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Spectral Analysis Implies Spectral Synthesis

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Abstract: In this paper we show that spectral analysis implies spectral synthesis for arbitrary varieties on locally compact Abelian groups that have no discrete subgroups of an infinite torsion-free rank.

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1. Introduction

Let $\mathcal{A}$ be a commutative topological algebra and X a topological vector module over $\mathcal{A}$. This means that X is a topological vector space and also a module over $\mathcal{A}$ so that the mapping $(a, x) \mapsto a \cdot x$ from $\mathcal{A} \times X$ to X is continuous in both variables. We say that *spectral analysis* holds for X, if every nonzero closed submodule of X contains a finite dimensional submodule. By simple linear algebra, this is equivalent to the property that every nonzero closed submodule contains a common eigenvector for $\mathcal{A}$. On the other hand, we say that *spectral synthesis* holds for X, if every closed submodule of X is the closure of the sum of all its finite dimensional submodules. Clearly, spectral synthesis implies spectral analysis.

This terminology comes from the classical results of L. Schwartz about spectral synthesis (see [1]). In that case, $\mathcal{A}$ is the *measure algebra* $\mathcal{M}_c(\mathbb{R})$: the space of all compactly supported complex Borel measures on $\mathbb{R}$ with the linear operations and with the convolution product. The topology of $\mathcal{M}_c(\mathbb{R})$ is the weak*-topology when $\mathcal{M}_c(\mathbb{R})$ is considered the topological dual of the space $\mathcal{C}(\mathbb{R})$, the space of all continuous complex valued functions on $\mathbb{R}$ equipped with the topology of uniform convergence on compact sets. Then, $\mathcal{C}(\mathbb{R})$ is a topological vector module over $\mathcal{M}_c(\mathbb{R})$, where the latter acts on the function space by the convolution of compactly supported measures with continuous functions. Here, the closed submodules are exactly the so-called *varieties*: these are the closed, translation invariant linear subspaces of $\mathcal{C}(\mathbb{R})$. The common eigenvectors of $\mathcal{M}_c(\mathbb{R})$ are exactly the exponential functions, and the finite-dimensional varieties are exactly those consisting of *exponential polynomials*, which are finite linear combinations of exponential functions with polynomial coefficients. Schwartz's famous theorem reads as follows:

Theorem 1 ([1]). *Spectral synthesis holds for* $\mathcal{C}(\mathbb{R})$.

From the pure algebraic point of view, varieties are the invariant subspaces of all translation operators. Spectral analysis for an invariant subspace means that there is an eigenvector in every invariant subspace, and spectral synthesis means that every invariant subspace has a spectral decomposition into—possibly infinitely many—finite dimensional eigenspaces, where the dimension corresponds to the "multiplicity" of the given eigenvector. Schwartz's theorem says that every invariant subspace of all translation operators has

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sufficiently many eigenvectors so that the corresponding eigenspaces provide a "spectral decomposition" for the invariant subspace.

The above setting is general enough to cover several interesting cases. In [2], M. Lefranc proved spectral synthesis on $\mathbb{Z}^n$ for each natural number n, where obviously $\mathcal{C}(\mathbb{Z}^n)$ is the set of all complex valued functions on $\mathbb{Z}^n$. A natural question is what about spectral synthesis for $\mathcal{C}(G)$, where G is an arbitrary locally compact Abelian group? In [3], the authors characterized those discrete Abelian groups, where spectral synthesis holds. On the other hand, it turned out that Schwartz's result cannot be generalized to several real variables: in [4], D. Gurevich presented a variety of $\mathbb{R}^2$ for which spectral analysis does not hold, and another one which contains finite dimensional subvarieties, but those do not span a dense subvariety. Spectral analysis and spectral synthesis results have been presented in [5–7].

A complete description of those locally compact Abelian groups where spectral synthesis holds for the space of all continuous functions was obtained in [8], where the present author proved the following two theorems:

Theorem 2. *Spectral synthesis holds on the compactly generated locally compact Abelian group if and only if it is topologically isomorphic to* $\mathbb{R}^a \times Z^b \times C$*, where* C *is compact, and* a, b *are nonnegative integers with* $a \leqslant 1$.

