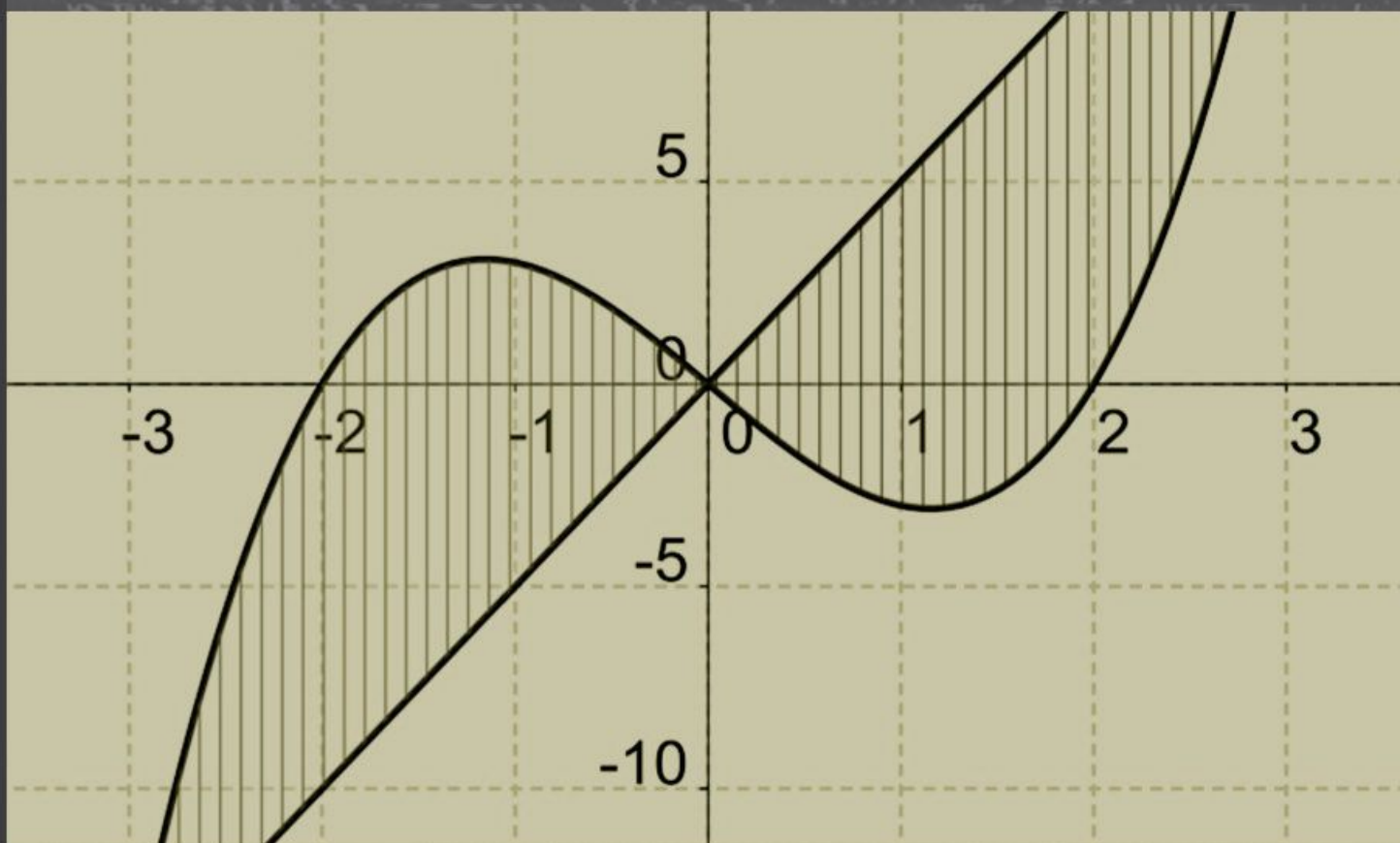


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Differential calculus and integral

for Engineers and Economists



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CALCULUS FOR
ENGINEERS AND ECONOMISTS



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Chapter 1

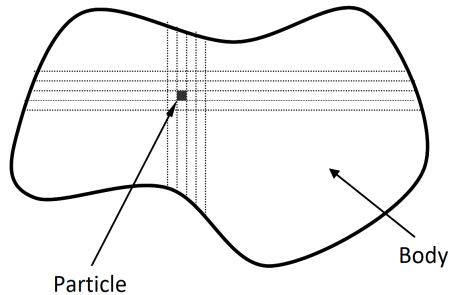
Differential calculus of one variable functions

1.1. Kinematics of a particle: description of the motion

Theoretical summary. If the dimensions of a body are negligible compared to the other dimensions in a mechanical problem then the body can be modeled as a *particle*.

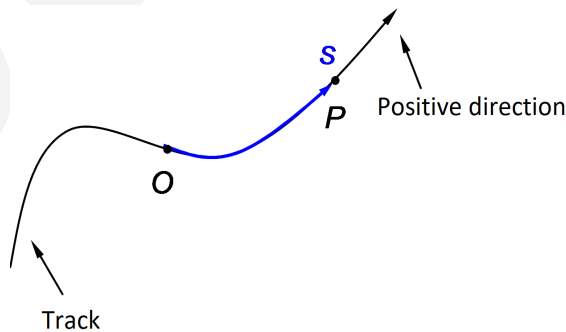
A particle is a geometric point with mass. (e.g. a car in the map of Hungary can be modeled as a particle.)

If a physical body is divided into parts the dimensions of which are negligible compared to the dimensions of the body then these parts can be also modeled as particles.



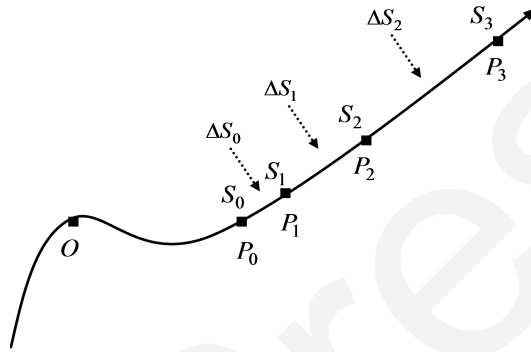
With the above idea we can apply our knowledge about particles for the mechanical analysis of physical bodies. The curve along which the particle moves is called *track* or *path*. If the track of the particle is known then its position can be defined as a scalar quantity.

For the definition of scalar position, we fix an origin O and the positive direction on the track.

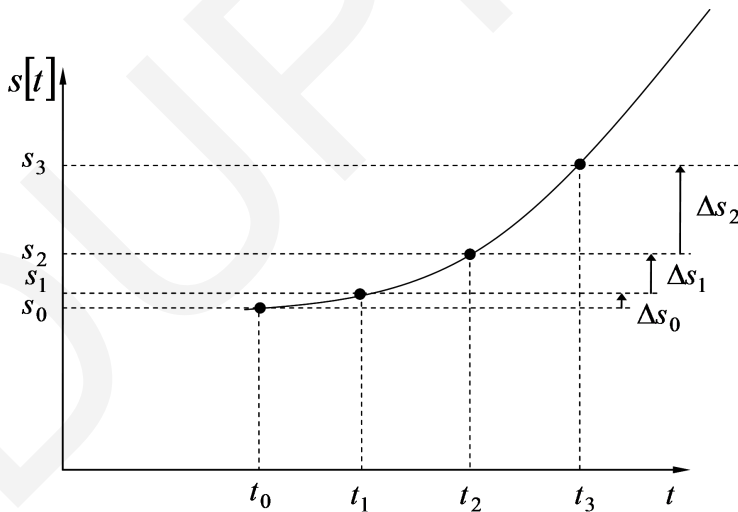


The position the particle (P) on the track can be unambiguously given by the signed arclength s measured from origin O . The above arc length is simply called *position* of the particle. The SI unit of position is meter.

Let's mark the position of a moving particle P on its track from time to time. The time difference between two neighbouring points is equally Δt .



Plot the position-time function of the particle:



The average velocity of the particle in the $[t_i; t_{i+1}]$ time interval is defined by the $\frac{\Delta s_i}{\Delta t}$ ratio.

Decreasing the length of the time interval (Δt) down to zero, we get the instantaneous velocity at moment t_i .

For example the instantaneous velocity at moment t_0 is

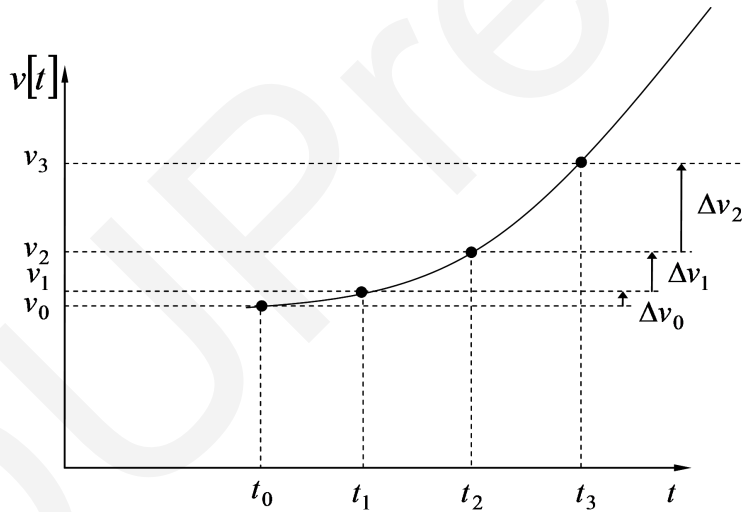
$$v(t_0) = \lim_{\Delta t \rightarrow 0} \frac{\Delta s_0}{\Delta t} = \lim_{t \rightarrow t_0} \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \dot{s}(t_0).$$

Consequently velocity-time function $v(t)$ is the time derivative of position-time function $s(t)$

$$v(t) = \frac{ds}{dt} = \dot{s}(t).$$

In mathematics it is well known that the derivative of a function at a given point is equal to the slope of the tangent line of the function at the same point. Physical meaning the slope of the function $s(t)$ at a point is the instantaneous velocity.

Plot the velocity-time function of the particle



The average acceleration of the particle in the $[t_i; t_{i+1}]$ time interval is defined by the $\frac{\Delta v_i}{\Delta t}$ ratio. Decreasing the lengths of the time interval (Δt) down to zero, we get the instantaneous acceleration at moment t_i . For example the instantaneous acceleration at moment t_0 is

$$a(t_0) = \lim_{\Delta t \rightarrow 0} \frac{\Delta v_0}{\Delta t} = \lim_{t \rightarrow t_0} \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \dot{v}(t_0).$$

Consequently, acceleration-time function $a(t)$ is the time derivative of the velocity-time function $v(t)$

$$a(t) = \frac{dv}{dt} = \dot{v}(t).$$

Physical meaning the slope of function $v(t)$ at a point is the instantaneous acceleration.

Applied mathematical knowledge. Limit calculation, differential calculus, derivatives of basic functions, extreme value calculus.

Sample problem 1. The position-time function of an electric driven race car – which is accelerating from rest – is given

$$s(t) = A \cdot \left[\ln(1 + e^{B(t+C)}) + \ln(1 + e^{-B(t+C)}) - D \cdot t \right] + E.$$

Data:

$$A = 250 \text{ [m];} \quad B = 0.145 \left[\frac{1}{\text{s}} \right]; \quad C = 6.494 \text{ [s];}$$

$$D = 0.064 \left[\frac{1}{\text{s}} \right]; \quad E = -400 \text{ [m].}$$



- Calculate the velocity-time function of the car.
- Find the acceleration-time function of the car.
- Give the value of position, velocity and acceleration 10 [s] after the starting of the car.
- Calculate the top speed of the car.

(The top speed is defined with formula $v_{\max} = \lim_{t \rightarrow \infty} v(t)$.)

- Calculate the acceleration of the car, if $t \rightarrow \infty$.

Solution:

a) Applying the derivation rule of composite functions, we get that

$$\frac{d}{dt}(\ln(1 + e^{B(t+C)})) = \frac{1}{1 + e^{B(t+C)}} \cdot e^{B(t+C)} \cdot B = B \cdot \frac{e^{B(t+C)}}{1 + e^{B(t+C)}}.$$

Similarly

$$\begin{aligned} \frac{d}{dt}(\ln(1 + e^{-B(t+C)})) &= \frac{1}{e^{-B(t+C)} + 1} \cdot e^{-B(t+C)} \cdot (-B) = \\ &= -B \cdot \frac{e^{-B(t+C)}}{1 + e^{-B(t+C)}}. \end{aligned}$$

Applying the formulas, the velocity-time function is

$$\begin{aligned} v(t) &= A \cdot \left[B \cdot \frac{e^{B(t+C)}}{1 + e^{B(t+C)}} - B \cdot \frac{e^{-B(t+C)}}{1 + e^{-B(t+C)}} - D \right] = \\ &= A \cdot \left[B \cdot \frac{e^{B(t+C)} \cdot e^{-B(t+C)}}{1 + e^{-B(t+C)}} - B \cdot \frac{e^{-B(t+C)}}{1 + e^{-B(t+C)}} - D \right] = \\ &= A \cdot \left[B \cdot \frac{1}{1 + e^{-B(t+C)}} - B \cdot \frac{e^{-B(t+C)}}{1 + e^{-B(t+C)}} - D \right] = \\ &= A \cdot B \cdot \frac{1 - e^{-B(t+C)}}{1 + e^{-B(t+C)}} - A \cdot D = \\ &= \left(36.25 \cdot \frac{1 - e^{-0.145 \cdot (t+6.494)}}{1 + e^{-0.145 \cdot (t+6.494)}} - 16 \right) \left[\frac{\text{m}}{\text{s}} \right]. \end{aligned}$$

b) Applying the fact, that the acceleration-time function is the time derivative of the velocity-time function, we get

$$\begin{aligned} \frac{d}{dt} \left(\frac{1 - e^{-B(t+C)}}{1 + e^{-B(t+C)}} \right) &= \frac{-e^{-B(t+C)} \cdot (-B) \cdot (1 + e^{-B(t+C)})}{(1 + e^{-B(t+C)})^2} - \\ &- \frac{(1 - e^{-B(t+C)}) \cdot e^{-B(t+C)} \cdot (-B)}{(1 + e^{-B(t+C)})^2} = \\ &= B \cdot \frac{e^{-B(t+C)} + e^{-2B(t+C)} + e^{-B(t+C)} - e^{-2B(t+C)}}{(1 + e^{-B(t+C)})^2} = \\ &= 2B \cdot \frac{e^{-B(t+C)}}{(1 + e^{-B(t+C)})^2}. \end{aligned}$$

Thus the acceleration-time function of the car is

$$\begin{aligned} a(t) &= \dot{v}(t) = 2A \cdot B^2 \cdot \frac{e^{-B(t+C)}}{(1 + e^{-B(t+C)})^2} = \\ &= 10.51 \cdot \frac{e^{-0.145 \cdot (t+6.494)}}{(1 + e^{-0.145 \cdot (t+6.494)})^2} \left[\frac{\text{m}}{\text{s}^2} \right]. \end{aligned}$$

c) The value of position 10 [s] after the starting of the car

$$\begin{aligned} s(10) &= 250 \cdot \left[\ln(1 + e^{0.145 \cdot (10+6.494)}) + \right. \\ &\quad \left. + \ln(1 + e^{-0.145 \cdot (10+6.494)}) - 0.064 \cdot 10 \right] - 400 = 81.67 \text{ [m]}. \end{aligned}$$

The value of velocity 10 [s] after the starting of the car is

$$v(10) = 36.25 \cdot \frac{1 - e^{-0.145 \cdot (10+6.494)}}{1 + e^{-0.145 \cdot (10+6.494)}} - 16 = 14.17 \left[\frac{\text{m}}{\text{s}} \right].$$

The value of acceleration 10 [s] after the starting of the car is

$$a(10) = 10.51 \cdot \frac{e^{-0.145 \cdot (10+6.494)}}{(1 + e^{-0.145 \cdot (10+6.494)})^2} = 0.81 \left[\frac{\text{m}}{\text{s}^2} \right].$$

d) The limit at infinity of formula $e^{-0.145 \cdot (t+6.494)}$

$$\lim_{t \rightarrow \infty} e^{-0.145 \cdot (t+6.494)} = 0.$$

Applying this result, the top speed is

$$\begin{aligned} v_{\max} &= \lim_{t \rightarrow \infty} \left(36.25 \cdot \frac{1 - e^{-0.145 \cdot (t+6.494)}}{1 + e^{-0.145 \cdot (t+6.494)}} - 16 \right) = \\ &= 36.25 - 16 = 20.25 \left[\frac{\text{m}}{\text{s}} \right]. \end{aligned}$$

e) Applying that

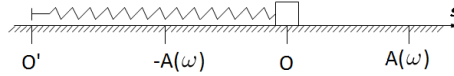
$$\lim_{t \rightarrow \infty} e^{-0.145 \cdot (t+6.494)} = 0$$

we get that the value of acceleration at infinity is

$$\begin{aligned} \lim_{t \rightarrow \infty} a(t) &= \lim_{t \rightarrow \infty} \left(10.51 \cdot \frac{e^{-0.145 \cdot (t+6.494)}}{(1 + e^{-0.145 \cdot (t+6.494)})^2} \right) = \\ &= 10.51 \cdot 0 = 0 \left[\frac{\text{m}}{\text{s}^2} \right]. \end{aligned}$$

Sample problem 2. The figure below illustrates a particle that is on a horizontal, frictionless surface:

$$a \cdot \sin(\omega \cdot t)$$



One end of the spring is attached to the particle, the other one is to point O' , which is forced to vibrate according to the function

$$s^*(t) = a \cdot \sin(\omega \cdot t).$$

As a result the particle will vibrate about point O . Assuming that the drag force (e.g. air resistance force) is linearly proportional to the speed of the particle, its position-time function is as follows

$$s(t) = A(\omega) \cdot \sin(\omega \cdot t + \varphi(\omega)).$$

Notations $A(\omega)$ and $\varphi(\omega)$ indicate that amplitude A and angle φ depend on frequency ω . It can be proved that the $A(\omega)$ and $\varphi(\omega)$ functions can be given as follows

$$A(\omega) = \frac{b}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4 \cdot \beta^2 \cdot \omega^2}}$$

$$\varphi(\omega) = \text{arctg} \left(\frac{2\beta \cdot \omega}{\omega_0^2 - \omega^2} \right).$$

The parameters in the formulas are as follows

$$b = \frac{k \cdot a}{m}; \quad \beta = \frac{c}{2m}; \quad \omega_0 = 2\pi \cdot f_0 = \sqrt{\frac{k}{m}}; \quad k = 1\,000 \left[\frac{\text{N}}{\text{m}} \right];$$

$$m = 1 \text{ [kg]}; \quad c = 20 \left[\frac{\text{N} \cdot \text{s}}{\text{m}} \right]; \quad a = 0.1 \text{ [m]},$$

where m is the mass of the particle, k is the stiffness of the spring, c and β are the drag constant and damping factor, f is the excitation frequency, f_0 is the natural frequency which consist of the spring and the body. (That is, the frequency with which the body vibrates if point O' is fixed.)

- Find the first derivative of function $A(\omega)$.
- Find the second derivative of function $A(\omega)$.
- Determine that value of variable ω at which the value of function $A(\omega)$ is maximum (ω_r).
- Determine that value of ω and $A(\omega)$ in question c), if $\beta = 0$.

- e) Sketch the graph of function $A(\omega)$.
- f) Give the monotonicity and convexity of function $\varphi(\omega)$ in the interval $[0; \omega_r]$, then plot the function.

Solution:

- a) Substituting the given values, we get that

$$b = 100 \left[\frac{\text{m}}{\text{s}^2} \right]; \quad \omega_0 = \sqrt{1000} \left[\frac{1}{\text{s}} \right]; \quad \beta = 10 \left[\frac{1}{\text{s}} \right].$$

Function $A(\omega)$ is

$$\begin{aligned} A(\omega) &= \frac{100}{\sqrt{(1000 - \omega^2)^2 + 400\omega^2}} = \frac{100}{\sqrt{10^6 - 2000\omega^2 + \omega^4 + 400\omega^2}} = \\ &= \frac{100}{\sqrt{10^6 - 1600\omega^2 + \omega^4}}. \end{aligned}$$

Transform it into an equivalent form

$$A(\omega) = 100 \cdot (10^6 - 1600\omega^2 + \omega^4)^{-\frac{1}{2}}.$$

The derivative of function $A(\omega)$ is

$$\begin{aligned} A'(\omega) &= -50 \cdot (10^6 - 1600\omega^2 + \omega^4)^{-\frac{3}{2}} \cdot (-3200\omega + 4\omega^3) = \\ &= -50 \cdot \frac{4\omega^3 - 3200\omega}{(10^6 - 1600\omega^2 + 10^6)^{\frac{3}{2}}} = \frac{-200\omega^3 + 160000\omega}{(\omega^4 - 1600\omega^2 + 10^6)^{\frac{3}{2}}}. \end{aligned}$$

- b) The second derivative of function $A(\omega)$ is

$$\begin{aligned} A''(\omega) &= \frac{(-600\omega^2 + 160000) \cdot (\omega^4 - 1600\omega^2 + 10^6)^{\frac{3}{2}}}{(\omega^4 - 1600\omega^2 + 10^6)^3} - \\ &- \frac{(-200\omega^3 + 160000\omega) \cdot \frac{3}{2} \cdot (\omega^4 - 1600\omega^2 + 10^6)^{\frac{1}{2}} \cdot (4\omega^3 - 3200\omega)}{(\omega^4 - 1600\omega^2 + 10^6)^3}. \end{aligned}$$

Simplifying the formula by expression

$$(\omega^4 - 1600\omega^2 + 10^6)^{\frac{1}{2}}$$

we get that

$$\begin{aligned} A''(\omega) &= \frac{(-600\omega^2 + 160\,000) \cdot (\omega^4 - 1\,600\omega^2 + 10^6)}{(\omega^4 - 1\,600\omega^2 + 10^6)^{\frac{5}{2}}} - \\ &= \frac{(-200\omega^3 + 160\,000\omega) \cdot \frac{3}{2} \cdot (4\omega^3 - 3\,200\omega)}{(\omega^4 - 1\,600\omega^2 + 10^6)^{\frac{5}{2}}} = \\ &= \frac{600\omega^6 - 800\,000\omega^4 - 88\,000\,000\omega^2 + 160\,000\,000\,000}{(\omega^4 - 1\,600\omega^2 + 1\,000\,000)^{\frac{5}{2}}}. \end{aligned}$$

- c) To get the extremum values of function $A(\omega)$ we have to solve equation $A'(\omega) = 0$:

$$\begin{aligned} \frac{-200\omega^3 + 160\,000\omega}{(\omega^4 - 1\,600\omega^2 + 1\,000\,000)^{\frac{3}{2}}} &= 0 \\ -200\omega^3 + 160\,000\omega &= 0 \\ -200\omega \cdot (\omega^2 - 800) &= 0. \end{aligned}$$

The solutions of the equation $A'(\omega) = 0$ are $\omega_1 = 0$ and $\omega_{2,3} = \pm\sqrt{800}$.

Since $A(\omega) \geq 0$, we get that $\omega_1 = 0$ or $\omega_2 = \sqrt{800}$.

Since

$$A''(0) = \frac{\sqrt{1\,000\,000}}{6\,250\,000} > 0$$

and

$$A''(\sqrt{800}) = -\frac{\sqrt{360\,000}}{405\,000} < 0,$$

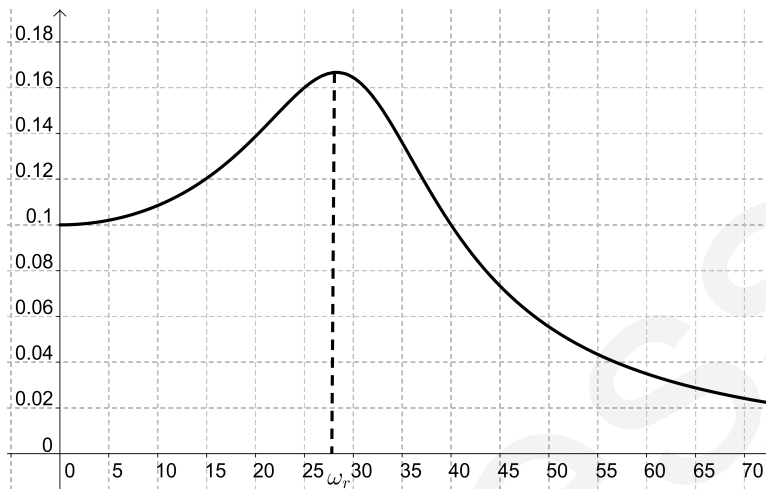
we get that $\omega_1 = 0 \left[\frac{1}{s}\right]$ is the position of local minimum and $\omega_2 = \sqrt{800} \left[\frac{1}{s}\right]$ is the one of the global maximum of function $A(\omega)$. Thus the value of ω at which the amplitude is maximum: $\omega_r = \sqrt{800} \left[\frac{1}{s}\right]$. Consequently, the value of maximum is

$$\begin{aligned} A(\omega_r) &= A(\sqrt{800}) = \frac{100}{\sqrt{40\,000 + 400 \cdot 800}} = \\ &= \frac{100}{\sqrt{360\,000}} = \frac{1}{6} \text{ [m]}. \end{aligned}$$

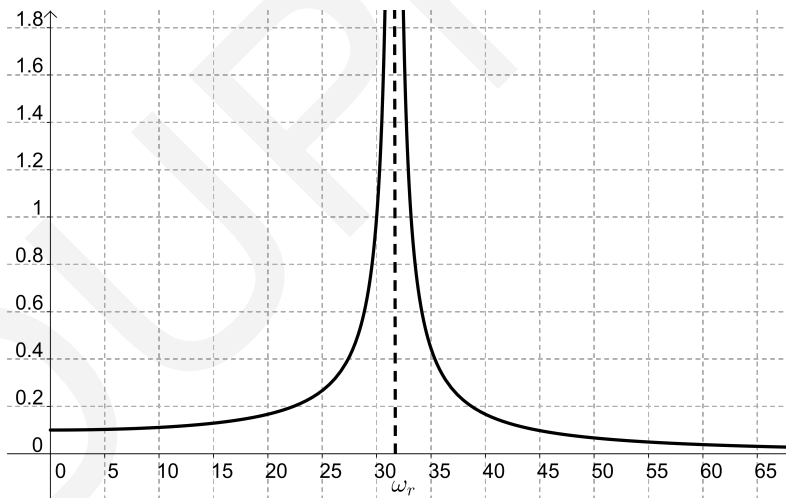
- d) If $\beta \rightarrow 0$ then $\omega_r \rightarrow \sqrt{1\,000}$ and $A(\omega_r) \rightarrow \infty$. Thus, when β is equal to zero, then ω_r is equal to the natural frequency of the system (ω_0).

At the above frequency (ω_0) the value of the amplitude is infinite. Frequency ω_r is called as the *resonance frequency*.

e) The graph of function $A(\omega)$ is:



The graph of function $A(\omega)$ if $\beta = 0$ is:



f) Substituting the data, we get that

$$\varphi(\omega) = \operatorname{arctg} \left(\frac{20 \cdot \omega}{1000 - \omega^2} \right).$$

The derivative of function $\varphi(\omega)$ is

$$\begin{aligned}\varphi'(\omega) &= \frac{1}{1 + \frac{400\omega^2}{(1000-\omega^2)^2}} \cdot \frac{20 \cdot (1000 - \omega^2) - 20\omega \cdot (-2\omega)}{(1000 - \omega^2)^2} = \\ &= \frac{(1000 - \omega^2)^2}{1000000 - 1600\omega^2 + \omega^4} \cdot \frac{20\omega^2 + 20000}{(1000 - \omega^2)^2} = \\ &= \frac{20 \cdot (\omega^2 + 1000)}{\omega^4 - 1600\omega^2 + 1000000}.\end{aligned}$$

If $\varphi'(\omega) > 0$ then the function $\varphi(\omega)$ is strictly monotonic. Furthermore

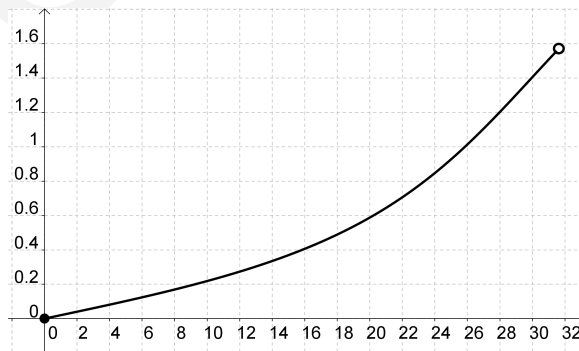
$$\begin{aligned}\varphi''(\omega) &= \frac{20 \cdot 2\omega \cdot (\omega^4 - 1600\omega^2 + 10^6)}{(\omega^4 - 1600\omega^2 + 10^6)^2} - \\ &- \frac{20 \cdot (\omega^2 + 1000) \cdot (4\omega^3 - 3200\omega)}{(\omega^4 - 1600\omega^2 + 10^6)^2} = \\ &= \frac{-40 \cdot \omega \cdot (\omega^4 + 2000 \cdot \omega^2 - 2600000)}{(\omega^4 - 1600 \cdot \omega^2 + 1000000)^2}\end{aligned}$$

thus $\varphi''(\omega) > 0$ for each $\omega \in [0; \sqrt{1000}]$, consequently $\varphi(\omega)$ is a convex function.

The limits of function $\varphi(\omega)$ at 0 and $\sqrt{1000}$ are

$$\begin{aligned}\lim_{\omega \rightarrow 0} \operatorname{arctg} \left(\frac{20 \cdot \omega}{1000 - \omega^2} \right) &= 0 \\ \lim_{\omega \rightarrow \sqrt{1000}} \operatorname{arctg} \left(\frac{20 \cdot \omega}{1000 - \omega^2} \right) &= \frac{\pi}{2}.\end{aligned}$$

The graph of function $\varphi(\omega)$ is:



Sample problem 3. A particle is moving in a straight line. Its position-time function is

$$s(t) = t^3 - 3t \text{ [m]},$$

where t is the time in seconds.

- Find expression for the particle's velocity-time function.
- Find expression for the particle's acceleration-time function.
- Calculate the position, velocity and acceleration of the particle at $t = 0$.
- Calculate the position, velocity and acceleration of the particle at $t = 1$.
- Calculate the position, velocity and acceleration of the particle at $t = 2$.
- Find the position of the particle when it changes direction of the motion.
- Given the time interval(s) in which the speed of the particle is increasing.
- What is the covered distance in $t \in [0; 2]$ time interval?
- Find the average velocity in time interval $[0; 1]$.
- Find the average acceleration in time interval $[0; 1]$.

Solution:

- The velocity-time function is the derivative of the position-time function

$$v(t) = \dot{s}(t) = 3t^2 - 3.$$

- The acceleration-time function is the derivative of the velocity-time function

$$a(t) = \dot{v}(t) = 6t.$$

- If $t = 0$ then

$$\begin{aligned} s(0) &= 0^3 - 3 \cdot 0 = 0 \text{ [m]} \\ v(0) &= 3 \cdot 0^2 - 3 = -3 \left[\frac{\text{m}}{\text{s}} \right] \\ a(0) &= 6 \cdot 0 = 0 \left[\frac{\text{m}}{\text{s}^2} \right]. \end{aligned}$$

- If $t = 1$ then

$$\begin{aligned} s(1) &= 1^3 - 3 \cdot 1 = -2 \text{ [m]} \\ v(1) &= 3 \cdot 1^2 - 3 = 0 \left[\frac{\text{m}}{\text{s}} \right] \\ a(1) &= 6 \cdot 1 = 6 \left[\frac{\text{m}}{\text{s}^2} \right]. \end{aligned}$$

e) If $t = 2$ then

$$s(2) = 2^3 - 3 \cdot 2 = 2 \text{ [m]}$$

$$v(2) = 3 \cdot 2^2 - 3 = 12 - 3 = 9 \left[\frac{\text{m}}{\text{s}} \right]$$

$$a(2) = 6 \cdot 2 = 12 \left[\frac{\text{m}}{\text{s}^2} \right].$$

f) The solution of equation $v(t) = 0$:

$$3t^2 - 3 = 0 \quad \Rightarrow \quad t = \pm 1.$$

Since the sign of function $v(t)$ is changed when $v(t) = 1 \left[\frac{\text{m}}{\text{s}} \right]$, a change in the direction of motion occurs at that instant. In our case

$$s(1) = 1^3 - 3 \cdot 1^2 = -2,$$

thus, when $s = -2 \text{ [m]}$ the direction of motion of the particle is changed.

g) Speed is increasing when $v(t)$ and $a(t)$ have the same sign. Since $a(t) \geq 0$ when $t \geq 0$ and $v(t) \geq 0$ when $t \geq 1$, we get that speed is increasing when $t \geq 1$.

h) The total covered distance is

$$|s(1) - s(0)| + |s(2) - s(1)| = |-2 - 0| + |2 - (-2)| = 6 \text{ [m]}.$$

i) Since $s(1) = -2$ and $s(0) = 0$ the average velocity on time interval $[0; 1] \text{ [s]}$ is

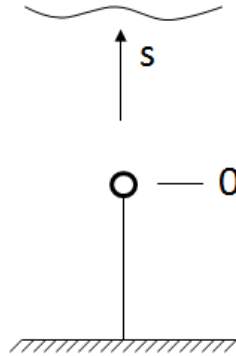
$$\frac{s(1) - s(0)}{1 - 0} = \frac{-2 - 0}{1 - 0} = -2 \left[\frac{\text{m}}{\text{s}} \right].$$

j) Since $v(1) = 0$ and $v(0) = -3$ the average velocity on time interval $[0; 1] \text{ [s]}$ is

$$\frac{v(1) - v(0)}{1 - 0} = \frac{0 - (-3)}{1 - 0} = 3 \left[\frac{\text{m}}{\text{s}} \right].$$

Problems

Problem 1.1.1. The figure below illustrates an underwater spherical buoy that is fixed to the seabed by a cable:



The buoy's density is smaller than the one of water, so the cable is tight and its position is vertical. Suddenly the cable get torn. After that the position-time function of the buoy is

$$s(t) = \frac{1}{A} \cdot \left(t + \frac{1}{A \cdot B} \cdot e^{-A \cdot B \cdot t} + \frac{1}{A \cdot B} \right) \text{ [m]},$$

where

$$A = \frac{4,5 \cdot \eta}{R^2 \cdot (\rho_{\text{water}} - \rho_{\text{buoy}}) \cdot g}; \quad B = g \cdot \frac{\rho_{\text{water}} - \rho_{\text{buoy}}}{\rho_{\text{water}}}.$$

In the formulas η is the dynamic viscosity of water, R is the radius of the buoy g is the magnitude of gravitational acceleration and ρ_{water} and ρ_{buoy} are the mass density of water and buoy respectively.

Data:

$$R = 0.01 \text{ [m]}; \quad \eta = 0.00102 \left[\frac{\text{kg}}{\text{m} \cdot \text{s}} \right]; \quad \rho_{\text{water}} = 1\,000 \left[\frac{\text{kg}}{\text{m}^3} \right];$$

$$\rho_{\text{buoy}} = 990 \left[\frac{\text{kg}}{\text{m}^3} \right]; \quad g = 9.81 \left[\frac{\text{m}}{\text{s}^2} \right].$$

- Substitute the data into the position-time function.
- Determine the velocity-time function of the buoy.
- Determine the acceleration-time function of the buoy.
- Find the value of functions $s(t)$, $v(t)$ and $a(t)$ at $t = 10$ [s].

e) Find the top speed of the buoy.

