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Generalized Berwald Manifolds of Dimensions 2 and 3

DISSERTATION FOR THE DEGREE OF DOCTOR OF PHILOSOPHY (PhD)

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Doctoral School of Mathematical and Computational Sciences
Debrecen, 2023

I hereby declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences at the University of Debrecen in order to obtain a PhD Degree in Natural Sciences at the University of Debrecen.

The results published in the thesis are not reported in any other PhD theses.

Debrecen, 2023.

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I hereby confirm that the candidate, Márk Oláh conducted his studies with my supervision within the Differential Geometry and Its Applications Doctoral Program of the Doctoral School of Mathematical and Computational Sciences between 2019 and 2023. The independent studies and research work of the candidate significantly contributed to the results published in the thesis.

I also declare that the results published in the thesis are not reported in any other theses.

I support the acceptance of the thesis.

Debrecen, 2023.

signature of the supervisor

On Generalized Berwald Manifolds of Dimensions 2 and 3

Dissertation submitted in partial fulfillment of the requirements for the doctoral (PhD) degree in Mathematics (Natural Sciences).

Written by Márk Oláh, certified Mathematician.

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List of featured publications and new results

chapter	topic	new results	publ.
Chapter 3	Randers spaces	• describing (the torsion components of) compatible linear connections • describing (the torsion components of) the extremal compatible linear connection • necessary and sufficient condition for a Randers space to admit compatible linear connections (new proof)	[1]
Chapter 4	generalized Berwald manifolds of dim. 2	• unicity and expression of compatible linear connections in 2D (new proofs) • the divergence representation of the Gauss curvature • a necessary topological condition for a manifold to admit compatible linear connections • a construction method and explicit examples for generalized Berwald structures	[2]
Chapter 5	generalized Berwald manifolds of dim. 3	• conditions for a 3-dimensional Finsler manifold to admit a unique / infinitely many compatible linear connections • describing (the torsion components of) compatible linear connections • describing (the torsion components of) the extremal compatible linear connection	[3]

[1] Cs. Vincze, M. Oláh, *On the extremal compatible linear connection of a Randers space*, Journal of Geometry **111** (19) (2020).

[2] Cs. Vincze, M. Oláh, L. M. Alabdulsada, *On the divergence representation of the Gauss curvature of Riemannian surfaces and its applications*, Rendiconti del Circolo Matematico di Palermo Series 2, **69** (2020), pp. 1–13.

[3] Cs. Vincze, M. Oláh, *On generalized Berwald manifolds of dimension three*, Publicationes Mathematicae Debrecen, **100** (2022), pp. 337–363.

Introduction

I believe one should always start a book with sharing the motivation and reasons that drove the author to take up a pen and the goals they hope to achieve with it. For a dissertation, one of these goals is obvious, but I also had another one in mind: to make the presentation of the material as simple and motivated as possible. I was aiming at students, not experts of the theory, first because the results covered here have already appeared in publications and there was no point in just repeating them, and second, I wanted to compile something which is, if not an enjoyable, then at least a tolerable read for someone encountering these topics for the first time.

Allow me to talk about my choice of style a bit. For me, Mathematics has never been about definitions, theorems and formulas; in my opinion, its beauty lies in the motivations, insights and connections between things that might seem unconnected at first sight. For me, a technical definition means nothing if I can't answer the question **why** it was defined this way and find its raison d'être beneath the set of funny letters and symbols. I don't feel like I understand a theorem or formula if I cannot imagine the geometric picture behind it. Of course, there is no real mathematical work without technical definitions and formulas and no one says there should be; however, they should only be the tools necessary to express the underlying ideas, and not the central objects in the spotlight. In this spirit, when introducing any new concept, I always tried to give some motivation and intuition, pose the right questions and answer them.

I have to apologize. Though I tried my best, I know I have fallen short on more than one occasions. Differential geometry is a vast and beautiful field of Mathematics where the objects we work with are mostly not that hard to imagine and understand, but the computations and tools can be rather difficult (at least long) and demanding. I attempted as much simplification as I could, but still, there are some nasty, lengthy computations and proofs in later chapters I could not avoid or omit, and where one can easily lose sight of the idea and the geometric picture. Finding an easier path remains a task for future researchers and students, I guess. Whenever things got too technical and scary, I tried to break the flow with some lighter material to let the Reader breathe (like Sections 3.1, 4.4, 4.6, 5.4 or 5.5).

When compiling this book, I wanted to make it as self-contained as possible. For this reason and to see the bigger picture, that is, for the Reader to be able to place the topic discussed here to its rightful place in the field of Differential Geometry, Chapter 1 starts from the very basics (the concept of a manifold) and covers all the fundamental notions of the theory: manifolds, the tangent structure, linear connections, tensors, parallelism, holonomy and metrics. These are well-known by anyone who has ever studied Differential Geometry, and when short on time, can be omitted. Also, for the same reason, I do not give proofs and details most of the time, only the motivation and main ideas; there are excellent textbooks covering these subjects.

The real 'plot' starts with the last two sections of Chapter 1 and continues throughout Chapter 2; this can be considered the heart of the book, as all the central notions, results and methods are presented here that are used in all of the later chapters. Trying to transplant the idea of metrical linear connections from Riemannian Geometry to the more general setting of Finsler Geometry, we define the compatibility of a linear connection to a given Finsler metric such that parallel translation with respect to this connection preserves the Finslerian length of tangent vectors, and, in particular, the unit spheres (indicatrices) of the metric. A Finsler manifold together with such a connection (notion of parallelism) is called generalized Berwald. These spaces, the Finslerian analogies of the Riemannian metric–Levi-Civita connection pairings, are the central subjects of our study.

Compatible linear connections may or may not exist on a Finsler manifold and they may or may not be unique. Our primary interest is finding all compatible linear connections to a given Finsler metric on a given manifold, and the main tool for this is the so-called compatibility equation (or CEQ for short) describing all of these connections in terms of their torsion components. In Chapter 2, we examine what form the CEQ takes and what we can do with it at a fixed point p to make it as simple as possible in the hope of obtaining a solution. We also describe the 'double' geometric structure of the tangent spaces, constituted by the original Finsler metric and an adequately chosen Riemannian metric. It turns out that in all the cases we have been able to solve so far, the solution of the CEQ heavily depended on understanding the indicatrices and some vector fields in the tangent spaces connected to these metrics. At the end of the chapter, we introduce the notion of the extremal compatible connection, a natural choice among the possibly infinite family of compatible connections, i.e. the Finslerian analogy of the Levi-Civita connection.

Solving the CEQ in general, i.e. describing all compatible linear connections to an arbitrary Finsler metric (or deciding if there are any) is a formidable quest and is yet to be conquered. So far, we only have results in some special cases, which give the contents of the last 3 chapters. Though these obviously depend heavily on Chapter 2, they are mostly independent from each other, apart from some facts we occasionally need (but these are always referenced accordingly).

Chapter 3, the first chapter containing actual results, is more of an example, as we saw fit to start with an explicit example for Finsler metrics, the CEQ and the solution method. First, we try to motivate why the notion of Finsler metrics in general is important, and introduce the most simple family of them, called Randers metrics, obtained by translating the indicatrices of a given Riemannian metric and related to a famous and everyday-life navigation problem. In the first half of the chapter, we examine these metrics in detail, describing their structure, indicatrices and the notion of compatibility for them. The main result of the chapter is the complete solution of the CEQ for them, that is, describing all compatible connections, in particular, the extremal one, and giving a new proof for a previously known necessary and sufficient condition for their existence. These results first appeared in our paper [**1**].

Instead of the metric, we can also specialize the form of the connections we are looking for. One such choice is the so-called semi-symmetric connections whose torsion has a special structure described by a suitable 1-form. If a Finsler metric admits any compatible linear connections of this form, it admits exactly one; we give a new proof for this well-known theorem in Chapter 4 based on [**4**], and using the fact that all connections in 2D are of this form, this handles the 2-dimensional case. The same result can be obtained by examining the CEQ directly, or by using some more 'advanced' tools; we describe these two alternative paths as well, based on [**5**]. To give topological conditions for the possibility of a 2-dimensional manifold being generalized Berwald, we use the divergence representation of the Gauss curvature, published in our paper [**2**]. Completing our study of this case, we end the chapter by constructing explicit examples of generalized Berwald manifolds out of the well-known Euclidean and hyperbolic planes.

In the final chapter, we investigate the case of 3 dimensions. Though the CEQ here is definitely more difficult than in the 2D or semi-symmetric cases, luckily, a well-known vector field appearing in it makes it relatively easy to describe the different possible cases and the solutions. The main result of this chapter is that the uniqueness/infinity of connections compatible to the metric is related to the rotational symmetry of its indicatrices, as we first proved it in [**3**]. We give the solution in both the undetermined and determined cases, and also describe the extremal connection.

And this is what we have learned so far about compatible linear connections and generalized Berwald manifolds. All the results so far are obtained by closely inspecting the geometry of the tangent spaces, and using this knowledge directly for the vector fields appearing in the CEQ. In these dimensions and special cases, this method works really well, but even in dimension 4, it seems to break down at once. To deal with higher dimensions or even solve the problem in general, we will most likely need a new approach with new tools and tricks. Hopefully, someone will find them one day.

Table of Contents

CHAPTER 1

Manifolds, metrics and compatible linear connections

In this chapter, we summarize the basic definitions, facts and notations we will use throughout this book. We start from the very basics (the concept of manifolds) and go through every step of building the spaces we are interested in (generalized Berwald manifolds), concentrating on the main ideas and geometrical motivation instead of the technical details (which belong to, and can be found in, any good textbook, such as [**11**], [**12**], [**9**] and [**7**]). We do this for two reasons: first, to try to make the presentation as self-contained as possible, and second, to give the Reader an imaginary map on which they can find the path leading to the subject studied here in the beautiful and vast landscape of Differential Geometry.

1.1. Manifolds

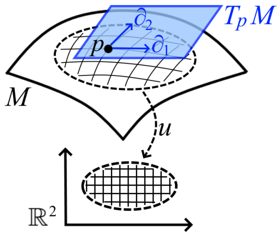
1.1.1. By an n-dimensional **manifold** we will always mean a smooth (differentiable) manifold without boundary, i.e. a topological space that is

- locally Euclidean: any point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$,
- second countable: its topology has a countable basis,
- Hausdorff: any two different points of the manifold can be separated by disjoint neighborhoods,
- and is endowed with a smooth structure (maximal smooth atlas), i.e. a set of charts that cover the whole manifold, are compatible (the transitions between them are smooth) and this set is maximal (contains every chart that is compatible to all of its elements).

We will adopt the following conventions:

- Whenever we say 'manifold' or 'space', it will refer to smooth manifolds.
- General points of manifolds will be denoted by p, q, ...
- Charts will be denoted by
$$u\colon U(\subseteq M) \to \mathbb{R}^n,\ u(p) = \big(u^1(p), \dots, u^n(p)\big) = (p^1, \dots, p^n).$$
- The dimension of a general manifold will be denoted by $n \in \mathbb{N}$, unless specified otherwise. Almost all of our manifolds will be of dimensions 2 or 3. We will call 2-dimensional manifolds **surfaces**.
- *Connectedness of manifolds will never be assumed*, unless specifically stated. On the other hand, *smoothness* of any object (curves, vector fields, differential forms, etc.) *will always be assumed.*

1.1.2. The **tangent space** of the manifold M at a point p (the set of all derivations at p of the algebra $C^\infty(M)$ of smooth functions, or, more intuitively but less precisely, the set of all tangent vectors of all possible curves through p) is denoted by T_pM. It is an n-dimensional vector space, and the coordinate vectors generated by a chart $(u^1, \dots, u^n)$ around p constitute a basis for it. We will denote this basis by one of the alternatives

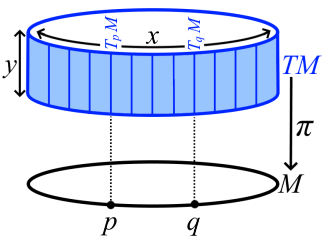


$$\left(\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}\right) \text{ or } (\partial_1, \dots, \partial_n).$$

A tangent vector $v_p \in T_pM$ can be written in this basis as

$$v_p = v^i \partial_{i|p}, \text{ where } v^i = v(u^i).$$

1.1.3. For the **tangent bundle** of M, we use the following notations:



- The **tangent manifold** (the disjoint union of all tangent spaces) is denoted by TM.
- The **footpoint projection** is defined by $\pi \colon TM \to M$, $\pi(v_p) = p$.

Local coordinates $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^n)$ on TM are obtained from local coordinates $(u^1, \dots, u^n)$ on M the following way:

- The $(x^1, \dots, x^n)$ are the location coordinates, telling us in which tangent space a tangent vector lives:

$$x^i(v_p) = u^i \circ \pi(v_p) = u^i(p) = p^i.$$

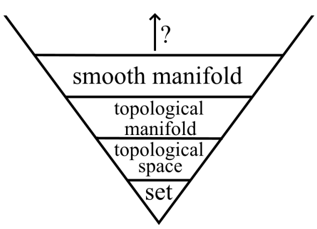
- The $(y^1, \dots, y^n)$ are the direction coordinates, telling us where the tangent vector points in its tangent space (more precisely, its coordinates in the basis induced by $(u^1, \dots, u^n)$):

$$y^i(v_p) = v_p(u^i) = v^i, \text{ where } v_p = v^i \partial_{i|p}.$$

The coordinate vectors (partial derivatives) will be denoted by

$$\widehat{\partial}_i := \frac{\partial}{\partial x^i} \qquad \text{and} \qquad \dot{\partial}_i := \frac{\partial}{\partial y^i}.$$

1.1.4. Manifolds are constructed to be the natural setting for geometrical investigations. But by itself, a manifold is just an analytic structure built on a topological one (see figure).



We can do calculus on it and consider isomorphisms that preserve some layers (bijections preserving the set structure, homeomorphisms preserving the topology and diffeomorphisms preserving the smooth structure), but a manifold itself does not carry any geometrical structure. To obtain one, we have to build our pyramid further, i.e. we need additional layers.

What questions do we need to answer to define a geometry?

1. *What are the points and lines and exactly how do they satisfy the standard geometrical axioms?*
   This gives us an incidence geometry: a very basic one, with points and lines, but with limited possibilities. For example, there is no way to distinguish diffeomorphic shapes: all triangles or quadrilaterals look the same. We cannot say anything about their sizes, because for that, for a richer geometry, we need a notion of measuring. So the other question is:
2. *How can we introduce metrical notions like distances, lengths and angles?*
   If we can answer these questions, we will be justified to call our structure a **geometry**.

1.2. Linear connections and parallelism

1.2.1. Let us tackle the first question: **what will be the points and lines of our geometry?** The first part is obvious, but what about lines? **What makes a line a line in Euclidean geometry, and how can we carry this notion over to an arbitrary manifold?**

First of all, let us clarify that by Euclidean lines we mean lines parameterized affinely (in the form $\boldsymbol{p}+t\boldsymbol{v}$). There are two ways we can characterize them:

- they are the curves minimizing the distance between points,
- they are the curves with zero acceleration, i.e. having parallel velocity vector fields (the paths of traveling objects with no outer force influencing them in classical Newtonian physics).

It is clear that generalizing the first property would require a notion of distance (and even then it would be hard to handle). So we need to work with the second one, which involves the notion of parallel vector fields, vector fields that do not change along a given curve (or, equivalently, in the direction of the velocity field of the curve). This requires a way of measuring the change of (differentiating) the given vector field along another. The question is: can it be done canonically on the manifold or does it require some additional structure?

The answer is: it cannot and it does. To compare the elements of a vector field to each other, we have to work with tangent vectors that live in different (tangent) spaces. In general, this cannot be done. Tangent spaces are just like bad neighbors: however 'close' they might be to each other, there is no connection (a canonical way to compare their elements) between them; if we want one, we have to introduce one. This will be called a linear connection, which mimics the properties of directional derivatives from Euclidean space.[1]

[1]The main difference is that in Euclidean space we can differentiate a vector field in the direction of a given **vector**, since it can be canonically translated ('pinned') to any point of the vector space. On manifolds, no such thing is possible (because tangent spaces are bad neighbors), therefore we need to choose a direction at each and every point (tangent space); in other words, we can only differentiate along a **vector field**.

1.2.1. Linear connections.

1.2.2. Definition. Let $\mathfrak{X}(M)$ denote the set of (smooth) vector fields on M. A **linear connection** is defined as a mapping

$$\nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \quad (X, Y) \mapsto \nabla_X Y$$

that satisfies the following properties:

(**LC1**) in the first variable it is tensorial, i.e. $C^\infty(M)$-linear:

$$\begin{aligned} \nabla_{X_1+X_2} Y &= \nabla_{X_1} Y + \nabla_{X_2} Y && \text{for any } X_1, X_2, Y \in \mathfrak{X}(M), \\ \nabla_{fX} Y &= f \cdot \nabla_X Y && \text{for any } f \in C^\infty(M),\ X, Y \in \mathfrak{X}(M); \end{aligned}$$

(**LC2**) in the second variable it is a derivation, i.e. $\mathbb{R}$-linear and satisfies a Leibniz rule:

$$\begin{aligned} \nabla_X (Y_1 + Y_2) &= \nabla_X Y_1 + \nabla_X Y_2 && \text{for any } X, Y_1, Y_2 \in \mathfrak{X}(M), \\ \nabla_X (\lambda Y) &= \lambda \cdot \nabla_X Y && \text{for any } \lambda \in \mathbb{R},\ X, Y \in \mathfrak{X}(M), \\ \nabla_X (fY) &= X(f) Y + f \nabla_X Y && \text{for any } f \in C^\infty(M),\ X, Y \in \mathfrak{X}(M). \end{aligned}$$

A linear connection is also called a **covariant derivative** sometimes.[2] $\nabla_X Y$ is the covariant derivative of the vector field Y along (in the direction of) the vector field X.

We summarize the basic facts about linear connections (the proofs can be found in any standard textbook such as [**12**] or [**9**]).

1.2.3. Lemma. *The value of $\nabla_X Y$ at a given point $p \in M$ depends only on the value of X at p (because of linearity in the first variable) and the values of Y in an arbitrary neighborhood of p (because of being a derivation in the second variable).*

1.2.4. Lemma. *Locally, in any chart, defining a linear connection ∇ is equivalent to choosing a family of smooth functions Γ_{ij}^k called **Christoffel symbols**. In detail,*

- *∇ determines its Christoffel symbols by the formula*

$$\boxed{\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k;} \tag{1}$$

- *an arbitrarily chosen family of Christoffel symbols determines ∇ (locally!) by the formula*

$$\boxed{\nabla_X Y = \left(X^i \partial_i (Y^k) + X^i Y^j \Gamma_{ij}^k \right) \partial_k, \ \textit{where } X = X^i \partial_i,\ Y = Y^i \partial_i.} \tag{2}$$

1.2.5. Corollary. *There exists a linear connection on any manifold.*

1.2.6. Example. The standard Euclidean connection on $\mathbb{R}^n$ is just the usual directional derivative:

$$\nabla_X Y = X(Y^i) \partial_i.$$

[2] Think about it as a differentiation rule changing from point to point, varying together with the vector field. However, this is not what the word 'covariant' actually stands for, that is related to coordinate changes.

1.2.2. From connections to geodesics.

1.2.7. From a linear connection we can build our geometry through the path highlighted above. The main steps are:

linear connection	$\sim$	measuring the change of a vector field in the direction of another
$\downarrow$		
linear connection along a curve	$\sim$	measuring the change of a vector field along a curve (in the direction of its velocity vector field)
$\downarrow$		
parallel translation along a curve	$\sim$	extending a tangent vector from a point to the whole curve such that the obtained vector field does not change along the curve
$\downarrow$		
parallel vector fields	$\sim$	vector fields along curves that do not change along the curve
$\downarrow$		
geodesics	$\sim$	curves whose velocity vector fields are parallel along them (they are the generalized lines)

In fact, the first three notions are just the different faces of the same thing: prescribing a linear connection is equivalent to prescribing the way of derivation or parallel translation along the curves of the manifold. (And thus any property of linear connections can be rephrased for these.) What we will need later is some additional properties of parallel vector fields and parallel translation.

1.2.8. Theorem. *Let us fix a linear connection ∇ on a manifold M and denote the connection it induces along a curve c by ∇_c.*

- *A vector field X along the curve c is parallel (does not change, i.e. $\nabla_c X = 0$) if and only if locally its components $X^k := y^k \circ X$ satisfy*

$$\boxed{X^{k\prime} = -X^j c^{i\prime} \Gamma^k_{ij} \circ c} \qquad k \in \{1, \ldots, n\}. \tag{3}$$

  *This first-order linear ODE is called the **differential equation of parallel vector fields**.*

- *Given a tangent vector v in a tangent space at some point of the curve c, it can uniquely be extended to a parallel vector field X along the whole curve (because of the linearity of the ODE above).*
  *The individual values $X(c(t))$ of this field are called the **parallel translates** of the initial vector v to the points of the curve, and the mappings φ taking all tangent vectors of the initial tangent space to the others are called **parallel translations/transports** (along c). They are linear isomorphisms between two tangent spaces.*

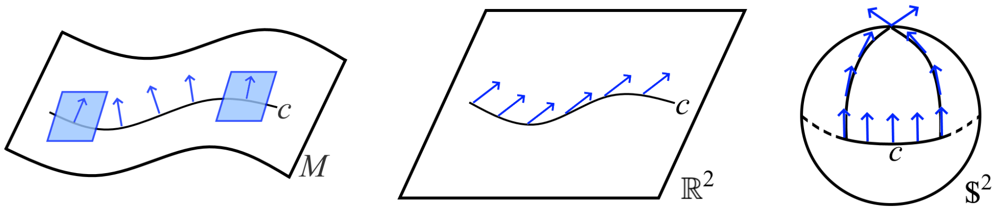
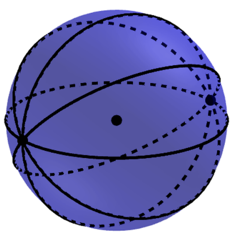


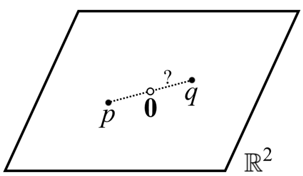
FIGURE 1. Parallel transports on an arbitrary and some well-known manifolds.

1.2.9. With all these tools at our disposal, we can define the general version of lines on manifolds, called **geodesics**, as curves whose velocity vector fields are parallel along the curve. We will not investigate them, so we don't go into details here, but perhaps it is worth mentioning that they can behave quite differently than what we are used to in Euclidean geometry. For example, let us consider the 'standard' linear connection on the 2-sphere inherited from $\mathbb{R}^3$,[3] where the geodesics are the great circles.



In this geometry, most of the standard axioms and properties will not hold:

- through two antipodal points we can draw infinitely many different lines (geodesics),
- any two different lines intersect in two points,
- after antipodal points no geodesic minimizes the distance between its points.



It is also not hard to construct an example where there are two different points that cannot be joined by any geodesic: take the plane with the standard Euclidean connection and remove the origin. There is a property called **completeness** that ensures this axiom does not fail; if we want to build a well-behaved geometry, completeness should always be satisfied.

The bottom line is: these geometries are very different from the Euclidean one, and most properties we are used to are satisfied only locally, or not even that way.

[3]When taking the covariant derivative of tangent vectors on the sphere, we view them as vectors in $\mathbb{R}^3$, take their derivative in $\mathbb{R}^3$, and consider only the tangential part of the result to the sphere.

1.2.3. Parallelism in general.

So far we have used linear connections to measure the change of vector fields and define their parallelism. But there are lots of other types of 'fields' on manifolds – **can we define a notion of parallelism for them, too?**

Yes, we can: a linear connection (defined for vector fields) can be used to measure the change of any tensor fields on the manifold.

1.2.10. DEFINITION. Let $\mathfrak{X}(M)$ and $\mathfrak{X}^*(M)$ denote the set (module) of (smooth) vector fields and 1-form fields on the manifold M, respectively. Then by a **tensor field** of type (k,l) we mean a $C^\infty(M)$-multilinear mapping

$$A\colon \underbrace{\mathfrak{X}^*(M)\times\cdots\times\mathfrak{X}^*(M)}_{k\text{ times}}\times\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{l\text{ times}}\to C^\infty(M).$$

We denote the set of such tensor fields on M by $\mathcal{T}_l^k(\mathfrak{X}(M))$.

1.2.11. REMARK. Let us note some things about tensors.

- Some special cases are:
  - $\mathcal{T}_0^0(\mathfrak{X}(M))$ is by definition $C^\infty(M)$,
  - $\mathcal{T}_0^1(\mathfrak{X}(M))=\mathfrak{X}(M)$ and $\mathcal{T}_1^0(\mathfrak{X}(M))=\mathfrak{X}^*(M)$,
  - and $\mathcal{T}_s^1(\mathfrak{X}(M))$ is canonically isomorphic to the set of multilinear mappings from $(\mathfrak{X}(M))^s$ to $\mathfrak{X}(M)$.
- We call these objects 'fields' because they are 'glued together' from their pointwise values, i.e. they can be considered pointwise by restricting the arguments. These pointwise values are tensors of the same type, but taking only vectors and covectors as variables. For example, the value of A at the point p is the tensor

  $$A_p\colon \underbrace{T_p^*M\times\cdots\times T_p^*M}_{k\text{ times}}\times\underbrace{T_pM\times\cdots\times T_pM}_{l\text{ times}}\to\mathbb{R}$$

  such that

  $$A\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)(p)=A_p\left(\omega^1|_p,\dots,\omega^k|_p,X_1|_p,\dots,X_l|_p\right).$$

  The set of such tensors will be denoted by $\mathcal{T}_l^k(T_pM)$.
- Choosing local coordinates $(u^1,\dots,u^n)$, we can use the (local) bases

  $$(\partial_1,\dots,\partial_n)\ \text{ and }\ \left(du^1,\dots,du^n\right)$$

  of $\mathfrak{X}(M)$ and $\mathfrak{X}^*(M)$, respectively, to write tensor fields locally as

  $$\begin{aligned}
  & A\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)\\
  =\ & A\left(\omega_{i_1}du^{i_1},\dots,\omega_{i_k}du^{i_k},X^{j_1}\partial_{j_1},\dots,X^{j_l}\partial_{j_l}\right)\\
  =\ & \omega_{i_1}\cdots\omega_{i_k}\cdot X^{j_1}\cdots X^{j_l}\cdot A\left(du^{i_1},\dots,du^{i_k},\partial_{j_1},\dots,\partial_{j_l}\right)\\
  =\ & \partial_{i_1}(\omega^1)\cdots\partial_{i_k}(\omega^k)\cdot du^{j_1}(X_1)\cdots du^{j_l}(X_l)\cdot A^{i_1\dots i_k}_{j_1\dots j_l}\\
  =\ & A^{i_1\dots i_k}_{j_1\dots j_l}\cdot\partial_{i_1}\otimes\cdots\otimes\partial_{i_k}\otimes du^{j_1}\otimes\cdots\otimes du^{j_l}\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right).
  \end{aligned}$$

In short,

$$A = A^{i_1 \dots i_k}_{j_1 \dots j_l} \cdot \partial_{i_1} \otimes \dots \otimes \partial_{i_k} \otimes du^{j_1} \otimes \dots \otimes du^{j_l},$$

where the functions

$$A^{i_1 \dots i_k}_{j_1 \dots j_l} := A\left(du^{i_1}, \dots, du^{i_k}, \partial_{j_1}, \dots, \partial_{j_l}\right)$$

are called the **tensor components** of A in the chosen coordinates.

Now we show that both faces of a linear connection can be extended naturally to tensor fields; the first one (covariant derivation) is a bit more abstract, but the second one (parallel translation) is much more intuitive.

1.2.12. Definition. Let ∇ be a linear connection. We define the **covariant derivative of tensor fields** along a vector field X, which will be tensor fields of the same type, the following way.[4]

- On $\mathcal{T}^0_0(\mathfrak{X}(M)) = C^\infty(M)$, ∇_X means ordinary differentiation:
$$\nabla_X f := Xf;$$
- on $\mathcal{T}^1_0(\mathfrak{X}(M)) = \mathfrak{X}(M)$, ∇_X is defined ab ovo;
- on $\mathcal{T}^0_1(\mathfrak{X}(M)) = \mathfrak{X}^*(M)$, we define ∇_X as
$$(\nabla_X \omega)(Y) := X(\omega(Y)) - \omega(\nabla_X Y),$$
so that writing $(\omega, Y) := \omega(Y)$, we have the Leibniz rule
$$\nabla_X(\omega, Y) = (\nabla_X \omega, Y) + (\omega, \nabla_X Y).$$
- Using these, we can define ∇_X for any (k,l)-type tensor field A as

$$\begin{aligned}(\nabla_X A)(\omega^1, \dots, \omega^k, Y_1, \dots, Y_l) := {} & X\left(A\left(\omega^1, \dots, \omega^k, Y_1, \dots, Y_l\right)\right) \\ & - \sum_{j=1}^{k} A\left(\omega^1, \dots, \nabla_X \omega^j, \dots, \omega^k, Y_1, \dots, Y_l\right) \\ & - \sum_{j=1}^{l} A\left(\omega^1, \dots, \omega^k, Y_1, \dots, \nabla_X Y_j, \dots, Y_l\right).\end{aligned}$$

1.2.13. Definition. The **total covariant derivative** of a tensor field A of type (k,l) with respect to a linear connection ∇ is a tensor field of type $(k, l+1)$, and is defined as

$$(\nabla A)\left(\omega^1, \dots, \omega^k, Y_1, \dots, Y_l, X\right) := (\nabla_X A)\left(\omega^1, \dots, \omega^k, Y_1, \dots, Y_l\right).$$

This gives a notion of 'change' for any tensor fields, and thus a possibility to talk about them being 'constant'.

1.2.14. Definition. We call a **tensor field** A **covariant constant** with respect to a linear connection ∇ if its total covariant derivative vanishes. Or, in other words, for any vector field X, $\nabla_X A \equiv 0$.

[4]It can be shown that it is the unique way of extension such that we get a so-called **tensor derivation** (which satisfies a Leibniz rule with the tensor product and commutes with all trace operators).

Let's see what happens if we treat ∇ as a rule for parallel translation.

1.2.15. Definition. Let ∇ be a linear connection, choose two arbitrary points $p, q \in M$ with a curve c connecting them and let φ denote the parallel translation along it from T_pM to T_qM (which is a linear isomorphism). Then φ induces a **parallel translation between** the **pointwise tensor spaces** $\mathcal{T}_l^k(T_pM)$ and $\mathcal{T}_l^k(T_qM)$ the following way. φ's action

- on a real number is that it leaves it constant;
- on tangent vectors is given ab ovo;
- on covectors is given by $(\varphi\omega_p)(v) := \omega_p(\varphi^{-1}(v))$, see figure 2.
- Then for $A_p \in \mathcal{T}_l^k(T_pM)$ we define its parallel translate to $\mathcal{T}_l^k(T_qM)$ as

$$\begin{aligned}(\varphi A_p)\left(\omega^1, \ldots, \omega^k, v_1, \ldots, v_l\right) :=\\ A_p\left(\varphi^{-1}(\omega^1), \ldots, \varphi^{-1}(\omega^k), \varphi^{-1}(v_1), \ldots, \varphi^{-1}(v_l)\right).\end{aligned} \tag{4}$$

1.2.16. Definition. We call a **tensor field parallel** with respect to ∇ if it is invariant under any parallel translation along any curve.

1.2.17. Statement. *For any tensor field A, the following are equivalent:*

(a) A is covariant constant,
(b) A is parallel,
(c) A's value is constant along parallel vector fields and covector fields,
(d) A can be recovered by parallel translations from its value at one point of every component of the manifold.

Proof. $\boxed{(b) \iff (c)}$ A being parallel means $A_q = \varphi A_p$ for any parallel translation φ. Then it takes the same value at any point on parallel covector and vector fields by definition. Conversely, if (c) is true, then the pointwise value A_q of A equals any parallel translate of some A_p, i.e. A is parallel.

$\boxed{(a) \iff (c)}$ If $\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l$ are parallel 1-form and vector frames along a curve c and $X = c'$, then $\nabla_X \omega^i = \nabla_X Y_j = 0$, so

$$(\nabla A)\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l, X\right) = X\left(A\left(\omega^1, \ldots, \omega^k, Y_1, \ldots, Y_l\right)\right).$$

Then (a) is true $\iff$ this expression equals 0 for any fields $\iff$ this equals 0 for any parallel frames along curves $\iff$ A is constant along parallel fields.

$\boxed{(b) \iff (d)}$ is straight-forward. $\square$

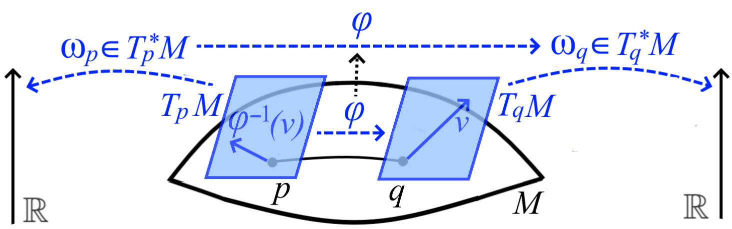


Figure 2. Parallel translation of 1-forms.

1.2.4. The holonomy of a connection.

1.2.18. We know that defining a linear connection is equivalent to prescribing the rule of parallel translation along any curve of the manifold. The latter is much easier to visualize and we can better understand the geometrical properties of the connection from it, but handling the family of all parallel translations along all the curves seems more difficult.

If we are familiar only with Euclidean geometry, we might think that a parallel translation is uniquely determined by the end points only (and independent of the curve connecting them). In general, this is not the case at all: choosing different curves between the same two points and parallel translating the same vector along them can yield different results.

For example, let us consider the 2-sphere with the linear connection (parallel translation) inherited from $\mathbb{R}^3$, take a parallel vector field along the equator, and translate its values to the whole sphere along the meridians. It is intuitively clear that as we travel from the equator towards the poles, the elements of the vector field start to 'incline inward' more and more. So, depending on along which meridian we travel, we will get different results at the poles (see the figure on the next page). We might even suspect that this is a consequence of the space being 'curved'.[5]

So how to describe the family of parallel translations? It turns out that the special case when we are translating vectors back to the same point along loops (closed, piecewise smooth curves) captures the essence of the connection.

1.2.19. Definition. For a given linear connection ∇, the parallel translations along all the loops based at a point p constitute a group (as linear isomorphisms of T_pM with the composition), called the **holonomy group** of the connection at p. It is denoted by $\mathrm{Hol}_p(\nabla)$.

Let us summarize some basic facts about these groups (see e.g. [8]).

1.2.20. Statement. *Let ∇ be a linear connection on a manifold M.*

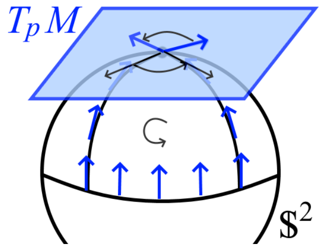
(a) The holonomy groups $\mathrm{Hol}_p(\nabla)$ are Lie subgroups of the general linear groups $GL(T_pM)$ of the tangent spaces. If M is orientable, then the elements have positive determinant, i.e. $\mathrm{Hol}_p(\nabla) \subseteq GL_+(T_pM)$.

(b) The holonomy groups at all the points of a connected component of M are conjugates of each other (in particular, isomorphic). Thus the holonomy is a global invariant of ∇ (at least on components), and for a connected manifold M we may simply write $\mathrm{Hol}(\nabla)$.

[5] And we are right, this is exactly the defining property of the curvature of the space. We will discuss it later.

1.2.21. Example. Let us examine the two most basic examples.

- In Euclidean space, since parallel translation does not depend on the path chosen, translating back to the same point always results in the same vector. In other words, here $\mathrm{Hol}(\nabla) = \{\mathrm{id}\}$.
- On the 2-sphere it is not hard to see that the result of the translation along a loop depends on the angle enclosed by the velocity of the curve at the starting and finishing points: the translation is exactly a rotation by this angle. Thus $\mathrm{Hol}(\nabla) = SO_2$.



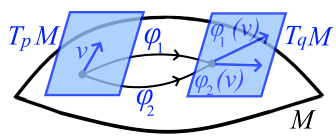
Now we investigate how the special case of the holonomy can tell us (almost) everything about parallel translations in general.

1.2.22. Lemma. *Let ∇ be a linear connection on a manifold M. Then parallel translation of tangent vectors between two points is path-independent $\iff$ the holonomy of the connection is trivial.*

Proof. If any parallel translation is path-independent, then it is true for translating back to the same point along loops. Since parallel translation along the constant loop is the identity, then so are all the other parallel translations along loops, i.e. $\mathrm{Hol}(\nabla) = \{\mathrm{id}\}$.

Let us suppose that parallel translation is not path-independent, i.e. there are points p and q with two curves between them such that the translations φ_1 and φ_2 along them yield different results for some vector v, i.e. $\varphi_1(v) \neq \varphi_2(v)$. Then it is easy to construct an element of the holonomy which is not the identity map.

For this, let us form a loop from these curves and consider the translation $\varphi_2^{-1} \circ \varphi_1$ along it (which is an element of $\mathrm{Hol}_p(\nabla)$). We see that it is not the identity: φ_2 is an isomorphism between T_pM and T_qM pairing v with $\varphi_2(v)$, but to have the identity map we should have



$$v \xrightarrow{\varphi_1} \varphi_1(v) \xrightarrow{\varphi_2^{-1}} v \iff \varphi_2^{-1}(\varphi_1(v)) = v \iff \varphi_1(v) = \varphi_2(v),$$

which is a contradiction. □

1.2.23. Remark. Note that the above must hold for all tangent vectors; if there is one vector whose parallel translation is path-independent, it only means that the elements of the holonomy have a fixed line (an eigenspace for 1) in common.

The above idea works for any geometrical object for which we can define parallel translation, so we can complete Statement 1.2.17. with yet another condition.

1.2.24. STATEMENT. *Let ∇ be a linear connection on a manifold M. For a tensor field A, the parallel translation of its pointwise value A_p is path-independent $\Longleftrightarrow$ the holonomy preserves A_p.*

PROOF. The proof is exactly the same as in the previous lemma. □

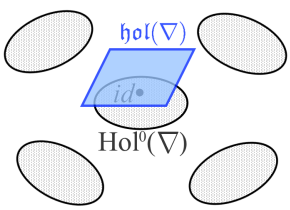
1.2.25. REMARK. Be careful: the above are **not** equivalent to A being parallel, as the (uniquely existing) parallel translated values might not coincide with the original values of the tensor field there, i.e. $\varphi A_p \neq A_q$.[6] What this result implies is that, considering a tensor at a single point only in each component of the manifold, if preserved by the holonomy, it can uniquely be extended to the whole manifold by parallel translations.

We conclude this section with some other definitions and technical things. First note that since $\mathrm{Hol}(\nabla)$ can be any kind of Lie group, it might be worth to restrict our attention to its identity component.

1.2.26. DEFINITION. If we restrict our attention to contractible loops only,[7] the parallel translations along them form a group at the given point, called the **restricted holonomy group** and denoted by $\mathrm{Hol}^0(\nabla)$. This group is exactly the identity component of $\mathrm{Hol}(\nabla)$, and thus a closed, connected normal subgroup of it.

1.2.27. REMARK. Note that though $\mathrm{Hol}^0(\nabla)$ is closed in $\mathrm{Hol}(\nabla)$, $\mathrm{Hol}(\nabla)$ is not necessarily closed in $GL(T_pM)$. So in general, $\mathrm{Hol}^0(\nabla)$ is not necessarily a closed subgroup of $GL(T_pM)$, which makes things slightly more difficult.

1.2.28. DEFINITION. The **holonomy algebra** of ∇ is the Lie algebra of the Lie groups $\mathrm{Hol}^0(\nabla)$ and $\mathrm{Hol}(\nabla)$, denoted by $\mathfrak{hol}(\nabla)$.[8]



1.2.29. REMARK. If we managed to compute $\mathfrak{hol}(\nabla)$ somehow, we would gain information about the restricted holonomy group $\mathrm{Hol}^0(\nabla)$ and we know that all the components of $\mathrm{Hol}(\nabla)$ are diffeomorphic to it. One more thing is also certain: the number of these components is countable, since there exists a surjective homomorphism from the (always countable) fundamental group of the manifold to the factor group $\mathrm{Hol}(\nabla)/\mathrm{Hol}^0(\nabla)$.

[6]Think about $\mathbb{R}^2$ with the usual Euclidean metric and connection whose holonomy is trivial. This holonomy preserves any value of any tensor (vector) field at any point, so they can be translated to all points of the plane path-independently, but we only obtain the original tensor (vector) field if it was parallel in the first place.

[7]They are loops that can be continuously deformed to a single point; more formally, they are path-homotopic to the constant loop.

[8]From the theory of Lie groups it is clear that the Lie algebra of a Lie group is determined by a neighborhood of its identity, in particular, the above Lie groups have the same Lie algebras.

1.2.5. Torsion and curvature.

Now that we have talked so much about tensors, it's time to introduce two of the most important ones: the torsion and the curvature.

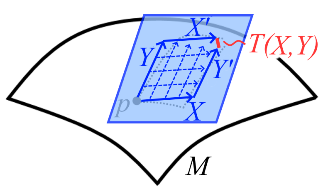
1.2.30. Definition. The **torsion tensor** of a linear connection ∇ is

$$\boxed{T\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M),\ T(X,Y):=\nabla_X Y-\nabla_Y X-[X,Y]\,.} \tag{5}$$

- In local coordinates $(u^1,\dots,u^n)$ we define the **torsion components** T_{ij}^k by the formula
$$T(\partial_i,\partial_j)=T_{ij}^k\partial_k. \tag{6}$$
With this, for any $X,Y\in\mathfrak{X}(M)$ we can write
$$T(X,Y)=T(X^i\partial_i,Y^j\partial_j)=X^iY^jT(\partial_i,\partial_j)=X^iY^jT_{ij}^k\partial_k.$$
- The connection between the torsion components and the Christoffel symbols in the same coordinates is given by
$$T_{ij}^k=\Gamma_{ij}^k-\Gamma_{ji}^k.$$
- Since T is anti-symmetric, so are its components in the lower indices:
$$T_{ij}^k=-T_{ji}^k.$$
- We say that ∇ is **torsion-free** if $T\equiv 0$. This is equivalent to all torsion components being 0 or the Christoffel symbols being symmetric in the lower indices: $\Gamma_{ij}^k=\Gamma_{ji}^k$.

1.2.31. Remark. Let's put some intuitive insight behind this tensor.

- At face value, we can motivate the definition with the following comparison. Remember that the Lie bracket $[X,Y]$ (which is a canonical data of a manifold, i.e. does not depend on any other additional structures) acts on a function f as
$$[X,Y]\,f=X(Yf)-Y(Xf).$$
Evaluating the torsion formula on a function f, we get
$$[X,Y]\,f=(\nabla_XY)(f)-(\nabla_YX)(f)-T(X,Y)(f).$$
So in a way, the torsion measures how differently the covariant derivation ∇ works from the canonical Lie derivation of the manifold; in this context, the torsion being zero means 'the closest analogy possible'.
- The true geometric meaning of the torsion is the following. If we consider the parallel translates of two tangent vectors $X,Y\in T_pM$ along each other, the resulting parallelogram-to-be might fail to close up by a vector, which is given by $T(X,Y)$. So geometrically, a connection being torsion-free means that these parallelograms actually close up.



1.2.32. Definition. The **curvature tensor** of a linear connection ∇ is

$$\boxed{\begin{aligned} R\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\times\mathfrak{X}(M) &\to \mathfrak{X}(M)\\ R(X,Y)Z &:= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.\end{aligned}} \tag{7}$$

- In local coordinates $(u^1,\ldots,u^n)$ the **curvature components** R^l_{ijk} are defined by

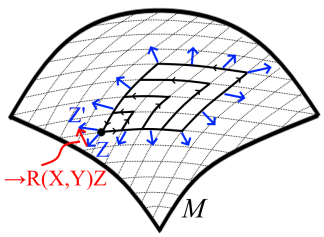
$$R(\partial_i,\partial_j)\partial_k = R^l_{ijk}\partial_l. \tag{8}$$

With this, for any $X,Y,Z\in\mathfrak{X}(M)$ we can write

$$R(X,Y)Z = R(X^i\partial_i, Y^j\partial_j)Z^k\partial_k = X^iY^jZ^kR(\partial_i,\partial_j)\partial_k = X^iY^jZ^kR^l_{ijk}\partial_l.$$

- The curvature tensor is anti-symmetric in the first two variables, and so are its components in the first two lower indices: $R^l_{ijk} = -R^l_{jik}$.
- The connection is called **flat** if its curvature is identically zero.

1.2.33. Remark. To see the geometric meaning of curvature, take two vector fields X,Y with vanishing Lie bracket. This implies that their flows must commute, in particular, we can construct a 'flow parallelogram' which follows the flow of Y, then X, then $-Y$, then $-X$ for some constant time back to the starting point. Depending on how big these flow parallelograms we choose, we obtain a family of holonomy transformations by translating the vectors $Z\in T_pM$ along these special loops back to T_pM.



Shrinking these paths to p and taking the derivative of the corresponding family of holonomy transformations, we get exactly the transformation $R(X,Y)$, for which $R(X,Y)Z$ gives the difference between the initial vector Z and its parallel translate around the 'infinitesimally small flow parallelogram'.

The above have another extremely important consequence: since the transformations $R(X,Y)$ are obtained as the derivative of a family of holonomy transformations at the identity, they are elements of the holonomy algebra! What may be more surprising is that $\mathfrak{hol}(\nabla)$ contains nothing else. This justifies the famous mathematical proverb: 'holonomy is curvature'.

1.2.34. Theorem (Ambrose–Singer holonomy theorem, [23]). *For a linear connection ∇ on a connected manifold M, the holonomy algebra $\mathfrak{hol}(\nabla)$ is spanned by all the curvature endomorphisms $R(X,Y)$ (for any pairs of tangent vectors at any points).*

1.2.35. Corollary. *In particular, because of* $\dim\mathrm{Hol}(\nabla) = \dim\mathfrak{hol}(\nabla)$, *we have the following.*

- $\mathrm{Hol}(\nabla)$ *is a discrete Lie group* $\iff \dim\mathrm{Hol}(\nabla) = 0 \iff \mathfrak{hol}(\nabla) = \{\mathbf{0}\} \iff R(X,Y)Z\equiv 0$;
- $\dim\mathrm{Hol}(\nabla) = n\geq 1 \iff \dim\mathfrak{hol}(\nabla) = n \geq 1 \iff$ *the curvature is nonzero somewhere.*

1.3. Metrics and measurements

1.3.1. The second ingredient for a geometry is an appropriate way of measuring. It is again clear that such a notion does not exist canonically on a manifold.[9] We need to introduce another level of structure again: a metric.[10] Let us think through what we need.

To measure the distances between points (without explicitly giving such a map), a reasonable choice is to define it as the infimum of the lengths of curves connecting them (at least on a connected manifold). But we don't have such a notion either, we have to define that, too. From elementary calculus, it is again clear that the length should be defined as the integral of the length of the velocity of the curve. That leads us to the point: to measure distances between points of the manifold (to have a metric on the manifold in the topological sense), one needs to be able to measure the tangent vectors in the tangent spaces (to have a metric on the tangent manifold).

