratic matrix of rank kn complied from n blocks in the diagonal wich are just the matrices $\|s^\alpha_\beta\|$. The set $\{sl\} = Sl$ of these matrices forms a group of transformations for matrix multiplication. Thus an equivalence class $\rho^{-1}(\Pi) = Sly_0 = \{sly_0 \mid sl \in Sl\}$. For $s^\alpha_\beta = \delta^\alpha_\beta$ we obtain the unit of the group Sl. If s^α_β is given such that $|s^\alpha_\beta - \delta^\alpha_\beta| < \varepsilon$ for each α, β except $\alpha = \beta = k$, then s^k_k can be chosen so that $\det |s^\alpha_\beta| = 1$ and this s^k_k is near $\delta^k_k = 1$. This implies that Sly_0 is a $k^2 - 1$

dimensional submanifold K of V^{kn}.

A connection H_ξ in ξ implies mappings $\pi^{-1}(x) \to \pi^{-1}(x+dx)$. A H_ξ will be called *special* (denoted by spH_ξ) if it takes equivalence classes of a fiber into equivalence classes of the other fiber (i.e. if it preserves equivalence classes). This means a condition for the coefficients $N_i^a(x,y)$ of the connection H_ξ.

Proposition. [7] *H_ξ is special (it preserves equivalence classes) if and only if at arbitrary x and y_0,*

$$N_i^a(x,y) = s^\alpha_\beta(x) N_i^{(\beta-1)n+j}(x,y_0) + s^\alpha_{\beta i}(x)(\overset{\beta}{v}_0)^j,$$
$$\forall y = sly_0,\ sl \in Sl \quad ;(a = (\alpha-1)n+j)$$

holds for any $y = sly_0$, $sl \in Sl$, where $s^\alpha_{\beta i}$ are partial derivatives of $s^\alpha_\beta(x)$.

We want to show that any *hom.* spH_ξ determines a *hom.* H_η, and any *hom.* H_η can be produced in this way from a set of spH_ξ.

Theorem. *hom.* $spH_\xi \iff$ *hom.* $H_\eta \left(\begin{matrix} \Longleftarrow & \textit{is } 1:1 \\ \Longrightarrow & \textit{is } 1:\infty \end{matrix}\right)$

The basic idea of our proof is shown on the following diagram

$$\begin{array}{llllr} \text{a)} & x & \rho\{y_0, y_1, \dots, y, \dots\} = \Pi & (x) & \\ & & H_\xi \downarrow & \downarrow H_\eta & (16) \\ \text{b)} & x+dx & \rho\{\tilde y_0, \tilde y_1, \dots, \tilde y, \dots\} = \tilde\Pi & (x+dx) & \end{array}$$

This means the following: ρ takes the elements of an equivalence class $\{y_0, y_1, \dots, y, \dots\} \subset \pi^{-1}(x) \simeq V^{kn}$ into an element $\Pi \in \bar\pi^{-1}(x) \subset \eta$ of the fiber of η over x. On the other hand the connection H_ξ implies $\pi^{-1}(x) \to \pi^{-1}(x+dx)$. The image of y in this mapping is denoted by $\tilde y$. Since H_ξ is special the images $\tilde y_0, \tilde y_1, \dots, \tilde y, \dots$ form an equivalence class again, i.e. $\rho\{\tilde y_0, \tilde y_1, \dots, \tilde y, \dots\}$ is a unique element $\tilde\Pi(x+dx)$ of $\bar\pi^{-1}(x+dx)$ in η. The correspondence of $\tilde\Pi(x+dx)$ to $\Pi(x)$ determines the connection H_η. H_η is again a splitting $T_u\overline{E} = V_u\overline{E} \oplus H_u\overline{E}$ ($u = (x,\Pi) \in \bar\pi^{-1}(x) \subset \overline{E}$) and spanned locally by the vectors $\bar\delta_i(u) = \frac{\partial}{\partial x^i}(u) - N_i^A(x,\Pi)\frac{\partial}{\partial \Pi^A}(u)$, $A = 1, \dots, N$ ($N = \dim GK_{n,k}$).

B We prove the followings:

I) $spH_\xi \Rightarrow H_\eta$

We have to show only that $\tilde\Pi(x+dx)$ has a part linear in dx. Using our previous notions and statements,

$$\tilde\Pi(x+dx) = \rho(sl\tilde y_0) = \rho(\tilde y) = \rho(\overset{1}{\tilde v}, \dots, \overset{k}{\tilde v}),$$

where $\overset{\alpha}{\tilde{v}}(x+dx) = H_\xi \overset{\alpha}{v}(x) = \overset{\alpha}{v} + d\overset{\alpha}{v}$ with components $(d\overset{\alpha}{v})^j = -N_i^{(\alpha-1)n+j}(x,y)dx^i$. Thus

$$\rho\left(\overset{1}{\tilde{v}}(x+dx),\dots,\overset{k}{\tilde{v}}(x+dx)\right) = \rho\left(\overset{1}{v}(x)+d\overset{1}{v},\dots,\overset{k}{v}(x)+d\overset{k}{v}\right)$$
$$= \overset{1}{v}+d\overset{1}{v}\wedge\cdots\wedge\overset{k}{v}+d\overset{k}{v} = \Pi(x)+N_i(x,\Pi)dx^i+o(dx),$$

where $o(x)$ means higher order terms in dx^i. In components

$$\tilde{\Pi}^A(x+dx) = \Pi^A(x) - N_i^A(x,\Pi)dx^i + o(x,\Pi,dx)$$

and H_η is determined by $N_i^A(x,\Pi)$.