(Top speed is defined as: $v_{\max} = \lim_{t \rightarrow \infty} v(t)$.)

Function $s(t)$ is the solution of the differential equation of motion of the buoy.

Solution:

a) Substituting the given data into the formulas, we get that

$$A = \frac{4.5 \cdot 0.00102}{(0.01)^2 \cdot (1000 - 900) \cdot 9.81} = 0.4679$$

and

$$B = 9.81 \cdot \frac{1000 - 900}{1000} = 0.981.$$

Thus

$$\frac{1}{A} = 2.137 \quad \text{and} \quad A \cdot B = 0.459 \quad \Rightarrow \quad \frac{1}{A \cdot B} = 2.178.$$

Finally, the position-time function is

$$s(t) = 2.137 \cdot (t + 2.179 \cdot e^{-0.4569t} - 2.179).$$

b) The velocity-time function is the time derivative of the position-time function

$$v(t) = \dot{s}(t) = \frac{1}{A} - \frac{A \cdot B}{A^2 \cdot B} \cdot e^{-A \cdot B \cdot t} = \frac{1}{A} \cdot (1 - e^{-A \cdot B \cdot t}) \left[\frac{\text{m}}{\text{s}} \right].$$

Substituting the values of constants A and B :

$$v(t) = 2.137 \cdot (1 - e^{-0.459t}) \left[\frac{\text{m}}{\text{s}} \right].$$

c) The acceleration-time function is the time derivative of the velocity-time function

$$a(t) = \dot{v}(t) = B \cdot e^{-A \cdot B \cdot t} = 0.981 \cdot e^{-0.459t} \left[\frac{\text{m}}{\text{s}^2} \right].$$

d) The values of functions $s(t)$, $v(t)$ and $a(t)$ at $t = 10$ [s] are

$$s(10) = 2.137 \cdot (10 + 2.179 \cdot e^{-0.4569 \cdot 10} - 2.179) = 16.76 \text{ [m]};$$

$$v(10) = 2.137 \cdot (1 - e^{-0.459 \cdot 10}) = 2.12 \left[\frac{\text{m}}{\text{s}} \right];$$

$$a(10) = 0.981 \cdot e^{-0.459 \cdot 10} = 0.00996 \left[\frac{\text{m}}{\text{s}^2} \right].$$

e) Since

$$\lim_{t \rightarrow \infty} e^{-0.0459t} = 0,$$

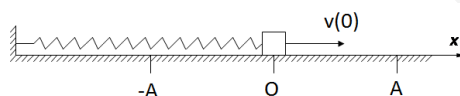
the top speed of the buoy is

$$v_{\max} = \lim_{t \rightarrow \infty} (2.1373 \cdot (1 - e^{-0.0459t})) = 2.1373 \left[\frac{\text{m}}{\text{s}} \right].$$

Problem 1.1.2. The figure below shows a particle attached to an ideal spring. When the particle is at point O the spring is unloaded, thus its extension is zero. The initial velocity of the particle is $v(0)$ and its position-time function is given by the following formula

$$s(t) = A \cdot \sin(\omega \cdot t + \varphi_0) \quad (t \geq 0).$$

Thus its motion is harmonic oscillation about point O .



Data:

$$s(0) = 0 \text{ [m]}; \quad v(0) = 10 \left[\frac{\text{m}}{\text{s}} \right]; \quad \omega = 30 \left[\frac{\text{rad}}{\text{s}} \right].$$

- Calculate the velocity-time function of the particle.
- Calculate its acceleration-time function.
- Determine the value of φ_0 and A ($A > 0$) from functions $s(t)$ and $v(t)$ using initial values $s(0)$ and $v(0)$.
- Give the $s(t)$, $v(t)$ and $a(t)$ functions applying the above results.

Solution:

- The velocity-time function of the particle is

$$v(t) = A \cdot \omega \cdot \cos(\omega \cdot t + \varphi_0) = 30A \cdot \cos(30 \cdot t + \varphi_0) \left[\frac{\text{m}}{\text{s}} \right].$$

- The acceleration-time function of the particle is

$$a(t) = -A \cdot \omega^2 \cdot \sin(\omega \cdot t + \varphi_0) = -900A \cdot \sin(30 \cdot t + \varphi_0) \left[\frac{\text{m}}{\text{s}^2} \right].$$

- Substituting $t = 0$ into functions $s(t)$ and $v(t)$ we get that

$$s(0) = A \cdot \sin \varphi_0$$

$$v(0) = 30A \cdot \cos \varphi_0.$$

Using that $s(0) = 0$ and $v(0) = 10 \left[\frac{\text{m}}{\text{s}} \right]$, we get that

$$\begin{aligned} 0 &= A \cdot \sin \varphi_0 \\ 10 &= 30A \cdot \cos \varphi_0. \end{aligned}$$

Since $A \neq 0$, from the first equation $\varphi_0 = 0$ or $\varphi_0 = \pi$. Thus from the second equation:

$$10 = 30A \quad \Rightarrow \quad A = \frac{1}{3} [\text{m}].$$

- d) Using the results, we get that the position-time, velocity-time and acceleration-time functions are

$$\begin{aligned} s(t) &= \frac{1}{3} \cdot \sin(30 \cdot t) [\text{m}]; \\ v(t) &= 10 \cdot \cos(30 \cdot t) \left[\frac{\text{m}}{\text{s}} \right]; \\ a(t) &= -300 \cdot \sin(30 \cdot t) \left[\frac{\text{m}}{\text{s}^2} \right]. \end{aligned}$$

Problem 1.1.3. A particle moves in a straight line according to the position-time function

$$s(t) = t^3 - 7t^2 + 11t - 3 [\text{cm}],$$

where t is the time in seconds.

- Find expression for the particle's velocity-time function.
- Find expression for the particle's acceleration-time function.
- Find the instant(s) when the velocity is equal to zero.
- Calculate the position, velocity and acceleration of the particle at $t = 0$.
- Calculate the position, velocity and acceleration of the particle at $t = 1$.
- Calculate the position, velocity and acceleration of the particle at $t = 2$.
- Determine the monotonicity of function $s(t)$.
- Calculate the covered distance in the $t \in [0; 3]$ time interval.

Solution:

- a) The velocity-time function is the time derivative of the position-time function

$$v(t) = \dot{s}(t) = 3t^2 - 14t + 11.$$

- b) The acceleration-time function is the time derivative of the velocity-time function

$$a(t) = \dot{v}(t) = 6t - 14.$$

- c) We have to solve the equation

$$3t^2 - 14t + 11 = 0.$$

Applying the quadratic formula, we get that

$$t_{1,2} = \frac{14 \pm \sqrt{196 - 132}}{6} = \frac{14 \pm 8}{6}.$$

Thus the solutions of the quadratic equation are $t = 1$ [s] or $t = \frac{11}{3}$ [s].

- d) If $t = 0$ then:

$$s(0) = 0^3 - 7 \cdot 0^2 + 11 \cdot 0 - 3 = -3 \text{ [cm];}$$

$$v(0) = 3 \cdot 0^2 - 14 \cdot 0 + 11 = 11 \left[\frac{\text{cm}}{\text{s}} \right];$$

$$a(0) = 6 \cdot 0 - 14 = -14 \left[\frac{\text{cm}}{\text{s}^2} \right].$$

Thus the particle is at a distance of 3 [cm] from point O on its left hand side and it is moving at a speed of 11 $\left[\frac{\text{cm}}{\text{s}} \right]$ in right direction.

- e) If $t = 1$, we get that

$$s(1) = 1^3 - 7 \cdot 1^2 + 11 \cdot 1 - 3 = 2 \text{ [cm];}$$

$$v(1) = 3 \cdot 1^2 - 14 \cdot 1 + 11 = 0 \left[\frac{\text{cm}}{\text{s}} \right];$$

$$a(1) = 6 \cdot 1 - 14 = -8 \left[\frac{\text{cm}}{\text{s}^2} \right].$$

- f) If $t = 2$, we get that

$$s(2) = 2^3 - 7 \cdot 2^2 + 11 \cdot 2 - 3 = -1 \text{ [cm];}$$

$$v(2) = 3 \cdot 2^2 - 14 \cdot 2 + 11 = -5 \left[\frac{\text{cm}}{\text{s}} \right];$$

$$a(2) = 6 \cdot 2 - 14 = -2 \left[\frac{\text{cm}}{\text{s}^2} \right].$$

- g) The monotonicity of the function $s(t)$ is

	$t < 1$	$t = 1$	$1 < t < \frac{11}{3}$	$t = \frac{11}{3}$	$t > \frac{11}{3}$
$\dot{s}(t)$	+	0	-	0	+
$s(t)$	↗	loc. max.	↘	loc. min.	↗

h) The total distance covered by the particle in the $[0; 3]$ time interval is

$$s = [s(1) - s(0)] + [s(3) - s(1)].$$

Since

$$s(1) - s(0) = 2 - (-3) = 5,$$

and

$$s(3) - s(1) = -6 - 2 = -8.$$

Consequently the total covered distance is

$$5 + 8 = 13 \text{ [m]}.$$

Problem 1.1.4. A particle is moving in a straight line according to the position-time function

$$s(t) = t^3 - 3t + 1 \text{ [m]},$$

where t is the time in seconds.

- Find expression for the particle's velocity-time function.
- Find expression for the particle's acceleration-time function.
- Calculate the position, velocity and acceleration of the particle at $t = 0$.
- Calculate the position, velocity and acceleration of the particle at $t = 1$.
- Calculate the position, velocity and acceleration of the particle at $t = 2$.
- Find the position of the particle when it changes direction of motion.
- Vizualize the motion of the particle.
- Given the time inreval(s) in which the speed of the particle is increasing.
- What is the covered distance in time interval $t \in [0; 2]$?
- Find the average velocity in time interval $[0; 1]$.
- Find the average acceleration in time interval $[0; 1]$.

Solution:

- a) The velocity-time function is the derivative of the position-time function

$$v(t) = \dot{s}(t) = 3t^2 - 3.$$

- b) The acceleration-time function is the derivative of the velocity-time function

$$a(t) = \dot{v}(t) = 6t.$$

c) If $t = 0$ then:

$$s(0) = 0^3 - 3 \cdot 0 + 1 = 1 \text{ [m];}$$

$$v(0) = 3 \cdot 0^2 - 3 = -3 \left[\frac{\text{m}}{\text{s}} \right];$$

$$a(0) = 6 \cdot 0 = 0 \left[\frac{\text{m}}{\text{s}^2} \right].$$

d) If $t = 1$ then:

$$s(1) = 1^3 - 3 \cdot 1 + 1 = -1 \text{ [m];}$$

$$v(1) = 3 \cdot 1^2 - 3 = 0 \left[\frac{\text{m}}{\text{s}} \right];$$

$$a(1) = 6 \cdot 1 = 6 \left[\frac{\text{m}}{\text{s}^2} \right].$$

e) If $t = 2$ then:

$$s(2) = 2^3 - 3 \cdot 2 + 1 = 3 \text{ [m];}$$

$$v(2) = 3 \cdot 2^2 - 3 = 12 - 3 = 9 \left[\frac{\text{m}}{\text{s}} \right];$$

$$a(2) = 6 \cdot 2 = 12 \left[\frac{\text{m}}{\text{s}^2} \right].$$

We get that the particle is 3 meter to the left of O , moving to the right at a speed of $9 \left[\frac{\text{m}}{\text{s}} \right]$. Thus the particle is at a distance of 3 [m] from point O on its right side.

f) The solution of equation $v(t) = 0$:

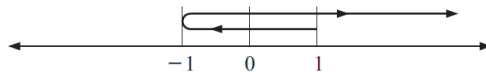
$$3t^2 - 3 = 0 \quad \Rightarrow \quad t = \pm 1.$$

Since the sign of $v(t)$ is changed when $v(t) = 1 \left[\frac{\text{m}}{\text{s}} \right]$, a change in the direction of motion occurs at that instant. In our case:

$$s(1) = 1^3 - 3 \cdot 1^2 + 1 = -1,$$

so, when $s = -1$ [m] the direction of motion of the particle is changed.

g) The motion of the particle can be visualized as follows



h) Speed is increasing when $v(t)$ and $a(t)$ have the same sign. Since $a(t) > 0$ when $t \geq 0$ and $v(t) > 0$ when $t > 1$, we get that speed is increasing when $t > 1$.

i) The total covered distance is

$$|s(1) - s(0)| + |s(2) - s(1)| = |-1 - 1| + |3 - (-1)| = 6 \text{ [m]}.$$

j) Since $s(1) = -1$ and $s(0) = 1$ the average velocity on the $[0; 1]$ [s] time interval is

$$\frac{s(1) - s(0)}{1 - 0} = \frac{-1 - 1}{1 - 0} = -2 \left[\frac{\text{m}}{\text{s}} \right].$$

k) Since $v(1) = 0$ and $v(0) = -3$ the average velocity on the $[0; 1]$ [s] time interval is

$$\frac{v(1) - v(0)}{1 - 0} = \frac{0 - (-3)}{1 - 0} = 3 \left[\frac{\text{m}}{\text{s}} \right].$$

Problem 1.1.5. A particle moving in a straight line, according to the position-time function

$$s(t) = t^2 - 4t + 2 \text{ [m]},$$

where t is the time in seconds.

- Find the average velocity in time interval $[2; 5]$ [s].
- Find expression for the particle's velocity-time function.
- Find the average acceleration in time interval $[2; 5]$ [s].
- Find formula for the particle's acceleration-time function.
- Calculate the position, velocity and acceleration of the particle at $t = 0$ and describe its motion at the above instant.
- Calculate the position, velocity and acceleration of the particle at $t = 1$ and describe its motion at the above instant.
- Describe the motion of the particle at $t = 2$ [s].
- Find the position of the particle at that moment when its direction of motion changes.
- Plot the position-time, velocity-time and acceleration-time functions in time interval $[0; 6]$ in the same coordinate system.

Solution:

a) Since

$$s(5) = 5^2 - 4 \cdot 5 + 2 = 25 - 20 + 2 = 7$$

and

$$s(2) = 2^2 - 4 \cdot 2 + 3 = 4 - 8 + 2 = -2$$

thus the average velocity in the $[2; 5]$ [s] time interval is

$$\frac{s(5) - s(2)}{5 - 2} = \frac{7 - (-2)}{3} = 3 \left[\frac{\text{m}}{\text{s}} \right].$$

- b) The velocity-time function is the derivative of the position-time function

$$v(t) = \dot{s}(t) = 2t - 4.$$

- c) Since

$$v(5) = 5 \cdot 2 - 4 = 6$$

and

$$v(2) = 2 \cdot 2 - 4 = 0$$

thus the average acceleration in the $[2; 5]$ [s] time interval

$$\frac{v(5) - v(2)}{5 - 2} = \frac{6 - 0}{3} = 2 \left[\frac{\text{m}}{\text{s}^2} \right].$$

- d) The acceleration-time function is the derivative of the velocity-time function

$$a(t) = \dot{v}(t) = 2.$$

- e) When $t = 0$:

$$s(0) = 0^2 - 4 \cdot 0 + 2 = 2 \text{ [m]};$$

$$v(0) = 2 \cdot 0^2 - 4 = -4 \left[\frac{\text{m}}{\text{s}} \right];$$

$$a(0) = 2 \left[\frac{\text{m}}{\text{s}^2} \right].$$

We get that the particle is at a distance of 2 meter to the right of point O and moving to the left at a speed of $4 \left[\frac{\text{m}}{\text{s}} \right]$.

- f) When $t = 1$:

$$s(1) = 1^2 - 4 \cdot 1 + 2 = -1 \text{ [m]};$$

$$v(1) = 2 \cdot 1^2 - 4 = -2 \left[\frac{\text{m}}{\text{s}} \right];$$

$$a(1) = 2 \left[\frac{\text{m}}{\text{s}^2} \right].$$

We get that the particle is at a distance of 0 [m] to the left of point O and moving to the left at a speed of $2 \left[\frac{\text{m}}{\text{s}} \right]$.

- g) When $t = 2$:

$$s(2) = 2^2 - 4 \cdot 2 + 2 = -2 \text{ [m]};$$

$$v(2) = 2 \cdot 2 - 4 = 0 \left[\frac{\text{m}}{\text{s}} \right];$$

$$a(2) = 2 \left[\frac{\text{m}}{\text{s}^2} \right].$$

h) The solution of equation $v(t) = 0$ is

$$2t - 4 = 0 \quad \Rightarrow \quad t = 2.$$

Since $v(t)$ changes sign when $v(t) = 2 \left[\frac{\text{m}}{\text{s}} \right]$, we get that

$$s(2) = -2,$$

so the particle changes direction on at point when it is at a distance of 2 [m] left to the point O .

i) The position-time function is

$$s(t) = t^2 - 4t + 2 = (t - 2)^2 - 4 + 2 = (t - 2)^2 - 2.$$

Thus the graph of function $s(t)$ is a parabola.

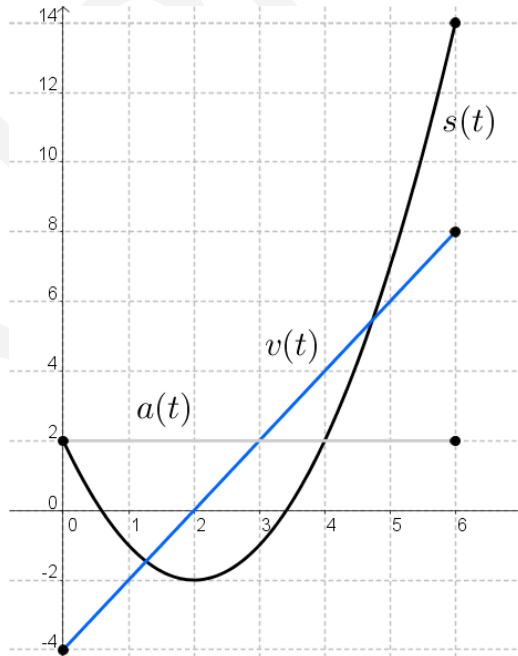
The velocity-time function is

$$v(t) = 2t - 4.$$

Thus the graph of function $v(t)$ is a line.

The acceleration-time function is a constant function.

The position-time, velocity-time and acceleration-time functions are



Problem 1.1.6. The position-time function of a particle is

$$s(t) = 100t + 200 \cdot e^{-\frac{t}{5}} \quad (t \geq 0).$$

- Find its velocity-time function.
- Find its acceleration-time function.
- Calculate its position, velocity and acceleration of the particle at $t = 0$.
- Find its top speed.
- Determine the convexity of velocity-time function.
- Sketch the graph of its velocity-time function.

Solution:

- a) The velocity-time function is

$$v(t) = \dot{s}(t) = 100 - \frac{200}{5} \cdot e^{-\frac{t}{5}} = 100 - 40 \cdot e^{-\frac{t}{5}} \left[\frac{\text{m}}{\text{s}} \right].$$

- b) The acceleration-time function is

$$a(t) = \dot{v}(t) = -\frac{40}{-5} \cdot e^{-\frac{t}{5}} = 8 \cdot e^{-\frac{t}{5}} \left[\frac{\text{m}}{\text{s}^2} \right].$$

- c) The initial values are

$$s(0) = 200 \text{ [m];}$$

$$v(0) = 100 - 40 = 60 \left[\frac{\text{m}}{\text{s}} \right];$$

$$a(0) = 8 \left[\frac{\text{m}}{\text{s}^2} \right].$$

- d) The top speed is

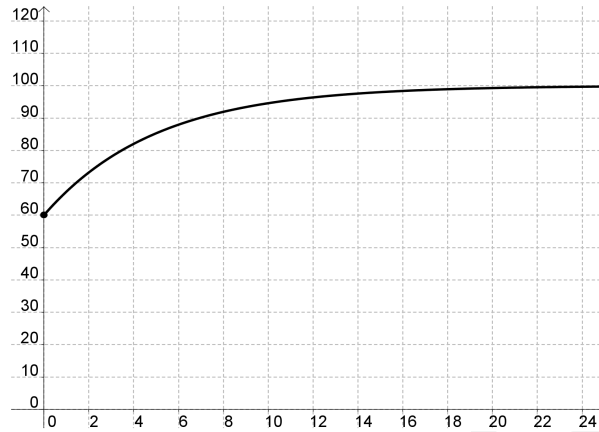
$$v_{\max} = \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} \left(100 - 40 \cdot e^{-\frac{t}{5}} \right) = 100 \left[\frac{\text{m}}{\text{s}} \right].$$

- e) The second derivative of function $v(t)$ is

$$\ddot{v}(t) = -1.6 \cdot e^{-\frac{t}{5}},$$

which is negative for all t , therefore function $v(t)$ is concave.

- f) The graph of the velocity-time function is:



Problem 1.1.7. An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is

$$s(t) = 2 \cos t + 3 \sin t \quad (t \geq 0),$$

where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)

- Find the velocity-time function.
- Find the acceleration-time function.
- Calculate the instant, when the speed is maximum.

Solution:

- a) The velocity-time function is

$$v(t) = \dot{s}(t) = -2 \sin t + 3 \cos t.$$

- b) The acceleration-time function is

$$a(t) = \dot{v}(t) = -2 \cos t - 3 \sin t.$$

- c) We have to solve equation $\dot{v}(t) = 0$, that is

$$-2 \cos t - 3 \sin t = 0.$$

Dividing both sides by $\cos t$:

$$-2 - 3 \operatorname{tg} t = 0 \quad \Rightarrow \quad \operatorname{tg} t = -\frac{2}{3}.$$

The solutions of the equation are

$$t = 0.8128\pi + k \cdot \pi \quad (k \in \mathbb{Z}).$$

Thus the top speed is

$$|v(0.8128\pi)| = 3.6057 \left[\frac{\text{m}}{\text{s}} \right].$$

Problem 1.1.8. The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle Discovery. A model for the velocity-time function of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$ s, is given by

$$v(t) = 0.0004t^3 - 0.0275t^2 + 7.2t - 1 \left[\frac{\text{m}}{\text{s}} \right].$$

Using this model, estimate the absolute maximum and minimum values of the acceleration of the shuttle between liftoff and the jettisoning of the boosters.

Solution:

First, we need to differentiate to find the acceleration-time function

$$a(t) = \dot{v}(t) = 0.0012t^2 - 0.055t + 7.2.$$

Its derivative is

$$\dot{a}(t) = 0.0024t - 0.055.$$

Zero of the function $\dot{a}(t)$ is

$$0.0024t - 0.055 = 0 \quad \Rightarrow \quad t = \frac{0.055}{0.0024} \approx 22.92.$$

Since

$$a(0) = 0.0012 \cdot 0^2 - 0.055 \cdot 0 + 7.2 = 7.2$$

$$a(22.92) = 0.0012 \cdot 22.92^2 - 0.055 \cdot 22.92 + 7.2 = 6.57$$

$$a(126) = 0.0012 \cdot 126^2 - 0.055 \cdot 126 + 7.2 = 19.32,$$

therefore, applying Closed Interval Theorem to continuous function on interval $[0; 126]$, we get that the minimum acceleration is about $6.57 \left[\frac{\text{m}}{\text{s}^2} \right]$ and the maximum acceleration is about $19.32 \left[\frac{\text{m}}{\text{s}^2} \right]$.

Problem 1.1.9. A ball is thrown vertically and its height above the ground is given by

$$s(t) = 1.2 + 28.1t - 4.9t^2$$

meters, where t is the time in seconds.

- From what distance above the ground was the ball released?
- Find $\dot{s}(t)$ and state what it represents.

- c) Find t when $\dot{s}(t) = 0$. What is the significance of this result?
 d) What is the maximum height reached by the ball?
 e) Find the ball's speed when released at $t = 2$ [s].
 f) How long will it take for the ball to hit the ground?
 g) What is the significance of $\ddot{s}(t)$?

Solution:

- a) The was $s(0) = 1.2$ [m] distance.
 b) The derivative of function $s(t)$ is

$$v(t) = \dot{s}(t) = 28.1 - 9.8t,$$

which represents the instantaneous velocity of the ball.

- c) We have to solve the equation $\dot{s}(t) = 0$, that is,

$$28.1 - 9.8t = 0 \quad \Rightarrow \quad t = 2.867 \text{ [s]}.$$

The ball has stopped and reached its maximum height at $t = 2.867$ [s].

- d) The maximum height is

$$s(2.867) = 1.2 + 28.1 \cdot 2.867 - 4.9 \cdot 2.867^2 = 41.49 \text{ [m]}.$$

- e) The velocity at $t = 2$ is

$$v(2) = 28.1 - 9.8 \cdot 2 = 28.1 - 19.6 = 8.5 \left[\frac{\text{m}}{\text{s}} \right].$$

- f) We have to solve the equation $s(t) = 0$, that is,

$$-4.9t^2 + 28.1t + 1.2 = 0.$$

Positive solution of the equation is $t = 5.777$ [s].

- g) The second derivative of function $s(t)$ is

$$\ddot{s}(t) = -9.8,$$

which is the rate of change of $v(t)$, that is, instantaneous acceleration.

Problem 1.1.10. A shell is accidentally fired vertically from a mortar at ground level and reaches the ground again after 14.2 seconds.

- a) Given that its height above the ground at any time t seconds is given by $s(t) = b \cdot t - 4.9t^2$ meters. Calculate the value of b .
 b) Find the initial velocity of the shell.

Solution:

a) Since $s(14.2) = 0$, the next equation holds

$$b \cdot 14.2 - 4.9 \cdot 14.2^2 = 0 \quad \Rightarrow \quad b = 69.58.$$

b) The velocity-time function is

$$v(t) = \dot{s}(t) = b - 9.8t = 69.58 - 9.8t,$$

thus the initial velocity is $v(0) = 69.58 \left[\frac{\text{m}}{\text{s}} \right]$.

Problem 1.1.11. Consider the position-time function of a sports car

$$s(t) = 100 \cdot [\ln(1 + e^t) + \ln(1 + e^{-t})] \text{ [m]},$$

where time is in seconds.

- Give the initial position of the car.
- Calculate the position at $t = 2$.
- Calculate the average velocity at time interval $[0; 2]$ [s].
- Sketch the graph of function $s(t)$.
- Find the velocity-time function of the sports car.
- Give the initial velocity of the car.
- Calculate the velocity at $t = 2$.
- Calculate the average acceleration at time interval $[0; 2]$ [s].
- Calculate the top speed of the car.
- Sketch the graph of the velocity-time function.
- Determine the acceleration-time function.
- Give the initial acceleration of the car.
- Calculate the acceleration at $t = 2$.
- Determine the limits of acceleration-time function at infinity.
- Sketch the graph of the acceleration-time function.

Solution:

a) The initial position is

$$s(0) = 100 \cdot (\ln 2 + \ln 2) = 138.6 \text{ [m]}.$$

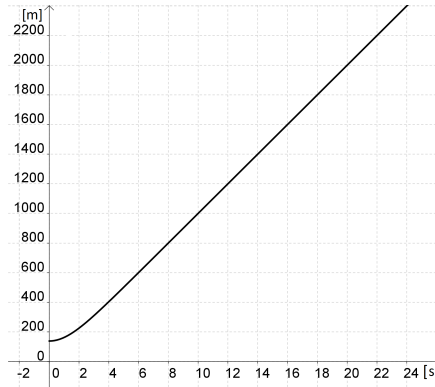
b) The position at $t = 2$ is

$$s(2) = 100 \cdot (\ln(1 + e^2) + \ln(1 + e^{-2})) = 225.4 \text{ [m]}.$$

c) The average velocity at time interval $[0; 2]$ is

$$\frac{s(2) - s(0)}{2 - 0} = \frac{225.4 - 138.6}{2} = 43.4 \left[\frac{\text{m}}{\text{s}} \right].$$

d) The graph of the position-time function is



e) The velocity-time function is

$$v(t) = \dot{s}(t) = 100 \cdot \left(\frac{e^t}{1 + e^t} - \frac{e^{-t}}{1 + e^{-t}} \right).$$

Transforming another form, we get that

$$v(t) = 100 \cdot \left(\frac{e^t}{1 + e^t} - \frac{e^{-t} \cdot e^t}{(1 + e^{-t}) \cdot e^t} \right) = 100 \cdot \frac{e^t - 1}{e^t + 1} \left[\frac{\text{m}}{\text{s}} \right].$$

f) The initial velocity of the sports car is

$$v(0) = 100 \cdot \frac{e^0 - 1}{e^0 + 1} = 0 \left[\frac{\text{m}}{\text{s}} \right].$$

g) The value of velocity-time function at $t = 2$ is

$$v(2) = 100 \cdot \frac{e^2 - 1}{e^2 + 1} = 76.16 \left[\frac{\text{m}}{\text{s}} \right].$$

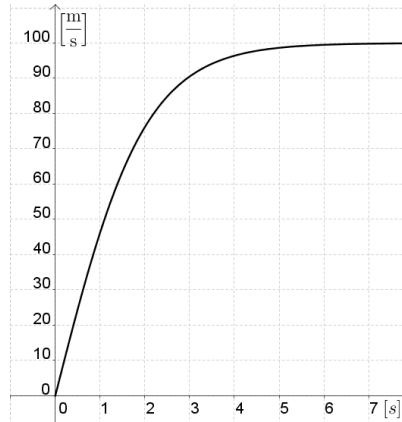
h) The average acceleration is

$$\frac{v(2) - v(0)}{2 - 0} = \frac{76.16 - 0}{2} = 38.08 \left[\frac{\text{m}}{\text{s}^2} \right].$$

i) Applying the L'Hospital Rule, we have the top speed is

$$v_{\max} = \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} 100 \cdot \frac{e^t - 1}{e^t + 1} = 100 \cdot \lim_{t \rightarrow \infty} \frac{e^t}{e^t} = 100 \left[\frac{\text{m}}{\text{s}} \right].$$

j) The graph of the velocity-time function is



k) The acceleration-time function is

$$\begin{aligned} a(t) = \dot{v}(t) &= 100 \cdot \frac{e^t \cdot (e^t + 1) - (e^t - 1) \cdot e^t}{(e^t + 1)^2} = \\ &= 100 \cdot \frac{e^{2t} + e^t - e^{2t} + e^t}{(e^t + 1)^2} = 100 \cdot \frac{2 \cdot e^t}{(e^t + 1)^2} = \frac{200 \cdot e^t}{(e^t + 1)^2} \left[\frac{\text{m}}{\text{s}^2} \right]. \end{aligned}$$

l) The initial acceleration is

$$a(0) = \frac{200 \cdot e^0}{(e^0 + 1)^2} = \frac{200}{4} = 50 \left[\frac{\text{m}}{\text{s}^2} \right].$$

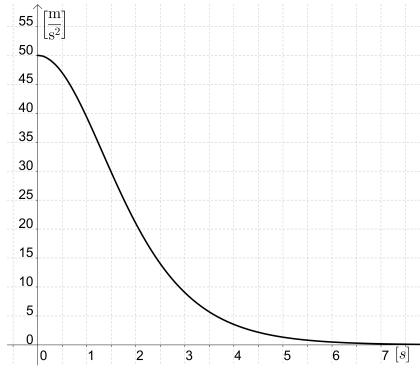
m) The acceleration at $t = 2$ is

$$a(2) = \frac{200 \cdot e^2}{(e^2 + 1)^2} = \frac{200}{4} = 21 \left[\frac{\text{m}}{\text{s}^2} \right].$$

n) Limit of acceleration-time function at infinity is

$$\begin{aligned} \lim_{t \rightarrow \infty} a(t) &= \lim_{t \rightarrow \infty} \frac{200 \cdot e^t}{(e^t + 1)^2} = \lim_{t \rightarrow \infty} \frac{200 \cdot e^t}{2 \cdot (e^t + 1) \cdot e^t} = \\ &= \lim_{t \rightarrow \infty} \frac{100}{e^t + 1} = 0. \end{aligned}$$

o) The graph of the acceleration-time function is



Problem 1.1.12. The position-time function of a particle is

$$s(t) = t + e^{-t} - 1 \text{ [m]},$$

where time is in seconds.

- Give the initial position of the particle.
- Calculate the position at $t = 2$.
- Calculate the average velocity at time interval $[0; 2]$ [s].
- Find the velocity-time function of the particle.
- Give the initial velocity of the particle.
- Calculate the velocity at $t = 2$.
- Calculate the average acceleration at time interval $[0; 2]$ [s].
- Calculate the top speed of the particle.
- Determine the acceleration-time function.
- Give the initial acceleration of the particle.
- Calculate the acceleration at $t = 2$.
- Determine the limits of acceleration-time function at infinity.

Solution:

- a) The initial position is

$$s(0) = 0 + e^0 - 1 = 0 \text{ [m]}.$$

- b) The position at $t = 2$ is

$$s(2) = 2 + e^{-2} - 1 = 1 + \frac{1}{e^2} = 1.14 \text{ [m]}.$$

c) The average velocity at time interval $[0; 2]$ is

$$\frac{s(2) - s(0)}{2 - 0} = \frac{1.14 - 0}{2} = 0.57 \left[\frac{\text{m}}{\text{s}} \right].$$

d) The velocity-time function is

$$v(t) = \dot{s}(t) = 1 - e^{-t} \left[\frac{\text{m}}{\text{s}} \right].$$

e) The initial velocity of the sports car is

$$v(0) = 1 - e^0 = 0 \left[\frac{\text{m}}{\text{s}} \right].$$

f) The value of velocity-time function at $t = 2$ is

$$v(2) = 1 - e^{-2} = 1 - \frac{1}{e^2} = 0.865 \left[\frac{\text{m}}{\text{s}} \right].$$

g) The average acceleration is

$$\frac{v(2) - v(0)}{2 - 0} = \frac{0.865 - 0}{2} = 0.4325 \left[\frac{\text{m}}{\text{s}^2} \right].$$

h) Top speed is

$$v_{\max} = \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} 1 - e^{-t} = 0.$$

i) The acceleration-time function is

$$a(t) = e^{-t} \left[\frac{\text{m}}{\text{s}^2} \right].$$

j) The initial acceleration is

$$a(0) = e^0 = 1 \left[\frac{\text{m}}{\text{s}^2} \right].$$

k) The acceleration at $t = 2$ is

$$a(2) = e^{-1} = \frac{1}{e^2} = 0.135 \left[\frac{\text{m}}{\text{s}^2} \right].$$

l) Limit of acceleration-time function at infinity is

$$\lim_{t \rightarrow \infty} a(t) = \lim_{t \rightarrow \infty} e^{-t} = 0.$$

Problem 1.1.13. The position-time function of a particle is

$$s(t) = (t + 1) \cdot e^{-t}.$$

Find its top speed.