1.3.1. Introducing metrics.

There are different kinds of metrics depending on what we choose for the tangent spaces (they are vector spaces, so we can choose inner products, norms, or even more general metrics, the so-called Minkowski norms) and how these change from point to point (i.e. what they depend on).

1.3.2. The simplest is the **Euclidean metric**: at all points of the vector space $\mathbb{R}^n$ (which is a very special manifold) we choose the same way to measure, the standard Euclidean inner product:

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \left\langle v^i \boldsymbol{e}_i, w^j \boldsymbol{e}_j \right\rangle := v^1 w^1 + \cdots + v^n w^n = \delta_{ij} v^i w^j.$$

This defines a norm, angle and distance by

$$\|\boldsymbol{v}\| := \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle} = \sqrt{(v^1)^2 + \cdots + (v^n)^2},$$

$$\cos\left(\measuredangle(\boldsymbol{v}, \boldsymbol{w})\right) := \frac{\langle \boldsymbol{v}, \boldsymbol{w} \rangle}{\|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\|},$$

$$d(\boldsymbol{v}, \boldsymbol{w}) := \|\boldsymbol{v} - \boldsymbol{w}\|.$$

Orthogonal and orthonormal vectors are also defined the usual way.

Here the way of measuring is the same in all tangent spaces (does not depend on anything) – this is only possible because we have a vector space where tangent spaces can be canonically identified.

[9] We may try to copy the metric of $\mathbb{R}^n$ to a manifold locally, on the domain of a chart. That would lead to metrical notions on that specific open subset, but nothing would guarantee that by choosing another chart, we would get the same metric.

[10] In the topological sense, a metric gives the distances between points (see metric spaces); geometrically, a metric does not directly give these distances, but measures tangent vectors instead.

1.3.3. A more general way is a **Riemannian metric**: here we still have inner products in all tangent spaces, but they may change from point to point in a smooth way.[11] Formally:

1.3.4. Definition. • A **Riemannian metric** g (mostly written as $\langle\cdot,\cdot\rangle$) is a 2-tensor field on a manifold M whose pointwise values g_p or $\langle\cdot,\cdot\rangle_p$ are inner products on the tangent spaces T_pM. In other words, g is

(**RM1**) symmetric: $\langle X, Y\rangle = \langle Y, X\rangle$ for any $X, Y \in \mathfrak{X}(M)$ and

(**RM2**) positive definite: $\langle X, X\rangle > 0$ for any $\mathbf{0} \neq X \in \mathfrak{X}(M)$.

- A **Riemannian manifold** is a pair (M, g), i.e. a manifold endowed with a Riemannian metric.
- The **local coordinate** (matrix) **representation** of a Riemannian metric g in local coordinates $(u^1, \dots, u^n)$ is

$$g_{ij} := \left\langle \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right\rangle,$$

and with this we can write

$$\langle X, Y\rangle = \left\langle X^i\partial_i, Y^j\partial_j\right\rangle = X^iY^j \left\langle\partial_i, \partial_j\right\rangle = g_{ij}X^iY^j \text{ for any } X, Y \in \mathfrak{X}(M)$$

or, in short, $g = g_{ij}du^i \otimes du^j$. The smoothness of g is equivalent to the smoothness of the component functions g_{ij}. g^{ij} denotes the inverse of g_{ij}.

In short, a Riemannian metric is a smoothly varying family of inner products in the tangent spaces, with ellipsoids as **indicatrices** (unit spheres). As such, the way of measuring depends on the position (in which tangent space the given vector lives, or where we stand on the manifold). Standard examples for Riemannian manifolds are embedded surfaces in $\mathbb{R}^3$ with their metric inherited from the Euclidean metric of $\mathbb{R}^3$.

1.3.5. The most general way of measuring is a **Finsler metric**: here we have a so-called Minkowski norm in each tangent space, and they vary smoothly from point to point. Formally:

1.3.6. Definition. • A **Finsler metric** on a manifold M is a continuous function $F\colon TM \to [0, \infty[$ satisfying the following conditions:

(**FM1**) F is smooth on the complement of the zero section, i.e. on TM without the origins of the tangent spaces (regularity),

(**FM2**) $F(tv) = tF(v)$ for all $t > 0$ (positive 1-homogeneity),

(**FM3**) the Hessian of the energy function $E = F^2/2$ of the Finsler metric, also called the **fundamental tensor**, i.e.

$$g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j}$$

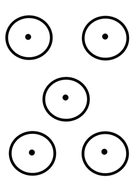
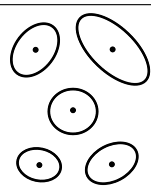
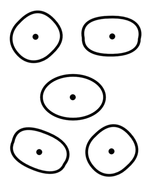
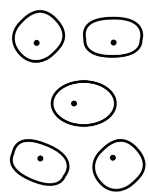
is positive definite at all nonzero elements $v \in T_pM$ (strong convexity).

[11] On a general manifold, it would not even make sense to say that we have 'the same' inner product in different tangent spaces.

- In other words, a Finsler metric is a smoothly varying family of Minkowski norms (length-measuring functions just like norms, but without central symmetry) in the tangent spaces (it can be checked that the above conditions imply positivity for $v \neq \mathbf{0}$ and the triangle inequality). If these are norms in the usual sense, i.e. F satisfies $F(v) = F(-v)$ (and then it is absolutely homogeneous), we say that the Finsler metric is **reversible**.
- A **Finsler manifold** (M, F) is a manifold endowed with a Finsler metric.
- With each Finsler metric comes naturally a family of Riemannian metrics: the g_{ij}'s of F define an inner product at any nonzero vector of any tangent space, so these constitute a Riemannian metric on every punctured tangent space $T_p^\circ M := T_pM \setminus \{\mathbf{0}\}$. These together are called the **Riemann–Finsler metric** (belonging to F).

If we look at the Finsler metric (or the Riemann–Finsler metric), we see that the way of measuring here changes not just from point to point, but even with the direction (the element of T_pM). We can say that the measuring depends on where we are and where (in which direction) we are going on the base manifold.

1.3.7. The Euclidean metric is obviously a special case of Riemannian metrics, and (the norms induced by) Riemannian metrics are special Finsler metrics. The most simple (non-Riemannian) examples of Finsler spaces are the so-called **Randers spaces**: they are obtained from Riemannian metrics by translating the metric in every tangent space (intuitively, we introduce some pointwise varying wind that blows away our Riemannian unit spheres or indicatrices). We will study them in more detail in Chapter 3.

	Euclidean metric	Riemannian metric	reversible Finsler metric	Finsler metric
locally (on T_pM)	inner product	inner product	norm	Minkowski norm
indicatrices	sphere	ellipsoid	boundary of cent. symm. strictly convex smooth body	boundary of general strictly convex smooth body
globally (over TM)	the same canonical inner product at any point	smoothly varying family of inner products	smoothly varying family of norms	smoothly varying family of Minkowski norms
measuring ...	is the same everywhere	changes with the position	changes with the position and direction	changes with the position and direction
				

An interesting question: **Do there exist a Riemannian and a (non-Riemannian) Finsler metric on any manifold?**

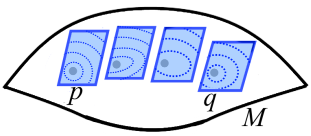
1.3.8. STATEMENT. *Any manifold can be equipped with a Riemannian and a (non-Riemannian) Finsler metric.*

PROOF. The existence of a Riemannian metric g can be proved easily: just pull back the standard Euclidean metric of $\mathbb{R}^n$ to the manifold through local charts and then blend the obtained local metrics together with the help of a smooth partition of unity. This is a standard exercise and is left to the Reader (or see Proposition 2.10. of [**9**], p. 43.)

It is also not hard to see that this Riemannian metric can be perturbed over a 'small' set to obtain a Randers metric, which is a non-Riemannian Finsler metric. We will present the details in Chapter 3. □

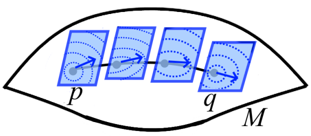
1.3.2. Measurements on manifolds.

1.3.9. Using the ideas from 1.3.1, on a connected manifold, a geometrical metric leads to a topological metric the following way.



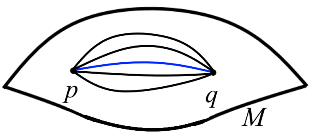
measuring the **length of tangent vectors**: geometrical metrics (Euclidean, Riemannian, Finslerian)

$\downarrow$



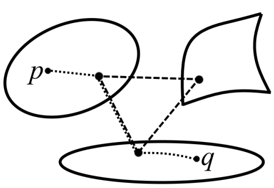
measuring the **length of curves**: integrating the length of velocity vectors:
$L(c) := \int \|c'\|$

$\downarrow$



measuring **distances:**
$d(p, q) :=$ infimum of lengths of (piecewise regular) curves between them

1.3.10. REMARK. But what to do with a non-connected manifold where not any two points can be connected with curves? We can still obtain a notion of distance from a geometrical metric there. The trick is to connect all the components with 'bridges' of length one, i.e. choose one point from each component and set their pairwise distances as 1.



Then the distance between points from different components is the sum of the distances of the points from the respective endpoints of the bridge connecting these components, plus the length of the bridge itself (which is 1). For the details, see Corollary 13.30. in [**11**].

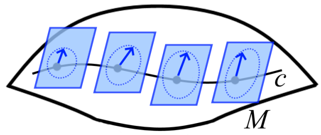
1.4. Relationships between metrics and connections

So far we have introduced two possible additional structures on a manifold: linear connections and metrics. The question is: **How can these two structures relate to each other in a nice and useful way?**

A connection gives us the notion of parallel vector fields, and a metric allows us to measure the length of the elements of these vector fields. It feels natural to require that if a vector field does not change (is parallel), then neither should its length. Throughout this book, such pairs of metrics and connections will be our primary interest.

1.4.1. Definition. Suppose that a manifold is endowed with a metric and a linear connection ∇. If all parallel vector fields (w.r.t. ∇) have constant length, then we say that ∇ is **metrical** or **compatible** to the metric.

From now on, we agree on the following convention:



- For Riemannian metrics, we say 'metrical';
- for Finsler metrics, we say 'compatible'.

We might ask the following questions:

1. *Given a metric, can we always find a linear connection ∇ that is metrical/compatible to it? If yes, how and how many? Is there a difference between the Riemannian and the Finslerian case?*

2. *Given a linear connection ∇, can we always find a metric such that ∇ is metrical/compatible to it? If yes, how and how many?*

1.4.1. Metrical linear connections in the Riemannian case and the Levi-Civita connection.

It turns out that the answer to the first question is very different in the cases of Riemannian and Finsler metrics: the former is much easier and is completely dealt with, but in the latter case only partial answers are known. In this section we only examine the case of Riemannian metrics. But first, let us rephrase the condition of metricity in terms of the different faces of a linear connection.

1.4.2. Theorem. *For a linear connection ∇, the condition of being metrical to a Riemannian metric g is equivalent to the following:*

- *∇ satisfies a Leibniz rule with the metric, i.e. for any $X, Y, Z \in \mathfrak{X}(M)$,*
$$X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\,;$$
- *∇_c (the linear connection along an arbitrary curve c) satisfies a Leibniz rule with the metric, i.e. for any vector fields $Y, Z \in \mathfrak{X}(c)$,*
$$\langle Y, Z\rangle' = \langle \nabla_c Y, Z\rangle + \langle Y, \nabla_c Z\rangle\,;$$
- *for any pair of parallel vector fields P, Q along the same curve, we have*
$$\langle P, Q\rangle = constant \iff \langle P, Q\rangle' = 0.$$

Now we can state and prove the main result of this section.

1.4.3. THEOREM. *On a Riemannian manifold (M, g) there exists an infinite family of linear connections metrical to the Riemannian metric g. These can be 'parameterized' by their torsion tensor, i.e. prescribing any value to the torsion uniquely determines one member of this family.*[12]

PROOF. **Unicity.** Assume first that there exists a linear connection ∇ that is metrical to g. Then, for any vector fields $X, Y, Z \in \mathfrak{X}(M)$, by permuting them cyclically, we have the following equations:

$$\begin{aligned} X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle Z,X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle . \end{aligned}$$

Note that every pair $\nabla_A B$ and $\nabla_B A$ appears twice. Using the torsion tensor, we can express one member of every such pair (the second one in each line above) as

$$\begin{aligned} T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] &\Longrightarrow \nabla_Y X = \nabla_X Y - T(X,Y) - [X,Y] \\ T(Y,Z) = \nabla_Y Z - \nabla_Z Y - [Y,Z] &\Longrightarrow \nabla_Z Y = \nabla_Y Z - T(Y,Z) - [Y,Z] \\ T(Z,X) = \nabla_Z X - \nabla_X Z - [Z,X] &\Longrightarrow \nabla_X Z = \nabla_Z X - T(Z,X) - [Z,X] . \end{aligned}$$

Substituting back to the three equations above we obtain

$$\begin{aligned} X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_Z X\rangle - \langle Y, T(Z,X)\rangle - \langle Y, [Z,X]\rangle \\ Y\langle Z,X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_X Y\rangle - \langle Z, T(X,Y)\rangle - \langle Z, [X,Y]\rangle \\ Z\langle X,Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Y Z\rangle - \langle X, T(Y,Z)\rangle - \langle X, [Y,Z]\rangle . \end{aligned}$$

Now comes the trick: if we add the first two equations and subtract the third, then of all the terms involving ∇ only one remains:

$$\begin{gathered} X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle = \\ 2\langle \nabla_X Y, Z\rangle + \langle X, T(Y,Z)\rangle - \langle Y, T(Z,X)\rangle - \langle Z, T(X,Y)\rangle \\ + \langle X, [Y,Z]\rangle - \langle Y, [Z,X]\rangle - \langle Z, [X,Y]\rangle . \end{gathered}$$

After some reordering we obtain the so-called **Koszul formula**:

$$\boxed{\begin{aligned} \langle \nabla_X Y, Z\rangle = 1/2\cdot\big(&X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &- \langle X, T(Y,Z)\rangle + \langle Y, T(Z,X)\rangle + \langle Z, T(X,Y)\rangle \\ &- \langle X, [Y,Z]\rangle + \langle Y, [Z,X]\rangle + \langle Z, [X,Y]\rangle\big). \end{aligned}} \tag{9}$$

What is on the right hand side? It depends on the metric, the Lie bracket of vector fields and the torsion of the connection. The first two are canonical (known) data of the manifold and by prescribing the value of the torsion, the whole right hand side becomes a known quantity. Therefore, it is not

[12] This result is a slight modification of the existence theorem of the Levi-Civita connection, found in most textbooks. The algorithm of the proof is known as the 'Christoffel process', which works just as well in this general setting.

hard to see that $\nabla_X Y$ is uniquely determined by it. For this, let us suppose that ∇^1 and ∇^2 are connections satisfying the same formula (with the same torsion). Since the right hand sides are the same, we have

$$\left\langle \nabla^1_X Y - \nabla^2_X Y, Z \right\rangle = 0.$$

But because Z is an arbitrary vector field and the metric is non-degenerate, this can only happen if $\nabla^1_X Y - \nabla^2_X Y = 0$ for any $X, Y \in \mathfrak{X}(M)$, i.e. $\nabla^1 = \nabla^2$.

The Christoffel symbols. For the proof of the existence (and later use), we compute the Christoffel symbols of a metrical connection ∇ with some prescribed torsion T. We will work in a coordinate frame $(\partial_1, \dots, \partial_n)$, so that the Lie brackets become automatically zero and the computations become easier.

Let us plug $X := \partial_i$, $Y := \partial_j$ and $Z := \partial_m$ into the formula (9). We get

$$\begin{aligned}
\langle \nabla_{\partial_i}\partial_j, \partial_m \rangle &= 1/2 \cdot \big(\partial_i \langle \partial_j, \partial_m \rangle + \partial_j \langle \partial_m, \partial_i \rangle - \partial_m \langle \partial_i, \partial_j \rangle \\
&\qquad - \langle \partial_i, T(\partial_j, \partial_m) \rangle + \langle \partial_j, T(\partial_m, \partial_i) \rangle + \langle \partial_m, T(\partial_i, \partial_j) \rangle \big) \\
\left\langle \Gamma^l_{ij}\partial_l, \partial_m \right\rangle &= 1/2 \cdot \big(\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij} \\
&\qquad - \left\langle \partial_i, T^l_{jm}\partial_l \right\rangle + \left\langle \partial_j, T^l_{mi}\partial_l \right\rangle + \left\langle \partial_m, T^l_{ij}\partial_l \right\rangle \big) \\
\Gamma^l_{ij} g_{lm} &= 1/2 \cdot \big(\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij} - T^l_{jm} g_{il} + T^l_{mi} g_{jl} + T^l_{ij} g_{ml} \big).
\end{aligned}$$

Using the notation $g_{ab;c} := \partial_c g_{ab}$, by multiplying with the inverse matrix g^{mk} of g_{lm} (so $g_{lm}g^{mk} = \delta^k_l$) we can express the Christoffel symbols as

$$\boxed{\Gamma^k_{ij} = \frac{1}{2} g^{mk} \big(- g_{ij;m} + g_{jm;i} + g_{mi;j} + T^l_{ij} g_{ml} - T^l_{jm} g_{il} + T^l_{mi} g_{jl} \big).} \tag{10}$$

Existence. If we define a connection with prescribed torsion T in each chart by the Christoffel symbols obtained in (10), the unicity result guarantees that it will be well-defined. What remains to be checked is that it is indeed metrical, but this can easily be seen (and not so surprising since it was the starting point of our construction). □

After our initial despair we now face another problem: we are now spoiled for choice with an enormous family of connections at hand. **If we only need one, which one should we favor?** The most natural candidate has been introduced by Levi-Civita and bears his name ever since.

1.4.4. THEOREM. *On any Riemannian manifold there exists exactly one metrical linear connection that is torsion-free (whose torsion is identically 0). It is called the* ***Levi-Civita*** *(or Riemannian)* ***connection*** *of the metric.*

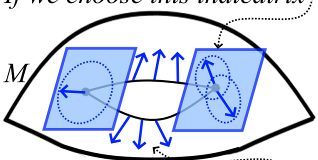
1.4.5. REMARK. We have already seen the geometric meaning of the torsion tensor, but we can consider these results as another reason which justifies its importance. Also, from the geometrical picture, it is also clear why torsion-freeness is a natural property to require.

1.4.2. Finding compatible metrics for given connections.

1.4.6. We might ask: what about the other direction, i.e. **for a given linear connection ∇, can we always find a metric such that ∇ is metrical/compatible to it?**

Our first intuition may lead us to believe that the answer is yes, at least on a connected manifold: just choose an arbitrary way of measuring in the tangent space at a fixed point p (in the Riemannian case, choose an inner product or a quadratic indicatrix; in the Finslerian case, a Minkowski norm or a convex body with smooth boundary) and consider its parallel translates to other points of the manifold. This way we should get a (smoothly varying) family of indicatrices at each point, i.e. a metric on the manifold, to which the connection is clearly metrical/compatible.

But is it really this simple? Unfortunately not. In [10] there is an explicit example of a linear connection on the torus with no Riemannian metric to which it is metrical (see page 211, Example 4.2.). So clearly there is a serious flaw in our argument. What is it?

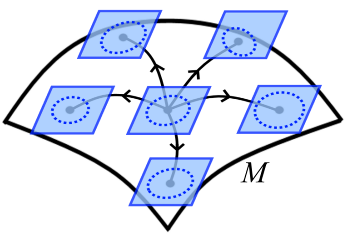


It is being forgetful about the dependence of parallel translation on the path chosen. It isn't a problem as long as the indicatrix as a set is preserved; but if not, that leads to multiple choices of indicatrices at points, and however we choose from them, along the omitted curves we will get parallel vector fields with non-constant length.

So, applying Statements 1.2.17. and 1.2.24. for the metric tensor, we can say the following.

1.4.7. Statement. *For a given linear connection ∇ on a **connected** manifold, the following conditions are equivalent:*

(a) there exists a Riemannian metric such that ∇ is metrical to it,
(b) there exists a Riemannian metric which is parallel/covariant constant with respect to ∇,
(c) all parallel translations between points preserve some inner product,
(d) the holonomy of the connection preserves some inner product, i.e. $\mathrm{Hol}(\nabla)$ *is a subgroup of O_n.*



Returning to our initial thoughts, we will make use of them the following way.

1.4.8. Lemma. *Given a metrical linear connection ∇ to a Riemannian metric on a **connected** manifold, we can reconstruct the metric from its value at a single point only.*

In the Finslerian case the situation is similar, but it is much harder to see what it means to preserve an indicatrix (Minkowski norm).

1.4.3. Compatible linear connections in the Finslerian case.

1.4.9. From now on, we examine the same questions in the Finslerian case as we did in the Riemannian setting. Namely:

1. For a given Finsler metric on a manifold, is there always a linear connection compatible to it? If not, can we describe the 'good' Finsler metrics or at least give some necessary or sufficient conditions?

2. If a Finsler metric does admit compatible linear connections, how many are there and how can we describe them?

3. Can we distinguish a 'canonical' choice in the above case (like the Levi-Civita connection in Riemannian Geometry)?

4. What about manifolds? We know that any manifold admits a non-Riemannian Finsler metric; does it admit one with a compatible linear connection? If not, can we give some conditions or at least some obstructions based on the manifold's topological or analytic properties?

There are only partial answers to these questions in the literature. We will try to add some new results to the list, but this topic is far from settled.

1.4.10. Definition. A **generalized Berwald manifold** (GBM for short) is a triplet (M, F, ∇), i.e. a Finsler manifold with a linear connection ∇ that is compatible to the metric.[13]

Two special cases we will encounter later are

- **generalized Berwald surfaces** that are just 2-dimensional GBMs;
- **Berwald manifolds/surfaces**, where the compatible linear connection is torsion-free.

GBMs are the Finslerian analogies of Riemannian manifolds with their metrical connections (and Berwald manifolds are the closest analogies possible to the Riemannian metric–Levi-Civita connection pairings, with the torsion-freeness property still holding).

All Riemannian manifolds are special GBMs, but what about the non-Riemannian ones? An interesting rigidity property is the following.

1.4.11. Statement (Rigidity of GBMs). *If a* ***connected*** *GBM is Riemannian (its indicatrix is quadratic) at one point, then it is Riemannian everywhere. Consequently, a* ***connected*** *GBM is either Riemannian (everywhere) or non-Riemannian everywhere.*

Proof. We have seen that any metric can be recovered from its value (indicatrix) at a single point only, if we have parallel translations preserving it. In this case, if one indicatrix is quadratic, so will be all the others, since a parallel translation is a linear operation preserving this property. □

[13]It is a fundamental result in Finsler geometry that any Finsler manifold admits a compatible connection with some other properties – but in general, it is not linear. Linearity of the connection and its parallel translation is the key property for GBMs. (For the meaning of nonlinear connections, see e.g. [7].)

1.5. Relationships between metrics and metrics

It is not just pairs of metrics and connections on a manifold that can relate to each other nicely – two metrics can also share an intimate bond, based on the family of linear connections metrical/compatible to them.

1.5.1. DEFINITION. Let F be a Finsler metric and γ be a Riemannian metric on the same manifold. We will call them **compatible** (or F compatible to γ) if any linear connection compatible to F is also metrical to γ. Or, in other words, if parallel translation preserves the Finslerian length, it must preserve the Riemannian length of the vectors, too.

Naturally, one might immediately pose the question if the existence of such a Riemannian metric is always guaranteed. And the answer is yes.

1.5.2. THEOREM. *For any Finsler metric F there is always a Riemannian metric γ such that they are compatible.*

PROOF. We sketch the construction of the averaged Riemannian metric.

- Consider the Riemann–Finsler metric g belonging to F (see Section 1.3.1). Remember, it consists of inner products in the tangent spaces $T_v(T_pM)$ at all points of all the punctured tangent spaces $T_p^\circ M$ (turning the latter into Riemannian manifolds).
- Consider the indicatrix hypersurfaces F_p of F in the tangent spaces.
- The volume form $d\mu = \sqrt{\det g_{ij}}\; dy^1 \wedge \ldots \wedge dy^n$ belonging to $T_p^\circ M$ as a Riemannian manifold induces a volume form on the indicatrix, namely

$$\mu = \sqrt{\det g_{ij}}\; \sum_{i=1}^{n} (-1)^{i-1} \frac{y^i}{F} dy^1 \wedge \ldots \wedge dy^{i-1} \wedge dy^{i+1} \ldots \wedge dy^n,$$

  called the **induced volume form** (on F_p).
- We define the **averaged Riemannian metric** on M by setting the inner product of two vector fields $X, Y \in \mathfrak{X}(M)$ at a point $p \in M$ as follows. Since T_pM is a vector space, we can pin the vectors $X_p, Y_p \in T_pM$ to any base point in T_pM; more precisely, consider their vertical lifts to the tangent spaces $T_v(T_pM)$, where we can take their inner products with respect to the Riemann–Finsler metric. This yields a numerical value at any nonzero element of T_pM; let us integrate these along the indicatrix with respect to the induced volume form. In short, γ is defined by

$$\boxed{\gamma_p(X,Y) := \int_{F_p} g(X_p^v, Y_p^v)\, \mu = X_p^i Y_p^j \int_{F_p} g_{ij}\, \mu.} \tag{11}$$

It is shown in [15] (Lemma 3.) that F is compatible to this metric. □

1.5.3. REMARK. The averaged Riemannian metric is of course not necessarily the best choice for a compatible Riemannian metric (since the integration is very difficult to compute in practice). In the case of Randers spaces, for example, there is a much more natural candidate, as we shall see in Chapter 3.

CHAPTER 2

Compatible linear connections and the compatibility equations

In this chapter we start our quest for the compatible linear connections on a given Finsler manifold. We formulate the equations characterizing them (the so-called compatibility equations) and transform these (using the method of Vincze) in a way such that it will not be completely hopeless to solve them, at least in lower dimensions or special cases. We also make our first steps towards the solution by inspecting the geometry of the tangent spaces and classifying the elements based on how they make the compatibility equations simpler. At the end of the chapter, we introduce the so-called extremal connection, invented by Vincze (see [**20**]) to take up the role of the Levi-Civita connection in the Finslerian setting.[1]

2.1. The compatibility equations

2.1.1. We now try to describe (all) compatible linear connections to a given Finsler metric F on a given manifold M. Recall that in terms of parallel translation determined by ∇, compatibility of ∇ means that parallel vector fields have constant Finslerian length, or in other words, the Finsler function is constant on parallel vector fields. That is, for any parallel vector field X along some curve c in M, in any chart we have

$$(F \circ X)' = (x^k \circ X)' \frac{\partial F}{\partial x^k} \circ X + (y^k \circ X)' \frac{\partial F}{\partial y^k} \circ X = 0.$$

Let us proceed the following way.

- First, to make the formulas more readable, let us use the standard abbreviations $(x^k \circ X)' = (u^k \circ \pi \circ X)' = (u^k \circ c)' =: c^{k\prime}$ and $y^k \circ X =: X^k$.
- Second, note that in the above form the presence of the connection we are trying to describe is not very visible. We can make it more so by writing out the ODE of parallel vector fields (see (3)) for X:

$$X^{k\prime} = -X^j c^{i\prime} \Gamma^k_{ij} \circ c = 0.$$

[1] In this chapter we summarize the basic ideas and results of [**20**] together with some observations from [**1**], [**3**] and [**4**].

So our equation takes the form

$$\begin{aligned} c^{k\prime}\frac{\partial F}{\partial x^k}\circ X - X^j c^{i\prime}\Gamma^k_{ij}\circ c\frac{\partial F}{\partial y^k}\circ X &= 0 \\ c^{i\prime}\left(\frac{\partial F}{\partial x^i} - y^j\Gamma^k_{ij}\circ\pi\frac{\partial F}{\partial y^k}\right)\circ X &= 0 \\ c^{i\prime}\left[\left(\frac{\partial}{\partial x^i} - y^j\Gamma^k_{ij}\circ\pi\frac{\partial}{\partial y^k}\right)(F)\right]\circ X &= 0. \end{aligned}$$

Since c and X are arbitrary, the vanishing of the left-hand side is equivalent to the vanishing of the term inside the brackets for all indices $i \in \{1,\dots,n\}$, which are vector fields (on TM) differentiating the Finsler function. We call these vector fields the horizontal vector fields determined by ∇.

In summary:

2.1.2. Theorem. *In any chart, a linear connection ∇ (through its Christoffel symbols) determines the* ***horizontal vector fields*** *$X^h_1,\dots,X^h_n$, where*

$$X^h_i := \frac{\partial}{\partial x^i} - y^j\Gamma^k_{ij}\circ\pi\frac{\partial}{\partial y^k}, \qquad i \in \{1,\dots,n\}. \tag{12}$$

∇ is compatible to the Finsler metric $F \iff$ the derivative of F vanishes along ∇'s horizontal vector fields (their integral curves), i.e.

$$\boxed{X^h_i F = \frac{\partial F}{\partial x^i} - y^j\Gamma^k_{ij}\circ\pi\frac{\partial F}{\partial y^k} = 0,} \qquad i \in \{1,\dots,n\}. \tag{13}$$

We will call this system of equations the ***compatibility equations*** *and denote it by CEQ (and the i-th member by CEQ-i).*

In the above form of the CEQ, we are looking for the compatible ∇s in terms of their Christoffel symbols. From now on, to make the equations easier to read, we use the shorthand notations for the partial derivatives and always typeset the unknown terms in bold, so we write (13) as

$$\widehat{\partial}_i F - y^j(\boldsymbol{\Gamma}^k_{ij}\circ\pi)\,\dot{\partial}_k F = 0, \qquad i \in \{1,\dots,n\}.$$

But this helps us quite little: it seems hopeless to solve this, even when F is relatively simple. In what follows we show that the situation is not so grim: by applying some clever transformations, (13) becomes easier to handle, and in later chapters we will be able to solve it, at least in special cases.

2.2. Transforming the compatibility equations

We will apply the tricks of Vincze (see e.g. [2], [1], [3], [20]) to make (13) easier to handle.

2.2.1. Trick 1. Choose a compatible Riemannian metric.

2.2.1. Recall that the Finsler metric is compatible to a Riemannian metric γ if any linear connection compatible to F is also metrical to γ, and such a γ can always be chosen (see Section 1.5).

The presence of γ is extremely useful: we don't have to start searching for the compatible $\boldsymbol{\nabla}$s in the dark; we only have to look for them among the well-behaved family of metrical linear connections to the Riemannian metric γ. This family is relatively easy to handle, since its members are uniquely determined by their torsion tensors – and this means that so are the linear connections compatible to F.

2.2.2. Remark. This also suggests a way to narrow the set of possible candidates for compatible linear connections to F. If we can find multiple different compatible Riemannian metrics, the compatible $\boldsymbol{\nabla}$s we are searching for must be metrical to *all* of them, i.e. must be in the intersection of their families of metrical linear connections.

It may sound good in theory, but things are not so simple: first, constructing compatible Riemannian metrics is not easy at all, and second, it is only guaranteed that compatible $\boldsymbol{\nabla}$s are *among* the elements of said intersection, but some (or most, or all) of them may not be compatible at all. So we could only narrow the possibilities, but finding the exact set of compatible connections would still remain a task.

2.2.3. From now on, with any Finsler metric F we consider an arbitrary (but previously given and fixed) compatible Riemannian metric γ. We denote

- its (uniquely existing) Levi-Civita connection by ∇^*,
- the (symmetric) Christoffel symbols of ∇^* by Γ^{k*}_{ij},
- the horizontal vector fields of ∇^* by

$$X_i^{h*} := \widehat{\partial}_i - y^j (\boldsymbol{\Gamma}^{k*}_{ij} \circ \pi)\, \dot{\partial}_k, \qquad i \in \{1, \dots, n\},$$

- the induced norm (reversible Finsler metric) and its energy function by

$$F^*(v) := \sqrt{\gamma(v,v)} \ \text{ and } \ E^*(v) := \frac{(F^*)^2(v)}{2} \qquad \text{for any } v \in TM.$$

2.2.2. Trick 2. Use the torsion components as unknowns.

2.2.4. At first, we wrote the CEQ in terms of the Christoffel symbols of the $\boldsymbol{\nabla}$s we're looking for. But as we have established in the previous section, they are also uniquely determined by their torsion tensors (or torsion components), meaning that we could use these as the new unknowns. Let us do that.

Since the $\boldsymbol{\nabla}$s we're looking for and the Levi-Civita connection ∇^* are both metrical to the Riemannian metric γ, we can apply the Christoffel process to both of them and determine their Christoffel symbols $\boldsymbol{\Gamma}_{ij}^k$ and Γ_{ij}^{k*} (in the same local coordinates). From formula (10) (and using the fact that ∇^* is torsion-free) we get

$$\begin{aligned}
\boldsymbol{\Gamma}_{ij}^k &= \frac{1}{2}\gamma^{mk}\big(- \gamma_{ij;m} + \gamma_{jm;i} + \gamma_{mi;j} + \boldsymbol{T}_{ij}^l\gamma_{ml} - \boldsymbol{T}_{jm}^l\gamma_{il} + \boldsymbol{T}_{mi}^l\gamma_{jl}\big) \\
\Gamma_{ij}^{k*} &= \frac{1}{2}\gamma^{mk}\big(- \gamma_{ij;m} + \gamma_{jm;i} + \gamma_{mi;j}\big).
\end{aligned}$$

Thus we can write the Christoffel symbols of $\boldsymbol{\nabla}$ in terms of its torsion components as

$$\begin{aligned}
\boldsymbol{\Gamma}_{ij}^k &= \Gamma_{ij}^{k*} + \frac{1}{2}\gamma^{mk}\big(\boldsymbol{T}_{ij}^l\gamma_{ml} - \boldsymbol{T}_{jm}^l\gamma_{il} + \boldsymbol{T}_{mi}^l\gamma_{jl}\big) \\
&= \Gamma_{ij}^{k*} + \frac{1}{2}\big(\boldsymbol{T}_{ij}^k - \boldsymbol{T}_{jm}^l\gamma^{mk}\gamma_{il} + \boldsymbol{T}_{mi}^l\gamma^{mk}\gamma_{jl}\big).
\end{aligned}$$

Plugging this back into (13), for any $i \in \{1, \dots, n\}$ we get the equation

$$\begin{aligned}
&\widehat{\partial}_i F - y^j\left(\Gamma_{ij}^{k*} + \frac{1}{2}\big(\boldsymbol{T}_{ij}^k - \boldsymbol{T}_{jm}^l\gamma^{mk}\gamma_{il} + \boldsymbol{T}_{mi}^l\gamma^{mk}\gamma_{jl}\big)\right)\circ\pi\cdot\dot{\partial}_k F = 0 \\
&X_i^{h*}F - \frac{1}{2}y^j\big(\boldsymbol{T}_{ij}^k - \boldsymbol{T}_{jm}^l\gamma^{mk}\gamma_{il} + \boldsymbol{T}_{mi}^l\gamma^{mk}\gamma_{jl}\big)\circ\pi\cdot\dot{\partial}_k F = 0 \qquad (14)\\
&y^j\big(-\boldsymbol{T}_{ij}^k + \boldsymbol{T}_{jm}^l\gamma^{mk}\gamma_{il} - \boldsymbol{T}_{mi}^l\gamma^{mk}\gamma_{jl}\big)\circ\pi\cdot\dot{\partial}_k F = -2X_i^{h*}F.
\end{aligned}$$

2.2.5. For the sake of simplicity, from now on we omit π from the notation and understand everything as pulled up to TM by π. Let us rearrange the left hand side with respect to the torsion components. Using their anti-symmetry, we only need to consider terms of the form $\boldsymbol{T}_{ab}^c$ where $a < b$. So the question is: if we wish to write the compatibility equations into the form

$$\sum_{a<b,c} \sigma_{ab;i}^c \boldsymbol{T}_{ab}^c = -2X_i^{h^*}F, \qquad i \in \{1, \dots, n\},$$

where the summation symbol means summing over the indices

$$\{(a,b,c) \in \mathbb{N}_n \times \mathbb{N}_n \times \mathbb{N}_n \,|\, a < b\}, \qquad \mathbb{N}_n := \{1, \dots, n\},$$

what will be the coefficients $\sigma^c_{ab;i}$ of the torsion components in the i-th equation? The answer is:

$$\begin{aligned}\sigma^c_{ab;i} &= -\left(\delta^a_i y^b - \delta^b_i y^a\right)\dot{\partial}_c F + \left(y^a\gamma^{bk} - y^b\gamma^{ak}\right)\gamma_{ic}\,\dot{\partial}_k F \\ &\quad -\left(\delta^b_i\gamma^{ak} - \delta^a_i\gamma^{bk}\right) y^j\,\gamma_{jc}\,\dot{\partial}_k F.\end{aligned} \tag{15}$$

2.2.3. Trick 3. Fix a point and use normal coordinates.

2.2.6. From now on, we consider the CEQ only at a fixed point $p \in M$. Having a Riemannian metric γ allows us to choose Riemannian normal coordinates around the point p, which simplifies the CEQ greatly:

- all $\Gamma^{k*}_{ij}(p)$ become zero, thus $X^{h*}_i(v) = \widehat{\partial}_i(v)$ for any $v \in T_pM$ and
- the induced coordinate vectors $(\partial_1|_p, \ldots, \partial_n|_p)$ constitute an orthonormal basis of T_pM and therefore $\gamma_{ij}(p) = \gamma^{ij}(p) = \delta^j_i$.[2]

In this case the coefficients $\sigma^c_{ab;i}$ from (15) take the form

$$\begin{aligned}\sigma^c_{ab;i} &= -\left(\delta^a_i y^b - \delta^b_i y^a\right)\dot{\partial}_c F + \left(y^a\delta^{bk} - y^b\delta^{ak}\right)\delta_{ic}\,\dot{\partial}_k F \\ &\quad -\left(\delta^b_i\delta^{ak} - \delta^a_i\delta^{bk}\right) y^j\,\delta_{jc}\,\dot{\partial}_k F \\ &= \left(\delta^b_i y^a - \delta^a_i y^b\right)\dot{\partial}_c F + \left(y^a\dot{\partial}_b F - y^b\dot{\partial}_a F\right)\delta^c_i + \left(\delta^a_i\dot{\partial}_b F - \delta^b_i\dot{\partial}_a F\right) y^c.\end{aligned}$$

Rearranging by the Kronecker deltas we obtain

$$\sigma^c_{ab;i} = \delta^a_i\left(y^c\dot{\partial}_b F - y^b\dot{\partial}_c F\right) + \delta^b_i\left(y^a\dot{\partial}_c F - y^c\dot{\partial}_a F\right) + \delta^c_i\left(y^a\dot{\partial}_b F - y^b\dot{\partial}_a F\right).$$

2.2.7. In summary: using an orthonormal basis in T_pM obtained from Riemannian normal coordinates (w.r.t. a compatible Riemannian metric), we will try to solve the CEQ **at a fixed point** p, in the form

$$\sum_{a<b,c}\sigma^c_{ab;i}\boldsymbol{T}^c_{ab} = -2\,\widehat{\partial}_i F, \qquad i \in \{1,\ldots,n\}. \tag{16}$$

The coefficients are

$$\sigma^c_{ab;i} = \delta^a_i f_{cb} + \delta^b_i f_{ac} + \delta^c_i f_{ab}, \tag{17}$$

where

$$f_{ij} = y^i\dot{\partial}_j F - y^j\dot{\partial}_i F = \left(y^i\frac{\partial}{\partial y^j} - y^j\frac{\partial}{\partial y^i}\right)(F). \tag{18}$$

If we can solve for the torsion, we get the compatible connections by

$$\boldsymbol{T}^c_{ab} \xrightarrow{\text{formula (10)}} \boldsymbol{\Gamma}^k_{ij} \xrightarrow{\text{formula (2)}} \boldsymbol{\nabla}. \tag{19}$$

[2]This, of course, is only possible in general at a fixed point p and not on the whole chart (that requires the metric to be flat).

2.3. The general form of the compatibility equations

2.3.1. Let us examine the **coefficients** $\sigma^c_{ab;i}$ of the CEQ (where $a < b$).

2.3.2. LEMMA. *If none of the indices a, b, c are equal to the row index i, then $\sigma^c_{ab;i} = 0$. Otherwise the following table contains all possible cases, where indices are separated according to their values (i.e. equal indices are in the same cell and different cells contain different values).*

	indices			the coefficients
1.	$i = a$	b	c	$\sigma^c_{ib;i} = f_{cb}$
2.	$i = a$	$b = c$		$\sigma^b_{ib;i} = 0$
3.	$i = b$	a	c	$\sigma^c_{ai;i} = f_{ac}$
4.	$i = b$	$a = c$		$\sigma^a_{ai;i} = 0$
5.	$i = c$	a	b	$\sigma^i_{ab;i} = f_{ab}$
6.	$i = a = c$	b		$\sigma^i_{ib;i} = 2f_{ib}$
7.	$i = b = c$	a		$\sigma^i_{ai;i} = 2f_{ai}$

We see that for a torsion component to appear in (CEQ-i), the row index i must be equal to one of the indices; if exactly one, then the remaining two indices must be different. We can summarize the rules in the following handy mnemonic.

To determine the coefficient $\sigma^c_{ab;i}$ of the torsion component $\boldsymbol{T}^c_{ab}$ in (CEQ-i), go through the indices in the order $a \nearrow c \searrow b$, omitting the first i you encounter. Then the coefficient is the $f_{..}$ with the remaining indices, multiplied by the total number of i's among the indices of $\boldsymbol{T}^c_{ab}$.[3]

For example, in large enough dimensions, the coefficient of

- $\boldsymbol{T}^6_{14}$ in (CEQ-5) is $\sigma^6_{14;5} = 0 \cdot f_{164} = 0$;
- $\boldsymbol{T}^7_{25}$ in (CEQ-7) is $\sigma^7_{25;7} = 1 \cdot f_{2\cancel{7}5} = f_{25}$;
- $\boldsymbol{T}^4_{24}$ in (CEQ-4) is $\sigma^4_{24;4} = 2 \cdot f_{2\cancel{4}4} = 2f_{24}$.

2.3.3. Let's now consider the torsion **components** $\boldsymbol{T}^c_{ab}$. From now on we will allow the lower indices to be in any order (as long as $a \neq b$). First note that there are two different kinds of torsion components:

- $\boldsymbol{T}^c_{ab}$ where all 3 indices are different, let's call these components **diverse**,
- and $\boldsymbol{T}^a_{ab} = -\boldsymbol{T}^a_{ba}$ where the upper index equals one of the lower indices, let's call these **repetitive**.

From the investigation of the coefficients we see that

- a diverse component $\boldsymbol{T}^c_{ab}$ appears in exactly 3 equations of the CEQ: (CEQ-a), (CEQ-b) and (CEQ-c) with coefficients

$$\sigma^c_{ab;a} = f_{cb}, \qquad \sigma^c_{ab;b} = f_{ac}, \qquad \sigma^c_{ab;c} = f_{ab};$$

[3]This also works in the case when none of the indices a, b, c are equal to i, because the meaningless f_{abc} is eliminated by the 0 multiplier.

- a repetitive component $\boldsymbol{T}^a_{ab}$ appears in exactly one equation of the CEQ, namely (CEQ-a), the row of the repeated index, with $\sigma^a_{ab;a} = 2f_{ab}$.

Looking at it from a reverse viewpoint,

- the indices of a diverse component's coefficient are always different from the row index;
- one of the indices of a repetitive component's coefficient is always equal to the row index.

2.3.4. Now let us fix an index i and consider the i-th compatibility **equation** (CEQ-i). We are going to rewrite it by grouping the terms according to the coefficients f_{ab} (instead of the torsion components). Since $a = b$ gives $f_{ab} = 0$, we will have $\binom{n}{2}$ terms. The question is: which torsion components are multiplied by a particular (fixed) coefficient f_{ab} in (CEQ-i)? We, again, have two cases:

- if the row index $i \notin \{a, b\}$, then f_{ab} only multiplies diverse coefficients, namely $\boldsymbol{T}^a_{ib}$, $\boldsymbol{T}^i_{ab}$ and $\boldsymbol{T}^b_{ai}$ (check it with the mnemonic!);
- if the row index $i \in \{a, b\}$, then f_{ab} only multiplies repetitive coefficients, namely, supposing $i = a (\neq b)$, the component $2\boldsymbol{T}^a_{ab} = 2\boldsymbol{T}^i_{ib}$.

Putting together all of the above, we obtain the following.

2.3.5. Theorem. *The form of the i-th compatibility equation at a fixed point p, with the notations and restrictions described in 2.2.7, is*

$$\boxed{\sum_a{}' \, 2 f_{ia} \boldsymbol{T}^i_{ia} + \sum_{a<b}{}' \, f_{ab} \left(\boldsymbol{T}^a_{ib} + \boldsymbol{T}^i_{ab} + \boldsymbol{T}^b_{ai} \right) = -2\, \widehat{\partial}_i F,} \tag{20}$$

where the primed summations mean summing for different indices (in the first one, $a \neq i$, and in the second one, a and b where $i \notin \{a, b\}$).

Notice these groups of 3 consisting of diverse components (and appearing in 3 equations); they are what really complicates the matter.

2.3.6. Example. In **dimension 2** we only have the torsion components $\boldsymbol{T}^1_{12}$ and $\boldsymbol{T}^2_{12}$, and the CEQ is of the form

$$\left.\begin{array}{rcl} f_{12}\boldsymbol{T}^1_{12} & = & -\widehat{\partial}_1 F \\ f_{12}\boldsymbol{T}^2_{12} & = & -\widehat{\partial}_2 F \end{array}\right\}. \tag{21}$$

In **dimension 3** we have a lot more unknown torsion components: $\boldsymbol{T}^1_{12}$, $\boldsymbol{T}^2_{12}$, $\boldsymbol{T}^3_{12}$, $\boldsymbol{T}^1_{13}$, $\boldsymbol{T}^2_{13}$, $\boldsymbol{T}^3_{13}$, $\boldsymbol{T}^1_{23}$, $\boldsymbol{T}^2_{23}$, $\boldsymbol{T}^3_{23}$. The CEQ (in matrix form) is

(22)

	f_{12}	f_{13}	f_{23}	RHS
1.	$2\boldsymbol{T}^1_{12}$	$2\boldsymbol{T}^1_{13}$	$\boldsymbol{T}^2_{13} + \boldsymbol{T}^1_{23} + \boldsymbol{T}^3_{21}$	$-2\,\widehat{\partial}_1 F$
2.	$2\boldsymbol{T}^2_{12}$	$\boldsymbol{T}^1_{23} + \boldsymbol{T}^2_{13} + \boldsymbol{T}^3_{12}$	$2\boldsymbol{T}^2_{23}$	$-2\,\widehat{\partial}_2 F$
3.	$\boldsymbol{T}^1_{32} + \boldsymbol{T}^3_{12} + \boldsymbol{T}^2_{13}$	$2\boldsymbol{T}^3_{13}$	$2\boldsymbol{T}^3_{23}$	$-2\,\widehat{\partial}_3 F$

We will return to these in later chapters.

