A geometrical characterization of commutative positive operator valued measures

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We show that a POV measure F on the Borel σ-algebra of the reals $\mathcal{B}(\mathbb{R})$ is commutative if and only if there exists a PV measure E on $\mathcal{B}(\mathbb{R})$ and, for every λ in the spectrum of E, a probability measure $\gamma_{(\cdot)}(\lambda)$ on $\mathcal{B}(\mathbb{R})$ such that the effect $F(\Delta)$ coincides with $\gamma_\Delta(A)$, where A is the self-adjoint operator associated to E. The relevance of this result to the theory of the sharp reconstruction is analyzed.
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I. INTRODUCTION

POV (positive operator valued) measures have been used to define generalized observables in quantum mechanics,[1–8] and are a consequence of the probabilistic structure of quantum mechanics. Their introduction can also be justified by the analysis of some ideal experiments which show that there are physical events which cannot be described by projection operators.[6] POV measures were also used to generalize Mackey's imprimitivity theorem[8] and to study the problem of the joint measurement of incompatible observables.[9,10]

As shown in Refs. 1–3 there exists a one-to-one correspondence between POV measures and affine maps $S \mapsto \mu_S^F(\cdot)$ from the set of states $\mathcal{S}$ into the set of probability measures on $\mathcal{B}(\mathbb{R})$. Moreover, this correspondence is determined by the relation $\mu_S^F(\Delta)=\mathrm{Tr}[SF(\Delta)]$. This allows one to interpret the number

$$\mu_S^F(\Delta) = \mathrm{Tr}[SF(\Delta)]$$

as the probability that the outcomes of a measurement of the observable F (corresponding to a POV measure F) is in Δ when the physical system is in the state $S \in \mathcal{S}$. We recall that an analogous relation holds for standard observables $E:\mathcal{B}(\mathbb{R}) \to \mathcal{E}(\mathcal{H})$, that is:

$$\mu_S^E(\Delta) = \mathrm{Tr}[SE(\Delta)].$$

Therefore POV measures are a generalization of the standard quantum observables and the problem of giving them a clear physical meaning should be dealt with. In order to address this problem several characterizations of commutative POV measures were found.[11–16]

In Ref. 13 it is proved that for each commutative POV measure F, there exists a unique Baire probability measure $\mu:\mathcal{P}(\mathbb{R},\mathcal{A}(F)) \to [0,1]$ [defined on the space of all the PV measures with values in the commutative von Neumann algebra $\mathcal{A}(F)$ generated by the set $\{F(\Delta)\}_{\Delta\in\mathcal{B}(\mathbb{R})}$] such that $F(\Delta)=\int_{\mathcal{P}(X,\mathcal{A}(F))} E(\Delta)\,\mathrm{d}\mu(E)$.

Naimark theorem[16] establishes that every POV measure F in a Hilbert space $\mathcal{H}$ can be dilated to a PV measure E^+ in an extended Hilbert space $\mathcal{H}^+$ such that F is the projection of E^+ on $\mathcal{H}$.

In Refs. 11 and 12 it is shown that: if $F:\mathcal{B}(\mathbb{R}) \to \mathcal{F}(\mathcal{H})$ is a commutative POV measure, then there exists a unique PV measure $E:\mathcal{P}_{\mathbb{R}} \to \mathcal{E}(\mathcal{H})$, where $\mathcal{P}_{\mathbb{R}}$ is the space of the probability measures on $\mathcal{B}(\mathbb{R})$, such that

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$$F(\Delta) = \int_{\mathcal{P}_{\mathbb{R}}} p(\Delta)E(\mathrm{d}p). \tag{1}$$

Here we follow a different approach, which stems from Refs. 14 and 15. In these papers, the outcomes of the measurement of a commutative POV measure are interpreted as deriving from a stochastic diffusion of the outcomes of the measurement of a particular PV measure. The starting point of this approach is a theorem due to von Neumann, which states that for any given set $\{F_i\}_{i\in I}$ of commuting self-adjoint operators there exists a self-adjoint operator A and a family of measurable functions $f_i(\lambda)$ such that $f_i(A)=F_i$ (see Theorem 1 below). von Neumann's theorem implies that a POV measure F is commutative if and only if there exists a self-adjoint operator A (corresponding to a PV measure E) and a family of functions $\{\mu_\Delta\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ such that $F(\Delta)=\mu_\Delta(A)$.

In Ref. 14, starting from a commutative POV measure with spectrum in [0, 1], the authors constructed explicitly a self-adjoint operator A with spectrum $\sigma(A)\subset[0,1]$ and, for every set $\Delta \in \mathcal{B}([0,1])$, a measurable function $\omega_\Delta(\lambda):\sigma(A)\to[0,1]$ such that $F(\Delta)=\omega_\Delta(A)$. Their construction uses a procedure analogous to the one used by Riesz and Nagy to prove von Neumann's theorem (see Ref. 21, Sec. 130).

In Ref. 15 it was shown that, for every real number λ, the application $\omega_{(\cdot)}(\lambda):\mathcal{B}([0,1]) \to[0,1]$, corresponding to the family of functions $\{\omega_\Delta(\lambda)\}_{\Delta\in\mathcal{B}([0,1])}$, defines an *additive* set function on a particular ring $\mathcal{R}(\mathcal{S})$ which generates $\mathcal{B}([0,1])$.

This result suggests to interpret the outcomes of the measurement of F as deriving from a randomization of the outcomes of the measurement of E. Indeed let us consider a pure state $S=P_\psi$. From $F(\Delta)=\omega_\Delta(A)$ we get

$$\mu_S^F(\Delta) = \int w_\Delta(\lambda)\|E((\lambda-\mathrm{d}\lambda,\lambda])\psi\|^2 = \int w_\Delta(\lambda)\mathrm{d}\mu_S^E, \tag{2}$$

where E is the PV measure corresponding to the operator A.

This relation shows that $w_\Delta(\lambda)$ could be interpreted, quoting from Ref. 15, "*as the probability that the outcome* λ *of E turns into an outcome in* Δ *for F*." The physical source of randomization could be the imprecision of the measuring apparatus, or some other cause.

The above interpretation is satisfactory if the application $\omega_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{B}([0,1])$. However, the problem of the σ-additivity of $\omega_{(\cdot)}(\lambda)$ was left open by the authors.

In this paper we follow a novel approach and give a general geometrical characterization of POV measures, without using constructive procedures. Specifically, we show that a POV measure $F:\mathcal{B}(\mathbb{R})\to\mathcal{F}(\mathcal{H})$ is commutative if and only if there exists a PV measure $E:\mathcal{B}(\mathbb{R})\to\mathcal{E}(\mathcal{H})$ (corresponding to a self-adjoint operator A) and, for every $\lambda\in\sigma(A)$, a *probability measure* $\gamma_{(\cdot)}\times(\lambda):\mathcal{B}(\mathbb{R})\to[0,1]$ such that $F(\Delta)=\gamma_\Delta(A)$ (see Theorem 2). This result is founded on the ground of von Neumann's theorem. Its relevance to the interpretation of the commutative POV measures proposed in Refs. 14 and 15 is that, repeating the reasoning used to get Eq. (2), we get

$$\mu_S^F(\Delta) = \int \gamma_\Delta(\lambda)\mathrm{d}\mu_S^E$$

so that the sharp reconstruction E and the POV measure F are related by a probability measure.