Theorem 3. *Spectral synthesis holds on the locally compact Abelian group* G *if and only if* G/B *is topologically isomorphic to* $\mathbb{R}^a \times Z^b \times F$*, where* B *is the subgroup of all compact elements in* G*,* F *is a discrete Abelian group of finite rank, and* a, b *are nonnegative integers with* $a \leqslant 1$.

In the last two theorems, the condition a is 0 or 1 is the consequence of Gurevich's counterexamples. In both examples, the given variety contains finite dimensional varieties, still, they do not span the whole variety. The point is, that we have to underline that the existence of finite dimensional subvarieties in a variety does not mean that spectral analysis holds for the variety: the requirement for spectral analysis is that all nonzero subvarieties must contain finite dimensional subvarieties. In Gurevich's examples, even spectral analysis fails to hold for the given varieties: not *every* nonzero subvariety includes finite dimensional subvarieties.

This observation leads to the following problem: is there a variety of locally compact Abelian groups for which spectral analysis holds, but spectral synthesis fails to hold? In the discrete case, the result in [3] says that spectral synthesis holds in **every variety** in the group if and only if the torsion-free rank of the group is finite, but in an earlier paper it has been proved that on a discrete Abelian group spectral analysis holds in **every variety** if and only if the torsion-free rank is less than the continuum (see, e.g., [9]). Here, the key is every variety. But what about a single variety? What is the exact connection between spectral analysis and spectral synthesis for a single variety? We know that synthesis implies analysis, but what about the converse? This question has not been answered yet, not even by the characterization theorems, nor by the counterexamples. The purpose of this paper is to present a somewhat unexpected result: in fact, spectral analysis and spectral synthesis are equivalent for any variety of locally compact Abelian groups, if the group does not contain a discrete subgroup of the infinite torsion-free rank. Roughly speaking, apart from the presence of discrete subgroups of the infinite torsion-free rank, every locally compact Abelian group has the property that if in a given variety each nonzero subvariety includes a common eigenvector then, in fact, the given variety possesses a spectral decomposition.

2. Preliminaries

Here, we summarize some known results we shall use in the subsequent paragraphs.

Given a locally compact Abelian group G, the continuous complex homomorphisms of G into the multiplicative group of nonzero complex numbers, resp. into the additive group of complex numbers are called *exponentials*, resp. *additive functions*. A product of additive functions is called a *monomial*, and a linear combination of monomials is called a *polynomial*. A product of an exponential and a polynomial is called an *exponential monomial*, and if the exponential is m, then we call the exponential monomial an *m-exponential monomial*. Hence, polynomials are exactly the 1-exponential monomials. Linear combinations of exponential monomials are called *exponential polynomials*. One-dimensional varieties are exactly those spanned by an exponential, and finite dimensional varieties are exactly those spanned by exponential monomials (see [10]). The *variety of the function* f in $\mathcal{C}(G)$, denoted by $\tau(f)$, is the intersection of all varieties including f.

Annihilators play an important role in spectral analysis and synthesis. For each closed ideal I in $\mathcal{M}_c(G)$ the *annihilator* $\operatorname{Ann} I$ is the set of all functions f in $\mathcal{C}(G)$ satistying $\mu * f = 0$ for each μ in I. The dual concept is the *annihilator* $\operatorname{Ann} V$ of the variety V in $\mathcal{C}(G)$, which is the set of all measures μ in $\mathcal{M}_c(G)$ satisfying $\mu * f = 0$ for each f in V. Clearly, $\operatorname{Ann} I$ is a variety in $\mathcal{C}(G)$ and $\operatorname{Ann} V$ is a closed ideal in $\mathcal{M}_c(G)$, further, we have

$$\operatorname{Ann}\operatorname{Ann} I = I \quad \text{and} \quad \operatorname{Ann}\operatorname{Ann} V = V.$$

(see [10] (Section 11.6), [11] (Section 1)).