2.4. The solution set of the compatibility equations

Now that we have transformed the CEQ into a friendlier form, i.e. into (16) and (20), let us think about what we got.

2.4.1. The solution at a fixed point.

2.4.1. Considered over the domain of a chart, (16) is

- a system of n functional equations,
- that must be satisfied for all elements $v \in U \subseteq T^\circ M := TM \setminus \{\mathbf{0}\}$,
- with unknown functions $\boldsymbol{T}_{ab}^c$ (pulled up to TM) constituting the torsion of the unknown $\boldsymbol{\nabla}$s.

After fixing a point $p \in M$, (16) becomes

- a system of n equations,
- that must be satisfied for all elements $v \in T_p^\circ M := T_pM \setminus \{\mathbf{0}\}$,
- with unknown numbers $\boldsymbol{T}_{ab}^c(p)$ constituting the value of the torsion of the unknown $\boldsymbol{\nabla}$s at p.

In other words, we get the solution for the torsion by solving this system for any v as it runs through $T_p^\circ M$, and taking the intersection of all these solutions. But what do we solve for an individual $v \in T_p^\circ M$? It is simply a system of (possibly inhomogeneous) linear equations.

2.4.2. The reason we can be happy with this (and why Vincze chose to introduce the torsion as the new unknown) is the following. Consider the torsion components not as a set of individual unknowns, but as the coordinates of one thing, the pointwise value of the torsion tensor, i.e. a (vector-valued) tensor taking its variables from $T_pM \times T_pM$. We can write

$$\begin{aligned}
\boldsymbol{T}_p(v,w) &= \boldsymbol{T}_p(v^i\partial_i, w^j\partial_j) = v^iw^j\boldsymbol{T}_{ij}^k(p)\partial_k \\
&= \sum_{i<j,k} \boldsymbol{T}_{ij}^k(p)v^iw^j\partial_k + \sum_{j<i,k} \boldsymbol{T}_{ij}^k(p)v^iw^j\partial_k \\
&= \sum_{i<j,k} \boldsymbol{T}_{ij}^k(p)v^iw^j\partial_k - \sum_{i<j,k} \boldsymbol{T}_{ij}^k(p)v^jw^i\partial_k \\
&= \sum_{i<j,k} \boldsymbol{T}_{ij}^k(p)\big(v^iw^j - v^jw^i\big)\partial_k \\
&= \sum_{i<j,k} \boldsymbol{T}_{ij}^k(p)\big(du^i(v)du^j(w) - du^j(v)du^i(w)\big)\partial_k,
\end{aligned}$$

or, concisely,

$$\boldsymbol{T}_p = \sum_{i<j,k} \boldsymbol{T}_{ij}^k(p)\; du_p^i \wedge du_p^j \otimes \left(\frac{\partial}{\partial u^k}\right)_p .$$

In other words, the pointwise value of the torsion is an element of the vector space $\wedge^2 T_p^*M \otimes T_pM$ of dimension $\binom{n}{2}n$ with the induced basis (in which the coordinates are the torsion components)

$$du_p^i \wedge du_p^j \otimes \left(\frac{\partial}{\partial u^k}\right)_p \qquad (1 \leq i < j \leq n,\; k \in \{1,\dots,n\}). \tag{23}$$

In summary, at a fixed $v \in T_p^\circ M$, we have to solve a linear system over a vector space, so the set of solutions $\boldsymbol{A}_p(v)$ (as the system may be inhomogeneous) is an affine subspace in $\wedge^2 T_p^* M \otimes T_p M$. To get the solutions of the system (for all values of v), we have to consider the intersection

$$\boldsymbol{A}_p = \bigcap_{v \in T_p^\circ M} \boldsymbol{A}_p(v),$$

which, as the intersection of affine subspaces, is itself an affine subspace in $\wedge^2 T_p^* M \otimes T_p M$.

2.4.2. The solution globally.

2.4.3. After solving the CEQ at one point $p \in M$, **what can we say about the global solution?**

- If $\boldsymbol{A}_p$ is empty, meaning that there is no solution at p, there can be no solution globally, i.e. there are no compatible $\boldsymbol{\nabla}$s to the Finsler metric.
- If $\boldsymbol{A}_p$ is not empty (there is a solution at p), that does not guarantee the existence of compatible $\boldsymbol{\nabla}$s, because there might be other points of the manifold with empty solution sets $\boldsymbol{A}_q$ or the pointwise solutions might not glue together well to constitute the torsion of a compatible $\boldsymbol{\nabla}$.
- If there are compatible $\boldsymbol{\nabla}$s on the manifold, then the pointwise solution sets must be non-empty and glue together well in the sense of the following theorems.

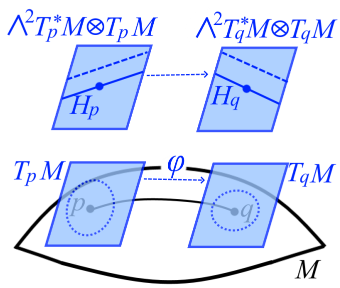
2.4.4. Statement. [20], [3] *Let (M, F) be a Finsler manifold and φ be a parallel translation induced by a compatible linear connection ∇ along some curve between two points p and q. Then we have the following.*

a) If γ is a compatible Riemannian metric, then φ is a linear isometry between T_pM and T_qM in both the Riemannian and Finslerian sense:

$$\gamma\big(\varphi(v), \varphi(w)\big) = \gamma(v, w) \qquad \text{for any } v, w \in T_pM, \tag{24}$$

$$F \circ \varphi = F, \ \text{ i.e. } F(\varphi(v)) = F(v) \qquad \text{for any } v \in T_pM. \tag{25}$$

b) Consider the affine subspaces $\boldsymbol{A}_p$ and $\boldsymbol{A}_q$ as the affine translates of their directional spaces $\boldsymbol{H}_p$ and $\boldsymbol{H}_q$ in $\wedge^2 T_p^* M \otimes T_p M$ and $\wedge^2 T_q^* M \otimes T_q M$, respectively. Then the parallel translation between these spaces induced by φ (see (4)) gives an isomorphism between $\boldsymbol{H}_p$ and $\boldsymbol{H}_q$; i.e. for any $\boldsymbol{T}_p \in \boldsymbol{H}_p$, $\boldsymbol{T}_q$ defined by



$$\boldsymbol{T}_q(v, w) := \varphi \circ \boldsymbol{T}_p(\varphi^{-1}(v), \varphi^{-1}(w)) \tag{26}$$

belongs to the directional space $\boldsymbol{H}_q$.

Thus, the mapping $p \in M \mapsto \boldsymbol{H}_p \subseteq \wedge^2 T_p^* M \otimes T_p M$ is a smooth distribution of constant rank of the torsion tensor bundle $\wedge^2 T^* M \otimes TM$ over the components of the manifold M.

PROOF. Part a) is straight-forward: since ∇ is compatible to F, it is also metrical to γ, and this means exactly that ∇'s parallel translation is a linear isometry in both senses.

For b), let $\big(Q^j_i\big)$ be the matrix representation of φ in the coordinate bases induced by the chart, i.e. $\varphi(\partial_k|_p) = Q^j_k \partial_j|_q$, and $P^i_j = \big(Q^j_i\big)^{-1}$. The directional space $\boldsymbol{H}_q$ is defined by

$$y^j(w)\big(-\boldsymbol{T}^k_{ij} + \boldsymbol{T}^l_{jm}\gamma^{mk}\gamma_{il} - \boldsymbol{T}^l_{mi}\gamma^{mk}\gamma_{jl}\big)(q)\,\dot{\partial}_k F(w) = 0, \qquad i \in \{1,\dots,n\}$$

(see formula (14)) where w runs through $T^\circ_q M$. Using the isomorphism φ between T_pM and T_qM, it is equivalent to writing $w = \varphi(v)$ and letting v run through $T^\circ_p M$, i.e. we can write

$$y^j \circ \varphi(v)\big(-\boldsymbol{T}^k_{ij} + \boldsymbol{T}^l_{jm}\gamma^{mk}\gamma_{il} - \boldsymbol{T}^l_{mi}\gamma^{mk}\gamma_{jl}\big)(q)\,\dot{\partial}_k F \circ \varphi(v) = 0,\ i \in \{1,\dots,n\}.$$

Using $\varphi^j = y^j \circ \varphi = Q^j_b y^b$ and $F \circ \varphi = F$ implying

$$\dot{\partial}_c F(v) = \dot{\partial}_c (F \circ \varphi)(v) = Q^j_c\,\dot{\partial}_j F \circ \varphi(v) \ \Rightarrow\ P^c_j\,\dot{\partial}_c F(v) = \dot{\partial}_j F \circ \varphi(v),$$

we can write the i-th equation equivalently as

$$y^b(v)\big(-\boldsymbol{T}^k_{ij}Q^j_b P^c_k + \boldsymbol{T}^l_{jm}Q^j_b P^c_k\gamma^{mk}\gamma_{il} - \boldsymbol{T}^l_{mi}Q^j_b P^c_k\gamma^{mk}\gamma_{jl}\big)(q)\,\dot{\partial}_c F(v) = 0.$$

Let us remember the following identities:

$$\gamma_{ab}(p) = \gamma\big(\partial_a|_p, \partial_b|_p\big) \overset{(24)}{=} \gamma\big(\varphi(\partial_a|_p), \varphi(\partial_b|_p)\big) = \gamma\big(Q^i_a\partial_i|_q, Q^j_b\partial_j|_q\big) = Q^i_a Q^j_b \gamma_{ij}(q)$$

$$\gamma^{ab}(p) = (\gamma_{ab}(p))^{-1} = P^a_i P^b_j \gamma^{ij}(q)$$

$$\begin{aligned} \gamma_{ab}(p) &= Q^i_a Q^j_b \gamma_{ij}(q) \implies Q^j_b\gamma_{ij}(q) = P^a_i\gamma_{ab}(p) \\ \gamma^{ab}(p) &= P^a_i P^b_j \gamma^{ij}(q) \implies P^b_j\gamma^{ij}(q) = Q^i_a\gamma^{ab}(p). \end{aligned}$$

Using them, we can make the transformation

$$y^b(v)\big(-\boldsymbol{T}^k_{ij}Q^j_b P^c_k + \boldsymbol{T}^l_{jm}Q^j_b \underline{P^c_k\gamma^{mk}}\gamma_{il} - \boldsymbol{T}^l_{mi}\underline{Q^j_b\gamma_{jl}}\ \underline{P^c_k\gamma^{mk}}\big)(q)\,\dot{\partial}_c F(v) = 0$$

$$y^b(v)\big(-\boldsymbol{T}^k_{ij}(q)Q^j_b P^c_k + \boldsymbol{T}^l_{jm}(q)Q^j_b Q^m_k\gamma^{ck}(p)\gamma_{il}(q) - \\ \boldsymbol{T}^l_{mi}(q)P^j_l\gamma_{jb}(p)Q^m_k\gamma^{ck}(p)\big)\,\dot{\partial}_c F(v) = 0.$$

Taking the product by the matrix (Q^i_a), the equivalent system of equations we obtain is

$$y^b(v)\big(-\boldsymbol{T}^k_{ij}(q)Q^i_a Q^j_b P^c_k + \boldsymbol{T}^l_{jm}(q)Q^j_b Q^m_k\gamma^{ck}(p)\underline{Q^i_a\gamma_{il}(q)} - \\ \boldsymbol{T}^l_{mi}(q)P^j_l Q^i_a Q^m_k\gamma^{ck}(p)\gamma_{jb}(p)\big)\,\dot{\partial}_c F(v) = 0$$

$$y^b(v)\big(-\boldsymbol{T}^k_{ij}(q)Q^i_a Q^j_b P^c_k + \boldsymbol{T}^l_{jm}(q)P^i_l Q^j_b Q^m_k\gamma^{ck}(p)\gamma_{ia}(p) - \\ \boldsymbol{T}^l_{mi}(q)P^j_l Q^i_a Q^m_k\gamma^{ck}(p)\gamma_{jb}(p)\big)\,\dot{\partial}_c F(v) = 0.$$

But observe that from (26) we have

$$\begin{aligned}\boldsymbol{T}^k_{ij}(q)\partial_k = \boldsymbol{T}_q(\partial_i,\partial_j) &\overset{(26)}{=} \varphi\big(\boldsymbol{T}_p\left(\varphi^{-1}(\partial_i),\varphi^{-1}(\partial_j)\right)\big) = \varphi\big(\boldsymbol{T}_p\left(P^a_i\partial_a, P^b_j\partial_b\right)\big)\\ &= \varphi\big(P^a_iP^b_j\boldsymbol{T}^c_{ab}(p)\partial_c\big) = \boldsymbol{T}^c_{ab}(p)P^a_iP^b_jQ^k_c\partial_k,\end{aligned}$$

i.e. $\boldsymbol{T}^k_{ij}(q) = \boldsymbol{T}^c_{ab}(p)P^a_iP^b_jQ^k_c$ or $\boldsymbol{T}^c_{ab}(p) = \boldsymbol{T}^k_{ij}(q)Q^i_aQ^j_bP^c_k$, so the equations are equivalent to

$$y^b(v)\big(-\boldsymbol{T}^c_{ab}+\boldsymbol{T}^i_{bk}\gamma^{ck}\gamma_{ia}-\boldsymbol{T}^j_{ka}\gamma^{ck}\gamma_{jb}\big)(p)\,\dot{\partial}_cF(v)=0, \qquad a\in\{1,\dots,n\}.$$

From this we see that at all points of a chart the directional spaces are isomorphic and can be spanned by smooth sections. It is easy to see that our claim holds over connected components, since this property can be 'transferred' between charts. (For any two points p and q in the same component, connect them with a curve, construct a finite cover for it by charts (use its compactness) and note that we can get from the chart around p to the chart around q through nonempty intersections (use the connectedness of the curve)). □

2.4.5. Theorem. [**20**], [**3**] *If we have a **connected** generalized Berwald manifold, then the mapping $p\in M\mapsto A_p\subseteq\wedge^2T^*_pM\otimes T_pM$ is a smooth affine distribution of constant rank of the torsion tensor bundle $\wedge^2T^*M\otimes TM$.*

Proof. Having a GBM means that the CEQ has global solutions, in particular, A_p is never empty. In the previous statement we saw that our claim is true for the homogeneous parts H_p. But this distribution is obtained from that one by a translation by the (smooth) torsion of a compatible linear connection: $A_p = T_p + H_p$, so our claim follows. □

2.5. The geometry of the tangent spaces

So far in this chapter we have concentrated on formulas so much that we almost forgot about the geometric picture. Now we make up for it and describe what the tangent space T_pM looks like at our fixed point $p\in M$.[4]

2.5.1. The double geometric structure of the tangent spaces.

2.5.1. Remember that we are working on a Finsler manifold (M,F), but we also chose a compatible Riemannian metric γ alongside the Finslerian one. Thus we have two different geometric structures on T_pM:

(1) **The Finsler structure**: the Finsler metric F restricted to T_pM is a Minkowski norm, so its level sets (and the indicatrix among them) are hypersurfaces that are the (smooth) boundaries of strictly convex bodies[5] containing the origin in their interior. We will call these Finslerian spheres, and the set $F_p := F^{-1}(1)\cap T_pM$ the Finslerian indicatrix.

[4] See [**3**] or [**20**].

[5] Strict convexity means that the line segments connecting any two points of the boundary are contained in the interior of the body. In particular, the boundary cannot contain any straight line segment (intuitively, it is curved everywhere, in every direction).

(2) **The Euclidean** (Riemannian) **structure**: the compatible Riemannian metric γ restricted to T_pM is an inner product that, in the orthonormal basis $(\partial/\partial u^1|_p, \ldots, \partial/\partial u^n|_p)$, is just the standard Euclidean metric. The level sets are Euclidean spheres, the Riemannian indicatrix R_p is just the Euclidean unit sphere.

2.5.2. By having a Euclidean structure we can work with inner products, norms, orthogonality and normal vectors. Whenever we will use these notions in T_pM, it will refer to the Riemannian metric $\gamma_p(v, w) =: \langle v, w\rangle$.

2.5.3. We also note that the CEQ (see the form (20)) is obviously 1^+-homogeneous in v.[6] Therefore, **when considering the CEQ, it is enough to let v run through the elements of one of the indicatrices.**

2.5.4. Let us determine the tangent hyperplanes of these spheres in T_pM.

(1) Since the Finslerian spheres are the level sets of F, the Euclidean gradient vector field of F, namely

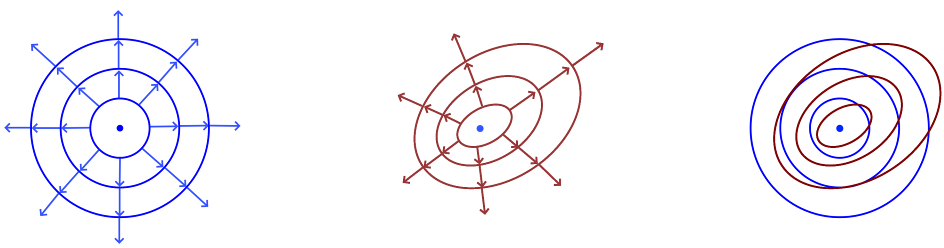
$$G := \mathrm{grad}(F) = \frac{\partial F}{\partial y^1}\frac{\partial}{\partial y^1} + \cdots + \frac{\partial F}{\partial y^n}\frac{\partial}{\partial y^n} \sim \left[\dot{\partial}_1 F, \ldots, \dot{\partial}_n F\right],$$

gives the (outer) normal vectors of them and their tangent hyperplanes.

(2) The (outer) normal vectors of the Euclidean spheres (and of their tangent hyperplanes) are given by the **radial** or **Liouville vector field**

$$C := y^1\frac{\partial}{\partial y^1} + \cdots + y^n\frac{\partial}{\partial y^n} \sim \left[y^1, \ldots, y^n\right], \quad \text{i.e.} \quad C_v \sim v.$$

For every $v \in T_p^\circ M$ there is one Finslerian and one Euclidean sphere passing through it with tangent hyperplanes $\mathcal{F}_v$ and $\mathcal{R}_v$, respectively.[7] The question is: **how can these two hyperplanes be related to each other?**



(A) The Euclidean spheres and the radial vector field C

(B) The Finslerian spheres and the vector field G

(C) The Euclidean and Finslerian spheres together

FIGURE 1. The two different kinds of level spheres and their outer normal vector fields in a 2-dimensional tangent space

[6]Since F is 1^+-homogeneous, differentiating with respect to y^i (on the LHS) decreases this degree by one, and differentiating with respect to x^i (on the RHS) leaves it the same.

[7]We mean these as affine subspaces at the given points. If we work with their linear translates (direction spaces), we will say so and will denote them by $\mathcal{LF}_v$ and $\mathcal{LR}_v$.

2.5.2. Useful vector fields.

2.5.5. We list here all the vector fields on $T_p^\circ M$ that will be used at some point [the coordinates are meant in the basis $(\dot\partial_1, \dots, \dot\partial_n)$].

- The Finslerian normal field $G_v := [\dot\partial_1 F(v), \dots, \dot\partial_n F(v)]$,
- the Riemannian normal field $C_v := [v^1, \dots, v^n] \sim v$,
- the coefficient fields $f_i(v) := [f_{i1}(v), f_{i2}(v), \dots, f_{in}(v)], \qquad i \in \{1, \dots, n\}$,
- the homogeneity field $H_v := \left[X_1^{h*}F(v), \dots, X_n^{h*}F(v)\right] = \left[\widehat\partial_1 F(v), \dots, \widehat\partial_n F(v)\right]$.

2.5.6. Lemma. *The vector fields G and C are nonzero everywhere.*

Proof. C is obviously nonzero if $v \neq \mathbf{0}$. If G was the zero vector at some point $v \in T_p^\circ M$, then, from Euler's homogeneity theorem,

$$0 = \langle C_v, \mathbf{0} \rangle = \langle C_v, G_v \rangle = v^i \dot\partial_i F(v) = F(v),$$

which contradicts the fact that $F(v) > 0$ if $v \neq \mathbf{0}$.[8] $\square$

2.5.7. Statement. *Considering all values of C or G, they span the whole space T_pM. In fact, it is enough to consider them over an arbitrarily small open neighborhood of the indicatrix.*

Proof. This is a well-known result from convex geometry and is true for the normal vector field of any smooth and strictly convex body over any open neighborhood U. Let S denote the linear span of all the normal vectors and I denote the intersection of all linear tangent hyperplanes at the points of U. A vector lies in I if and only if it is orthogonal to all the normal vectors, or, in other words, I and S are orthogonal complements of each other. In particular, $\dim(I) + \dim(S) = \dim(T_pM) = n$.

Let us suppose that $\dim(S) \le n-1$, or, equivalently, $\dim(I) \ge 1$. This means that all the tangent hyperplanes considered have a directional line in common. In other words, over the set U, we can find a tangential vector field to the convex body that is constant (in T_pM considered as $\mathbb{R}^n$). But the integral curves of such vector fields are straight line segments and they must run on the boundary, contradicting the strict convexity of the body. $\square$

2.5.8. Lemma. *For all $v \in T_p^\circ M$,* $\operatorname{span}(f_1(v), \dots, f_n(v)) \subseteq \operatorname{span}(G_v, C_v)$.

Proof. Observe that the f_i can be written as

$$f_i = \begin{bmatrix} f_{i1} \\ f_{i2} \\ \vdots \\ f_{in} \end{bmatrix} = \begin{bmatrix} y^i \dot\partial_1 F - y^1 \dot\partial_i F \\ y^i \dot\partial_2 F - y^2 \dot\partial_i F \\ \vdots \\ y^i \dot\partial_n F - y^n \dot\partial_i F \end{bmatrix} = y^i \begin{bmatrix} \dot\partial_1 F \\ \dot\partial_2 F \\ \vdots \\ \dot\partial_n F \end{bmatrix} - \dot\partial_i F \begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix},$$

or, in short,

$$f_i = y^i \cdot G - \dot\partial_i F \cdot C. \tag{27}$$ $\square$

[8] From this it also follows that C and G cannot be orthogonal to each other anywhere.

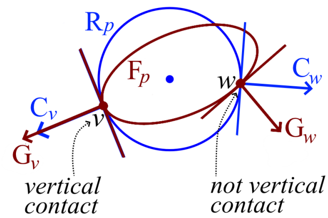
2.5.3. Vertical and horizontal contact points.

Let us examine how the tangent hyperplanes of the two kinds of spheres passing through a point $v \in T_p^\circ M$, considered as affine subspaces of T_pM at v, can relate to each other. The most simple case is when they coincide.

2.5.9. Definition. We call an element $v \in T_p^\circ M$ a **vertical contact point** (of F and γ) if $\mathcal{F}_v = \mathcal{R}_v$, i.e. the Finslerian and Euclidean spheres passing through it have the same tangent hyperplanes at v; or, equivalently, if their normal vectors G_v and C_v are parallel. The tangent space T_pM is called vertical contact if all of its nonzero elements are vertical contact points.

We note some easy facts about vertical contact points.

2.5.10. Statement. *We can classify the elements of $T_p^\circ M$ as follows:*



- *vertical contact points where the two tangent hyperplanes coincide;*
- *not vertical contact points where the two tangent hyperplanes intersect each other in an affine subspace of dimension $n-2$.*

Proof. If $v \in T_p^\circ M$ is not vertical contact, i.e. one of the two tangent hyperplanes has an element not contained in the other, then their sum gives the whole T_pM. So using the Grassmann formula for the dimension of the intersection of affine subspaces with nonempty intersection,

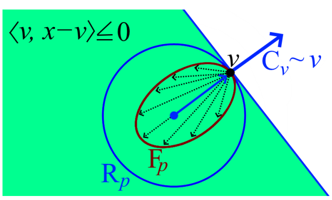
$$\begin{aligned} \dim(K \cap L) &= \dim(K) + \dim(L) - \dim(K+L) \\ &= (n-1) + (n-1) - n = n-2. \end{aligned}$$

□

2.5.11. Statement. *Any tangent space has a vertical contact point.*

Proof. Consider the Finslerian indicatrix F_p and choose one of its elements v with maximal Euclidean norm (by compactness, such a v exists). We will show that v is a vertical contact point.

Taking the inner product of any element $x \in F_p$ with v, i.e.



$$\langle v, x\rangle = \|v\| \cdot \|x\| \cdot \cos(\measuredangle(v,x)),$$

since v has maximal Euclidean norm, we have

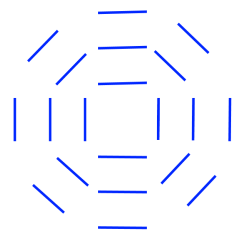
$$\begin{aligned} \langle v, x\rangle &\le \langle v, v\rangle \\ \langle v, x-v\rangle = \langle v, x\rangle - \langle v, v\rangle &\le 0. \end{aligned}$$

This means that F_p lies in one of the half-spaces of T_pM determined by the hyperplane going through v with normal vector v, or in other words, this hyperplane supports F_p at v. But because F_p is smooth and strictly convex, this hyperplane is exactly its tangent hyperplane, and it is obviously equal to the Euclidean one (both their normal vectors are v). □

2.5.12. Statement. *If T_pM is a vertical contact tangent space, then F is quadratic at the point p. If the manifold is a* ***connected*** *GBM, then F is quadratic everywhere, i.e. it is induced by a Riemannian metric.*

Proof. If T_pM is a vertical contact tangent space, then the tangent hyperplanes of the Riemannian and Finslerian spheres are the same at every point $v \in T_p^\circ M$.

We can consider their family as a smooth distribution on $T_p^\circ M$ of dimension $n-1$, and then both the families of Riemannian and Finslerian spheres constitute a foliation of it. Then they are obviously equal, meaning that the Finslerian indicatrix is a scalar multiple of the Riemannian indicatrix, and is therefore itself quadratic.



If we have a connected GBM, then by using the rigidity property (see Statement 1.4.11.), having a quadratic indicatrix at one point implies the metric being Riemannian. □

Another notion we will use for the points of T_pM is the following.

2.5.13. Definition. We call an element $v \in T_p^\circ M$ a **horizontal contact point** (of F and γ) if

$$H_v = \left[X_1^{h*}F(v), \dots, X_n^{h*}F(v)\right] = \left[\widehat{\partial}_1 F(v), \dots, \widehat{\partial}_n F(v)\right] = \mathbf{0},$$

i.e. the derivative of the Finsler function vanishes at v in the direction of all the horizontal vector fields determined by γ. T_pM is called horizontal contact if all of its nonzero elements are horizontal contact points.

2.6. Contact points and the compatibility equations

Let us return to the CEQ in its form (16). Remember that we obtain its solution at p by solving it at all elements $v \in T_p^\circ M$ and taking their intersection. Now we look at what happens at the individual v's.

2.6.1. Lemma. *$v \in T_p^\circ M$ is a horizontal contact point $\iff$ the right hand side of the CEQ at v is zero, i.e. we have a homogeneous linear system.*

Proof. This is exactly how we defined horizontal contact points. □

2.6.2. Corollary. *If T_pM is a horizontal contact tangent space, then the CEQ has a solution at the point p, namely $T \equiv 0$.*

Proof. If all elements $v \in T_p^\circ M$ are horizontal contact, then for all of them we get homogeneous linear systems. □

2.6.3. Statement. *A Finsler manifold is a classical Berwald manifold (a GBM with a torsion-free compatible linear connection) $\iff$ all the tangent spaces are horizontal contact.*

Proof. We have a classical Berwald manifold with a torsion-free compatible ∇ if and only if $T \equiv 0$ is a solution of the CEQ at every point. But this is equivalent to the CEQ being homogeneous at every $v \in T^\circ M$. □

2.6.4. Lemma. *$v \in T_p^\circ M$ is a vertical contact point $\Longleftrightarrow$ all the f_{ij}, and thus the whole left-hand side of the CEQ is zero at v.*

Proof. At a fixed point v, all coefficients $f_{ij} = 0 \Longleftrightarrow$ all coefficient fields $f_i = \mathbf{0}$. But v being vertically contact means $G_v = \lambda C_v$ for some nonzero $\lambda \in \mathbb{R}$. From (27) we get

$$f_i(v) = v^i \cdot \lambda C_v - \dot{\partial}_i F(v) \cdot C_v = v^i \cdot \lambda C_v - \lambda v^i \cdot C_v = 0, \qquad i \in \{1, \dots, n\}.$$

Conversely, if all the coefficient fields are zero at v, then, choosing an index for which $v^i \neq 0$, from (27) we get

$$f_i(v) = v^i \cdot G_v - \dot{\partial}_i F(v) \cdot C_v = 0 \implies G_v = \frac{\dot{\partial}_i F(v)}{v^i} \cdot C_v,$$

i.e. G_v and C_v are parallel, and thus v is vertically contact. □

2.6.5. Corollary. *In order for the CEQ to have a solution, i.e. to have a GBM, all vertical contact points must be horizontal contact.*

2.6.6. Remark. Vertical contact points can be used for checking the existence of solutions (they can be obstructions against it if they are not horizontally contact), but when the existence is known, these points are quite useless, because they give the equations $0 = 0$. Therefore, when looking for the solution of the CEQ, we need to work with not vertically contact points.

For the sake of completeness (and later use), let's continue the examination of the coefficient fields.

2.6.7. Lemma. *At a not vertically contact $v \in T_p^\circ M$, at least one of the coefficients $f_{ij}(v)$ is nonzero, and the vectors $f_i(v)$ and $f_j(v)$ belonging to these indices are linearly independent. The same is true at the points of some adequately small neighborhood of v.*

Proof. The first claim is immediate from Lemma 2.6.4, and if there is an $f_{ij}(v)$ different from zero, there are two vectors $f_i(v)$ and $f_j(v)$ also different from zero. Since the matrix

$$\begin{bmatrix} f_i \\ f_j \end{bmatrix} = \begin{bmatrix} f_{i1} & \dots & 0 & \dots & f_{ij} & \dots & f_{in} \\ f_{j1} & \dots & f_{ji} & \dots & 0 & \dots & f_{jn} \end{bmatrix}$$

has rank 2 at v (choose the i-th and j-th columns), they are linearly independent. The last claim follows from the continuity of f_{ij}. □

2.6.8. Corollary. *At a point $v \in T_p^\circ M$ the coefficient vector fields together span the subspace*

$$\operatorname{span}(f_1(v), \dots, f_n(v)) = \begin{cases} \{\mathbf{0}\} & \text{if } v \text{ is vertical contact,} \\ \operatorname{span}(G_v, C_v) & \text{if } v \text{ is not vertical contact.} \end{cases}$$

2.7. The extremal compatible linear connection

Since we know that the pointwise solutions of the CEQ are affine subspaces, it is clear that in general nothing guarantees the existence of compatible ∇s, or, at the other extreme, there might exist a large family of them. The latter case leads us back to our third initial question: **on a GBM, which compatible linear connection should we choose as a canonical one?** A reasonable choice is the so-called extremal connection, introduced in [20].

2.7.1. Bundle metrics induced by a Riemannian metric.

2.7.1. We are going to describe how a Riemannian metric on a manifold, which measures tangent vectors as

$$\langle v, w\rangle_{T_pM} = \left\langle v^i\partial_i, w^j\partial_j\right\rangle_{T_pM} = v^iw^j\left\langle\partial_i,\partial_j\right\rangle_{T_pM} = g_{ij}v^iw^j,$$

induces a so-called bundle metric, i.e. a smoothly varying family of inner products on the fibers, on any tensor bundles (see e.g. [12]).

- The most simple case is the cotangent bundle: it is well-known that a Riemannian metric gives an isomorphism between T_pM and T_p^*M, namely

$$X = X^i\partial_i \in T_pM \mapsto \widehat{X} = X_jdu^j = g_{ij}X^idu^j \in T_p^*M,$$

 and through it the metric is carried over to T_p^*M as

$$\left\langle \widehat{X}, \widehat{Y}\right\rangle_{T_p^*M} := \langle X, Y\rangle_{T_pM},$$

 or in coordinates,

$$\langle\alpha,\beta\rangle_{T_p^*M} = \left\langle\alpha_idu^i,\beta_jdu^j\right\rangle_{T_p^*M} = g^{ij}\alpha_i\beta_j.$$

- Since all the tensor bundles are built from elements of T_pM and T_p^*M, bundle metrics on them can also be built from the metrics introduced on these spaces. For example, we can define

$$\langle\alpha\otimes v,\beta\otimes w\rangle_{T_p^*M\otimes T_pM} := \langle\alpha,\beta\rangle_{T_p^*M}\cdot\langle v,w\rangle_{T_pM}.$$

- The case we need is the torsion tensor bundle $\wedge^2T^*M\otimes TM$. Here we can define

$$\begin{aligned}
&\langle T, S\rangle_{\wedge^2T_p^*M\otimes T_pM}\\
&= \left\langle \sum_{a<b,c} T_{ab}^c(p)\,du^a\wedge du^b\otimes\partial_c, \sum_{i<j,k} S_{ij}^k(p)\,du^i\wedge du^j\otimes\partial_k\right\rangle_{\wedge^2T_p^*M\otimes T_pM}\\
&= \sum_{a<b,c}\ \sum_{i<j,k} T_{ab}^c(p)\,S_{ij}^k(p)\left\langle du^a\wedge du^b\otimes\partial_c, du^i\wedge du^j\otimes\partial_k\right\rangle_{\wedge^2T_p^*M\otimes T_pM}\\
&= \sum_{a<b,c}\ \sum_{i<j,k} T_{ab}^c(p)\,S_{ij}^k(p)\left\langle du^a\wedge du^b, du^i\wedge du^j\right\rangle_{\wedge^2T_p^*M}\cdot\langle\partial_c,\partial_k\rangle_{T_pM}.
\end{aligned}$$

Since the middle term is

$$
\begin{aligned}
&\left\langle du^a \wedge du^b, du^i \wedge du^j \right\rangle_{\wedge^2 T_p^* M} \\
= \; & \left\langle \frac{1}{2}\left(du^a \otimes du^b - du^b \otimes du^a\right), \frac{1}{2}\left(du^i \otimes du^j - du^j \otimes du^i\right) \right\rangle_{\wedge^2 T_p^* M} \\
= \; & \frac{1}{4}\Big(\left\langle du^a \otimes du^b, du^i \otimes du^j \right\rangle_{T_p^*M \otimes T_p^*M} - \left\langle du^a \otimes du^b, du^j \otimes du^i \right\rangle_{T_p^*M \otimes T_p^*M} \\
& - \left\langle du^b \otimes du^a, du^i \otimes du^j \right\rangle_{T_p^*M \otimes T_p^*M} + \left\langle du^b \otimes du^a, du^j \otimes du^i \right\rangle_{T_p^*M \otimes T_p^*M} \Big) \\
= \; & \frac{1}{4}\left(g^{ai} g^{bj} - g^{aj} g^{bi} - g^{bi} g^{aj} + g^{bj} g^{ai}\right) = \frac{1}{2}\left(g^{ai} g^{bj} - g^{bi} g^{aj}\right),
\end{aligned}
$$

we have

$$
\langle T, S \rangle_{\wedge^2 T_p^* M \otimes T_p M} = \frac{1}{2} \sum_{a<b,c} \sum_{i<j,k} \left(T^c_{ab}\, S^k_{ij} \left(g^{ai} g^{bj} - g^{bi} g^{aj}\right) g_{ck} \right)(p).
$$

Using an orthonormal basis at the point $p \in M$, this becomes

$$
\begin{aligned}
\langle T, S \rangle_{\wedge^2 T_p^* M \otimes T_p M} &= \frac{1}{2} \sum_{a<b,c} \sum_{i<j,k} T^c_{ab}(p)\, S^k_{ij}(p) \left(\delta^{ai}\delta^{bj} - \delta^{bi}\delta^{aj}\right) \delta_{ck} \\
&= \frac{1}{2} \sum_{a<b,c} \left(T^c_{ab}\, S^c_{ab} - T^c_{ab} S^c_{ba}\right)(p) = \sum_{a<b,c} T^c_{ab}(p) S^c_{ab}(p).
\end{aligned}
$$

Thus we can make the following definition.

2.7.2. Definition. Let (M, γ) be a Riemannian manifold and choose local coordinates such that the coordinate vectors $(\partial/\partial u^1|_p, \ldots, \partial/\partial u^n|_p)$ form an orthonormal basis at the point $p \in M$. We introduce an inner product on $\wedge^2 T_p^* M \otimes T_p M$ the following way.

If $T = \sum_{i<j,k} T^k_{ij}\, du^i \wedge du^j \otimes \dfrac{\partial}{\partial u^k}$, then we define

$$
\boxed{\langle T_p, S_p \rangle := \sum_{i<j,k} T^k_{ij}(p)\, S^k_{ij}(p) \qquad \text{and} \qquad \|T_p\|^2 = \sum_{i<j,k} T^k_{ij}(p)^2.} \tag{28}
$$

These inner products constitute a bundle metric for $\wedge^2 T^* M \otimes TM$ (called the **bundle metric induced by** γ).

With this inner product the induced basis

$$
du^i \wedge du^j \otimes \frac{\partial}{\partial u^k} \quad (1 \le i < j \le n, k = 1, \ldots, n)
$$

is an orthonormal basis of $\wedge^2 T_p^* M \otimes T_p M$.

Since this metric is induced by a Riemannian metric on the manifold, if a map preserves the latter, its induced map should preserve the former. We are of course talking about the induced parallel translations on tensor spaces.

2.7.3. Statement. *The correspondence*

$$T_q(v,w) := \varphi \circ T_p(\varphi^{-1}(v), \varphi^{-1}(w))$$

introduced in (26) *between the direction spaces* H_p *and* H_q *induced by the parallel translation* φ *is not only a linear isomorphism, but also an isometry, preserving the inner product* (28)*.*

Proof. It is enough to see that φ takes an orthonormal basis of $\wedge^2 T_p^* M \otimes T_p M$ to an orthonormal basis of $\wedge^2 T_q^* M \otimes T_q M$, and this is easy to check. Considering an orthonormal basis of $T_p M$, since φ is a Riemannian isometry, its image is an orthonormal basis of $T_q M$. By the definition of the bundle metric, their induced bases in $\wedge^2 T_p^* M \otimes T_p M$ and $\wedge^2 T_q^* M \otimes T_q M$ are also orthonormal; but they are also each other's images by the induced translation (26). $\square$

2.7.2. The extremal compatible connection.

If we can measure in the vector space of the pointwise values of the possible torsion tensors, then we have a way of comparing them.

2.7.4. Definition. Let (M, F) be a GBM with a compatible Riemannian metric γ. Then the **extremal compatible linear connection** (with respect to γ) is the uniquely determined compatible linear connection whose torsion has the minimal pointwise norm everywhere with respect to the bundle metric (28) induced by γ on $\wedge^2 T^* M \otimes TM$. We will denote this connection by ∇^0 (or we could write ∇^0_γ to emphasize the role of γ).

2.7.5. Remark. Here we should think through some things.

- **Why is it unique?** Since the pointwise solution sets A_p of the CEQ are affine subspaces of the vector spaces $\wedge^2 T_p^* M \otimes T_p M$, they satisfy the unique nearest point property: if there is a solution, the one with minimal norm (the nearest point to the origin) is uniquely determined (it is the point of A_p whose position vector is orthogonal to A_p).
- **In what sense is it unique?** Only as long as the compatible Riemannian metric γ is fixed: choosing another γ may yield a different Euclidean structure in the tangent spaces, a different notion of orthogonality and thus a different nearest point in A_p to the origin.

2.7.6. Remark. A more subtle question is: **when does it exist?**

The situation is similar as with the solution subspaces A_p themselves: even if they are nonempty at every point, and thus we can take the closest elements and define a torsion tensor-to-be from the pointwise values, nothing guarantees its smoothness (or even continuity), meaning that we don't necessarily get a global solution of the CEQ which would determine a compatible linear connection.

But provided that solutions, i.e. compatible linear connections do exist (the manifold is a GBM), then the solution spaces A_p vary smoothly (as we have seen in Theorem 2.4.5), and so do their elements with minimal torsion norm. So from them we get a continuous and smooth section of $\wedge^2 T^*M \otimes TM$, and thus this indeed determines a compatible linear connection, the extremal one.

2.7.7. Remark. Let us also note the following use of the extremal connection. Determining whether or not a space can be a GBM means solving the CEQ, which is not easy. But if, for a particular metric, we can give some general formula for the extremal connection (when it exists), then we don't have to solve the CEQ generally; we only have to check whether the connection that *should be* the extremal compatible connection is indeed compatible. If it is not, then the space is not a GBM (if it were, it would have an extremal connection, and that would be this one); if it is, then the space is of course a GBM.

2.7.8. Remark. There is just one problem left: computing the extremal connection in practice. In [**20**] Vincze gave an algorithm, which can be used (in theory) to obtain the torsion of it in finitely many steps. Finding an explicit formula, however, is still an open question.

Finally, we summarize some basic observations.

2.7.9. Lemma. *If on a GBM there is a torsion-free compatible linear connection, then that is the extremal one.*

2.7.10. Corollary. *If all the tangent spaces are horizontally contact, then the extremal connection exists and it is torsion-free.*

Proof. Horizontally contact tangent spaces obviously imply that the space is a classical Berwald manifold; see Statement 2.6.3. □

2.7.11. Corollary. *If a* ***connected*** *GBM has one vertically contact tangent space, then the extremal connection is the Levi-Civita connection of the compatible Riemannian metric.*

Proof. We saw in Statement 2.5.12 that in this case the Finsler metric is actually Riemannian everywhere and it is a multiple of the norm induced by the compatible Riemannian metric γ. So it has the same family of compatible connections as γ, among which the one with minimal torsion norm is the Levi-Civita connection (which is actually torsion-free). □

CHAPTER 3

An example: Randers spaces

Finding compatible linear connections to a given Finsler metric on a given manifold (or deciding if there are any such connections) is not an easy exercise. Solving the problem in general, i.e. finding *all* compatible linear connections for *any* Finsler metric on a manifold of *any* dimensions (giving the general solution of the compatibility equations) is a much more formidable task and is yet to be solved.

Hoping to gather some insight and to develop some techniques, it seems logical to attack the problem in special cases first. There are 3 things we can specialize: the form of the metric we are given, the type of connections we are looking for and the dimension of the space we are working on. There are some results in all of these directions:

- easy-to-handle metrics are, for example, Randers metrics (or more generally, (α, β) metrics) and m-th root metrics (where some exponent of the metric is just a polynomial of the direction coordinates);
- linear connections easy to work with are the so-called semi-symmetric connections, whose case is completely solved, see the next chapter;
- the cases of dimensions 2 and 3 (for general metrics and connections) are also settled, and the rest of this book will be dedicated to them.

But first, in this chapter, we are following the first path and consider the special family of Randers metrics that are obtained by 'translating' the Riemannian structure. In a way, these Finsler metrics are the ones standing closest to the Riemannian case, and thus the easiest to carry out computations for.[1] In the first half of the chapter, we give some motivation why Randers metrics (or Finsler metrics in general) are indeed useful and natural, by introducing a famous problem leading to their definition. Then, before doing our thing and considering compatible connections, we try to explore the basic structure of Randers metrics.

In the second half, we deal with the CEQ: we give the complete solution for Randers spaces, determine which torsion components are uniquely determined or freely chosen, find the extremal connection with minimal torsion norm and give a new proof for a well-known result on Randers spaces being generalized Berwald.[2]

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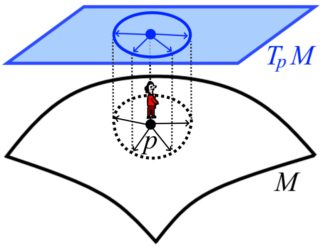
[1] In differential geometry, calculations tend to be long and difficult, so it's not that uncommon to use these spaces as easy (counter-)examples or for testing conjectures.

[2] In this chapter, we present the results of [**1**].

3.1. Do we really need Finsler metrics?

3.1.1. In Chapter 1, we introduced 2 types of metrics on manifolds: Riemannian and Finslerian ones. In the Riemannian case, the metric consists of a pointwise changing family of inner products, or in other words, the family of indicatrices comprises ellipsoids, changing from point to point.



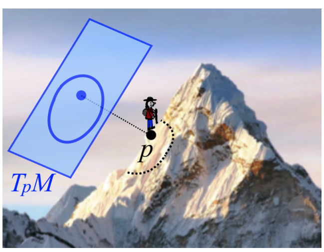
Switching to orthonormal coordinates, the inner product and the indicatrix become the Euclidean inner product and unit sphere, respectively, at least at some fixed point of the manifold. For example, in 2D, standing at this point and planning to go for a stroll, the points we can reach in a fixed (small) amount of time constitute a circle.

It's not hard to imagine that such a construction describes a well-behaved notion of distance, which is in accordance with our common sense, but can also be used to model various problems in science (of which perhaps the most famous is the theory of relativity by Albert Einstein, which uses a bit more general notion of Riemannian metrics).

3.1.2. On the other hand, a Finsler metric is a pointwise changing family of Minkowski (not necessarily centrally symmetric) norms, and whatever coordinate change we may apply, the indicatrix in general will remain the boundary of a general, strictly convex smooth body, not even centrally symmetric, let alone quadratic. This seems like a generalization too far gone and not corresponding to any practical use. But in reality, it's the exact opposite: even some very basic examples lead to the notion of Finsler metrics, and the number of scientific applications also grows by the day.

Two situations where Finsler metrics arise naturally are the following.

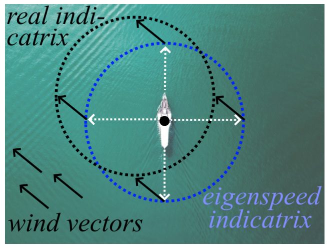
- **Walking on a hill:** Suppose you are on a hillside and measure distance as "the amount of time it takes to get there" (we'll call it time distance), using the same effort for walking all the time. What is your indicatrix at the point you are standing at, i.e. how far can you get in a fixed amount of time (let's say 1 minute) in each direction?

  It's much easier to walk downhill: with the same effort, you can walk a lot farther in that direction. Going upwards is definitely harder due to gravity hindering you. Thus, your indicatrix looks different in each direction and it is "stretched" downwards, i.e. you have a non-reversible Finsler metric. (These are called **slope metrics**, see e.g. [**28**].)

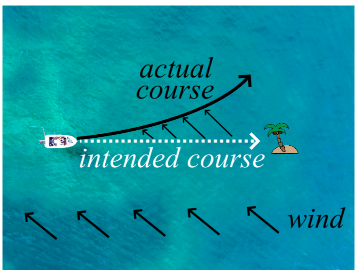
  When you are walking on the hillside, maybe you don't think about Finsler metrics in your head. But, for example, if you had to plan the path of a Mars rover on a hilly terrain, your knowledge about slope metrics could really come in handy.

- **Zermelo's navigation problem:** Imagine you are on a ship in the middle of the ocean, and the ship's engine is working at maximum capacity all the time (giving the same amount of power to move the ship). If we use the time distance again, at the point we are at, the indicatrix (the set of points on the sea we can reach in unit time, let's say 1 second) is a circle, so the metric is Riemannian – if there is no outer force influencing the path of the ship. But usually, this is not the case, and there is some wind altering our course.