It is worth mentioning that this result is close to the one obtained by Holevo[11,12] but, at variance with Theorem 2, the PV measure in (1) is not defined on $\mathcal{B}(\mathbb{R})$ but on the space of the probability measures on $\mathcal{B}(\mathbb{R})$. It would be interesting to check if there are relationships between the two characterizations.

The paper is organized as follows. In Sec. II we give some basic definitions and state the classical von Neumann's theorem. Then, in Sec. III, we prove the main result of the paper, Theorem 2. In Sec. IV we prove Proposition 2 which will be used in the last section where we discuss the relevance of Theorem 2 to the theory of the sharp reconstruction. In the appendices we give some technical results which are used in the paper.

II. PRELIMINARIES

In this section we fix the basic notation and terminology, and state von Neumann's theorem, which will be the starting point of the paper.

We denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra of $\mathbb{R}$, and by $\mathcal{F}(\mathcal{H})$ the space of bounded, positive self-adjoint operators acting on the Hilbert space $\mathcal{H}$. A POV measure is defined as follows:

Definition 1: A POV (positive operator valued) measure is a mapping $F:\mathcal{B}(\mathbb{R})\to\mathcal{F}(\mathcal{H})$ such that:

1. $F(\Delta)\geqslant 0$ *for all* $\Delta\in\mathcal{B}(\mathbb{R})$*;*
2. *if* $\{\Delta_n\}$ *is a countable family of disjoint sets in* $\mathcal{B}(\mathbb{R})$ *then*

$$F\left(\bigcup_{n=1}^{\infty}\Delta_n\right)=\sum_{n=1}^{\infty}F(\Delta_n),$$

where the series converges in the weak operator topology;
The POV measure is said to be normalized if one has also

3. $F(\mathbb{R})=1$.

Definition 2: A PV (projection valued) measure is a normalized POV measure which is orthogonal, that is, such that

4. $F(\Delta_1)F(\Delta_2)=0$ *if* $\Delta_1\cap\Delta_2=\emptyset$.

If this is the case we have $F(\Delta)=F(\Delta)^2$. Then, $F(\Delta)$ is a projection operator for every $\Delta\in\mathcal{B}(\mathbb{R})$ and the PV measure is denoted by $E:\mathcal{B}(\mathbb{R})\to\mathcal{E}(\mathcal{H})$, where $\mathcal{E}(\mathcal{H})$ is the space of all the projection operators of the Hilbert space $\mathcal{H}$. We recall that to each PV measure it is possible to associate a measure $\langle E(\Delta)x,x\rangle$ for every $x\in\mathcal{H}$.

Definition 3: A POV measure F is said to be commutative if the following relations are valid:

$$[F(\Delta_1),F(\Delta_2)]=0;\quad \forall\,\Delta_1,\Delta_2\in\mathcal{B}(\mathbb{R}).$$

We recall that PV measures are also named *sharp* or *standard* observables and, by the spectral theorem,[17] they are in a one to one correspondence with the self-adjoint operators, while nonorthogonal normalized POV measures are also named *unsharp* observables.

In what follows we shall always refer to normalized POV measures defined on $\mathcal{B}(\mathbb{R})$.

Next we proceed to define the spectrum of a POV measure.

Definition 4, see Ref. 18: Given a POV measure F we define the co-spectrum of F as the open set

$$\rho(F)=\cup\{\Delta:\Delta \textit{ is open},\ F(\Delta)=0\}$$

and the spectrum $\sigma(F)$ of F as the complement of $\rho(F)$.

The spectrum of a POV measure is characterized by the following proposition:

Proposition 1: The spectrum $\sigma(F)$ of F is closed and coincides with the set $\{\lambda\in\mathbb{R}:\forall\,\delta>0,F((\lambda-\delta,\lambda+\delta))\neq 0\}$.

Proof: Let us suppose $\lambda\in\sigma(F)$. Then, $\lambda\notin\rho(F)$ and $F(\Delta)\neq\mathbf{0}$ for every open set Δ. In particular,

$$F((\lambda-\delta,\lambda+\delta))\neq 0$$

for every $\delta>0$. On the converse, let us suppose that λ is so that $F((\lambda-\delta,\lambda+\delta))\neq 0$ for every $\delta>0$. For every open set Δ containing λ it must exist a number $\delta>0$ such that the open interval $(\lambda-\delta,\lambda+\delta)$ is contained in Δ. By items 1 and 2 of definition 1 we get

$$0<F((\lambda-\delta,\lambda+\delta))\leqslant F(\Delta)$$

then, $\lambda\notin\rho(F)$, hence $\lambda\in\sigma(F)$. □

We shall use the term "measurable" for the Borel measurable functions and consider the class $\mathcal{M}([0,1])$ of measurable functions defined on $[0, 1]$ which are bounded or are equal almost everywhere (a.e.), with respect to a PV measure E, to a bounded measurable function. Thus $\mathcal{M}([0,1])$ is the set of functions such that

$$E\text{-ess}\sup_{t\in[0,1]}|f(t)| = \inf_{E(\Delta)=I}\sup_{t\in\Delta}|f(t)| < +\infty.$$

The norm in $\mathcal{M}([0,1])$ is $\|f\|=E-\text{ess}\sup_{t\in[0,1]}|f(t)|$. Moreover we shall always consider equivalence classes of measurable functions. We say that two functions f and g are equivalent if $f=g$ a.e. with respect to a PV measure E, that is a.e. with respect to all the measures $\langle E(\Delta)x,x\rangle$, where x runs through all the vectors of the space $\mathcal{H}$.

As it is well known,[17,22] if a measurable function f is defined on the spectrum $\sigma(A)$ of a self-adjoint operator A, one has $f(A)=\int f(\lambda)E((\lambda-\mathrm{d}\lambda,\lambda])$ where E is the PV measure corresponding to A.

We shall denote by $\mathbf{0}$ and $\mathbf{1}$ the null and the identity operators, respectively.

Theorem 1 (von Neumann[19–21]): *For any family $\{F_i\}_{i\in\mathcal{I}}$ of self-adjoint operators acting on a Hilbert space $\mathcal{H}$ such that $[F_i,F_j]=\mathbf{0}$ for all $i,j\in\mathcal{I}$, there exist*

a) a bounded self-adjoint operator A;
b) a family of measurable functions $\{f_i:\mathbb{R}\to R\}_{i\in\mathcal{I}}$ such that for every $i\in\mathcal{I}$,

$$F_i = f_i(A) = \int_{-\infty}^{\infty} f_i(\lambda)E((\lambda-d\lambda,\lambda]) = \int_{-\infty}^{\infty} f_i(\lambda)\mathrm{d}E_\lambda$$

where E is the PV measure corresponding to A.