Solution:

The velocity-time function is

$$\begin{aligned} v(t) = \dot{s}(t) &= e^{-t} + (t + 1) \cdot e^{-t} \cdot (-1) = \\ &= e^{-t} \cdot (1 - t - 1) = -t \cdot e^{-t}. \end{aligned}$$

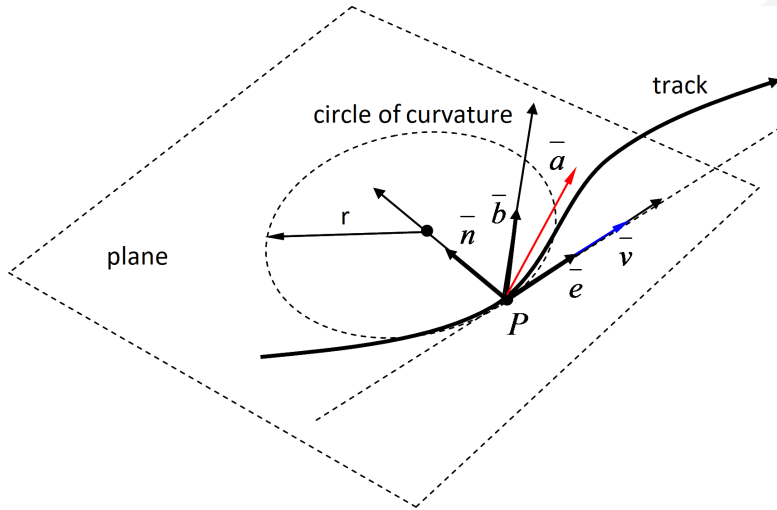
The top speed is

$$v_{\max} = \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} -t \cdot e^{-t} = \lim_{t \rightarrow \infty} -\frac{t}{e^t} = \lim_{t \rightarrow \infty} -\frac{1}{e^t} = 0.$$

1.2. Kinetics of a particle: constrained motion on a 2D or 3D curve

Theoretical summary. If the particle is forced to move on a given 2D or 3D curve, it is practical to describe its motion in *Frenet-Serret frame*.

The Frenet-Serret frame consists of three basis vectors which are perpendicular to each other (vectors \bar{e} , \bar{n} and \bar{b}).



Unit vector \bar{e} is tangential to the track and its sense is the same as the one of the track. Unit vector \bar{n} points towards the centre of curvature. The *binormal unit vector* is defined by the following formula:

$$\bar{b} = \bar{e} \times \bar{n}.$$

The velocity and acceleration of the particle expressed by basis vectors \bar{e} and \bar{n} :

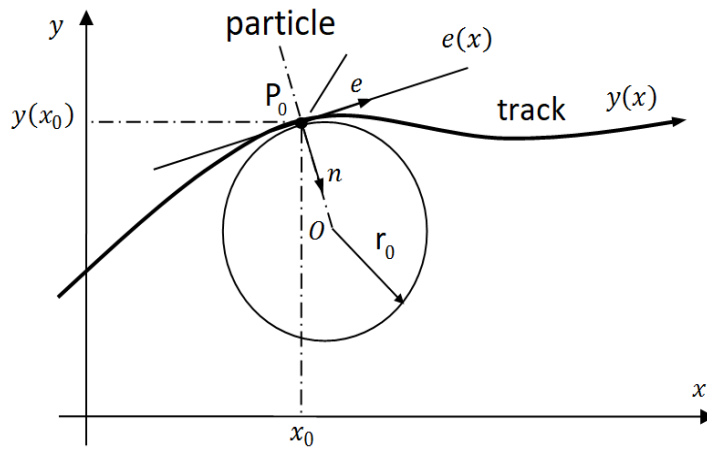
$$\begin{aligned} \bar{v} &= v \cdot \bar{e} \\ \bar{a} &= a \cdot \bar{e} + \frac{v^2}{r} \cdot \bar{n}. \end{aligned}$$

In the above formulas, r is the radius of curvature, v and a are the scalar velocity and acceleration of the particle. The acceleration lies in the plane of basis vectors \bar{e} and \bar{n} , so the binormal acceleration is zero. When the particle is moving on a 2D curve and the equation of its track is given in explicit form,

the radius of curvature at point x_0 is calculated as

$$r_o = \left| \frac{\left(1 + (y'(x_0))^2\right)^{\frac{3}{2}}}{y''(x_0)} \right|.$$

The figure shows the track of a particle and also the tangent line and circle of curvature of the track at point P_0 :



The equation of the tangent line of the track at point P_0 can be given as

$$e(x) = y(x_0) + y'(x_0) \cdot (x - x_0).$$

In Frenet-frame the equation of motion of the particle can be written as

$$\sum_i \bar{F}_i = m \cdot \bar{a}.$$

That is

$$\begin{pmatrix} \sum_i F_{ie} \\ \sum_i F_{in} \\ \sum_i F_{ib} \end{pmatrix} = m \cdot \begin{pmatrix} a \\ \frac{v^2}{r} \\ 0 \end{pmatrix}.$$

Thus we get the scalar equations is

$$\begin{aligned}\sum_i F_{ie} &= m \cdot a \\ \sum_i F_{in} &= m \cdot \frac{v^2}{r} \\ \sum_i F_{ib} &= 0.\end{aligned}$$

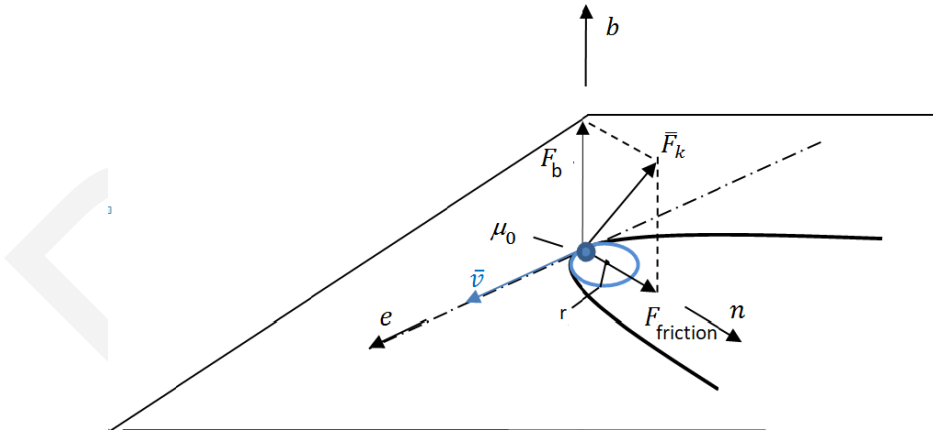
Study of *drifting*:

Solving the equations of motion in Frenet-frame the kinetic study of drifting becomes possible. The problem of drifting can be summarized in the following question:

What is the maximum speed of a vehicle in a horizontal road bend at which it will not drift away from the road perpendicularly to its motion direction. The vehicle will not drift, if and only if

$$|F_{friction}| \leq \mu_0 \cdot F_b.$$

In the inequality above, $F_{friction}$ and F_b are the normal and binormal components of the reaction force that the road exerts on the wheels of the car, μ_0 is the coefficient of static friction between the road and the wheels.

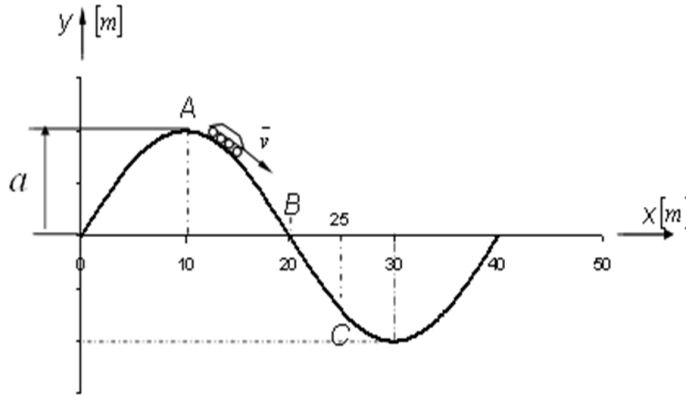


Applied mathematical knowledge. Differential calculus, differentiation rules, equation of the tangent line and circle of curvature of a 2D curve.

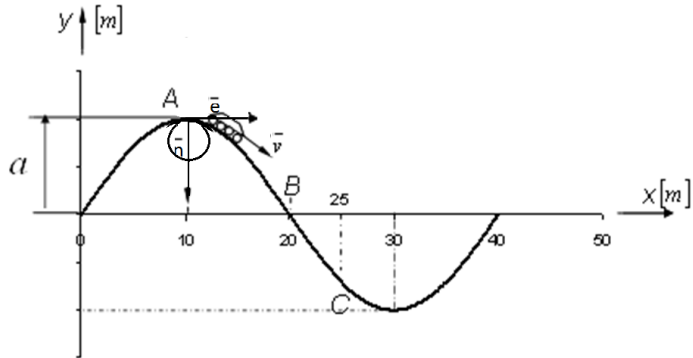
Sample problem 1. The figure shows a roller coaster. The equation of the railway of the roller coaster in the given coordinate system is

$$y = a \cdot \sin(b \cdot x)$$

shape.



- Determine the value of parameter b from the coordinates of point B .
- According to the standards for roller coasters, the angle between the path and the horizontal plane can not be higher than 60° . Determine the maximal allowed value of parameter a .
- Find the coordinates of A of the path when the value of parameter a is maximal.
- Find the coordinates of C of the path when the value of parameter a is maximal.
- Determine the first derivative of function $y(x)$.
- Determine the values of first derivative of function $y(x)$ at point A .
- Determine the values of first derivative of function $y(x)$ at point C .
- Determine the second derivative of function $y(x)$.
- Determine the values of second derivative of function $y(x)$ at point A .
- Determine the values of second derivative of function $y(x)$ at point C .
- Determine the radius of curvature of the path at point A .
- Determine the radius of curvature of the path at point C .
- Calculate the magnitude of the reaction force that the railway exerts on the carriage at point A in normal direction if the velocity magnitude and mass of the carriage is $v_A = 5 \left[\frac{\text{m}}{\text{s}} \right]$ and 500 [kg] , respectively.

Solution:


- a) The coordinates of point B are $x_B = 20$ [m] and $y_B = 0$ [m]. Following that, we get that

$$0 = a \cdot \sin(20b) \quad \Rightarrow \quad 0 = \sin(20b).$$

Consequently, the value of b is

$$20b = \pi \quad \Rightarrow \quad b = \frac{\pi}{20}.$$

The equation of the path is

$$y = a \cdot \sin\left(\frac{\pi}{20}x\right).$$

- b) The slope of the track as a function of variable a can be given as

$$m(x) = y'(x) = \frac{a \cdot \pi}{20} \cdot \cos\left(\frac{\pi}{20}x\right).$$

The slope at angle 60° is

$$m = \operatorname{tg} 60^\circ = \sqrt{3}.$$

The condition $m \leq \sqrt{3}$ has to be fulfilled thus we have to solve the inequality

$$\frac{a \cdot \pi}{20} \cdot \cos\left(\frac{\pi}{20}x\right) \leq \sqrt{3}.$$

It follows from the above inequality that

$$\frac{a \cdot \pi}{20} \cdot \cos\left(\frac{\pi}{20}x\right) \leq \sqrt{3} \quad \Rightarrow \quad \frac{a \cdot \pi}{20} \leq \sqrt{3},$$

thus the value of a is

$$a \leq \frac{20 \cdot \sqrt{3}}{\pi} \approx 11.03.$$

c) When $a = 11.03$, we get that

$$y(x) = \frac{20 \cdot \sqrt{3}}{\pi} \cdot \sin\left(\frac{\pi}{20}x\right).$$

Then

$$\begin{aligned} y(10) &= \frac{20 \cdot \sqrt{3}}{\pi} \cdot \sin\left(\frac{\pi}{20} \cdot 10\right) = \\ &= \frac{20 \cdot \sqrt{3}}{\pi} \cdot \sin\left(\frac{\pi}{2}\right) = \frac{20 \cdot \sqrt{3}}{\pi}. \end{aligned}$$

Thus the coordinates of point A are

$$A = \left(10; \frac{20 \cdot \sqrt{3}}{\pi}\right).$$

d) Since

$$\begin{aligned} y(25) &= \frac{20 \cdot \sqrt{3}}{\pi} \cdot \sin\left(\frac{\pi}{20} \cdot 25\right) = \\ &= \frac{20 \cdot \sqrt{3}}{\pi} \cdot \sin\left(\frac{5\pi}{4}\right) = -\frac{20 \cdot \sqrt{3}}{\pi} \cdot \frac{\sqrt{2}}{2} = -\frac{10 \cdot \sqrt{6}}{\pi} \end{aligned}$$

thus the coordinates of point C are

$$C = \left(25; -\frac{10 \cdot \sqrt{6}}{\pi}\right).$$

e) The derivative of function $y(x)$ is

$$y'(x) = \sqrt{3} \cdot \cos\left(\frac{\pi}{20}x\right).$$

f) The derivative at point A is

$$y'(10) = \sqrt{3} \cdot \cos\left(\frac{\pi}{20} \cdot 10\right) = \sqrt{3} \cdot \cos\left(\frac{\pi}{2}\right) = 0.$$

g) The derivative at point C is

$$y'(25) = \sqrt{3} \cdot \cos\left(\frac{\pi}{20} \cdot 25\right) = \sqrt{3} \cdot \cos\left(\frac{5\pi}{4}\right) = \sqrt{3} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{2}.$$

h) The second derivative of the function $y(x)$ is

$$y''(x) = -\frac{\sqrt{3} \cdot \pi}{20} \cdot \sin\left(\frac{\pi}{20}x\right).$$

i) The second derivative at point A is

$$\begin{aligned} y''(10) &= -\frac{\sqrt{3} \cdot \pi}{20} \cdot \sin\left(\frac{\pi}{20} \cdot 10\right) = \\ &= -\frac{\sqrt{3} \cdot \pi}{20} \cdot \sin\left(\frac{\pi}{2}\right) = -\frac{\sqrt{3} \cdot \pi}{20}. \end{aligned}$$

j) The second derivative at point C is

$$\begin{aligned} y''(25) &= -\frac{\pi}{20} \cdot \sin\left(\frac{\pi}{20} \cdot 25\right) = \\ &= -\frac{\pi}{20} \cdot \sin\left(\frac{5\pi}{4}\right) = \frac{\pi \cdot \sqrt{2}}{40}. \end{aligned}$$

k) The radius of curvature at point A is

$$r_A = \left| \frac{\left(1 + (y'(10))^2\right)^{\frac{3}{2}}}{y''(10)} \right| = \frac{20}{\sqrt{3} \cdot \pi} \approx 3.68.$$

l) The radius of curvature at point C is

$$\begin{aligned} r_C &= \left| \frac{\left(1 + (y'(25))^2\right)^{\frac{3}{2}}}{y''(25)} \right| = \left| \frac{\left(1 + \frac{3}{2}\right)^{\frac{3}{2}}}{\frac{\sqrt{2} \cdot \pi}{40}} \right| = \\ &= \left(\frac{5}{2}\right)^{\frac{3}{2}} \cdot \frac{40}{\sqrt{2} \cdot \pi} \approx 35.59. \end{aligned}$$

m) From the equation above

$$\sum_i F_{in} = m \cdot g - F_n = m \cdot \frac{v_A^2}{r_A},$$

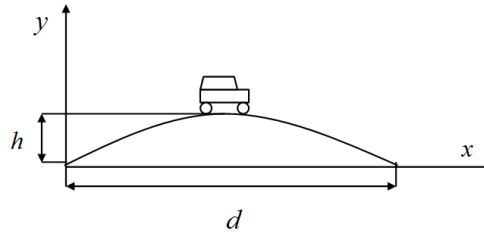
we get that

$$\begin{aligned} F_n &= m \cdot g - m \cdot \frac{v_A^2}{r_A} = 500 \cdot 9.81 - 500 \cdot \frac{5^2}{3.68} = \\ &= 500 \cdot (9.81 - 6.79) = 1510 \text{ [N]}. \end{aligned}$$

Sample problem 2. The figure illustrates a bridge. In the coordinate system, which is given in the figure, the equation of the bridge is

$$y = a \cdot x^2 + b \cdot x + c.$$

During the design, we would like to maximize parameter h , which is the height of the bridge. The width of the bridge is $d = 80$ [m].



- Express parameters a , b and c with the known d and unknown h parameters.
- The incline angle of the roadway should not exceed 30° anywhere. Determine the maximum allowed height of the bridge h_{\max} .
- Determine the radius of curvature of the bridge at its middle point at the maximum value of h .

Solution:

- Points $(0; 0)$, $(80; 0)$ and $(40; h)$ are on the parabola. Substituting their coordinates into the equation of the parabola, we get the following system of equations:

$$c = 0$$

$$6\,400a + 80b = 0$$

$$1\,600a + 40b = h.$$

If we express b from the second equation, we get that

$$b = -80a.$$

Substituting the above formula into the third equation, we get that

$$1\,600a - 3\,200a = h \quad \Rightarrow \quad a = -\frac{h}{1\,600}.$$

Thus, the solution of the system of equations are

$$a = -\frac{h}{1\,600} \quad b = \frac{h}{20}.$$

The equation of the track is

$$y = -\frac{h}{1\,600} \cdot x^2 + \frac{h}{20} \cdot x.$$

- b) The derivative of the above function gives the slope of the track at each point. The derivative of function $y(x)$ is

$$y'(x) = -\frac{h}{800} \cdot x + \frac{h}{20} \quad (x \in [0; 80]).$$

The maximum value of the slope at $x = 0$ is

$$y'(0) = \frac{h}{20}.$$

Since the incline angle should not exceed 30° , we get that

$$\frac{h}{20} \leq \operatorname{tg} 30^\circ \quad \Rightarrow \quad \frac{h}{20} \leq \frac{\sqrt{3}}{3}.$$

If we solve the inequality above, we get that

$$h \leq 20 \cdot \frac{\sqrt{3}}{3} \approx 11.55 \text{ [m]}.$$

Thus the maximum height of the bridge is 11.55 [m].

- c) The derivative of function $y(x)$ is

$$y'(x) = -\frac{h}{800} \cdot x + \frac{h}{20}.$$

Thus, we get that

$$y'(40) = -\frac{h}{800} \cdot 40 + \frac{h}{20} = 0.$$

The second derivative of the function $y(x)$ is

$$y''(x) = -\frac{h}{800}.$$

Thus, we get that

$$y''(40) = -\frac{h}{800}.$$

The radius of the curvature at the middle point of the bridge is

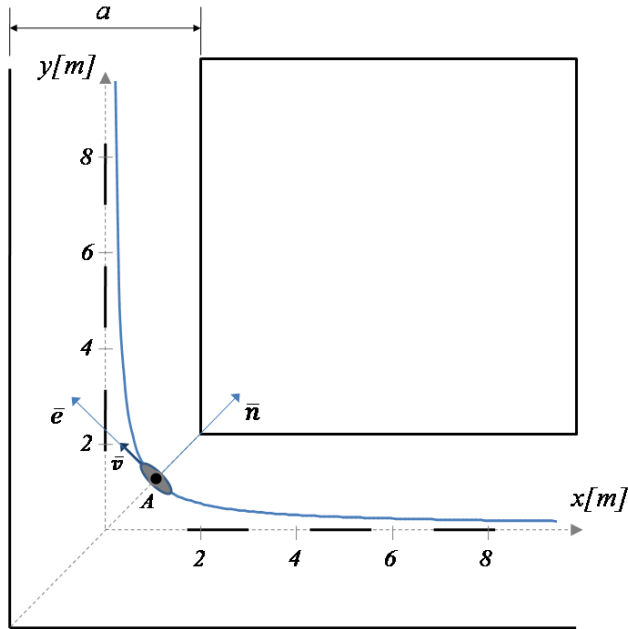
$$r_{\text{mid}} = \left| \frac{\left[1 + (y'(40))^2\right]^{\frac{3}{2}}}{y''(40)} \right| = \left| -\frac{800}{h} \right|.$$

If we substitute the data, we get that the radius is

$$r_{\text{mid}} = \frac{800}{11.55} = 69.26 \text{ [m]}.$$

Sample problem 3. The figure shows the track of a roller blader. The width of the road the roller blader is travelling on is a . The speed of the roller blader is constant v . The equation of its track in the given coordinate system is

$$y(x) = \frac{1}{x}.$$



Data:

$$m = 50 \text{ [kg]}; \quad \mu_0 = 0.1; \quad g = 9.81 \left[\frac{\text{m}}{\text{s}^2} \right]; \quad a = 4 \text{ [m]}.$$

- Calculate the radius of curvature at point A of the track.
- Calculate the maximum speed the roller blader can travel at point A without drifting.

Solution:

- The first derivative of function $y(x)$ is

$$y'(x) = -1 \cdot x^{-2} = -\frac{1}{x^2}.$$

Its second derivative is

$$y''(x) = 2 \cdot x^{-3} = \frac{2}{x^3}.$$

The first and second derivatives of $y(x)$ at $x = 1$ are

$$y'(1) = -\frac{1}{1^2} = -1, \quad y''(1) = \frac{2}{1^3} = 2.$$

The radius of curvature at point A is

$$r_A = \left| \frac{[1 + (y'(1))^2]^{\frac{3}{2}}}{y''(1)} \right| = \left| \frac{[1 + (-1)^2]^{\frac{3}{2}}}{2} \right| = \sqrt{2} \text{ [m]}.$$

b) The equation of motion of the roller blader at point A is

$$\sum_i \bar{F}_i = m \cdot g + \bar{F}_k = m \cdot \bar{a}_A.$$

The motion of equation in *Frenet-Serret frame* is

$$\begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix} + \begin{pmatrix} 0 \\ F_{friction} \\ F_b \end{pmatrix} = m \cdot \begin{pmatrix} a_e \\ a_n \\ a_b \end{pmatrix} = m \cdot \begin{pmatrix} a_A \\ \frac{v_A^2}{r_A} \\ 0 \end{pmatrix}.$$

Thus we get the following equations:

$$\begin{aligned} 0 &= m \cdot a_e = m \cdot a_A \\ F_{friction} &= m \cdot a_n = m \cdot \frac{v_A^2}{r_A} \\ 0 &= m \cdot a_b = -m \cdot g + F_b. \end{aligned}$$

Since $m \neq 0$, from the first equation, we get that $a_A = 0$. From the last equation, we get that $F_b = m \cdot g$. The roller blader does not drift if and only if

$$|F_{friction}| \leq \mu_0 \cdot F_b = \mu_0 \cdot m \cdot g.$$

Expressing v from the equation above:

$$\begin{aligned} m \cdot \frac{v_A^2}{r_A} &\leq \mu_0 \cdot m \cdot g \\ v_A &\leq \sqrt{\mu_0 \cdot g \cdot r_A}. \end{aligned}$$

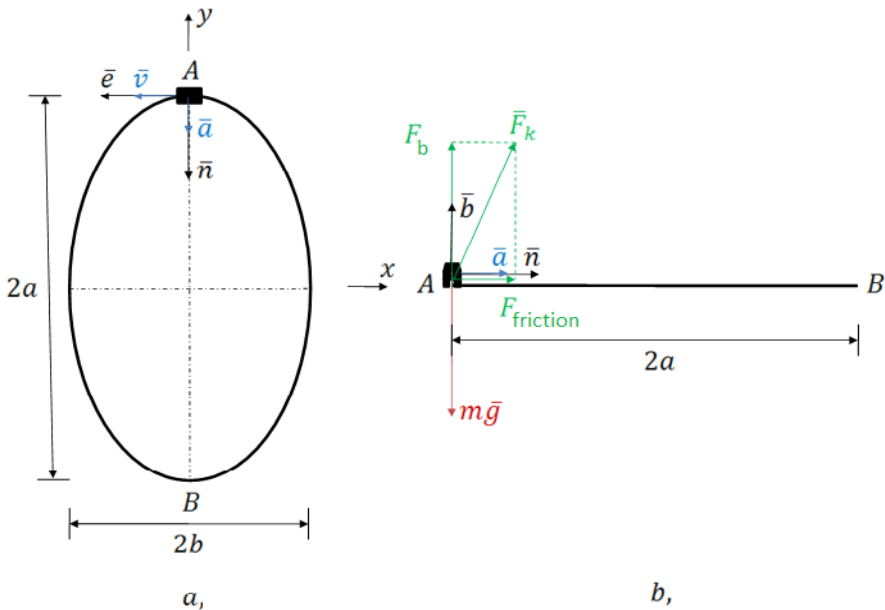
Thus, the value of top speed is

$$v_{Amax} = \sqrt{\mu_0 \cdot g \cdot r_A},$$

where r_A is the value of the radius of curvature at point A . Substituting the data, we get that

$$v_{\max} = \sqrt{0.1 \cdot 9.81 \cdot \sqrt{2}} \approx 1.18 \left[\frac{\text{m}}{\text{s}} \right].$$

Sample problem 4. Part a and part b of the figure shows the overlap and side view of an ellipse-shaped go-kart track. The go-kart travels at a constant speed v on the track.



Data:

$$m = 100 \text{ [kg]}; \quad \mu_0 = 0.2; \quad g = 9.81; \quad \left[\frac{\text{m}}{\text{s}^2} \right]; \quad a = 50 \text{ [m]}; \quad b = 30 \text{ [m]}.$$

Calculate the maximum speed the go-kart can travel at point A without drifting.

Solution:

Part b of the figure shows the forces acting on the go-kart. The equation of motion of the go-kart at point A is

$$\sum_i \vec{F}_i = m \cdot \vec{g} + \vec{F}_k = m \cdot \vec{a}_A.$$

The motion of equation according to the *Frenet-Serret frame* is

$$\begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix} + \begin{pmatrix} 0 \\ F_{friction} \\ F_b \end{pmatrix} = m \cdot \begin{pmatrix} a_e \\ a_n \\ a_b \end{pmatrix} = m \cdot \begin{pmatrix} a_A \\ \frac{v_A^2}{r_A} \\ 0 \end{pmatrix}.$$

Thus we get the following equations:

$$\begin{aligned} \sum_i F_{ie} &= 0 = m \cdot a_A \\ \sum_i F_{in} &= F_{friction} = m \cdot \frac{v_A^2}{r_A} \\ \sum_i F_{ib} &= -m \cdot g + F_b = 0. \end{aligned}$$

Since $m \neq 0$, from the first equation, we get that $a_A = 0$.
From the last equation, we get that

$$F_b = m \cdot g.$$

The go-kart does not drift, if and only if

$$|F_{friction}| \leq \mu_0 \cdot F_b.$$

Applying $F_{friction} > 0$, we get that

$$m \cdot \frac{v_A^2}{r_A} \leq \mu_0 \cdot m \cdot g.$$

Expressing v from the equation above:

$$v_A \leq \sqrt{\mu_0 \cdot g \cdot r_A}.$$

Thus, the value of top speed is

$$v_{Amax} = \sqrt{\mu_0 \cdot g \cdot r_A},$$

where r_A is the radius of curvature at point A . The radius of the curvature can be calculated as

$$r_A = \left| \frac{[1 + (y'(0))^2]^{\frac{3}{2}}}{y''(0)} \right|,$$

where $y(x)$ can be expressed from the following equation of the ellipse:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

Assuming that $y(x) \geq 0$, we get that

$$y(x) = a \cdot \sqrt{1 - \frac{x^2}{b^2}}.$$

Converted $y(x)$ to an equivalent form, we get that

$$y(x) = a \cdot \left(1 - \frac{x^2}{b^2}\right)^{\frac{1}{2}}.$$

The first derivative of function $y(x)$ is

$$y'(x) = a \cdot \frac{1}{2} \cdot \left(1 - \frac{x^2}{b^2}\right)^{-\frac{1}{2}} \cdot \left(\frac{-2x}{b^2}\right).$$

Applying algebraic identities, we get that

$$y'(x) = -\frac{a}{b^2} \cdot \left(1 - \frac{x^2}{b^2}\right)^{-\frac{1}{2}} = -\frac{a}{b^2} \cdot x \cdot \frac{x}{\sqrt{1 - \frac{x^2}{b^2}}}.$$

The second derivative of the function $y(x)$ is

$$y''(x) = -\frac{a}{b^2} \cdot \left[\left(1 - \frac{x^2}{b^2}\right)^{-\frac{1}{2}} + x \cdot \left(-\frac{1}{2}\right) \cdot \left(1 - \frac{x^2}{b^2}\right)^{-\frac{3}{2}} \cdot \frac{-2x}{b^2} \right].$$

Applying algebraic identities, we get that

$$y''(x) = -\frac{a}{b^2} \cdot \left[\frac{1}{\sqrt{1 - \frac{x^2}{b^2}}} + \frac{x}{\sqrt{1 - \frac{x^2}{b^2}} \cdot \left(1 - \frac{x^2}{b^2}\right)} \right].$$

Substituting the values of parameters a and b , the first derivative of function $y(x)$ at $x = 0$ is

$$y'(0) = -\frac{50}{30^2} \cdot \frac{0}{\sqrt{1 - \frac{0^2}{30^2}}} = 0.$$

Substituting the values of parameters a and b , the second derivative of function $y(x)$ at $x = 0$ is

$$y''(0) = -\frac{50}{30^2} \cdot \left[\frac{1}{\sqrt{1 - \frac{0^2}{30^2}}} + \frac{0}{\sqrt{1 - \frac{0^2}{30^2}} \cdot \left(1 - \frac{0^2}{30^2}\right)} \right] = -\frac{11}{198}.$$

The radius of curvature at point A is

$$r_A = \left| \frac{\left[1 + (y'(0))^2\right]^{\frac{3}{2}}}{y''(0)} \right|.$$

Substituting the values of $y'(0)$ and $y''(0)$, we get that

$$r_A = \left| \frac{\left[1 + 0^2\right]^{\frac{3}{2}}}{-\frac{11}{198}} \right| = \frac{198}{11} = 18 \text{ [m]}.$$

Applying the calculated values, we get that

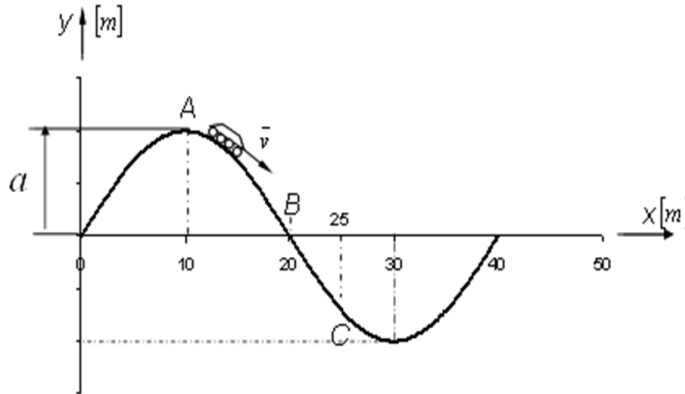
$$v_{A\max} = \sqrt{\mu_0 \cdot g \cdot r_A} = \sqrt{0.2 \cdot 9.81 \cdot 18} = \sqrt{35.316} \approx 5.94 \left[\frac{\text{m}}{\text{s}} \right].$$

Problems

Problem 1.2.1. The figure shows a roller coaster. The equation of the railway of the roller coaster in the given coordinate system is

$$y = a \cdot \sin(b \cdot x)$$

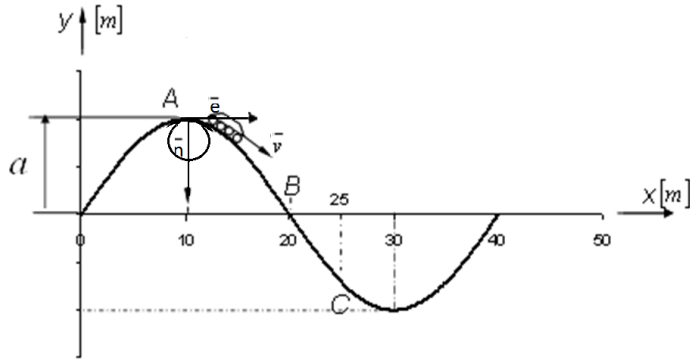
shape.



- Determine the value of parameter b from the coordinates of point B .
- According to the standards for roller coasters, the angle between the path and the horizontal plane can not be higher than 45° . Determine the maximum allowed value of parameter a .
- Find the coordinates of A of the path when the value of parameter a is maximal.
- Find the coordinates of C of the path when the value of parameter a is maximal.
- Determine the first derivative of function $y(x)$.
- Determine the values of first derivative of function $y(x)$ at point A .
- Determine the values of first derivative of function $y(x)$ at point C .
- Determine the second derivative of function $y(x)$.
- Determine the values of second derivative of function $y(x)$ at point A .
- Determine the values of second derivative of function $y(x)$ at point C .
- Determine the radius of curvature of the path at point A .
- Determine the radius of curvature of the path at point C .

- m) Calculate the magnitude of the reaction force that the railway exerts on the carriage at point A in normal direction if the velocity magnitude and mass of the carriage are $v_A = 5 \left[\frac{m}{s} \right]$ and 500 [kg] , respectively.

Solution:



- a) The coordinates of point B are $x_B = 20 \text{ [m]}$ and $y_B = 0 \text{ [m]}$. Following that, we get that

$$0 = a \cdot \sin(20b) \quad \Rightarrow \quad 0 = \sin(20b).$$

Consequently, the value of b is

$$20b = \pi \quad \Rightarrow \quad b = \frac{\pi}{20}.$$

The equation of the path is

$$y = a \cdot \sin\left(\frac{\pi}{20}x\right).$$

- b) The slope of the track as a function of variable a can be given as

$$m(x) = y'(x) = \frac{a \cdot \pi}{20} \cdot \cos\left(\frac{\pi}{20}x\right).$$

The slope at angle 45° is

$$m = \text{tg } 45^\circ = 1.$$

The following condition has to be fulfilled:

$$m \leq 1,$$

thus we have to solve the inequality

$$\frac{a \cdot \pi}{20} \cdot \cos\left(\frac{\pi}{20}x\right) \leq 1.$$

It follows from the above inequality that

$$\frac{a \cdot \pi}{20} \cdot \cos\left(\frac{\pi}{20}x\right) \leq \sqrt{3} \quad \Rightarrow \quad \frac{a \cdot \pi}{20} \leq 1,$$

thus the value of a is

$$a \leq \frac{20}{\pi} \approx 6.37.$$

c) When $a = 6.37$, we get that

$$y(x) = \frac{20}{\pi} \cdot \sin\left(\frac{\pi}{20}x\right).$$

Then

$$\begin{aligned} y(10) &= \frac{20}{\pi} \cdot \sin\left(\frac{\pi}{20} \cdot 10\right) = \\ &= \frac{20}{\pi} \cdot \sin\left(\frac{\pi}{2}\right) = \frac{20}{\pi}. \end{aligned}$$

Thus the coordinates of point A are

$$A = \left(10; \frac{20}{\pi}\right).$$

d) Since

$$\begin{aligned} y(25) &= \frac{20}{\pi} \cdot \sin\left(\frac{\pi}{20} \cdot 25\right) = \\ &= \frac{20}{\pi} \cdot \sin\left(\frac{5\pi}{4}\right) = -\frac{20}{\pi} \cdot \frac{\sqrt{2}}{2} = -\frac{10 \cdot \sqrt{2}}{\pi} \end{aligned}$$

then the coordinates of point C are

$$C = \left(25; -\frac{10 \cdot \sqrt{2}}{\pi}\right).$$

e) The derivative of function $y(x)$ is

$$y'(x) = \cos\left(\frac{\pi}{20}x\right).$$

f) The derivative at point A is

$$y'(10) = \cos\left(\frac{\pi}{20} \cdot 10\right) = \cos\left(\frac{\pi}{2}\right) = 0.$$

g) The derivative at point C is

$$y'(25) = \cos\left(\frac{\pi}{20} \cdot 25\right) = \cos\left(\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

h) The second derivative of the function $y(x)$ is

$$y''(x) = -\frac{\pi}{20} \cdot \sin\left(\frac{\pi}{20}x\right).$$

i) The second derivative at point A is

$$y''(10) = -\frac{\pi}{20} \cdot \sin\left(\frac{\pi}{20} \cdot 10\right) = -\frac{\pi}{20} \cdot \sin\left(\frac{\pi}{2}\right) = \frac{-\pi}{20}.$$

j) The second derivative at point C is

$$y''(25) = -\frac{\sqrt{3} \cdot \pi}{20} \cdot \sin\left(\frac{\pi}{20} \cdot 25\right) = -\frac{\sqrt{3} \cdot \pi}{20} \cdot \sin\left(\frac{5\pi}{4}\right) = \frac{\sqrt{6}}{20}.$$

k) The radius of curvature at point A is

$$r_A = \left| \frac{\left(1 + (y'(10))^2\right)^{\frac{3}{2}}}{y''(10)} \right| = \frac{20}{\pi} \approx 6.37.$$

l) The radius of curvature at point C is

$$\begin{aligned} r_C &= \left| \frac{\left(1 + (y'(25))^2\right)^{\frac{3}{2}}}{y''(25)} \right| = \left| \frac{\left(1 + \frac{1}{2}\right)^{\frac{3}{2}}}{\frac{\sqrt{2} \cdot \pi}{40}} \right| = \\ &= \left(\frac{3}{2}\right)^{\frac{3}{2}} \cdot \frac{40}{\sqrt{2} \cdot \pi} \approx 16.54. \end{aligned}$$

m) From the equation above

$$\sum_i F_{in} = m \cdot g - F_n = m \cdot \frac{v_A^2}{r_A},$$

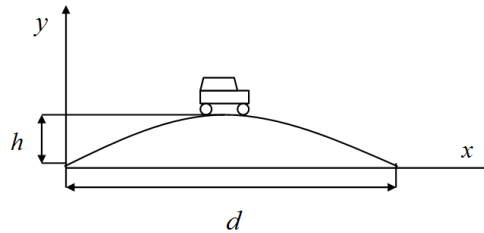
we get that

$$\begin{aligned} F_n &= m \cdot g - m \cdot \frac{v_A^2}{r_A} = 500 \cdot 9.81 - 500 \cdot \frac{5^2}{6.37} = \\ &= 500 \cdot (9.81 - 3.92) = 2945 \text{ [N]}. \end{aligned}$$

Problem 1.2.2. The figure illustrates a bridge. In the coordinate system, which is given in the figure, the equation of the bridge is

$$y = a \cdot x^2 + b \cdot x + c.$$

During designing, we would like to maximize parameter h , which is the height of the bridge. The width of the bridge is $d = 100$ [m].