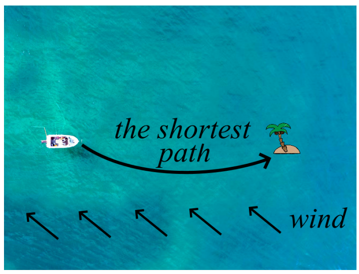
  

  Thus, in this case, the set of points we can reach in unit time is a translate of a circle, meaning that we have a non-Riemannian Finsler metric. Of course, the wind can vary from point to point (it is described by a vector field) and this construction (translating the Riemannian spheres at each point) can be made precise in any dimensions. These metrics are called **Randers metrics**.

The presence of the wind also complicates the work of the helmsman (or a navigator): if they want to arrive at a given destination point, they cannot get there along a line (Riemannian geodesic), because if they try, the wind will veer the ship off course. They always have to adjust the direction accordingly. Or, they could take the wind into account in advance, and choose a direction which leads there in the shortest time possible – this is called Zermelo's problem. Solving this amounts to finding the geodesics of the Randers metric described above. [25]



(A) When we don't count with the wind…



(B) and when we do.

Hopefully, after these really elementary everyday examples it is not that surprising that Finsler metrics have significant applications in various fields of science. Just to name a few:

- in optics, to describe the time distance for light, traveling through an anisotropic (changing from direction to direction) medium;
- in seismology, for modeling the spread of earthquakes;
- in ecology, to describe the spreading of wildfires;
- also in ecology, to measure distances between the states of evolution for some population.

For more, see [7] or [24].

3.2. The definition of a Randers metric

It is now time to define these metrics properly.

3.2.1. Definition. [7] A Finsler metric on a manifold M is called a **Randers metric** if it has the special form

$$\boxed{F(x,y) = \alpha(x,y) + \beta(x,y),} \tag{29}$$

where
- α is a norm induced by some Riemannian metric a on the manifold M
- and β is a 1-form field, i.e. a smoothly varying family of linear functionals on the tangent spaces.

We will call a the **Riemannian part** and β the **perturbating term** of the metric.

When working in coordinates, as usual, we consider the components (depending on the position on M) pulled up to TM by π, i.e. we use the coordinates x instead of u. In detail, we write
- $a_{ij}(x)$ for the components of the Riemannian part, so we have
$$\alpha(x,y) = \sqrt{a\left((x,y),(x,y)\right)} = \sqrt{a_{ij}(x)\, y^i\, y^j}$$
- and $\beta(x,y) = \beta_i(x)\, y^i$ for the perturbating term.

Sometimes we will work with the dual vector field $\beta^\sharp$ (**perturbating vector field**), which is defined by $\beta(x,y) = a(\beta^\sharp(x), y)$ and, in coordinates, it has the form
$$\beta^\sharp(x) = \beta^i(x)\, \dot\partial_i = a^{ij}(x)\, \beta_j(x)\, \dot\partial_i.$$

3.2.2. Remark. First note that a Randers metric is absolutely homogeneous (meaning that $F(x,\lambda v) = |\lambda|\, F(x,y)$ for all $\lambda \in \mathbb{R}$) if and only if the 1-form β is identically zero at all points, which, in turn, is equivalent to F being Riemannian. In what follows, we will always suppose without further emphasis that β is not identically zero, and thus the Randers metric is non-Riemannian.

The first thing that comes to mind after reading the definition is this question: **what can the 1-form β be?** Since a nonzero linear functional takes both positive and negative values on T_pM, we cannot choose β carelessly and expect the resulting function $\alpha + \beta$ to be a Finsler metric; to preserve the positivity, for example, β has to be 'small' in some sense. In addition to that, we also have to address the issue of strong convexity.

First, let us clarify that by the norm of β, we mean the norm induced by the Riemannian metric a and pulled to the space of covectors, that is,
$$\|\beta\| = \sqrt{\beta_i\, \beta^i} = \sqrt{\beta_i\, a^{ij}\, \beta_j}.$$
It is obviously equal to $\left\|\beta^\sharp\right\|$ (see Section 2.7).

3.2.3. STATEMENT. [7] *For any Riemannian metric a and 1-form β, $F = \alpha + \beta$ is positive for any nonzero tangent vector $\Longleftrightarrow$ $\|\beta(x)\| < 1$ everywhere.*

PROOF. $\boxed{\Rightarrow}$ Supposing the positivity, for any $v \in T^\circ M$ we have

$$\begin{aligned} F(x,y) = \alpha(x,y) + \beta(x,y) &> 0 \\ \alpha(x,y) &> -\beta(x,y) \\ \sqrt{a_{ij}(x)\, y^i\, y^j} &> -\beta_i(x)\, y^i. \end{aligned}$$

In particular, this must hold for the special choice of the vector whose coordinates are $y^i = -\beta^i(x)$. By substituting, we get

$$\begin{aligned} \sqrt{a_{ij}\,(-\beta^i)\,(-\beta^j)} &> -\beta_i\,(-\beta^i) \\ \sqrt{\|\beta^\sharp\|^2} &> \|\beta\|^2 \\ \|\beta\| &> \|\beta\|^2 \\ 1 &> \|\beta\|\,. \end{aligned}$$

$\boxed{\Rightarrow}$ Supposing $\|\beta\| = \left\|\beta^\sharp\right\| < 1$, the Cauchy–Schwarz inequality for the vectors $\beta^\sharp$ and $v = y$ yields

$$\begin{aligned} \left|\left\langle \beta^\sharp, y\right\rangle\right| &< \|y\| \cdot \left\|\beta^\sharp\right\| < \|y\| \\ \left|a_{ij}\,\beta^i\, y^j\right| &< \|y\| \\ \left|a_{ij}\, a^{ik}\,\beta_k\, y^j\right| &< \|y\| \\ -\beta_i\, y^i \le \left|\beta_i\, y^i\right| &< \sqrt{a_{ij}\, y^i\, y^j} \\ 0 &< \sqrt{a_{ij}\, y^i\, y^j} + \beta_i\, y^i = F(x,y). \end{aligned}$$

And this is what we wanted to prove. □

Now, about the issue of the strong convexity of F, or, the positive definiteness of the fundamental tensor

$$g_{ij} = \frac{1}{2}\,\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

it turns out that it is also assured by the same condition on the norm of the perturbating term!

3.2.4. THEOREM. [7] *For any Riemannian metric a and 1-form β, if $\|\beta(x)\| < 1$ everywhere, then $F = \alpha + \beta$ is strongly convex.*

PROOF. The result is a consequence of the determinant formula

$$\det(g_{ij}) = \left(\frac{F}{\alpha}\right)^{n+1} \det(a_{ij})$$

between the fundamental tensor and the Riemannian part of a Randers metric (for a proof, see [7]). If F is positive, then both sides are positive for any nonzero tangent vector.

Now consider the metric family

$$F_\varepsilon := \sqrt{a_{ij}\, y^i\, y^j} + \varepsilon\, \beta_i\, y^i, \qquad 0 \le \varepsilon \le 1.$$

This 'connects' the Riemannian metric a and the Randers metric $F = \alpha + \beta$ in a sense, as for $\varepsilon = 0$, $F_0 = \alpha$ and for $\varepsilon = 1$, $F_1 = F$. Also, everything here depends smoothly (and continuously) on the parameter ε.

Using the hypothesis $\|\beta(x)\| < 1$ and the fact that $0 \le \varepsilon \le 1$, the norm of the perturbating term of any of the metrics F_ε is strictly less than 1, making them all positive (by the previous statement). Applying the determinant formula for them, we can see that the determinant of their fundamental tensor is always positive. This does not imply the positivity of the eigenvalues yet, but they certainly cannot be zero. We can also use the following facts.

- For $\varepsilon = 0$, when $F_0 = \alpha$, the fundamental tensor has the same eigenvalues as the Riemannian part, which are all positive, so it is positive definite.
- Since the metrics depend continuously on the parameter ε, then so do their fundamental tensors and their eigenvalues. But then, because we started from positive values, they can never switch to negative as ε changes, or else they would be zero somewhere in between, which can never happen.

In conclusion, all the metrics F_ε, in particular, $F_1 = F$ have a positive definite fundamental tensor. □

Finally, we give the proof promised for Theorem 1.3.8 in Chapter 1.

3.2.5. Statement. *There exists a non-Riemannian Randers metric on any manifold M.*

Proof. First, remember that (as we have proved back then) there always exists a Riemannian metric on any manifold. We only have to perturb it with a nonzero 1-form β over some 'small' set.

Define β such that it is nonzero only in the 'middle' of a bounded chart U and zero on its 'outer regions' and everywhere outside it. It can be done by setting one of the coordinate functions of β as a bump function whose value is 1 at some point and its support is a compact subset properly contained in the chart, and setting all other coordinate functions to 0. This way β extends smoothly to the whole manifold as the 0-form outside the chart.

We are ready, we have a Randers metric, if we can guarantee $\|\beta(x)\| < 1$ everywhere. Outside U there is no problem, but for it to hold also on the inside, we need an extra step. Consider $\|\beta(x)\|$ at all points of its support; by the compactness and continuity, it can be strictly bounded from above by some positive constant K. Introducing the 1-form

$$\widetilde{\beta} := \frac{1}{\sqrt{K}}\, \beta,$$

this will be an adequate choice for the perturbating term, since

$$\left\|\widetilde{\beta}\right\| = a^{ij} \widetilde{\beta}_i \widetilde{\beta}_j = a^{ij} \frac{\beta_i}{\sqrt{K}} \frac{\beta_j}{\sqrt{K}} = \frac{1}{K} a^{ij} \beta_i \beta_j = \frac{\|\beta\|}{K} < 1. \qquad \square$$

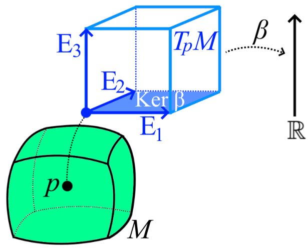
3.3. Adapted coordinates for Randers metrics

Doing calculations for Randers metrics, since they stand close to Riemannian metrics, is not that difficult and usually can be carried out even by hand. But, if we only want to work in one particular tangent space at some fixed point of the manifold (like we do when solving the CEQ), it is worth choosing coordinates in which both the Riemannian part and the perturbating term take a simple form.

3.3.1. Consider the Randers metric $F = \alpha + \beta$ with Riemannian part a and perturbating 1-form field β. To avoid the Riemannian case, β must be nonzero, meaning that it's not the identically zero 1-form/covector/real-valued linear functional on at least one tangent space, but it can be zero for some (or most) of them (where the original Riemannian metric is not ruined or blown away by the wind).

For us, this case is irrelevant: since we are only interested in (connected) generalized Berwald manifolds, by their rigidity property (see Statement 1.4.11), either they are Riemannian everywhere or not Riemannian anywhere. Thus, in the non-Riemannian case, if we hope to find compatible linear connections for a given Randers metric, it cannot be Riemannian anywhere, or equivalently, β cannot be the zero 1-form at any point of the manifold.

Take some fixed point $p \in M$ on the base manifold. The image of the covector β over T_pM is the 1-dimensional real line (since it's not the zero covector), so its kernel is a linear subspace of dimension $n-1$ inside T_pM. Let us choose an orthonormal basis $E_1, \ldots, E_{n-1}$ from this subspace. To obtain an orthonormal basis for the whole T_pM, choose E_n accordingly (we have 2 possibilities).



3.3.2. This orthonormal basis (like any other) uniquely determines a Riemannian normal neighborhood with normal coordinates around the point p, which we will refer to as **adapted normal coordinates**. In these, the metric has the following properties at p.

- For the components of the Riemannian metric a, all the partial derivatives $\widehat{\partial}_i a_{ij}$ vanish at p, and in addition, we have $a_{ij}(p) = a^{ij}(p) = \delta_{ij}$, so it is just the usual Euclidean metric on T_pM:
$$\alpha(x, y) = \sqrt{(y^1)^2 + \cdots + (y^n)^2}.$$
- The components of the perturbating term are $\beta_1(p) = \cdots = \beta_{n-1}(p) = 0$ and $\beta_n(p) \neq 0$. We also have $\|\beta\| = \beta_n$ because of $\|\beta\|^2 = a^{ij}\,\beta_i\,\beta_j = \delta^{ij}\,\beta_i\,\beta_j = (\beta_1)^2 + \cdots + (\beta_n)^2 = (\beta_n)^2$. Thus, in short, we can write

$$\beta_i = \delta_i^n \,\|\beta\|\,. \tag{30}$$

3.3.3. STATEMENT. *Given a Randers metric $F = \alpha + \beta$ on a manifold M and a point $p \in M$ with adapted coordinates around it, the level spheres $F = c$ in T_pM are described by*

$$\boxed{\sum_{i=1}^{n-1}(y^i)^2 + \lambda\left(y^n + \frac{c\,\|\beta\|}{\lambda}\right)^2 = \frac{c^2}{\lambda},} \tag{31}$$

where $\lambda = 1 - \|\beta\|^2$.

PROOF. In adapted coordinates, $F(x, y) = c$ becomes

$$\begin{aligned}
\sqrt{(y^1)^2 + \cdots + (y^n)^2} + \delta_i^n \|\beta\|\; y^i &= c \\
\sqrt{(y^1)^2 + \cdots + (y^n)^2} &= c - \|\beta\|\, y^n \\
(y^1)^2 + \cdots + (y^n)^2 &= c^2 - 2c\,\|\beta\|\, y^n + \|\beta\|^2 (y^n)^2 \\
\sum_{i=1}^{n-1}(y^i)^2 + \left(1 - \|\beta\|^2\right)(y^n)^2 + 2c\,\|\beta\|\, y^n &= c^2.
\end{aligned}$$

On the LHS, the last two terms give

$$\begin{aligned}
\lambda(y^n)^2 + 2c\,\|\beta\|\, y^n &= \lambda\left[(y^n)^2 + \frac{2c\,\|\beta\|\, y^n}{\lambda}\right] \\
&= \lambda\left[\left(y^n + \frac{c\,\|\beta\|}{\lambda}\right)^2 - \frac{c^2\,\|\beta\|^2}{\lambda^2}\right].
\end{aligned}$$

By rearranging, we get the form given above. □

3.3.4. REMARK. In adapted coordinates, the Randers level sphere $F = c$ is an n-dimensional ellipsoid with center $[0, \ldots, 0, -c\,\|\beta\|/\lambda]$ having the coordinate axes as its own axes with half-axis lengths $(1, \ldots, 1, 1/\sqrt{\lambda})$. This is important for us for 2 reasons:

- We see that the indicatrix is rotationally symmetric w.r.t. any plane generated by 2 coordinate axes chosen from $y^1, \ldots, y^{n-1}$ (because the equation is symmetric in the norm on these 2 coordinates). We will use this fact in Chapter 5, when we return to Randers metrics.
- Since in these coordinates a is the Euclidean metric and its indicatrix is the Euclidean unit sphere, we see that the Randers indicatrix is *not* just a translate of this (which is a common rookie misconception), it's also stretched along the n-th coordinate axis.

3.4. Compatibility for Randers metrics

In the remaining part of the chapter, we try to answer the questions posed in Section 1.4.3 in the special case of Randers metrics, concerning the existence of compatible linear connections. Before starting, we summarize some basic results.

First, we need a compatible Riemannian metric to the given Randers metric $F = \alpha + \beta$. We know that the so-called averaged metric is always a good choice (see Section 1.5), but in case of a Randers metric, where a Riemannian metric is given ab ovo, one wonders whether we could also use that one. It turns out that the answer is yes.

3.4.1. Statement. [**17**] *If $F = \alpha + \beta$ is a Randers metric and ∇ is a linear connection compatible to it, then both the Riemannian and the perturbating parts are parallel/covariant constant w.r.t. ∇, i.e. $\nabla a = 0$ and $\nabla\beta = 0$.*

Proof. We have to show that any parallel translation φ w.r.t. ∇ preserves both the Riemannian and the perturbating parts, if it preserves F itself. Compatibility of ∇ to F means that for any $v \in TM$, we have

$$F(\varphi(v)) = F(v)$$
$$\alpha\left(\varphi(v)\right) + \beta(\varphi(v)) = \alpha(v) + \beta(v).$$

Repeating this for the vector $-v$ yields

$$\alpha\left(\varphi(-v)\right) + \beta(\varphi(-v)) = \alpha(-v) + \beta(-v)$$
$$\alpha\left(\varphi(v)\right) - \beta(\varphi(v)) = \alpha(v) - \beta(v).$$

Adding the equations from above, we get

$$\alpha\left(\varphi(v)\right) = \alpha(v),$$

which, by the polarization formula, is equivalent to

$$a(\varphi(v), \varphi(w)) = a(v, w).$$

Subtracting instead of adding gives

$$\beta(\varphi(v)) = \beta(v).$$

With this, we have proved both claims of the Statement. □

3.4.2. Corollary. [**17**] *For any Randers metric $F = \alpha + \beta$, its Riemannian part a is always compatible to it.*

3.4.3. Remark. This means that we hit two birds with one stone: the adapted coordinates making the form of the Randers metric simpler can also be used as the Riemannian normal coordinates described in 2.2.7, so we don't have to choose which side of the CEQ to simplify.

In [**17**], the following two criteria are given for Randers manifolds being generalized Berwald.

3.4.4. Corollary. [17] *A Randers metric $F = \alpha + \beta$ is a GBM $\iff$ there exists a linear connection ∇ on the manifold such that $\nabla a = 0$ and $\nabla \beta = 0$.*

Proof. The '$\Rightarrow$' part is proved above; the other direction is immediate (if both parts are preserved, then so is the metric itself). $\square$

3.4.5. Theorem. [17] *A **connected** Randers manifold is a GBM $\iff$ $\|\beta\|$ is constant.*

Proof. [17] $\boxed{\Rightarrow}$ If the manifold is a GBM, let us choose a compatible ∇. By the above statement, it obviously preserves the Riemannian length of the perturbating term, i.e. $\|\beta\|$ is constant.

$\boxed{\Leftarrow}$ Suppose that β, and, consequently, its dual vector field $\beta^\sharp$ has constant length and let ∇^* denote the Levi-Civita connection of the Riemannian part $a = \langle ., . \rangle$. We will show that the connection

$$\nabla_X Y = \nabla^*_X Y + \frac{\left\langle \nabla^*_X \beta^\sharp, Y \right\rangle \beta^\sharp - \left\langle Y, \beta^\sharp \right\rangle \nabla^*_X \beta^\sharp}{\|\beta^\sharp\|^2}$$

preserves both a and β, and thus the Randers metric itself.

- To check that ∇ is metrical to a, let's compute

$$\begin{aligned}
&\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\
&= \langle \nabla^*_X Y, Z \rangle + \frac{\left\langle \nabla^*_X \beta^\sharp, Y \right\rangle \left\langle \beta^\sharp, Z \right\rangle - \left\langle Y, \beta^\sharp \right\rangle \left\langle \nabla^*_X \beta^\sharp, Z \right\rangle}{\|\beta^\sharp\|^2} \\
&\quad + \langle Y, \nabla^*_X Z \rangle + \frac{\left\langle \nabla^*_X \beta^\sharp, Z \right\rangle \left\langle Y, \beta^\sharp \right\rangle - \left\langle Z, \beta^\sharp \right\rangle \left\langle Y, \nabla^*_X \beta^\sharp \right\rangle}{\|\beta^\sharp\|^2} \\
&= \langle \nabla^*_X Y, Z \rangle + \langle Y, \nabla^*_X Z \rangle = X \langle Y, Z \rangle ,
\end{aligned}$$

  which is exactly what we wanted to prove.
- For the 1-form β, or its dual vector field $\beta^\sharp$, first note that having constant length means that

$$\frac{1}{2} X \|\beta^\sharp\|^2 = \frac{1}{2} X \left\langle \beta^\sharp, \beta^\sharp \right\rangle = \left\langle \nabla_X \beta^\sharp, \beta^\sharp \right\rangle = 0.$$

  Using this, we have

$$\begin{aligned}
\nabla_X \beta^\sharp &= \nabla^*_X \beta^\sharp + \frac{\left\langle \nabla^*_X \beta^\sharp, \beta^\sharp \right\rangle \beta^\sharp - \left\langle \beta^\sharp, \beta^\sharp \right\rangle \nabla^*_X \beta^\sharp}{\|\beta^\sharp\|^2} \\
&= \nabla^*_X \beta^\sharp + 0 - \nabla^*_X \beta^\sharp = 0.
\end{aligned}$$

This shows that the ∇ defined above is indeed compatible to the Randers metric. $\square$

3.5. The compatibility equations for Randers metrics

3.5.1. With all the basic knowledge about Randers metrics at our disposal, we can finally write and solve the compatibility equations for them.[3] We work inside one tangent space T_pM only and use adapted normal coordinates around the point p. According to (20), the general form of the i-th equation of the CEQ at p is

$$\text{(CEQ-}i\text{)} \qquad \sum_a{}' \, 2f_{ia}\boldsymbol{T}^i_{ia} + \sum_{a<b}{}' \, f_{ab}\left(\boldsymbol{T}^a_{ib} + \boldsymbol{T}^i_{ab} + \boldsymbol{T}^b_{ai}\right) = -2\,\widehat{\partial}_i F,$$

where primed summations mean summing for different indices (in the first one, $a \neq i$, and in the second one, a and b where $i \notin \{a, b\}$) and $\boldsymbol{T}^c_{ab}$ are the torsion components of the compatible linear connections we are hunting for.

3.5.2. For the LHS, we can do the following computations.

- First, in the adapted normal coordinates, the Randers metric at the point p takes the form

$$\begin{aligned} F(x, y) &= \sqrt{a_{ij}\, y^i\, y^j} + \beta_i(x)\, y^i = \sqrt{\delta_{ij}\, y^i\, y^j} + \beta_n(x)\, y^n \\ &= \sqrt{(y^1)^2 + \cdots + (y^n)^2} + \|\beta(x)\|\, y^n. \end{aligned}$$

- The partial derivatives of F w.r.t. the directional coordinates are

$$\dot{\partial}_k F(x, y) = \frac{y^k}{\sqrt{\sum_{i=1}^n (y^i)^2}} + \delta^n_k\, \|\beta(x)\|\,.$$

- The coefficients of the torsion components on the LHS are

$$\begin{aligned} f_{ab} &= y^a\, \dot{\partial}_b F - y^b\, \dot{\partial}_a F \\ &= y^a\left(\frac{y^b}{\sqrt{\sum_{i=1}^n (y^i)^2}} + \delta^n_b\, \|\beta\|\right) - y^b\left(\frac{y^a}{\sqrt{\sum_{i=1}^n (y^i)^2}} + \delta^n_a\, \|\beta\|\right) \\ &= \left(y^a\delta^n_b - y^b\delta^n_a\right)\|\beta\|\,. \end{aligned}$$

  From this, we see that in order for f_{ab} to be different from zero, one of its indices must be equal to n (and exactly one, because of the antisymmetry). The two cases are:

$$f_{nb} = -y^b\,\|\beta\| \quad (b \neq n) \qquad \text{and} \qquad f_{an} = y^a\,\|\beta\| \quad (a \neq n).$$

  For later use, we introduce the notations

$$(32) \qquad \widetilde{f}_{nb} = -y^b \quad (b \neq n) \qquad \text{and} \qquad \widetilde{f}_{an} = y^a \quad (a \neq n).$$

[3] In the remainder of the chapter, we present the results of [1].

3.5.3. Let's look at the RHS.

- First note that, since we are working in Riemannian normal coordinates, the partial derivatives of the a_{ij} at the point p w.r.t. the positional coordinates are all zero, so, consequently,
$$\widehat{\partial}_i \alpha = 0.$$
- The partial derivatives of F w.r.t. the positional coordinates are thus
$$\widehat{\partial}_i F = \widehat{\partial}_i \alpha + \widehat{\partial}_i \beta = \widehat{\partial}_i \beta = \widehat{\partial}_i (\beta_j y^j) = y^j \, \widehat{\partial}_i \beta_j.$$
- Introducing the notation
$$\boxed{C_{j;i}(x) := -\frac{\widehat{\partial}_i \beta_j}{\|\beta\|}(x),} \tag{33}$$
we can rewrite the RHS of (CEQ-i) as
$$2 \|\beta\| \sum_{j=1}^{n} y^j \, C_{j;i}.$$

All in all, we can write the equations of the CEQ the following way.

3.5.4. THEOREM. [1] *With the above notations and restrictions, for a Randers metric $F = \alpha + \beta$, the CEQ takes the form*

$$\boxed{\begin{array}{lll} (\textit{CEQ-R-i}) & 2y^i \boldsymbol{T}^i_{in} + \displaystyle\sum_{j \notin \{i,n\}} y^j \left(\boldsymbol{T}^j_{in} + \boldsymbol{T}^i_{jn} + \boldsymbol{T}^n_{ji} \right) = 2 \sum_{j=1}^{n} y^j \, C_{j;i} & (i \neq n); \\ (\textit{CEQ-R-n}) & \displaystyle\sum_{a=1}^{n-1} y^a \boldsymbol{T}^n_{an} = \sum_{j=1}^{n} y^j \, C_{j;n}. & \end{array}}$$

PROOF. If $i < n$, the only nonzero coefficient for the repetitive components is $2f_{in} = 2y^i \|\beta\|$, and for the diverse components they are $f_{an} = y^a \|\beta\|$ with $a \notin \{i, n\}$. An index change $a \sim j$ and division by $\|\beta\|$ yields the above form.

For $i = n$, since n never appears among the indices of the coefficients for the diverse components, they are all absent from this equation. On the other hand, all coefficients for the repetitive torsion components survive, because $2f_{na} = -2y^a \|\beta\|$. Changing the lower indices of the components to the natural order eliminates the minus sign. We can divide by $2\|\beta\|$ to achieve the above form. $\square$

The summation symbols on the right hand sides are included to make the comparison of the two sides easier. Let's solve these equations!

3.6. The solution and solvability of the CEQ

3.6.1. Note that (CEQ-R) is linear in the directional coordinate functions $y^1, \dots, y^n$, thus, to examine the solvability and determine the solution, we just have to compare the corresponding coefficients on the two sides. Also, we can see that y^n is always absent from the LHS of the equations. Basically, these two observations are enough to give a necessary and sufficient condition for the solvability and determine which torsion components are uniquely determined or freely chosen.

3.6.1. Expressing the repetitive torsion components.

We first deal with the repetitive torsion components whose upper index equals one of the lower indices. If none of their indices are equal to n, then they are absent from (CEQ-R) and freely chosen. Otherwise, each of them appears in the CEQ exactly once, and a quick glance at (CEQ-R) suggests they might be uniquely expressible.

3.6.2. Statement (Solution of (CEQ-R-n)). *(CEQ-R-n) is solvable $\iff$ $C_{n;n} = 0$. In this case, the repetitive torsion components with the index n repeated are uniquely determined:*

$$\boldsymbol{T}^n_{an}(p) = C_{a;n}(x) \overset{(33)}{=} -\frac{\widehat{\partial}_n \beta_a}{\|\beta\|}(x), \qquad a \in \{1, \dots, n-1\}. \tag{34}$$

Proof. First note that these components only appear in (CEQ-R-n). By 3.6.1, solvability of this equation is equivalent to all coordinate functions $y^1, \dots, y^n$ having the same coefficients on both sides. Since y^n is missing from the LHS, its coefficient on the other side, namely $C_{n;n}$, must also be zero. Comparing the coefficients for $y^1, \dots, y^{n-1}$, we get the only possible value for the components $\boldsymbol{T}^n_{an}$. $\square$

For the remaining repetitive components, we have the following result.

3.6.3. Statement (Solvability of the (CEQ-R-i), $i \neq n$). *For any $i \in \{1, \dots, n-1\}$, (CEQ-R-i) is solvable $\iff$ $C_{n;i} = 0$. In this case, the repetitive torsion components with an index $a \neq n$ repeated are uniquely determined:*

$$\boldsymbol{T}^a_{an}(p) = C_{a;a}(x) \overset{(33)}{=} -\frac{\widehat{\partial}_a \beta_a}{\|\beta\|}(x), \qquad a \in \{1, \dots, n-1\}. \tag{35}$$

Proof. First note that $\boldsymbol{T}^a_{an} = \boldsymbol{T}^i_{in}$ is only present in (CEQ-R-i), and its coefficient y^i does not appear with any other torsion components in this equation. By comparing the coefficients for y^n, we get the solvability condition, and for y^i, we obtain the formula for $\boldsymbol{T}^i_{in}$. $\square$

3.6.2. Expressing the diverse torsion components.

3.6.4. In case of Randers metrics, we can classify diverse torsion components (having three different indices) into 2 categories:

- diverse components not containing the index n never appear in the CEQ, so they can be freely chosen;
- the ones having an index equal to n appear in the CEQ in groups of 3, where the indices are the same values permuted. Let's take a closer look at these triplets.

3.6.5. LEMMA. *For any indices $a < b < n$, the components of the triplet $\boldsymbol{T}^n_{ab}, \boldsymbol{T}^b_{an}, \boldsymbol{T}^a_{bn}$ appear only in the equations (CEQ-R-a) and (CEQ-R-b), and*

$$\begin{aligned} \dot{\partial}_a(\textit{CEQ-R-b}) &\Longrightarrow \quad \boldsymbol{T}^n_{ab} + \boldsymbol{T}^a_{bn} + \boldsymbol{T}^b_{an} = 2C_{a;b} \\ \dot{\partial}_b(\textit{CEQ-R-a}) &\Longrightarrow -\boldsymbol{T}^n_{ab} + \boldsymbol{T}^a_{bn} + \boldsymbol{T}^b_{an} = 2C_{b;a}. \end{aligned} \tag{36}$$

PROOF. In general (see Section 2.3), triplets of diverse components with indices $a < b < n$ can only appear in the equations (CEQ-a), (CEQ-b) and (CEQ-n). Now, in the Randers case, we have the following.

- They are absent from the last equation (CEQ-R-n).
- In (CEQ-R-a), they appear as $\widetilde{f}_{bn}(-\boldsymbol{T}^n_{ab} + \boldsymbol{T}^a_{bn} + \boldsymbol{T}^b_{an}) \overset{(32)}{=} y^b(-\boldsymbol{T}^n_{ab} + \boldsymbol{T}^a_{bn} + \boldsymbol{T}^b_{an})$ (check it with the mnemonic!), and they are the only components with this coefficient.
- In (CEQ-R-b), they appear as $\widetilde{f}_{an}(\boldsymbol{T}^n_{ab} + \boldsymbol{T}^a_{bn} + \boldsymbol{T}^b_{an}) \overset{(32)}{=} y^a(\boldsymbol{T}^n_{ab} + \boldsymbol{T}^a_{bn} + \boldsymbol{T}^b_{an})$ (check it with the mnemonic!), and they are the only components with this coefficient.

By comparing the coefficients of y^b in (CEQ-R-a) and y^a in (CEQ-R-b) (which is the same as differentiating), we get the result stated above. □

Thus we can classify diverse torsion components as follows.

3.6.6. STATEMENT (The classification of diverse torsion components).

- *Diverse components with indices all different from n are freely chosen.*
- *The two diverse components having indices $a < b < n$, with n as a lower index, satisfy the equation*

$$\boldsymbol{T}^b_{an}(p) + \boldsymbol{T}^a_{bn}(p) = C_{a;b}(x) + C_{b;a}(x) \overset{(33)}{=} -\frac{\widehat{\partial}_a\beta_b + \widehat{\partial}_b\beta_a}{\|\beta\|}(x). \tag{37}$$

- *Diverse components with n as an upper index are uniquely determined:*

$$\boldsymbol{T}^n_{ab}(p) = C_{a;b}(x) - C_{b;a}(x) \overset{(33)}{=} \frac{\widehat{\partial}_a\beta_b - \widehat{\partial}_b\beta_a}{\|\beta\|}(x). \tag{38}$$

PROOF. We have already seen that the first type of diverse components do not appear in the equations of the CEQ. For the triplets with indices $a < b < n$, we have the two equations (36) from the previous lemma. Adding them yields formula (37), and substituting this back to one of the equations of (36), we immediately obtain (38). □

In summary, we have proved the following.

3.6.7. Theorem (The solution of the CEQ for Randers spaces, [1]). *Suppose that a non-Riemannian Randers manifold (M, F) is a GBM, fix a point $p \in M$ and use adapted coordinates around it. Then at this point, the torsion components $\boldsymbol{T}^c_{ab}(p)$ of compatible linear connections are given by the following table.*

type	*indices*	*formula*	*no. of comp.'s*
any	$n \notin \{a, b, c\}$	$\boldsymbol{T}^c_{ab}$ *is freely chosen*	$(n-1)\binom{n-1}{2}$
repetitive	$a \neq n$	$\boldsymbol{T}^n_{an} = C_{a;n}$	$n-1$
	$a \neq n$	$\boldsymbol{T}^a_{an} = C_{a;a}$	$n-1$
diverse	$a < b < n$	$\boldsymbol{T}^b_{an} + \boldsymbol{T}^a_{bn} = C_{a;b} + C_{b;a}$	$2\binom{n-1}{2}$
	$a < b < n$	$\boldsymbol{T}^n_{ab} = C_{a;b} - C_{b;a}$	$\binom{n-1}{2}$

Here, the quantities $C_{j;i}$ are defined by (33). From these, we can retrieve the connections by (19).

3.6.8. Corollary. *We can see that out of the total number of $n\binom{n}{2}$ torsion components, we have quite a large number of free choices from the first and fourth cases (where one of each pair can be freely chosen), together*

$$(n-1)\binom{n-1}{2} + \binom{n-1}{2} = n\binom{n-1}{2}$$

possibilities.[4] *In other words, we can freely choose the*

$$n\binom{n-1}{2} \Big/ n\binom{n}{2} = \frac{(n-1)(n-2)}{2} \cdot \frac{2}{n(n-1)} = \frac{n-2}{n}$$

portion of all torsion components (which tends to 1 as $n \to \infty$).

3.6.9. Example. For low dimensions, the values are the following.

dim.	no of comp.'s	no. of free comp.'s	ratio of free comp.'s
2D	2	0	0
3D	9	3	$\frac{1}{3} = 33.\dot{3}\%$
4D	24	12	$\frac{1}{2} = 50\%$
5D	50	30	$\frac{3}{5} = 60\%$

[4]The formula works for 2D as well, interpreting the binomial coefficient $\binom{1}{2}$ as 0.

In particular, we have the following torsion components.

- In 2D, $\boldsymbol{T}^1_{12} = C_{1;1}$ and $\boldsymbol{T}^2_{12} = C_{1;2}$ (so if there is a solution, it is unique).[5]
- In 3D,
  - $\boldsymbol{T}^1_{12}$ and $\boldsymbol{T}^2_{12}$ are freely chosen,
  - $\boldsymbol{T}^3_{13} = C_{1;3}, \;\; \boldsymbol{T}^3_{23} = C_{2;3}$,
  - $\boldsymbol{T}^1_{13} = C_{1;1}, \;\; \boldsymbol{T}^2_{23} = C_{2;2}$,
  - $\boldsymbol{T}^2_{13} + \boldsymbol{T}^1_{23} = C_{1;2} + C_{2;1}$ (one of them is freely chosen),
  - $\boldsymbol{T}^3_{12} = C_{1;2} - C_{2;1}$.
- In 4D,
  - $\boldsymbol{T}^1_{12}, \boldsymbol{T}^2_{12}, \boldsymbol{T}^3_{12}, \boldsymbol{T}^1_{13}, \boldsymbol{T}^2_{13}, \boldsymbol{T}^3_{13}, \boldsymbol{T}^1_{23}, \boldsymbol{T}^2_{23}, \boldsymbol{T}^3_{23}$ are freely chosen,
  - $\boldsymbol{T}^4_{14} = C_{1;4}, \;\; \boldsymbol{T}^4_{24} = C_{2;4}, \;\; \boldsymbol{T}^4_{34} = C_{3;4}$,
  - $\boldsymbol{T}^1_{14} = C_{1;1}, \;\; \boldsymbol{T}^2_{24} = C_{2;2}, \;\; \boldsymbol{T}^3_{34} = C_{3;4}$
  - $\boldsymbol{T}^2_{14} + \boldsymbol{T}^1_{24} = C_{1;2} + C_{2;1}, \;\; \boldsymbol{T}^3_{14} + \boldsymbol{T}^1_{34} = C_{1;3} + C_{3;1}, \;\; \boldsymbol{T}^3_{24} + \boldsymbol{T}^2_{34} = C_{2;3} + C_{3;2}$ (one of each pair is freely chosen),
  - $\boldsymbol{T}^4_{12} = C_{1;2} - C_{2;1}, \;\; \boldsymbol{T}^4_{13} = C_{1;3} - C_{3;1}, \;\; \boldsymbol{T}^4_{23} = C_{2;3} - C_{3;2}$.

3.7. The solvability of the CEQ

We have already investigated the solvability of the CEQ in the Randers case and formulated necessary and sufficient conditions. Using these, we can give an alternative proof for the result on the norm of the perturbating term.

3.7.1. THEOREM. [17] *A* ***connected*** *Randers manifold is a GBM* $\Longleftrightarrow$ *the norm* $\|\beta\|$ *of the perturbating term is constant.*

PROOF. [1] The norm $\|\beta\|$ is constant if and only if

$$\widehat{\partial}_i \, \|\beta\|^2 = \frac{\partial_i \, a^{jk} \, \beta_j \, \beta_k}{\partial x^i} = \frac{\partial_i a^{jk}}{\partial x^i} \, \beta_j \, \beta_k + a^{jk} \, \frac{\partial_i \beta_j}{\partial x^i} \, \beta_k + a^{jk} \, \beta_j \, \frac{\partial_i \beta_k}{\partial x^i} = 0$$

for any $i \in \{1, \dots, n\}$ at any point of the manifold. Fixing a point p and using adapted coordinates where the partial derivatives of a^{jk} vanish, $a^{jk} = \delta^{jk}$ and $\beta_i = \delta_i^n \beta_n = \delta_i^n \, \|\beta\|$, the above become

$$\widehat{\partial}_i \, \|\beta\|^2 = 0 + \delta^{jk} \, \frac{\partial_i \beta_j}{\partial x^i} \, \delta_k^n \, \|\beta\| + \delta^{jk} \, \delta_j^n \, \|\beta\| \, \frac{\partial_i \beta_k}{\partial x^i} = 2 \, \|\beta\| \, \frac{\partial_i \beta_n}{\partial x^i} = 2 \, \|\beta\| \, \widehat{\partial}_i \beta_n = 0.$$

Thus, at p, $\|\beta\|$ being constant is equivalent to $\widehat{\partial}_i \beta_n = 0$ for all $i \in \{1, \dots, n\}$.

On the other hand, in Statement 3.6.2 and 3.6.3 we verified that the solvability of (CEQ-R) is equivalent to

$$C_{n;i} \overset{(33)}{=} -\frac{\widehat{\partial}_i \beta_n}{\|\beta\|} = 0,$$

or $\widehat{\partial}_i \beta_n = 0$, for all $i \in \{1, \dots, n\}$. □

3.7.2. REMARK. In case of non-Riemannian Randers GBMs, where β is nonzero everywhere, for the constant norm we actually have $0 < \|\beta\| < 1$.

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[5] In the next chapter we will show that this is true for any Finsler metric in 2D.

3.8. The extremal connection on Randers spaces

The only thing left to do is to find the extremal compatible linear connection, defined in Section 2.7.2, whose torsion has minimal norm among all the other compatible linear connections. Since we are working with orthonormal bases, this norm is just the squared sum of the components under the square root (see (28)).

3.8.1. THEOREM (The extremal connection of a Randers space, [1]). *Suppose that a non-Riemannian Randers manifold (M, F) is a GBM, fix a point $p \in M$ and use adapted coordinates around it. Then at this point, the torsion components $T^c_{ab}(p)$ of the extremal compatible linear connection are given by the following table.*

type	*indices*	*formula*	*no. of comp.'s*
any	$n \notin \{a, b, c\}$	$\boldsymbol{T}^c_{ab} = 0$	$(n-1)\binom{n-1}{2}$
repetitive	$a \neq n$	$\boldsymbol{T}^n_{an} = C_{a;n}$	$n-1$
	$a \neq n$	$\boldsymbol{T}^a_{an} = C_{a;a}$	$n-1$
diverse	$a < b < n$	$\boldsymbol{T}^b_{an} = \boldsymbol{T}^a_{bn} = \dfrac{C_{a;b} + C_{b;a}}{2}$	$2\binom{n-1}{2}$
	$a < b < n$	$\boldsymbol{T}^n_{ab} = C_{a;b} - C_{b;a}$	$\binom{n-1}{2}$

Here, the quantities $C_{j;i}$ are defined by (33). From these, we can retrieve the connection by (19).

PROOF. The uniquely determined torsion components are given by Theorem 3.6.7 and we have to set the ones freely chosen such that their squared sum is minimal.

- In the first case, this means setting every component to zero.
- In the fourth case, where in general

$$\boldsymbol{T}^b_{an} + \boldsymbol{T}^a_{bn} = C_{a;b} + C_{b;a} (= \text{constant}),$$

we use that the minimum problem $x^2 + (c - x)^2 \to \min$ is solved by $x = y = c/2$. □

3.8.2. REMARK. The extremal connection can always be introduced on a Randers space by the formulas of the theorem. However, it is not compatible to the Randers metric in general. The necessary and sufficient condition for such a linear connection to be compatible to the metric is that $\|\beta\|$ be constant on all components of the manifold.

CHAPTER 4

2-dimensional generalized Berwald manifolds

In this chapter we study generalized Berwald manifolds of dimension 2. This is the most simple case possible and it can (and has been) dealt with using different tools, depending on what aspects of the manifold we wish to use. We show three different paths:

- one that is based on our work with the compatibility equations,
- one that uses classical tools of Finsler geometry,
- and one that is a special case of the work of Vincze on compatible linear connections with a so-called semi-symmetric torsion tensor.

But all roads lead to the same result: a 2-dimensional non-Riemannian Finsler manifold (Finsler surface) is either not generalized Berwald, or if it is, then its compatible linear connection is uniquely determined and can be expressed from the canonical data of the metric (in various ways).

In the second half of the chapter, using the divergence representation of the Gauss curvature, we investigate what types of surfaces can admit a generalized Berwald structure. We give an easy way of constructing such a metric compatible to a given Riemannian one and as an illustration, we re-metrize some well-known Riemannian manifolds.[1]

4.1. Solving the compatibility equations in dimension 2

4.1.1. In Chapter 2 we discussed how to search for the $\boldsymbol{\nabla}$s compatible to a Finsler metric F in terms of their torsion components $\boldsymbol{T}^c_{ab}$ (and how to reconstruct $\boldsymbol{\nabla}$ from them). In 2 dimensions there are only two of these components, and by choosing a compatible Riemannian metric γ to F, fixing a point p and choosing normal coordinates at p, the CEQ becomes (see (21))

$$\text{(CEQ-2D)} \qquad \left.\begin{array}{rcl} f_{12}\boldsymbol{T}^1_{12} &=& -\widehat{\partial}_1 F \\ f_{12}\boldsymbol{T}^2_{12} &=& -\widehat{\partial}_2 F \end{array}\right\} \iff \boxed{\left.\begin{array}{rcl} f_{21}\boldsymbol{T}^1_{12} &=& \widehat{\partial}_1 F \\ f_{21}\boldsymbol{T}^2_{12} &=& \widehat{\partial}_2 F \end{array}\right\}}$$

where

- $f_{21} = y^2\dot{\partial}_1 F - y^1\dot{\partial}_2 F = y^2\dfrac{\partial F}{\partial y^1} - y^1\dfrac{\partial F}{\partial y^2}$,
- $\widehat{\partial}_i F(v) = \dfrac{\partial F}{\partial x^i}(v)$ are the horizontal partial derivatives of F
- and we consider these at all points $v \in T^\circ_p M$ (or just one of the indicatrices R_p or F_p).

[1] In this chapter, we present the results of [2], with some facts used from [4] and [5].

This is what we need to solve for the torsion components – and that is easy. Observe that the solution (and solvability) depends mainly on the coefficient function f_{21}.

4.1.2. THEOREM (The CEQ in 2D locally). *Let* (M, F) *be a* ***Finsler surface*** *and consider the CEQ in the form of* (CEQ-2D) ***at a fixed point p****. There are two possible cases:*

(a) If f_{21} is identically zero, then F is quadratic at p. In this case,
- *if T_pM is not horizontal contact, then the CEQ has no solution at p;*
- *if T_pM is horizontal contact, then the CEQ is arbitrarily solvable at p.*

(b) If f_{21} is ***not*** *identically zero, then F is* ***not*** *quadratic at p. In this case,*
- *either the CEQ has no solution at p*
- *or it has a unique solution at p.*

PROOF. (a) Since we only have one coefficient now, by Lemma 2.6.4, $f_{21}(v) = 0 \iff v$ is vertically contact. Thus $f_{21} \equiv 0$ is equivalent to the whole T_pM being vertically contact. In this case, by Statement 2.5.12, F is quadratic at p. Since the left hand side of the CEQ is identically zero, there are solutions if and only if this is also true for the right hand side, and in this case, all equations are vacant (0=0).

(b) If $f_{21}(v) \neq 0$ at some (not vertically contact) $v \in T_p^\circ M$, then we can divide both equations to express the only possible solution:

$$\boxed{\boldsymbol{T}^1_{12} = \frac{\widehat{\partial}_1 F(v)}{f_{21}(v)} \quad \text{and} \quad \boldsymbol{T}^2_{12} = \frac{\widehat{\partial}_2 F(v)}{f_{21}(v)}.} \tag{39}$$

This is indeed a solution if the expressions on the RHS are independent of the (not vertical contact) v chosen and all vertical contact points are horizontal contact. □

4.1.3. REMARK. Observe that in part (a), F being Riemannian at one point does not imply it being Riemannian everywhere (and the right hand sides being identically 0); think about a Randers metric obtained by perturbing a Riemannian metric over some small region. But, the case of Riemannian metrics is included in part (a), when the pointwise solution is, of course, arbitrary.

4.1.4. THEOREM (The CEQ in 2D globally). *For a* ***connected non-Riemannian Finsler surface*** (M, F)*, there are two possibilities:*

(a) either there exists no compatible linear connection on it (it is not a GBM)

(b) or it is a GBM with a uniquely determined compatible linear connection.

PROOF. By the rigidity property in Statement 1.4.11, a non-Riemannian connected GBM is not Riemannian (quadratic) anywhere. So, if there is a solution of the CEQ (a compatible ∇), by the local version of the theorem it must be unique at any point. □

4.2. Using the classical tools of Finsler geometry

It is possible to describe the compatible linear connections of 2-dimensional GBMs using the classical tools of Finsler geometry, used already by the father of the theory of GBMs, V. Wagner (see [**21**]). These are the following.[2]

4.2.1. Let (M, F) be a Finsler manifold with energy function E, Riemann–Finsler metric g_{ij} and its inverse g^{ij} (see Section 1.3.1).