It follows that an unsharp observable $F:\mathcal{B}(\mathbb{R})\to\mathcal{F}(\mathcal{H})$ is commutative if and only if there exist:[14]

- *a sharp observable $E:\mathcal{B}(\mathbb{R})\to\mathcal{E}(\mathcal{H})$ (which corresponds to a self-adjoint operator $\mathbf{0}<A<\mathbf{1}$),*
- *a family of set functions*

$$\omega_{(\cdot)}(\lambda):\mathcal{B}(\mathbb{R})\to[0,1], \quad \Delta\mapsto\omega_\Delta(\lambda), \quad \lambda\in\sigma(A)$$

such that $F(\Delta)=\omega_\Delta(A)$.

Indeed the family $\{F(\Delta),\Delta\in\mathcal{B}(\mathbb{R})\}$ is a commutative family of bounded positive operators and von Neumann's theorem can be applied. Conversely, if a self-adjoint operator A exists such that all the operators $F(\Delta)$ are functions of A, then the family $\{F(\Delta),\Delta\in\mathcal{B}(\mathbb{R})\}$ is commutative (see Ref. 19, theorem 2, Sec. 75).

In what follows whenever a POV measure F, a self-adjoint operator A and an application $\mu_{(\cdot)}(\lambda)$ are such that $F(\Delta)=\mu_\Delta(A)$ we shall say that the triple $(F,A,\mu_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem.

III. A CHARACTERIZATION OF COMMUTATIVE POV MEASURES

Let us consider a family of functions $\mathcal{F}=\{\mu_\Delta(\lambda):[0,1]\to[0,1]\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ and the corresponding family of set functions $\mathcal{F}_S=\{\mu_{(\cdot)}(\lambda):\mathcal{B}(\mathbb{R})\to[0,1],\lambda\in\sigma(A)\}$ and assume that the triple $(F,A,\mu_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem. The aim of the present section is to show that the family of functions $\{\mu_\Delta(\lambda):[0,1]\to[0,1]\}_{\Delta\in\mathcal{B}(\mathbb{R})}$ can always be chosen in such a way that the corresponding set functions $\mu_{(\cdot)}(\lambda)\in\mathcal{F}_S$ are probability measures.

Theorem 2 (Main result): *A normalized POV measure $F:\mathcal{B}(\mathbb{R})\to\mathcal{F}(\mathcal{H})$ is commutative if and only if there exist an application $\mu_{(\cdot)}(\lambda):\mathcal{B}(\mathbb{R})\to[0,1]$ and a PV measure $E:\mathcal{B}(\mathbb{R})\to\mathcal{E}(\mathcal{H})$ such that:*

1. *the application* $\mu_{(\cdot)}(\lambda):\mathcal{B}(\mathbb{R})\to[0,1]$ *is a probability measure for every* $\lambda\in\sigma(A)$;
2. $F(\Delta)=\mu_\Delta(A)=\int\mu_\Delta(\lambda)E((\lambda-\mathrm{d}\lambda,\lambda])$,

where A is the self-adjoint operator associated to the PV measure E.

To prove this theorem we need two lemmas:

Lemma 1: For any commutative POV measure F, it is possible to find a couple $(A,\omega_{(\cdot)}(\lambda))$ *such that the triple* $(F,A,\omega_{(\cdot)}(\lambda))$ *satisfies the thesis of von Neumann theorem and* $\omega_{(\cdot)}(\lambda)$ *is additive on an appropriate ring which generates the Borel* σ*-algebra* $\mathcal{B}(\mathbb{R})$.

Proof: Following Ref. 14 we shall consider, without loss of generality, POV measures with a bounded spectrum contained in [0, 1] so that $\Delta\in\mathcal{B}([0,1])$ (see Appendix A). To prove the lemma let us consider a family of set functions

$$\gamma_{(\cdot)}(\lambda):\mathcal{B}([0,1])\to[0,1],\quad \Delta\mapsto\mu_\Delta(\lambda),\quad \lambda\in\sigma(A)$$

such that the triple $(F,A,\gamma_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem. We have that $\gamma_{[0,1]}(A)=\mathbf{1}$. Moreover the function $\omega_{[0,1]}(\lambda)\coloneqq 1$ is such that $\omega_{[0,1]}(A)=F([0,1])$ so that we can replace $\gamma_{[0,1]}(\lambda)$ with $\omega_{[0,1]}(\lambda)$.

We proceed by showing that it is possible to replace the functions $\gamma_{[0,1/2]}(\lambda)$ and $\gamma_{(1/2,1]}(\lambda)$ with two measurable functions $\omega_{[0,1/2]}(\lambda)$ and $\omega_{(1/2,1]}(\lambda)$ such that:

1) $\omega_{[0,1/2]}(\lambda)+\omega_{(1/2,1]}(\lambda)=\omega_{[0,1]}(\lambda)$ for all λ;
2) $\omega_{[0,1/2]}(A)=F([0,1/2])$ and $\omega_{(1/2,1]}(A)=F((1/2,1])$.

We have

$$\begin{aligned}\int\omega_{[0,1]}(\lambda)\mathrm{d}E_\lambda &= F([0,1]) = F([0,1/2]) + F((1/2,1])\\ &= \int\gamma_{[0,1/2]}(\lambda)\mathrm{d}E_\lambda + \int\gamma_{(1/2,1]}(\lambda)\mathrm{d}E_\lambda = \int[\gamma_{[0,1/2]}(\lambda)+\gamma_{(1/2,1]}(\lambda)]\mathrm{d}E_\lambda\end{aligned}$$

so that, by Ref. 22, Corollary 9 (see also Ref. 21, Chap. IX),

$$\gamma_{[0,1/2]}(\lambda)+\gamma_{(1/2,1]}(\lambda)=\omega_{[0,1]}(\lambda)\ \text{a.e. with respect to } E.$$

By setting

$$\omega_{(1/2,1]}(\lambda)=\omega_{[0,1]}(\lambda)-\gamma_{[0,1/2]}(\lambda)$$

$$\omega_{[0,1/2]}(\lambda)=\gamma_{[0,1/2]}(\lambda)$$

we get two measurable functions $\omega_{[0,1/2]}(\lambda)$ and $\omega_{(1/2,1]}(\lambda)$ such that items 1) and 2) are satisfied and which differ from functions $\gamma_{[0,1/2]}(\lambda)$ and $\gamma_{(1/2,1]}(\lambda)$ only in a set of zero E-measure.

We proceed by showing that it is possible to replace the functions $\gamma_{[0,1/4]}(\lambda)$ and $\gamma_{(1/4,1/2]}(\lambda)$ with two measurable functions $\omega_{[0,1/4]}(\lambda)$ and $\omega_{(1/4,1/2]}(\lambda)$ such that:

1′) $\omega_{[0,1/4]}(\lambda)+\omega_{(1/4,1/2]}(\lambda)=\omega_{[0,1/2]}(\lambda)$ for all λ;
2′) $\omega_{[0,1/4]}(A)=F([0,1/4])$ and $\omega_{(1/4,1/2]}(A)=F((1/4,1/2])$.