Theorem 4. *Let G be a locally compact Abelian group. The following statements are equivalent.*

1. *M is a closed maximal ideal in $\mathcal{M}_c(G)$.*
2. *The residue ring of M is the complex field.*
3. *$\operatorname{Ann} M$ is a one-dimensional variety in $\mathcal{C}(G)$.*
4. *M is the annihilator of the variety of a (unique) exponential.*
5. *M is the closure of the ideal generated by all measures $\delta_{-y} - m(y)\delta_0$ for y in G, where m is a (unique) exponential.*

(See [11] (Section 2), [10]). For the exponential m we write $M_m = \operatorname{Ann} \tau(m)$.

Corollary 1. *Let G be a locally compact Abelian group and V a variety on G. The following statements are equivalent.*

1. *Spectral analysis holds for the variety V.*
2. *Every maximal ideal in the residue ring of* $\operatorname{Ann} V$ *is closed.*

Theorem 5. *Let G be a locally compact Abelian group and M_m a closed maximal ideal in $\mathcal{M}_c(G)$. The following statements are equivalent.*

1. *The function f in $\mathcal{C}(G)$ is an m-exponential monomial.*
2. *The variety $\tau(f)$ is finite dimensional, and there is a natural number k such that M_m^k is in* $\operatorname{Ann} \tau(f)$.

(See [11] (Theorem 12, Theorem 13)).

Corollary 2. *Let G be a locally compact Abelian group and V a variety on G. The following statements are equivalent.*

1. *Spectral synthesis holds for the variety V.*
2. *The annihilator* $\operatorname{Ann} V$ *is the intersection of all closed ideals I satisfying* $\operatorname{Ann} V \subseteq I$ *whose residue ring is a local Artin ring.*
3. *We have*

$$\mathrm{Ann}\, V = \bigcap_{\mathrm{Ann}\, V \subseteq M_m} \bigcap_{k=0}^{\infty} M_m^{k+1},$$

where the first intersection is extended to all exponentials m in V for which $\mathcal{M}_c(G)/M_m^{k+1}$ is finite dimensional, for each k, and the second intersection is extended for those values of k for which $\mathrm{Ann}\, V \subseteq M_m^{k+1}$.

Spectral synthesis is also related to another, more general concept of polynomials: the so-called *generalized polynomials*. Given an exponential maximal ideal M_m in the measure algebra $\mathcal{M}_c(G)$ of the locally compact Abelian group G, the functions in $\mathrm{Ann}\, M_m^{k+1}$ are called *generalized exponential monomials of degree at most k*, for each natural number k, which are called *generalized polynomials*, if $m = 1$. For $k = 0$ these are the constant multiples of m. If $k = 1$, then every generalized exponential monomial of degree at most k has the form $x \mapsto (a(x) + b)m(x)$, where a is additive and b is a complex number; in fact, this is an m-exponential monomial. However, for $k \geqslant 2$ there may exist generalized exponential monomials of degree at most k, which are not exponential monomials, as their variety is infinite dimensional (see [10] (Sections 12.5 and 12.6)). For the variety of a generalized exponential monomial of a degree of at least 2, spectral analysis clearly holds, but this variety is not synthesizable (see [10] (Corollary 15.2.1)). The existence of generalized exponential monomials which are not exponential monomials depends on the torsion-free rank of a discrete Abelian group. We recall that an Abelian group is *torsion-free*, if every nonzero element of the group is of infinite order. In any Abelian group, all the elements of finite order form a subgroup, the *torsion subgroup*, and the corresponding factor group is torsion-free. The rank of this torsion-free group is called the *torsion-free rank* of the group. In fact, on a discrete Abelian group, every generalized exponential polynomial is an exponential polynomial if and only if the torsion-free rank of the group is finite (see [10] (Corollary 13.2.2)). It follows that, when studying the relation between spectral analysis and spectral synthesis on a variety, it is reasonable to assume that the underlying locally compact Abelian group does not contain a discrete subgroup of infinite torsion-free rank. This is equivalent to the assumption that $\mathrm{Hom}\,(G, \mathbb{R})$ is a finite dimensional vector space. In other words, there are only a finite number of linearly independent real characters on G. We note that the torsion-free rank is the dimension of $\mathrm{Hom}\,(G, \mathbb{R})$ as a real vector space.