- The **Cartan tensor**
$$C_{ijk} := \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$$
measures how 'non-Riemannian' the metric is (F is Riemannian $\Longleftrightarrow$ $C_{ijk} \equiv 0$).
- The **geodesic spray coefficients** are defined by
$$G^l := \frac{1}{2}g^{lm}\left(y^k \frac{\partial^2 E}{\partial y^m \partial x^k} - \frac{\partial E}{\partial x^m}\right)$$
and we write
$$G^l_i := \frac{\partial G^l}{\partial y^i}, \qquad G^l_{ij} := \frac{\partial G^l_i}{\partial y^j} \qquad \text{and} \qquad G^l_{ijk} := \frac{\partial G^l_{ij}}{\partial y^k}.$$
G^l_{ijk} is called the **mixed curvature**.
- Define the vector fields
$$V := \frac{\partial F}{\partial y^1}\frac{\partial}{\partial y^2} - \frac{\partial F}{\partial y^2}\frac{\partial}{\partial y^1} \qquad \text{and} \qquad V_0 := \frac{1}{\sqrt{g(V, V)}}V.$$
Since $V_0 F = VF = 0$, i.e. F is constant along the integral curves of V_0, these integral curves are the level curves of F, with the Finslerian indicatrix among them. If we parameterize it this way, i.e.
$$V_0 \circ c_p(\theta) = c_p'(\theta),$$
then c_p is called the **central affine arcwise parametrization** of the indicatrix curve and the parameter θ is the **central affine arclength** of the indicatrix.
- The **main scalar** of the Finsler surface is defined by
$$\lambda := V_0^j V_0^k V_0^l C_{jkl}.$$
The vanishing of λ characterizes when the surface is Riemannian. Evaluating λ along the indicatrix gives
$$\lambda \circ c_p(\theta) = \left(\ln \sqrt{\det g_{ij}} \circ c_p\right)'(\theta).$$
It can be interpreted as the 'central affine curvature'.
- We also have the so-called **Berwald frame**, a local frame on $T^\circ M$, but we don't go into the details here; for this and all the above, see [**5**].

[2] We state here the main result of [**5**], but since discussing it in detail would be a huge detour from the path we are taking in this book, we just list the tools used.

After a huge amount of calculations, we can obtain the following result describing compatible $\boldsymbol{\nabla}$s.

4.2.2. Theorem. [5] *Let us define*

$$\alpha_i := \frac{1}{2} F V_0^j V_0^k G^l_{ijk} \frac{\partial F}{\partial y^l} \qquad \text{and} \qquad \omega_i := V_0^j V_0^k G^l_{ijk} g\left(V_0, \frac{\partial}{\partial y^l}\right).$$

Then the Christoffel symbols of compatible linear connections of a ***connected non-Riemannian generalized Berwald surface*** *are*

$$\boxed{\begin{aligned} \Gamma^1_{ij} \circ \pi &= G^1_{ij} - \frac{\partial f_i}{\partial y^j}\frac{\partial F}{\partial y^2} - f_i \frac{\partial^2 F}{\partial y^j \partial y^2}, \\ \Gamma^2_{ij} \circ \pi &= G^2_{ij} + \frac{\partial f_i}{\partial y^j}\frac{\partial F}{\partial y^1} + f_i \frac{\partial^2 F}{\partial y^j \partial y^1} \end{aligned} \quad (i, j = 1, 2).} \tag{40}$$

Here, the 1-homogeneous functions f_1, f_2 are uniquely determined by

$$f_i \circ c_p(t) = \frac{1}{\sqrt{g(V,V) \circ c_p(t)}} \left(\int_0^t \alpha_i \circ c_p(\theta)\, \mathrm{d}\theta + k_i(p) \right) \quad (i = 1, 2)$$

and the integration constants $k_i(p)$ (depending only on the position) satisfy the equations

$$\begin{aligned} k_i(p)(\lambda \circ c_p)'(t) &= \omega_i \circ c_p(t) + (\alpha_i \circ c_p)'(t) \\ &\quad - (\lambda \circ c_p)'(t) \int_0^t \alpha_i \circ c_p(\theta)\, \mathrm{d}\theta - \lambda \circ c_p(t) \alpha_i \circ c_p(t) \quad (i = 1, 2). \end{aligned}$$

4.2.3. Corollary. *For a* ***connected non-Riemannian Finsler surface*** (M, F)*, there are two possibilities:*

(a) either there exists no compatible linear connection on it (it is not a GBM)
(b) or it is a GBM with a uniquely determined compatible linear connection.

Proof. If there exists a compatible linear connection, its Christoffel symbols are of the form (40). Here on the right hand side, almost everything is a uniquely determined canonical data of the manifold. The only question is what values the integration constants $k_1(p), k_2(p)$ can take.

We can see that it depends on whether we can divide by the expression $(\lambda \circ c_p)'(t)$ in the equation determining them. This is not possible only if the main scalar λ is constant along the whole indicatrix, but in this case, it can be shown that λ is constant 0, and the indicatrix is quadratic at this point (and by the rigidity property, everywhere). Thus, in the non-Riemannian case, $(\lambda \circ c_p)'(t)$ is nonzero somewhere, and division gives the only possible value for $k_i(p)$. $\square$

4.3. Linear connections with semi-symmetric torsion

Though the general theory of GBMs, independently of the dimension, is still unsolved, significant progress has been made in some special cases. When starting in this direction, one has two choices: either consider only special types of Finsler metrics (as we did so in Chapter 3) or restrict the set of compatible ∇s we are searching for. Now we follow the second path and consider linear connections with a special type of torsion.[3]

4.3.1. Since the torsion tensor T is a vector valued (0,2)-tensor, or, equivalently, a (1,2)-tensor, its trace $\widetilde{T}$ is a (0,1)-tensor, i.e. a differential form. Denoting the dimension of the space by n as usual, we can decompose the torsion tensor as

$$T(X,Y) := T_1(X,Y) + T_2(X,Y), \tag{41}$$

where

- $T_2(X,Y) := \dfrac{1}{n-1}\left(\widetilde{T}(Y)X - \widetilde{T}(X)Y\right)$ is the part with the trace $\widetilde{T}$
- and $T_1(X,Y)$ is the traceless part.

T_1 can be further decomposed into two components A_1 (the totally antisymmetric part) and S_1, writing

$$T(X,Y) := A_1(X,Y) + S_1(X,Y) + T_2(X,Y), \tag{42}$$

which gives 8 possible cases depending on which terms are identically zero or not (indexed by triplets $(x,y,z) \in (\mathbb{Z}_2)^3$ accordingly, where zero means the vanishing and 1 the surviving of the terms from (42)). This second decomposition for us is irrelevant, because we are only interested in the following case.

4.3.2. Definition. We call the torsion tensor T **semi-symmetric**[4] if it is of the form

$$\boxed{T(X,Y) = \rho(Y)X - \rho(X)Y} \tag{43}$$

for some differential 1-form ρ on the base manifold. That is, if we consider the decomposition of T as in (41), its traceless part T_1 vanishes.

4.3.3. Remark. • The 1-form ρ in (43) is actually

$$\rho(X) = \frac{1}{n-1}\widetilde{T}(X).$$

- Since $T(\partial_i, \partial_j) = \rho(\partial_j)\partial_i - \rho(\partial_i)\partial_j = \rho_j\partial_i - \rho_i\partial_j = \left(\delta_i^k\rho_j - \delta_j^k\rho_i\right)\partial_k$, the torsion components are

$$T_{ij}^k = \delta_i^k\rho_j - \delta_j^k\rho_i. \tag{44}$$

---

[3] See for example [22], [16] and [18].

[4] Following Vincze in [16] and [18].

We now summarize some basic facts from [16].

4.3.4. Statement. *Suppose that (M, F) is a Finsler manifold with a compatible Riemannian metric γ, and ∇ is a compatible linear connection with semi-symmetric torsion of the form (43). Then it is related to the Levi-Civita connection ∇^* of γ through the formula*

$$\boxed{\nabla_X Y = \nabla^*_X Y + \rho(Y)X - \gamma(X,Y)\rho^\sharp,} \tag{45}$$

where $\rho^\sharp$ is the dual vector field of ρ; i.e. $\gamma\left(\rho^\sharp, X\right) = \rho(X)$.

Proof. For simplicity, let us write $\langle X, Y\rangle := \gamma(X, Y)$. By the compatibility of γ to F, both the connections ∇ and ∇^* are metrical to γ. So using formula (9) for both of them (and the torsion-freeness of ∇^*) we get

$$\begin{aligned} \langle \nabla_X Y, Z\rangle &= 1/2\cdot\big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &\quad - \langle X, T(Y,Z)\rangle + \langle Y, T(Z,X)\rangle + \langle Z, T(X,Y)\rangle \\ &\quad - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle\big); \end{aligned}$$

$$\begin{aligned} \langle \nabla^*_X Y, Z\rangle &= 1/2\cdot\big(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle \\ &\quad - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle\big). \end{aligned}$$

From this we see that

$$\langle \nabla_X Y, Z\rangle - \langle \nabla^*_X Y, Z\rangle = \frac{1}{2}\big(- \langle X, T(Y,Z)\rangle + \langle Y, T(Z,X)\rangle + \langle Z, T(X,Y)\rangle\big).$$

Using that the torsion T of ∇ is of the form $T(X,Y) = \rho(Y)X - \rho(X)Y$, by substituting we obtain on the right hand side

$$\begin{aligned} &\frac{1}{2}\big(- \langle X, \rho(Z)Y - \rho(Y)Z\rangle + \langle Y, \rho(X)Z - \rho(Z)X\rangle + \langle Z, \rho(Y)X - \rho(X)Y\rangle\big) \\ &\quad = \frac{1}{2}\big(- \rho(Z)\langle X,Y\rangle + \rho(Y)\langle X,Z\rangle + \rho(X)\langle Y,Z\rangle \\ &\qquad -\rho(Z)\langle X,Y\rangle + \rho(Y)\langle X,Z\rangle - \rho(X)\langle Y,Z\rangle\big) \\ &\quad = \rho(Y)\langle X,Z\rangle - \rho(Z)\langle X,Y\rangle = \rho(Y)\langle X,Z\rangle - \left\langle \rho^\sharp, Z\right\rangle\langle X,Y\rangle \\ &\quad = \langle \rho(Y)X, Z\rangle - \left\langle \langle X,Y\rangle\rho^\sharp, Z\right\rangle. \end{aligned}$$

Comparing this with the left hand side above, since Z is arbitrary, we immediately get formula (45). □

4.3.5. Corollary. *In particular, we have*

$$\begin{aligned} \nabla_{\partial_i}\partial_j &= \nabla^*_{\partial_i}\partial_j + \rho(\partial_j)\partial_i - \gamma(\partial_i,\partial_j)\rho^\sharp \\ &= \Gamma^{k*}_{ij}\partial_k + \rho_j\partial_i - \gamma_{ij}\rho^k\partial_k = \left(\Gamma^{k*}_{ij} + \delta^k_i\rho_j - \gamma_{ij}\rho^k\right)\partial_k, \end{aligned}$$

so the Christoffel symbols of ∇ and ∇^* are related via

$$\boxed{\Gamma^k_{ij} = \Gamma^{k*}_{ij} + \delta^k_i\rho_j - \gamma_{ij}\rho^k.} \tag{46}$$

The main result of this topic is the following.

4.3.6. Theorem (Unicity of semi-symmetric connections). *Any connected non-Riemannian Finsler manifold admits at most one compatible linear connection with semi-symmetric torsion.*

The original proof is due to Vincze: he gave an explicit formula for this ∇ by integrating some differential forms (canonical data) of the manifold; see Section 3 of [**16**] or Theorem 2.10. of [**18**]. We present here an elementary proof given in [**4**] (along with some solvability conditions and formulas).

Proof. First note that, as we are aiming for a unicity result, it is enough to consider the homogeneous version of the CEQ; let's call it H-CEQ. If we can prove that its only solution is the zero vector, then we are done. Second, using the isomorphism between T_pM and T_p^*M given by the Riemannian (Euclidean) structure, we can consider the covector ρ from the torsion of the compatible ∇ as a vector in T_pM, having the same coordinates $\rho^i = \rho_i$ in the dual basis.

Let's rewrite the H-CEQ with the vector ρ as the new unknown. We have seen in (44) that $T_{ij}^k = \delta_i^k \rho_j - \delta_j^k \rho_i$, so

- diverse torsion components (having 3 different indices) are zero
- and repetitive torsion components (with the upper index equal to one of the lower indices) are $T_{ij}^i = \delta_i^i \rho_j - \delta_j^i \rho_i = \rho_j = \rho^j$ $(j \neq i)$.

Substituting these into the CEQ in the form (20), dividing by 2 and homogenizing, the H-CEQ takes the form

(47)
$$\begin{array}{ccccc|c}
\rho^1 & \rho^2 & \rho^3 & \cdots & \rho^n & \text{RHS} \\
\hline
0 & f_{12} & f_{13} & \cdots & f_{1n} & 0 \\
f_{21} & 0 & f_{23} & \cdots & f_{2n} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
f_{n1} & f_{n2} & f_{n3} & \cdots & 0 & 0
\end{array}$$

Using the coefficient vector fields defined in 2.5.5 and considering ρ as a vector in T_pM, this can be written as

(H-CEQ-SS)
$$\left.\begin{array}{rcl}
\langle f_1, \rho \rangle & = & 0 \\
 & \vdots & \\
\langle f_n, \rho \rangle & = & 0
\end{array}\right\}.$$

We have to solve this system of linear equations as v ranges over $T_p^\circ M$. At a fixed v, solving it means finding the orthogonal complement of $\operatorname{span}(f_1(v), \ldots, f_n(v))$. By Corollary 2.6.8,

- in case of a vertical contact v, this set is the whole T_pM,
- in case of a not vertically contact v, this set is the orthogonal complement of $\operatorname{span}(G_v, C_v)$, which is exactly $\mathcal{LF}_v \cap \mathcal{LR}_v$, the intersection of the Finslerian and Riemannian linear tangent hyperplanes at v.

We get the solution of (H-CEQ-SS) at p by taking the intersection of all the solutions at the particular v's. In fact, by the 1^+-homogeneity in v, it actually suffices for v to run through the elements of one of the indicatrices; let's consider the Riemannian one (which is the usual sphere).

- If all the elements of T_pM are vertical contact, then by Statement 2.5.12 the metric is Riemannian at p. In case of a GBM, this would mean a Riemannian metric everywhere (by the rigidity property), which we excluded from our investigations.
- If there is one not vertically contact element v, there is one on the Riemannian indicatrix (by homogeneity) and also a neighborhood U of v consisting of not vertically contact elements (by continuity). It is enough to consider (H-CEQ-SS) over this neighborhood: at every v, for the solution vectors ρ we have
$$\rho \in \bigcap_{v\in U} (\mathcal{LF}_v \cap \mathcal{LR}_v) \subseteq \bigcap_{v\in U} \mathcal{LR}_v.$$
But it is obvious (or see Statement 2.5.7) that this intersection contains only the zero vector, so the H-CEQ has $\rho = 0$ as its only solution at p. Consequently, the CEQ has either 0 or 1 solution at p, and therefore everywhere. □

Now observe how the theory of these special compatible ∇s with semi-symmetric torsion describes the case of 2D completely.

4.3.7. Statement. *In dimension 2, every linear connection has semi-symmetric torsion.*

Proof. A simple computation in coordinates yields
$$\begin{aligned} T(X,Y) &= T(X^i\partial_i, Y^j\partial_j) = X^iY^jT^k_{ij}\partial_k \\ &= X^1Y^2T^k_{12}\partial_k + X^2Y^1T^k_{21}\partial_k \\ &= (X^1Y^2 - X^2Y^1)T^k_{12}\partial_k \\ &= (X^1Y^2 - X^2Y^1)\left(T^1_{12}\partial_1 + T^2_{12}\partial_2\right). \end{aligned}$$
On the other hand
$$\begin{aligned} \rho(Y)X - \rho(X)Y &= \rho(Y^i\partial_i)X^k\partial_k - \rho(X^i\partial_i)Y^k\partial_k \\ &= \left(\rho_iY^iX^k - \rho_iX^iY^k\right)\partial_k \\ &= (X^2Y^1 - X^1Y^2)\rho_1\partial_2 + (X^1Y^2 - X^2Y^1)\rho_2\partial_1 \\ &= (X^1Y^2 - X^2Y^1)\left(\rho_2\partial_1 - \rho_1\partial_2\right). \end{aligned}$$
So we can set $\rho_1 = -T^2_{12} = T^2_{21}$ and $\rho_2 = T^1_{12}$. □

4.3.8. Corollary. *For a* ***connected non-Riemannian Finsler surface*** *(M, F), there are two possibilities:*

(a) either there exists no compatible linear connection on it (it is not a GBM)
(b) or it is a GBM with a uniquely determined compatible linear connection (whose torsion is semi-symmetric).

4.4. The holonomy and curvature of 2-dimensional manifolds

4.4.1. Riemannian holonomy and curvature.

A Riemannian manifold is naturally equipped with its Levi-Civita connection ∇^*, and by Riemannian curvature and holonomy, we mean the curvature tensor and holonomy group of ∇^*.

4.4.1. STATEMENT. *For a Riemannian manifold,* $\mathrm{Hol}(\nabla^*)$ *is a Lie subgroup of the orthogonal group O_n (for an orientable manifold, SO_n). It is not necessarily a closed subgroup (just as in the general case), but the restricted holonomy group* $\mathrm{Hol}^0(\nabla^*)$ *is always closed in O_n.*

PROOF. In general, the holonomy is a subgroup of $GL_n(T_pM)$, but because the connection preserves the metric, it is a subgroup of O_n. Closedness of $\mathrm{Hol}^0(\nabla^*)$ is a theorem of Borel and Lichnerowicz, see [27]. □

4.4.2. REMARK. All groups have been classified that can arise as the holonomy of certain Riemannian manifolds. It's called Berger's list [26].

In Riemannian geometry it's more common to use the so-called sectional and Gauss curvatures instead of the curvature tensor itself.

4.4.3. DEFINITION. The **Riemannian curvature tensor** of a Riemannian metric γ is defined as

$$R^*(X,Y,Z,W) := \gamma(R^*(X,Y)Z,W). \tag{48}$$

The **sectional curvature** of a Riemannian manifold (M,γ) is a function that assigns a number to each 2-plane of all tangent spaces such that if X, Y is a basis for it, then

$$\kappa^*(X,Y) := \frac{R^*(X,Y,Y,X)}{\|X\|^2\,\|Y\|^2 - \langle X,Y\rangle^2}. \tag{49}$$

It is easy to see that this notion is independent of the basis chosen.

For a 2-dimensional manifold (a surface), κ^* is a function assigning a value to each tangent space; in this case, it is called the **Gauss curvature**.

4.4.4. It is an interesting question to examine what values the curvature, the sectional/Gauss curvature in particular, can take for Riemannian manifolds. It turns out that anything is possible: there are Riemannian manifolds with constant sectional curvature 0, 1 and -1 (or any other number) in any dimensions. They are the models of the three main geometries and are called the **model spaces**:

- for the Euclidean space $\mathbb{R}^n$ with the usual Euclidean metric, $R^* \equiv \kappa^* \equiv 0$ (Euclidean geometry);
- for the spheres $\mathbb{S}^n$ with the inherited metric from $\mathbb{R}^n$, $\kappa^* \equiv 1$ (spherical and projective geometry);
- for the upper half-space $\mathbb{H}^n := \{(u^1,\dots,u^n) \in \mathbb{R}^n \mid u^n > 0\}$ with the metric $\gamma_{ij} := \dfrac{1}{(u^n)^2}\delta_{ij}$, $\kappa^* \equiv -1$ (hyperbolic geometry).

4.4.2. Non-Riemannian GBMs' holonomy and curvature.

What changes for non-Riemannian generalized Berwald manifolds?

4.4.5. STATEMENT. *Let* (M, F, ∇) *be a* ***connected non-Riemannian GBM*** *with compatible Riemannian metric* γ*. Then* $\mathrm{Hol}(\nabla)$ *is a Lie subgroup of the orthogonal group* O_n *of* γ *that has no dense orbits on the Riemannian spheres (its closure is not transitive on these spheres).*

PROOF. First, by the compatibility of F and γ, ∇'s parallel translation preserves not only F, but also γ. In particular, elements of the holonomy of ∇ preserve γ, i.e. they are in O_n.

Now suppose that, under the elements of $\mathrm{Hol}(\nabla)$, some vector $v \in T_pM$ has a dense orbit on some Riemannian sphere. By the homogeneity of both metrics, we may suppose that $F(v) = 1$, i.e. v is an element of the Finslerian indicatrix F_p. But ∇ being compatible to F means the holonomy preserving F_p, so all the elements of v's orbit must be in F_p. Thus the Finslerian indicatrix contains a dense subset of a Riemannian sphere. Being closed, it must also contain their limit points, and therefore the whole sphere, so the Finsler metric is actually Riemannian (at one point, and thus everywhere by the rigidity property). □

4.4.6. COROLLARY. *The holonomy group of a* ***connected non-Riemannian*** *generalized Berwald* ***surface*** (M, F, ∇) *is always finite.*

PROOF. Let us consider the closure of the holonomy group. It is a closed subgroup in O_2 that is not transitive on the Riemannian spheres. By a result of Vincze (see [19], Theorem 4., Lemma 9. and Corollary 8.) such a group is either reducible (i.e. has non-trivial invariant subspaces along whom it can be splitted to a direct sum of lower-dimensional subgroups) or finite. But let us notice that in 2D, if a subgroup of the orthogonal group is reducible, then it must be finite, since there are only two 1-dimensional orthogonal transformations: id and −id. So the closure of the holonomy group, and thus the holonomy itself is finite. □

4.4.7. COROLLARY. *The curvature of a* ***connected non-Riemannian*** *generalized Berwald* ***surface*** (M, F, ∇) *is always identically zero.*

PROOF. Since the holonomy group is finite, in particular, a discrete Lie group, its dimension is 0, and thus the holonomy algebra is trivial. Using the Ambrose–Singer theorem, since the holonomy algebra is generated by the curvature endomorphisms, all of them must be zero, too. □

Let us note an interesting little consequence of this.

4.4.8. COROLLARY. *Let* (M, γ) *be a connected Riemannian* ***surface****. If there exists a* ***non-Riemannian*** *generalized Berwald metric on* M *compatible to* γ*, then* γ *admits a metrical linear connection of zero curvature.*

4.5. The divergence representation of the Gauss curvature

So far we have examined whether given Finsler surfaces are generalized Berwald or not. Now we turn to a much more sophisticated question: **given just a surface, when can it be equipped with a Finsler metric that turns it into a GBM?**[5] To give a (partial) answer, we need to investigate the curvature of metrical/compatible linear connections.

4.5.1. Lemma. [2] *Let (M,γ) be a Riemannian* ***surface*** *with Levi-Civita connection ∇^*. Then the curvature R of any metrical linear connection ∇ is related the curvature R^* of ∇^* via*

$$R(X,Y)Z - R^*(X,Y)Z =$$
$$\left((\nabla^*_X\rho)(Z) - \rho(X)\rho(Z) + \gamma(X,Z)\left\|\rho^\sharp\right\|^2\right)Y + \rho(X)\gamma(Y,Z)\rho^\sharp + \gamma(X,Z)\nabla^*_Y\rho^\sharp$$
$$-\left((\nabla^*_Y\rho)(Z) - \rho(Y)\rho(Z) + \gamma(Y,Z)\left\|\rho^\sharp\right\|^2\right)X - \rho(Y)\gamma(X,Z)\rho^\sharp - \gamma(Y,Z)\nabla^*_X\rho^\sharp,$$

where X,Y are arbitrary commuting vector fields (i.e. $[X,Y]=0$).

Proof. Choose vector fields X,Y with vanishing Lie brackets and write $\langle X,Y\rangle := \gamma(X,Y)$. Remember that in 2D, according to Statement 4.3.7, any linear connection has semi-symmetric torsion, and by being metrical, according to Statement 4.3.4, ∇ is related to the Levi-Civita connection via

$$\nabla_X Y = \nabla^*_X Y + \rho(Y)X - \langle X,Y\rangle\,\rho^\sharp.$$

All we need to do is substitute this formula to the curvature tensor of ∇.

$$
\begin{aligned}
& R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z)\\
= \;& \nabla_X\left(\nabla^*_Y Z + \rho(Z)Y - \langle Y,Z\rangle\rho^\sharp\right) - \nabla_Y\left(\nabla^*_X Z + \rho(Z)X - \langle X,Z\rangle\rho^\sharp\right)\\
= \;& \nabla^*_X\left(\nabla^*_Y Z + \rho(Z)Y - \langle Y,Z\rangle\rho^\sharp\right)\\
+ \;& \rho\left(\nabla^*_Y Z + \rho(Z)Y - \langle Y,Z\rangle\rho^\sharp\right)X\\
- \;& \left\langle X, \nabla^*_Y Z + \rho(Z)Y - \langle Y,Z\rangle\rho^\sharp\right\rangle\rho^\sharp\\
- \;& \nabla^*_Y\left(\nabla^*_X Z + \rho(Z)X - \langle X,Z\rangle\rho^\sharp\right)\\
- \;& \rho\left(\nabla^*_X Z + \rho(Z)X - \langle X,Z\rangle\rho^\sharp\right)Y\\
+ \;& \left\langle Y, \nabla^*_X Z + \rho(Z)X - \langle X,Z\rangle\rho^\sharp\right\rangle\rho^\sharp\\
= \;& \nabla^*_X\nabla^*_Y Z \underline{+\nabla^*_X(\rho(Z)Y) - \nabla^*_X\left(\langle Y,Z\rangle\rho^\sharp\right)}\\
+ \;& \left(\rho(\nabla^*_Y Z) + \rho(Y)\rho(Z) - \langle Y,Z\rangle\rho\left(\rho^\sharp\right)\right)X\\
- \;& \underline{\langle X,\nabla^*_Y Z\rangle\rho^\sharp}\; \cancel{-\rho(Z)\langle X,Y\rangle\rho^\sharp} + \rho(X)\langle Y,Z\rangle\rho^\sharp\\
- \;& \nabla^*_Y\nabla^*_X Z \underline{-\nabla^*_Y(\rho(Z)X) + \nabla^*_Y\left(\langle X,Z\rangle\rho^\sharp\right)}\\
- \;& \left(\rho(\nabla^*_X Z) + \rho(X)\rho(Z) - \langle X,Z\rangle\rho\left(\rho^\sharp\right)\right)Y\\
+ \;& \underline{\langle Y,\nabla^*_X Z\rangle\rho^\sharp}\; \cancel{+\rho(Z)\langle X,Y\rangle\rho^\sharp} - \rho(Y)\langle X,Z\rangle\rho^\sharp.
\end{aligned}
$$

[5] From now on, we present the results of [2].

Let us index the above terms just as if they were in a type $(6,3)$ matrix for easier reference, and continue transforming them the following way.

- $(1,1)$ and $(4,1)$ give $\nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z = R^*(X,Y)Z$.
- In $(2,3)$ and $(5,3)$, we have $\rho(\rho^\sharp) = \langle \rho^\sharp, \rho^\sharp \rangle = \left\| \rho^\sharp \right\|^2$.
- From the underlined terms we get

$$\begin{aligned} & \nabla^*_X (\rho(Z)Y) - \nabla^*_X \left(\langle Y, Z \rangle \rho^\sharp \right) + \langle Y, \nabla^*_X Z \rangle \rho^\sharp \\ - \; & \nabla^*_Y (\rho(Z)X) + \nabla^*_Y \left(\langle X, Z \rangle \rho^\sharp \right) - \langle X, \nabla^*_Y Z \rangle \rho^\sharp \\ = \; & X(\rho(Z))Y + \underline{\rho(Z)\nabla^*_X Y} - \underline{\langle \nabla^*_X Y, Z \rangle \rho^\sharp} \\ - \; & \cancel{\langle Y, \nabla^*_X Z \rangle \rho^\sharp} - \langle Y, Z \rangle \nabla^*_X \rho^\sharp + \cancel{\langle Y, \nabla^*_X Z \rangle \rho^\sharp} \\ - \; & Y(\rho(Z))X - \underline{\rho(Z)\nabla^*_Y X} + \underline{\langle \nabla^*_Y X, Z \rangle \rho^\sharp} \\ + \; & \cancel{\langle X, \nabla^*_Y Z \rangle \rho^\sharp} + \langle X, Z \rangle \nabla^*_Y \rho^\sharp - \cancel{\langle X, \nabla^*_Y Z \rangle \rho^\sharp} \end{aligned}$$

Here the underlined items become zero because of the torsion-freeness of ∇^*, and the terms $X(\rho(Z))Y$ and $-Y(\rho(Z))X$ can be combined with terms $(5,1)$ and $(2,1)$, respectively, to get

$$\begin{aligned} X(\rho(Z))Y - \rho(\nabla^*_X Z)Y &= \quad (\nabla^*_X \rho)(Z)Y \\ -Y(\rho(Z))X + \rho(\nabla^*_Y Z)X &= -(\nabla^*_Y \rho)(Z)X \end{aligned}$$

So what remains is exactly the formula of the lemma. □

We are ready to state and prove the various forms of the divergence representation of the Gauss curvature.

4.5.2. Theorem (The divergence representation of the Gauss curvature, first version [2]). *Let (M,γ) be a Riemannian* ***surface*** *with Levi-Civita connection ∇^* and Gauss curvature κ^*. Then, for any metrical linear connection ∇, we have*

$$\kappa^* = \operatorname{div}^*(\rho^\sharp) + R(X,Y,Y,X) \tag{50}$$

for any commuting orthonormal frame X, Y. On the right hand side,

- *$\rho^\sharp$ is the dual vector field of the 1-form ρ from the torsion of ∇, if we write it in the form of (43),*
- *div^* is the divergence operator, i.e. $\operatorname{div}^*(\rho^\sharp) = \gamma\left(\nabla^*_X \rho^\sharp, X\right) + \gamma\left(\nabla^*_Y \rho^\sharp, Y\right)$*
- *and R is the Riemannian curvature tensor of ∇.*

Proof. Let us fix a point p and choose vector fields X, Y such that their Lie bracket vanishes and they form an orthonormal basis at p, i.e. writing $\langle X, Y \rangle := \gamma(X,Y)$, we have $\langle X, Y \rangle = 0$, $\langle X, X \rangle = \langle Y, Y \rangle = 1$.

Let us compare the Riemannian curvature tensors of ∇ and ∇^*. Plugging $Z = Y$ into the formula of Lemma 4.5.1. and taking the inner product of both sides with X, at the point p we get

$$\begin{aligned}
&\langle R(X,Y)Y,X\rangle - \langle R^*(X,Y)Y,X\rangle \\
&= \left(\left(\nabla^*_X \rho\right)(Y) - \rho(X)\rho(Y) + \langle X,Y\rangle \left\|\rho^\sharp\right\|^2 \right) \langle Y,X\rangle \\
&\quad + \rho(X)\langle Y,Y\rangle \left\langle \rho^\sharp, X\right\rangle + \langle X,Y\rangle \left\langle \nabla^*_Y \rho^\sharp, X\right\rangle \\
&\quad - \left(\left(\nabla^*_Y \rho\right)(Y) - \rho(Y)\rho(Y) + \langle Y,Y\rangle \left\|\rho^\sharp\right\|^2 \right) \langle X,X\rangle \\
&\quad - \rho(Y)\langle X,Y\rangle \left\langle \rho^\sharp, X\right\rangle - \langle Y,Y\rangle \left\langle \nabla^*_X \rho^\sharp, X\right\rangle \\
&= \rho^2(X) - \left(\nabla^*_Y \rho\right)(Y) + \rho^2(Y) - \left\|\rho^\sharp\right\|^2 - \left\langle \nabla^*_X \rho^\sharp, X\right\rangle .
\end{aligned}$$

What do we have here?

- On the left hand side, $\langle R^*(X,Y)Y,X\rangle = R^*(X,Y,Y,X)$ is exactly the Gauss curvature κ^* of γ (since X,Y is an orthonormal basis at p).
- Because of the orthonormality of X and Y, $\left\|\rho^\sharp\right\|^2$ on the right hand side equals the squared sum of its coordinates in this basis, which is exactly $\left\langle \rho^\sharp, X\right\rangle^2 + \left\langle \rho^\sharp, Y\right\rangle^2$, or, equivalently, $\left\|\rho^\sharp\right\|^2 = \rho^2(X) + \rho^2(Y)$.
- Furthermore, we can compute

$$\begin{aligned}
\left(\nabla^*_Y \rho\right)(Y) &= Y\left(\rho(Y)\right) - \rho(\nabla^*_Y Y) = Y\left\langle \rho^\sharp, Y\right\rangle - \rho(\nabla^*_Y Y) \\
&= \left\langle \nabla^*_Y \rho^\sharp, Y\right\rangle + \left\langle \rho^\sharp, \nabla^*_Y Y\right\rangle - \rho(\nabla^*_Y Y) = \left\langle \nabla^*_Y \rho^\sharp, Y\right\rangle .
\end{aligned}$$

  So what remains, after multiplying by -1, is

$$-R(X,Y,Y,X) + \kappa^* = \left\langle \nabla^*_X \rho^\sharp, X\right\rangle + \left\langle \nabla^*_Y \rho^\sharp, Y\right\rangle ,$$

where the right hand side is exactly the divergence of $\rho^\sharp$. □

Formula (50) says that the metric, through its Gauss curvature, places a restriction on any metrical linear connection ∇ on the surface: its divergence and Riemannian curvature must obey (50). As a consequence, we have the following.

4.5.3. Corollary (The divergence representation of the Gauss curvature, second version [**2**]). *Let (M,γ) be a Riemannian* ***surface*** *with Levi-Civita connection ∇^*. Then the surface admits a metrical linear connection ∇ of zero curvature $\Longleftrightarrow$ the Gauss curvature of the surface can be represented as the divergence of some vector field, i.e.*

$$\kappa^* = \operatorname{div}^*(\rho^\sharp). \tag{51}$$

Here, $\rho^\sharp$ is the dual vector field of the 1-form coming from the semi-symmetric torsion of ∇ in the form of (43).

Proof. $\boxed{\Rightarrow}$ If ∇ is a metrical linear connection, according to the first version of the theorem, it satisfies (50). But since now the curvature of ∇ is zero, the formula becomes (51).

$\boxed{\Leftarrow}$ Suppose there exists a vector field $\rho^\sharp$ satisfying (51). Any such vector field uniquely determines a metrical linear connection the following way:

$$\rho^\sharp \text{ v. field} \xleftrightarrow{\text{dual}} \rho \text{ 1-form field} \xleftrightarrow{(43)} T \text{ torsion} \xleftrightarrow{(10)} \nabla\ \textbf{metrical} \text{ lin. conn.}$$

For this ∇,

- on the one hand, (51) holds by assumption,
- on the other hand, (50) holds as it does for any metrical linear connection.

So, consequently, the curvature of ∇ is zero. □

4.5.4. Remark. Let us note two things.

- The second half of the proof gives us a construction for linear connections with the desired properties from the vector field $\rho^\sharp$. As a shortcut, we could have used formula (46). We will return to this shortly.
- For surfaces whose Gauss curvature is constant zero, the everywhere-zero vector field $\mathbf{0}$ is a trivial solution for (51). The ∇ constructed from it is just the Levi-Civita connection (because of $\rho^\sharp \equiv 0 \implies T \equiv 0$), so for 'interesting' ∇s we should find a nontrivial solution for (51).

We can also apply our results for GBMs.

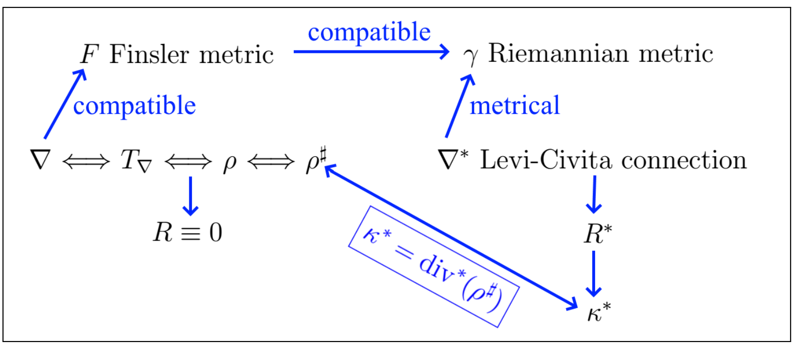
4.5.5. Corollary (The divergence representation of the Gauss curvature, third version [2]). *Let (M, F, ∇) be a generalized Berwald* ***surface*** *and γ a compatible Riemannian metric with Levi-Civita connection ∇^*. Then the Gauss curvature of γ is the divergence of the vector field $\rho^\sharp$ dual to the 1-form ρ coming from the semi-symmetric torsion of ∇ in the form of* (43)*:*

(52)
$$\kappa^* = \operatorname{div}^* \rho^\sharp.$$

Proof. About ∇ we know that

- it is metrical to γ (by the compatibility of F and γ)
- and its curvature is zero, as it is true for any compatible linear connections of generalized Berwald surfaces (see Corollary 4.4.7.).

So, we can use the '$\Rightarrow$' part of the second version of the theorem. □



4.5.6. Remark. Note that the other direction '$\Leftarrow$' cannot be transferred to the case of GBMs: if (51) holds, we can construct a linear connection ∇ of zero curvature and metrical to γ, but nothing guarantees its compatibility to F. What we can use it for is to *define* a Finsler metric which is preserved by ∇, if it is possible. We will return to this shortly.

4.6. A topological obstruction for 2-dimensional GBMs

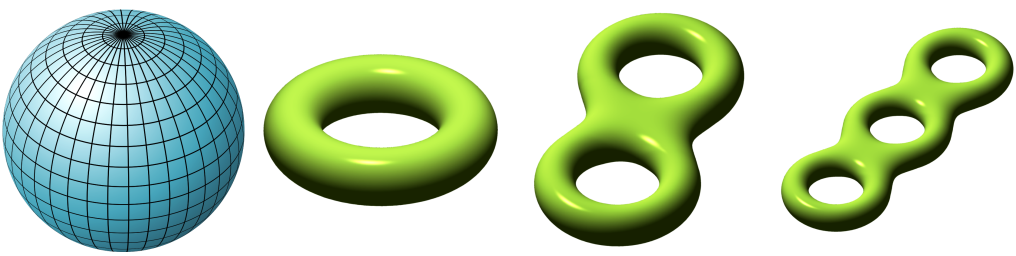
To answer the question we posed at the beginning of the previous section, let us first recall what types of surfaces (2-dimensional manifolds without boundary) there are to choose from.

Topologically, we can classify compact surfaces the following way.[6]

4.6.1. THEOREM (Topological classification of compact surfaces, [13]). *Up to* ***homeomorphism****, every* ***connected compact surface*** *can be uniquely described by two pieces of data: whether it is orientable or not and its Euler characteristic. Namely, we have the following cases.*

- *In the* ***orientable case****, there are the 2-sphere and the g-fold tori obtained from it by attaching g handles; more precisely, for $g = 0$ we have the 2-sphere and for any other $g \in \mathbb{N}$ the connected sum of g tori. g is called the genus of the surface and the Euler characteristic χ is obtained by $\chi = 2 - 2g$:*

	surface	genus	Euler characteristic
$\mathbb{S}^2$	2-sphere	$g = 0$	$\chi = 2$
$\mathbb{T}$	torus	$g = 1$	$\chi = 2 - 2 \cdot 1 = 0$
$\mathbb{T}\#\mathbb{T}$	double torus	$g = 2$	$\chi = 2 - 2 \cdot 2 = -2$
$\mathbb{T}\#\mathbb{T}\#\mathbb{T}$	triple torus	$g = 3$	$\chi = 2 - 2 \cdot 3 = -4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$



(A) $\mathbb{S}^2$ (B) $\mathbb{T}$ (C) $\mathbb{T}\#\mathbb{T}$ (D) $\mathbb{T}\#\mathbb{T}\#\mathbb{T}$

FIGURE 1. Some well-known surfaces (source: Wikipedia).

- *In the* ***non-orientable case****, there are the connected sums of $k \geq 1$ projective planes, with Euler characteristic $\chi = 2 - k$.*

Geometrically, surfaces can be classified according to the following list.

4.6.2. THEOREM (Uniformization theorem, [12]). *Every* ***connected surface*** *is* ***diffeomorphic*** *to a quotient (by a discrete group of isometries acting freely and properly discontinuously) of one of the 3 model spaces:*

- *the Euclidean plane,*
- *the 2-sphere or*
- *the hyperbolic plane.*

[6] The non-compact case is much more difficult, see [30].

With Corollary 4.5.5, we can immediately rule out a lot of spaces that can never carry generalized Berwald structures.

4.6.3. Theorem. *Let M be a connected, compact and orientable surface. If it is the base manifold of a non-Riemannian generalized Berwald surface, then its Euler characteristic is zero.*

Proof. Let us integrate both sides of the divergence representation (52) over the whole manifold M:

$$\int_M \kappa^* = \int_M \operatorname{div}^*(\rho^\sharp).$$

- By the Gauss–Bonnet theorem, the left hand side equals $2\pi \cdot \chi(M)$.
- By the divergence theorem, the right hand side equals $\int_{\partial M} \left\langle \rho^\sharp, N \right\rangle$ with N as the outward-pointing unit normal vector field. But our manifolds have no boundary, so this integral is 0. $\square$

4.6.4. Corollary. *Of all the compact, connected and orientable surfaces, none can be the base manifold of a non-Riemannian GBM except for the torus.*

This also rules out one of the model spaces, the sphere $\mathbb{S}^2$; the other two will have a generalized Berwald structure, and we will present examples in the following sections.

4.7. Constructing generalized Berwald surfaces

4.7.1. We have already seen that, given a non-Riemannian generalized Berwald manifold, we can always construct a Riemannian metric compatible to it (for example, the averaged Riemannian metric, see Section 1.5.) This process can be viewed as keeping the compatible ∇ (the notion of parallelism), but replacing the original non-Riemannian geometry (metric structure) of the manifold with a Riemannian one.

We might ask: **what about the other direction?** What if we start with a Riemannian metric on a manifold, and would like to replace it with a non-Riemannian generalized Berwald metric, keeping the notion of parallelism (i.e. requiring that metrical linear connections be compatible to the new metric).

In general, it is clearly not possible for multiple reasons:

- First, we already know that there are manifolds (the 2-sphere, for example) that do not admit any non-Riemannian generalized Berwald structures.
- Second, compatibility of the metrics in this reverse direction is more tricky: it is definitely not true that all metrical linear connections to the Riemannian metric (of whom there is always an infinite family) will be compatible to the Finsler metric (in particular, because in 2 dimensions there can only be one such connection, at most).

4.7.2. What we can do is the following: if we wish to replace the Riemannian metric γ on a given connected manifold with a non-Riemannian generalized Berwald metric F, choose ***one*** metrical linear connection ∇ as the candidate for the compatible connection of F (the notion of parallelism to be kept). Since we want ∇ to preserve the (not yet defined) new metric F, we should define F accordingly: choose a non-Riemannian indicatrix at an arbitrary point and parallel translate it to all other points by ∇. The indicatrix-family obtained this way will define a non-Riemannian Finsler metric F for which ∇ is compatible by definition.

However, there are some traps carefully hidden in the above 'sketch':

1. First, we should at least make sure that there are no topological obstructions that immediately rule out the possibility of turning the manifold into a GBM, to not work in vain.

2. Then we should be careful about how we choose the candidate ∇ for the compatible linear connection of the GBM. Is the Levi-Civita connection the best choice? Is it a possible choice at all?

3. How to find a non-Riemannian indicatrix that can be translated to the whole manifold to constitute a Finsler metric? Remember, the possibility of this is equivalent to the parallel translations of the indicatrix being path-independent, or the holonomy $\mathrm{Hol}(\nabla)$ preserving it. So: how to find a non-quadratic indicatrix preserved by the holonomy of ∇?

4. Finally, we would like to have a GBM that is compatible to the original Riemannian metric. However, the above process only guarantees that the designated ∇ will be both compatible to F and metrical to γ. But there might be other compatible linear connections to F – what guarantees their metricity to γ?

4.7.3. In general, the questions above are not so easy to answer. In 2 dimensions, they are much easier.

1. In 2D, a topological condition for the existence of a generalized Berwald metric, at least for compact and orientable surfaces, is that the Euler characteristic of the surface be zero (see Theorem 4.6.3.).[7]

2. In 2D, the curvature of the compatible ∇ of a non-Riemannian GBM is always zero (Corollary 4.4.7.). Thus, the candidate metrical connection ∇ must have zero curvature, and we have seen in Corollary 4.5.3. that such a ∇ exists if and only if the Gauss curvature of the Riemannian metric can be represented as the divergence of a vector field. What's more, in the proof we gave a way to construct ∇ from this vector field.

3a. For the third question, first note that if ∇ will be compatible to some non-Riemannian metric F, its holonomy must be finite (Corollary 4.4.6.). This is not automatically guaranteed by the vanishing curvature (only that the holonomy is a discrete Lie group); for that to hold, the holonomy must be compact, i.e. closed in O_2. If the manifold's fundamental group is finite, this is automatically guaranteed.

[7] We might wonder if this condition is also sufficient; we will return to this later.

3b. The answer to the third question is polyellipses (generalized conics): if the holonomy of ∇ has no dense orbits on the Euclidean spheres, then it always preserves some non-quadratic pollyellipse (generalized conic) whose translates will constitute a non-Riemannian GBM. Polyellipses/polyellipsoids are the direct generalizations of classical ellipses/ellipsoids by allowing an arbitrary number of focal points, and they are defined as the set of points whose average distance from them is some constant.[8] The reason why they are easy to work with in this setting is that they are completely described by their focal set (and average distance): if the holonomy preserves the focal set, it preserves the polyellipse itself. Thus, to choose an invariant polyellipse for an indicatrix, choose one whose focal set is an orbit under the holonomy (or union of orbits). One also needs to think through that this way we can always construct polyellipses that are non-quadratic and are smooth (have no singularities); in [**19**] it is shown that this is always possible.

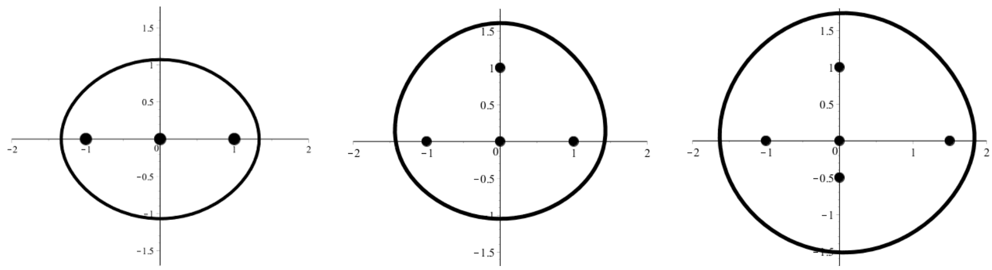


FIGURE 2. Some polyellipses.

4. The question of the compatibility of the metrics is trivial in 2D: it could only be ruined by a linear connection compatible to F and different from ∇. But, by the unicity result in 2D, there are no such connections. Thus all compatible linear connections (namely, only ∇) compatible to F are also metrical to γ, making these metrics compatible.