- Express parameters a , b and c with the known d and unknown h parameters.
- The incline angle of the roadway should not exceed 30° anywhere. Determine the maximum allowed height of the bridge h_{\max} .
- Determine the radius of curvature of the bridge at its middle point at the maximum value of h .

Solution:

- Points $(0; 0)$, $(100; 0)$ and $(50; h)$ are on the parabola. Substituting their coordinates into the equation of the parabola, we get the following system of equations:

$$\begin{aligned} c &= 0 \\ 10\,000a + 100b &= 0 \\ 2\,500a + 50b &= h. \end{aligned}$$

If we express b from the second equation, we get that

$$b = -100a.$$

Substituting the above formula into the third equation, we get that

$$2\,500a - 5\,000a = h \quad \Rightarrow \quad a = -\frac{h}{2\,500}.$$

Thus, the solution of the system of equations is

$$a = -\frac{h}{2\,500} \quad b = \frac{h}{25}.$$

The equation of the track is

$$y = -\frac{h}{2500} \cdot x^2 + \frac{h}{25} \cdot x.$$

- b) The derivative of the above function gives the slope of the track at each point. The derivative of function $y(x)$ is

$$y'(x) = -\frac{h}{800} \cdot x + \frac{h}{25} \quad (x \in [0; 100]).$$

The maximum value of the slope at $x = 0$ is

$$y'(0) = \frac{h}{25}.$$

Since the incline angle should not exceed 30° , we get that

$$\frac{h}{25} \leq \operatorname{tg} 45^\circ \quad \Rightarrow \quad \frac{h}{25} \leq \frac{\sqrt{2}}{2}.$$

If we solve the inequality above, we get that

$$h \leq 25 \cdot \frac{\sqrt{2}}{2} \approx 17.68 \text{ [m]}.$$

Thus the maximum height of the bridge is 17.68 [m].

- c) The derivative of $y(x)$ is

$$y'(x) = -\frac{h}{1250} \cdot x + \frac{h}{25}$$

and its second derivative is

$$y''(x) = -\frac{h}{1250}.$$

The first and second derivatives at $x = 50$ are

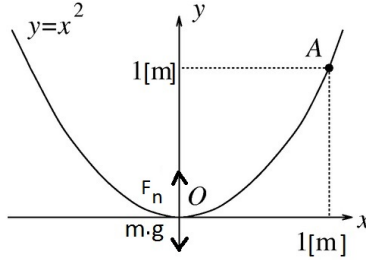
$$y'(50) = 0; \quad y''(50) = -\frac{h}{1250}.$$

The radius of curvature at the middle point of the bridge is

$$r_{\text{mid}} = \left| \frac{[1 + (y'(50))^2]^{\frac{3}{2}}}{y''(50)} \right| = \left| -\frac{1250}{h} \right| = \frac{1250}{17.68} = 70.7 \text{ [m]}.$$

Problem 1.2.3. A particle slips down from point A of the ideally smooth surfaced parabolic groove in the figure. Calculate the magnitude of the reaction force that the groove exerts on the particle at point O . The mass of the particle is 0.1 [kg].

Solution:



Applying the work-energy theorem, the magnitude of the velocity of the particle at point O can be calculate as

$$v_o = \sqrt{2 \cdot g \cdot h},$$

where h is the distance between point A and axis x . The magnitude of the normal acceleration of the particle at point O is

$$\frac{v_o^2}{r_o},$$

where r_o is the radius of curvature at point O . The equation of motion in normal direction is

$$\sum_i F_{in} = F_n - m \cdot g = m \cdot \frac{v_o^2}{r_o}.$$

Thus the magnitude of reaction force at point O is

$$F_n = m \cdot g + \frac{2 \cdot m \cdot g \cdot h}{r_o}.$$

The equation of the parabola is $f(x) = x^2$.

The first and second derivatives of function $f(x)$ are

$$f'(x) = 2x$$

$$f''(x) = 2.$$

The radius of curvature as a function of x is

$$r_o = \left| \frac{[1 + (f'(x))^2]^{\frac{3}{2}}}{f''(x)} \right| = \left| \frac{[1 + (2x)^2]^{\frac{3}{2}}}{2} \right|.$$

The radius of curvature at $x = 0$ is

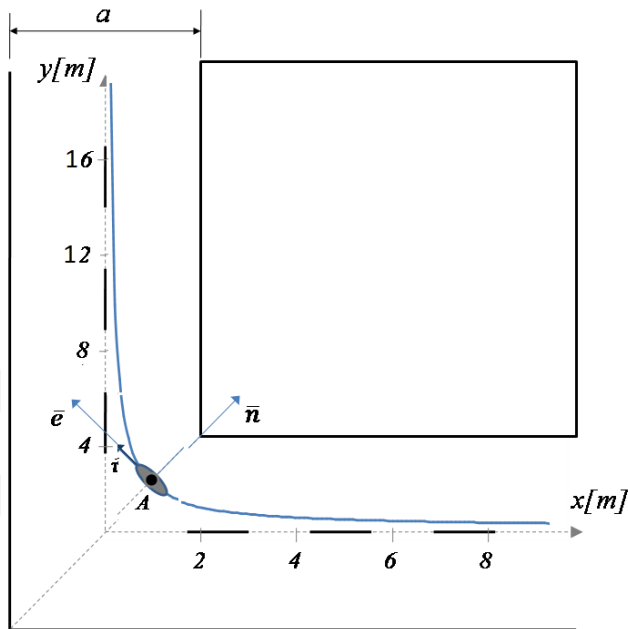
$$r_o = \frac{[1 + (2 \cdot 0)^2]^{\frac{3}{2}}}{2} = \frac{1}{2}.$$

Applying the results above, we get that

$$F_n = m \cdot g + \frac{2 \cdot m \cdot g \cdot h}{r_o} = 0.1 \cdot 10 + \frac{2 \cdot 0.1 \cdot 10 \cdot 1}{0.5} = 5 \text{ [N]}.$$

Problem 1.2.4. The figure shows the track of a roller blader. The width of the road the roller blader is travelling on is a . The speed of the roller blader is constant v . The equation of its track in the given coordinate system is

$$y(x) = \frac{1}{x^2}.$$



Data:

$$m = 50 \text{ [kg]}; \quad \mu_0 = 0.1; \quad g = 9.81 \left[\frac{\text{m}}{\text{s}^2} \right]; \quad a = 4 \text{ [m]}.$$

a) Calculate the radius of curvature at point A of the track.

- b) Calculate the maximum speed the roller blader can travel at point A without drifting.

Solution:

- a) The first derivative of function $y(x)$ is

$$y'(x) = -2 \cdot x^{-3} = -\frac{2}{x^3}.$$

Its second derivative is

$$y''(x) = 6 \cdot x^{-4} = \frac{6}{x^4}.$$

The first and second derivatives of function $y(x)$ at $x = 1$ are

$$y'(1) = -\frac{2}{1^3} = -2, \quad y''(1) = \frac{6}{1^4} = 6.$$

The radius of curvature at point A is

$$r_A = \left| \frac{[1 + (y'(1))^2]^{\frac{3}{2}}}{y''(1)} \right| = \left| \frac{[1 + (-2)^2]^{\frac{3}{2}}}{6} \right| = \frac{\sqrt{125}}{6} \text{ [m]}.$$

- b) The equation of motion of the roller blader at point A is

$$\sum_i \bar{F}_i = m \cdot g + \bar{F}_k = m \cdot \bar{a}_A.$$

The motion of equation in *Frenet-Serret frame* is

$$\begin{pmatrix} 0 \\ 0 \\ -m \cdot g \end{pmatrix} + \begin{pmatrix} 0 \\ F_{friction} \\ F_b \end{pmatrix} = m \cdot \begin{pmatrix} a_e \\ a_n \\ a_b \end{pmatrix} = m \cdot \begin{pmatrix} a_A \\ \frac{v_A^2}{r_A} \\ 0 \end{pmatrix}.$$

Thus we get the following equations:

$$0 = m \cdot a_e = m \cdot a_A$$

$$F_{friction} = m \cdot a_n = m \cdot \frac{v^2}{r_A}$$

$$0 = m \cdot a_b = -m \cdot g + F_b.$$

Since $m \neq 0$, from the first equation, we get that $a_A = 0$. From the last equation, we get that $F_b = m \cdot g$. The roller blader does not drift if and only if

$$|F_{friction}| \leq \mu_0 \cdot F_b = \mu_0 \cdot m \cdot g$$

Expressing v from the equation above:

$$m \cdot \frac{v_A^2}{r_A} \leq \mu_0 \cdot m \cdot g$$

$$v_A \leq \sqrt{\mu_0 \cdot g \cdot r_A}.$$

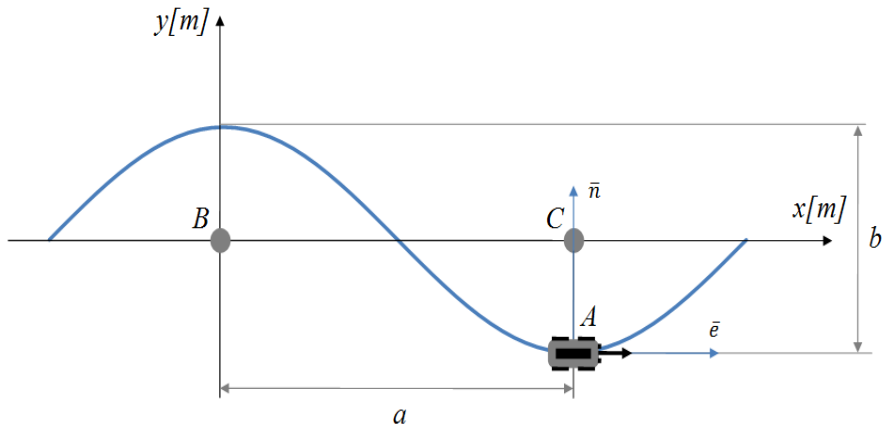
Thus, the value of top speed is

$$v_{\max} = \sqrt{\mu_0 \cdot g \cdot r_A},$$

where r_A is the value of the radius of curvature at point A . Substituting the data, we get that

$$v_{\max} = \sqrt{0.1 \cdot 9.81 \cdot 1.86} \approx 1.35 \left[\frac{\text{m}}{\text{s}} \right].$$

Problem 1.2.5. The figure shows the slalom track of a sports car. The distance between buoys B and C is a . The sports car is travelling at a constant speed v . See the figure below:



Data:

$$m = 800 \text{ [kg]}; \quad \mu_0 = 0.2; \quad g = 9.81 \left[\frac{\text{m}}{\text{s}^2} \right];$$

$$a = 15 \text{ [m]}; \quad b = 10 \text{ [m]}.$$

a) The equation of the track is given as

$$y(x) = D \cdot \cos(E \cdot x).$$

Determine the value of parameteres D and E .

b) Determine the radius of curvature of the track at point A .

c) Calculate the maximum speed the car can travel at point A without drifting.

Solution:

- a) Since $\frac{b}{2} = 5$, the point $(0; 5)$ is on the graph of $y(x)$, hence the next equation holds

$$5 = D \cdot \cos 0 \quad \Rightarrow \quad D = 5.$$

Since $a = 15$ and $\frac{b}{2} = 5$, then the coordinates of point A are $(15; 5)$. Therefore

$$-5 = 5 \cdot \cos(15E) \quad \Rightarrow \quad -1 = \cos(15E),$$

thus $E = \frac{\pi}{15}$.

Consequently, the equation of the track is

$$y(x) = 5 \cdot \cos\left(\frac{\pi}{15}x\right).$$

- b) The first derivative of function $y(x)$ is

$$y'(x) = -\frac{\pi}{3} \cdot \sin\left(\frac{\pi}{15}x\right).$$

Its second derivative is

$$y''(x) = -\frac{\pi^2}{45} \cdot \cos\left(\frac{\pi}{15}x\right).$$

The first and second derivatives of function $y(x)$ at $x = 15$ are

$$y'(15) = 0; \quad y''(15) = \frac{\pi^2}{45}.$$

The radius of curvature at point A is

$$r_A = \left| \frac{[1 + (y'(15))^2]^{\frac{3}{2}}}{y''(15)} \right| = \left| \frac{[1 + 0^2]^{\frac{3}{2}}}{\frac{\pi^2}{45}} \right|,$$

that is,

$$r_A = \frac{45}{\pi^2} \approx 4.56 \text{ [m]}.$$

- c) The maximum speed at point A is

$$v_{\max} = \sqrt{\mu_0 \cdot g \cdot r_A} = \sqrt{0.2 \cdot 9.81 \cdot 4.56} \approx 2.99 \left[\frac{\text{m}}{\text{s}} \right].$$

Problem 1.2.6. A car is travelling at a constant speed v on the track, which equation is given as

$$y(x) = \sqrt{20x - x^2}.$$

The coefficient of friction is 0.2. Determine the maximum speed the car can travel at point $A = (10; 10)$ without drifting.

Solution:

The first derivative of function $y(x)$ is

$$y'(x) = \frac{1}{2} \cdot (20x - x^2)^{-\frac{1}{2}} \cdot (20 - 2x) = \frac{10 - x}{\sqrt{20x - x^2}}.$$

The second derivative of function $y(x)$ is

$$\begin{aligned} y''(x) &= \frac{-1 \cdot \sqrt{20x - x^2} - (10 - x) \cdot \frac{1}{2} \cdot (20x - x^2)^{-\frac{1}{2}} \cdot (20 - 2x)}{20x - x^2} = \\ &= \frac{-\sqrt{20x - x^2} - \frac{(10-x)^2}{\sqrt{20x-x^2}}}{20x - x^2}. \end{aligned}$$

Transforming in equivalent form, we have

$$y''(x) = \frac{-20x + x^2 - (10 - x)^2}{(20x - x^2) \cdot \sqrt{20x - x^2}} = \frac{-100}{(20x - x^2) \cdot \sqrt{20x - x^2}}.$$

The value of $y'(x)$ and $y''(x)$ at $x = 10$ are

$$\begin{aligned} y'(10) &= \frac{10 - 10}{\sqrt{20 \cdot 10 - 10^2}} = 0 \\ y''(10) &= \frac{-100}{(20 \cdot 10 - 10^2) \cdot \sqrt{20 \cdot 10 - 10^2}} = -\frac{1}{10}. \end{aligned}$$

Applying this results, we get that

$$r_A = \left| \frac{[1 + (y'(10))^2]^{\frac{3}{2}}}{y''(10)} \right| = \left| \frac{[1 + 0^2]^{\frac{3}{2}}}{-\frac{1}{10}} \right| = 10.$$

The maximum speed at point A is

$$v_{\max} = \sqrt{\mu_0 \cdot g \cdot r_A} = \sqrt{0.2 \cdot 9.81 \cdot 10} \approx 4.43 \left[\frac{\text{m}}{\text{s}} \right].$$

Problem 1.2.7. A roller blader is travelling at a constant speed v on the track, which equation is given as

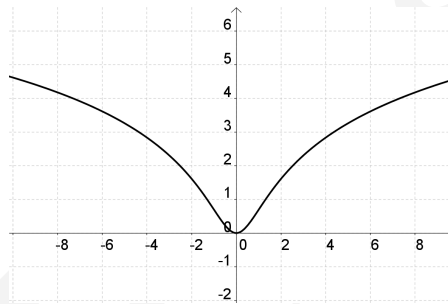
$$y(x) = \ln(1 + x^2).$$

The coefficient of friction is 0.2.

- Sketch the graph of function $y(x)$.
- Determine the maximum speed the roller blader can travel at point $A = (0; 0)$ without drifting.

Solution:

- The graph of function $y(x)$ is



- The first derivative of function $y(x)$ is

$$y'(x) = \frac{1}{1+x^2} \cdot 2x = \frac{2x}{1+x^2}.$$

The second derivative of function $y(x)$ is

$$\begin{aligned} y''(x) &= \frac{2 \cdot (1+x^2) - 2x \cdot 2x}{(1+x^2)^2} = \\ &= \frac{2 + 2x^2 - 4x^2}{(1+x^2)^2} = \frac{2 - 2x^2}{(1+x^2)^2}. \end{aligned}$$

The value of $y'(x)$ and $y''(x)$ at $x = 0$ are

$$y'(0) = \frac{2 \cdot 0}{1+0^2} = 0$$

$$y''(0) = \frac{2 - 2 \cdot 0^2}{(1+0^2)^2} = 2.$$

Applying this results, we get that

$$r_A = \left| \frac{[1 + (y'(0))^2]^{\frac{3}{2}}}{y''(0)} \right| = \left| \frac{[1 + 0^2]^{\frac{3}{2}}}{2} \right| = \frac{1}{2}.$$

The maximum speed at point A is

$$v_{\max} = \sqrt{\mu_0 \cdot g \cdot r_A} = \sqrt{0.2 \cdot 9.81 \cdot \frac{1}{2}} \approx 0.99 \left[\frac{\text{m}}{\text{s}} \right].$$

Problem 1.2.8. A roller blader is travelling at a constant speed v on the track, which equation is given as

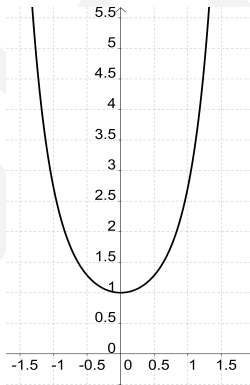
$$y(x) = e^{x^2}.$$

The coefficient of friction is 0.2.

- Sketch the graph of function $y(x)$.
- Determine the maximum speed the roller blader can travel at point $A = (0; 1)$ without drifting.

Solution:

- The graph of function $y(x)$:



- The first derivative of function $y(x)$ is

$$y'(x) = 2x \cdot e^{x^2}.$$

The second derivative of function $y(x)$ is

$$y''(x) = 2 \cdot e^{x^2} + 2x \cdot e^{x^2} \cdot 2x = e^{x^2} \cdot (2 + 4x^2).$$

The value of $y'(x)$ and $y''(x)$ at $x = 0$ are

$$y'(0) = 2 \cdot 0 \cdot e^{0^2} = 0$$

$$y''(0) = e^{0^2} \cdot (2 + 4 \cdot 0^2) = 2.$$

Applying this results, we get that

$$r_A = \left| \frac{[1 + (y'(0))^2]^{\frac{3}{2}}}{y''(0)} \right| = \left| \frac{[1 + 0^2]^{\frac{3}{2}}}{2} \right| = \frac{1}{2}.$$

The maximum speed at point A is

$$v_{\max} = \sqrt{\mu_0 \cdot g \cdot r_A} = \sqrt{0.2 \cdot 9.81 \cdot \frac{1}{2}} \approx 0.99 \left[\frac{\text{m}}{\text{s}} \right].$$

1.3. Application of the least squares method

Theoretical summary. Least squares method has a wide range of applications for the regression of data point series. The model function, which is applied for the regression usually has theoretical basis.

In the following we deal with the simplest case, when there is only one unknown parameter in the model function, so it can be written in the following form:

$$f(x) = f(x, a).$$

When applying the procedure our aim is to determine that value of parameter a at which the model function gives the best fit to the given series of data points.

Applying the method the aim is to find the minimum of the following function:

$$H(a) = \sum_{k=1}^n (y_k - f(x_k, a))^2$$

for given points of data:

$$(x_1; y_1), (x_2; y_2), \dots, (x_n; y_n).$$

This method is called as *the least square method*.

The minimum value can be found by applying the methods that were presented in the section about extreme value calculus of real functions.

Applied mathematical knowledge. Rules of derivation, extreme value calculus, plotting of real functions, numerical method, Newton-Raphson formula, application of mathematical software.

Sample problem 1. We intend to measure the electric resistance of the stator coil of an electric motor. In course of the measurements electric current with different intensities is flowing through the coil and the voltage drop across the coil is measured. The measurement results are summarized in the table:

I[A]	1	2	3	4	5.5	6	9.2	10
U[V]	0.0166	0.033	0.05	0.066	0.092	0.099	0.153	0.165

Determine the electric resistance of the stator coil applying the least squares method and Ohm's law.

Solution:

Ohm's law can be written in the following form:

$$V = R \cdot I,$$

where I is the intensity of current flowing through the resistor (coil), V is the voltage through the resistor (across the coil) and R is the electric resistance of the resistor (coil). In the given situation the model function can be written as:

$$V(I) = V(I, R) = R \cdot I.$$

Substituting the data, for function $H(R)$ (see in the theoretical summary), we get that:

$$\begin{aligned} H(R) = & (0.0166 - R)^2 + (0.033 - 2R)^2 + (0.05 - 3R)^2 + \\ & + (0.066 - 4R)^2 + (0.092 - 5.5R)^2 + (0.099 - 6R)^2 + \\ & + (0.153 - 9.2R)^2 + (0.165 - 10R)^2. \end{aligned}$$

The derivative of the function above is:

$$\begin{aligned} H'(R) = & -2 \cdot (0.0166 - R) - 4 \cdot (0.033 - 2R) - 6 \cdot (0.05 - 3R) - \\ & - 8 \cdot (0.066 - 4R) - 11 \cdot (0.092 - 5.5R) - \\ & - 12 \cdot (0.099 - 6R) - 18.4 \cdot (0.153 - 9.2R) - \\ & - 20 \cdot (0.165 - 10R) = -9.3084 + 561.78R. \end{aligned}$$

The minimum value of function $H(R)$ can be found, where its derivative function is zero, thus we have to solve the equation below:

$$-9.3084 + 561.78R = 0 \quad \Rightarrow \quad R \approx 0.0166 [\Omega].$$

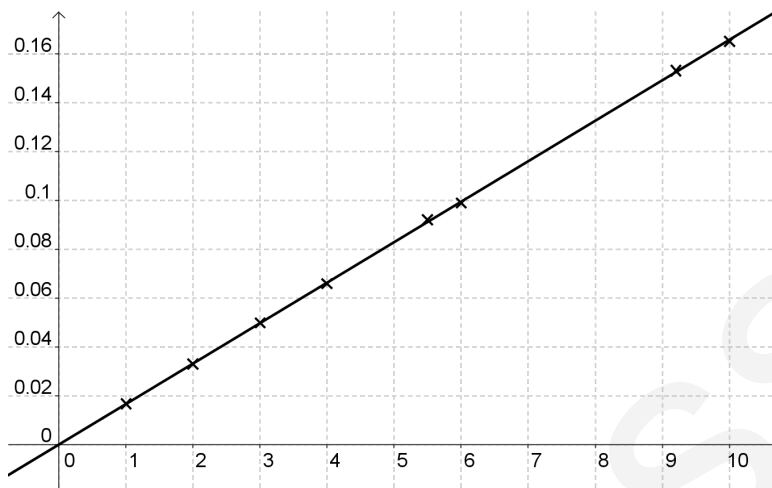
Since

$$H''(R) = 561.78 > 0,$$

thus function H has a minimum value at $R \approx 0.0166$. Thus function $V(I)$ is

$$V(I) = 0.0166 \cdot I.$$

The fitted data point series is presented in the figure:



Sample problem 2. In course of a nuclear reactor accident radioactive material was spread into the atmosphere. In a research institute (where nobody knows about the accident) at a distance of 200 km from the reactor the increased radioactivity in the air is observed. Because of that five measurements were performed by the researchers in the institute. The time difference between the consequent measurements was one day. The obtained results are shown in the table:

t[day]	0	1	2	3	4
A[Bq/m ³]	10.00	9.17	8.41	7.72	7.10

Remark: the activity is 1 [Bq/m³], if there are 1 radioactive decay in 1 m³ air in one second.

From the data series, applying the rule of radioactive decay and also the least squares method, the researchers can be determined the half life of the unknown radioactive isotope. Determine the unknown half-life time and give the isotope.

Solution:

The rule of radioactive decay is

$$A(t) = A(0) \cdot \left(\frac{1}{2}\right)^{\frac{t}{T}},$$

where $A(0)$ and $A(t)$ are the activities measured at times 0 and t , and T is the half-life of the isotope. In the recent problem $A(0) = 10$ [Bq/m³], thus the rule

can be written in the following form:

$$A(t) = 10 \cdot \left(\frac{1}{2}\right)^{\frac{t}{T}}.$$

Let's consider the $H(t)$ function:

$$\begin{aligned} H(T) = & \left(9.17 - 10 \cdot \left(\frac{1}{2}\right)^{\frac{1}{T}}\right)^2 + \left(8.41 - 10 \cdot \left(\frac{1}{2}\right)^{\frac{2}{T}}\right)^2 + \\ & + \left(7.72 - 10 \cdot \left(\frac{1}{2}\right)^{\frac{3}{T}}\right)^2 + \left(7.1 - 10 \cdot \left(\frac{1}{2}\right)^{\frac{4}{T}}\right)^2. \end{aligned}$$

The derivative of function $H(T)$ is

$$\begin{aligned} H'(T) = & 2 \cdot \left(9.17 - 10 \cdot \left(\frac{1}{2}\right)^{\frac{1}{T}}\right) \cdot \left(-10 \cdot \left(\frac{1}{2}\right)^{\frac{1}{T}} \cdot \ln \frac{1}{2} \cdot \left(-\frac{1}{T^2}\right)\right) + \\ & + 2 \cdot \left(8.41 - 10 \cdot \left(\frac{1}{2}\right)^{\frac{2}{T}}\right) \cdot \left(-10 \cdot \left(\frac{1}{2}\right)^{\frac{2}{T}} \cdot \ln \frac{1}{2} \cdot \left(-\frac{1}{T^2}\right)\right) + \\ & + 2 \cdot \left(7.72 - 10 \cdot \left(\frac{1}{2}\right)^{\frac{3}{T}}\right) \cdot \left(-10 \cdot \left(\frac{1}{2}\right)^{\frac{3}{T}} \cdot \ln \frac{1}{2} \cdot \left(-\frac{1}{T^2}\right)\right) + \\ & + 2 \cdot \left(7.1 - 10 \cdot \left(\frac{1}{2}\right)^{\frac{4}{T}}\right) \cdot \left(-10 \cdot \left(\frac{1}{2}\right)^{\frac{4}{T}} \cdot \ln \frac{1}{2} \cdot \left(-\frac{1}{T^2}\right)\right). \end{aligned}$$

The solution of the equation $H'(T) = 0$ can be found with a mathematics software (Maple, GeoGebra, Mathematica, Wolfram Alpha, etc.) or with a numerical method (like Newton-Raphson method). Finally the half-life is

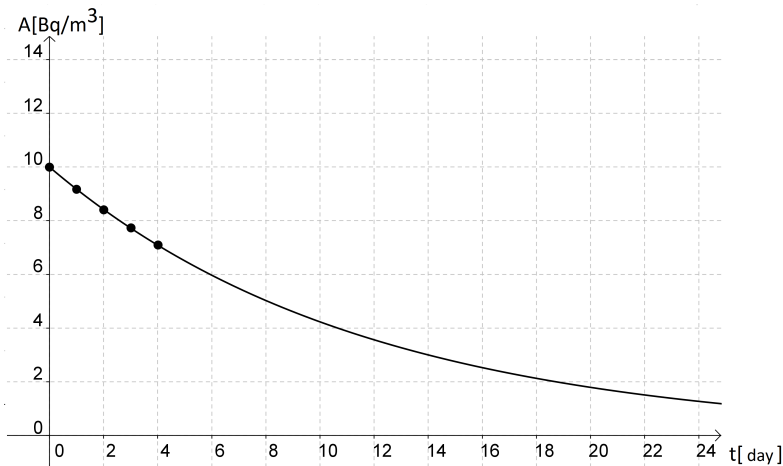
$$T \approx 8.06 \text{ [day]}.$$

Consequently the unknown radioactive isotope is ^{131}I .

Thus function $A(t)$ is

$$A(t) = 10 \cdot \left(\frac{1}{2}\right)^{\frac{t}{T}} = 10 \cdot \left(\frac{1}{2}\right)^{\frac{t}{8.06}}.$$

In the figure the result of fitting (applying the least squares method) is presented:



Sample problem 3. The propagation of bacteria can be modelled by the function

$$N(t) = N(t, a) = a \cdot 2^t.$$

To determine the number of bacteria, we perform measurements on consecutive days. The results are summarized in the table:

t[day]	1	2	3	4
N[thousand pieces]	1.995	4.050	7.943	16.093

Applying the least squares method determine unknown parameter a and plot the function.

Solution:

Function H is

$$H(a) = (1.995 - 2a)^2 + (4.050 - 4a)^2 + (7.943 - 8a)^2 + (16.093 - 16a)^2.$$

The derivative of function H is

$$H'(a) = 2 \cdot (1.995 - 2a) \cdot (-2) + 2 \cdot (4.050 - 4a) \cdot (-4) + 2 \cdot (7.943 - 8a) \cdot (-8) + 2 \cdot (16.093 - 16a) \cdot (-16).$$

Simplifying the formula above, we get that

$$H'(a) = 680a - 682.44.$$

We can determine the value of parameter a from the equation:

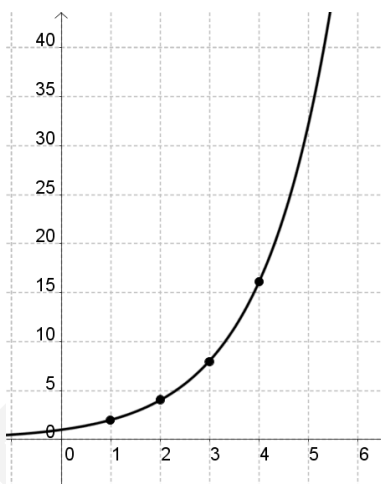
$$680a - 682.44 = 0 \quad \Rightarrow \quad a = \frac{682.44}{680} = 1.004.$$

Since

$$H''(a) = 680 > 0,$$

function H has a minimum at the above value, thus function N is

$$N(t) = 1.004 \cdot 2^t.$$



Problems

Problem 1.3.1. The propagation of bacteria can be modelled by the function

$$N(t) = N(t, a) = a \cdot 2^t.$$

To determine the number of bacteria, we perform measurements on consecutive days. The results are summarized in the table:

t[day]	1	2	3	4
N[thousand pieces]	3	6	10	14

Applying the least squares method, determine unknown parameter a , then plot the function.

Solution:

Function H is

$$H(a) = (3 - 2a)^2 + (6 - 4a)^2 + (10 - 8a)^2 + (14 - 16a)^2.$$

The derivative of function H is

$$\begin{aligned} H'(a) &= 2 \cdot (3 - 2a) \cdot (-2) + 2 \cdot (6 - 4a) \cdot (-4) + \\ &+ 2 \cdot (10 - 8a) \cdot (-8) + 2 \cdot (14 - 16a) \cdot (-16). \end{aligned}$$

Simplifying the formula above, we get that

$$H'(a) = 680a - 668.$$

We can determine the value of parameter a from the equation

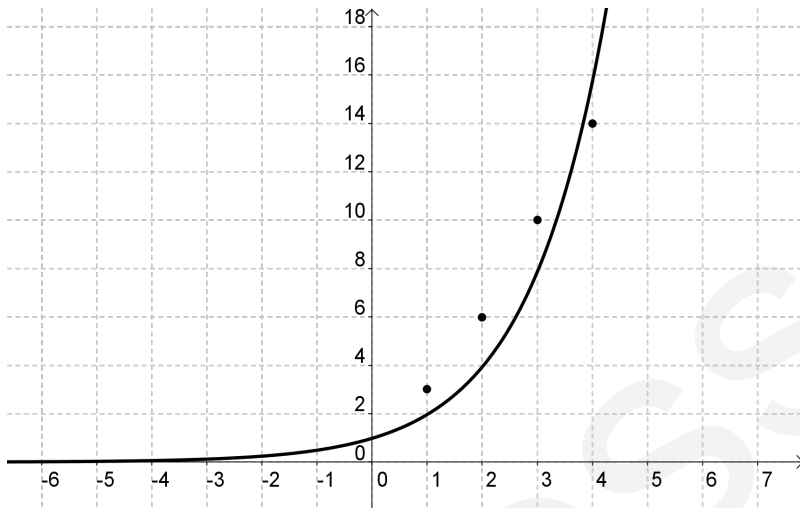
$$680a - 668 = 0 \quad \Rightarrow \quad 680a = 668 \quad \Rightarrow \quad a = \frac{668}{680} = \frac{167}{170}.$$

Since

$$H''(a) = 680 > 0,$$

function H has a minimum at the above value, thus function N is

$$N(t) = \frac{167}{170} \cdot 2^t.$$



Problem 1.3.2. The annual profit of a dynamically developing company has been registered in the first four years after its establishment. The obtained data are summarized in the table:

t[year]	1	2	3	4
P[m HUF]	3	6	12	14

In the table HUF means Hungarian Forint.