We have

$$\int \omega_{[0,1/2]}(\lambda)\mathrm{d}E_\lambda = F([0,1/2]) = F([0,1/4]) + F((1/4,1/2])$$
$$= \int \gamma_{[0,1/4]}(\lambda)\mathrm{d}E_\lambda + \int \gamma_{(1/4,1/2]}(\lambda)\mathrm{d}E_\lambda = \int [\gamma_{[0,1/4]}(\lambda) + \gamma_{(1/4,1/2]}(\lambda)]\mathrm{d}E_\lambda,$$

so that

$$\gamma_{[0,1/4]}(\lambda) + \gamma_{(1/4,1/2]}(\lambda) = \omega_{[0,1/2]}(\lambda) \text{ a.e. with respect to } E.$$

Let us consider the set $N \subset [0,1]$ such that $\gamma_{[0,1/4]}(\lambda) > \omega_{[0,1/2]}(\lambda)$. Since N is a null set, there exists a Borel set $\bar{\Delta}$ such that $N \subset \bar{\Delta}$ and $E(\bar{\Delta})=0$. Now we define the function

$$\bar{\gamma}_{[0,1/4]}(\lambda) := \begin{cases} \gamma_{[0,1/4]}(\lambda) & \text{if } \lambda \notin \bar{\Delta} \\ 0 & \text{if } \lambda \in \bar{\Delta}, \end{cases}$$

which is less than or equal to $\omega_{[0,1/2]}(\lambda)$. Moreover $\bar{\gamma}_{[0,1/4]}(\lambda)$ is measurable. Indeed by the measurability of $\gamma_{[0,1/4]}(\lambda)$ we have that, for every $x \in [0,1]$,

$$\{\lambda \in [0,1] | \bar{\gamma}_{[0,1/4]}(\lambda) \leqslant x\} = \{\lambda \in [0,1] \setminus \bar{\Delta} | \gamma_{[0,1/4]}(\lambda) \leqslant x\} \cup \bar{\Delta}$$
$$= [\{\lambda \in [0,1] | \gamma_{[0,1/4]}(\lambda) \leqslant x\} \cap \{[0,1] \setminus \bar{\Delta}\}] \cup \bar{\Delta} \in \mathcal{B}([0,1]).$$

By setting

$$\omega_{[0,1/4]}(\lambda) = \bar{\gamma}_{[0,1/4]}(\lambda),$$

$$\omega_{(1/4,1/2]}(\lambda) = \omega_{[0,1/2]}(\lambda) - \bar{\gamma}_{[0,1/4]}(\lambda),$$

we get two functions such that items $1'$) and $2'$) are satisfied and which differ from the functions $\gamma_{[0,1/4]}(\lambda)$ and $\gamma_{(1/4,1/2]}(\lambda)$ only in a set of zero E-measure.

Analogously, starting from the functions $\gamma_{(1/2,3/4]}(\lambda)$ and $\gamma_{(3/4,1]}(\lambda)$, we can define two measurable functions $\omega_{(1/2,3/4]}(\lambda)$ and $\omega_{(3/4,1]}(\lambda)$ such that

$1''$) $\omega_{(1/2,3/4]}(\lambda)+\omega_{(3/4,1]}(\lambda)=\omega_{(1/2,1]}(\lambda)$;

$2''$) $\omega_{(1/2,3/4]}(A)=F((1/2,3/4])$, and $\omega_{(3/4,1]}(A)=F((3/4,1])$.

Iterating the procedure we shall finally obtain a family of functions $\{\omega_\Delta(\lambda):[0,1] \to [0,1]\}_{\Delta \in \mathcal{S}}$ where $\mathcal{S}$ is the semi-ring:

$$\mathcal{S} = \{[0,1/2^{n-1}],(k/2^{n-1},k+1/2^{n-1}]|,k=1,2,\ldots,2^{n-1}-1,n \in \mathbb{N}\}.$$

Moreover, for every given λ, the set function $\omega_{(\cdot)}(\lambda):\mathcal{S} \to [0,1]$ is additive on the semi-ring $\mathcal{S}$.

Now, for any λ, it is possible to extend $\omega_{(\cdot)}(\lambda):\mathcal{S} \to [0,1]$ to the ring $\mathcal{R}(\mathcal{S})$ generated by $\mathcal{S}$. The extension $\omega_{(\cdot)}(\lambda):\mathcal{R}(\mathcal{S}) \to [0,1]$ is an additive set function on $\mathcal{R}(\mathcal{S})$ such that $\omega_\Delta(A)=F(\Delta)$ (see Ref. 24). □

The following lemma concerns the definition and the monotonicity property of $\omega_{[0,t)}(\lambda)$ as a function of t. In particular it will be proved that $\omega_{[0,t)}(\lambda)$ is nondecreasing with respect to t and such that $\omega_{[0,t)}(A)=F[0,t)$.

In order to define $\omega_{[0,t)}(\lambda)$ we use the fact that it is always possible to decompose $[0,t)$ into the union of a disjoint sequence of intervals from $\mathcal{S}$.

Definition 5: For every $t \in (0,1]$ we define $\omega_{[0,t)}(\lambda)=\Sigma_{j=1}^{\infty}\omega_{\Delta_j}(\lambda)$ where $\{\Delta_j:\Delta_j \in \mathcal{S}\}_{j\in\mathbb{N}}$ is a decomposition of the interval $[0,t)$ with the following property: for every $x \in (0,t)$ there is a closed interval $[0,a]$: $a \geqslant x$ such that $[0,a]=\cup_{j=1}^{n}\Delta_j$. The set of the decompositions so defined is denoted by $\mathcal{D}$.

The series $\Sigma_{j=1}^{\infty}\omega_{\Delta_j}(\lambda)$ used to define the function $\omega_{[0,t)}(\lambda)$ is convergent. Indeed for every $p \in \mathbb{N}$, $\cup_{j=1}^{p}\Delta_j \subset [0,t) \subset [0,1]$ and then, using the additivity of the measure on $\mathcal{R}(\mathcal{S})$, $\Sigma_{j=1}^{p}\omega_{\Delta_j}(\lambda) \leqslant \omega_{[0,1]}$. Moreover, as shown in Appendix B, $\omega_{[0,t)}(\lambda)$ is well defined.

Lemma 2. The set function $\omega_{[0,t)}(\lambda)$, *is nondecreasing with respect to t and* $\omega_{[0,t)}(A) = F([0,t))$.