In summary, for a connected Riemannian surface, the construction of a compatible non-Riemannian generalized Berwald metric (if it is possible):

solve $\kappa^* = \operatorname{div}^*(\rho^\sharp) \longrightarrow \rho^\sharp$ vector field

$\xrightarrow{(46)} \nabla$ **metrical** linear connection $\xrightarrow{1.2.7.}$ notion of parallelism

$\downarrow$

choose a non-quadratic polyellipse at $p \in M$, invariant under $\mathrm{Hol}(\nabla)$

$\longrightarrow$ parallel translate this to all points of M, using $\nabla \longrightarrow$

as an indicatrix family, these together constitute a non-Riemannian

Finsler metric F compatible to γ

[8] We even allow the focal set to be infinite; in this case, average distances are taken to be integrals and the resulting sets are called generalized conics.

4.8. The Euclidean plane as a generalized Berwald surface

Let us now illustrate the process described in the previous section by replacing the usual Euclidean (Riemannian) metric

$$\gamma_{ij} = \delta_{ij}$$

on the plane $\mathbb{R}^2 = \{(u^1, u^2) \mid u^1, u^2 \in \mathbb{R}\}$ with a non-Riemannian generalized Berwald metric compatible to it.

Step 1. *Represent the Gauss curvature as divergence.*

Now the Gauss curvature is identically zero, and using the formula

$$\operatorname{div}^*(X) = \frac{1}{\sqrt{\det \gamma_{ij}}} \left[\frac{\partial \left(\sqrt{\det \gamma_{ij}} X^1\right)}{\partial u^1} + \frac{\partial \left(\sqrt{\det \gamma_{ij}} X^2\right)}{\partial u^2} \right], \tag{53}$$

and the fact that $\sqrt{\det \gamma_{ij}} = 1$, the divergence of a vector field is just

$$\operatorname{div}^*(X) = \frac{\partial X^1}{\partial u^1} + \frac{\partial X^2}{\partial u^2}.$$

So, we need to find vector fields

$$\rho^\sharp = \rho^1 \frac{\partial}{\partial u^1} + \rho^2 \frac{\partial}{\partial u^2}$$

(apart from the obvious $\mathbf{0}$) that solve the equation

$$0 = \frac{\partial \rho^1}{\partial u^1} + \frac{\partial \rho^2}{\partial u^2}.$$

Let us notice that the vanishing of the divergence of $\rho^\sharp$ is equivalent to the vanishing of the curl (rotation) of its rotated vector field

$$(\rho^\sharp)^\perp := \rho^2 \frac{\partial}{\partial u^1} - \rho^1 \frac{\partial}{\partial u^2}.$$

And these we can describe by finding global potential functions for them, i.e. some smooth function $f \colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\rho^2 = \frac{\partial f}{\partial u^1} \qquad \text{and} \qquad \rho^1 = -\frac{\partial f}{\partial u^2}. \tag{54}$$

Step 2. *Describe the linear connection we get from these vector fields.*

The coordinates of the 1-form field

$$\rho = \rho_1 du^1 + \rho_2 du^2$$

dual to $\rho^\sharp$ are given by

$$\rho_i = \delta_{ik} \rho^k = \rho^i,$$

so we have

$$\rho_1 = \rho^1 = -\frac{\partial f}{\partial u^2} \qquad \text{and} \qquad \rho_2 = \rho^2 = \frac{\partial f}{\partial u^1}.$$

To compute the Christoffel symbols of ∇ whose torsion is semi-symmetric with the 1-form ρ, we can use the formula

$$\Gamma_{ij}^k = \Gamma_{ij}^{k*} + \delta_i^k \rho_j - \gamma_{ij}\rho^k$$

(see (46)) to skip the step of computing the torsion. Because the Christoffel symbols of the canonical Euclidean metric are all 0, this now becomes

$$\Gamma_{ij}^k = \delta_i^k \rho_j - \delta_{ij}\rho_k.$$

In detail,

$$\boxed{\begin{array}{lcl@{\qquad\qquad\qquad}lcl}
\Gamma_{11}^1 & = & \rho_1 - \rho_1 = 0 & \Gamma_{11}^2 & = & -\rho_2 \\
\Gamma_{12}^1 & = & \rho_2 & \Gamma_{12}^2 & = & 0 \\
\Gamma_{21}^1 & = & 0 & \Gamma_{21}^2 & = & \rho_1 \\
\Gamma_{22}^1 & = & -\rho_1 & \Gamma_{22}^2 & = & \rho_2 - \rho_2 = 0
\end{array}}$$

Step 3. *Describe parallel vector fields.*

If X is a parallel vector field with respect to ∇ along a curve $c\colon [0,1] \to \mathbb{R}^2$, then, using the ODE of parallel vector fields (see (3)), we get

$$\begin{aligned}
(X^1)' &= -\left(\Gamma_{ij}^1 \circ c\right)(c^i)'X^j = -X^2\left(\rho_2 \circ c \cdot (c^1)' - \rho_1 \circ c \cdot (c^2)'\right) \\
&= -X^2\left(\frac{\partial f}{\partial u^1} \circ c \cdot (c^1)' + \frac{\partial f}{\partial u^2} \circ c \cdot (c^2)'\right) \\
(X^2)' &= -\left(\Gamma_{ij}^2 \circ c\right)(c^i)'X^j = -X^1\left(-\rho_2 \circ c \cdot (c^1)' + \rho_1 \circ c \cdot (c^2)'\right) \\
&= X^1\left(\frac{\partial f}{\partial u^1} \circ c \cdot (c^1)' + \frac{\partial f}{\partial u^2} \circ c \cdot (c^2)'\right)
\end{aligned}$$

Writing $\varphi = f \circ c$, this becomes

$$\begin{aligned}
(X^1)' &= -X^2 \cdot \varphi' \\
(X^2)' &= X^1 \cdot \varphi'.
\end{aligned} \tag{55}$$

So what does the parallel vector field X look like? In polar coordinates, we can write

$$X(t) = r_0\left(\cos(\theta(t))\left(\frac{\partial}{\partial u^1}\right)_{c(t)} + \sin(\theta(t))\left(\frac{\partial}{\partial u^2}\right)_{c(t)}\right),$$

where, as a consequence of the metricity and parallelism, the Euclidean norm r_0 is constant and

$$\left.\begin{aligned}
(X^1)'(t) &= -r_0 \sin(\theta(t))\,\theta'(t) = -X^2(t)\theta'(t) \\
(X^2)'(t) &= r_0 \cos(\theta(t))\,\theta'(t) = X^1(t)\theta'(t)
\end{aligned}\right\}$$

$$\xRightarrow{(55)} \theta'(t) = \varphi'(t) \Longrightarrow \theta(t) = \varphi(t) + \varphi_0.$$

Thus, the general form of a parallel vector field with respect to ∇ along the curve $c\colon [0,1] \to \mathbb{R}^2$ is

$$\boxed{\begin{gathered} X(t) = r_0 \left(\cos\left(\varphi(t) + \varphi_0\right) \left(\frac{\partial}{\partial u^1}\right)_{c(t)} + \sin\left(\varphi(t) + \varphi_0\right) \left(\frac{\partial}{\partial u^2}\right)_{c(t)} \right), \\ \text{where } \varphi(t) = f(c(t)). \end{gathered}}$$

<u>**Step 4.**</u> *Construct the new metric.*

First, it is clear that the holonomy of ∇ is trivial:

$$c(0) = c(1) \Longrightarrow \varphi(0) = \varphi(1) \Longrightarrow X(0) = X(1).$$

So we can choose an arbitrary non-Riemannian indicatrix at any chosen point; parallel translations with respect to ∇ will spread it out to the whole plane, to constitute a non-Riemannian Finsler metric with compatible linear connection ∇.

4.9. The hyperbolic plane as a generalized Berwald surface

Let us now consider the hyperbolic plane in its upper half-plane model, i.e. the metric

$$\gamma_{ij} := \frac{1}{(u^2)^2}\delta_{ij}$$

on the upper half-plane $\mathbb{H}^2 = \{(u^1, u^2) \mid u^2 > 0\}$. We will again replace it with a non-Riemannian generalized Berwald metric compatible to it.

<u>**Step 1.**</u> *Represent the Gauss curvature as divergence.*

Now the Gauss curvature is constant -1, and using formula (53) and the fact that $\sqrt{\det \gamma_{ij}} = \sqrt{1/(u^2)^4} = 1/(u^2)^2$, the divergence of a vector field is

$$\begin{aligned} \operatorname{div}^*(X) &= (u^2)^2 \left[\frac{\partial}{\partial u^1}\left(\frac{1}{(u^2)^2} X^1\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{(u^2)^2} X^2\right) \right] \\ &= \frac{\partial X^1}{\partial u^1} + \frac{\partial X^2}{\partial u^2} - \frac{2}{u^2} X^2. \end{aligned}$$

So, we need to find vector fields

$$\rho^\sharp = \rho^1 \frac{\partial}{\partial u^1} + \rho^2 \frac{\partial}{\partial u^2}$$

that solve the equation

$$-1 = \frac{\partial \rho^1}{\partial u^1} + \frac{\partial \rho^2}{\partial u^2} - \frac{2}{u^2}\rho^2.$$

Note that with some clever transformations, we can write this as

$$\begin{aligned} 0 &= \frac{\partial \rho^1}{\partial u^1} + \frac{\partial \rho^2}{\partial u^2} - \frac{2}{u^2}\rho^2 + 1 \\ &= \frac{1}{(u^2)^2}\frac{\partial \rho^1}{\partial u^1} + \frac{1}{(u^2)^2}\frac{\partial \rho^2}{\partial u^2} - \frac{2}{(u^2)^3}\rho^2 + \frac{1}{(u^2)^2} \\ &= \frac{\partial}{\partial u^1}\left(\frac{1}{(u^2)^2}\rho^1\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{(u^2)^2}\rho^2 - \frac{1}{u^2}\right) \\ &=: \frac{\partial}{\partial u^1}(\tilde{\rho}^1) + \frac{\partial}{\partial u^2}(\tilde{\rho}^2). \end{aligned}$$

Thus, solving the divergence representation in the hyperbolic case actually leads back to solving it in the Euclidean case, and the general form of the solution, using (54), is

$$\begin{aligned} \tilde{\rho}^1 &= \frac{1}{(u^2)^2}\rho^1 &&= -\frac{\partial f}{\partial u^2} &&\Longrightarrow &\rho^1 &= -(u^2)^2\frac{\partial f}{\partial u^2}, \\ \tilde{\rho}^2 &= \frac{1}{(u^2)^2}\rho^2 - \frac{1}{u^2} &&= \frac{\partial f}{\partial u^1} &&\Longrightarrow &\rho^2 &= (u^2)^2\frac{\partial f}{\partial u^1} + u^2 \end{aligned}$$

for some smooth function $f\colon \mathbb{H}^2 \to \mathbb{R}$.

Step 2. *Describe the linear connection we get from these vector fields.*

The coordinates of the 1-form field

$$\rho = \rho_1 du^1 + \rho_2 du^2$$

dual to $\rho^\sharp$ are given by

$$\rho_i = \gamma_{ik}\rho^k = \frac{1}{(u^2)^2}\delta_{ik}\rho^k = \frac{1}{(u^2)^2}\rho^i,$$

so we have

$$\rho_1 = \frac{1}{(u^2)^2}\rho^1 = -\frac{\partial f}{\partial u^2} \qquad \text{and} \qquad \rho_2 = \frac{1}{(u^2)^2}\rho^2 = \frac{\partial f}{\partial u^1} + \frac{1}{u^2}.$$

To compute the Christoffel symbols of ∇ whose torsion is semi-symmetric with the 1-form ρ, we can use the formula

$$\Gamma_{ij}^k = \Gamma_{ij}^{k*} + \delta_i^k\rho_j - \gamma_{ij}\rho^k$$

(see (46)) which now becomes

$$\Gamma_{ij}^k = \Gamma_{ij}^{k*} + \delta_i^k\rho_j - \frac{1}{(u^2)^2}\delta_{ij}\rho^k = \Gamma_{ij}^{k*} + \delta_i^k\rho_j - \delta_{ij}\rho_k.$$

Since the Christoffel symbols of the hyperbolic metric are

$$\begin{aligned} &\Gamma_{11}^{1*} = 0, \quad &&\Gamma_{12}^{1*} = -\frac{1}{u^2}, \quad &&\Gamma_{21}^{1*} = -\frac{1}{u^2}, \quad &&\Gamma_{22}^{1*} = 0, \\ &\Gamma_{11}^{2*} = \frac{1}{u^2}, \quad &&\Gamma_{12}^{2*} = 0, \quad &&\Gamma_{21}^{2*} = 0, \quad &&\Gamma_{22}^{2*} = -\frac{1}{u^2}, \end{aligned}$$

the Christoffel symbols of ∇ are given by

$$\boxed{\begin{array}{lll lll}
\Gamma_{11}^1 & = & \rho_1 - \rho_1 = 0 & \Gamma_{11}^2 & = & \dfrac{1}{u^2} - \rho_2 = -\dfrac{\partial f}{\partial u^1} \\
\Gamma_{12}^1 & = & -\dfrac{1}{u^2} + \rho_2 = \dfrac{\partial f}{\partial u^1} & \Gamma_{12}^2 & = & 0 \\
\Gamma_{21}^1 & = & -\dfrac{1}{u^2} & \Gamma_{21}^2 & = & \rho_1 = -\dfrac{\partial f}{\partial u^2} \\
\Gamma_{22}^1 & = & -\rho_1 = \dfrac{\partial f}{\partial u^2} & \Gamma_{22}^2 & = & -\dfrac{1}{u^2} + \rho_2 - \rho_2 = -\dfrac{1}{u^2}.
\end{array}}$$

Step 3. *Describe parallel vector fields.*

If X is a parallel vector field with respect to ∇ along a curve $c\colon [0,1] \to \mathbb{H}^2$, then, using the ODE of parallel vector fields (see (3)), we get

$$\begin{aligned}
(X^1)' &= -\left(\Gamma_{ij}^1 \circ c\right)(c^i)'X^j = 0 \\
&= -\frac{\partial f}{\partial u^1} \circ c \cdot (c^1)'X^2 + \frac{1}{c^2}(c^2)'X^1 - \frac{\partial f}{\partial u^2} \circ c \cdot (c^2)'X^2 \\
&= X^1 \frac{(c^2)'}{c^2} - X^2 \left(\frac{\partial f}{\partial u^1} \circ c \cdot (c^1)' + \frac{\partial f}{\partial u^2} \circ c \cdot (c^2)'\right), \\
(X^2)' &= -\left(\Gamma_{ij}^2 \circ c\right)(c^i)'X^j = 0 \\
&= \frac{\partial f}{\partial u^1} \circ c \cdot (c^1)'X^1 + \frac{\partial f}{\partial u^2} \circ c \cdot (c^2)'X^1 + \frac{1}{c^2}(c^2)'X^2 \\
&= X^1 \left(\frac{\partial f}{\partial u^1} \circ c \cdot (c^1)' + \frac{\partial f}{\partial u^2} \circ c \cdot (c^2)'\right) + X^2 \frac{(c^2)'}{c^2}.
\end{aligned}$$

Writing $\varphi = f \circ c$, this becomes

$$\begin{aligned}
(X^1)' &= X^1 \cdot \frac{(c^2)'}{c^2} - X^2 \cdot \varphi' \\
(X^2)' &= X^1 \cdot \varphi' + X^2 \cdot \frac{(c^2)'}{c^2}.
\end{aligned} \tag{56}$$

So what does the parallel vector field X look like? In polar coordinates with respect to the orthonormal bases

$$\left(u^2 \frac{\partial}{\partial u^1}, u^2 \frac{\partial}{\partial u^2}\right)$$

in the tangent spaces, we can write

$$X(t) = r_0 c^2(t) \left(\cos(\theta(t)) \left(\frac{\partial}{\partial u^1}\right)_{c(t)} + \sin(\theta(t)) \left(\frac{\partial}{\partial u^2}\right)_{c(t)}\right),$$

where, as a consequence of the metricity and parallelism, r_0 is a constant and because of

$$(X^1)'(t) = r_0(c^2)'(t)\cos(\theta(t)) - r_0c^2(t)\sin(\theta(t))\,\theta'(t) = X^1\frac{(c^2)'}{c^2} - X^2(t)\theta'(t)$$

$$(X^2)'(t) = r_0(c^2)'(t)\sin(\theta(t)) + r_0c^2(t)\cos(\theta(t))\,\theta'(t) = X^2\frac{(c^2)'}{c^2} + X^1(t)\theta'(t)$$

from (56) it follows that

$$\theta'(t) = \varphi'(t) \Longrightarrow \theta(t) = \varphi(t) + \varphi_0.$$

Thus, the general form of a parallel vector field with respect to ∇ along the curve $c\colon [0,1] \to \mathbb{H}^2$ is

$$\boxed{\begin{gathered} X(t) = r_0c^2(t)\left(\cos(\varphi(t)+\varphi_0)\left(\frac{\partial}{\partial u^1}\right)_{c(t)} + \sin(\varphi(t)+\varphi_0)\left(\frac{\partial}{\partial u^2}\right)_{c(t)}\right), \\ \text{where } \varphi(t) = f(c(t)). \end{gathered}}$$

Step 4. *Construct the new metric.*

First, it is clear that the holonomy of ∇ is trivial:

$$c(0) = c(1) \Longrightarrow \varphi(0) = \varphi(1) \Longrightarrow X(0) = X(1).$$

So we can choose an arbitrary non-Riemannian indicatrix at any chosen point; parallel translations with respect to ∇ will spread it out to the whole upper half-plane, to constitute a non-Riemannian Finsler metric with compatible linear connection ∇.

CHAPTER 5

3-dimensional generalized Berwald manifolds

In this chapter, we continue our investigation on generalized Berwald manifolds, but now in 3 dimensions. In dimension 2, the CEQ was relatively simple and easy to solve, so we might expect that increasing the dimension just by 1 does not complicate things dramatically. Unfortunately, this is not the case: in 3D the CEQ becomes much more complicated because of the considerably larger number of torsion components and the appearance of 'torsion triplets'. What helps us in this unlucky situation is the lucky fact that for $n = 3$, the dimension is equal to the number of the coefficients f_{ab}, so – instead of the 9-dimensional space of the torsion components – we can work inside the 3-dimensional vector space T_pM. After finding the solutions as vectors in T_pM, it's not that difficult to get the components $\boldsymbol{T}^c_{ab}$ of the compatible linear connection we are searching for.

We will see that in 3D, a generalized Berwald manifold admits either a unique compatible linear connection (whose form we will describe) or an infinite family of them, in which case the indicatrices of the metric must be Euclidean surfaces of revolution. We can also find the extremal connection among this family of compatible connections. As an application, we show how we can apply the results of the chapter for the familiar family of Randers metrics.[1]

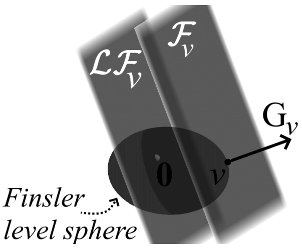
5.1. The geometry of the tangent spaces in 3 dimensions

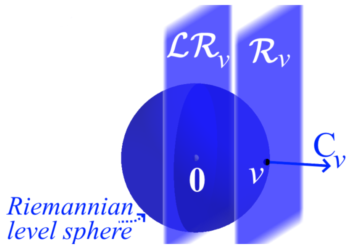
Before facing the CEQ, let us summarize what we know about the geometry of the tangent spaces T_pM of a 3-dimensional Finsler manifold. Of course, everything we have discussed in Chapter 2 for general dimensions, also applies here.

5.1.1. Remember, we chose a Riemannian metric γ compatible to the given Finsler metric F, fixed a point p and used Riemannian normal coordinates around it. This way, the induced basis $(\partial_1|_p, \partial_2|_p, \partial_3|_p)$ of T_pM (and thus the 'lifted' bases $(\dot{\partial}_1|_v, \dot{\partial}_2|_v, \dot{\partial}_3|_v)$ of $T_vT_p^\circ M$) are orthonormal, so working in T_pM with γ is just the same as working inside $\mathbb{R}^3$. Whenever we talk about the coordinates of a vector in T_pM or a vector field on T_pM, we mean them in these coordinates. The following table summarizes the tools and notations we have for the Finslerian and Riemannian structure of T_pM.

[1] In this chapter, we present the results of [3].

Finsler metric		**Riemannian metric**
Finslerian spheres	level sets	Euclidean spheres
$\mathcal{F}_v$	tangent planes at $v \in T_p^\circ M$	$\mathcal{R}_v$
$\mathcal{LF}_v := \mathcal{F}_v - v$	linear tangent planes at $v \in T_p^\circ M$	$\mathcal{LR}_v := \mathcal{R}_v - v$
$G_v := \operatorname{grad} F(v) = [\dot{\partial}_1 F, \dots, \dot{\partial}_n F](v)$	normal vector fields (w.r.t. γ)	$C_v := v = [v^1, \dots, v^n]$





By the classification of the elements of $T_p^\circ M$,

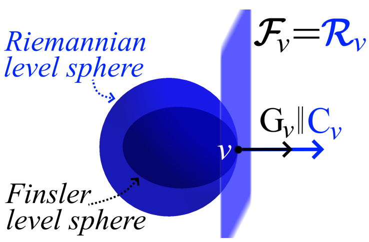
- $v \in T_p^\circ M$ is vertically contact $\iff$ $G_v \parallel C_v$ $\iff$ $\mathcal{F}_v = \mathcal{R}_v$;
- $v \in T_p^\circ M$ is **not** vertically contact $\iff$ $G_v \nparallel C_v$ $\iff$ $\mathcal{F}_v \cap \mathcal{R}_v$ is a 1-dimensional affine subspace, i.e. a line.

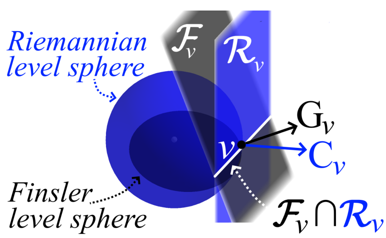
But in 3D, these notions are closely related to a certain vector field.

5.1.2. Lemma. *Introducing the cross product*

$$C \times G = \begin{vmatrix} \dot{\partial}_1 & \dot{\partial}_2 & \dot{\partial}_3 \\ y^1 & y^2 & y^3 \\ \dot{\partial}_1 F & \dot{\partial}_2 F & \dot{\partial}_3 F \end{vmatrix} = \begin{bmatrix} y^2 \dot{\partial}_3 F - y^3 \dot{\partial}_2 F \\ y^3 \dot{\partial}_1 F - y^1 \dot{\partial}_3 F \\ y^1 \dot{\partial}_2 F - y^2 \dot{\partial}_1 F \end{bmatrix} = \begin{bmatrix} f_{23} \\ f_{31} \\ f_{12} \end{bmatrix}, \tag{57}$$

- *$v \in T_p^\circ M$ is vertically contact* $\iff$ *$C_v \times G_v = \mathbf{0}$;*
- *if v is not vertically contact, the nonzero vector $C_v \times G_v$ gives the direction of $\mathcal{F}_v \cap \mathcal{R}_v$ (since it's orthogonal to both C_v and G_v, it lies in both kinds of tangent planes).*





We can also give an alternative proof for Statement 2.5.11.

5.1.3. Statement. *In 3D, T_pM always has a vertical contact point.*

Proof. Let us suppose indirectly that T_pM has no vertical contact points. In this case the vector field $C \times G$ is a (continuous) nonvanishing tangential vector field to the 2-dimensional Euclidean sphere, which (according to the famous hedgehog theorem) cannot exist. $\square$

5.2. The compatibility equations in 3 dimensions

5.2.1. In (22) of Chapter 2, we have written the CEQ in 3D as

	$f_{12}(v)$	$f_{13}(v)$	$f_{23}(v)$	RHS
1.	$2\boldsymbol{T}^1_{12}$	$2\boldsymbol{T}^1_{13}$	$\boldsymbol{T}^2_{13}+\boldsymbol{T}^1_{23}+\boldsymbol{T}^3_{21}$	$-2\,\widehat{\partial}_1 F(v)$
2.	$2\boldsymbol{T}^2_{12}$	$\boldsymbol{T}^1_{23}+\boldsymbol{T}^2_{13}+\boldsymbol{T}^3_{12}$	$2\boldsymbol{T}^2_{23}$	$-2\,\widehat{\partial}_2 F(v)$
3.	$\boldsymbol{T}^1_{32}+\boldsymbol{T}^3_{12}+\boldsymbol{T}^2_{13}$	$2\boldsymbol{T}^3_{13}$	$2\boldsymbol{T}^3_{23}$	$-2\,\widehat{\partial}_3 F(v)$

Notice that the three coefficients are exactly the coordinates of the vector field $C \times G$ defined in (57)! So we can rewrite the equations of the CEQ to contain this vector as

$$\text{(CEQ-3D-1)} \qquad \left\langle C_v \times G_v, \begin{bmatrix} -\boldsymbol{T}^3_{12}+\boldsymbol{T}^2_{13}+\boldsymbol{T}^1_{23} \\ -2\boldsymbol{T}^1_{13} \\ 2\boldsymbol{T}^1_{12} \end{bmatrix} \right\rangle = -2\,\widehat{\partial}_1 F(v),$$

$$\text{(CEQ-3D-2)} \qquad \left\langle C_v \times G_v, \begin{bmatrix} 2\boldsymbol{T}^2_{23} \\ -\boldsymbol{T}^3_{12}-\boldsymbol{T}^2_{13}-\boldsymbol{T}^1_{23} \\ 2\boldsymbol{T}^2_{12} \end{bmatrix} \right\rangle = -2\,\widehat{\partial}_2 F(v),$$

$$\text{(CEQ-3D-3)} \qquad \left\langle C_v \times G_v, \begin{bmatrix} 2\boldsymbol{T}^3_{23} \\ -2\boldsymbol{T}^3_{13} \\ \boldsymbol{T}^3_{12}+\boldsymbol{T}^2_{13}-\boldsymbol{T}^1_{23} \end{bmatrix} \right\rangle = -2\,\widehat{\partial}_3 F(v).$$

5.2.2. For a moment, we can forget about what the unknowns actually are. In this form, we can think about the equations of the CEQ as

$$\text{(CEQ-3D-}i\text{)} \qquad \langle C_v \times G_v, \overrightarrow{\boldsymbol{s}_i} \rangle = b_i(v), \qquad i \in \{1,2,3\},$$

i.e. 3 linear equations with unknown vectors $\overrightarrow{\boldsymbol{s}_1}, \overrightarrow{\boldsymbol{s}_2}, \overrightarrow{\boldsymbol{s}_3}$ in T_pM. Since these equations only differ from each other in their inhomogeneous parts, we can handle them together by considering only the homogeneous version

$$\text{(H-CEQ-3D)} \qquad \langle C_v \times G_v, \overrightarrow{\boldsymbol{s}} \rangle = 0.$$

To solve this, we have to find the vectors $\overrightarrow{\boldsymbol{s}}$ of T_pM orthogonal to all the vectors $C_v \times G_v$ (or, equivalently, orthogonal to their linear span), as v runs through the elements of $T_p^\circ M$. It is also clear that we only have to consider not vertically contact elements (otherwise $C_v \times G_v = \mathbf{0}$).

In summary: first consider the homogeneous version (H-CEQ-3D). Its solution set, which we will call the common **directional space**, is a linear subspace that is the orthogonal complement of span$(C_v \times G_v)$ inside T_pM. The solution sets of the original, possibly inhomogeneous equations (CEQ-3D-1), (CEQ-3D-2), (CEQ-3D-3) of the CEQ, which we will call the **solution spaces**, are either empty or affine translates of the common directional space (and therefore parallel to it and to each other).

5.3. The possible solutions of the CEQ

5.3.1. We have seen that basically, the 'size' of the CEQ's solution set depends on how many dimensions the vectors $C_v \times G_v$ can span. More precisely, denoting

$$\mathrm{DS}(C \times G) := \dim \left(\operatorname*{span}_{v \in T_p^\circ M} (C_v \times G_v) \right),$$

the dimension of the common directional space is $3 - \mathrm{DS}(C \times G)$, so we have 4 possibilities for it: $0, 1, 2$ or 3. The question is: **when can these actually happen (what is the geometry behind them), or are all of them even possible?** Before attempting an answer, let us describe the integral curves of this field.

5.3.2. Statement. *The integral curves of the vector field $C \times G$ in $T_p^\circ M$ are always running in the intersection of the Finslerian and the Riemannian (Euclidean) spheres.*

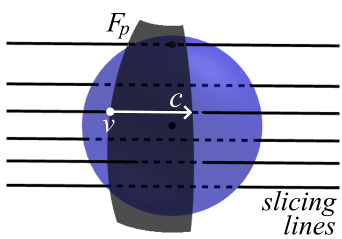
Proof. At vertically contact points, our claim is trivial ($C_v \times G_v = \mathbf{0}$, the integral curve is constant, i.e. the point v itself, which is in the intersection of some Finslerian and Euclidean spheres). Consider a point v in the open domain of all not vertically contact elements (if there are any) and denote by c the integral curve starting from it. Since $C \times G$ is tangential to both the Finslerian and Euclidean spheres going through v, c will remain on both of them. □

5.3.3. Now let's take a closer look at the possible values for $\mathrm{DS}(C \times G)$!

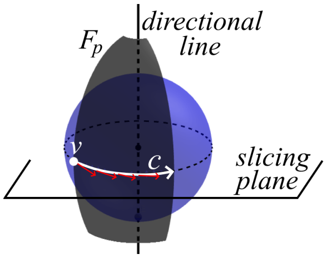
- $\boxed{\mathrm{DS}(C \times G) = 0}$ In this case, the vector field $C \times G$ is identically zero on the whole $T_p^\circ M$. This means that T_pM is a vertical contact tangent space, so, by Statement 2.5.12, the Finsler metric F is quadratic at p. Since we are concerned with connected GBMs, by the rigidity property, such a GBM is Riemannian everywhere and of no interest to us.

- $\boxed{\mathrm{DS}(C \times G) = 1}$ If the vectors $C_v \times G_v$ together span a 1-dimensional line S in T_pM, then, considering them pinned to their respective 'home points', they all lie in one of the affine parallel translates of S; let's call these lines, slicing up T_pM parallelly, slicing lines. Take a point v from the open domain of not vertically contact elements and consider the (non-trivial) integral curve c starting from it.

Since $C \times G$ is tangential to (lies in) the slicing line going through v, c remains in this line, i.e. it is a line segment. Also, by the above statement, it lies in the intersection of the two kinds of spheres. But obviously, strictly convex bodies like the sphere cannot contain any line segments on their boundaries, so this case can never happen.

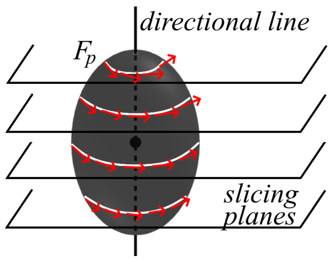


• $\boxed{\mathrm{DS}(C \times G) = 2}$ If the vectors $C_v \times G_v$ together span a 2-dimensional plane S in T_pM, then, considering them pinned to their respective 'home points', they all lie in one of the affine parallel translates of S; let's call these planes, slicing up T_pM parallelly, slicing planes. Take a point v of the Finslerian indicatrix from the open domain of not vertically contact elements and consider the (non-trivial) integral curve c starting from it.



By the above statement, c lies in the intersection of the two kinds of spheres, in particular, it is part of a Euclidean sphere. Also, since $C \times G$ is tangential to (lies in) the slicing plane going through v, c remains in this plane. Thus, over this domain, all the slices of the Finslerian indicatrix are circular arcs from a Euclidean sphere around the origin (for different slices, these spheres can, of course, be different).

What happens at vertically contact points? If they are all just accumulation points of this domain, then the slices are all circles by the closedness of F_p. If there are open regions consisting of vertically contact points, by their nature (or see the proof of Statement 2.5.12), they guarantee that the Finslerian indicatrix there also coincides with the Euclidean sphere.



In summary: if the vectors $C_v \times G_v$ span a 2-dimensional plane S, then the slices of the Finslerian indicatrix F_p parallel to it are Euclidean circles with centers on the orthogonal complement of S. In other words, F_p is a Euclidean surface of revolution, whose axis is the directional space, that is, a line through the origin which is the solution set of (H-CEQ-3D).

Note two things:

1. **The converse is also true:** if the (non-Riemannian) Finslerian indicatrix is rotationally symmetric with an axis through the origin, its intersections with the Euclidean spheres are circles perpendicular to the axis, and thus so are all the vectors $C \times G$, spanning only 2 dimensions.
2. **It is important that the axis go through the origin.** It is possible to have a (non-Riemannian) Finslerian indicatrix that is rotationally symmetric with no linear axes, but in that case the intersections with the Euclidean spheres (around the origin) cannot lie in parallel planes (otherwise we are in the case from above), so the vectors $C_v \times G_v$ must span 3 dimensions.

• $\boxed{\mathrm{DS}(C \times G) = 3}$ If the vectors $C_v \times G_v$ together span the whole T_pM, then their orthogonal complement, the solution of (H-CEQ-3D), is the zero vector alone.

Let us summarize the observations we have made above.

5.3.4. Statement. *We have the following cases for the dimension of the common directional space, i.e. the solution set of (H-CEQ-3D):*

dim. of span of $C_v \times G_v$	dim. of directional space	solution spaces of the equations (CEQ-3D-i)	geometry
0	3	T_pM or $\emptyset$	F_p is quadratic
1	impossible		
2	1	3 parallel lines or $\emptyset$	F_p is a revolution surface with the directional line as axis
3	0	3 points or $\emptyset$	F_p does not have a rotational axis through the origin

We considered the solutions here as vectors in T_pM, but it's easy to see (or see the next sections) that the uniqueness/infinity of the solutions is the same thing either we consider them as vectors or as the torsion components of the compatible linear connections we are looking for.

Now we are ready to describe all possibilities completely.

5.3.5. Theorem (Main theorem of 3-dimensional GBMs [3]). *If M is a connected non-Riemannian generalized Berwald manifold of dimension three, then we have the following possible cases.*

- ***Undetermined case (UDC):*** *The space admits infinitely many compatible linear connections if and only if the Finslerian indicatrix is a Euclidean surface of revolution at some and therefore all points of the manifold, with rotational axes going through the origins of the tangent spaces. These axes are generated by a globally well-defined (nowhere vanishing) vector field $D \in \mathfrak{X}(M)$, whose values have length 1, are in the orthogonal complement of the span of $C_v \times G_v$ in each T_pM and are parallel (covariant constant) with respect to any compatible linear connection. One of these connections is given by*

$$\nabla_X Y = \nabla^*_X Y + \frac{\langle \nabla^*_X D, Y\rangle D - \langle Y, D\rangle \nabla^*_X D}{\langle D, D\rangle}. \tag{58}$$

- ***Determined case (DC):*** *The space admits a unique compatible linear connection that is flat. It is given by*

$$\nabla_X Y = \nabla^*_X Y - \rho(X) \times Y, \tag{59}$$

where ρ is an endomorphism of $\mathfrak{X}(M)$ satisfying

$$R^*(X,Y)Z = \Big((\nabla^*_X \rho)(Y) - (\nabla^*_Y \rho)(X) - \rho(X) \times \rho(Y) \Big) \times Z$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

PROOF. **Undetermined case (UDC)**

Having more than one compatible linear connection means having at least one point p of the manifold where the CEQ has more than one solution. For a non-Riemannian GBM (which is not quadratic anywhere), it means that *at this point*, by the statement above, the dimension of the directional space is 1, and the Finslerian indicatrix is a Euclidean surface of revolution whose axis is the orthogonal complement of the span of the vectors $C_v \times G_v$. But it's also easy to see that for any GBM, if the Finslerian indicatrix is rotationally symmetric at one point, it is rotationally symmetric everywhere, since all the indicatrices are (linear) parallel translates of each other. So are their axis vectors, which means exactly that the vector field D constituted by them is invariant under any compatible linear connection. (Since we have a non-Riemannian generalized Berwald manifold, the rotational axes are uniquely determined, see [**19**] (Corollary 8) or the next Section.)

As for the ∇ defined in (58), we can check that ∇

- *is indeed a linear connection;*
- *preserves the axis:* $\nabla D = 0$, i.e. D is covariant constant with respect to it, because for all vector fields $X \in \mathfrak{X}(M)$,
$$\nabla_X D = \nabla^*_X D + \frac{\langle \nabla^*_X D, D\rangle\, D - \langle D, D\rangle\, \nabla^*_X D}{\langle D, D\rangle} = \frac{\langle \nabla^*_X D, D\rangle\, D}{\langle D, D\rangle},$$
and since D has constant length 1, then (by the metricity of ∇^* to γ) we have
$$0 = X\,\langle D, D\rangle = 2\,\langle \nabla^*_X D, D\rangle\,,$$
giving $\nabla_X D = 0$ for all $X \in \mathfrak{X}(M)$;
- *is metrical to the Riemannian metric and preserves all Euclidean spheres and circles,* because for any $X, Y, Z \in \mathfrak{X}(M)$,
$$\begin{aligned}\langle \nabla_X Y, Z\rangle &= \langle \nabla^*_X Y, Z\rangle + \frac{\langle \nabla^*_X D, Y\rangle\,\langle D, Z\rangle - \langle Y, D\rangle\,\langle \nabla^*_X D, Z\rangle}{\langle D, D\rangle},\\ \langle Y, \nabla_X Z\rangle &= \langle Y, \nabla^*_X Z\rangle + \frac{\langle \nabla^*_X D, Z\rangle\,\langle Y, D\rangle - \langle Z, D\rangle\,\langle Y, \nabla^*_X D\rangle}{\langle D, D\rangle}\end{aligned}$$
and adding these and using the metricity of ∇^* yields
$$\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = \langle \nabla^*_X Y, Z\rangle + \langle Y, \nabla^*_X Z\rangle = X\,\langle Y, Z\rangle\,;$$
- *is compatible to the Finsler metric,* preserving the indicatrices, since for a rotationally symmetric body, its enough to preserve the axis and the Euclidean circles.

Determined case (DC)

If the compatible linear connection is unique, then the solution of (H-CEQ-3D) is just the zero vector at all points of the manifold, and therefore, by the above statement, the indicatrices are not rotationally symmetric (at least not with an axis through the origin). It is not hard to see that in this case, the holonomy of the connection must be trivial.

Suppose, on the contrary, that the holonomy algebra is at least 1-dimensional. Then the holonomy group (which consists of orthogonal transformations, as we have seen in Statement 4.4.5) contains a connected, 1-dimensional Lie subgroup of $SO(3)$, that is, the group of rotations around some linear axis.[2] But this would mean that the indicatrices are preserved by these, i.e. they are rotationally symmetric, which gives a contradiction. Thus, the holonomy algebra must be trivial, and by the Ambrose–Singer theorem, the curvature is identically zero.

Now let's try to find a formula for this flat connection ∇! First, consider its so-called difference tensor with the Levi-Civita connection of the compatible Riemannian metric γ, which is the type $(1,2)$ tensor field

$$A(X,Y) := \nabla^*_X Y - \nabla_X Y.$$

Since the Levi-Civita connection is metrical to γ, we have

$$X\langle Y,Z\rangle = \langle \nabla^*_X Y, Z\rangle + \langle Y, \nabla^*_X Z\rangle .$$

∇ must also be metrical to γ, so we must also have

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle ,$$

which is equivalent to

$$\begin{aligned}
\langle \nabla^*_X Y, Z\rangle + \langle Y, \nabla^*_X Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\
\langle \nabla^*_X Y, Z\rangle - \langle \nabla_X Y, Z\rangle &= \langle Y, \nabla_X Z\rangle - \langle Y, \nabla^*_X Z\rangle \\
\langle A(X,Y), Z\rangle &= -\langle A(X,Z), Y\rangle .
\end{aligned}$$

Fixing the first variable, the skew-symmetry implies that the transformation can be represented as a cross product with some fixed vector (depending on X), that is, $A(X,Y) = \rho(X)\times Y$ with some endomorphism ρ of $\mathfrak{X}(M)$, so

$$\nabla_X Y = \nabla^*_X Y - \rho(X)\times Y.$$

[2] All the elements of $SO(3)$ are rotations. In a 1-dimensional rotation group, if it is connected, the rotations between *id* and any nontrivial element obviously generate all rotations around the given circle.

Plugging in $Y \sim \nabla_Y Z$, we obtain

$$\begin{aligned}\nabla_X \nabla_Y Z &= \nabla^*_X \nabla_Y Z - \rho(X) \times \nabla_Y Z \\ &= \nabla^*_X \Big(\nabla^*_Y Z - \rho(Y) \times Z \Big) - \rho(X) \times \Big(\nabla^*_Y Z - \rho(Y) \times Z \Big) \\ &= \nabla^*_X \nabla^*_Y Z - \Big((\nabla^*_X \rho)(Y) + \rho(\nabla^*_X Y) \Big) \times Z - \rho(Y) \times \nabla^*_X Z \\ &\quad - \rho(X) \times \nabla^*_Y Z + \rho(X) \times \Big(\rho(Y) \times Z \Big).\end{aligned}$$

Similarly,

$$\begin{aligned}\nabla_Y \nabla_X Z = \nabla^*_Y \nabla^*_X Z - \Big((\nabla^*_Y \rho)(X) + \rho(\nabla^*_Y X) \Big) \times Z - \rho(X) \times \nabla^*_Y Z \\ - \rho(Y) \times \nabla^*_X Z + \rho(Y) \times \Big(\rho(X) \times Z \Big).\end{aligned}$$

To compute $R(X, Y)Z$, suppose, for the sake of simplicity, that $[X, Y] = 0$. Then, using

- the torsion-freeness of ∇^* implying $\nabla^*_X Y - \nabla^*_Y X = 0$
- and the Jacobi identity saying that

$$\rho(X) \times \Big(\rho(Y) \times Z \Big) + \rho(Y) \times \Big(Z \times \rho(X) \Big) + Z \times \Big(\rho(X) \times \rho(Y) \Big) = 0$$

$$\rho(X) \times \Big(\rho(Y) \times Z \Big) - \rho(Y) \times \Big(\rho(X) \times Z \Big) = \Big(\rho(X) \times \rho(Y) \Big) \times Z,$$

we get the formula

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z =$$

$$R^*(X, Y)Z - (\nabla^*_X \rho)(Y) \times Z + (\nabla^*_Y \rho)(X) \times Z + \Big(\rho(X) \times \rho(Y) \Big) \times Z.$$

Since ∇ is flat, the formula for the curvature of the Levi-Civita connection follows immediately. $\square$

5.3.6. Remark. Note that the condition on the holonomy algebra (and the curvature) is not an 'if and only if' statement; obviously, the holonomy of a compatible linear connection can always be trivial, not just in the determined case. Also, to stress one final time, revolution surfaces with axes **not** going through the origin, belong to the determined case, and their holonomy is trivial! If it was at least 1-dimensional, by the argument of the proof, it would have another axis through the origin. On the other hand, it may be worth to think through what revolution axes Finslerian indicatrices can have in 3 dimensions.

5.4. The possible revolution axes of indicatrices in 3D

To take a break before starting the more demanding calculations, let's examine how many revolution axes a Finslerian indicatrix can have in 3 dimensions and what it means for the metric.

0 axes It is a general, not rotationally symmetric metric. The space is either not a GBM or admits a unique compatible ∇.

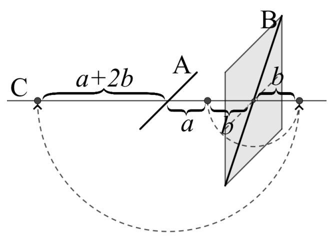
1 axis It is a rotationally symmetric metric, which – if it is a GBM – admits infinitely many compatible ∇s if and only if the axis goes through the origin.

2 axes For two lines A and B in $\mathbb{R}^3$, there are 3 possibilities:

- *they are parallel:* By applying two rotations by π around these axes, together they would yield a parallel translation. But for the indicatrix, having these lines as revolution axes would mean that it is itself translation-invariant, which is clearly impossible.

- *they are skew (not intersecting):* Let C be the line intersecting both axes orthogonally and consider the rotation by π around A followed by the rotation by π around B.

  This transformation leaves both the indicatrix and the line C invariant, but translates the points of C by twice the distance of the two axes. Thus, taking an intersection point of the indicatrix with C and iterating this transformation, we can take this point as far as we wish: the indicatrix is unbounded! This, also, is an impossible case.



- *they are intersecting at some point:* Translating everything to the origin, it's easy to see that the rotations around these axes generate the whole $SO(3)$, as you can reach any point on the Euclidean sphere if you are allowed to rotate around two different main circles. Thus the indicatrix contains, and therefore itself is, a translate of a Euclidean sphere. But then it has infinitely many axes, so this case is also impossible.

In summary, there is no Finslerian indicatrix with exactly 2 rotation axes.

3 or more axes By the above, among 3 axes, no pair can be parallel or skew to each other, so any pair must be intersecting. Taking such two axes, they generate the Euclidean sphere. All other lines through this intersection point are also axes, and they are the only ones. The metric is Riemannian if and only if the intersection point is the origin, and if not, it is a Randers space. Since one of the axes goes through the origin (connect it with the center), if it is a GBM, there must be infinitely many compatible connections (and there are, as we have seen it in Chapter 3).

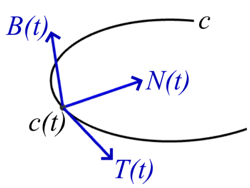
5.5. The geometry of spherical curves in 3 dimensions

After thoroughly examining the possible solutions of the CEQ, our next goal will be to explicitly determine them. For this, we are going to need the apparatus of elementary differential geometry; let's revise them quickly!

5.5.1. For a **smooth curve** $c\colon I \to \mathbb{R}^3$, we have the following notions.

- c' is called the **velocity** and c'' is the **acceleration** of the curve.
- c is **regular** if $c'(t) \neq \mathbf{0}$ anywhere, **biregular** if $(c'(t), c''(t))$ are linearly independent everywhere.
- $\|c'\|$ is called the **speed** of the curve, and its integral between two parameters is the **(arc) length**. The parametrization is called **natural** if the speed is $\|c'\| \equiv 1$ and the arc length between two points is just the difference of the parameters. From now on, curves with natural parametrization will be denoted here by $\widetilde{c}$.

5.5.2. At any point of a *biregular* curve c, $(c'(t), c''(t), c'(t) \times c''(t))$ constitutes a basis of $T_{c(t)}\mathbb{R}^3$, possibly changing from point to point; it is called a moving frame. From this, we can construct an orthonormal frame called the **Frenet frame**, whose basis vectors are



- the **unit tangent vector** $T(t) := \dfrac{1}{\|c'(t)\|} c'(t)$;
- the **unit normal vector** N, for which $(T(t), N(t))$ and $(c'(t), c''(t))$ have the same orientation and span the same plane;
- the **unit binormal vector** B which is the cross product $B = T \times N$.

The change of this frame is described by the **Frenet equations**

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \|c'\|\,\kappa & 0 \\ -\|c'\|\,\kappa & 0 & \|c'\|\,\tau \\ 0 & -\|c'\|\,\tau & 0 \end{bmatrix} \cdot \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{60}$$

where

- $\kappa\colon I \to \mathbb{R}$ is called the **curvature** of c; it is always positive and measures how much c deviates from lying in a straight line (being a line segment);
- $\tau\colon I \to \mathbb{R}$ is called the **torsion** of c and measures how much c deviates from lying in a plane (being a plane curve).