The experts are assuming a linear relationship between the profit and time. On the basis of the above assumption, applying the least squares method, determine the unknown parameter m in the function below, and then estimate the profit of the company five years after its establishment:

$$P(t) = P(t, m) = m \cdot t.$$

Solution:

Substituting the obtain data function $H(m)$ is

$$H(m) = (3 - m)^2 + (6 - 2m)^2 + (12 - 3m)^2 + (14 - 4m)^2.$$

The derivative of the function above is

$$\begin{aligned} H'(m) &= 2 \cdot (3 - m) \cdot (-1) + 2 \cdot (6 - 2m) \cdot (-2) + \\ &+ 2 \cdot (12 - 3m) \cdot (-3) + 2 \cdot (14 - 4m) \cdot (-4). \end{aligned}$$

Simplifying the above function, we get that

$$H'(m) = 60m - 214.$$

The zero of the function above

$$60m - 214 = 0 \quad \Rightarrow \quad 60m = 214 \quad \Rightarrow \quad m = \frac{214}{60} = \frac{107}{30}.$$

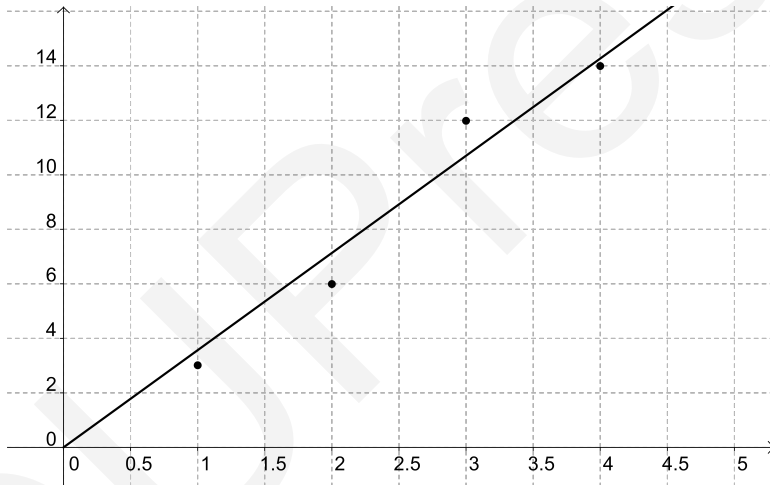
Consequently the value of unknown parameter m is

$$m = \frac{107}{30}$$

thus function $P(t)$ is

$$P(t) = \frac{107}{30} \cdot t.$$

The graph of function $P(t)$ is



If $t = 5$ years, we get that

$$P(5) = \frac{107}{30} \cdot 5 = \frac{107}{6} \approx 17.83.$$

Thus the profit after five years is 17.83 million Forints.

Problem 1.3.3. The table shows the average carbon dioxide level in the atmosphere, measured in parts per million at Hawaii from 2008 to 2016:

[year]	2008	2010	2012	2014	2016
[CO ₂ level]	369	373	377	381	385

Use the data in the table to find the unknown parameter a in the model function:

$$f(x) = a \cdot x + 370.$$

Solution:

Let's consider the following points:

$$A = (0; 369); B = (2; 373); C = (4; 377); D = (6; 381); E = (8; 385).$$

Substituting the obtained data function $H(a)$ is

$$\begin{aligned} H(a) &= (369 - 370)^2 + (373 - 2a - 370)^2 + (377 - 4a - 370)^2 + \\ &+ (381 - 6a - 370)^2 + (385 - 8a - 370)^2 = \\ &= 1 + (3 - 2a)^2 + (7 - 4a)^2 + (11 - 6a)^2 + (15 - 8a)^2. \end{aligned}$$

The derivative of function $H(a)$ is

$$\begin{aligned} H'(a) &= 2 \cdot (3 - 2a) \cdot (-2) + 2 \cdot (7 - 4a) \cdot (-4) + \\ &+ 2 \cdot (11 - 6a) \cdot (-6) + 2 \cdot (15 - 8a) \cdot (-8). \end{aligned}$$

Simplifying function $H'(a)$, we get that

$$H'(a) = 240a - 440.$$

The zero of the function above:

$$240a - 440 = 0 \quad \Rightarrow \quad 240a = 440 \quad \Rightarrow \quad a = \frac{11}{6}.$$

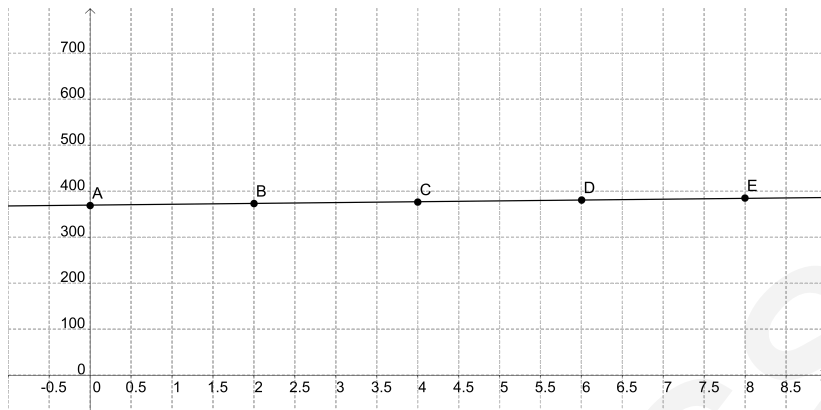
Consequently the value of unknown parameter a is

$$a = \frac{11}{6}$$

thus function $f(x)$ is

$$f(x) = \frac{11}{6} \cdot x + 370.$$

The graph of function $f(x)$ is



Problem 1.3.4. The table shows data for the population (in millions) of the world in the 20th and 21st century:

Year	Population [million]
1900	1 650
1910	1 750
1920	1 860
1930	2 070
1940	2 300
1950	2 560
1960	3 040
1970	3 710
1980	4 450
1990	5 280
2000	6 080
2010	6 870

The pattern of the data point series suggests exponential growth. Find the unknown parameter a in the model function:

$$P(t) = a \cdot 1.015^t + 1650.$$

Solution:

Let's consider the following points:

$$A = (0; 1650);$$

$$B = (10; 1750);$$

$$C = (20; 1860);$$

$$D = (30; 2070);$$

$$E = (40; 2300);$$

$$F = (50; 2560);$$

$$G = (60; 3040);$$

$$H = (70; 3710);$$

$$I = (80; 4450);$$

$$J = (90; 5280);$$

$$K = (100; 6080);$$

$$L = (110; 6870).$$

Substituting the obtained data function $H(a)$ is

$$\begin{aligned} H(a) = & (1650 - a \cdot 1.015^0 - 1650)^2 + (1750 - a \cdot 1.015^{10} - 1650)^2 + \\ & + (1860 - a \cdot 1.015^{20} - 1650)^2 + (2070 - a \cdot 1.015^{30} - 1650)^2 + \\ & + (2300 - a \cdot 1.015^{40} - 1650)^2 + (2560 - a \cdot 1.015^{50} - 1650)^2 + \\ & + (3040 - a \cdot 1.015^{60} - 1650)^2 + (3710 - a \cdot 1.015^{70} - 1650)^2 + \\ & + (4450 - a \cdot 1.015^{80} - 1650)^2 + (5280 - a \cdot 1.015^{90} - 1650)^2 + \\ & + (6080 - a \cdot 1.015^{100} - 1650)^2 + (6870 - a \cdot 1.015^{110} - 1650)^2. \end{aligned}$$

Simplifying the function above, we get that

$$\begin{aligned} H(a) = & a^2 + (100 - a \cdot 1.015^{10})^2 + (210 - a \cdot 1.015^{20})^2 + \\ & + (420 - a \cdot 1.015^{30})^2 + (650 - a \cdot 1.015^{40})^2 + \\ & + (910 - a \cdot 1.015^{50})^2 + (1390 - a \cdot 1.015^{60})^2 + \\ & + (2060 - a \cdot 1.015^{70})^2 + (2800 - a \cdot 1.015^{80})^2 + \\ & + (3630 - a \cdot 1.015^{90})^2 + (4430 - a \cdot 1.015^{100})^2 + \\ & + (5220 - a \cdot 1.015^{110})^2. \end{aligned}$$

The derivative of the function above is

$$\begin{aligned}
 H'(a) = & 2a + 2 \cdot (100 - a \cdot 1.015^{10}) \cdot (-1.015^{10}) + \\
 & + 2 \cdot (210 - a \cdot 1.015^{20}) \cdot (-1.015^{20}) + \\
 & + 2 \cdot (420 - a \cdot 1.015^{30}) \cdot (-1.015^{30}) + \\
 & + 2 \cdot (650 - a \cdot 1.015^{40}) \cdot (-1.015^{40}) + \\
 & + 2 \cdot (910 - a \cdot 1.015^{50}) \cdot (-1.015^{50}) + \\
 & + 2 \cdot (1390 - a \cdot 1.015^{60}) \cdot (-1.015^{60}) + \\
 & + 2 \cdot (2060 - a \cdot 1.015^{70}) \cdot (-1.015^{70}) + \\
 & + 2 \cdot (2800 - a \cdot 1.015^{80}) \cdot (-1.015^{80}) + \\
 & + 2 \cdot (3630 - a \cdot 1.015^{90}) \cdot (-1.015^{90}) + \\
 & + 2 \cdot (4430 - a \cdot 1.015^{100}) \cdot (-1.015^{100}) + \\
 & + 2 \cdot (5220 - a \cdot 1.015^{110}) \cdot (-1.015^{110}).
 \end{aligned}$$

Simplifying the above function, we get that

$$\begin{aligned}
 H'(a) = & 2a + 2 \cdot (100 - 1.161a) \cdot (-1.161) + \\
 & + 2 \cdot (210 - a \cdot 1.347) \cdot (-1.347) + \\
 & + 2 \cdot (420 - a \cdot 1.563) \cdot (-1.563) + \\
 & + 2 \cdot (650 - a \cdot 1.814) \cdot (-1.814) + \\
 & + 2 \cdot (910 - a \cdot 2.105) \cdot (-2.105) + \\
 & + 2 \cdot (1390 - a \cdot 2.443) \cdot (-2.443) + \\
 & + 2 \cdot (2060 - a \cdot 2.835) \cdot (-2.835) + \\
 & + 2 \cdot (2800 - a \cdot 3.291) \cdot (-3.291) + \\
 & + 2 \cdot (3630 - a \cdot 3.819) \cdot (-3.819) + \\
 & + 2 \cdot (4430 - a \cdot 4.432) \cdot (-4.432) + \\
 & + 2 \cdot (5220 - a \cdot 5.144) \cdot (-5.144).
 \end{aligned}$$

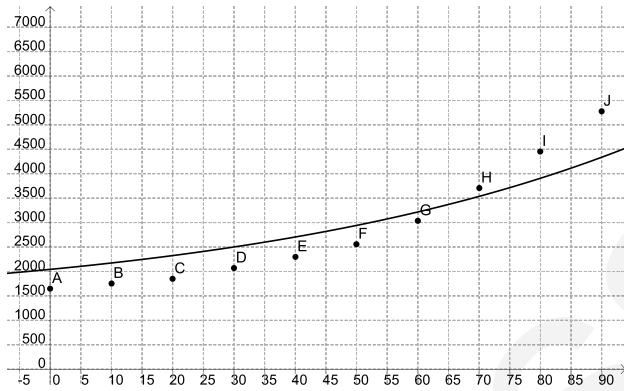
Expanding expression, we get that

$$H'(a) = 377.743a - 289337.$$

The zero of the function above is

$$199.702a - 165898 = 0 \quad \Rightarrow \quad a \approx 830.$$

The graph of function $P(t)$ is



Approximation of data is not good so in the next problem we use another model.

Problem 1.3.5. Table shows data for the population (in millions) of the world in the 20th and 21st century:

Year	Population [million]
1900	1 650
1910	1 750
1920	1 860
1930	2 070
1940	2 300
1950	2 560
1960	3 040
1970	3 710
1980	4 450
1990	5 280
2000	6 080
2010	6 870

The pattern of the data point series suggests exponential growth. Find the unknown parameter a in the model

$$P(t) = a \cdot 1.015^t.$$

Solution:

Let's consider the following points:

$$\begin{aligned} A &= (0; 1\ 650); & B &= (10; 1\ 750); \\ C &= (20; 1\ 860); & D &= (30; 2\ 070); \\ E &= (40; 2\ 300); & F &= (50; 2\ 560); \\ G &= (60; 3\ 040); & H &= (70; 3\ 710); \\ I &= (80; 4\ 450); & J &= (90; 5\ 280); \\ K &= (100; 6\ 080); & L &= (110; 6\ 870). \end{aligned}$$

Consider the function $H(a)$, such that

$$H(a) = \sum_{k=1}^{12} (y_k - P(x_k, a))^2,$$

where for given points of data:

$$A = (x_1; y_1), B = (x_2; y_2), \dots, L = (x_{12}; y_{12}).$$

Substituting the obtain data function $H(a)$ is

$$\begin{aligned} H(a) &= (1\ 650 - a \cdot 1.015^0)^2 + (1\ 750 - a \cdot 1.015^{10})^2 + \\ &+ (1\ 860 - a \cdot 1.015^{20})^2 + (2\ 070 - a \cdot 1.015^{30})^2 + \\ &+ (2\ 300 - a \cdot 1.015^{40})^2 + (2\ 560 - a \cdot 1.015^{50})^2 + \\ &+ (3\ 040 - a \cdot 1.015^{60})^2 + (3\ 710 - a \cdot 1.015^{70})^2 + \\ &+ (4\ 450 - a \cdot 1.015^{80})^2 + (5\ 280 - a \cdot 1.015^{90})^2 + \\ &+ (6\ 080 - a \cdot 1.015^{100})^2 + (6\ 870 - a \cdot 1.015^{110})^2. \end{aligned}$$

The derivative of the function above is

$$\begin{aligned}
 H'(a) = & 2a + 2 \cdot (1750 - a \cdot 1.015^{10}) \cdot (-1.015^{10}) + \\
 & + 2 \cdot (1860 - a \cdot 1.015^{20}) \cdot (-1.015^{20}) + \\
 & + 2 \cdot (2070 - a \cdot 1.015^{30}) \cdot (-1.015^{30}) + \\
 & + 2 \cdot (2300 - a \cdot 1.015^{40}) \cdot (-1.015^{40}) + \\
 & + 2 \cdot (2560 - a \cdot 1.015^{50}) \cdot (-1.015^{50}) + \\
 & + 2 \cdot (3040 - a \cdot 1.015^{60}) \cdot (-1.015^{60}) + \\
 & + 2 \cdot (3710 - a \cdot 1.015^{70}) \cdot (-1.015^{70}) + \\
 & + 2 \cdot (4450 - a \cdot 1.015^{80}) \cdot (-1.015^{80}) + \\
 & + 2 \cdot (5280 - a \cdot 1.015^{90}) \cdot (-1.015^{90}) + \\
 & + 2 \cdot (6080 - a \cdot 1.015^{100}) \cdot (-1.015^{100}) + \\
 & + 2 \cdot (6870 - a \cdot 1.015^{110}) \cdot (-1.015^{110}).
 \end{aligned}$$

Simplifying the above function, we get that

$$\begin{aligned}
 H'(a) = & 2a + 2 \cdot (1750 - 1.161a) \cdot (-1.161) + \\
 & + 2 \cdot (1860 - a \cdot 1.347) \cdot (-1.347) + \\
 & + 2 \cdot (2070 - a \cdot 1.563) \cdot (-1.563) + \\
 & + 2 \cdot (2300 - a \cdot 1.814) \cdot (-1.814) + \\
 & + 2 \cdot (2560 - a \cdot 2.105) \cdot (-2.105) + \\
 & + 2 \cdot (3040 - a \cdot 2.443) \cdot (-2.443) + \\
 & + 2 \cdot (3710 - a \cdot 2.835) \cdot (-2.835) + \\
 & + 2 \cdot (4450 - a \cdot 3.291) \cdot (-3.291) + \\
 & + 2 \cdot (5280 - a \cdot 3.819) \cdot (-3.819) + \\
 & + 2 \cdot (6080 - a \cdot 4.432) \cdot (-4.432) + \\
 & + 2 \cdot (6870 - a \cdot 5.144) \cdot (-5.144).
 \end{aligned}$$

Expanding expression, we get that

$$H'(a) = 186.491a - 249201.$$

The zero of the function above is

$$186.491a - 249201 = 0 \quad \Rightarrow \quad a \approx 1336.26.$$

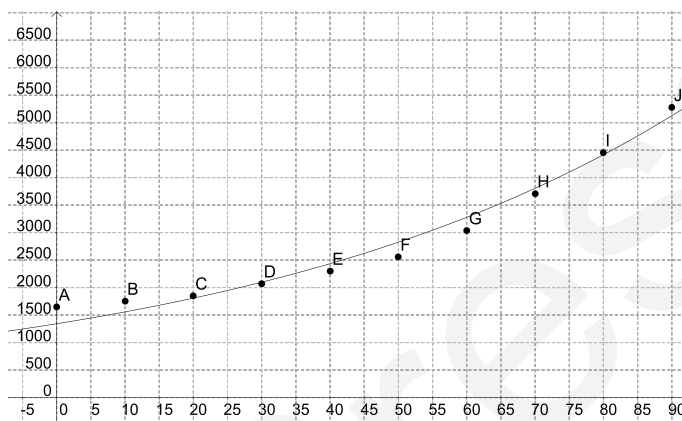
Consequently the value of unknown parameter a is

$$a = 1\,336.26$$

thus function $P(t)$ is

$$P(t) = 1\,336.26 \cdot 1.015^t.$$

The graph of function $P(t)$ is



Problem 1.3.6. We intend to measure the electric resistance of the stator coil of an electric motor. In course of the measurements electric current with different intensities is flowing through the coil and the voltage drop across the coil is measured. The measurement results are summarized in the table:

I[A]	1	2	3	4	5
U[V]	0.01	0.022	0.03	0.041	0.049

Determine the electric resistance of the stator coil applying the least squares method and Ohm's law.

Solution:

Ohm's law can be written in the following form:

$$V = R \cdot I,$$

where I is the intensity of current flowing through the resistor (coil), V is the voltage through the resistor (across the coil) and R is the electric resistance of the resistor (coil). In the given situation the model function can be written as:

$$V(I) = V(I, R) = R \cdot I.$$

Substituting the data, for function $H(R)$ (see in the theoretical summary), we get that:

$$H(R) = (0.01 - R)^2 + (0.022 - 2R)^2 + (0.03 - 3R)^2 + (0.041 - 4R)^2 + (0.049 - 5R)^2.$$

The derivative of the function above is:

$$H'(R) = -2 \cdot (0.01 - R) - 4 \cdot (0.022 - 2R) - 6 \cdot (0.03 - 3R) - 8 \cdot (0.041 - 4R) - 10 \cdot (0.049 - 5R).$$

The minimum value of function $H(R)$ can be found, where its derivative function is zero, thus we have to solve the equation

$$-0.01106 + 110R = 0 \quad \Rightarrow \quad R \approx 0.01 \text{ } [\Omega].$$

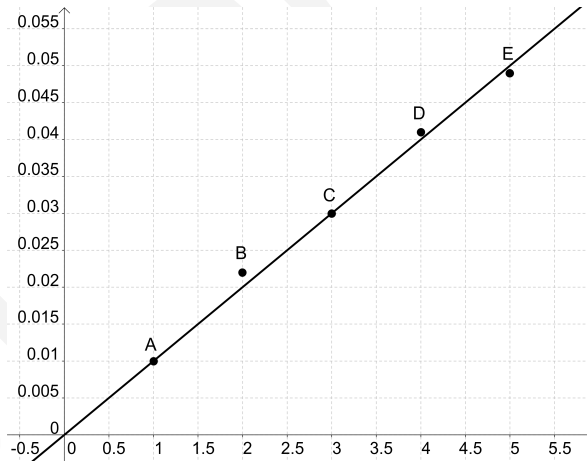
Since

$$H''(R) = 110 > 0,$$

thus function H has a minimum value at $R \approx 0.01$. Thus function $V(I)$ is

$$V(I) = 0.01 \cdot I.$$

The fitted data point series is presented in the figure:



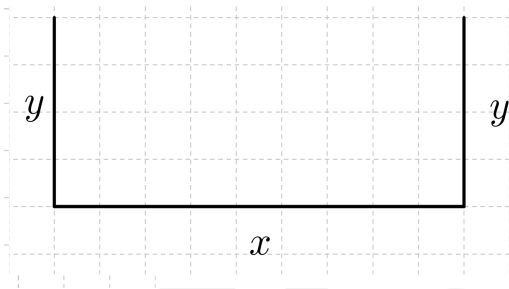
1.4. Extremum problems in engineering practice

Applied mathematical knowledge. Rules of derivation, extreme value calculus, function analysis

Sample problem 1. A farmer has 400 [m] of fencing and wants to fence off a rectangular field that borders a straight river. If needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution:

Let's denote the sides of the rectangle by x and y .



The total length of the fence is

$$P = 2y + x.$$

Using the equation $P = 400$, we get

$$2y + x = 400 \quad \Rightarrow \quad x = 400 - 2y.$$

The area of the rectangle is

$$A = x \cdot y = (400 - 2y) \cdot y = 400y - 2y^2.$$

Thus the function we intend to maximize is

$$A(y) = 400y - 2y^2.$$

Note that $y \geq 0$ and $y \leq 200$, thus the domain of function A is the $[0; 200]$ interval.

The derivative of function A is

$$A'(y) = 400 - 4y,$$

so to find the extreme values we have to solve the equation $A'(y) = 0$, that is

$$400 - 4y = 0.$$

The solution of the equation is $y = 100$.

The maximum value of A can be found at 100 or at the end points of the interval which is the domain of function A . Since

$$A(0) = 0; A(100) = 40\,000 - 20\,000 = 20\,000; A(200) = 0,$$

the Closed Interval Method gives the maximum value as

$$A(100) = 20\,000.$$

Alternatively, we could have observed that

$$A''(y) = -4 < 0$$

for all y , so A is always concave downward and the local maximum at $y = 100$ must be an absolute maximum. Thus the dimensions of the rectangular field are 100 and 200 meters. The maximum area is

$$A = x \cdot y = 100 \cdot 200 = 20\,000 \text{ [m}^2\text{]}.$$

Sample problem 2. The quantity of water in a water storage is given as a function of time t is

$$V(t) = 2t^3 - 21t^2 + 60t \text{ [hl]},$$

where t is in the interval $[0; 6]$ and t is the number of days since the beginning of the observation. Note that 1 hl = 100 [l].

- How much fluid was in the storage at the beginning of the observation, one day after the observation and at the end of the observation?
- Calculate the rate of change of the quantity of water.
- What is the rate of change at $t = 1$ and $t = 2$?
- Give the time interval when the function is increasing and the one when it is decreasing.
- Find the global minimum and global maximum of function $V(t)$.
- Sketch the graph of function $V(t)$.

Solution:

- a) The beginning of the observation is

$$V(0) = 2 \cdot 0^3 - 21 \cdot 0^2 + 60 \cdot 0 = 0 \text{ [hl]}.$$

At the end of the observation is

$$V(6) = 2 \cdot 6^3 - 21 \cdot 6^2 + 60 \cdot 6 = 36 \text{ [hl]}.$$

One day after the beginning of the observation

$$V(1) = 2 \cdot 1^3 - 21 \cdot 1^2 + 60 \cdot 1 = 41 \text{ [hl]}.$$

b) The rate of the change of function $V(t)$ is

$$\dot{V}(t) = 6t^2 - 42t + 60.$$

c) The rate of change at $t = 1$ is

$$\dot{V}(1) = 6 \cdot 1^2 - 42 \cdot 1 + 60 = 24 \left[\frac{\text{hl}}{\text{day}} \right].$$

The rate of change at $t = 2$ is

$$\dot{V}(2) = 6 \cdot 2^2 - 42 \cdot 2 + 60 = 0 \left[\frac{\text{hl}}{\text{day}} \right].$$

d) We have to solve the equation below

$$6t^2 - 42t + 60 = 0.$$

Simplifying the equation above, we get that

$$t^2 - 7t + 10 = 0.$$

Applying the quadratic formula, we get that

$$t_{1,2} = \frac{7 \pm \sqrt{49 - 40}}{2} = \frac{7 \pm 3}{2},$$

that is $t = 2$ [day] or $t = 5$ [day]. The sign of the first derivative of function $V(t)$ is shown in the table below

	$t < 2$	$t = 2$	$2 < t < 5$	$t = 5$	$t > 5$
$V'(t)$	+	0	-	0	+
$V(t)$	↗	loc. max.	↘	loc. min.	↗

e) The values of function V at $t = 2$ and $t = 5$ are

$$V(2) = 2 \cdot 2^3 - 21 \cdot 2^2 + 60 \cdot 2 = 52 \text{ [hl]}$$

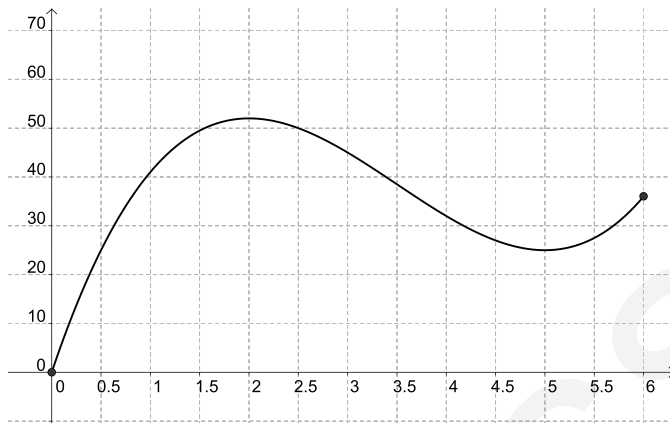
$$V(5) = 2 \cdot 5^3 - 21 \cdot 5^2 + 60 \cdot 5 = 25 \text{ [hl]}$$

and

$$V(0) = 0; V(6) = 36,$$

thus the minimum value is 0 [hl], the maximum value is 52 [hl].

f) The graph of function $V(t)$ is

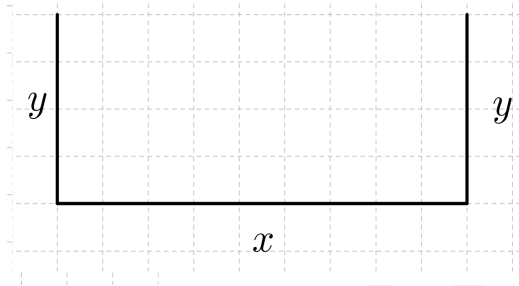


Problems

Problem 1.4.1. A farmer has 800 [m] of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

Solution:

Let's denote the sides of the rectangle by x and y .



The total length of the fence is

$$P = 2y + x.$$

From the previous equation, we get that

$$2y + x = 800 \quad \Rightarrow \quad x = 800 - 2y.$$

The area of the rectangle is

$$A = x \cdot y = (800 - 2y) \cdot y = 800y - 2y^2.$$

Thus the function we intend to maximize is

$$A(y) = 800y - 2y^2.$$

Note that $y \geq 0$ and $y \leq 400$, thus the domain of function A is the $[0; 400]$ interval.

The derivative of function A is

$$A'(y) = 800 - 4y,$$

so to find the extremum values we have to solve the equation $A'(y) = 0$, that is,

$$800 - 4y = 0.$$

The solution of the equation above is $y = 200$.

The maximum value of A can be found at 200 or at the end points of the interval which is the domain of function A . Since

$$A(0) = 0; A(200) = 160\,000 - 80\,000 = 80\,000; A(400) = 0,$$

the Closed Interval Method gives the maximum value as

$$A(200) = 80\,000.$$

Alternatively,

$$A''(y) = -4 < 0$$

for all y , so A is always concave downward and the local maximum at $y = 200$ is an absolute maximum. Thus the dimensions of the rectangular field are 200 and 400 meters. The maximum area is

$$A = x \cdot y = 400 \cdot 200 = 80\,000 \text{ [m}^2\text{]}.$$

Problem 1.4.2. Engineers are designing wagons for carrying coal in a straight railway tunnel with parabolic shaped cross section. The height of the tunnel is nine meters, while its width is six meters at its bottom. If the aim of the engineers is to design wagons with the maximum possible cross section, this way transporting the maximum amount of coal at a given wagon length, what would be the height and width of the wagons?

Solution:

Regarding the text of the problem the parabola goes through points $(-3; 0)$, $(0; 9)$ and $(3; 0)$ and the function which is describing it can be written as

$$f(x) = ax^2 + bx + c.$$

Substituting the coordinates of the points above into function $f(x)$, we get the system of equations

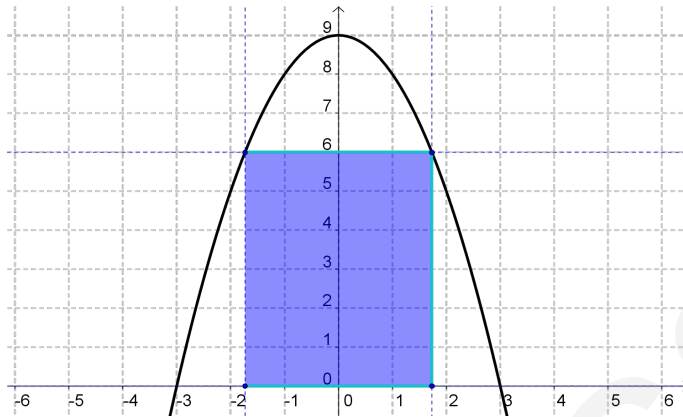
$$0 = a \cdot (-3)^2 + b \cdot (-3) + c$$

$$9 = a \cdot 0^2 + b \cdot 0 + c$$

$$0 = a \cdot 3^2 + b \cdot 3 + c.$$

Solving the system of equations, we get that $c = 9$, $b = 0$ and $a = -1$. Consequently, the function which describes the parabola is

$$f(x) = -x^2 + 9.$$



The cross section area of the wagon with width of $2x$ is

$$A(x) = 2x \cdot (-x^2 + 9) = -2x^3 + 18x.$$

The derivative of the function $A(x)$ is

$$A'(x) = -6x^2 + 18.$$

The zeros of the function are $\pm\sqrt{3}$. Since

$$A''(x) = -12x \quad \Rightarrow \quad A''(\sqrt{3}) < 0,$$

we have that function A has a local and global maximum. Consequently, the width of the wagon is

$$2x \approx 3.46 \text{ [m]}$$

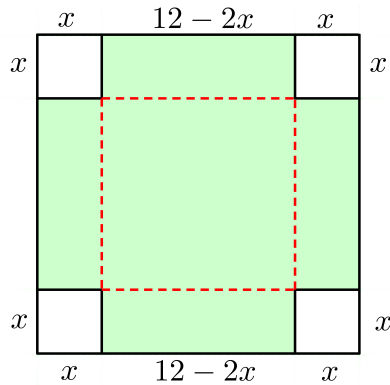
and its height is

$$f(\sqrt{3}) = -3 + 9 = 6 \text{ [m]}.$$

Problem 1.4.3. An open box is made by cutting congruent squares from the corners of a 12 [cm] by 12 [cm] cardboard sheet. How large should the squares be so that the box has a maximum volume? What is the maximum capacity of the box?

Solution:

Let x be a side of the square in [cm].



The volume of the box is

$$V(x) = (12 - 2x)^2 \cdot x = (144 - 48x + 4x^2) \cdot x = 4x^3 - 48x^2 + 144x.$$

The domain of function $V(x)$ is interval $[0; 6]$.

The derivative of function $V(x)$ is

$$V'(x) = 12x^2 - 96x + 144,$$

thus we have to solve the equation

$$12x^2 - 96x + 144 = 0.$$

If we simplify this, we get that

$$x^2 - 8x + 12 = 0.$$

Applying the quadratic formula

$$x_{1,2} = \frac{8 \pm \sqrt{64 - 4 \cdot 12}}{2} = \frac{8 \pm 4}{2},$$

that is $x_1 = 6$ or $x_2 = 2$. The second derivative of function $V(x)$ is

$$V''(x) = 24x - 96 \quad \Rightarrow \quad V''(2) = -48 < 0, \quad V''(6) = 48 > 0.$$

If $x = 2$ then $V''(2) < 0$ hence at $x = 2$ function $V(x)$ has a maximum.

The maximum volume is

$$V = 4 \cdot 2^3 - 96 \cdot 2^2 + 144 \cdot 2 = 128 \text{ [cm}^3\text{]}.$$

Problem 1.4.4. Engineers are designing a cylindrical can to hold 1 litre of car oil, with the minimum surface area in order to minimize costs. Find the optimal dimensions of the can.

Solution:

Let's denote the radius of the base and the height of the can by r and h respectively. Then the volume of the cylindrical can is

$$V = r^2 \cdot \pi \cdot h.$$

Since the volume is $V = 1000 \text{ cm}^3$, we get that

$$1000 = r^2 \cdot \pi \cdot h \quad \Rightarrow \quad h = \frac{1000}{\pi \cdot r^2}.$$

The surface area of the can is

$$A = 2 \cdot r^2 \cdot \pi + 2 \cdot r \cdot \pi \cdot h.$$

The surface area expressed by radius r :

$$A(r) = 2 \cdot r^2 \cdot \pi + 2 \cdot r \cdot \pi \cdot \frac{1000}{\pi \cdot r^2} = 2 \cdot r^2 \cdot \pi + \frac{2000}{r}.$$

The domain of function $A(r)$ is interval $]0; \infty[$.

The derivative of function $A(r)$ is

$$A'(r) = 4 \cdot r \cdot \pi - \frac{2000}{r^2}.$$

We have to solve the following equation

$$4 \cdot r \cdot \pi - \frac{2000}{r^2} = 0.$$

Expressing r , we get that

$$4r^3 \cdot \pi - 2000 = 0 \quad \Rightarrow \quad r = \sqrt[3]{\frac{500}{\pi}} \approx 5.42$$

The second derivative of function $A(r)$ is

$$A''(r) = 4\pi + \frac{4000}{r^3} \quad (r > 0).$$

Since function $A''(r) > 0$, hence function $V(r)$ is convex for all $r > 0$, thus the global minimum is at $r \approx 5.42$. The value of h is

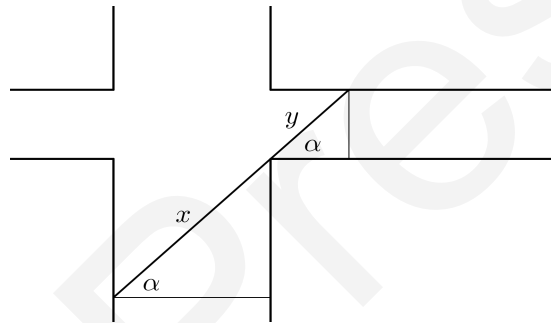
$$h = \frac{1000}{\pi \cdot r^2} \approx \frac{1000}{\pi \cdot 5.42^2} = 10.84 \text{ [cm]}.$$

Substituting r and h into A , we get that

$$A = 2 \cdot 5.42^2 \cdot \pi + 5.42 \cdot \pi \cdot 10.84 \approx 368.97 \text{ [cm}^2\text{]}.$$

Problem 1.4.5. A steel pipe is being carried in a corridor with width of 2.4 meters. At the end of the corridor there is right-angled turn into a narrower hallway which is 1.6 meters wide. What is the length of the longest pipe that can be carried horizontally around the corner?