Proof. Let us consider the two intervals $[0,t)$ and $[0,t')$ where $t<t'$ and let us decompose them as $\{\Delta_k;\Delta_k \in \mathcal{S}\}_{k\in\mathbb{N}}$ and $\{\Delta'_k;\Delta'_k \in \mathcal{S}\}_{k\in\mathbb{N}}$, respectively. Following Definition 5 it is possible to find $m \in \mathbb{N}$ so that

$$[0,t) \subset [0,b] = \bigcup_{k=1}^{m} \Delta'_k \subset [0,t'),$$

with $[0,b] \in \mathcal{R}(\mathcal{S})$. We can write:

$$[0,t') = \left[\bigcup_{k=1}^{m} \Delta'_k\right] \cup \left[\bigcup_{k=m+1}^{\infty} \Delta'_k\right].$$

By the additivity of $\omega_{(\cdot)}(\lambda)$ on $\mathcal{R}(\mathcal{S})$ we get:

$$\omega_{[0,t')}(\lambda) = \sum_{k=1}^{m} \omega_{\Delta'_k}(\lambda) + \sum_{k=m+1}^{\infty} \omega_{\Delta'_k}(\lambda) = \omega_{[0,b]}(\lambda) + \sum_{k=m+1}^{\infty} \omega_{\Delta'_k}(\lambda) \geqslant \omega_{[0,b)}(\lambda).$$

Using again the additivity of the measure and the relation

$$\bigcup_{k=1}^{p} \Delta_k \subset \bigcup_{k=1}^{\infty} \Delta_k = [0,t) \subset [0,b]; \qquad \forall\, p \in \mathbb{N},$$

we finally have

$$\omega_{(\cup_{k=1}^{p}\Delta_k)}(\lambda) \leqslant \omega_{[0,b]}(\lambda), \qquad \forall\, p \in \mathbb{N},$$

$$\omega_{[0,b]}(\lambda) \geqslant \omega_{[0,t)}(\lambda),$$

which ends the proof of the first part of the proposition. The second assertion is proved as follows: $F([0,t)) = F(\cup_j \Delta_j) = \Sigma_j F(\Delta_j) = \Sigma_j \int \omega_{\Delta_j}(\lambda) dE_\lambda$ and, by Theorem VII.2.d in Ref. 17, we get

$$F([0,t)) = \int \sum_j \omega_{\Delta_j}(\lambda) \mathrm{d}E_\lambda = \omega_{[0,t)}(A).$$

□

Now we are ready to prove the main theorem of this paper.

Proof of Theorem 2. Let $(F, A, \omega_{(\cdot)}(\lambda))$ satisfy the thesis of von Neumann's theorem. By Lemma 1, the family of set functions $\{\omega_{(\cdot)}(\lambda), \lambda \in \sigma(A)\}$ can be chosen in such a way that, for each $\lambda \in \sigma(A)$, $\omega_{(\cdot)}(\lambda)$ is an additive set function. Moreover, Lemma 2 allows us to define a function $\omega_{[0,t)}(\lambda)$ which is nondecreasing with respect to t and such that $\omega_{[0,t)}(A) = F([0,t))$.

Now, starting from $\omega_t := \omega_{[0,t)}(\lambda)$, it is possible to define a distribution function $\mu_t(\lambda)$ continuous from the left as follows:

$$\mu_t(\lambda) = \begin{cases} \omega_{t^-}(\lambda) & \text{if } 0 < t \leqslant 1 \\ 0 & \text{if } t \leqslant 0 \\ 1 & \text{if } t > 1, \end{cases}$$

where $\omega_{t^-}(\lambda) = \lim_{x\to t^-} \omega_{[0,t)}(\lambda)$.

In its turn, for every λ, starting from the left continuous function $\mu_t(\lambda)$ it is possible to define the set function $\mu_{[a,b)}(\lambda)=\mu_b-\mu_a$, $[a,b)\subset(-\infty,\infty)$ which can be extended to a σ-additive probability measure (see Ref. 23, p. 97) $\mu_{(\cdot)}(\lambda):\mathcal{B}(\mathbb{R})\to[0,1]$ such that

$$\mu_\Delta(\lambda)=0 \quad \text{if } \Delta\cap[0,1]=\emptyset.$$

We want to prove that:

$$\mu_t(A)=F([0,1]\cap(-\infty,t)) \tag{3}$$

and that the probability measure $\mu_{(\cdot)}(\lambda):\mathcal{B}([0,1])\to[0,1]$, which is the restriction of $\mu_{(\cdot)}\times(\lambda):\mathcal{B}(\mathbb{R})\to[0,1]$ to $\mathcal{B}([0,1])$, is such that

$$\mu_\Delta(A)=F(\Delta), \quad \Delta\in\mathcal{B}([0,1]). \tag{4}$$

First we prove (3). If $t\in(0,1]$, we have:

$$\mu_t(A)=\int \mu_t(\lambda)\mathrm{d}E_\lambda=\int [\lim_{x\to t^-}\omega_x(\lambda)]\mathrm{d}E_\lambda.$$

Notice that the integral exists thanks to the measurability of $\mu_t(\lambda)$ as a function of λ (see Remark 2 below). Let us consider a sequence $\{x_n\}$ such that $x_n\to t$ from the left. We have $\mu_t(A)=\int[\lim_{x\to t^-}\omega_x(\lambda)]\mathrm{d}E_\lambda=\int[\lim_{n\to\infty}\omega_{x_n}(\lambda)]\mathrm{d}E_\lambda$. By the continuity of F and theorem 11 in Ref. 18 we get

$$\mu_t(A)=\int [\lim_{n\to\infty}\omega_{x_n}(\lambda)]\mathrm{d}E_\lambda=\lim_{n\to\infty}\int \omega_{x_n}(\lambda)\mathrm{d}E_\lambda=\lim_{n\to\infty}F([0,x_n))=F([0,t))=F([0,1]\cap(-\infty,t)).$$

Moreover, if $t\leqslant 0$ then $\mu_t(A)=\mathbf{0}=F(\emptyset)=F([0,1]\cap(-\infty,t))$, while, if $t>1$ then $\mu_t(A)=\mathbf{1}=F([0,1])=F([0,1]\cap(-\infty,t))$.

It remains to prove that the probability measure $\mu_{(\cdot)}(\lambda):\mathcal{B}(\mathbb{R})\to[0,1]$ generated by $\mu_t(\lambda)$ is such that $\mu_\Delta(A)=F(\Delta)$, $\Delta\in\mathcal{B}([0,1])$. Let us consider the POV measure $F':\mathcal{B}(\mathbb{R})\to\mathcal{F}(\mathcal{H})$ such that $F'(\Delta)=F(\Delta\cap[0,1])$. Notice that F is the restriction of F' to $\mathcal{B}([0,1])$ and $F'(\Delta)=\mathbf{0}$ if $\Delta\cap[0,1]=\emptyset$.