Theorem 6. *Let G be a locally compact Abelian group that does not contain a discrete subgroup of infinite torsion-free rank. Then, every generalized exponential monomial is an exponential monomial on G. Conversely, if every generalized exponential monomial is an exponential monomial on G, then G does not contain a discrete subgroup of infinite torsion-free rank.*

Proof. Assume that G is a locally compact Abelian group that does not contain a discrete subgroup of infinite torsion-free rank. Let B denote the closed subgroup of all compact elements in G. Clearly, every additive function is zero on B; consequently, every generalized polynomial is constant on B. It follows that every generalized exponential monomial on B is a constant multiple of an exponential. Let $\Phi : G \to G/B$ denote the natural homomorphism. It follows that φ is a generalized exponential monomial on G/B if and only if $\varphi \circ \Phi$ is a generalized exponential monomial on G. On the other hand, G/B is topologically isomorphic to $\mathbb{R}^n \times F$, where n is a natural number and F is a discrete torsion-free Abelian group (see [12] ((24.35) Corollary)). By our assumption, F is of finite rank, hence, it is a homomorphic image of $\mathbb{Z}^k$ for some natural number k, which implies that there are only infinitely many linearly independent additive functions on $\mathbb{R}^n \times F$. It follows that G has the same property.

For the converse, we refer to [10] (Corollary 13.2.2). □

3. Spectral Analysis Implies Spectral Synthesis

Given a locally compact Abelian group G a *representation of the measure algebra* $\mathcal{M}_c(G)$ on the locally convex topological vector space X is a continuous algebra homomorphism of $\mathcal{M}_c(G)$ onto an algebra of the continuous linear operators of X. Here, X is called the *representation space*. In order to avoid trivial cases we always assume that the measure δ_0 is mapped onto the identity operator. Every representation of $\mathcal{M}_c(G)$ makes a natural vector module structure on X with the action $x \mapsto \pi(\mu) \cdot x$, where π is the representation and $\pi(\mu)$ is the linear operator on X corresponding to μ. By the *canonical representation* of $\mathcal{M}_c(X)$ we mean the one with the representation space $\mathcal{C}(X)$ and μ is mapped onto the linear operator $f \mapsto \mu * f$ for each μ in $\mathcal{M}_c(G)$ and f in $\mathcal{C}(G)$. Given a variety V in $\mathcal{C}(G)$, the *canonical representation of* $\mathcal{M}_c(G)$ *on the variety* V is the one where the representation space is V, and the linear operators $f \mapsto \mu * f$ are restricted to V.

Lemma 1. *Let π be a representation of $\mathcal{M}_c(G)$ on the locally convex topological vector space X. Then,* $\operatorname{Ker}\pi$ *is a closed ideal in $\mathcal{M}_c(G)$. Conversely, every closed closed ideal in $\mathcal{M}_c(G)$ is the kernel of a representation of $\mathcal{M}_c(G)$.*

Proof. Clearly, $\operatorname{Ker}\pi$ is a closed linear space in $\mathcal{M}_c(G)$. As π is an algebra homomorphism, it follows that $\operatorname{Ker}\pi$ is closed under multiplication. Further, for each μ in $\operatorname{Ker}\pi$ and ν in $\mathcal{M}_c(G)$, we have

$$\pi(\mu * \nu)(x) = \pi(\mu)(x) \cdot \pi(\nu)(x) = 0,$$

which shows that $\operatorname{Ker}\pi$ is a closed ideal. Conversely, let I be a closed ideal in $\mathcal{M}_c(G)$, then $\mathcal{A} = \mathcal{M}_c(G)/I$ is a locally convex topological vector space which is also a topological algebra. Clearly, the natural mapping Φ of $\mathcal{M}_c(G)$ onto $\mathcal{A}$ is an algebra homomorphism. On the other hand, if V denotes the annihilator of I in $\mathcal{C}(G)$, then V is a variety, and $\mathcal{A}$ is topologically isomorphic to the measure algebra of V. As $\mathcal{M}_c(G)/I$ is topologically isomorphic to $\mathcal{A}$, for any topological isomorphism ι from $\mathcal{M}_c(G)/I$ to $\mathcal{A}$ the mapping $\pi = \iota \circ \Phi$ is a representation of $\mathcal{M}_c(G)$ on V with $\operatorname{Ker}\pi = I$. □