Solution:



According to the figure

$$x = \frac{2.4}{\cos \alpha}; \quad y = \frac{1.6}{\sin \alpha}.$$

The length of the pipe is

$$d(\alpha) = x + y = \frac{2.4}{\cos \alpha} + \frac{1.6}{\sin \alpha} \quad \left(\alpha \in \left] 0; \frac{\pi}{2} \right[\right).$$

The derivative of function $d(\alpha)$ is

$$\begin{aligned} d'(\alpha) &= \frac{2.4 \sin \alpha}{\cos^2 \alpha} - \frac{1.6 \cos \alpha}{\sin^2 \alpha} = \\ &= \frac{2.4 \sin^3 \alpha - 1.6 \cos^3 \alpha}{\sin^2 \alpha \cos^2 \alpha}. \end{aligned}$$

Function $d'(\alpha)$ is zero, if and only if,

$$2.4 \sin^3 \alpha - 1.6 \cos^3 \alpha = 0.$$

Since $\operatorname{tg} \alpha = \frac{\sin \alpha}{\cos \alpha}$, we have

$$2.4 \sin^3 \alpha - 1.6 \cos^3 \alpha = 0$$

$$2.4 \sin^3 \alpha = 1.6 \cos^3 \alpha$$

$$\operatorname{tg}^3 \alpha = \frac{2}{3}$$

$$\operatorname{tg} \alpha = \sqrt[3]{\frac{2}{3}}$$

Applying function arctg , we get that

$$\alpha \approx 41^\circ.$$

Thus

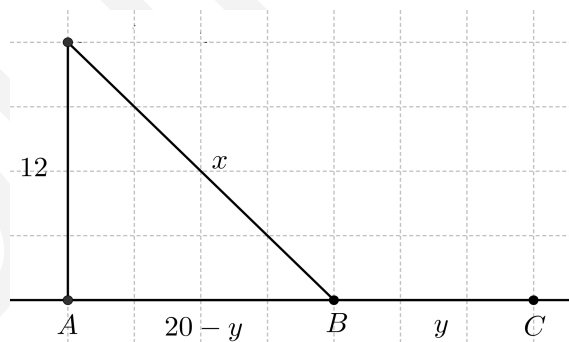
$$x + y = \frac{2.4}{\cos 41^\circ} + \frac{1.6}{\sin 41^\circ} \approx 3.2 + 2.4 \text{ [m]}.$$

The length of the longest pipe that can be carried horizontally around the corner is 5.6 meters.

Problem 1.4.6. A drilling rig 12 km offshore is to be connected by pipe to a refinery onshore, 20 km straight down the coast from the rig. If underwater pipe costs $500\,000 \left[\frac{\$}{\text{km}} \right]$ and landbased pipe costs $300\,000 \left[\frac{\$}{\text{km}} \right]$. What combination of the two will give the least expensive connection?

Solution:

Consider the figure below:



Applying the Pythagorean theorem, we have

$$x^2 = 12^2 + (20 - y)^2.$$

Expressing x from the equation above, we get that

$$x = \sqrt{144 + (20 - y)^2}.$$

The cost of pipeline in dollar is

$$C = 500\,000x + 300\,000y.$$

After substituting, we get that

$$C(y) = 500\,000 \cdot \sqrt{144 + (20 - y)^2} + 300\,000y.$$

Applying algebraic identities, we get that

$$C(y) = 500\,000 \cdot (144 + (20 - y)^2)^{\frac{1}{2}} + 300\,000y.$$

The first derivative of function $C(y)$ is

$$C'(y) = 500\,000 \cdot \frac{1}{2} (144 + (20 - y)^2)^{-\frac{1}{2}} \cdot 2(20 - y) \cdot (-1) + 300\,000.$$

Simplifying the above function, we get that

$$C'(y) = -500\,000 \cdot \frac{20 - y}{\sqrt{144 + (20 - y)^2}} + 300\,000.$$

The zeros of $C'(y)$ are the solutions of the following equation

$$-500\,000 \cdot \frac{20 - y}{\sqrt{144 + (20 - y)^2}} + 300\,000 = 0.$$

Solving the above equation, we get that

$$500\,000 \cdot (20 - y) = 300\,000 \sqrt{144 + (20 - y)^2}$$

$$\frac{5}{3} \cdot (20 - y) = \sqrt{144 + (20 - y)^2}$$

$$\frac{25}{9} \cdot (20 - y)^2 = 144 + (20 - y)^2$$

$$\frac{16}{9} \cdot (20 - y)^2 = 144.$$

The solutions of the equation are $y_1 = 11$ and $y_2 = 29$. Since $y = 29$ is not a solution of the problem, we get that $y = 11$.

Since,

$$C(0) = 11\,661\,900;$$

$$C(11) = 10\,800\,000;$$

$$C(20) = 12\,000\,000$$

the minimum cost is

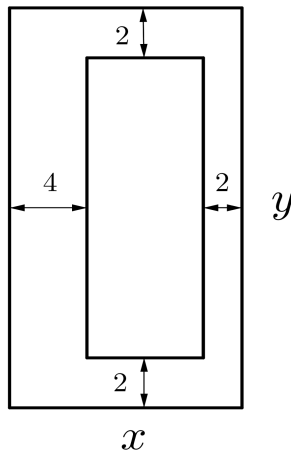
$$C(11) = 10\,800\,000 \text{ dollars.}$$

The least expensive connection costs 10 800 000 dollars and we achieve it by running the line underwater to the point on shore 11 [km] from the refinery.

Problem 1.4.7. The editor of a publishing house is designing the layout of a book. The planned margins are 2 [cm] wide on the top, on the bottom and on the outer edge of the pages, but the inner margins have to be 4 cm wide because of the binding. The total area of a page is 600 [cm²]. How should the editor set the dimensions of the pages in order to have the maximal printing area?

Solution:

Let's denote the sides of the page by x and y .



The area of the paper is

$$x \cdot y = 600.$$

Expressing variable y , we have

$$y = \frac{600}{x}.$$

The printing area is

$$\begin{aligned} f(x) &= (x - 6) \cdot (y - 4) = xy - 4x - 6y + 24 = \\ &= 600 - 4x - \frac{3600}{x} + 24 = 624 - 4x - \frac{3600}{x}. \end{aligned}$$

The domain of function $f(x)$ is $x \in [6; 150]$.

The first derivative of function $f(x)$ is

$$f'(x) = -4 + \frac{3600}{x^2}.$$

Solving the equation $f'(x) = 0$, we get that

$$-4 + \frac{3600}{x^2} = 0 \quad \Rightarrow \quad \frac{3600}{x^2} = 4 \quad \Rightarrow \quad x^2 = 900,$$

The zeros of function $f'(x)$ are $x = \pm 30$. Since $x \in [6; 150]$, the only solution is $x = 30$.

The second derivative of function $f(x)$ is

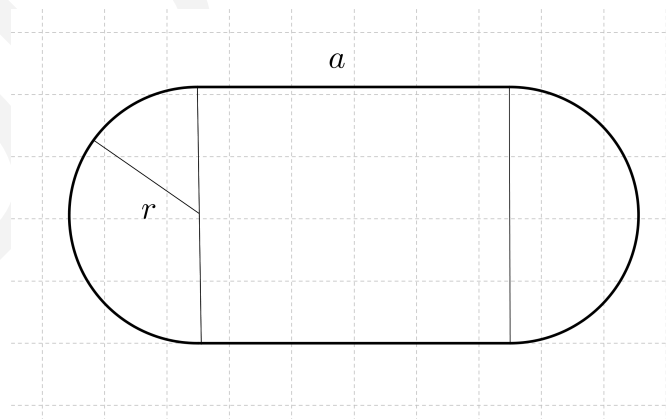
$$f''(x) = -\frac{7200}{x^3} \quad \Rightarrow \quad f''(30) = -\frac{7200}{30^3} < 0,$$

thus at $x = 30$, there is a maximum. Consequently side y is

$$y = \frac{600}{30} = 20.$$

Thus the dimensions of the optimal page are 30 [cm] and 20 [cm], respectively.

Problem 1.4.8. Find the maximum area of the flower bed shown in figure



if its perimeter is 400 meters.

Solution:

The formula for the perimeter is

$$P = 2 \cdot r \cdot \pi + 2 \cdot a.$$

The perimeter is 400 meters, thus

$$400 = 2 \cdot r \cdot \pi + 2 \cdot a.$$

It follows that

$$a = 200 - r \cdot \pi.$$

The area of the flower bed is

$$A(r) = 2 \cdot r \cdot a = 2 \cdot r \cdot (200 - r \cdot \pi) = 400r - 2 \cdot r^2 \cdot \pi.$$

The first derivative of function $A(r)$ is

$$A'(r) = 400 - 4 \cdot r \cdot \pi.$$

The zero of function $A'(r)$ is $r = \frac{100}{\pi}$.

The second derivative of function $A(r)$ is

$$A''(r) = -4\pi < 0,$$

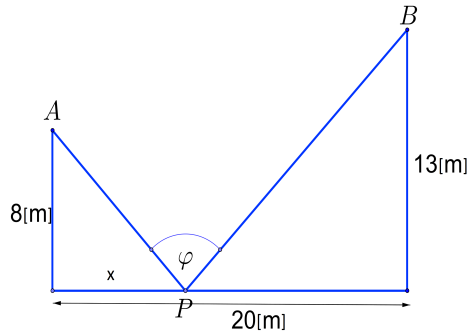
consequently at $r = \frac{100}{\pi}$, there is a global maximum. The value of parameter a is

$$a = 200 - r \cdot \pi = 200 - \frac{100}{\pi} \cdot \pi = 100.$$

The maximum area is

$$A = 2 \cdot r \cdot a = 2 \cdot \frac{100}{\pi} \cdot 100 = 6\,366 \text{ [m}^2\text{]}.$$

Problem 1.4.9. Two poles, one of them is 8 meters tall and the other one is 15 meters tall, are 20 meters apart. A length of wire is attached to the top of each pole and it is also staked to the ground somewhere between the two poles.



- Determine function $\varphi(x)$, where x is the distance of P from the 8 meters tall pole.
- Calculate the value of φ at $x = 0$ [m].
- Calculate the value of φ at $x = 20$ [m].
- Give derivative of function $\varphi(x)$.
- Where should the wire be staked so that the angle formed by the two pieces of wire at the stake be the maximum?
- Plot the graph of function $\varphi(x)$ in the interval $[0; 20]$ [m].

Solution:

- Let α be the angle, which is between AP hypotenuse and horizontal leg. Let β be the angle, which is between PB hypotenuse and horizontal plane.

Since

$$\operatorname{tg} \alpha = \frac{8}{x}; \quad \operatorname{tg} \beta = \frac{13}{20 - x},$$

using identity

$$\varphi = \pi - \alpha - \beta,$$

we get that

$$\begin{aligned} \varphi(x) &= \pi - \operatorname{arctg} \left(\frac{8}{x} \right) - \operatorname{arctg} \left(\frac{13}{20 - x} \right) = \\ &= \left(\frac{\pi}{2} - \operatorname{arctg} \left(\frac{8}{x} \right) \right) + \left(\frac{\pi}{2} - \operatorname{arctg} \left(\frac{13}{20 - x} \right) \right). \end{aligned}$$

Applying the trigonometric identity

$$\operatorname{arctg} \left(\frac{1}{x} \right) = \frac{\pi}{2} - \operatorname{arctg} x,$$

we get that

$$\varphi(x) = \operatorname{arctg}\left(\frac{x}{8}\right) + \operatorname{arctg}\left(\frac{13}{20-x}\right).$$

b) If $x = 0$ then

$$\varphi \approx 56.98^\circ (\approx 0.9939 \text{ rad}).$$

c) If $x = 20$ then

$$\varphi \approx 68.2^\circ (\approx 1.1897 \text{ rad}).$$

d) Applying the differential rule of composite functions

$$\varphi'(x) = \frac{1}{1 + \frac{x^2}{64}} \cdot \frac{1}{8} + \frac{1}{1 + \frac{400-40x+x^2}{169}} \cdot \left(-\frac{1}{13}\right).$$

Applying algebraic identities, we get that

$$\begin{aligned} \varphi'(x) &= \frac{1}{8 + \frac{x^2}{8}} - \frac{1}{13 + \frac{400-40x+x^2}{13}} = \\ &= \frac{8}{64 + x^2} - \frac{13}{569 - 40x + x^2} = \\ &= \frac{-5x^2 - 320x + 3720}{(64 + x^2) \cdot (569 - 40x + x^2)} = \\ &= \frac{5 \cdot (744 - 64x - x^2)}{(x^2 + 64) \cdot (x^2 - 40x + 569)}. \end{aligned}$$

e) We have to solve the equation $\varphi'(x) = 0$, that is,

$$\frac{5 \cdot (744 - 64x - x^2)}{(x^2 + 64) \cdot (x^2 - 40x + 569)} = 0.$$

The equation equivalent to

$$-x^2 - 64x + 744 = 0.$$

Solving the quadratic equation above, we get that

$$x_{1,2} = \frac{64 \pm \sqrt{64^2 + 4 \cdot 744}}{-2} \approx \frac{64 \pm 84.1}{-2}.$$

The solutions of the equation above are $x \approx 10.05$ and $x \approx -74.05$.

Since $x > 0$, we get that $x \approx 10.05$. The inequality

$$x^2 + 64 > 0,$$

implies the denominator of the function $\varphi'(x)$ is positive for each x .

On the other hand the discriminant of the polynom $x^2 - 40x + 569$ is

$$D = 40^2 - 4 \cdot 569 = 1600 - 2276 = -676 < 0,$$

thus inequality

$$x^2 - 40x + 569 > 0$$

holds for all real x .

That is,

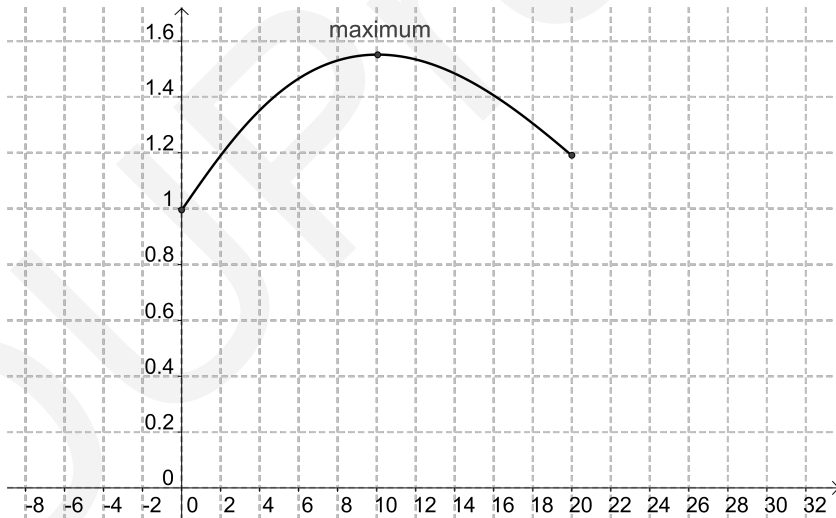
- if $0 \leq x \leq 10.05$, then $\varphi'(x) \geq 0$;
- if $10.05 \leq x \leq 20$, then $\varphi'(x) \leq 0$,

thus at $x = 10.05$, there is a global maximum.

The maximum value of φ is

$$\varphi(10.05) \approx 51.48^\circ + 37.43^\circ = 88.91^\circ \approx 1.51 \text{ rad.}$$

f) The graph of function $\varphi(x)$ is



Problem 1.4.10. The population of bacteria (P) in thousands at a time t in hours is modelled by

$$P(t) = 10 + e^t - 3t \quad (t \geq 0).$$

a) Find the initial population of bacteria.

- b) Find the function $P'(t)$.
- c) Find the time at which the bacteria are growing at a rate of 6 million per hour.
- d) Find the function $P''(t)$ and explain the physical significance of this quantity.
- e) Find the minimum number of bacteria, justifying that it is a minimum.

Solution:

- a) Since

$$P(0) = 10 + e^0 - 3 \cdot 0 = 11,$$

then the initial population of bacteria is 11 0000.

- b) The derivative of function $P(t)$ is

$$P'(t) = e^t - 3.$$

- c) We have to solve the equation $P'(t) = 6\,000$, that is,

$$6\,000 = e^t - 3.$$

Solution of the equation is $t = \ln 6\,003$, that is, $t = 8.7$.

- d) The second derivative of function $P(t)$ is

$$P''(t) = e^t,$$

which is the rate of change of the growth rate of the bacteria.

- e) We have to solve the equation $P'(t) = 0$, that is,

$$e^t - 3 = 0 \quad \Rightarrow \quad e^t = 3,$$

thus $t = \ln 3 = 1.099$.

Second derivative of function $P(t)$ is positive for all t , thus P has a minimum at $t = 1.099$.

Since

$$P(\ln 3) = 10 + e^{\ln 3} - 3 \cdot \ln 3 = 9.704,$$

the minimum number of bacteria is 9 704.

Problem 1.4.11. A paper aeroplane of weight $w > 1$ will travel at a constant speed of $1 - \frac{1}{w}$ [$\frac{\text{m}}{\text{s}}$] for a time of $\frac{5}{w}$ [s]. What weight will achieve the maximum distance travelled?

Solution:

The covered distance is

$$s(w) = \left(1 - \frac{1}{\sqrt{w}}\right) \cdot \frac{5}{w} = \frac{5w^{\frac{1}{2}} - 5}{w^{\frac{3}{2}}}.$$

Derivative of function $s(w)$ is

$$s'(w) = \frac{\frac{5}{2} \cdot w^{-\frac{1}{2}} \cdot w^{\frac{3}{2}} - (5w^{\frac{1}{2}} - 5) \cdot \frac{3}{2} \cdot w^{\frac{1}{2}}}{w^3}.$$

Applying algebraic identities, we get that

$$s'(w) = \frac{\frac{5}{2} \cdot w - \frac{15}{2} \cdot w + \frac{15}{2} \cdot w^{\frac{1}{2}}}{w^3} = \frac{-5w + \frac{15}{2} \cdot \sqrt{w}}{w^3}.$$

We have to solve the equation $s'(w) = 0$, that is,

$$\frac{-5w + \frac{15}{2} \cdot \sqrt{w}}{w^3} = 0 \quad \Rightarrow \quad -5w + \frac{15}{2} \cdot \sqrt{w} = 0.$$

Factorizing the equation, we have

$$\sqrt{w} \cdot \left(-5\sqrt{w} + \frac{15}{2}\right).$$

Since $w \neq 0$, thus

$$-10\sqrt{w} + 15 = 0,$$

hence $w = 2.25$.

Second derivative of function $s(w)$ is

$$\begin{aligned} s''(w) &= \frac{\left(-5 + \frac{15}{4\sqrt{w}}\right) \cdot w^3 - (5w + \frac{15}{2}\sqrt{w}) \cdot 3w^2}{w^6} = \\ &= \frac{-5w + \frac{15}{4}\sqrt{w} - 5w - \frac{15}{2}\sqrt{w}}{w^4} = \frac{-10w - \frac{15}{4}\sqrt{w}}{w^4}. \end{aligned}$$

Since $w > 1$, then $s''(w) < 0$, consequently, function $s(w)$ has a global maximum at $w = 2.25$.

1.5. Differential calculus in economics

Theoretical summary. Calculus is applied in basic economic theory in marginal analysis. Economists analyze how small changes, for example increasing the production of a product by a single unit, affect profits, and costs. Marginal analysis quantifies the benefits of performing such an action against the costs.

When the benefits or profits, exceed the cost of the action, you can proceed on this course until this balance changes. The break-even point occurs when the production costs and the total revenues the amount of income generated before any deductions are made, are the same.

Notations in this section:

- x : number of units produced or sold;
- $P(x)$: *demand* as a function of amount of units x ;
- $S(x)$: *supply* as a function of amount of units x ;
- $R(x)$: *total revenue* from selling x amount of units;
- FC : *fix cost*;
- $VC(x)$: *variable cost*;
- $AFC = \frac{FC}{x}$: *average fix cost*;
- $AVC(x) = \frac{VC(x)}{x}$: *average variable cost*;
- $AC(x) = \frac{C(x)}{x}$: *average total cost*;
- $C(x)$: *total cost* of producing x amount of units;
- $\Pi(x)$: *profit* in selling x amount of units;
- $R'(x)$: *marginal revenue*, which is the extra revenue for selling one extra unit;
- $MC(x) = C'(x)$: *marginal cost* or *marginal cost function*, which is the extra cost for selling one extra unit;
- $M\Pi(x) = \Pi'(x)$: *marginal profit* or *marginal profit function*, which is the extra profit for selling one additional unit;
- $MR(x) = R'(x)$: *marginal revenue* or *marginal revenue function*.

It is clearly, that

- $C(x) = FC + VC(x)$;
- $AC(x) = AFC + AVC(x)$.

Generally the cost function $C(x)$ is a third-degree polynomial function, that is,

$$C(x) = ax^3 + bx^2 + cx + d,$$

where d is the fixed cost.

In economics, *demand* is the quantity of a commodity or a service that people are willing or able to buy at a certain price, per unit of time.

In economics, *supply* is the amount of something that firms, consumers, labourers, providers of financial assets, or other economic agents are willing to provide to the marketplace.

The market will reach *equilibrium*, when the quantity demanded and the quantity supplied are equal.

In economics, *elasticity* is the measurement of how an economic variable responds to a change in another. In economics, elasticity is used to determine how changes in product demand and supply related to changes in consumer income or the producer's price.

Since

$$\Delta f(x) = f(x + \Delta x) - f(x),$$

the relative change is

$$\frac{\Delta f(x)}{f(x)} = \frac{f(x + \Delta x) - f(x)}{f(x)}.$$

That is,

$$\frac{\frac{\Delta f(x)}{f(x)}}{\frac{\Delta x}{x}} = \frac{x}{f(x)} \cdot \frac{\Delta f(x)}{\Delta x} = \frac{x}{f(x)} \cdot \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Price increasing one percent if $\Delta x = \frac{x}{100}$. Then

$$\frac{\Delta x}{x} = \frac{1}{100}.$$

The elasticity of function $f(x)$ is the limit of the function

$$\frac{x}{f(x)} \cdot \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Applying the definition of the differential quotient, we get that

$$E(x) = \frac{x}{f(x)} \cdot f'(x).$$

Applied mathematical knowledge. Rules of differentiation, monotonicity, optimum of the functions, convexity.

Sample problem 1. The cost of manufacturing fishing poles, in thousand units, is modeled by

$$C(x) = x^3 - 10x^2 + 20x \quad (0 \leq x \leq 10).$$

The revenue function is modeled by

$$R(x) = 7x + 3.$$

- Find the fix cost function.
- Find the variable cost function.
- Find the profit function.
- Find the production level that maximize profits.
- Calculate the marginal cost function.
- Find the production level, if it exists, that minimizes cost.
- Find the average cost function.
- Find the production level, if it exists, that minimizes average cost.

Solution:

a) Fix cost function is 0.

b) Variable cost function is

$$VC(x) = x^3 - 10x^2 + 20x.$$

c) The profit function is

$$\begin{aligned} \Pi(x) &= R(x) - C(x) = 7x + 3 - (x^3 - 10x^2 + 20x) = \\ &= -x^3 + 10x^2 - 13x + 3. \end{aligned}$$

d) We have to solve the equation

$$\Pi'(x) = 0,$$

that is

$$-3x^2 + 20x - 13 = 0.$$

Applying the quadratic form, we get that

$$x_{1,2} = \frac{-20 \pm \sqrt{400 - 12 \cdot 13}}{-6}.$$

The solutions of the $\Pi'(x) = 0$ equation are $x_1 \approx 0.73$ and $x_2 \approx 5.94$. The second derivative of function $\Pi(x)$ is

$$\Pi''(x) = -6x + 20.$$

Since $\Pi''(5.94) < 0$, maximum profit occurs at a production level of 5 490 fishing poles.

e) The marginal cost function is

$$MC(x) = C'(x) = 3x^2 - 20x + 20.$$

f) We have to solve the equation

$$C'(x) = 0,$$

that is

$$3x^2 - 20x + 20 = 0.$$

Applying the quadratic form, we get that

$$x_{1,2} = \frac{20 \pm \sqrt{400 - 240}}{6} \approx \frac{20 \pm 12.65}{6}.$$

The solutions of the quadratic equation are $x_1 \approx 5.44$ or $x_2 \approx 1.225$. Since

$$C''(x) = 6x - 10$$

and

$$C''(5.44) = 6 \cdot 5.44 - 10 > 0,$$

so we get that minimum cost occurs at a production level of approximately 5 440 fishing poles.

g) The average cost function is

$$AC(x) = \frac{C(x)}{x} = \frac{x^3 - 10x^2 + 20x}{x} = x^2 - 10x + 20$$

h) The derivative of the average cost is

$$AC'(x) = 2x - 10.$$

The solution of the equation $2x - 10 = 0$ is $x = 5$, that is the production level, when the average cost is minimum $x = 5$.

Sample problem 2. The total cost function is

$$C(x) = x^3 - 165x^2 + 10\,000x + 20\,000 \quad (0 \leq x < 200).$$

Find the marginal cost. Find the extremum value(s) and inflection point(s) of the function $C(x)$ and sketch the graph of function.

Solution:

The marginal cost is

$$C'(x) = 3x^2 - 330x + 10\,000.$$

The equation $C'(x) = 0$ is

$$3x^2 - 330x + 10\,000 = 0.$$

The determinant of the previous equation is

$$D = 330^2 - 120\,000 = 108\,900 - 120\,000 = -11\,100 < 0,$$

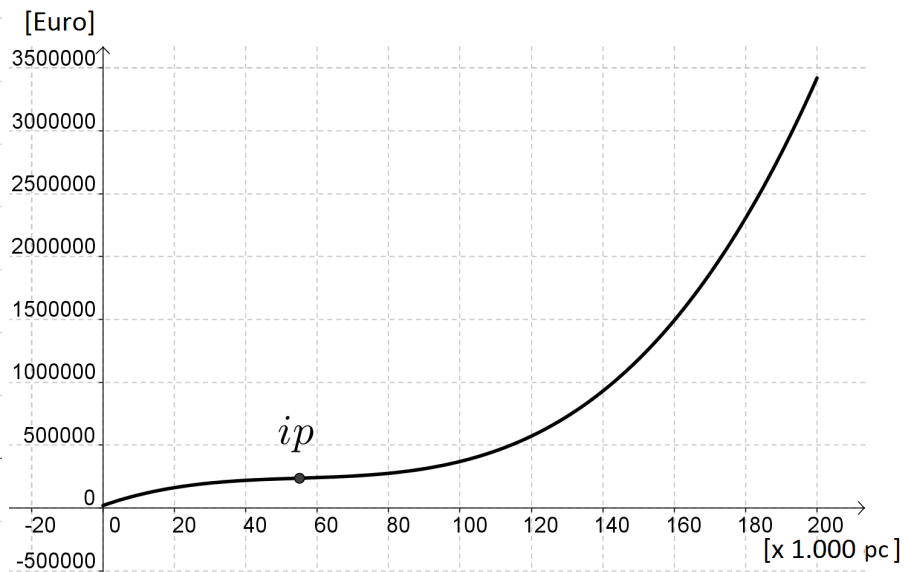
thus the equation $C'(x) = 0$ has no solution, thus the total cost function $C(x)$ is strictly monotonic increasing. The second derivative of function $C(x)$ is

$$C''(x) = 6x - 330.$$

The solution of equation $C''(x) = 0$ is $x = 55$. The next table includes the sign of the $C''(x)$ function:

	$x < 55$	$x = 55$	$x > 55$
$C''(x)$	-	0	+
$C(x)$	concave	inflection point	convex

Since $C(0) = 20\,000$, thus the fix cost is 20 000 euro.



Sample problem 3. Suppose that the demand function is

$$f(x) = \frac{100}{x+5}.$$

Find the elasticity function. The price of the product is 5 dollar. How many percent change the demand, when increase the price 1 percent or decrease the price 3 percent.

Solution:

The derivative of function f is

$$f'(x) = \frac{-100}{(x+5)^2}.$$

The elasticity function is

$$E(x) = \frac{x}{f(x)} f'(x) = \frac{x}{\frac{100}{x+5}} \cdot \frac{-100}{(x+5)^2} = x \cdot \frac{x+5}{100} \cdot \frac{-100}{(x+5)^2} = \frac{-x}{x+5}.$$

The price of the product is 5 dollar, thus we calculate $E(5)$:

$$E(5) = \frac{-5}{5+5} = -\frac{1}{2}.$$

If the price increase with 1 percent then the demand decrease with 0.5 percent.
If the price decrease with 3 percent, then the demand increase with $3 \cdot 0.5 = 1.5$ percent.

Problems

Problem 1.5.1. The revenue function is

$$R(x) = 4\sqrt{x}.$$

The total cost function is

$$C(x) = 2x^2.$$

The units is in thousand pieces. The revenue and cost is in thousand dollars.

Calculate the number of units, when the profit is maximal. Find the maximum value of profit function.

Solution:

The profit function is

$$\Pi(x) = R(x) - C(x) = 4\sqrt{x} - 2x^2.$$

The derivative of function $\Pi(x)$ is

$$\Pi'(x) = 4 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} - 4x = \frac{2}{\sqrt{x}} - 4x.$$

The solution of equation $\Pi'(x) = 0$ is

$$\frac{2}{\sqrt{x}} - 4x = 0 \quad \Rightarrow \quad 2 = 4x \cdot \sqrt{x}.$$

Applying the algebraic identity, we get that

$$4 = 16x^3 \quad \Rightarrow \quad x \approx 0.63.$$

The second derivative of the function $\Pi(x)$ is

$$\Pi''(x) = -x^{-\frac{3}{2}} - 4 < 0$$

thus the company has to produce 630 units. Since

$$\Pi(0.63) \approx 2.38,$$

the maximal profit is 2 380 dollar.

Problem 1.5.2. A three-sided fence is to be built next to a straight section of a river, which forms the fourth side of a rectangular region. The enclosed area is equal to $100 \text{ [m}^2\text{]}$. The cost the side, which parallel of the river is 6 dollar and the cost of sides, which orthogonal the river are 3 dollar. Find the dimensions of the corresponding enclosure.

Solution:

Let the sides of the rectangular be a and b . Then the area is

$$A = a \cdot b,$$

thus

$$100 = a \cdot b \quad \Rightarrow \quad b = \frac{100}{a}.$$

The cost of the fence is

$$C(a) = 6a + 6b = 6a + \frac{600}{a}.$$

The derivative of function $C(a)$ is

$$C'(a) = 6 - \frac{600}{a^2}.$$

We have to solve the equation $C'(a) = 0$.

The solutions of this equation are

$$6 - \frac{600}{a^2} = 0 \quad \Rightarrow \quad 6a^2 = 600 \quad \Rightarrow \quad a = \pm 10.$$

Since $a > 0$, we get that $a = 10$.

The second derivative of function $C(a)$ is

$$C''(a) = \frac{1.200}{a^3} \quad \Rightarrow \quad C''(10) = 1.2 > 0,$$

thus $a = 10$ is the minimum of function $C(a)$. Then

$$b = \frac{100}{10} = 10,$$

furthermore

$$C = 6 \cdot 10 + 6 \cdot 10 = 120,$$

thus the minimal cost is 120 dollar.

Problem 1.5.3. Consider the revenue function is

$$R(x) = x \cdot \sqrt{6900 - 0.2x},$$

where x is the number of units.

Calculate the number of units to be sold to maximizing the revenue. Find the maximum value of revenue function.

Solution:

The derivative of function $R(x)$ is

$$\begin{aligned} R'(x) &= \sqrt{6\,900 - 0.2x} + x \cdot \frac{1}{2} \cdot (6\,900 - 0.2x)^{-\frac{1}{2}} \cdot (-0.2) = \\ &= \sqrt{6\,900 - 0.2x} - \frac{0.1x}{\sqrt{6\,900 - 0.2x}}. \end{aligned}$$

The solution of equation $B'(x) = 0$ is

$$\sqrt{6\,900 - 0.2x} - \frac{0.1x}{\sqrt{6\,900 - 0.2x}} = 0.$$

Writing the equation in an equivalent form

$$6\,900 - 0.3x = 0,$$

that is $x = 23\,000$. The table of the sign of function $B'(x)$ is below:

	$0 < x < 23\,000$	$x = 23\,000$	$x > 23\,000$
$R'(x)$	+	0	-
$R(x)$	↗	loc. max.	↘

We get that the company has to produce 23 000 units. Then the maximal revenue is

$$R(23\,000) = 23\,000 \cdot \sqrt{6\,900 - 0.2 \cdot 23\,000} \approx 1\,103\,041.$$

Problem 1.5.4. The demand function of the company is

$$P(x) = 42 - 0.001x$$

and the cost function is

$$C(x) = 30x + 1\,200,$$

where x is the number of units demanded.

- Give the marginal cost function.
- Determine the revenue function.
- Find the marginal revenue function.
- Give the profit function.
- Determine the marginal profit function.
- Calculate the profit for 1 000 units.
- Calculate the number of units, when the profit has maximum.

Solution:

a) Marginal cost (function) is

$$MC(x) = C'(x) = -0.001.$$

b) The revenue function is

$$R(x) = x \cdot P(x) = x \cdot (42 - 0.001x) = 42x - 0.001x^2.$$

c) The marginal revenue function is $MR(x) = R'(x) = 42 - 0.002x$.

d) The profit function is

$$\begin{aligned} \Pi(x) &= R(x) - C(x) = 42x - 0.001x^2 - (30x + 1200) = \\ &= -0.001x^2 + 12x - 1200. \end{aligned}$$

e) The marginal profit function is

$$\Pi'(x) = -0.002x + 12.$$

f) The profit for 1 000 units is

$$\Pi(1000) = -0.001 \cdot 1000^2 + 12 \cdot 1000 - 1200 = 9800.$$

g) The solution of equation $\Pi'(x) = 0$ is

$$-0.002x + 12 = 0 \quad \Rightarrow \quad x = 6000.$$

Since

$$\Pi''(x) = -0.02 < 0,$$

the profit is maximal, when the company produces 6 000 units. Then the profit is

$$\Pi(6000) = -0.001 \cdot 6000^2 + 12 \cdot 6000 - 1200 = 34800.$$

Problem 1.5.5. Suppose that the total cost, in dollars, of producing x cell phones is

$$C(x) = 6000 - 30x + x^3.$$

Find the minimum average cost.

Solution:

The average cost function is

$$AC(x) = \frac{6000}{x} - 30 + 3x^2.$$

The derivative of $AC(x)$ function is

$$AC'(x) = -\frac{6000}{x^2} + 6x.$$

The solution of equation $AC'(x) = 0$ is

$$-\frac{6\,000}{x^2} + 6x = 0 \quad \Rightarrow \quad -6\,000 + 6x^3 = 0,$$

furthermore

$$AC''(x) = \frac{12\,000}{x^3} + 6$$

thus

$$AC''(10) = \frac{12\,000}{1\,000} + 6 = 18 > 0$$

that is function $AC(x)$ has a minimum at $x = 10$. Thus the company has to produce 10 units cell phones.

Problem 1.5.6. A production facility is capable of producing 60 000 widgets in a day and the total daily cost of producing x widgets in a day is given by

$$C(x) = 250\,000 + 0.08x + \frac{200\,000\,000}{x}.$$

How many widgets per day should they produce in order to minimize production costs?