As a consequence of (3),

$$\begin{aligned}F'([t,\bar{t}))&=F'((-\infty,\bar{t}))-F'((-\infty,t))=F([0,1]\cap(-\infty,\bar{t}))-F([0,1]\cap(-\infty,t))\\&=\mu_{\bar{t}}(A)-\mu_t(A)=\mu_{[t,\bar{t})}(A)\end{aligned}$$

so that $F'(\Delta)=\mu_\Delta(A)$ for every set Δ in the ring $\mathcal{P}$ generated by sets of the kind $[t,\bar{t})\subset(-\infty,\infty)$. Moreover by the uniqueness of the extensions of $F':\mathcal{P}\to\mathcal{F}(\mathcal{H})$ to the σ-algebra $\mathcal{B}(\mathbb{R})$ (see Ref. 18, theorem 7), it follows that the probability measure $\mu_{(\cdot)}(\lambda):\mathcal{B}(\mathbb{R})\to[0,1]$ generated by the distribution function $\mu_t(\lambda)$ and the POV measure $F':\mathcal{B}(\mathbb{R})\to\mathcal{F}(\mathcal{H})$ are such that $F'(\Delta)=\mu_\Delta(A)$ for every $\Delta\in\mathcal{B}(\mathbb{R})$, hence $\mu_\Delta(A)=F(\Delta)$ for every $\Delta\in\mathcal{B}([0,1])$. □

Remark 1: The argument used to prove Theorem 2, applies to every pair $(B,\omega^B_{(\cdot)}(\lambda))$ *such that the triple* $(F,B,\omega^B_{(\cdot)}(\lambda))$ *satisfies the thesis of von Neumann's theorem. Therefore we can state the following corollary:*

Corollary 1: If the triple $(F,B,\omega^B_{(\cdot)}(\lambda))$ *satisfies the thesis of von Neumann theorem then there exists a probability measure* $\mu^B_{(\cdot)}(\lambda)$, *equivalent to* $\omega^B_{(\cdot)}(\lambda)$ *(that is, for every* $\Delta\in\mathcal{B}([0,1])$, $\mu^B_\Delta(\lambda)=\omega^B_\Delta(\lambda)$ *a.e.), such that the triple* $(F,B,\mu^B_{(\cdot)}(\lambda))$ *satisfies the thesis of von Neumann's theorem.*

Remark 2: The function $\mu_t(\cdot)$ *defined in the proof of Theorem 2 is measurable because it is the limit of a monotone sequence of measurable functions.*[17,23–25] *Indeed, for each* λ, *the function* $\omega_x(\lambda)$ *is bounded and nondecreasing with respect to x, then the limit* $\omega_{t^-}(\lambda)$ *exists and we have*

$\mu_t(\lambda)=\omega_{t^-}(\lambda)=\lim_{x\to t^-}\omega_x(\lambda)=\lim_{n\to\infty}\ \omega_{x_n}(\lambda)$ *where* $\{x_n\}$ *is an increasing sequence of numbers such that* $x_n\to t$ *from the left.*

IV. A NECESSARY AND SUFFICIENT CONDITION FOR THE σ-ADDITIVITY OF $\omega_{(\cdot)}(\lambda)$

In the proof of Theorem 2 we started from the *additive* set function $\omega_{(\cdot)}(\lambda):\mathcal{R}(\mathcal{S})\to[0,1]$ such that $\omega_\Delta(A)=F(\Delta)$, $\Delta\in\mathcal{R}(\mathcal{S})$ and proved the existence of a probability measure $\mu_{(\cdot)}(\lambda):\mathcal{B}([0,1])\to[0,1]$ such that the triple $(F,A,\mu_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem. Clearly, if $\omega_{(\cdot)}(\lambda):\mathcal{R}(\mathcal{S})\to[0,1]$ were σ-additive on $\mathcal{R}(\mathcal{S})$ it could be directly extended to a probability measure $\mu_{(\cdot)}(\lambda)$ on $\mathcal{B}([0,1])$ such that $\mu_\Delta(A)=F(\Delta)$. Now we want to show that the additive set function $\omega_{(\cdot)}(\lambda):\mathcal{R}(\mathcal{S})\to[0,1]$ is σ-additive if and only if $\omega_\Delta(\lambda)=\mu_\Delta(\lambda)$ for every $\Delta\in\mathcal{R}(\mathcal{S})$.

Proposition 2: The additive set function $\omega_{(\cdot)}(\lambda):\mathcal{R}(\mathcal{S})\to[0,1]$ *is* σ*-additive on* $\mathcal{R}(\mathcal{S})$ *if and only if* $\omega_\Delta(\lambda)=\mu_\Delta(\lambda)$ *for every* $\Delta\in\mathcal{R}(\mathcal{S})$, *where* $\mu_{(\cdot)}(\lambda)$ *is the probability measure introduced in the proof of Theorem 2.*

Proof: Let us assume that, for any λ, $\omega_{(\cdot)}(\lambda)$ is σ-additive on the ring $\mathcal{R}(\mathcal{S})$. Then, by Caratheodory theorem (see Ref. 24 p. 88, and Ref. 25), it can be uniquely extended to a σ-additive measure $\nu_{(\cdot)}(\lambda)$ on $\mathcal{B}([0,1])$. Now, we can define (see Theorem 4.8 of Ref. 24 and p. 88 of Ref. 23), a distribution function continuous from the left (see Definition 9 in Appendix D) as follows:

$$\nu_t(\lambda)=\begin{cases}\nu_{([0,t))}(\lambda) & t\in(0,1]\\ 1 & t>1\\ 0 & t\leqslant 0.\end{cases}$$

By Theorem 4 in Appendix D, it follows that the measure $\nu_{(\cdot)}(\lambda):\mathcal{B}(\mathbb{R})\to[0,1]$ generated by $\nu_t(\lambda)$ coincides with $\nu_{(\cdot)}(\lambda):\mathcal{B}([0,1])\to[0,1]$ on $\mathcal{B}([0,1])$ so that it is an extension of $\omega_{(\cdot)}(\lambda)$ to $\mathcal{B}(\mathbb{R})$. Moreover $\nu_\Delta(\lambda)=0$ if $\Delta\cap[0,1]=\emptyset$. Therefore, by the σ-additivity of the measure and by Definition 5, we have

$$\nu_{[0,t)}(\lambda)=\sum_{k=1}^{\infty}\nu_{\Delta_k}(\lambda)=\sum_{k=1}^{\infty}\omega_{\Delta_k}(\lambda)$$

for every decomposition

$$\cup_k\Delta_k=[0,t),\quad \Delta_k\in\mathcal{S}.$$

Hence,

$$\omega_{[0,t)}(\lambda)=\nu_{[0,t)}(\lambda),\quad t\in(0,1]$$

and $\omega_{[0,t)}(\lambda)$ is continuous from the left in (0,1]. Therefore we get

$$\nu_t(\lambda)=\mu_t(\lambda)=\begin{cases}\omega_{([0,t))}(\lambda) & t\in(0,1]\\ 1 & t>1\\ 0 & t\leqslant 0,\end{cases}\tag{5}$$

which means that the probability measure $\mu_{(\cdot)}(\lambda)$ corresponding to $\mu_t(\lambda)$ coincides with the probability measure $\nu_{(\cdot)}(\lambda)$ corresponding to $\nu_t(\lambda)$. Hence, $\mu_\Delta(\lambda)=\nu_\Delta(\lambda)=\omega_\Delta(\lambda)$ on $\mathcal{R}(\mathcal{S})$. The converse is obvious because if it were $\mu_\Delta(\lambda)=\omega_\Delta(\lambda)$ for every $\Delta\in\mathcal{R}(\mathcal{S})$, then $\omega_{(\cdot)}(\lambda)$ would be σ-additive on $\mathcal{R}(\mathcal{S})$. □

V. SHARP RECONSTRUCTION AND UNIQUENESS OF THE CHARACTERIZATION

This section is devoted to the analysis of the implications of Theorem 2 to the theory of the sharp reconstruction. Moreover we show (Theorem 3) that the result obtained in Ref. 15 implies the uniqueness (in a sense specified below) of the characterization in Theorem 2.