II) *hom.* $spH_\xi \Rightarrow$ *hom.* H_η

Homogeneity of H_η means that $H_\eta y = \tilde{y}$ implies $H_\eta ty = t\tilde{y}$ $(t \in R)$. According to (16) $\tilde{\Pi} = H_\eta\Pi = \rho \cdot H_\xi \cdot \rho^{-1}\Pi$. H_ξ is hom. according to our assumption, and ρ and ρ^{-1} are so according to what has been explained in sections 1.A and B. Hence $H_\eta\Pi = \tilde{\Pi}$ implies $H_\eta t\Pi = t\tilde{\Pi}$, i.e. H_η is homogeneous.

III) $H_\eta \Rightarrow H_\xi$

Consider a $\Pi_0 \in \bar{\pi}^{-1}(x_0) \subset \overline{E}$. Then $\rho^{-1}(\Pi_0) = \{y_0, y_1, \dots, y, \dots\} = Sly_0 = \{sly_0 \mid sl \in Sl\}$. It is a k^2-1 dimensional submanifold $K \subset \pi^{-1}(x-0) \simeq V^{kn}$ according to the third paragraph of this section 2. We order to $H_\eta(x_0,\Pi_0)$ a $H_\xi(x_0,y)$ at each $y \in \rho^{-1}(\Pi_0)$.

Choose a curve $\gamma(t) \subset M$, $\dot{\gamma}(t_0) \neq 0$ through $x_0 = \gamma(t_0)$. We denote by

$${}_\eta P_{\gamma;t_0,t}\Pi_0 = \Pi(t) \tag{17}$$

the horizontal lift of $\gamma(t)$ to Π_0 in $\overline{E}$ with respect to H_η. Then $\bar{\pi}(\Pi(t)) = \gamma(t)$ and

$$\rho^{-1}(\Pi(t)) =: K(t) \tag{18}$$

is a k^2 dimensional submanifold of E for which $\pi K(t) = x(t)$. Let $z(t) = (x(t); y(t)) = (x(t); \overset{1}{v}(t), \dots, \overset{k}{v}(t))$ $y(t_0) = y_0$ be a curve in $\rho^{-1}(\Pi(t))$. Thus

$$\dot{z}(t) \in T_{z(t)}K(t). \tag{19}$$

$z(t)$ is not uniquely determined, but any $y(t)$ can be represented in the form $y(t) = sl(t)y_0(t)$ with a fixed curve $y_0(t) \subset \rho^{-1}(\Pi(t))$ and with an appropriate 1-parameter set of transformations $sl(t)$. Since $z(t)$ is not determined uniquely consequently neither is $\dot{z}(t_0)$, but the difference between two such $\dot{z}_1(t_0)$ and

$\dot{z}_2(t_0)$ $(z_1(t), z_2(t) \subset \rho^{-1}(\Pi(t)))$ is always a vector A of $T_{z(t_0)}K : \dot{z}_1(t_0) - \dot{z}_2(t_0) = A \in T_{z(t_0)}K$. It is easy to see that for given $z(t) \subset \rho^{-1}(\Pi(t))$ and $A \in T_{z(t_0)}K$ one can find $z^*(t) \subset \rho^{-1}(\Pi(t))$, $z^*(t_0) = z(t_0)$ such that $\dot{z}(t_0) - \dot{z}^*(t_0) = A$. We remark that because of $d\pi\dot{z}(t_0) = \dot{x}(0) \neq 0$ $\dot{z}(t_0)$ is never vertical.

Let now $\gamma_i(t)$ be n curves in M through $x_0 = \gamma_i(t_0)$ having linearly independent tangents $\dot{\gamma}_i(t_0)$. Replacing $\gamma(t)$ by these $\gamma_i(t)$ in the construction of the last paragraph, we obtain curves $z_i(t)$ $z_i(t_0) = z_0$. Then $\dot{z}_1(t_0), \ldots, \dot{z}_n(t_0)$ span a subspace of $T_{z_0}E$ which we define as $H_\xi(z_0)$. Since $\rho^{-1}(\overline{E}) = E$, performing this construction for each $u = (x, \Pi) \in \overline{E}$, we obtain a H_ξ.