Solution:

Here we need to minimize the cost subject to the constraint that x must be in the range $0 \leq x \leq 60\,000$. The first derivative of the function $C(x)$ is

$$C'(x) = 0.08 - \frac{200\,000\,000}{x^2}.$$

Zeros of the function $C'(x)$ are the solutions of $C'(x) = 0$ equation

$$\begin{aligned} 0.08 - \frac{200\,000\,000}{x^2} &= 0 \\ 0.08x^2 &= 200\,000\,000. \end{aligned}$$

Since $x > 0$, we get that

$$x = \pm 50\,000.$$

Clearly the negative value does not make any sense in this setting and so we have a single critical point in the range of possible solutions 50 000. The second derivatives of the function C is

$$C''(x) = \frac{400\,000\,000}{x^3}.$$

As long as $x > 0$, the second derivative is

$$C''(50\,000) = \frac{400\,000\,000}{50\,000^3}$$

positive and thus, in the range of possible solutions the function is always concave up and so producing 50 000 widgets will yield the absolute minimum production cost.

Problem 1.5.7. An apartment complex has 250 apartments to rent. If they rent x apartments then their monthly profit, in dollars, is given by

$$P(x) = -8x^2 + 3\,200x - 80\,000.$$

How many apartments should they rent in order to maximize their profit?

Solution:

All that we are really being asked to do here is to maximize the profit subject to the constraint that x must be in the range $0 \leq x \leq 250$. The first derivative of P is

$$P'(x) = -16x + 3\,200.$$

The zero of function P' is

$$-16x + 3\,200 = 0 \quad \Rightarrow \quad x = 200.$$

Since the profit function is continuous and we have an interval with finite bounds we can find the maximum value by simply plugging in the only critical point that we have and the end points of the range:

$$P(0) = -8 \cdot 0^3 + 3\,200 \cdot 0 - 80\,000$$

$$P(200) = -8 \cdot 200^2 + 3\,200 \cdot 200 - 80\,000 = 240\,000$$

$$P(250) = -8 \cdot 250^2 + 3\,200 \cdot 250 - 80\,000 = 220\,000.$$

The profit function has maximum, if they only rent out 200 of the apartments instead of all 250 of them.

Problem 1.5.8. An airline company is operating a direct flight between two European cities. The flights are leaving if there are at least 10 passengers and no plane can carry more than 36 passengers. Looking for improving the utilization of its flights the company is considering the following option: in case of 20 or less passengers the price of the ticket is 16 000 HUF per passenger. If, however, there are more than 20 passengers then the price is calculated as follows: the difference between the actual number of passengers and 20 is multiplied by 400 and the originally 16 000 HUF fare is decreased by this product. Let $P(x)$ be the income of the company if there are x passengers accepted on a flight. Find a formula for function $P(x)$. Find also the domain of the function P . How many passengers shall the company carry on a flight in order to maximize its income on the given flight and how much is the maximal income?

Solution:

If $10 \leq x \leq 20$ and x is an integer number then

$$P(x) = 16\,000x.$$

If $20 < x \leq 36$ and x is an integer number, then

$$P(x) = 16\,000x - 400 \cdot (x - 20) \cdot x = -400x^2 + 24\,000x.$$

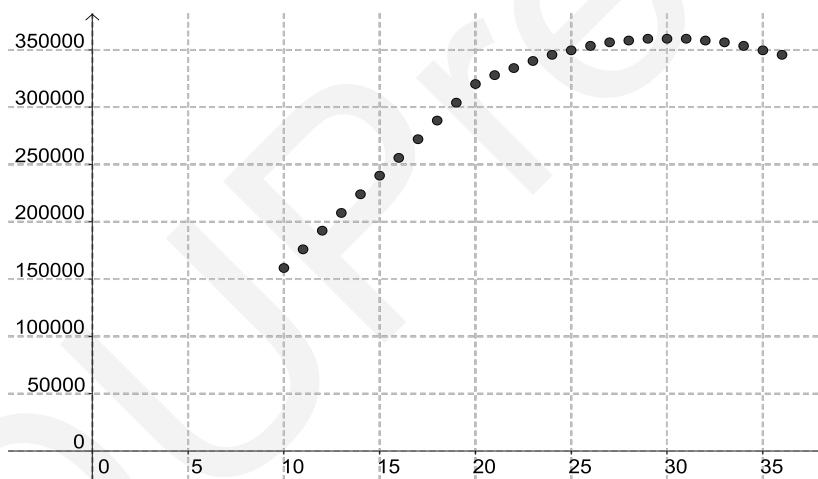
Let U and V be

$$U = \{x \in \mathbb{N} \mid 10 \leq x \leq 20\}$$

$$V = \{x \in \mathbb{N} \mid 21 \leq x \leq 36\}.$$

Function $P(x)$ is

$$P(x) = \begin{cases} 16\,000x, & \text{ha } x \in U \\ -400x^2 + 24\,000x, & \text{ha } x \in V \end{cases}.$$



Function $P(x)$ is not differentiable. Consider the extension of the domain of function $P(x)$ to interval, that is, and the next function

$$P^*(x) = \begin{cases} 16\,000x, & \text{if } 10 \leq x \leq 20 \\ -400x^2 + 24\,000x, & \text{if } 20 < x \leq 36. \end{cases}$$

Then function $P^*(x)$ is differentiable. On the interval $[10; 20]$ the maximum value of P^* is

$$16\,000 \cdot 20 = 320\,000.$$

The first derivative of function P^* on the interval $]10; 36[$ is

$$-800x + 24\,000,$$

whose zero is $x = 30$. The value of function P^* at $x = 30$

$$P(30) = -400 \cdot 30^3 + 24\,000 \cdot 30 = 360\,000.$$

Global maximum place is $x = 30$. Global maximum value is

$$P(30) = 360\,000.$$

The income will be maximal if there are 30 passengers. The maximal income is 360 000 HUF.

Problem 1.5.9. A cosmetics company manufactures a certain kind of cream in large quantities. The total monthly production is between 100 [kg] and 700 [kg]. By contract, it is all sold to a wholesale merchant in the same month. The contract also states that the merchant is to pay a price of

$$f(x) = 36 - 0.03x$$

euros per kilogram for the cream. The monthly production costs (expenses) also depend on the quantity of cream manufactured. The total monthly cost related to cream production, is given by the cost function

$$C(x) = 0.0001x^3 - 30.12x + 13\,000.$$

- Calculate the quantity of cream in kilograms for which the total monthly revenue (income) has a maximum.
- What is the maximum monthly income?
- What is the maximum monthly profit that the company can make by manufacturing cream?
- For what quantity of cream (in kilograms) will that profit be achieved?

Solution:

- a) The demand function is

$$f(x) = x \cdot (36 - 0.03x) = -0.03x^2 + 36x.$$

- b) First derivative of the demand function is

$$f'(x) = -0.06x + 36,$$

which has zero at $x = 600$. This is the only one solution of the derivative function, thus $x = 600$ the global maximum place. Then

$$f(600) = -0.03 \cdot 600^2 + 36 \cdot 600 = 10\,800 \text{ euro.}$$

c) Profit function is

$$\begin{aligned} P(x) &= -0.03x^2 + 36x - (0.0001x^3 - 30.12x + 13\,000) = \\ &= -0.0001x^3 - 0.03x^2 + 66.12x - 13\,000. \end{aligned}$$

d) Domain of profit function is $[100; 700]$.

First derivative of function P is

$$P'(x) = -0.0003x^2 - 0.06x + 66.12.$$

Zero of function $P(x)$ is the solution of the equation

$$-0.0003x^2 - 0.06x + 66.12 = 0.$$

This equation equivalent with equation

$$x^2 + 200x - 220.400 = 0.$$

Applying quadratic formula, we get that

$$x_{1,2} = \frac{-200 \pm \sqrt{200^2 - 4 \cdot (-220.400)}}{2} = \frac{-200 \pm 960}{2}.$$

That is $x_1 = -580$ and $x_2 = 380$. The values of P at $x = 0$, $x = 380$ and $x = 700$ are

$$N(0) = -13.000 \quad N(380) = 2306.4 \quad N(700) = -15.716,$$

thus maximum place is 380, maximum value is 2 306, 4. Greatest monthly profit is achieved by selling 380 kg of cream, and its value is 2306.4 euros.

Problem 1.5.10. The cost function is

$$f(x) = 40x + 8\,000$$

and the number of units is 200. How many percent change the cost, if we decrease the cost with 2 percent.

Solution:

The derivative of function $f(x)$ is

$$f'(x) = 40.$$

The elasticity function of function $f(x)$ is

$$E(x) = \frac{x}{f(x)} \cdot f'(x) = \frac{x}{40x + 8000} \cdot 40 = \frac{40x}{40x + 8000}.$$

The $E(200)$ value is

$$E(200) = \frac{40 \cdot 200}{40 \cdot 200 + 8000} = \frac{1}{2}.$$

Thus the cost decrease with 0.5 percent.

Problem 1.5.11. The demand function is

$$f(x) = x^2 \cdot e^{-x^2}.$$

Calculate the elasticity function at $x = 2$.

Solution:

The derivative of function $f(x)$ is

$$f'(x) = 2xe^{-x^2} + x^2 \cdot e^{-x^2} \cdot (-2x) = e^{-x^2} \cdot (2x - 2x^3).$$

The elasticity function is

$$\begin{aligned} E(x) &= \frac{x}{f(x)} \cdot f'(x) = \frac{x}{x^2 \cdot e^{-x^2}} \cdot e^{-x^2} \cdot (2x - 2x^3) = \\ &= \frac{2x - 2x^3}{x} = 2 - 2x^2. \end{aligned}$$

The elasticity function at $x = 2$ is

$$E(2) = 2 - 2 \cdot 2^2 = -6.$$

If we increase the price with one percent, the demand decreases with 6 percent.

Problem 1.5.12. A company has the demand function

$$f(x) = 20 - x.$$

The cost function is

$$C(q) = q^3 + 4q + 10.$$

The cost is $C(q)$ million forints and q is number of units in thousands.

The supply function is

$$S(x) = 4x.$$

- How much does it cost to produce one thousand products?
- How much does it cost to produce two thousand products?
- How much does it cost to produce three thousand products?
- Calculate the fix cost.
- Calculate the variable cost.

- f) Find the average cost.
- g) Calculate the average fix cost.
- h) Find the average variable cost.
- i) Sketch the graph of cost function.
- j) Find the inverse of the demand function.
- k) Find the revenue function.
- l) Find the profit function.
- m) Sketch the graph of the demand function.
- n) Sketch the graph of the revenue function.
- o) Find the quantity, when has maximum value of the revenue function.
- p) Find the maximum value of the revenue function.
- q) Calculate the marginal cost function.
- r) Calculate the marginal revenue function.
- s) Calculate the marginal profit function.
- t) Find the quantity, when the value of profit function is maximal.
- u) Find the maximum value of the profit function.
- v) Calculate the unit price, when the value of the profit function is maximal.
- w) Sketch the graph of the profit function.
- x) Sketch the graph of the demand function and supply function in the same coordinate-system.
- y) Calculate the equilibrium price.
- z) Calculate the equilibrium price.
- α) Calculate the elasticity of the demand function, when the value of the profit function is maximal.
- β) Sketch the graph of the elasticity function of the demand function.

Solution:

- a) Since

$$C(1) = 1^3 + 4 \cdot 1 + 10 = 15,$$

thus the cost is 15 000 000 HUF.

- b) Since

$$C(2) = 2^3 + 4 \cdot 2 + 10 = 26,$$

thus the cost is 26 000 000 HUF.

c) Since

$$C(3) = 3^3 + 4 \cdot 3 + 10 = 49,$$

thus the cost is 49 000 000 HUF.

d) The fix cost is

$$FC = C(0) = 0^3 + 4 \cdot 0 + 10 = 10,$$

that is 10 000 000 HUF.

e) The variable cost is

$$VC(q) = q^3 + 4q.$$

f) The average cost is

$$AC(q) = \frac{C(q)}{q} = \frac{q^3 + 4q + 10}{q} = q^2 + 4 + \frac{10}{q}.$$

g) The average fix cost is

$$AFC(q) = \frac{FC}{q} = \frac{10}{q}.$$

h) The average variable cost is

$$AVC(q) = \frac{VC(q)}{q} = q^2 + 4.$$

i) The derivative of function $C(q)$ is

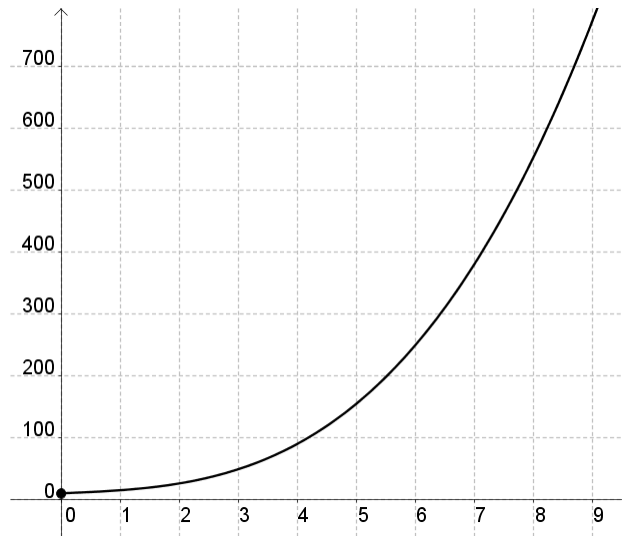
$$C'(q) = 3q^2 + 4 > 0,$$

thus function $C(q)$ is strictly monotonic increasing.

The second derivative of function $C(q)$ is

$$C''(q) = 6q.$$

The zero of the function above is $q = 0$, thus $C(q)$ at point $(0; C(0))$ has an inflexion point of the function $C(q)$. Function $C(q)$ is convex for all $x \geq 0$.



j) The inverse of the demand function is

$$f^{-1}(q) = 20 - q.$$

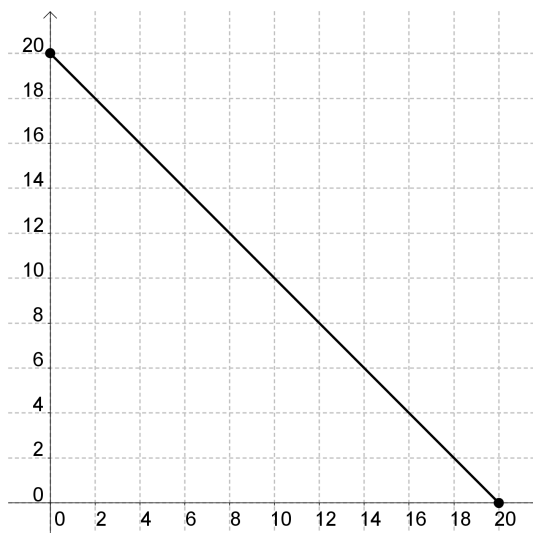
k) The revenue function is

$$R(q) = q \cdot (20 - q) = 20q - q^2.$$

l) The profit function is

$$\begin{aligned} \Pi(q) &= R(q) - C(q) = 20q - q^2 - (q^3 + 4q + 10) = \\ &= 20q - q^2 - q^3 - 4q - 10 = -q^3 - q^2 + 16q - 10. \end{aligned}$$

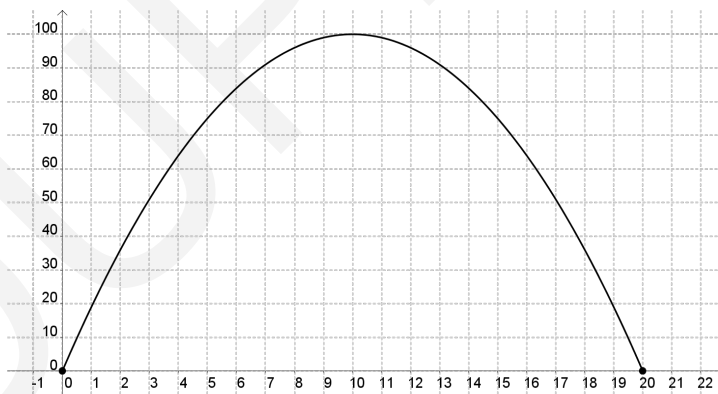
m) The graph of the demand function is



n) Since

$$\begin{aligned} R(q) &= 20q - q^2 = -(q^2 - 20q) = -((q - 10)^2 - 100) = \\ &= -(q - 10)^2 + 100, \end{aligned}$$

thus the graph of the revenue function is



o) The maximum of the revenue function has at $q = 10$.

p) Since

$$R(10) = 100,$$

thus the maximum value of the revenue function is 100 000 000 HUF.

q) The marginal cost function is

$$C'(q) = 3q^2 + 4.$$

r) The marginal revenue function is

$$R'(q) = -2q + 20.$$

s) The marginal profit function is

$$\Pi'(q) = -3q^2 - 2q + 16.$$

t) We have to solve equation $\Pi'(q) = 0$:

$$-3q^2 - 2q + 16 = 0.$$

Applying the quadratic formula, we get that

$$q_{1,2} = \frac{2 \pm \sqrt{4 + 192}}{-6} = \frac{2 \pm 14}{-6}.$$

thus $q_1 = -\frac{8}{3}$ or $q = 2$. According to the domain of the profit function, we get that $q = 2$.

Since

$$\Pi''(q) = -6q - 2$$

and

$$\Pi''(2) = -6 \cdot 2 - 2 = -14 < 0,$$

we get that, the maximum value of the profit function has at $q = 2$ 000.

u) The value of profit function at $q = 2$

$$\Pi(2) = -2^3 - 2^2 + 16 \cdot 2 - 10 = -8 - 4 + 32 - 10 = 12,$$

thus the maximal profit is 12 000 000 HUF.

v) If $q = 2$ then

$$f^{-1}(2) = 20 - 2 = 18,$$

thus the unit price is 18 000 HUF.

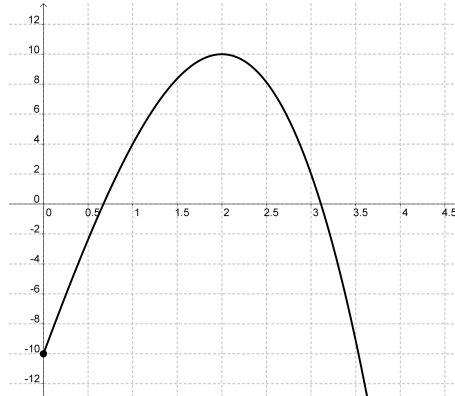
w) The zeros of the profit function are $q_1 = -\frac{8}{3}$; $q_2 = 2$. The sign of the derivative of the profit function are

q	$]0; 2[$	2	$]2; \infty[$
$\Pi'(q)$	+	0	-
$\Pi(q)$	\nearrow	loc. max.	\searrow
$\Pi(q)$		12	

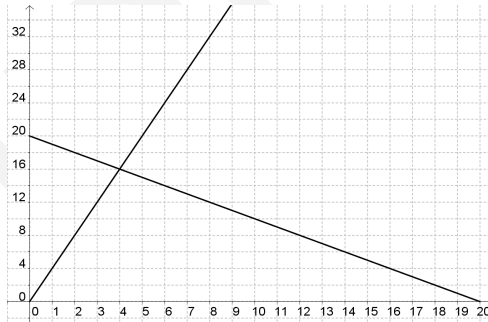
The second derivative of the profit function is

$$\Pi''(q) = -6q - 2 < 0$$

thus the profit function is a concave function. The graph of the profit function is



x) The graphs of the demand and supply functions are



y) The equilibrium price is the solution of the equation below

$$20 - x = 4x$$

thus $x = 4$, that is 4 000 HUF.

z) The equilibrium quantity is

$$S(4) = f(4) = 16,$$

that is 16 000 pieces.

α) The elasticity function is

$$E(x) = \frac{x}{f(x)} \cdot f'(x) = \frac{x}{20-x} \cdot (-1) = -\frac{x}{20-x}.$$

If $x = 18$ then

$$E(18) = -\frac{18}{20-18} = -9.$$

Since

$$|E(18)| = 9 > 1,$$

then the demand function is elastic. If the unit price increases by 1%, the demand decreases by 9%.

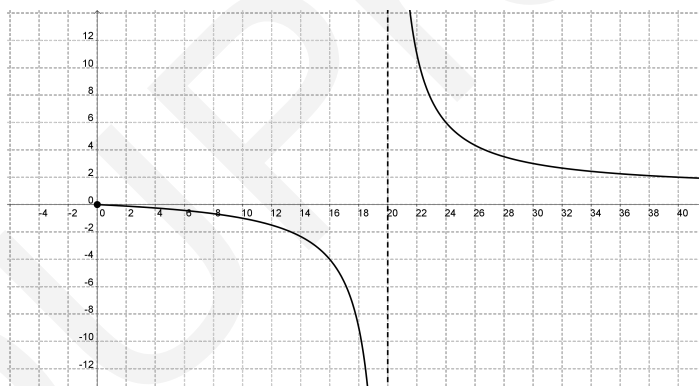
β) The elasticity function is

$$E(x) = \frac{x}{x-20}.$$

Applying algebraic transformations, we get that

$$E(x) = \frac{x-20+20}{x-20} = 1 + \frac{20}{x-20}.$$

The graph of the function is



Problem 1.5.13. The estimated future profits of a small business are given by

$$\Pi(t) = 2t^2 - 12t + 118$$

thousand dollars, where t is the time in years from now.

- What is the current annual profit?
- Find the function $\Pi'(t)$.
- What is the significance of $\Pi'(t)$?

- d) When will the profit increase?
- e) When will the profit decrease?
- f) What is the minimum profit and when does it occur?
- g) Find $\Pi'(t)$ at $t = 4$.

Solution:

- a) Since

$$\Pi(0) = 2 \cdot 0^2 - 12 \cdot 0 + 118 = 118,$$

the current annual profit is 118 000\$.

- b) The function $\Pi'(t)$ is

$$\Pi'(t) = 4t - 12.$$

- c) Function $\Pi'(t)$ is the rate of change in profit with time.

- d) We have to solve the inequality $\Pi'(t) \leq 0$, that is

$$4t - 12 \leq 0 \quad \Rightarrow \quad t \leq 3.$$

- e) We have to solve the inequality $\Pi'(t) \geq 0$, that is

$$4t - 12 \geq 0 \quad \Rightarrow \quad t \geq 3.$$

- f) We have to solve the equation $\Pi'(t) = 0$, that is, $t = 3$.

Since $\Pi''(t) = 4 > 0$, thus the profit function has a minimum at $t = 3$.

- g) The value of function $\Pi'(t)$ at $t = 4$ is

$$\Pi'(4) = 4 \cdot 4 - 12 = 4.$$

Chapter 2

Integral calculus of one variable functions

2.1. Application of indefinite integral for economic problems

Theoretical summary. If the marginal revenue function for a manufacturer's product is $MR(q)$ then by integrating this function and using an initial condition $R(0) = 0$, we can find the revenue function.

Revenue function is given by the general relationship

$$R(q) = p \cdot q,$$

where p is the price per unit.

If the marginal cost function for a manufacturer's product is $MC(q)$ then by integrating this function and using an initial condition $C(0) = FC$, we can find the cost function.

Applied mathematical knowledge. Primitive function, indefinite integral, definite integral, integral with initial conditions.

Sample problem. If the marginal revenue function for a manufacturer's product is

$$MR(q) = 2000 - 20q - 3q^2.$$

- Find the revenue function.
- Find the inverse of the demand function.
- Find the demand function.
- Sketch the graph of the demand function.

Solution:

- By integrating the marginal revenue function and using an initial condition $R(0) = 0$, we can find the revenue function. Since

$$\begin{aligned} R(q) &= \int 2000 - 20q - 3q^2 \, dq = \\ &= \int 2000 \, dq - \int 20q \, dq - \int 3q^2 \, dq = \\ &= 2000q - 10q^2 - q^3 + c \end{aligned}$$

and $R(0) = 0$, thus

$$2000 \cdot 0 - 10 \cdot 0^2 - 0^3 + c = 0,$$

we get that $c = 0$. Consequently, the revenue function is

$$R(q) = -q^3 - 10q^2 + 2000q.$$

b) The revenue function is

$$R(q) = p \cdot q,$$

where p is the unit price. If the demand function is $f(q)$ then

$$p = f^{-1}(q).$$

Thus

$$R(q) = f^{-1}(q) \cdot q,$$

consequently the inverse of the demand function is

$$f^{-1}(q) = \frac{R(q)}{q} = \frac{-q^3 - 10q^2 + 2000q}{q} = -q^2 - 10q + 2000.$$

c) Demand function is the inverse of the inverse demand function.

Since

$$f^{-1}(f(x)) = x,$$

we get that

$$-(f(x))^2 - 10 \cdot f(x) + 2000 = x.$$

We have to solve the equation to $f(x)$, that is,

$$-(f(x))^2 - 10 \cdot f(x) + 2000 - x = 0 \quad (x \geq 0).$$

Applying the quadratic formula, we get that

$$f(x)_{1,2} = \frac{10 \pm \sqrt{100 - 4 \cdot (-1) \cdot (2000 - x)}}{-2} = \frac{10 \pm \sqrt{8100 - 4x}}{-2}.$$

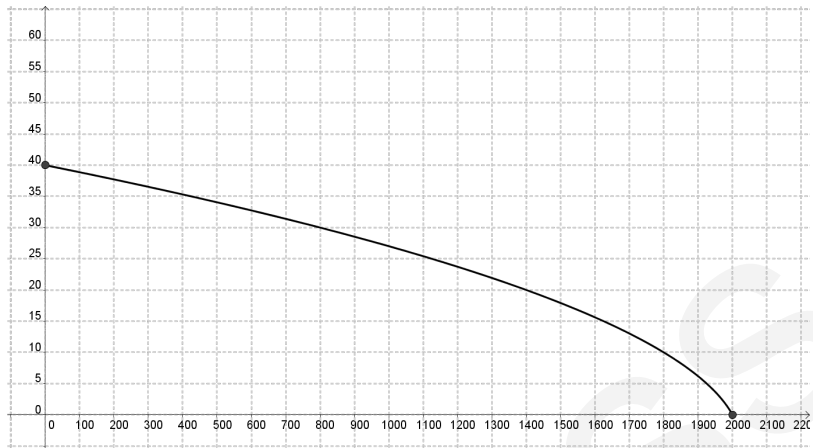
Applying algebraic transformations, we get that

$$f(x)_{1,2} = \frac{10 \pm 2 \cdot \sqrt{2025 - x}}{-2} = -5 \pm \sqrt{2025 - x}.$$

Since $x \geq 0$ and $f(x) \geq 0$, we get that

$$f(x) = -5 + \sqrt{2025 - x}.$$

d) The graph of the demand function is



Problems

Problem 2.1.1. In the manufacture of a product, fixed costs per week are 4 000 dollars. If the marginal cost function is

$$MC(q) = 0.000001 \cdot (0.002q^2 - 25q + 0.2)$$

in dollars of producing q kilogram of product per week. Find the cost of producing 10 000 kilogram in 1 week.

Solution:

Since $MC(q)$ is the derivative of the total cost, we get that

$$\begin{aligned} C(q) &= \int 0.000001 \cdot (0.002q^2 - 25q + 0.2) \, dq = \\ &= 0.000001 \cdot \int (0.002q^2 - 25q + 0.2) \, dq = \\ &= 0.000001 \cdot \left(\int 0.002q^2 \, dq - \int 25q \, dq + \int 0.2 \, dq \right) = \\ &= 0.000001 \cdot \left(0.002 \cdot \frac{q^3}{3} - 25 \cdot \frac{q^2}{2} + 0.2q \right) + c. \end{aligned}$$

The fixed costs are constant regardless of output. Therefore, when $q = 0$

$$C(0) = 4\,000,$$

thus

$$0.000001 \cdot \left(0.002 \cdot \frac{0^3}{3} - 25 \cdot \frac{0^2}{2} + 0.2 \cdot 0 \right) + c \quad \Rightarrow \quad c = 4\,000.$$

Thus the cost function is

$$C(q) = 0.000001 \cdot \left(0.002 \cdot \frac{q^3}{3} - 25 \cdot \frac{q^2}{2} + 0.2q \right) + 4\,000.$$

From the function above, we get that

$$\begin{aligned} C(10\,000) &= 0.000001 \cdot \left(0.002 \cdot \frac{10\,000^3}{3} - 25 \cdot \frac{10\,000^2}{2} + 0.2 \cdot 10\,000 \right) + \\ &+ 4\,000 = 5416.67 \text{ \$}. \end{aligned}$$

Problem 2.1.2. The sole producer of a product has determined that the marginal revenue function is

$$MR(q) = 100 - 3q^2$$

thousand HUF.

- Find the revenue function.
- Determine the inverse demand function.
- Find the demand function.
- Determine the elasticity of the demand function.
- If increases by 3% the 75 HUF unit price, how it changes the demand?

Solution:

- a) By integrating the marginal revenue function and using an initial condition $R(0) = 0$, we can find the revenue function. Since

$$\begin{aligned} R(q) &= \int 100 - 3q^2 \, dq = \int 100 \, dq - \int 3q^2 \, dq = \\ &= 100q - q^3 + c. \end{aligned}$$

Since $R(0) = 0$, we get that

$$-100 \cdot 0 - 0^3 + c = 0 \quad \Rightarrow \quad c = 0,$$

thus the revenue function is

$$R(q) = 100q - q^3.$$

- b) If $f^{-1}(q)$ is the inverse demand function, we get that

$$R(q) = q \cdot f^{-1}(q),$$

thus the inverse demand function is

$$f^{-1}(q) = \frac{R(q)}{q} = \frac{-q^3 + 100q}{q} = -q^2 + 100.$$

- c) The demand function is the inverse of the inverse demand function.

Since

$$f^{-1}(f(x)) = x,$$

thus

$$-(f(x))^2 + 100 = x.$$

Since $x \geq 0$ and $f(x) \geq 0$ we get that

$$f(x) = \sqrt{100 - x}.$$

d) The derivative of the demand function is

$$f'(x) = \frac{-1}{2 \cdot \sqrt{100-x}},$$

thus the elasticity function is

$$E(x) = \frac{x}{f(x)} \cdot f'(x) = \frac{x}{\sqrt{100-x}} \cdot \frac{-1}{2 \cdot \sqrt{100-x}}.$$

Applying algebraic transformations, we get that

$$E(x) = \frac{-x}{200-2x}.$$

e) If $x = 75$ then

$$E(75) = \frac{-75}{200-2 \cdot 75} = -1.5.$$

If the unit price is 75 HUF and increases by 3%, the demand decreases by $3 \cdot 1.5 = 4.5\%$.

Problem 2.1.3. A manufacturer has determined that the marginal cost function is

$$MC(q) = 0.003q^2 - 0.4q + 40,$$

where q is the number of units produced. The fixed costs are $FC = 5\,000$ dollars. Determine the average cost of producing $q = 100$ units.

Solution:

Since $MC(q)$ is the derivative of the total cost, we get that

$$\begin{aligned} C(q) &= \int (0.003q^2 - 0.4q + 40) dq = \\ &= \int 0.003q^2 dq - \int 0.4q dq + \int 40 dq = \\ &= 0.001q^3 - 0.2q^2 + 40q + c. \end{aligned}$$

Since $C(0) = 5\,000$, we get that

$$c = 5\,000.$$

Thus the cost function is

$$C(q) = 0.001q^3 - 0.2q^2 + 40q + 5\,000.$$

The average cost function is

$$AC(q) = \frac{C(q)}{q} = 0.001q^2 - 0.2q + 40 + \frac{5000}{q}.$$

If $q = 500$, the average cost is

$$AC(500) = 0.001 \cdot 500^2 - 0.2 \cdot 500 + 40 + \frac{5000}{500} = 80.$$

Problem 2.1.4. The marginal cost function for a manufacturer's product is given by

$$MC(q) = 10 - \frac{100}{q+10},$$

where C is the total cost in dollars if q units are produced. When 100 units are produced, the average cost is 50 dollar per unit, that is,

$$AC(100) = 50.$$

To the nearest dollar, determine the manufacturer's fixed cost.

Solution:

Since $MC(q)$ is the derivative of the total cost, we get that

$$\begin{aligned} C(q) &= \int \left(10 - \frac{100}{q+10} \right) dq = \\ &= \int 10 dq - \int \frac{100}{q+10} dq = \\ &= 10q - 100 \cdot \ln(q+10) + c. \end{aligned}$$

Thus, the average cost function is

$$AC(q) = \frac{C(q)}{q} = 10 - 100 \cdot \frac{\ln(q+10)}{q} + \frac{c}{q}.$$

Since $AC(100) = 50$, thus

$$50 = 10 - \frac{100 \cdot \ln 110}{100} + \frac{c}{100}.$$

Solving the above equation, we get that

$$c = 400 + 100 \cdot \ln 110 \approx 870.$$

Since

$$FC = C(0) = c,$$

the fixed cost is

$$FC = 870.$$

Problem 2.1.5. Suppose that the sole producer of a product has determined that the marginal revenue function is

$$MR(q) = 10 - \frac{1}{16}q$$

thousand HUF.

- Find the revenue function.
- Find the revenue at $q = 5$.
- Sketch the graph of the revenue function.

Solution:

- By integrating the marginal revenue function and using an initial condition $R(0) = 0$, we can find the revenue function. Since

$$\begin{aligned} R(q) &= \int 10 - \frac{1}{16}q \, dq = \int 10 \, dq - \int \frac{1}{16}q \, dq = \\ &= 10q - \frac{1}{32}q^2 + c. \end{aligned}$$

Since $R(0) = 0$, we get that

$$10 \cdot 0 - \frac{1}{32} \cdot 0^2 + c = 0 \quad \Rightarrow \quad c = 0,$$

thus the revenue function is

$$R(q) = 10q - \frac{1}{32}q^2.$$

- The revenue at $q = 5$ is

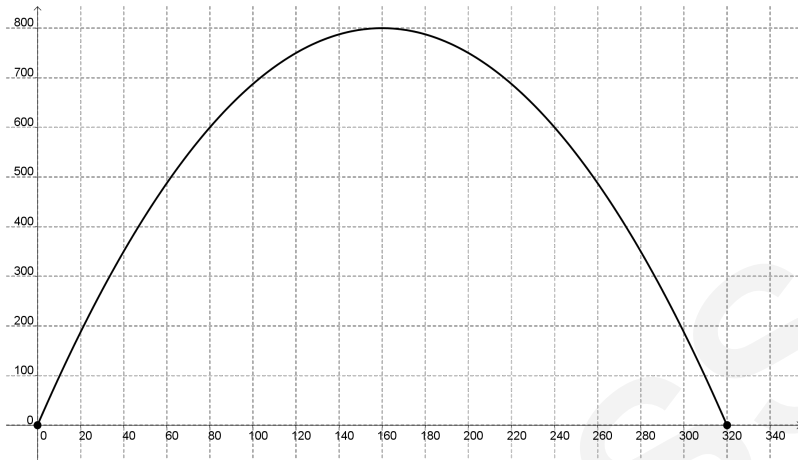
$$R(5) = 10 \cdot 5 - \frac{1}{32} \cdot 5^2 = 49.21875.$$

Thus the revenue is 49 218.75 HUF.