For the curvature and torsion, we have the formulas

$$\kappa = \frac{\|c' \times c''\|}{\|c'\|^3} \qquad \text{and} \qquad \tau = \frac{\det(c', c'', c''')}{\|c' \times c''\|^2}. \tag{61}$$

From the formula for the torsion, we can see that the following is true.

5.5.3. Lemma. *For any biregular curve c,*

$$c \text{ is a plane curve} \iff \tau \equiv 0 \iff \det(c', c'', c''') \equiv 0 \iff c''' \text{ is a linear combination of } c' \text{ and } c''.$$

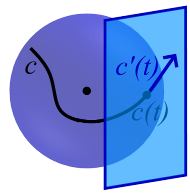
5.5.4. If $\widetilde{c}$ has natural parametrization, i.e. $\|\widetilde{c}'\| \equiv 1$, then

- $\widetilde{c}'' \perp \widetilde{c}'$ everywhere, because differentiating $\langle \widetilde{c}', \widetilde{c}' \rangle = 1$ gives $\langle \widetilde{c}'', \widetilde{c}' \rangle = 0$;
- $T = \widetilde{c}', \ N = \dfrac{\widetilde{c}''}{\|\widetilde{c}''\|}, \ B = \dfrac{\widetilde{c}' \times \widetilde{c}''}{\|\widetilde{c}''\|}$
- and for the curvature and torsion, we have the formulas

$$\kappa = \|\widetilde{c}''\| \qquad \text{and} \qquad \tau = \frac{\det(\widetilde{c}', \widetilde{c}'', \widetilde{c}''')}{\|\widetilde{c}''\|^2}. \tag{62}$$

5.5.5. In the remainder of the chapter, we will work with the integral curves of $C \times G$, which are running on the Euclidean spheres, and as such, are examples of spherical curves.

5.5.6. Definition. A **spherical curve** is a smooth curve $c \colon I \to \mathbb{R}^3$ for which $\|c(t)\|$ is constant ($\neq 0$), meaning that c runs on a Euclidean sphere with radius $\|c(t)\| > 0$, centered at the origin. As such, all its velocity vectors are contained in the corresponding tangent planes of this sphere.



5.5.7. Lemma. *For any spherical curve c, $c'(t) \perp c(t)$ everywhere.*

Proof 1. The velocity vector $c'(t)$ lies in the tangent plane of the sphere at $c(t)$; but all elements of this plane are orthogonal to the vector $c(t)$. □

Proof 2. Since $\|c(t)\|^2 = \langle c(t), c(t) \rangle =$ constant, by differentiating,

$$2 \langle c'(t), c(t) \rangle = 0.$$ □

5.5.8. Corollary. *For any **regular** spherical curve c, $c(t)$ and $c'(t)$ are linearly independent everywhere.*

5.5.9. Lemma. *For any **regular** spherical curve c, $c'(t)$ and $c''(t)$ are linearly independent everywhere.*

Proof. Differentiating the equation $\langle c'(t), c(t) \rangle = 0$ again, we have

$$\langle c''(t), c(t) \rangle + \langle c'(t), c'(t) \rangle = 0.$$

Rearranging this and using the regularity of the curve gives

$$\langle c''(t), c(t) \rangle = -\|c'(t)\|^2 \neq 0.$$

Geometrically, this means that the vectors $c(t)$ and $c''(t)$ cannot be orthogonal to each other (in fact, they enclose an angle greater than 90°), thus $c''(t)$ does not lie in the tangent plane of the sphere at $c(t)$. But $c'(t)$ does, so $c''(t)$ is not parallel to it, i.e. they are linearly independent. □

5.5.10. Corollary. *A **regular** spherical curve is automatically biregular, so $(c'(t), c''(t), c'(t) \times c''(t))$ constitute a basis at all of its points, and it has a Frenet frame.*

5.6. The solution vectors of the 3-dimensional CEQ in general

5.6.1. In this section, supposing the space is a GBM, we compute the solutions of the CEQ. To avoid the Riemannian case, we suppose that not all elements of $T_p^\circ M$ are vertically contact, consider only the open domain of not vertically contact elements[3] and the (non-trivial) integral curves of the (nowhere zero) vector field $C \times G$ over this domain. Let us choose a point v and denote by c the integral curve starting from it, that is,

$$c\colon I \to T_pM, \qquad c(0) = v, \qquad c'(t) = (C \times G)_{c(t)}.$$

As we have established it in Statement 5.3.2, c is a (regular) spherical curve running in the intersection of the Finslerian and Riemannian spheres.

5.6.2. Consider the equations (CEQ-3D-i) only over the points $v = c(t)$, where $C_v \times G_v = (C \times G)_{c(t)} = c'(t)$. Differentiating and using the fact that the unknown vectors do not depend on v, at $t = 0$ we get the two equations

$$\text{(D-CEQ-3D-}i\text{)} \qquad \left\langle c'(0), \overrightarrow{\boldsymbol{s}_i} \right\rangle = b_i(v), \qquad \left\langle c''(0), \overrightarrow{\boldsymbol{s}_i} \right\rangle = (b_i \circ c)'(0)$$

from each of them, where

$$b_i(v) = -2\, \widehat{\partial}_i F(v) = -2\, X_i^{h*} F(v), \qquad i \in \{1,2,3\},$$

and $c'(0)$ and $c''(0)$ are linearly independent, as shown in Lemma 5.5.9.

5.6.3. Statement (The solution vectors in terms of $c', c'', c' \times c''$). *Let c denote the integral curve of $C \times G$ starting from a not vertical contact element $v \in T_p^\circ M$. Then the solution vectors $\overrightarrow{\boldsymbol{s}_1}, \overrightarrow{\boldsymbol{s}_2}, \overrightarrow{\boldsymbol{s}_3}$ of (CEQ-3D-i) in T_pM, if there are any, are of the form*

$$\begin{bmatrix} \overrightarrow{\boldsymbol{s}_1} \\ \overrightarrow{\boldsymbol{s}_2} \\ \overrightarrow{\boldsymbol{s}_3} \end{bmatrix} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} \cdot \begin{bmatrix} c'(0) \\ c''(0) \\ c'(0) \times c''(0) \end{bmatrix}, \tag{63}$$

where the coefficients in the first two columns are uniquely determined:

- $\omega_{i1} = \dfrac{-2\, \widehat{\partial}_i F(v)}{\|c'(0)\|^2} + 2 \dfrac{\langle c'(0), c''(0) \rangle}{\|c'(0)\|^5\, \kappa^2(0)} \left(\dfrac{\widehat{\partial}_i F \circ c}{\|c'\|} \right)' (0), \qquad i \in \{1,2,3\},$
- $\omega_{i2} = \dfrac{-2}{\|c'(0)\|^3\, \kappa^2(0)} \left(\dfrac{\widehat{\partial}_i F \circ c}{\|c'\|} \right)' (0), \qquad i \in \{1,2,3\}.$

Here, κ denotes the curvature of c.

Proof. By Corollary 5.5.10, $(c'(t), c''(t), c'(t) \times c''(t))$ constitute a basis at all the points of c, in particular, at $c(0) = v$. We are looking for the coordinates of the solution vectors in this basis, i.e.

$$\overrightarrow{\boldsymbol{s}_i} = \omega_{i1}\, c'(0) + \omega_{i2}\, c''(0) + \omega_{i3}\, c'(0) \times c''(0), \qquad i \in \{1,2,3\}.$$

[3]At vertically contact elements, the left-hand side of the CEQ is zero, and supposing we have solutions, they must also be horizontally contact.

Plugging these back into the equations of (D-CEQ-3D-i), we get

$$\left.\begin{aligned} \omega_{i1}\left\|c'(0)\right\|^2 + \omega_{i2}\left\langle c'(0), c''(0)\right\rangle &= b_i(v) \\ \omega_{i1}\left\langle c''(0), c'(0)\right\rangle + \omega_{i2}\left\|c''(0)\right\|^2 &= (b_i\circ c)'(0) \end{aligned}\right\},$$

or, in matrix form,

$$\begin{bmatrix} \left\|c'(0)\right\|^2 & \left\langle c'(0), c''(0)\right\rangle \\ \left\langle c''(0), c'(0)\right\rangle & \left\|c''(0)\right\|^2 \end{bmatrix} \cdot \begin{bmatrix} \omega_{i1} \\ \omega_{i2} \end{bmatrix} = \begin{bmatrix} b_i(v) \\ (b_i\circ c)'(0) \end{bmatrix}.$$

Note that the coefficient matrix is the Gram matrix of the independent vectors $c'(0)$ and $c''(0)$, and as such, its determinant is the squared area of the parallelogram spanned by them, that is,

$$\left\|c'(0)\right\|^2 \cdot \left\|c''(0)\right\|^2 - \left\langle c'(0), c''(0)\right\rangle^2 = \left\|c'(0)\times c''(0)\right\|^2. \tag{64}$$

Multiplying by the inverse,

$$\begin{bmatrix} \omega_{i1} \\ \omega_{i2} \end{bmatrix} = \frac{1}{\left\|c'(0)\times c''(0)\right\|^2} \begin{bmatrix} \left\|c''(0)\right\|^2 & -\left\langle c'(0), c''(0)\right\rangle \\ -\left\langle c'(0), c''(0)\right\rangle & \left\|c'(0)\right\|^2 \end{bmatrix} \cdot \begin{bmatrix} b_i(v) \\ (b_i\circ c)'(0) \end{bmatrix}.$$

Basically, we are ready because we have expressed the coordinates, but to obtain the neater form presented in the Statement, let's work on them a bit more. First, observe that

$$D := \left(\frac{b_i\circ c}{\left\|c'\right\|}\right)'(0) = \frac{(b_i\circ c)'(0)\left\|c'(0)\right\|^2 - b_i(v)\left\langle c'(0), c''(0)\right\rangle}{\left\|c'(0)\right\|^3},$$

and rearranging it gives

$$(b_i\circ c)'(0) = \frac{D\left\|c'(0)\right\|^3 + b_i(v)\left\langle c'(0), c''(0)\right\rangle}{\left\|c'(0)\right\|^2}.$$

Plugging this back to the formulas for the coefficients obtained above, simplifying, using (64) and (61) for the curvature, we obtain

$$\begin{aligned} \omega_{i1} &= \frac{\left\|c''(0)\right\|^2 b_i(v)}{\left\|c'(0)\times c''(0)\right\|^2} - \left\langle c'(0), c''(0)\right\rangle \frac{D\left\|c'(0)\right\|^3 + b_i(v)\left\langle c'(0), c''(0)\right\rangle}{\left\|c'(0)\times c''(0)\right\|^2\left\|c'(0)\right\|^2} \\ &= \frac{\left(\left\|c'(0)\right\|^2\left\|c''(0)\right\|^2 - \left\langle c'(0), c''(0)\right\rangle^2\right) b_i(v)}{\left\|c'(0)\times c''(0)\right\|^2\left\|c'(0)\right\|^2} - \frac{\left\langle c'(0), c''(0)\right\rangle D}{\left\|c'(0)\right\|^5 \dfrac{\left\|c'(0)\times c''(0)\right\|^2}{\left\|c'(0)\right\|^6}} \\ &= \frac{b_i(v)}{\left\|c'(0)\right\|^2} - \frac{\left\langle c'(0), c''(0)\right\rangle D}{\left\|c'(0)\right\|^5 \kappa^2(0)} \end{aligned}$$

and

$$\omega_{i2} = \|c'(0)\|^2 \frac{D\,\|c'(0)\|^3 + b_i(v)\,\langle c'(0), c''(0)\rangle}{\|c'(0)\times c''(0)\|^2\,\|c'(0)\|^2} - \frac{\langle c'(0), c''(0)\rangle\, b_i(v)}{\|c'(0)\times c''(0)\|^2}$$
$$= \frac{D}{\|c'(0)\|^3\,\dfrac{\|c'(0)\times c''(0)\|^2}{\|c'(0)\|^6}} = \frac{D}{\|c'(0)\|^3\,\kappa^2(0)}.$$

Writing the value of D and $b_i(v)$ back, we obtain the desired formulas. □

This process can be repeated if we normalize the vector field $C\times G$ (and thus obtain the intersection curves with a natural parametrization) and use the Frenet frame.

5.6.4. Corollary (The solution vectors in the Frenet frame). *Let $\widetilde{c}$ denote the integral curve of the normalized vector field*

$$\frac{1}{\|C\times G\|}\,C\times G$$

starting from a not vertical contact element $v\in T_p^\circ M$. Then, using the Frenet frame (T,N,B) along $\widetilde{c}$, the solution vectors $\overrightarrow{\boldsymbol{s}_1}, \overrightarrow{\boldsymbol{s}_2}, \overrightarrow{\boldsymbol{s}_3}$ of (CEQ-3D-i) in T_pM, if there are any, are of the form

$$\begin{bmatrix}\overrightarrow{\boldsymbol{s}_1}\\ \overrightarrow{\boldsymbol{s}_2}\\ \overrightarrow{\boldsymbol{s}_3}\end{bmatrix} = \begin{bmatrix}\widetilde{\omega}_{11} & \widetilde{\omega}_{12} & \widetilde{\omega}_{13}\\ \widetilde{\omega}_{21} & \widetilde{\omega}_{22} & \widetilde{\omega}_{23}\\ \widetilde{\omega}_{31} & \widetilde{\omega}_{32} & \widetilde{\omega}_{33}\end{bmatrix}\cdot\begin{bmatrix}T(0)\\ N(0)\\ B(0)\end{bmatrix}, \tag{65}$$

where the coefficients in the first two columns are uniquely determined:

$$\widetilde{\omega}_{i1} = \widetilde{b}_i(v),\quad \widetilde{\omega}_{i2} = \frac{(\widetilde{b}_i\circ\widetilde{c})'(0)}{\kappa(0)},\quad \widetilde{b}_i = \frac{b_i}{\|C\times G\|} = \frac{-2\,\widehat{\partial}_i F}{\|C\times G\|},\quad i\in\{1,2,3\}.$$

Proof. The computations are the same as before, but much easier due to the orthonormality of the Frenet frame. First, by normalizing the equations of (D-CEQ-3D-i), we get

$$\left\langle \widetilde{c}\,'(0), \overrightarrow{\boldsymbol{s}_i}\right\rangle = \widetilde{b}_i(v),\qquad \left\langle \widetilde{c}\,''(0), \overrightarrow{\boldsymbol{s}_i}\right\rangle = (\widetilde{b}_i\circ\widetilde{c})'(0).$$

We are looking for the coordinates of the solution vectors in the Frenet basis:

$$\overrightarrow{\boldsymbol{s}_i} = \widetilde{\omega}_{i1}\,T(0) + \widetilde{\omega}_{i2}\,N(0) + \widetilde{\omega}_{i3}\,B(0),\qquad i\in\{1,2,3\}.$$

Plugging these back into the normalized equations, we get

$$\left.\begin{aligned}\widetilde{\omega}_{i1}\,\langle \widetilde{c}\,'(0), T(0)\rangle + \widetilde{\omega}_{i2}\,\langle \widetilde{c}\,'(0), N(0)\rangle &= \widetilde{b}_i(v)\\ \widetilde{\omega}_{i1}\,\langle \widetilde{c}\,''(0), T(0)\rangle + \widetilde{\omega}_{i2}\,\langle \widetilde{c}\,''(0), N(0)\rangle &= (\widetilde{b}_i\circ\widetilde{c})'(0)\end{aligned}\right\}.$$

As our vector field is normalized and the integral curve has a natural parametrization, $\widetilde{c}' = T$ and $\widetilde{c}'' = T'$. We can get the values of the inner products using the orthonormality and the Frenet equations in (60):

$$\begin{aligned} &\langle \widetilde{c}'(0), T(0)\rangle = \langle T(0), T(0)\rangle = 1, && \langle T(0), N(0)\rangle = 0, \\ &\langle \widetilde{c}''(0), T(0)\rangle = \langle T'(0), T(0)\rangle = 0, && \langle \widetilde{c}''(0), N(0)\rangle = \langle T'(0), N(0)\rangle = \kappa. \end{aligned}$$

From here, the coordinates are obtained easily. □

5.6.5. Remark. According to these results, the coefficients ω_{i1} and ω_{i2} (or $\widetilde{\omega}_{i1}$ and $\widetilde{\omega}_{i2}$) are uniquely determined and they are related to the curvature of the integral curve of (the normalized) $C \times G$ (in a more general sense: the derivatives of the integral curve up to order two). In other words, the determined part of the solution is lying in the osculating plane of the curve c (or $\widetilde{c}$) at the starting point. Therefore, in the undetermined case, the freedom appears in the coefficients ω_{i3} (or $\widetilde{\omega}_{i3}$), which are the coordinates belonging to the binormal direction (directional line).

Finally, we present a sufficient condition for the unicity of the solution.

5.6.6. Corollary (A sufficient condition for the uniqueness of the solution). *Let $\widetilde{c}$ denote the integral curve of the normalized vector field*

$$\frac{1}{\|C \times G\|} C \times G$$

starting from a not vertical contact element $v \in T_p^\circ M$. If $\widetilde{c}$ has a non-vanishing torsion at the starting point, then the solution vectors $\overrightarrow{\boldsymbol{s}_1}, \overrightarrow{\boldsymbol{s}_2}, \overrightarrow{\boldsymbol{s}_3}$ of (CEQ-3D-i) in T_pM, if there are any, are uniquely determined, and they can be expressed in the Frenet frame (T, N, B) along $\widetilde{c}$ as

$$\begin{bmatrix} \overrightarrow{\boldsymbol{s}_1} \\ \overrightarrow{\boldsymbol{s}_2} \\ \overrightarrow{\boldsymbol{s}_3} \end{bmatrix} = \begin{bmatrix} \widetilde{\omega}_{11} & \widetilde{\omega}_{12} & \widetilde{\omega}_{13} \\ \widetilde{\omega}_{21} & \widetilde{\omega}_{22} & \widetilde{\omega}_{23} \\ \widetilde{\omega}_{31} & \widetilde{\omega}_{32} & \widetilde{\omega}_{33} \end{bmatrix} \cdot \begin{bmatrix} T(0) \\ N(0) \\ B(0) \end{bmatrix}, \tag{66}$$

where

$$\widetilde{\omega}_{i1} = \widetilde{b}_i(v), \qquad \widetilde{\omega}_{i2} = \frac{(\widetilde{b}_i \circ \widetilde{c})'(0)}{\kappa(0)},$$

$$\widetilde{\omega}_{i3} = \frac{(\widetilde{b}_i \circ \widetilde{c})''(0)\,\kappa(0) - (\widetilde{b}_i \circ \widetilde{c})'(0)\,\kappa'(0) + \widetilde{b}_i(v)\,\kappa^3(0)}{\kappa^2(0)\,\tau(0)},$$

$$\widetilde{b}_i = \frac{b_i}{\|C \times G\|} = \frac{-2\,\widehat{\partial}_i F}{\|C \times G\|}, \quad i \in \{1, 2, 3\}.$$

Proof. We have already determined the values of the coordinates $\widetilde{\omega}_{i1}$ and $\widetilde{\omega}_{i2}$ in the previous Corollary. To get the missing ones, let us continue differentiating the CEQ. From (D-CEQ-3D-i), we obtain

$$\left\langle \widetilde{c}'(0), \overrightarrow{\boldsymbol{s}_i} \right\rangle = \widetilde{b}_i(v), \quad \left\langle \widetilde{c}''(0), \overrightarrow{\boldsymbol{s}_i} \right\rangle = (\widetilde{b}_i \circ \widetilde{c})'(0), \quad \left\langle \widetilde{c}'''(0), \overrightarrow{\boldsymbol{s}_i} \right\rangle = (\widetilde{b}_i \circ \widetilde{c})''(0).$$

By Lemma 5.5.3, the torsion of $\widetilde{c}$ being nonzero is equivalent to $(\widetilde{c}', \widetilde{c}'', \widetilde{c}''')$ being linearly independent, so these equations indeed determine the coefficients $\widetilde{\omega}_{i3}$ uniquely.

To express them, write

$$\overrightarrow{\boldsymbol{s}_i} = \widetilde{\omega}_{i1}\, T(0) + \widetilde{\omega}_{i2}\, N(0) + \widetilde{\omega}_{i3}\, B(0), \qquad i \in \{1, 2, 3\},$$

and plug this back to the third equation, to get

$$\widetilde{\omega}_{i1} \langle \widetilde{c}'''(0), T(0) \rangle + \widetilde{\omega}_{i2} \langle \widetilde{c}'''(0), N(0) \rangle + \widetilde{\omega}_{i3} \langle \widetilde{c}'''(0), B(0) \rangle = (\widetilde{b}_i \circ \widetilde{c})''(0). \tag{67}$$

Since $\widetilde{c}$ has a natural parametrization, we can use the formulas of Subsection 5.5.4. Differentiating $\langle \widetilde{c}'', \widetilde{c}' \rangle = 0$ yields

$$\langle \widetilde{c}''', \widetilde{c}' \rangle + \langle \widetilde{c}'', \widetilde{c}'' \rangle = 0 \implies \langle \widetilde{c}''', \widetilde{c}' \rangle = -\langle \widetilde{c}'', \widetilde{c}'' \rangle = -\kappa^2,$$

and by differentiating $\langle \widetilde{c}'', \widetilde{c}'' \rangle = \kappa^2$, we get

$$\langle \widetilde{c}''', \widetilde{c}'' \rangle = \kappa\, \kappa'.$$

With the help of these, we can compute the inner products as

$$\begin{aligned}
\langle \widetilde{c}'''(0), T(0) \rangle &= \langle \widetilde{c}'''(0), \widetilde{c}'(0) \rangle = -\kappa^2(0), \\
\langle \widetilde{c}'''(0), N(0) \rangle &= \left\langle \widetilde{c}'''(0), \frac{\widetilde{c}''(0)}{\|\widetilde{c}''(0)\|} \right\rangle = \frac{\kappa(0)\, \kappa'(0)}{\kappa(0)} = \kappa'(0), \\
\langle \widetilde{c}'''(0), B(0) \rangle &= \left\langle \widetilde{c}'''(0), \frac{\widetilde{c}'(0) \times \widetilde{c}''(0)}{\|\widetilde{c}''(0)\|} \right\rangle = \kappa(0)\, \tau(0).
\end{aligned}$$

Putting everything together, rearranging (67) yields

$$\begin{aligned}
\widetilde{\omega}_{i3} &= \frac{(\widetilde{b}_i \circ \widetilde{c})''(0) - \widetilde{\omega}_{i1} \langle \widetilde{c}'''(0), T(0) \rangle - \widetilde{\omega}_{i2} \langle \widetilde{c}'''(0), N(0) \rangle}{\langle \widetilde{c}'''(0), B(0) \rangle} \\
&= \frac{(\widetilde{b}_i \circ \widetilde{c})''(0) + \widetilde{b}_i(v) \kappa^2(0) - \dfrac{(\widetilde{b}_i \circ \widetilde{c})'(0)}{\kappa(0)} \cdot \kappa'(0)}{\kappa(0)\, \tau(0)}.
\end{aligned}$$

This is exactly the formula we wished to prove. □

5.6.7. Remark. The above result says that the two kinds of torsions we are concerned with, i.e. the torsion of the integral curves of the normalized version of $C \times G$ (from elementary differential geometry) and the torsion of the compatible linear connections we are looking for, are strongly related, apart from having the same name.

5.7. Compatible linear connections in the undetermined case

In the previous section, we computed the solution vectors of the CEQ, so what remains to determine are the torsion components of compatible linear connections themselves. We first deal with the undetermined case. Supposing there are solutions, we know that

- all vertical contact points of $T_p^\circ M$ are also horizontally contact,
- there are not vertically contact points and at these points together, the vectors $C_v \times G_v$ span 2 dimensions,
- the solution lines $\overrightarrow{\boldsymbol{s}_1}, \overrightarrow{\boldsymbol{s}_2}, \overrightarrow{\boldsymbol{s}_3}$ of (CEQ-3D-i) are three parallel lines in T_pM, whose common direction is the orthogonal complement of $\operatorname{span}(C \times G)$, or, equivalently, their direction is given by

$$B(0) = \frac{\widetilde{c}'(0) \times \widetilde{c}''(0)}{\|\widetilde{c}''(0)\|},$$

where B is the binormal vector of the integral curve $\widetilde{c}$ of the normalized field of $C \times G$, starting from a not vertically contact element $v = \widetilde{c}(0)$.

5.7.1. Let us compute the torsion components of the compatible linear connections. To do this, first introduce the notation

$$\widetilde{\partial}_i F := -\frac{\widehat{\partial}_i F}{\|C \times G\|}.$$

Going back to the equations (CEQ-3D-1), (CEQ-3D-2) and (CEQ-3D-3), we can write the solution lines in the form

$$\overrightarrow{\boldsymbol{s}_1} = \begin{bmatrix} -\boldsymbol{T}^3_{12} + \boldsymbol{T}^2_{13} + \boldsymbol{T}^1_{23} \\ -2\boldsymbol{T}^1_{13} \\ 2\boldsymbol{T}^1_{12} \end{bmatrix}(p) = 2\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} + 2s\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix},$$

$$\overrightarrow{\boldsymbol{s}_2} = \begin{bmatrix} 2\boldsymbol{T}^2_{23} \\ -\boldsymbol{T}^3_{12} - \boldsymbol{T}^2_{13} - \boldsymbol{T}^1_{23} \\ 2\boldsymbol{T}^2_{12} \end{bmatrix}(p) = 2\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} + 2t\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix},$$

$$\overrightarrow{\boldsymbol{s}_3} = \begin{bmatrix} 2\boldsymbol{T}^3_{23} \\ -2\boldsymbol{T}^3_{13} \\ \boldsymbol{T}^3_{12} + \boldsymbol{T}^2_{13} - \boldsymbol{T}^1_{23} \end{bmatrix}(p) = 2\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} + 2u\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix},$$

where

- the common directional line is

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} := B(0) = \frac{\widetilde{c}'(0) \times \widetilde{c}''(0)}{\|\widetilde{c}''(0)\|} = \frac{1}{\kappa(0)}\begin{bmatrix} \widetilde{c^2}'(0)\,\widetilde{c^3}''(0) - \widetilde{c^3}'(0)\,\widetilde{c^2}''(0) \\ \widetilde{c^3}'(0)\,\widetilde{c^1}''(0) - \widetilde{c^1}'(0)\,\widetilde{c^3}''(0) \\ \widetilde{c^1}'(0)\,\widetilde{c^2}''(0) - \widetilde{c^2}'(0)\,\widetilde{c^1}''(0) \end{bmatrix};$$

- the determined parts, using

$$
\begin{aligned}
\widetilde{\omega}_{i1}\, T(0) + \widetilde{\omega}_{i2}\, N(0) &= \widetilde{b}_i(v)\, \widetilde{c}\,'(0) + \frac{(\widetilde{b}_i \circ \widetilde{c})'(0)}{\kappa(0)} \frac{\widetilde{c}\,''(0)}{\kappa(0)} \\
&= 2\left(\widetilde{\partial}_i F(v)\, \widetilde{c}\,'(0) + \frac{(\widetilde{\partial}_i F \circ \widetilde{c})'(0)}{\kappa^2(0)} \widetilde{c}\,''(0) \right),
\end{aligned} \tag{68}
$$

are of the form

$$
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} := \frac{\widetilde{\omega}_{11}\, T(0) + \widetilde{\omega}_{12}\, N(0)}{2} = \widetilde{\partial}_1 F(v) \begin{bmatrix} \widetilde{c^1}'(0) \\ \widetilde{c^2}'(0) \\ \widetilde{c^3}'(0) \end{bmatrix} + \frac{(\widetilde{\partial}_1 F \circ \widetilde{c})'(0)}{\kappa^2(0)} \begin{bmatrix} \widetilde{c^1}''(0) \\ \widetilde{c^2}''(0) \\ \widetilde{c^3}''(0) \end{bmatrix},
$$

$$
\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} := \frac{\widetilde{\omega}_{21}\, T(0) + \widetilde{\omega}_{22}\, N(0)}{2} = \widetilde{\partial}_2 F(v) \begin{bmatrix} \widetilde{c^1}'(0) \\ \widetilde{c^2}'(0) \\ \widetilde{c^3}'(0) \end{bmatrix} + \frac{(\widetilde{\partial}_2 F \circ \widetilde{c})'(0)}{\kappa^2(0)} \begin{bmatrix} \widetilde{c^1}''(0) \\ \widetilde{c^2}''(0) \\ \widetilde{c^3}''(0) \end{bmatrix},
$$

$$
\begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} := \frac{\widetilde{\omega}_{31}\, T(0) + \widetilde{\omega}_{32}\, N(0)}{2} = \widetilde{\partial}_3 F(v) \begin{bmatrix} \widetilde{c^1}'(0) \\ \widetilde{c^2}'(0) \\ \widetilde{c^3}'(0) \end{bmatrix} + \frac{(\widetilde{\partial}_3 F \circ \widetilde{c})'(0)}{\kappa^2(0)} \begin{bmatrix} \widetilde{c^1}''(0) \\ \widetilde{c^2}''(0) \\ \widetilde{c^3}''(0) \end{bmatrix},
$$

- and the free parameters are

$$
s = \frac{1}{2}\widetilde{\omega}_{13}, \qquad t = \frac{1}{2}\widetilde{\omega}_{23}, \qquad u = \frac{1}{2}\widetilde{\omega}_{33}. \tag{69}
$$

Repetitive torsion components (with the upper index equal to one of the lower indices) can be expressed easily:

$$
\begin{aligned}
&\boldsymbol{T}^1_{13}(p) = -P_2 - sD_2, \qquad && \boldsymbol{T}^2_{23}(p) = Q_1 + tD_1, \qquad && \boldsymbol{T}^3_{23}(p) = R_1 + uD_1, \\
&\boldsymbol{T}^1_{12}(p) = P_3 + sD_3, && \boldsymbol{T}^2_{12}(p) = Q_3 + tD_3, && \boldsymbol{T}^3_{13}(p) = -R_2 - uD_2.
\end{aligned}
$$

For the diverse components (having 3 different indices and appearing in groups of 3) we have the linear system

$$
\begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}^3_{12} \\ \boldsymbol{T}^2_{13} \\ \boldsymbol{T}^1_{23} \end{bmatrix} (p) = 2 \begin{bmatrix} P_1 + sD_1 \\ Q_2 + tD_2 \\ R_3 + uD_3 \end{bmatrix}.
$$

The coefficient matrix here is invertible, and the solution is

$$
\begin{bmatrix} \boldsymbol{T}^3_{12} \\ \boldsymbol{T}^2_{13} \\ \boldsymbol{T}^1_{23} \end{bmatrix} (p) = \frac{1}{2} \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} 2 \begin{bmatrix} P_1 + sD_1 \\ Q_2 + tD_2 \\ R_3 + uD_3 \end{bmatrix} = \begin{bmatrix} -P_1 - Q_2 - sD_1 - tD_2 \\ P_1 + R_3 + sD_1 + uD_3 \\ -Q_2 - R_3 - tD_2 - uD_3 \end{bmatrix}.
$$

Let's summarize these results.

5.7.2. THEOREM (The solutions in the UDC case in 3D). *At a point p of a 3-dimensional GBM, under the assumptions of Subsection 2.2.7, if the vectors $C \times G$ span 2 dimensions, then the torsion components of the linear connections compatible to the Finsler metric F are*

$$\boxed{\begin{array}{c} \begin{aligned} \boldsymbol{T}^1_{13}(p) &= -P_2 - sD_2, & \boldsymbol{T}^2_{23}(p) &= Q_1 + tD_1, & \boldsymbol{T}^3_{23}(p) &= \quad R_1 + uD_1, \\ \boldsymbol{T}^1_{12}(p) &= \quad P_3 + sD_3, & \boldsymbol{T}^2_{12}(p) &= Q_3 + tD_3, & \boldsymbol{T}^3_{13}(p) &= -R_2 - uD_2, \end{aligned} \\ \begin{aligned} \boldsymbol{T}^3_{12}(p) &= -P_1 - Q_2 - sD_1 - tD_2, \\ \boldsymbol{T}^2_{13}(p) &= \quad P_1 + R_3 + sD_1 + uD_3, \\ \boldsymbol{T}^1_{23}(p) &= -Q_2 - R_3 - tD_2 - uD_3, \end{aligned} \end{array}}$$

where s, t, u are free parameters and all the other quantities are defined in Subsection 5.7.1. From these, we can retrieve the connections by (19).

5.7.3. Let's find **the extremal connection**, defined in Section 2.7.2, whose torsion has minimal norm among all the other compatible linear connections. Since we are working with orthonormal bases, this norm is just the squared sum of the components under the square root (see (28)).

5.7.4. REMARK. Before starting, let us note the following. We transformed the compatibility equations such that we got the torsion components from three lines of a 3-dimensional Euclidean space. If the components had been exactly the coordinates of these lines, minimizing their squared sum would just mean finding the closest elements of these lines to the origin. But the coordinates are linear combinations of the torsion components, so these choices would **not** give us the extremal connection, as we can easily check by a direct computation.

Using the formulas of Theorem 5.7.2, the norm of the torsion is

$$\begin{gathered} \|T_p\|^2 = (T^1_{12})^2(p) + (T^2_{12})^2(p) + (T^3_{12})^2(p) + (T^1_{13})^2(p) \\ +(T^2_{13})^2(p) + (T^3_{13})^2(p) + (T^1_{23})^2(p) + (T^2_{23})^2(p) + (T^3_{23})^2(p) = \\ as^2 + bt^2 + cu^2 + ds + et + fu + gst + hsu + itu + j, \end{gathered}$$

where the coefficients are

$$\begin{gathered} a := 2D_1^2 + D_2^2 + D_3^2, \qquad b := D_1^2 + 2D_2^2 + D_3^2, \qquad c := D_1^2 + D_2^2 + 2D_3^2, \\ d := 4D_1P_1 + 2D_1Q_2 + 2D_1R_3 + 2D_2P_2 + 2D_3P_3, \\ e := 2D_1Q_1 + 2D_2P_1 + 4D_2Q_2 + 2D_2R_3 + 2D_3Q_3, \\ f := 2D_1R_1 + 2D_2R_2 + 2D_3P_1 + 2D_3Q_2 + 4D_3R_3, \\ g := 2D_1D_2, \qquad h := 2D_1D_3, \qquad i := 2D_2D_3, \\ j := 2P_1^2 + P_2^2 + P_3^2 + Q_1^2 + 2Q_2^2 + Q_3^2 + R_1^2 + R_2^2 + 2R_3^2 + 2P_1Q_2 + 2P_1R_3 + 2Q_2R_3, \end{gathered}$$

using the notations introduced in Subsection 5.7.1. We will also need

$$\begin{aligned} k &:= D_1^2 + 2D_2^2 + 2D_3^2, & l &:= 2D_1^2 + D_2^2 + 2D_3^2, \\ m &:= 2D_1^2 + 2D_2^2 + D_3^2, & \|D\|^2 &:= D_1^2 + D_2^2 + D_3^2. \end{aligned}$$

Note that $\|D\|, a, b, c, k, l, m > 0$, because D is a nonzero vector.

Considering $\|T_p\|^2$ as a function of the variables s, t, u, we are looking for its global minimum point. To find the critical points, consider the system

$$\left.\begin{aligned}\frac{\partial \|T_p\|^2}{\partial s} &= 2as + d + gt + hu = 0\\ \frac{\partial \|T_p\|^2}{\partial t} &= 2bt + e + gs + iu = 0\\ \frac{\partial \|T_p\|^2}{\partial u} &= 2cu + f + hs + it = 0.\end{aligned}\right\}$$

It is a linear system for s, t, u, whose matrix form is

$$\begin{bmatrix} 2a & g & h \\ g & 2b & i \\ h & i & 2c \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} = \begin{bmatrix} -d \\ -e \\ -f \end{bmatrix}. \tag{70}$$

Obviously the Hessian H is the same at every point as the coefficient matrix, whose corner minors are

$$\begin{aligned} \Delta_1 &= 2a > 0,\\ \Delta_2 &= 4m\|D\|^2 > 0,\\ \det(H) = \Delta_3 &= 16\|D\|^6 > 0. \end{aligned}$$

This means that the matrix H is positive definite, and consequently, invertible. Thus we have only one critical point, which is a global minimum.

To determine this point, note that the inverse of the coefficient matrix is

$$H^{-1} = \frac{1}{8\|D\|^4} \begin{bmatrix} 2k & -g & -h \\ -g & 2l & -i \\ -h & -i & 2m \end{bmatrix},$$

so the parameters of the global minimum are

$$\begin{bmatrix} s^0 \\ t^0 \\ u^0 \end{bmatrix} = H^{-1} \begin{bmatrix} -d \\ -e \\ -f \end{bmatrix} = \frac{1}{8\|D\|^4} \begin{bmatrix} -2dk + eg + fh \\ dg - 2el + fi \\ dh + ei - 2fm \end{bmatrix}.$$

In summary, we have the following result.

5.7.5. Theorem (The extremal connection in the UDC case in 3D). *At a point p of a 3-dimensional GBM, under the assumptions of Subsection 2.2.7, if the vectors $C \times G$ span 2 dimensions, then the torsion components of the extremal compatible linear connection, given in the form as in Theorem 5.7.2, belong to the parameters*

$$s^0 = \frac{-2dk + eg + fh}{8\|D\|^4}, \quad t^0 = \frac{dg - 2el + fi}{8\|D\|^4}, \quad u^0 = \frac{dh + ei - 2fm}{8\|D\|^4},$$

with the quantities here defined in Subsection 5.7.3. From these, we can retrieve the extremal connection by formula (19).

5.8. Compatible linear connections in the determined case

Finally, we turn our attention to the determined case and compute the torsion components of the uniquely existing compatible linear connection. In this case,

- all vertical contact points of $T_p^\circ M$ must also be horizontally contact,
- there are not vertically contact points and at these points together, the vectors $C_v \times G_v$ span 3 dimensions,
- the solution spaces of (CEQ-3D-i) (the values of $\overrightarrow{\boldsymbol{s}_1}, \overrightarrow{\boldsymbol{s}_2}, \overrightarrow{\boldsymbol{s}_3}$) are just three points of T_pM.

5.8.1. Let $\widetilde{c}$ denote the integral curve of the normalized vector field

$$\frac{1}{\|C \times G\|} C \times G$$

starting from a not vertical contact element $v \in T_p^\circ M$ and (T, N, B) the Frenet frame along $\widetilde{c}$. Writing the solutions in the form

$$\begin{bmatrix} \overrightarrow{\boldsymbol{s}_1} \\ \overrightarrow{\boldsymbol{s}_2} \\ \overrightarrow{\boldsymbol{s}_3} \end{bmatrix} = \begin{bmatrix} \widetilde{\omega}_{11} & \widetilde{\omega}_{12} & \widetilde{\omega}_{13} \\ \widetilde{\omega}_{21} & \widetilde{\omega}_{22} & \widetilde{\omega}_{23} \\ \widetilde{\omega}_{31} & \widetilde{\omega}_{32} & \widetilde{\omega}_{33} \end{bmatrix} \cdot \begin{bmatrix} T(0) \\ N(0) \\ B(0) \end{bmatrix},$$

all the coefficients of the first two columns are uniquely determined (see Corollary 5.6.4), and the determined parts are

$$\Delta_v := \widetilde{\omega}_{i1}\, T(0) + \widetilde{\omega}_{i2}\, N(0) = 2\left(\widetilde{\partial}_i F(v)\, \widetilde{c}\,'(0) + \frac{(\widetilde{\partial}_i F \circ \widetilde{c})'(0)}{\kappa^2(0)} \widetilde{c}\,''(0)\right)$$

(see formula (68)), so we only have to compute the missing coefficients $\widetilde{\omega}_{i3}$ ($i \in \{1, 2, 3\}$).

Consider the non-zero binormal vector

$$B(0) = \frac{\widetilde{c}\,'(0) \times \widetilde{c}\,''(0)}{\|\widetilde{c}\,''(0)\|} = \frac{\widetilde{c}\,'(0) \times \widetilde{c}\,''(0)}{\kappa(0)}$$

at the starting point $v = \widetilde{c}(0)$ of the curve. This vector is perpendicular to both $\widetilde{c}\,'(0)$ and $\widetilde{c}\,''(0)$, and, in the undetermined case, all other values of $C \times G$ (because this is the rotation axis of the indicatrix, so all values of $C \times G$ lie in the plane perpendicular to it). But in the determined case, since the vectors $C \times G$ do not remain in this plane, there is some (not vertically contact) point $w \neq v$ in $T_p^\circ M$ such that

$$\langle C_w \times G_w, B(0) \rangle \neq 0.$$

So, considering (CEQ-3D-i) at this point w, writing the solution vectors in the above form, we get

$$\begin{gathered} \left\langle C_w \times G_w, \overrightarrow{\boldsymbol{s}_i} \right\rangle = b_i(w) \\ \langle C_w \times G_w, \Delta_v + \widetilde{\omega}_{i3} B(0) \rangle = b_i(w) \\ \langle C_w \times G_w, \Delta_v \rangle + \widetilde{\omega}_{i3} \langle C_w \times G_w, B(0) \rangle = b_i(w), \end{gathered}$$

so, by rearranging,

$$\begin{aligned}
\widetilde{\omega}_{i3} &= \frac{b_i(w) - \langle C_w \times G_w, \Delta_v\rangle}{\langle C_w \times G_w, B(0)\rangle} \\
&= \frac{b_i(w) - \langle C_w \times G_w, \widetilde{\omega}_{i1}\, T(0) + \widetilde{\omega}_{i2}\, N(0)\rangle}{\langle C_w \times G_w, B(0)\rangle} \\
&= -2\,\frac{\widehat{\partial}_i F(w) + \left\langle C_w \times G_w,\ \widetilde{\partial}_i F(v)\,\widetilde{c}\,'(0) + \dfrac{(\widetilde{\partial}_i F \circ \widetilde{c})'(0)}{\kappa^2(0)}\widetilde{c}\,''(0)\right\rangle}{\left\langle C_w \times G_w, \dfrac{\widetilde{c}\,'(0) \times \widetilde{c}\,''(0)}{\kappa(0)}\right\rangle}.
\end{aligned}$$

To obtain the torsion components of the uniquely existing compatible $\boldsymbol{\nabla}$, we just have to plug these into the formulas of Theorem 5.7.2, inside the parameters s, t, u. In detail, we have the following result.

5.8.2. THEOREM (The solution in the DC case in 3D). *At a point p of a 3-dimensional GBM, under the assumptions of Subsection 2.2.7, if the vectors $C\times G$ span 3 dimensions, then the torsion components of the uniquely determined compatible linear connection are given by substituting*

$$\boxed{\begin{gathered}
\widetilde{\omega}_{i3} = -2\,\frac{\widehat{\partial}_i F(w) + \left\langle C_w \times G_w,\ \widetilde{\partial}_i F(v)\widetilde{c}\,'(0) + \dfrac{(\widetilde{\partial}_i F \circ \widetilde{c})'(0)}{\kappa^2(0)}\widetilde{c}\,''(0)\right\rangle}{\left\langle C_w \times G_w, \dfrac{\widetilde{c}\,'(0) \times \widetilde{c}\,''(0)}{\kappa(0)}\right\rangle}, \\[2ex]
s = \frac{1}{2}\widetilde{\omega}_{13}, \qquad t = \frac{1}{2}\widetilde{\omega}_{23}, \qquad u = \frac{1}{2}\widetilde{\omega}_{33}
\end{gathered}}$$

to the formulas of Theorem 5.7.2, where

- *$\widetilde{c}$ is the integral curve of the normalized vector field*

$$\frac{1}{\|C \times G\|}\, C \times G,$$

starting at a not vertically contact point $v = \widetilde{c}(0)$ of $T_p^\circ M$,

- *κ is the curvature of $\widetilde{c}$,*
- *$w \neq v$ is a not vertically contact point in $T_p^\circ M$ where*

$$\langle C_w \times G_w, \widetilde{c}\,'(0) \times \widetilde{c}\,''(0)\rangle \neq 0$$

- *and* $\widetilde{\partial}_i F := -\dfrac{\widehat{\partial}_i F}{\|C \times G\|}$.

The connection itself can be retrieved by formula (19).

5.9. An example: 3-dimensional Randers spaces

To illustrate the solution process presented in this chapter, we apply it for the familiar Randers spaces introduced in Chapter 3. There we defined Randers metrics $F = \alpha + \beta$ and adapted normal coordinates for them around a fixed point p of the manifold. In 3D, in these coordinates the metric at p has the form

$$F(x,y) = \sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2} + \|\beta(x)\|\, y^3 = \|y\| + \|\beta(x)\|\, y^3,$$

and its indicatrix in T_pM is described by the equation (see (31))

$$(y^1)^2 + (y^2)^2 + \lambda\left(y^3 + \frac{\|\beta\|}{\lambda}\right)^2 = \frac{1}{\lambda} \qquad (\lambda = 1 - \|\beta\|^2).$$

This is an ellipsoid with center $[0, 0, -\|\beta\|/\lambda]$ that has the coordinate axes as its own axes. It is rotationally symmetric in y^1 and y^2, so Randers metrics must belong to the undetermined case (UDC) and have infinitely many compatible linear connections (if there are any). The directional line will be the rotation axis belonging to y^3 with direction vector $D = [0, 0, 1]$. Also, as a compatible Riemannian metric, we use the Riemannian part a of F.

5.9.1. The vector field $C \times G$ and its integral curves. Since the partial derivatives of F with respect to the directional coordinates are

$$\dot{\partial}_k F(x,y) = \frac{y^k}{\sqrt{(y^1)^2 + (y^2)^2 + (y^3)^2}} + \delta_k^3 \|\beta\| = \frac{y^k}{\|y\|} + \delta_k^3 \|\beta\|,$$

the Euclidean gradient vector field of F is given by

$$G_y = \frac{1}{\|y\|}\left[y^1, y^2, y^3\right] + [0, 0, \|\beta\|] = \frac{C_y}{\|y\|} + \|\beta\|\,[0, 0, 1]$$

From this, the cross product $C \times G$ is computed as

$$C_y \times G_y = \frac{C_y \times C_y}{\|y\|} + \|\beta\|\, C_y \times [0, 0, 1] = \|\beta\| \begin{vmatrix} \dot{\partial}_1 & \dot{\partial}_2 & \dot{\partial}_3 \\ y^1 & y^2 & y^3 \\ 0 & 0 & 1 \end{vmatrix} = \|\beta\| \begin{bmatrix} y^2 \\ -y^1 \\ 0 \end{bmatrix}.$$

Therefore, an element $v \in T_pM$ is vertical contact if and only if $v = [0, 0, v^3]$, i.e. the vertical contact points are the elements of the y^3 coordinate axis.