In Ref. 14, starting from a commutative POV measure F, the authors constructed a PV measure E (corresponding to a self-adjoint operator A) called the sharp reconstruction of F, and an application $\omega_{(\cdot)}(\lambda):\mathcal{B}([0,1])\to[0,1]$ which connects A and F by means of relation $\omega_\Delta(A)=F(\Delta)$, proving that the triple $(F,A,\omega_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem. In Ref. 15 it was proven that for every given $\lambda\in\sigma(A)$ the set function $\omega_{(\cdot)}(\lambda)$ is *additive* on $\mathcal{R}(S)$. By Corollary 1 it follows that, for every given $\lambda\in\sigma(A)$, there exists a probability measure $\mu_{(\cdot)}(\lambda):\mathcal{B}([0,1])\to[0,1]$ such that the triple $(F,A,\mu_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem. Indeed, thanks to the additivity of $\omega_{(\cdot)}(\lambda)$ on $\mathcal{R}(S)$, it is sufficient to apply the reasoning in the proof of Theorem 2 to the functions $\omega_\Delta(\lambda)$, $\Delta\in\mathcal{R}(S)$.

Moreover, by proposition 2, $\omega_{(\cdot)}(\lambda)$ is σ-additive on $\mathcal{R}(S)$ if and only if $\omega_\Delta(\lambda)=\mu_\Delta(\lambda)$ for every $\Delta\in\mathcal{R}(S)$ so that if $\omega_{(\cdot)}(\lambda)$ is not σ-additive on $\mathcal{R}(S)$ it would be interesting to check whether there are conditions, on the construction of the functions $\omega_\Delta(\lambda)$, $\Delta\in\mathcal{R}(S)$, which, added to conditions C and D in Ref. 14, imply the σ-additivity of $\omega_{(\cdot)}(\lambda)$ on $\mathcal{R}(S)$. In Ref. 15 it was proven that the sharp reconstruction A is unique in the following sense: if the triples $(F,A,\omega_{(\cdot)}\times(\lambda))$ and $(F,B,\omega^B_{(\cdot)}(\lambda))$ satisfy the thesis of von Neumann's theorem then there exists a function $g:[0,1]\to[0,1]$ such that $A=g(B)$, so that, among the set of all sharp observables which satisfy the thesis of von Neumann's theorem, the sharp reconstruction A is the only one (modulo a bijection) which is related to the others by means of a relation of the kind $A=g(B)$. Due to the last relation, the operators A and B commute. This allowed the authors to consistently interpret the outcomes of measurements of the unsharp observable F as deriving from a randomization of the outcomes of measurements of the sharp reconstruction A (see Ref. 15). By Theorem 2 of the present work and by the main result of Ref. 15 we get the following theorem:

Theorem 3: *A POV measure $F:\mathcal{B}(\mathbb{R})\to\mathcal{F}(\mathcal{H})$ is commutative if and only if there exist a PV measure $E^B:\mathcal{B}(\mathbb{R})\to\mathcal{E}(\mathcal{H})$ (corresponding to a self-adjoint operator B) and, for every $\lambda\in\sigma(B)$, a probability measure $\mu^B_{(\cdot)}(\lambda):\mathcal{B}(R)\to[0,1]$ such that the triple $(F,B,\mu^B_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem. The sharp reconstruction A is the only one (modulo a bijection) which is related to the others as follows:*

for every triple $(F,B,\mu^B_{(\cdot)}(\lambda))$ satisfying the thesis of von Neumann's theorem there exists a function $g:[0,1]\to[0,1]$ such that $A=g(B)$.

APPENDIX A: POV MEASURES WITH SPECTRUM IN [0, 1]

In this appendix we show[14] that, without loss of generality, we can restrict ourselves to POV measures with spectrum in [0, 1].

Definition 6: Given the POV measure F, the application $\omega_{(\cdot)}(\lambda):\mathcal{B}(\mathbb{R})\to[0,1]$ and a bijective function $f:(0,1)\to\mathbb{R}$ we denote by $\bar{F}$ the POV measure $\Delta\in\mathcal{B}([0,1])\to F(f(\Delta))$ and by $\bar{\omega}_{(\cdot)}(\lambda)$ the application $\Delta\in\mathcal{B}([0,1])\to\omega_{f(\Delta)}(\lambda)$.

Notice that the POV measure $\bar{F}$ has spectrum in [0, 1].

Proposition 3: Given a POV measure F, and a self-adjoint operator A, the triple $(F,A,\omega_{(\cdot)}\times(\lambda))$ satisfies the thesis of von Neumann's theorem if and only if the triple $(\bar{F},A,\bar{\omega}_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem too.

Proof: If $(F,A,\omega_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem then $\bar{F}(f^{-1}(\Delta))=F(\Delta)=\omega_{(\Delta)}(A)=\bar{\omega}_{f^{-1}(\Delta)}(A)$ for every $\Delta\in\mathcal{B}(\mathbb{R})$. Notice that f establishes a one to one correspondence between Borel sets in $\mathcal{B}([0,1])$ and Borel set in $\mathcal{B}(\mathbb{R})$. Therefore the triple $(\bar{F},A,\bar{\omega}_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann theorem. Now let us suppose that the triple $(\bar{F},A,\bar{\omega}_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann theorem. Then we have $F(f(\Delta))=\bar{F}(\Delta)=\bar{\omega}_{(\Delta)}(A)=\omega_{f(\Delta)}(A)$ for every $\Delta \in \mathcal{B}([0,1])$ which means that $(F,A,\omega_{(\cdot)}(\lambda))$ satisfies the thesis of von Neumann's theorem. □

APPENDIX B: THE NONDECREASING REAL FUNCTION $\omega_{[0,t)}(\lambda)$

In this appendix we prove the following lemma:

Lemma 3: The sum in Definition 5 does not depend on the decomposition of the interval $[0,t)$.