Let π_1, π_2 be two representations of $\mathcal{M}_c(G)$. We say that π_2 *is a subrepresentation of* π_1, if $\operatorname{Ker}\pi_1 \subseteq \operatorname{Ker}\pi_2$. For instance, if V_1, V_2 are varieties on G, then π_{V_1} is a subrepresentation of π_{V_2} if and only if $V_1 \subseteq V_2$.

Lemma 2. *Let V be a variety on G. Then, spectral analysis holds for V if and only if every maximal ideal in $\mathcal{M}_c(G)$, which includes the kernel of some nonzero subrepresentation of the canonical representation of $\mathcal{M}_c(G)$ on V, is closed.*

Proof. By Corollary 1, spectral analysis holds for a variety V if and only if every maximal ideal, which includes $\operatorname{Ann}V$, is closed. Suppose that spectral analysis holds for V, and π is a subrepresentation of π_V, the canonical representation of $\mathcal{M}_c(G)$ on V. The measure μ is in $\operatorname{Ker}\pi_V$ if and only if $\mu * f = 0$ for each f in V, hence $\operatorname{Ker}\pi_V$ is the annihilator of V. Assume that M is a maximal ideal in $\mathcal{M}_c(G)$ with $M \supseteq \operatorname{Ker}\pi$. Then, we have $M \supseteq \operatorname{Ker}\pi_V = \operatorname{Ann}V$, consequently, M is closed.

The converse is obvious, as π_V is a subrepresentation of itself, hence every maximal ideal including $\operatorname{Ker}\pi_V = \operatorname{Ann}V$ is closed, which is equivalent to spectral analysis for V. □

Let V be a variety on G and $W \subseteq V$ be a subvariety. The natural representation of $\mathcal{M}_c(G)$ on V/W is defined as follows: for each f in V we let

$$\pi_{V/W}(\mu) \cdot (f + W) = \mu * f + W.$$

Obviously, $\pi_{V/W}(\mu)\cdot(f+W)$ is in V/W, and $\pi_{V/W}(\mu)$ is a linear operator on V/W, further, $\pi_{V/W}:\mu\mapsto\mu_V$ is a representation of $\mathcal{M}_c(G)$ on V/W.

It is obvious that $\pi_{V/W}$ is a subrepresentation of π_V. Indeed, if μ is in $\operatorname{Ker}\pi_V$, then $\pi_V(\mu)=0$, that is, $\mu*f=\pi_V(\mu)*f=0$ for each f in V. Then, we have $\pi_{V/W}(\mu)(f+W)=\mu*f+W$, which is in W, hence, μ is in $\operatorname{Ker}\pi_{V/W}$.

Our main result follows.

Theorem 7. *Let G be a locally compact Abelian group that has no discrete subgroup of infinite torsion-free rank. Spectral synthesis holds for a variety in $\mathcal{C}(G)$ if and only if spectral analysis holds for it.*

Proof. If spectral synthesis holds for a variety, then obviously spectral analysis holds for it.