IV) *H_ξ defined in III) is independent of the choice of γ_i.*

We denote

$$_\eta P_{\gamma_i;t_0,t}\Pi_0 = \Pi_i(t), \qquad \rho^{-1}\Pi_i(t) = K_i(t).$$

Then each $\dot{z}_i(t_0)$ is determined similarly to $\dot{z}(t_0)$ in III) up to a vector $A_i \in K_i(t_0) = \rho^{-1}\Pi_0 = K$ only, where K is independent of the index i. Hence $H_\xi(z_0)$ may be any n-dimensional linear subspace S of the vector space $W \equiv [K, \dot{z}_1(t_0), \ldots, \dot{z}_n(t_0)] \subset T_{z_0}E$ spanned by K, $\dot{z}_1(t_0), \ldots, \dot{z}_n(t_0)$ with the property $d\pi S = T_{x_0}M$. Our construction in III) determines $H_\xi(z_0)$ up to this uncertainty. If we replace one of the $\gamma_i(t)$ in III), say $\gamma_n(t)$, by an arbitrary other $\gamma(t)$ through x_0 (such that $\dot{\gamma}(t_0)$ is linearly independent of the remaining $n-1$ $\dot{\gamma}_i(t_0)$), then we obtain an n-dimensional subspace $\bar{S} = [\dot{z}_1(t_0), \ldots, \dot{z}_{n-1}(t_0), \dot{z}(t_0) + A] \subset W$, $A \in K$ such that $d\pi\bar{S} = T_{x_0}M$. Hence $\bar{S}$ is an element (for every $A \in K$) of the set constructed as $H_\xi(z_0)$ in III).

V) *H_η defined in III) preserves equivalence classes.*

$K(t_0)$ and $K(t_1)$ are equivalence classes in $\pi^{-1}(x(t_0))$ and $\pi^{-1}(x(t_1))$ resp. We want to show that the parallel translated $\hat{K}(t_0, t_1)$ of the elements of the equivalence class $K(t_0)$ along γ from $x(t_0) = x_0$ to $x(t_1)$ according to H_ξ defined in III) coincide with the equivalence class $K(t_1)$ i.e. $\hat{K}(t_0, t_1) = K(t_1)$.

The horizontal lift $hl_{z_0}\gamma(t) = z(t)$ of $\gamma(t)$ to $(x_0, y_0) = z_0 \in K(t_0)$ according to H_ξ defined in III) is also the parallel translated $_\xi P_{\gamma;t_0,t}y_0 = \hat{z}(t) \in \hat{K}(t_0, t)$ of y_0 along $\gamma(t)$ with respect to H_ξ. Then $\dot{z}(t) \in H_\xi(z(t))$, and according to IV) and the construction of H_ξ in III) we get also $\dot{z}(t) \in T_{z(t)}K(t)$ (see (19)). Hence $z(t) \subset K(t)$, $z(t) = \hat{z}(t)$ and because of $\hat{K}(t_0, t_1) = \{\hat{z}(t_1)\}$ we get $\hat{K}(t_0, t_1) \subset K(t_1)$. Considering an arbitrary element $z_1 \in K(t_1)$ and starting our previous consideration with a $z_0 = {}_\xi P_{\gamma;t_1,t_0}z_1$ we obtain $\hat{K}(t_0, t_1) = K(t_1)$. This means that H_ξ defined in II) is a spH_ξ.

VI) *hom.* $H_\eta \Rightarrow$ *hom.* H_η

From the diagram (16) $H_\xi = \rho^{-1} \cdot H_\eta \cdot \rho$. Then H_η is homogeneous by our assumption, and ρ and ρ^{-1} are so according to their explanations in section 1.A and B. Hence also H_ξ is homogeneous.

VII) $H_\eta \xrightarrow{\text{III)}} H_\xi \xrightarrow{\text{I)}} H_\eta$
i.e. the H_ξ defined by H_η according to III) induces the original H_η.

Indeed, the H_ξ defined by H_η according to III) yields $\hat{K}(t_0, t_0 + dt) = K(t_0 + dt)$ as just shown in V). Let us consider an element of $K(t_0) : (x_0, y_0) = z_0 \in K(t_0) = \rho^{-1}(\Pi_0)$. Then

$$\begin{aligned} H_\xi z_0 = \tilde{z}_0 \in \hat{K}(t_0, t_0 + dt) = K(t_0 + dt) = \rho^{-1}\tilde{\Pi}_0 &= \rho^{-1} H_\eta \Pi_0 \\ &= \rho^{-1} H_\eta \rho z_0. \end{aligned}$$

From this $\{H_\xi z \mid z \in \sigma^{-1}\Pi\} \equiv H_\xi \rho^{-1}\Pi = \rho^{-1} H_\eta \Pi$ and hence $H_\eta \Pi_0 = \rho H_\xi \rho^{-1} \Pi_0$. But this gives the construction of the H_η from H_ξ in I) using the diagram (16). This means: the H_η constructed from H_ξ according to I) is just the H_η which induces H_ξ.

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