In order to prove Lemma 3 the following simple facts will be used:

1) if the interval $((k'-1)/2^{q-1},k'/2^{q-1}=b]$ is an element of the decomposition $\{\Delta_j\}_{j\in\mathbb{N}}$ of $[0,t)$ it is not possible that intervals of the kind $((j-1)/2^{p-1},j/2^{p-1}=b]$ with $p\neq q$ be elements of the decomposition too. Otherwise $\{\Delta_j\}_{j\in\mathbb{N}}$ would not be a disjoint family of sets;
2) the extrema of the sets $\Delta\in\mathcal{S}$ define the set of points:

$$A=\left\{\frac{k}{2^{n-1}}k,n\in\mathbb{N}:k\leq 2^{n-1}\right\};$$

if $a\in A$ and $a\in((i-1)/2^{n-1},i/2^{n-1}]\in\mathcal{S}$ it follows that $a=r/2^{l-1}$ for some $r,l\in\mathbb{N}$ such that $r\leq 2^{l-1}$ and $n\leq l$. Therefore every set $\Delta\in\mathcal{S}$ having $a=r/2^{l-1}$ as one of its extrema must be contained in $((i-1)/2^{n-1},i/2^{n-1}]$;
3) every point $a\in A$, with the exception of 0, is contained in [0, 1] as the right extreme of one interval $\Delta\in\mathcal{S}$

Definition 7: For any decomposition $d_t=\{\Delta_j\}_{j\in\mathbb{N}}\in\mathcal{D}$ *of* $[0,t)$ *and for any* $x\in(0,t)$ *we define the set*

$$\mathcal{M}_{d_t}(x)=\left\{b\,\middle|\,x\in[0,b]=\bigcup_{l=1}^{m}\Delta_l,\Delta_l\in d_t\right\}.$$

Proposition 4: Let us consider two decompositions $d_t=\{\Delta_l\}_{l\in\mathbb{N}}$, $\tilde{d}_t=\{\tilde{\Delta}_k\}_{k\in\mathbb{N}}\in\mathcal{D}$ *of* $[0,t)$. *If for any* $0<x<t$ *it is* $\mathcal{M}_{d_t}(x)\cap\mathcal{M}_{\tilde{d}_t}(x)\neq\emptyset$ *then sequences* $\{\sum_{l=1}^{n}\omega_{\Delta_l}(\lambda)\}_{n\in\mathbb{N}}$ *and* $\{\sum_{k=1}^{n}\omega_{\tilde{\Delta}_k}(\lambda)\}_{n\in\mathbb{N}}$ *converge to the same limit*.

Proof: Let us choose a sequence of numbers $\{x_i;0\leq x_i<t\}$ such that $x_i\to t$. For every x_i there exists a number $b_i\in\mathcal{M}_{d_t}(x_i)\cap\mathcal{M}_{\tilde{d}_t}(x_i)$ such that $x_i\leq b_i<t$, hence a sequence $\{b_i\}$ such that $b_i\to t$ is defined. We have

$$\lim_{i\to\infty} b_i=t,$$

$$\lim_{i\to\infty} n(i)=\lim_{i\to\infty} m(i)=\infty,$$

$$[0,b_i]=\bigcup_{l=1}^{m_i}\Delta_l=\bigcup_{k=1}^{n_i}\tilde{\Delta}_k,$$

$$\sum_{l=1}^{m_i}\omega_{\Delta_l}(\lambda)=\sum_{k=1}^{n_i}\omega_{\tilde{\Delta}_k}(\lambda)\ \forall\, i,$$

hence

$$\lim_{i\to\infty}\sum_{l=1}^{m_i}\omega_{\Delta_l}(\lambda)=\lim_{i\to\infty}\sum_{k=1}^{n_i}\omega_{\tilde{\Delta}_k}(\lambda)$$

which proves the existence of two subsequences $\{\Sigma_{l=1}^{m_i}\omega_{\Delta_l}(\lambda)\}_{m_i\in\mathbb{N}}$ and $\{\Sigma_{k=1}^{n_i}\omega_{\tilde{\Delta}_k}(\lambda)\}_{n_i\in\mathbb{N}}$ of sequences $\{\Sigma_{l=1}^{n}\omega_{\Delta_l}(\lambda)\}_{n\in\mathbb{N}}$ and $\{\Sigma_{k=1}^{n}\omega_{\tilde{\Delta}_k}(\lambda)\}_{n\in\mathbb{N}}$, respectively, converging to the same limit. $\square$

Now we are ready to prove Lemma 3.

Proof of Lemma 3: Let us consider two decompositions $\{\Delta_l\}_{l\in\mathbb{N}},\{\tilde{\Delta}_k\}_{k\in\mathbb{N}}\in\mathcal{D}$ of $[0,t)$. The proposition is proved if we prove that for every $0<x<1$ it is $\mathcal{M}_{d_t}(x)\cap\mathcal{M}_{\tilde{d}_t}(x)\neq\emptyset$.

For any x, with $0<x<t$, we denote by b the minimum of $\mathcal{M}_{d_t}(x)$ and by $\tilde{b}$ the minimum of $\mathcal{M}_{\tilde{d}_t}(x)$. The existence of the minimum comes from the definition of the class $\mathcal{D}$. If $b\neq\tilde{b}$ we suppose (without loss of generality) $b<\tilde{b}$.

Let $((k'-1)/2^{q-1},k'/2^{q-1}=\tilde{b}]\in\{\tilde{\Delta}_k\}_{k\in\mathbb{N}}$ be the interval of decomposition $\{\tilde{\Delta}_k\}_{k\in\mathbb{N}}$ having $\tilde{b}$ as the right extreme. Then, considering that $\tilde{b}$ is the minimum of set $\mathcal{M}_{\tilde{d}_t}(x)$ and that sets $\tilde{\Delta}_k$ are disjoint, we get $x\in((k'-1)/2^{q-1},k'/2^{q-1}=\tilde{b}]$. Let us consider the number $\bar{a}=\max\{a\in\mathcal{M}_{d_t}(x):[0,a]\subset[0,\tilde{b}]\}$. We have $x\leq\bar{a}<\tilde{b}$ and then $\bar{a}\in((k'-1)/2^{q-1},k'/2^{q-1}=\tilde{b}]$. Following item 2), we have that every interval of decomposition $\{\Delta_l\}_{l\in\mathbb{N}}$ having $\bar{a}$ as the left extreme is contained in $((k'-1)/2^{q-1},k'/2^{q-1}=\tilde{b}]$. In other words for every $\Delta_l\in\{\Delta_l\}_{l\in\mathbb{N}}\subset\mathcal{S}$ such that $\Delta_l=(\bar{a},y]$ $(y\in[0,1))$ it must be $y\leq\tilde{b}$. Considering that $\bar{a}$ is the maximum of $\mathcal{M}_{d_t}(x)$, that $\tilde{b}\in[0,1]$ and by item 3), we get $y=\tilde{b}$. This proves that $\mathcal{M}_{d_t}(x)\cap\mathcal{M}_{\tilde{d}_t}(x)\neq\emptyset$. The thesis derives from Proposition 4. $\square$

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