For the converse, let V be a variety and suppose that spectral analysis holds for V. Let V_e denote the closed subspace generated by all exponential monomials in V: then V_e is a variety. Proving by contradiction, we assume that V_e is a proper subvariety in V and f is in V but not in V_e. Let π_{V/V_e} denote the natural representation of $\mathcal{M}_c(G)$ on the space V/V_e. Then, π_{V/V_e} is a subrepresentation of the canonical representation π_V of $\mathcal{M}_c(G)$ on V. By spectral analysis on V, we find that every maximal ideal in $\mathcal{M}_c(G)$ which includes $\operatorname{Ker}\pi_{V/V_e}$, is closed. If V/V_e is nonzero, then, by spectral analysis, there exists at least one maximal ideal M_0 including $\operatorname{Ker}\pi_{V/V_e}$. The annihilator of M_0 in V/V_e is a subvariety of V/V_e. It is easy to see, that every subvariety of V/V_e is of the form W/V_e with some subvariety W of V which includes V_e. So, we find $\operatorname{Ann}M_0=W/V_e$ in V/V_e. The closed maximal ideal M_0 is generated by the measures $\delta_{-y}-m_0(y)\delta_0$ with y in G, where m_0 is an exponential on G, which is in W (see, e.g., [11] (Section 2)). It follows that if $f+V_e$ is in W/V_e, then we have that $(\delta_{-y}-m_0(y)\delta_0)\cdot(f+V_e)$ is zero in W/V_e, i.e., the function $(\delta_{-y}-m_0(y))*f$ is in V_e, for each y in G. As spectral synthesis obviously holds for V_e, we have that the annihilator of V_e satisfies

$$\operatorname{Ann}V_e=\bigcap_{\operatorname{Ann}V_e\subseteq M_m}\ \bigcap_{k=0}^{\infty}M_m^{k+1},$$

where the first intersection extends to all exponentials m in V, by Theorem 6. Each function of M_0*f is annihilated by $\operatorname{Ann}V_e$, hence f is annihilated by $\operatorname{Ann}V_e*M_0$. This means that f is in $\operatorname{Ann}(\operatorname{Ann}V_e*M_0)$, that is, f is in

$$\sum_m\sum_{k=0}^{\infty}\operatorname{Ann}(M_m^{k+1}*M_0)=\sum_{m\neq m_0}\sum_{k=0}^{\infty}\operatorname{Ann}(M_m^{k+1}\cap M_0)+\sum_{k=1}^{\infty}\operatorname{Ann}M_0^{k+1}=$$

$$\sum_{m\neq m_0}\sum_{k=0}^{\infty}\operatorname{Ann}M_m^{k+1}+\operatorname{Ann}M_0+\sum_{k=1}^{\infty}\operatorname{Ann}M_0^{k+1}=$$

$$\sum_{m\neq m_0}\sum_{k=0}^{\infty}\operatorname{Ann}M_m^{k+1}+\sum_{k=1}^{\infty}\operatorname{Ann}M_0^{k}=\sum_m\sum_{k=0}^{\infty}\operatorname{Ann}M_m^{k+1}.$$

The elements of the variety $\operatorname{Ann}M_m^{k+1}$ are m-exponential monomials. Consequently, the elements of the variety $\operatorname{Ann}(\operatorname{Ann}V_e*M_0)$ are limits of exponential polynomials in V. We conclude that f is a limit of exponential polynomials, hence it is in V_e, a contradiction. Our theorem is proved. □

4. Conclusions

Every variety on a locally compact Abelian group G can be realized as the solution space of a system of convolution equations:

$$\mu * f = 0 \quad \text{for each} \quad \mu \in I,$$

where I is a subset in the measure algebra $\mathcal{M}_c(G)$. The solution space is exactly $\operatorname{Ann} I$. *Synthesizability* of such a system means that every solution can be uniformly approximated on compact sets by linear combinations of *exponential monomial solutions*. *Spectral synthesis* for such a system means that every extension of the system is synthesizable: by adding any new equations to the system any solution of that new system can be approximated by exponential polynomial solutions of *the extended system*. It is easy to see that if an exponential monomial is a solution of such a system, then the corresponding exponential itself is a solution as well. Finally, *spectral analysis* for such a system of convolution equations means that not just the system, but *every nontrivial extension of the system* has exponential solutions. We mean by "trivial extension" a system which has only the trivial solution. Our main theorem shows that if the underlying group has no discrete subgroup with infinitely many linearly independent real characters, then if every nontrivial extension of a system of convolution equations has an exponential solution, then every such extension has the property that all solutions can be uniformly approximated on compact sets by exponential monomial solutions. We note that the proof given by J. P. Kahane in [13] for L. Schwartz's original theorem is based on the statement that spectral analysis implies spectral synthesis for varieties on the reals.

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