To simplify the formalism let us consider the vector field

$$\frac{C_y \times G_y}{\|\beta\|} = \begin{bmatrix} y^2 \\ -y^1 \\ 0 \end{bmatrix}$$

instead of the normalized version of $C \times G$. The integral curves of this field, starting from any not vertically contact element $v = [v^1, v^2, v^3] \in T_p^\circ M$, give the parametrization of the intersection of the Euclidean and Randers spheres. As the latter are of the form described above, these intersection curves are circles in planes parallel to the coordinate plane $[y^1, y^2]$. Thus, they and their derivatives are of the form

$$c(t) = \begin{bmatrix} r\sin(t+t_0) \\ r\cos(t+t_0) \\ v^3 \end{bmatrix}, \qquad c(0) = \begin{bmatrix} r\sin(t_0) \\ r\cos(t_0) \\ v^3 \end{bmatrix} = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = v,$$

$$c'(t) = \begin{bmatrix} r\cos(t+t_0) \\ -r\sin(t+t_0) \\ 0 \end{bmatrix}, \qquad c'(0) = \begin{bmatrix} r\cos(t_0) \\ -r\sin(t_0) \\ 0 \end{bmatrix} = \begin{bmatrix} v^2 \\ -v^1 \\ 0 \end{bmatrix} = \frac{C_v \times G_v}{\|\beta\|},$$

$$c''(t) = \begin{bmatrix} -r\sin(t+t_0) \\ -r\cos(t+t_0) \\ 0 \end{bmatrix}, \qquad c''(0) = \begin{bmatrix} -r\sin(t_0) \\ -r\cos(t_0) \\ 0 \end{bmatrix} = \begin{bmatrix} -v^1 \\ -v^2 \\ 0 \end{bmatrix}.$$

5.9.2. The compatibility equations. Remember (or see Subsection 3.5.3 in Chapter 3) that for Randers metrics, we have

$$\widehat{\partial}_i F = \widehat{\partial}_i \alpha + \widehat{\partial}_i \beta = \widehat{\partial}_i \beta = \widehat{\partial}_i (\beta_j y^j) = y^j\, \widehat{\partial}_i \beta_j,$$

and we can write the RHS of the CEQ in the form

$$2\,\|\beta\|\, y^j\, C_{j;i}, \qquad \text{where} \qquad C_{j;i}(x) := -\frac{\widehat{\partial}_i \beta_j}{\|\beta\|}(x).$$

Thus, for the solution vectors $\overrightarrow{\boldsymbol{s}_1}$, $\overrightarrow{\boldsymbol{s}_2}$, $\overrightarrow{\boldsymbol{s}_3}$ we have the equations

$$\text{(CEQ-3D-R-}i\text{)} \qquad \left\langle \frac{C \times G}{\|\beta\|}, \overrightarrow{\boldsymbol{s}_i} \right\rangle = 2\, y^j\, C_{j;i}, \qquad i \in \{1,2,3\}.$$

Considering them along the integral curves c of $C \times G / \|\beta\|$, they and their differentiated versions take the form

$$\langle c'(0), \overrightarrow{\boldsymbol{s}_i} \rangle = \left\langle \begin{bmatrix} y^2 \\ -y^1 \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{s}_i^1 \\ \boldsymbol{s}_i^2 \\ \boldsymbol{s}_i^3 \end{bmatrix} \right\rangle = \ y^2 \boldsymbol{s}_i^1 - y^1 \boldsymbol{s}_i^2 = \ 2\, y^j\, C_{j;i}, \qquad i \in \{1,2,3\},$$

$$\langle c''(0), \overrightarrow{\boldsymbol{s}_i} \rangle = \left\langle \begin{bmatrix} -y^1 \\ -y^2 \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{s}_i^1 \\ \boldsymbol{s}_i^2 \\ \boldsymbol{s}_i^3 \end{bmatrix} \right\rangle = -y^1 \boldsymbol{s}_i^1 - y^2 \boldsymbol{s}_i^2 = 2\,(c^j)'\, C_{j;i}, \qquad i \in \{1,2,3\}.$$

At not vertically contact points, none of the coefficients y^1 and y^2 are zero.

5.9.3. The directional line. The vectors $\overrightarrow{\boldsymbol{s}}$ of the common directional space are the solution set of the homogenized system

$$\left.\begin{aligned} \langle c'(0), \overrightarrow{\boldsymbol{s}} \rangle &= 0 \\ \langle c''(0), \overrightarrow{\boldsymbol{s}} \rangle &= 0 \end{aligned}\right\} \iff \left.\begin{aligned} y^2 \boldsymbol{s}^1 - y^1 \boldsymbol{s}^2 &= 0 \\ -y^1 \boldsymbol{s}^1 - y^2 \boldsymbol{s}^2 &= 0 \end{aligned}\right\} \iff \begin{bmatrix} y^2 & -y^1 \\ y^1 & y^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{s}^1 \\ \boldsymbol{s}^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

at any not vertically contact point. It's easy to see that at these points the determinant $(y^1)^2 + (y^2)^2$ of the coefficient matrix is nonzero, so the directional space is 1-dimensional and consists of the elements $\overrightarrow{\boldsymbol{s}} = [0, 0, \boldsymbol{s}^3]$ with $\boldsymbol{s}^3$ arbitrarily chosen. In other words, it is the y^3 axis with direction vector $D = [0, 0, 1]$.

5.9.4. The solution lines. We can write the CEQ in detail as

$$y^2\boldsymbol{s}^1_i - y^1\boldsymbol{s}^2_i = 2\left(C_{1;i}\,y^1 + C_{2;i}\,y^2 + C_{3;i}\,y^3\right), \qquad i \in \{1,2,3\}.$$

Comparing the coefficients of y^1, y^2 and y^3 on the two sides shows that the compatibility equations have solutions if and only if $C_{3;i} = 0$, which is equivalent to the perturbating term having constant length (see Theorem 3.7.1). Furthermore, the determined parts are $\boldsymbol{s}^1_i = 2C_{2;i}$ and $\boldsymbol{s}^2_i = -2C_{1;i}$.

5.9.5. The torsion components of compatible linear connections. To use the formulas of Theorem 5.7.2, we need to substitute

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} C_{2;1} \\ -C_{1;1} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} C_{2;2} \\ -C_{1;2} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \begin{bmatrix} C_{2;3} \\ -C_{1;3} \\ 0 \end{bmatrix}.$$

Because of $D_1 = D_2 = 0$, the following components are uniquely determined:

$$\boldsymbol{T}^1_{13}(p) = -P_2 - sD_2 = C_{1;1}, \qquad \boldsymbol{T}^2_{23}(p) = Q_1 + tD_1 = C_{2;2}$$
$$\boldsymbol{T}^3_{23}(p) = R_1 + uD_1 = C_{2;3}, \qquad \boldsymbol{T}^3_{13}(p) = -R_2 - uD_2 = C_{1;3},$$
$$\boldsymbol{T}^3_{12}(p) = -P_1 - Q_2 - sD_1 - tD_2 = C_{1;2} - C_{2;1}.$$

The freely chosen components can be expressed as

$$\boldsymbol{T}^1_{12}(p) = P_3 + sD_3 = s, \qquad \boldsymbol{T}^2_{12}(p) = Q_3 + tD_3 = t,$$
$$\boldsymbol{T}^2_{13}(p) = P_1 + R_3 + sD_1 + uD_3 = C_{2;1} + u,$$
$$\boldsymbol{T}^1_{23}(p) = -Q_2 - R_3 - tD_2 - uD_3 = C_{1;2} - u.$$

5.9.6. The torsion components of the extremal connection.

$$d = 4D_1P_1 + 2D_1Q_2 + 2D_1R_3 + 2D_2P_2 + 2D_3P_3 = 0,$$
$$e = 2D_1Q_1 + 2D_2P_1 + 4D_2Q_2 + 2D_2R_3 + 2D_3Q_3 = 0,$$
$$f = 2D_1R_1 + 2D_2R_2 + 2D_3P_1 + 2D_3Q_2 + 4D_3R_3 = 2D_3(P_1 + Q_2) = 2C_{2;1} - 2C_{1;2},$$
$$g = 2D_1D_2 = 0, \qquad h = 2D_1D_3 = 0, \qquad i = 2D_2D_3 = 0,$$
$$k = D_1^2 + 2D_2^2 + 2D_3^2 = 2, \qquad l = 2D_1^2 + D_2^2 + 2D_3^2 = 2,$$
$$m = 2D_1^2 + 2D_2^2 + D_3^2 = 1, \qquad \|D\|^2 = D_1^2 + D_2^2 + D_3^2 = 1,$$

imply, using Theorem 5.7.5, that the coefficients of the extremal compatible linear connection belong to the parameters

$$s^0 = \frac{-2dk + eg + fh}{8\,\|D\|^4} = 0, \qquad t^0 = \frac{dg - 2el + fi}{8\,\|D\|^4} = 0,$$
$$u^0 = \frac{dh + ei - 2fm}{8\,\|D\|^4} = \frac{4C_{1;2} - 4C_{2;1}}{8} = \frac{C_{1;2} - C_{2;1}}{2}.$$

Consequently, the free components take the values

$$\boldsymbol{T}^1_{12}(p) = s^0 = 0, \qquad \boldsymbol{T}^2_{13}(p) = C_{2;1} + u^0 = C_{2;1} + \frac{C_{1;2} - C_{2;1}}{2} = \frac{C_{1;2} + C_{2;1}}{2},$$
$$\boldsymbol{T}^2_{12}(p) = t^0 = 0, \qquad \boldsymbol{T}^1_{23}(p) = C_{1;2} - u^0 = C_{1;2} - \frac{C_{1;2} - C_{2;1}}{2} = \frac{C_{1;2} + C_{2;1}}{2}.$$

CHAPTER 6

Appendix

6.1. Summary in English

In differential geometry, two of the most important tools on manifolds are linear connections and metrics.

A **linear connection** ∇ defines a way to compare tangent vectors living in different tangent spaces at different points of the manifold, and thus enables us to differentiate one vector field along another. Defining a linear connection is actually equivalent to prescribing the rule for (linear) **parallel translation** from one tangent space to any other along any curve. A vector field along a curve is called **parallel** if it does not change along (the velocity vector field of) the curve.

To be able to take measurements on the manifold, we need to define a way to measure the length of tangent vectors first – this is the notion of a **metric** in the geometrical sense. A **Riemannian metric** γ is a smoothly varying family of inner products in the tangent spaces, with ellipsoids as indicatrices (unit spheres). In the case of a **Finsler metric** F, we have a Minkowski (not necessarily centrally symmetric) norm in each tangent space with general, strictly convex smooth bodies as indicatrices, which vary smoothly from point to point. A manifold endowed with a Riemannian/Finsler metric is called a **Riemannian/Finsler manifold**. Finsler manifolds can be considered as the direct generalizations of Riemannian manifolds, but they behave quite differently in many situations.

Given both a linear connection ∇ and a metric on the same manifold, ∇ gives us the notion of parallel vector fields, and the metric allows us to measure the length of the elements of these vector fields. It feels natural to require that if a vector field does not change (is parallel), then neither should its length. In this case, we say that ∇ is **metrical** to the metric (if it is Riemannian) or **compatible** to the metric (if it is Finslerian). The following questions arise naturally:

Q: *Given a metric, can we always find a linear connection ∇ metrical/compatible to it? If yes, how and how many? Is there a difference between the Riemannian and the Finslerian case?*

The Riemannian case is quite easy and has been solved for more than a century now. On a Riemannian manifold, there exists an infinite family of linear connections metrical to the Riemannian metric γ. These can be 'parameterized' by their torsion tensor (given by the so-called Koszul formula), i.e. prescribing any value to the torsion uniquely determines one member of this family. We usually consider the one with identically zero torsion, called the **Levi-Civita connection** of the metric.

Unfortunately, the Finslerian case is incomparably more difficult and is still unsolved today. In this dissertation, we were looking for the answers to the following questions:

1. *For a given Finsler metric on a manifold, is there always a linear connection compatible to it? If not, can we describe the 'good' Finsler metrics or at least give some necessary or sufficient conditions?*

2. *If a Finsler metric does admit compatible linear connections, how many are there and how can we describe them?*

3. *Can we distinguish a 'canonical' choice in the above case (like the Levi-Civita connection in Riemannian Geometry)?*

4. *What about manifolds? We know that any manifold admits a non-Riemannian Finsler metric; does it admit one with a compatible linear connection? If not, can we give some conditions or at least some obstructions based on the manifold's topological or analytic properties?*

In the literature, some of these questions are already answered (we have examples for Finsler metrics admitting 0, 1 or infinitely many compatible linear connections), but we also added new results to the list.

A Finsler manifold admitting a compatible linear connection ∇ is called a **generalized Berwald manifold**, or **GBM** for short. All Riemannian manifolds are special GBMs, but we exclude these and only consider non-Riemannian Finsler manifolds and GBMs. Two fundamental facts are used very often:

- The **rigidity property** says that a *connected* GBM is either Riemannian (everywhere) or non-Riemannian everywhere.
- Vincze proved in [**15**] that for any Finsler metric F there exists a Riemannian metric γ compatible to it, meaning that any linear connection compatible to F is also metrical to γ.

Let's introduce the notations

$$\widehat{\partial}_i := \frac{\partial}{\partial x^i}, \qquad \dot{\partial}_i := \frac{\partial}{\partial y^i}, \qquad f_{ij} = y^i \dot{\partial}_j F - y^j \dot{\partial}_i F.$$

Compatible $\boldsymbol{\nabla}$s to a given Finsler metric F are the solutions of the system of **compatibility equations**, or **CEQ** for short. With the Christoffel symbols $\boldsymbol{\Gamma}_{ij}^k$ as unknowns, these equations are

$$\widehat{\partial}_i F - y^j (\boldsymbol{\Gamma}_{ij}^k \circ \pi)\, \dot{\partial}_k F = 0, \qquad i \in \{1, \ldots, n\}. \tag{71}$$

Using the torsion components $\boldsymbol{T}_{ab}^c$ as the new unknowns, choosing a fixed compatible Riemannian metric γ and considering everything in the tangent space T_pM at a fixed point p only, with Riemannian normal coordinates around p, the form of the CEQ we investigated and solved is

$$\boxed{(72)\qquad \sum_{a}{}' \, 2 f_{ia} \boldsymbol{T}^i_{ia} + \sum_{a<b}{}' \, f_{ab} \left(\boldsymbol{T}^a_{ib} + \boldsymbol{T}^i_{ab} + \boldsymbol{T}^b_{ai} \right) = -2 \widehat{\partial}_i F, \qquad i \in \{1, \dots, n\}.}$$

Note two things:

- Compatible $\boldsymbol{\nabla}$s are uniquely determined by their torsion, so, after solving the CEQ for the $\boldsymbol{T}^c_{ab}$, the connections can be retrieved using formula (19).
- We stress one more time that these equations are considered for the elements $v \in T_p^\circ M := T_pM \setminus \{\mathbf{0}\}$ **at a fixed point** $p \in M$ **only**. For a fixed value of v, this is a linear system for the unknown numbers $\boldsymbol{T}^c_{ab}(p)$, whose possible values (as vectors) are from the vector space $\wedge^2 T_p^* M \otimes T_p M$ of dimension $\binom{n}{2} n$ (or most of the time, with some tricks, they can be considered as elements of T_pM itself). Thus, the solution set (after considering all values of v) is the intersection of affine subspaces, i.e. an affine subspace of $\wedge^2 T_p^* M \otimes T_p M$ (or sometimes T_pM). This is, of course, the solution only at one point, and solvability here does not imply the existence of global solutions, but considering this for any point of the manifold, we can learn much about those.

In [**20**], Vincze also introduces the notion of the **extremal compatible linear connection** (extremal connection from now on), which can be considered as a canonical choice for a compatible ∇ on a GBM (the Finslerian analogy of the Levi-Civita connection). It is defined as the unique compatible connection whose torsion has the minimal pointwise norm everywhere with respect to the so-called bundle metric induced by the compatible Riemannian metric γ (Definitions 28 and 2.7.4).

In all cases where we were able to solve the CEQ and find the extremal connection, we used the 'double' geometric structure of T_pM, consisting of the Finsler and Riemannian (Euclidean) metrics and and their normal vector fields G and C, respectively. We also needed the notion of vertical and horizontal contact points, introduced by Vincze in [**20**]:

	role in the CEQ	geometrical role
vertical contact points	the coefficients (the whole LHS) of the CEQ disappears	the Finslerian and Euclidean normal vectors are parallel
horizontal contact points	the RHS of the CEQ disappears (the CEQ becomes homogeneous)	the derivative of the Finsler function vanishes along the horizontal directions $\widehat{\partial}_i$

- Vertical contact points can serve as obstructions against the existence of solutions: they must also be horizontally contact to have solutions. If this is satisfied, they are otherwise useless, giving the equations $0 = 0$. If all points of T_pM are vertically contact, then F is Riemannian at p.
- At horizontally contact points, the CEQ becomes homogeneous. If all points of T_pM are horizontally contact, then $T \equiv 0$ is a trivial solution at p.

In Chapters 3, 4, 5 of the dissertation, we solved the CEQ and find the extremal connection in special cases.

In Chapter 3, we solved the problem in the special case of **Randers metrics**, i.e. Finsler metrics of the form

$$F(x, y) = \alpha(x, y) + \beta(x, y),$$

where α is a norm induced by some Riemannian metric a (**Riemannian part**) and β is a 1-form field (**perturbating part**) on the manifold M. To simplify the CEQ, a is used as the compatible Riemannian metric and we use **adapted normal coordinates**, meaning that the Riemannian normal coordinates around $p \in M$ are chosen in a way such that the first $n-1$ coordinate vectors span the kernel of β.

We gave a new proof for a previous result of Vincze (see [**17**]) about the existence of compatible $\boldsymbol{\nabla}$s on Randers spaces, which is equivalent to the perturbating term having constant length (Theorem 3.7.1). The main results of the chapter (from [**1**]) are the expression of the torsion components of all compatible linear connections on Randers spaces (if any) and the extremal connection among them (Theorems 3.6.7 and 3.8.1).

In Chapter 4, we investigated 2-dimensional generalized Berwald manifolds. It is well-known that 2-dimensional connected non-Riemannian Finsler manifolds (surfaces) either don't admit any compatible linear connections, or admit a unique one. In the first half of the chapter, we presented three different ways to prove this and to obtain formulas for this connection.

- The first method is to consider the CEQ directly:

$$\text{(CEQ-2D)} \qquad \left.\begin{array}{rcl} f_{21}\boldsymbol{T}^1_{12} & = & \widehat{\partial}_1 F \\ f_{21}\boldsymbol{T}^2_{12} & = & \widehat{\partial}_2 F \end{array}\right\}.$$

  From this, it can be seen that in the non-Riemannian case, if there is a solution, it must be unique and its torsion at p is

$$\boxed{\boldsymbol{T}^1_{12} = \frac{\widehat{\partial}_1 F(v)}{f_{21}(v)} \quad \text{and} \quad \boldsymbol{T}^2_{12} = \frac{\widehat{\partial}_2 F(v)}{f_{21}(v)}.} \tag{73}$$

- The second method uses classical tools of Finsler geometry and is only mentioned briefly (see [**5**]).
- The third method is based on the fact that, in 2D, the torsion of any linear connection is semi-symmetric, meaning that it can be written in the form

$$T(X, Y) = \rho(Y)X - \rho(X)Y \tag{74}$$

  for some differential 1-form ρ on the base manifold. This, together with the theorem of Vincze (from [**16**], for which we gave a new elementary proof in [4]) stating that any Finsler metric admits at most one such connection, yields the unicity result.

In the second half of the chapter, we presented the following new results/proofs from [2]. First we presented three versions of the divergence representation of the Gauss curvature (Theorem 4.5.2 and Corollary 4.5.3 and 4.5.5), which states that on a generalized Berwald surface the Gauss curvature of the compatible Riemannian metric is the divergence of the vector field dual to the 1-form coming from the semi-symmetric torsion of the uniquely existing compatible connection. By integrating this formula, we got a topological obstruction theorem: for any connected, compact and orientable surface, for it to admit a non-Riemannian generalized Berwald structure, its Euler characteristic must be zero (Theorem 4.6.3). This immediately ruled out the possibility of turning any compact, connected and orientable surface into a non-Riemannian GBM (with the 2-sphere among them), except for the torus. We also showed that the Euclidean and hyperbolic planes (the 2 model spaces along the 2-sphere) can also carry non-Riemannian generalized Berwald structures, and described this method (which also uses the divergence representation of the Gauss curvature).

In the last chapter, we considered the case of 3-dimensional generalized Berwald manifolds. In 3D, by regrouping the torsion components with respect to the coefficients, the CEQ can be written into the form

$$\big\langle C_v \times G_v, \overrightarrow{\boldsymbol{s}_i} \big\rangle = -2\,\widehat{\partial}_i F(v), \qquad i \in \{1,2,3\}, \tag{CEQ-3D-i}$$

i.e. 3 linear equations where

- the vector field $C \times G$ describes how F and γ are related:
  - ▶ $v \in T_p^{\circ}M$ is vertically contact $\iff$ $C_v \times G_v = \mathbf{0}$.
  - ▶ If v is not vertically contact, the nonzero vector $C_v \times G_v$ gives the direction of the line $\mathcal{F}_v \cap \mathcal{R}_v$. Consequently, the integral curves of the vector field $C \times G$ in $T_p^{\circ}M$ are always running in the intersection of the Finslerian and the Riemannian (Euclidean) spheres.
- the unknowns $\overrightarrow{\boldsymbol{s}_1}, \overrightarrow{\boldsymbol{s}_2}, \overrightarrow{\boldsymbol{s}_3}$ are vectors in T_pM, and a solution for these uniquely determines a solution for the torsion.

Since these equations only differ from each other in their inhomogeneous parts, we can handle them together by considering only the homogeneous version

$$\big\langle C_v \times G_v, \overrightarrow{\boldsymbol{s}} \big\rangle = 0. \tag{H-CEQ-3D}$$

So first, we consider the homogeneous version (H-CEQ-3D). Its solution set (the common **directional space**) is a linear subspace that is the orthogonal complement of $\operatorname{span}(C_v \times G_v)$ inside T_pM. The solution sets of the original, possibly inhomogeneous equations (CEQ-3D-1), (CEQ-3D-2), (CEQ-3D-3) of the CEQ (the **solution spaces**) are either empty or affine translates of the directional space (and therefore parallel to it and to each other). Therefore, the different cases for the solution depend on how many dimensions the vectors $C_v \times G_v$ can span, and we classified the possible cases based on this, and described the solution in each case. In more detail, the main results (from [**3**]) are the following.

For a connected non-Riemannian generalized Berwald manifold of dimension three, there are two possibilities (Theorem 5.3.5). If the vectors $C_v \times G_v$ span 2 dimensions, then there are infinitely many compatible linear connections, and the Finslerian indicatrices are Euclidean surfaces of revolution around (linear) axes generated by a globally well-defined (nowhere vanishing) vector field, whose values have length 1, are in the orthogonal complement of the span of $C_v \times G_v$ in each T_pM and are parallel (covariant constant) with respect to any compatible linear connection (undetermined case). If the vectors $C_v \times G_v$ span 3 dimensions, then there is a unique flat compatible linear connection (determined case).

We also solved the CEQ, using the integral curves of the vector field $C \times G$ (normalized). We described the torsion components of compatible linear connections in the undetermined case (Theorem 5.7.2) and the extremal connection among them (Theorem 5.7.5), and the uniquely existing connection in the determined case (Theorem 5.8.2).

6.2. Summary in Hungarian (Összefoglaló magyar nyelven)

A differenciálgeometria legfontosabb eszközei sokaságokon a lineáris konnexiók és a metrikák.

Egy ∇ **lineáris konnexió** lehetővé teszi a különböző pontokbeli érintőterekben lévő érintővektorok összehasonlítását, és így egy vektormező differenciálását egy másik mentén. Egy lineáris konnexió megadása a sokaságon ekvivalens a **párhuzamos eltolás** működésének előírásával minden egyes görbe mentén. Egy vektormezőt **párhuzamos**nak hívunk egy görbe mentén, ha nem változik a görbe (sebességvektormezője) mentén.

Hogy a sokaságon méréseket tudjunk végezni, az érintővektorok mérését kell először értelmeznünk – geometriailag ezt jelenti egy **metrika** megadása. Egy γ **Riemann-metrika** lényegében belső szorzatok egy simán változó családja az érintőtereken, melynek indikátrixai (egységgömbjei) ellipszoidok. Egy F **Finsler-metrika** minden érintőtéren egy Minkowski-normából (nem feltétlenül középpontosan szimmetrikus normából) áll, melyek simán változnak pontról pontra, és melyeknek indikátrixai általános, szigorúan konvex testek sima határai. Ha egy sokaságot egy Riemann- vagy Finsler-metrikával látunk el, akkor ezek együttesét **Riemann-/Finsler-sokaság**nak nevezzük. A Finsler-sokaságok a Riemann-sokaságok közvetlen általánosításai, de sok tekintetben különböznek tőlük.

Amennyiben egy sokaságon egyszerre rendelkezésünkre áll egy ∇ lineáris konnexió és egy metrika is, tekinthetjük a ∇-ra nézve párhuzamos vektormezőket, és ezek hosszát a metrika segítségével meg is mérhetjük. Természetes követelménynek érezzük, hogy ha egy vektormező párhuzamos (nem változik), akkor a hossza is legyen állandó a görbe mentén. Ez esetben azt mondjuk, hogy ∇ **metrikus** a (Riemann-)metrikához vagy **kompatibilis** a (Finsler-)metrikához. Rögtön felmerülhetnek bennünk a következő kérdések:

K: *Egy adott metrikához vajon mindig létezik hozzá metrikus/kompatibilis lineáris konnexió? Ha igen, vajon mennyi? Hogy lehet ezeket megtalálni? Van-e különbség a Riemann- és Finsler-metrikák esete között?*

A Riemann-eset nem túl nehéz, és már több mint egy évszázada megoldott. Egy Riemann-sokaságon végtelen sok metrikus lineáris konnexió létezik, amelyeket a torziótenzoruk segítségével "paraméterezhetünk" (ezt adja meg a Koszul-formula), azaz a torziónak bármilyen értéket előírva, az egyértelműen meghatároz egy metrikus konnexiót ezek közül. Többnyire azzal szokás dolgozni, amelyiknek torziója azonosan nulla, ezt hívjuk a sokaság **Lévi-Civita konnexió**jának.

Sajnos a Finsler-eset jóval bonyolultabb, és máig megoldatlan probléma. Ebben az esetben az alábbi kérdéseket tehetjük fel:

1. *Egy adott Finsler-metrikához vajon mindig létezik kompatibilis lineáris konnexió? Ha nem, tudunk valamilyen (legalább szükséges vagy elégséges) feltételt adni a létezésre?*

2. *Ha egy Finsler-metrikához léteznek kompatibilis lineáris konnexiók, vajon hány ilyen van, és hogyan tudjuk őket megtalálni?*

3. *A fenti esetben ki tudunk-e választani egy olyan kitüntetett konnexiót, mint a Riemann-esetben a Lévi-Civita konnexió?*

4. *Mi a helyzet, ha csak sokaságokat tekintünk? Minden sokaságon létezik (nem-Riemann) Finsler-metrika, de vajon van-e biztosan legalább egy, amelyhez létezik kompatibilis lineáris konnexió? Ha nem, tudunk-e valamilyen feltételt adni a létezésre a sokaság topologikus/analitikus tulajdonságai alapján?*

E kérdések egy részére a válaszok már ismertek a szakirodalomban (vannak példák olyan Finsler-sokaságokra, amelyeken 0, 1 vagy végtelen sok kompatibilis ∇ létezik). A disszertációban ezen eredmények listájához próbáltunk újabbakat hozzáadni.

Ha egy Finsler-sokaságon létezik egy ∇ kompatibilis lineáris konnexió, akkor **általánosított Berwald-sokaság**nak hívjuk (röviden **GBM**). Minden Riemann-sokaság triviális példa, így csak nem-Riemann Finsler-sokaságokat és GBM-eket vizsgálunk. Gyakran használjuk az alábbi elemi tulajdonságokat:

- A **merevségi tétel** szerint egy összefüggő GBM vagy minden pontban Riemann, vagy egyetlen pontban sem az.
- Minden F Finsler-metrikához létezik egy hozzá kompatibilis γ Riemann-metrika; ezt azt jelenti, hogy minden, F-hez kompatibilis lineáris konnexió metrikus γ-hoz (Vincze, [**15**]).

Vezessük be az alábbi jelöléseket:

$$\widehat{\partial}_i := \frac{\partial}{\partial x^i}, \qquad \dot{\partial}_i := \frac{\partial}{\partial y^i}, \qquad f_{ij} = y^i \dot{\partial}_j F - y^j \dot{\partial}_i F.$$

Egy adott F Finsler-metrikához kompatibilis $\boldsymbol{\nabla}$-kat az ún. **kompatibilitási egyenletek** (röviden **CEQ**) megoldásaiként kapjuk. A $\boldsymbol{\Gamma}_{ij}^k$ Christoffel-szimbólumokkal felírva, ezek az egyenletek az alábbiak:

$$\widehat{\partial}_i F - y^j (\boldsymbol{\Gamma}_{ij}^k \circ \pi)\, \dot{\partial}_k F = 0, \qquad i \in \{1, \dots, n\}. \tag{75}$$

Választva egy kompatibilis γ Riemann-metrikát, használhatjuk a T_{ab}^c torziókomponenseket ismeretlenként. A p pontbeli érintőteret rögzítve és áttérve p körüli Riemann-normálkoordinátákra, a CEQ alakja

$$\boxed{\sum_a{}' \, 2 f_{ia} \boldsymbol{T}_{ia}^i + \sum_{a<b}{}' \, f_{ab} \left(\boldsymbol{T}_{ib}^a + \boldsymbol{T}_{ab}^i + \boldsymbol{T}_{ai}^b \right) = -2 \widehat{\partial}_i F, \qquad i \in \{1, \dots, n\}.} \tag{76}$$

A megoldással kapcsolatban a következőket tudjuk.

- A kompatibilis $\boldsymbol{\nabla}$-kat a torziójuk egyértelműen meghatározza, így, ha megoldjuk a CEQ-et a torziókomponensekre, azokból visszanyerhetjük magukat a konnexiókat a (19) formula segítségével.
- A fenti egyenleteket **csak egy rögzített** $p \in M$ **pontbeli érintőtér** $v \in T_p^\circ M := T_pM \setminus \{\mathbf{0}\}$ **elemeire** tekintjük. Egy darab v-re egy lineáris egyenletrendszert kapunk $\boldsymbol{T}_{ab}^c(p)$ ismeretlen számokkal, melyeket vektorként tekintve, értékeik az $\binom{n}{2}n$-dimenziós $\wedge^2 T_p^*M \otimes T_pM$ vektortérből kerülhetnek ki (ügyes átalakítások után, a megoldásvektorok többnyire T_pM elemei lesznek). Így az összes v-re tekintett megoldás, mint affin alterek metszete, maga is egy affin altér $\wedge^2 T_p^*M \otimes T_pM$-ben (vagy T_pM-ben). Mivel csak egy rögzített pontban dolgozunk, az itteni megoldhatóság természetesen nem garantálja globális megoldások létezését, de a sokaság minden pontját sorra véve, így is sokat megtudhatunk azokról.

[**20**]-ban Vincze bevezette az extremális kompatibilis lineáris konnexiók fogalmát (mostantól röviden **extremális konnexió**), egy “kanonikus” kompatibilis konnexiót egy GBM esetén (a Riemann-sokaságok Lévi-Civita konnexiójának mintájára). Az extremális konnexió definíció szerint az az egyértelműen létező kompatibilis ∇, amelynek torziója minden pontban a lehető legkisebb hosszal rendelkezik a kompatibilis Riemann-metrika által indukált nyalábmetrikával mérve (l. a 28 és 2.7.4. Definíciókat).

Minden esetben, amikor meg tudtuk oldali a CEQ-et és megtalálni az extremális konnexiót, T_pM “kettős” geometriai struktúráját használtuk, mely az adott Finsler-metrikából és a hozzá kompatibilis Riemann-metrikából áll (mely utóbbi a hagyományos euklideszi metrikává válik normálkoordináták választása után), illetve ezek normálvektormezőit, melyeket G és C jelöl. Használtuk továbbá a vertikális és horizontális érintkezési pontok fogalmát, melyeket Vincze definiált [**20**]-ban.

	szerepe a CEQ-ben	geometriai jelentés
vertikális érintkezési pont	az együtthatók (a teljes bal oldal) eltűnik	a Finsler- és Riemann-normálvektorok párhuzamosak
horizontális érintkezési pont	a CEQ jobb oldala eltűnik (homogén lesz)	F deriváltja eltűnik a $\widehat{\partial}_i$ horizontális irányok mentén

- A vertikális érintkezési pontok meggátolhatják a megoldások létezését, amennyiben nem horizontális érintkezési pontok is egyszerre. Ha azok, akkor azonban feleslegesek, mert itt a CEQ a $0 = 0$ egyenletet adja. Ha T_pM minden eleme vertikális érintkezési pont, akkor F Riemann p-ben.
- Horizontális érintkezési pontokban a CEQ homogén. Ha T_pM minden eleme horizontális érintkezési pont, akkor $T \equiv 0$ triviális megoldás p-ben.

A disszertáció 3., 4. és 5. fejezeteiben megoldottuk a CEQ-et és megkerestük az extremális konnexiót különböző speciális esetekben.

A 3. fejezetben megoldottuk a fenti problémákat Randers-metrikák esetén. Ezek olyan Finsler-metrikák, melyek alakja

$$F(x,y) = \alpha(x,y) + \beta(x,y),$$

ahol α egy a Riemann-metrikából származó normafüggvény (**Riemann rész**) és β egy kovektormező (**perturbáló rész**). A CEQ egyszerűsítése végett a tölti be a kompatibilis Riemann-metrika szerepét, és **adaptált normálkoordinátáka**t, azaz olyan Riemann-normálkoordinátákat használunk p körül, amelyben az első $n-1$ koordinátavektor feszíti ki β magterét.

A fejezetben új bizonyítást adtunk Vincze egy korábbi eredményére (l. [**17**]) kompatibilis $\boldsymbol{\nabla}$-k létezéséről Randers-tereken, ami azzal ekvivalens, hogy a perturbáló kovektormező hossza állandó (3.7.1 Tétel). A fejezet fő eredményei ([**1**] alapján) az összes kompatibilis lineáris konnexió meghatározása a torziójuk segítségével, köztük az extremális konnexióé (3.6.7 és 3.8.1 Tételek).

A 4. fejezetben 2-dimenziós GBM-eket vizsgáltunk. Ismert, hogy minden 2-dimenziós, összefüggő, nem-Riemann Finsler-sokaságon (felületen) vagy nem létezik egyetlen kompatibilis $\boldsymbol{\nabla}$ sem, vagy pontosan egy létezik. A fejezet első felében három különböző módszert mutattunk be ennek igazolására és a konnexió meghatározására.

- Az első módszer az, hogy magát a CEQ-et vizsgáljuk:

$$\text{(CEQ-2D)} \qquad \left.\begin{aligned} f_{21}\boldsymbol{T}^1_{12} &= \widehat{\partial}_1 F \\ f_{21}\boldsymbol{T}^2_{12} &= \widehat{\partial}_2 F \end{aligned}\right\}.$$

  Ebből egyszerűen látható, hogy a nem-Riemann esetben, ha van megoldás, akkor az egyértelmű és a következő alakú p-ben:

$$\text{(77)} \qquad \boxed{\boldsymbol{T}^1_{12} = \frac{\widehat{\partial}_1 F(v)}{f_{21}(v)} \quad \text{and} \quad \boldsymbol{T}^2_{12} = \frac{\widehat{\partial}_2 F(v)}{f_{21}(v)}.}$$

- A második módszer klasszikus Finsler-geometriai eszközöket használ, ezeket csak megemlítettük (l. [**5**]).
- A harmadik módszer alapja, hogy 2D-ben minden lineáris konnexió torziója szemiszimmetrikus, azaz

$$\text{(78)} \qquad T(X,Y) = \rho(Y)X - \rho(X)Y$$

  alakban írható valamilyen ρ kovektormező segítségével a sokaságon. Ebből, Vincze azon eredményével együtt ([**16**]-ből, melyre új, elemi bizonyítást adtunk [**4**]-ben), mely szerint minden Finsler-metrikához legfeljebb egy ilyen lineáris konnexió létezik, rögtön következik az egyértelműség.

A fejezet második felében az alábbi új eredményeket/bizonyításokat mutattuk be [2]-ből. Először bemutattuk a Gauss-görbület divergencia-reprezentációjának három különböző változatát (4.5.2 Tétel és 4.5.3 és 4.5.5 Következmények), mely szerint egy általánosított Berwald-felületen a kompatibilis Riemann-metrika Gauss-görbülete éppen annak a vektormezőnek a divergenciája, amelynek duális kovektormezője az egyértelműen létező kompatibilis konnexió szemiszimmetrikus torziójában szerepel. Ezen formula integrálásával az alábbi topológiai obstrukciós eredményt nyertük: hogy egy összefüggő, kompakt és irányítható felületen létezzen nem-Riemann általánosított Berwald-struktúra, ahhoz a felület Euler-karakterisztikájának zérusnak kell lennie (4.6.3 Tétel). Ez rögtön kizárja, hogy a kompakt, összefüggő és irányítható felületeken (közöttük a gömbön) létezhessen nem-Riemann általánosított Berwald-struktúra, kivéve a tóruszt. Megmutattuk továbbá, hogy az euklideszi és hiperbolikus síkokon (a 2 modelltéren a gömb mellett) szintén létezik nem-Riemann általánosított Berwald-struktúra, és megadtuk ezek konstrukcióját (amely szintén a Gauss-görbület divergencia-reprezentációjára támaszkodik).

Az utolsó fejezetben a 3-dimenziós általánosított Berwald-sokaságok esetét vizsgáltuk. 3D-ben, a változók újracsoportosításával az együtthatóik szerint, a CEQ az alábbi alakban írható:

$$\left\langle C_v \times G_v, \overrightarrow{\boldsymbol{s}_i} \right\rangle = -2\,\widehat{\partial}_i F(v), \qquad i \in \{1,2,3\}, \tag{CEQ-3D-i}$$

azaz 3 lineáris egyenletet kapunk, ahol

- a $C \times G$ vektormező az F és γ metrikák kapcsolatát írja le:
  - ▶ $v \in T_p^\circ M$ vertikális érintkezési pont $\iff$ $C_v \times G_v = \mathbf{0}$.
  - ▶ Ha v nem vertikális érintkezési pont, akkor a nemzérus $C_v \times G_v$ vektor az $\mathcal{F}_v \cap \mathcal{R}_v$ egyenes irányát adja. Következésképpen, $C \times G$ integrálgörbéi $T_p^\circ M$-ben mindig a Finsler- és Riemann-gömbök metszetében futnak.
- az $\overrightarrow{\boldsymbol{s}_1}, \overrightarrow{\boldsymbol{s}_2}, \overrightarrow{\boldsymbol{s}_3}$ ismeretlenek pedig T_pM-beli vektorok, és minden megoldás ezekre egyértelműen meghatároz egy megoldást a torziókomponensek értékeire is.

Mivel ezek az egyenletek csak a jobb oldali inhomogén részükben különböznek egymástól, együtt kezelhetjük őket, ha csak a homogén részüket tekintjük:

$$\left\langle C_v \times G_v, \overrightarrow{\boldsymbol{s}} \right\rangle = 0. \tag{H-CEQ-3D}$$

Tehát először meg kell oldanunk a (H-CEQ-3D) homogén verziót, melynek megoldáshalmaza (a közös **iránytér**) egy lineáris altér, méghozzá span$(C_v \times G_v)$ ortogonális komplementere T_pM-ben. Az eredeti, esetlegesen inhomogén (CEQ-3D-1), (CEQ-3D-2), (CEQ-3D-3) egyenletek megoldáshalmazai (a **megoldásterek**) vagy üresek, vagy az iránytér affin eltoltjai (és így azzal és egymással is párhuzamosak). Tehát a lehetséges megoldások típusa lényegében attól függ, hány dimenziót feszítenek ki a $C_v \times G_v$ vektorok együttesen, és ez alapján leírtuk a lehetséges különböző eseteket, és azok mindegyikében a konkrét megoldást. A fő eredmények ([**3**]-ból) a következők.

Egy 3-dimenziós összefüggő, nem-Riemann általánosított Berwald-sokaság esetén az alábbi esetek lehetségesek (5.3.5 Tétel). Ha a $C_v \times G_v$ vektorok 2 dimenziót generálnak, akkor végtelen sok kompatibilis lineáris konnexió létezik, és a Finsler-indikátrixok euklideszi forgásfelületek, melyek (lineáris) tengelyeinek irányát egy globálisan jóldefiniált (sehol sem eltűnő) vektormező adja, melynek elemei konstans 1 hosszúak, minden T_pM-ben $\mathrm{span}(C_v \times G_v)$ ortogonális komplementerében találhatók és bármely kompatibilis ∇-ra nézve párhuzamosak (határozatlan eset). Ha a $C_v \times G_v$ vektorok 3 dimenziót generálnak, akkor egyértelműen létezik egy kompatibilis lineáris konnexió zérus görbülettel (határozott eset).

A fejezet második részében megoldottuk a CEQ-et, a $C \times G$ vektormező (illetve annak normalizált verziójának) integrálgörbéit felhasználva. Meghatároztuk a kompatibilis lineáris konnexiók torzióját a határozatlan esetben (5.7.2 Tétel), köztük az extremális konnexióval (5.7.5 Tétel), illetve a határozott esetben létező egyetlen kompatibilis ∇-t (5.8.2 Tétel).

6.3. List of Symbols and Abbreviations

not.	meaning and most common use	p.
a, α	Riemannian part of a Randers metric	48
β	perturbating part (1-form) of a Randers metric	48
c	curve in a manifold M	–
C	the normal vector field of the Riemannian (Euclidean) spheres in T_pM	36, 37
$C^\infty(M)$	the algebra of smooth real-valued functions on a manifold M	2
CEQ	compatibility equations	26
CEQ-2D	compatibility equations in dimension 2	62
CEQ-3D	compatibility equations in dimension 3	88
CEQ-SS	compatibility equations for linear connections with semi-symmetric torsion	68
$C_{j;i}$	$C_{j;i}(x) = -\widehat{\partial}_i\beta_j / \left\|\beta\right\|(x)$ for a Randers metric $F = \alpha + \beta$	56
∂_i	coordinate vectors (partial derivatives) for the coordinates $u^1, \ldots, u^n$ on the manifold: $\partial_i := \partial/\partial u^i$	2
$\widehat{\partial}_i$	coordinate vectors (partial derivatives) for the coordinates $x^1, \ldots, x^n$ on TM: $\widehat{\partial}_i := \partial/\partial x^i$	2
$\dot{\partial}_i$	coordinate vectors (partial derivatives) for the coordinates $y^1, \ldots, y^n$ on TM: $\dot{\partial}_i := \partial/\partial y^i$	2
$\widetilde{\partial}_i F$	$\widetilde{\partial}_i F := -\widehat{\partial}_i F / \left\|C \times G\right\|$	103
$du^1, \ldots, du^n$	coordinate 1-form fields on a maifold	7
E	energy function of a Finsler metric F	16
F	Finsler metric	16
$\mathcal{F}_v$	the tangent hyperplane of the Finslerian sphere passing through $v \in T_p^\circ M$	36
f_{ab}	$f_{ab} = y^a\dot{\partial}_b F - y^b\dot{\partial}_a F$ (coefficients in the CEQ)	29
f_i	$f_i := [f_{i1}, f_{i2}, \ldots, f_{in}]$ (coefficient vector fields)	37
G	$G := \mathrm{grad}(F)$ (the normal vector field of the Finslerian spheres in T_pM)	36, 37

not.	meaning and most common use	p.
g_{ij}, g^{ij}	components (and their inverse) of a general Riemannian metric / the fundamental tensor of a Finsler metric F	16
$\gamma, \gamma_{ij}, \gamma^{ij}$	the compatible Riemannian metric (and its components and their inverse) to a Finsler metric F	24, 27
Γ_{ij}^k	Christoffel symbols of a linear connection ∇	4
Γ_{ij}^{k*}	Christoffel symbols of the Levi-Civita connection ∇^* of the compatible Riemannian metric γ	27
GBM	generalized Berwald manifold	23
H	$H := \left[X_1^{h*}F, \dots, X_n^{h*}F\right] = \left[\widehat{\partial}_1 F, \dots, \widehat{\partial}_n F\right]$ (the homogeneity vector field)	37
H-CEQ-…	the homogeneous version of the CEQ	68, 88
$\mathrm{Hol}(\nabla)$	holonomy group of the connection ∇	10
$\mathrm{Hol}^0(\nabla)$	restricted holonomy group of the connection ∇	12
$\mathfrak{hol}(\nabla)$	holonomy algebra of the connection ∇	12
κ	curvature of a curve c (Chp. 5)	96
κ^*	Gauss curvature of a (compatible) Riemannian metric (Chp. 4)	70
M	(smooth) manifold of dimension n	1
∇	linear connection on a manifold (usually compatible to a given Finsler metric F)	4
∇_c	linear connection along a curve c	5
∇^*	the Levi-Civita connection of the compatible Riemannian metric γ	27
$p, q, \dots$	points of a manifold	–
φ	parallel translation of tangent vectors (or tensors) along a curve in a manifold	5, 9
π	the footpoint projection in the tangent bundle	2
R, R_{ijk}^l	the curvature tensor and its components of a linear connection ∇	14
R^*, R_{ijk}^{l*}	the curvature tensor and its components of a (compatible) Riemannian metric	70

not.	meaning and most common use	p.
ρ	the 1-form in a semi-symmetric torsion tensor (usually of a compatible linear connection)	66
$\sigma^c_{ab;i}$	coefficients in the CEQ	29
$\mathcal{R}_v$	the tangent hyperplane of the Riemannian (Euclidean) sphere passing through $v \in T^\circ_p M$	36
τ	torsion of a curve c (Chp. 5)	96
T, T^c_{ab}	the torsion tensor and its components of a linear connection ∇	13
$\boldsymbol{T}^c_{ab}$	the unknown torsion components of compatible linear connections	26
$\mathcal{T}^k_l(\mathfrak{X}(M))$	(k,l)-type tensor field on a manifold M	7
$\mathcal{T}^k_l(T_pM)$	(k,l)-type tensor on T_pM	7
T_pM	the tangent space of a manifold M at a point p	2
$T^\circ_p M$	$T^\circ_p M := T_pM \setminus \{\mathbf{0}\}$	37
TM	tangent manifold of a manifold M	2
(U,u)	local chart on a manifold	1
$u^1,\dots,u^n$	local coordinates on a manifold	2
v, v_p, w, w_p	tangent vectors at a point p of a manifold	2
X, Y, Z	vector fields on a manifold	–
$x^1,\dots,x^n$	induced local location coordinates on TM	2
X^h_i	horizontal vector fields of a linear connection ∇ (usually compatible to F)	26
X^{h*}_i	the horizontal vector fields of the compatible Riemannian metric	27
$\mathfrak{X}(M)$	the set (module) of vector fields on a manifold M	4, 7
$\mathfrak{X}^*(M)$	the set (module) of 1-form fields on a manifold M	7
$y^1,\dots,y^n$	induced local direction coordinates on TM	2

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