- Since

$$\begin{aligned} R(q) &= 10q - \frac{1}{32}q^2 = -\frac{1}{32} \cdot (q^2 - 320q) = \\ &= -\frac{1}{32} \cdot ((q - 160)^2 - 160^2) = -\frac{1}{32} \cdot (q - 160)^2 + 800. \end{aligned}$$

The graph of the revenue function is



Problem 2.1.6. Suppose that the marginal profit function of a company is

$$M\Pi(q) = -10 + \frac{450}{q}$$

and $\Pi(400) = 4\,000$.

- Find the profit function.
- Calculate the profit, if $q = 900$.

Solution:

- By integrating the marginal profit function and using a condition

$$\Pi(400) = 4\,000,$$

we can find the profit function. Since

$$\begin{aligned} \pi(q) &= \int -10 + \frac{450}{\sqrt{q}} \, dq = \int -10 \, dq + \int \frac{450}{\sqrt{q}} \, dq = \\ &= -10q + \int 450 \cdot q^{-\frac{1}{2}} \, dq = -10q + 450 \cdot \frac{q^{\frac{1}{2}}}{\frac{1}{2}} + c = \\ &= -10q + 900 \cdot \sqrt{q} + c. \end{aligned}$$

Since $\Pi(400) = 4\,000$, we get that

$$4\,000 = -4\,000 + 18\,000 + c \quad \Rightarrow \quad c = -10\,000,$$

thus the profit function is

$$\Pi(q) = -10q + 900 \cdot \sqrt{q} - 10\,000.$$

b) If $q = 900$ then

$$\Pi(500) = -10 \cdot 900 + 900 \cdot \sqrt{900} - 10\,000 = 8\,000.$$

Problem 2.1.7. If the marginal revenue function for a manufacturer's product is

$$MR(q) = 4\,000 - 40q - 6q^2.$$

- Find the revenue function.
- Find the inverse of the demand function.
- Find the demand function.

Solution:

- By integrating the marginal revenue function and using an initial condition $R(0) = 0$, we can find the revenue function. Since

$$\begin{aligned} R(q) &= \int 4\,000 - 40q - 6q^2 \, dq = \\ &= \int 4\,000 \, dq - \int 40q \, dq - \int 6q^2 \, dq = \\ &= 4\,000q - 20q^2 - 2q^3 + c \end{aligned}$$

and $R(0) = 0$, thus

$$4\,000 \cdot 0 - 20 \cdot 0^2 - 2 \cdot 0^3 + c = 0,$$

we get that $c = 0$. Consequently, the revenue function is

$$R(q) = -2q^3 - 20q^2 + 4\,000q.$$

- The revenue function is

$$R(q) = p \cdot q,$$

where p is the unit price. If the demand function is $f(q)$ then

$$p = f^{-1}(q).$$

Thus

$$R(q) = f^{-1}(q) \cdot q,$$

consequently the inverse of the demand function is

$$f^{-1}(q) = \frac{R(q)}{q} = \frac{-2q^3 - 20q^2 + 4\,000q}{q} = -2q^2 - 20q + 4\,000.$$

c) Demand function is the inverse of the inverse demand function.

Since

$$f^{-1}(f(x)) = x,$$

we get that

$$-2 \cdot (f(x))^2 - 20 \cdot f(x) + 4000 = x.$$

We have to solve the equation to $f(x)$, that is,

$$-2 \cdot (f(x))^2 - 20 \cdot f(x) + 4000 - x = 0 \quad (x \geq 0).$$

Applying the quadratic formula, we get that

$$f(x)_{1,2} = \frac{20 \pm \sqrt{400 - 4 \cdot (-2) \cdot (4000 - x)}}{-4} = \frac{10 \pm \sqrt{8000 - 4x}}{-2}.$$

Applying algebraic transformations, we get that

$$f(x)_{1,2} = \frac{10 \pm 2 \cdot \sqrt{2025 - x}}{-2} = -5 \pm \sqrt{2025 - x}.$$

Since $x \geq 0$ and $f(x) \geq 0$, we get that

$$f(x) = -5 + \sqrt{2025 - x}.$$

2.2. Area between the graphs of two functions

Theoretical summary. If $f(x)$ and $g(x)$ are continuous functions, such that $g(x) \geq f(x)$ in interval $[a; b]$, then the area between the graphs of the functions from a to b can be calculated as:

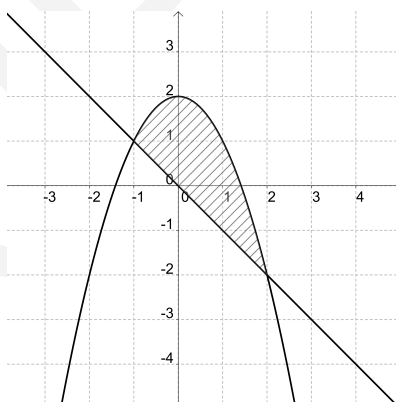
$$A = \int_a^b (g(x) - f(x)) dx.$$

Before applying the formula above it is practical the sketch the graphs of the functions. This way we can see that which one is the upper ($g(x)$) and lower ($f(x)$) one. It also helps you to find the limits of integration if the area is not already known. To be able to do that it may be necessary to find the intersection points of the graphs of the $f(x)$ and $g(x)$ functions, thus we have to solve the equation $f(x) = g(x)$. After that we can integrate the $g(x) - f(x)$ function between solutions x_1 and x_2 .

Sample problem. Find the area of the region enclosed by functions $f(x) = 2 - x^2$ and $g(x) = -x$.

Solution:

First, we sketch the graphs of the functions:



The limits of integration can be found by solving the equation

$$2 - x^2 = -x.$$

Applying algebraic transformation, we get that

$$x^2 - x - 2 = 0.$$

Solving the quadratic equation below, we get that

$$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}.$$

Thus the solutions of the equation above are $x_1 = -1$ and $x_2 = 2$.

Consequently, the region runs from $x = -1$ to $x = 2$, thus the limits of integration are -1 and 2 .

The area between the graphs of the functions is

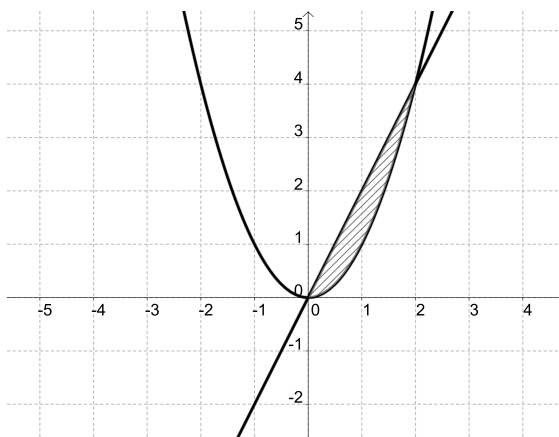
$$\begin{aligned} A &= \int_{-1}^2 2 - x^2 - (-x) \, dx = \left[2x - \frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^2 = \\ &= \left(4 - \frac{8}{3} + 2 \right) - \left(-2 + \frac{1}{3} + \frac{1}{2} \right) = \\ &= 6 - \frac{8}{3} + 2 - \frac{5}{6} = \frac{9}{2}. \end{aligned}$$

Problems

Problem 2.2.1. Find the area of the region enclosed by functions $f(x) = x^2$ and $g(x) = 2x$.

Solution:

First, sketch the graphs of the functions:



The limits of integration can be found by solving the equation $f(x) = g(x)$, that is,

$$x^2 = 2x$$

$$x^2 - 2x = 0$$

$$x \cdot (x - 2) = 0.$$

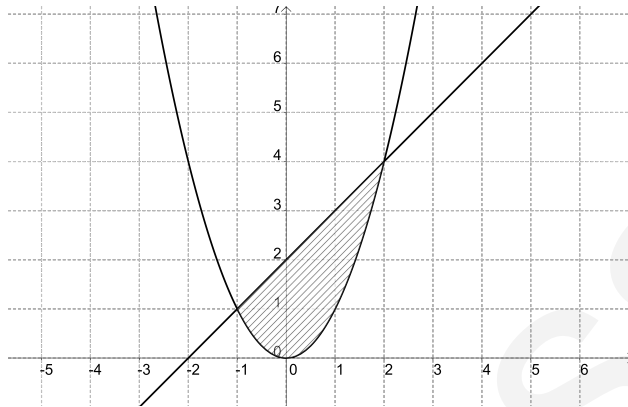
Thus solutions of the equation above are $x = 0$ and $x = 2$. The area between the graphs of the functions is

$$\begin{aligned} A &= \int_0^2 2x - x^2 \, dx = \left[x^2 - \frac{x^3}{3} \right]_0^2 = \\ &= 4 - \frac{8}{3} = \frac{12 - 8}{3} = \frac{4}{3}. \end{aligned}$$

Problem 2.2.2. Find the area of the region enclosed by functions $f(x) = x^2$ and $g(x) = x + 2$.

Solution:

First, we sketch the graphs of the functions



The limits of integration can be found by solving the equation $f(x) = g(x)$, that is,

$$x^2 = x + 2.$$

Transform the equation into the following form, we get that

$$x^2 - x - 2 = 0.$$

Applying the quadratic formula, we get that

$$x_{1,2} = \frac{1 \pm \sqrt{1 + 8}}{2} = \frac{1 \pm 3}{2},$$

thus $x_1 = -1$ or $x_2 = 2$.

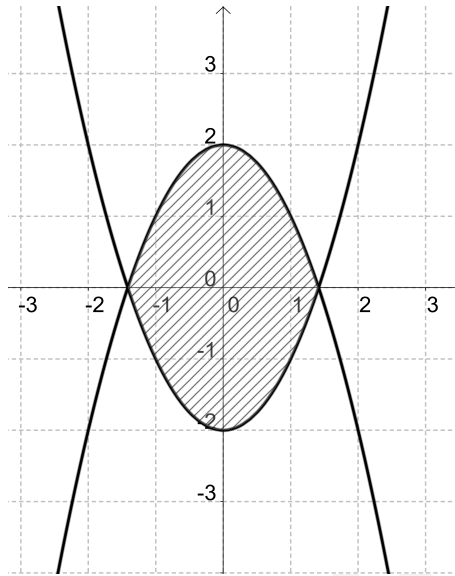
The area is

$$\begin{aligned} A &= \int_{-1}^2 x + 2 - x^2 \, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 = \\ &= \left(\frac{2^2}{2} + 2 \cdot 2 - \frac{2^3}{3} \right) - \left(\frac{(-1)^2}{2} + 2 \cdot (-1) - \frac{(-1)^3}{3} \right) = 4.5. \end{aligned}$$

Problem 2.2.3. Find the area of the region enclosed by functions $f(x) = x^2 - 2$ and $g(x) = 2 - x^2$.

Solution:

First, we sketch the graphs of the functions:



The limits of integration can be found by solving the equation $f(x) = g(x)$, that is,

$$\begin{aligned}x^2 - 2 &= 2 - x^2 \\x^2 &= 4,\end{aligned}$$

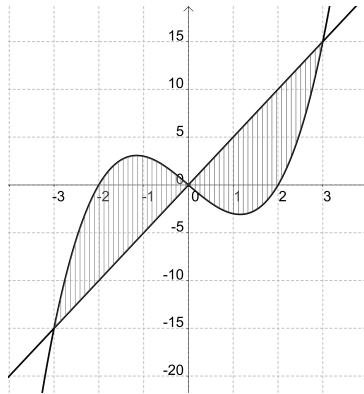
thus $x = -2$ or $x = 2$. The area is

$$\begin{aligned}A &= \int_{-2}^2 (2 - x^2) - (x^2 - 2) \, dx = \int_{-2}^2 (4 - 2x^2) \, dx = \\&= \left[4x - \frac{2x^3}{3} \right]_{-2}^2 = \\&= \left(4 \cdot 2 - 2 \cdot \frac{2^3}{3} \right) - \left(4 \cdot (-2) - \frac{(-2)^3}{3} \right) = \\&= \frac{8}{3} - \left(-\frac{8}{3} \right) = \frac{16}{3}.\end{aligned}$$

Problem 2.2.4. Find the area of the region enclosed by functions $f(x) = x^3 - 4x$ and $g(x) = 5x$.

Solution:

First, we sketch the graphs of the functions:



The limits of integration can be found by solving the equation $f(x) = g(x)$, that is,

$$x^3 - 4x - 5x = 0.$$

Factorizing the equation, we get that

$$x \cdot (x^2 - 9) = 0 \quad \Rightarrow \quad x \cdot (x - 3) \cdot (x + 3) = 0.$$

Solutions of the equation are $x_1 = -3$; $x_2 = 0$; $x_3 = 3$.

Since

$$\begin{aligned} A_1 &= \int_0^3 (5x - (x^3 - 4x)) dx = \int_0^3 (9x - x^3) dx = \left[9 \cdot \frac{x^2}{2} - \frac{x^4}{4} \right]_0^3 = \\ &= \frac{81}{2} - \frac{81}{4} = \frac{81}{4}, \end{aligned}$$

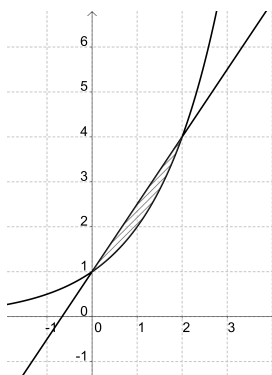
thus the area is

$$A = 2 \cdot T_1 = 2 \cdot \frac{81}{4} = \frac{81}{2} = 40.5.$$

Problem 2.2.5. Find the area of the region enclosed by functions $f(x) = 2^x$ and $g(x) = \frac{3}{2}x + 1$.

Solution:

First, we sketch the graphs of the functions:



The limits of integration can be found by solving the equation $f(x) = g(x)$, that is,

$$2^x = \frac{3}{2}x + 1.$$

The solutions of the equation are $x_1 = 0$ and $x_2 = 2$.

The area is

$$\begin{aligned} T &= \int_0^2 \left(\frac{3}{2}x + 1 - 2^x \right) dx = \left[\frac{3}{2} \cdot \frac{x^2}{2} + x - \frac{2^x}{\ln 2} \right]_0^2 = \\ &= \left(3 + 2 - \frac{4}{\ln 2} \right) + \frac{1}{\ln 2} = 5 - \frac{3}{\ln 2} \approx 0.6719. \end{aligned}$$

2.3. Volume of solids of revolution

Theoretical summary. The geometric body generated by rotating a plane region about an axis in its plane is called as a *solid of revolution*. If the graph of continuous function $f(x)$ is rotated between a and b , then the volume of the geometric body is

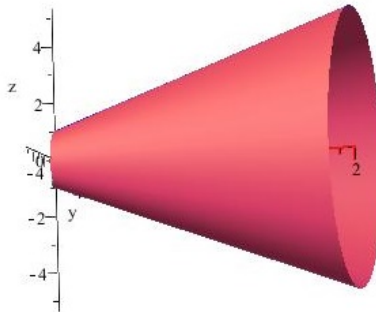
$$V = \pi \cdot \int_a^b (f(x))^2 dx.$$

Applied mathematical knowledge. Riemann integral, primitive functions, integration by parts.

Sample problem. The region between the graph of function $f(x) = 2x + 1$ ($x \in [0; 2]$) and the x -axis is rotated about the x -axis to generate a geometric body. Find its volume.

Solution:

Rotating the graph of function $f(x)$ about axis x , we get the following geometric body:



The volume of the body is

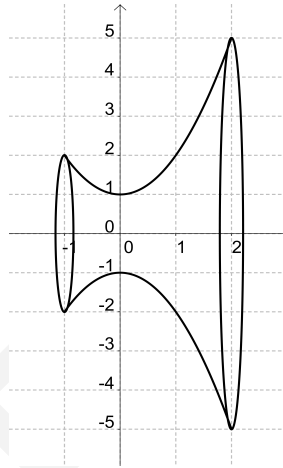
$$\begin{aligned} V &= \pi \cdot \int_0^2 (2x + 1)^2 dx = \pi \cdot \left[\frac{(2x + 1)^3}{6} \right]_0^2 = \\ &= \pi \cdot \left(\frac{(2 \cdot 2 + 1)^3}{6} \right) - \pi \cdot \left(\frac{(2 \cdot 0 + 1)^3}{6} \right) = \\ &= \pi \cdot \frac{124}{6} = \frac{62}{3} \cdot \pi. \end{aligned}$$

Problems

Problem 2.3.1. The region between the graph of function $f(x) = x^2 + 1$ ($x \in [-1; 2]$) and the x -axis is rotated about the x -axis to generate a geometric body. Find its volume.

Solution:

Rotating the graph of function $f(x)$ about axis x , we get the following geometric body:



Since

$$(f(x))^2 = (x^2 + 1)^2 = x^4 + 2x^2 + 1,$$

thus

$$\begin{aligned} \int_{-1}^2 (x^2 + 1)^2 dx &= \int_{-1}^2 x^4 + 2x^2 + 1 dx = \\ &= \left[\frac{x^5}{5} + 2 \cdot \frac{x^3}{3} + x \right]_{-1}^2 = \\ &= \left(\frac{2^5}{5} + 2 \cdot \frac{2^3}{3} + 2 \right) - \left(\frac{(-1)^5}{5} + 2 \cdot \frac{(-1)^3}{3} + (-1) \right) = \\ &= \left(\frac{32}{5} + \frac{16}{3} + 2 \right) - \left(-\frac{1}{5} - \frac{2}{3} - 1 \right) = \\ &= \frac{33}{5} + \frac{18}{3} + 3 = \frac{78}{5}. \end{aligned}$$

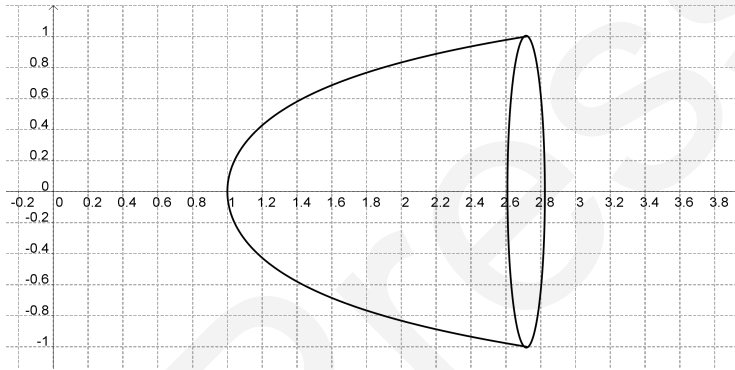
Consequently, the volume of solid of revolution is

$$V = \frac{78}{5} \cdot \pi \approx 49.$$

Problem 2.3.2. The region between the graph of function $f(x) = \sqrt{\ln x}$ ($x \in [1; e]$) and the x-axis is rotated about the x-axis to generate a geometric body. Find its volume.

Solution:

The geometric body is shown in the figure:



Since

$$(f(x))^2 = (\sqrt{\ln x})^2 = \ln x,$$

thus integration by parts, we get that

$$\begin{aligned} \int_1^e \ln x \, dx &= [x \cdot \ln x]_1^e - \int_1^e x \cdot \frac{1}{x} \, dx = \\ &= [x \cdot \ln x - x]_1^e = \\ &= e \cdot \ln e - e - (1 \cdot \ln 1 - 1) = 1. \end{aligned}$$

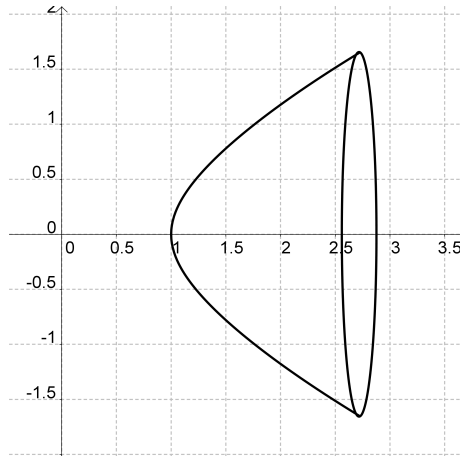
Consequently, the volume of solid of revolution is

$$V = \pi \approx 3.14.$$

Problem 2.3.3. The region between the graph of function $f(x) = \sqrt{x \cdot \ln x}$ ($x \in [1; e]$) and the x-axis is rotated about the x-axis to generate a geometric body. Find its volume.

Solution:

The geometric body is shown in the figure:



Since

$$(f(x))^2 = (\sqrt{x \cdot \ln x})^2 = x \cdot \ln x,$$

thus integration by parts, we get that

$$\begin{aligned} \int_1^e x \cdot \ln x \, dx &= \left[\frac{x^2}{2} \cdot \ln x \right]_1^e - \int_1^e \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \\ &= \left[\frac{x^2}{2} \cdot \ln x - \frac{x^2}{4} \right]_1^e = \\ &= \frac{e^2}{2} \cdot \ln e - \frac{e^2}{4} - \left(\frac{1}{2} \cdot \ln 1 - \frac{1}{4} \right) = \\ &= \frac{e^2 + 1}{4}. \end{aligned}$$

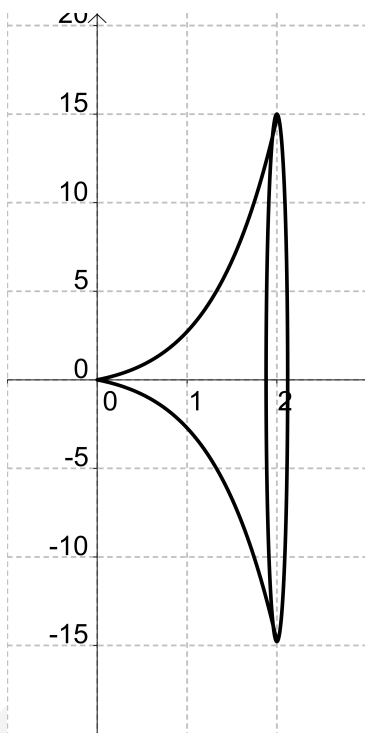
Consequently, the volume of the solid of revolution is

$$V = \pi \cdot \frac{e^2 + 1}{4} \approx 6.58.$$

Problem 2.3.4. The region between the graph of function $f(x) = x \cdot e^x$ ($x \in [0; 2]$) and the x -axis is rotated about the x -axis to generate a geometric body. Find its volume.

Solution:

The geometric body is shown in the figure:



Since

$$(f(x))^2 = (x \cdot e^x)^2 = x^2 \cdot e^{2x},$$

thus integration by parts, we get that

$$\begin{aligned} \int_0^2 x^2 \cdot e^{2x} dx &= \left[\frac{x^2 \cdot e^{2x}}{2} \right]_0^2 - \int_0^2 x \cdot e^{2x} dx = \\ &= \left[\frac{x^2 \cdot e^{2x}}{2} \right]_0^2 - \left(\frac{x \cdot e^{2x}}{2} - \int_0^2 \frac{e^{2x}}{2} dx \right) = \\ &= \left[\frac{x^2 \cdot e^{2x}}{2} \right]_0^2 - \left[\frac{x \cdot e^{2x}}{2} - \frac{e^{2x}}{4} \right]_0^2 = \\ &= \left[\frac{(2x^2 - 2x + 1) \cdot e^{2x}}{4} \right]_0^2 = \\ &= \frac{(8 - 4 + 1) \cdot e^4 - 1}{4} = \frac{5e^4 - 1}{4}. \end{aligned}$$

Consequently, the volume of the solid of revolution is

$$V = \pi \cdot \frac{5e^4 - 1}{4} \approx 213.62.$$

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2.4. Surface area of revolutions

Theoretical summary. A surface of revolution is formed when a curve is rotated about a line.

If function $f(x)$ is positive and has a continuous derivative, we define the *surface area* of the surface obtained by rotating the graph of function $f(x)$, $x \in [a; b]$ about the x -axis as

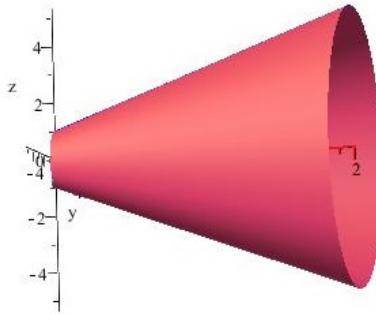
$$S = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + (f'(x))^2} dx.$$

Applied mathematical knowledge. Riemann integral, primitive functions, integral by parts, integral by substitution.

Sample problem. The region between the graph of function $f(x) = 2x + 1$ ($x \in [0; 2]$) and the x -axis is rotated about the x -axis to generate a geometric body. Find its surface area.

Solution:

The geometric body is shown in the figure below:



The surface area of revolution is

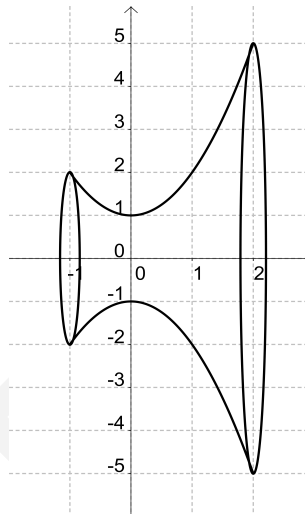
$$\begin{aligned} S &= 2\pi \cdot \int_0^2 (2x + 1) \cdot \sqrt{5} dx = 2\pi \cdot \sqrt{5} \left[\frac{(2x + 1)^2}{2 \cdot 2} \right]_0^2 = \\ &= 2\pi \cdot \left(\frac{(2 \cdot 2 + 1)^2}{4} \right) - 2\pi \cdot \left(\frac{(2 \cdot 0 + 1)^2}{4} \right) = \\ &= 2\pi \cdot 6 \cdot \sqrt{5} = 12\pi \cdot \sqrt{5} \approx 84.3. \end{aligned}$$

Problems

Problem 2.4.1. The region between the graph of function $f(x) = x^2 + 1$ ($x \in [-1; 2]$) and the x -axis is rotated about the x -axis to generate a geometric body. Find its surface area.

Solution:

The geometric body is shown in the figure:



The derivative of function $f(x)$ is

$$f'(x) = 2x.$$

We have to calculate the definite integral

$$2\pi \cdot \int_{-1}^2 (x^2 + 1) \cdot \sqrt{1 + 4x^2} dx.$$

First, we determine a primitive function of

$$(x^2 + 1) \cdot \sqrt{1 + 4x^2}.$$

Using the substitution $2x = \sinh t$, we get that

$$x = \frac{1}{2} \cdot \sinh t$$

and

$$\frac{dx}{dt} = \frac{1}{2} \cdot \cosh t.$$

After substituting, we get that

$$\begin{aligned} I &= \int (x^2 + 1) \cdot \sqrt{1 + 4x^2} dx = \\ &= \int \left(\frac{1}{4} \cdot \sinh^2 t + 1 \right) \cdot \sqrt{1 + \sinh^2 t} \cdot \frac{1}{2} \cdot \cosh t dt. \end{aligned}$$

Applying hyperbolic identity

$$\cosh^2 t - \sinh^2 t = 1 \quad \Rightarrow \quad \cosh^2 t = 1 + \sinh^2 t,$$

we get that

$$I = \int \left(\frac{1}{4} \cdot \sinh^2 t + 1 \right) \cdot \frac{1}{2} \cdot \cosh^2 t dt.$$

Expanding and applying the elementary properties of primitive function, we get that

$$I = \frac{1}{8} \cdot \int \sinh^2 t \cdot \cosh^2 t dt + \frac{1}{2} \cdot \int \cosh^2 t dt.$$

Applying the hyperbolic identity

$$\sinh 2t = 2 \cdot \sinh t \cdot \cosh t \quad \Rightarrow \quad \sinh^2 2t = 4 \cdot \sinh^2 t \cdot \cosh^2 t,$$

we get that

$$\sinh^2 t \cdot \cosh^2 t = \frac{\sinh^2 2t}{4},$$

thus

$$I = \frac{1}{32} \cdot \int \sinh^2 2t dt + \frac{1}{4} \cdot \int t + \frac{\sinh 2t}{2} dt.$$

Since

$$\cosh^2 2t - \sinh^2 2t = 1$$

$$\cosh^2 2t + \sinh^2 2t = \cosh 4t$$

then

$$\sinh^2 2t = \frac{\cosh 4t - 1}{2},$$

thus

$$\begin{aligned} I &= \frac{1}{32} \cdot \int \frac{\cosh 4t - 1}{2} dt + \frac{1}{4} \cdot \int t + \frac{\sinh 2t}{2} dt = \\ &= \frac{1}{64} \cdot \left(\frac{\sinh 4t}{4} - t \right) + \frac{1}{4} \cdot \left(t + \frac{\sinh 2t}{2} \right) + c = \\ &= \frac{1}{256} \cdot \sinh 4t + \frac{13}{64} t + \frac{1}{8} \sinh 2t + c, \end{aligned}$$

where $c \in \mathbb{R}$.

Since

$$t = \operatorname{arsh} 2x$$

then

$$I = \frac{1}{256} \cdot \sinh (4 \cdot (\operatorname{arsh} 2x)) + \frac{13}{64} \cdot \operatorname{arsh} 2x + \frac{1}{8} \cdot \sinh (2 \cdot \operatorname{arsh} 2x) + c.$$

Thus

$$\begin{aligned} &\int_{-1}^2 (x^2 + 1) \cdot \sqrt{1 + 4x^2} dx = \\ &= \left[\frac{1}{256} \cdot \sinh (4 \cdot (\operatorname{arsh} 2x)) + \frac{13}{64} \cdot \operatorname{arsh} 2x + \frac{1}{8} \cdot \sinh (2 \cdot \operatorname{arsh} 2x) \right]_{-1}^2 = \\ &= \left(\frac{\sinh (4 \cdot (\operatorname{arsh} 4))}{256} + \frac{13}{64} \cdot \operatorname{arsh} 4 + \frac{\sinh (2 \cdot \operatorname{arsh} 4)}{8} \right) - \\ &- \left(\frac{\sinh (4 \cdot (\operatorname{arsh} (-2)))}{256} + \frac{13}{64} \cdot \operatorname{arsh} (-2) + \frac{\sinh (2 \cdot \operatorname{arsh} (-2))}{8} \right) \approx \\ &\approx 15.2. \end{aligned}$$

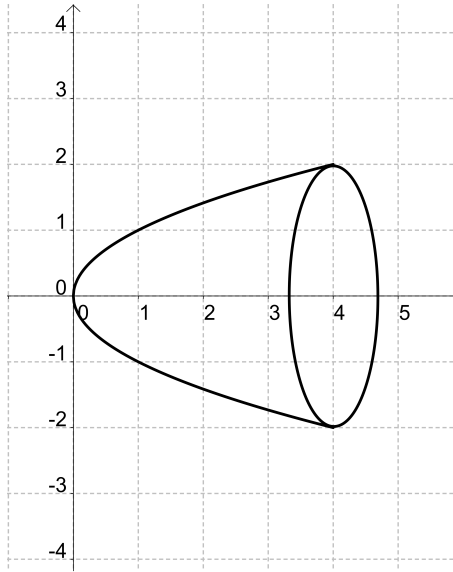
Surface area of revolution is

$$S = 2\pi \cdot 15.2 \approx 95.5.$$

Problem 2.4.2. The region between the graph of function $f(x) = \sqrt{x}$ ($x \in [0; 4]$) and the x -axis is rotated about the x -axis to generate a geometric body. Find its surface area.

Solution:

The geometric body is shown in the figure:



The derivative of function $f(x)$ is

$$f'(x) = \frac{1}{2 \cdot \sqrt{x}}.$$

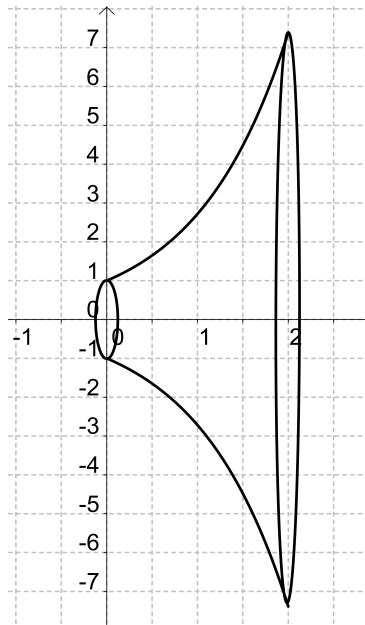
Surface area of revolution is

$$\begin{aligned} S &= 2\pi \cdot \int_0^4 \sqrt{x} \cdot \sqrt{1 + \frac{1}{4x}} dx = 2\pi \cdot \int_0^4 \sqrt{x + \frac{1}{4}} dx = \left[\left(x + \frac{1}{4}\right)^{\frac{3}{2}} \cdot \frac{2}{3} \right]_0^4 = \\ &= \left(\frac{17}{4}\right)^{\frac{3}{2}} \cdot \frac{2}{3} - \left(\frac{1}{4}\right)^{\frac{3}{2}} \cdot \frac{2}{3} = \frac{17 \cdot \sqrt{17}}{8} \cdot \frac{2}{3} - \frac{1}{8} \cdot \frac{2}{3} = \\ &= \frac{17 \cdot \sqrt{17} - 1}{6} \cdot \pi \approx 36.18. \end{aligned}$$

Problem 2.4.3. The region between the graph of function $f(x) = e^x$ ($x \in [0; 2]$) and the x -axis is rotated about the x -axis to generate a geometric body. Find its surface area.

Solution:

The geometric body is shown in the figure below:



The derivative of function $f(x)$ is

$$f'(x) = e^x.$$

Surface area of revolution is

$$S = 2\pi \cdot \int_0^2 f(x) \cdot \sqrt{1 + (f'(x))^2} dx.$$

That is,

$$S = 2\pi \cdot \int_0^2 e^x \cdot \sqrt{1 + e^{2x}} dx.$$

First, we have to find a primitive function of function

$$e^x \cdot \sqrt{1 + e^{2x}}.$$

Let

$$I = \int e^x \cdot \sqrt{1 + e^{2x}} dx.$$

Substituting $e^x = t$, we get that

$$x = \ln t \quad \Rightarrow \quad \frac{dx}{dt} = \frac{1}{t},$$

thus

$$I = \int t \cdot \sqrt{1+t^2} \cdot \frac{1}{t} dt = \int \sqrt{1+t^2} dt.$$

Substituting $t = \sinh y$, we get that

$$\frac{dt}{dy} = \cosh y$$

and

$$y = \operatorname{arsinh} t$$

thus

$$I = \int \sqrt{1 + \sinh^2 y} \cdot \cosh y dy = \int \cosh^2 y.$$

Applying hyperbolic identities

$$\cosh^2 y - \sinh^2 y = 1$$

$$\cosh^2 y + \sinh^2 y = \cosh 2y,$$

we get that

$$\cosh^2 y = \frac{1 + \cosh 2y}{2}.$$

Thus

$$I = \frac{1}{2} \cdot \left(y + \frac{\sinh 2y}{2} \right).$$

Since

$$\begin{aligned} \sinh 2y &= 2 \cdot \sinh y \cdot \cosh y = 2 \cdot \sinh(\operatorname{arsh} t) \cdot \cosh(\operatorname{arsh} t) = \\ &= 2 \cdot t \cdot \sqrt{1 + \cosh^2(\operatorname{arcosh} t)} = 2 \cdot t \cdot \sqrt{1 + t^2}, \end{aligned}$$

then

$$I = \frac{1}{2} \cdot \left(\operatorname{arsh} t + t \cdot \sqrt{1 + t^2} \right).$$

Using that $t = e^x$, it follows that

$$I = \frac{\operatorname{arsh} e^x}{2} + \frac{e^x \cdot \sqrt{1 + e^{2x}}}{2},$$

thus

$$S = 2\pi \cdot \left[\frac{\operatorname{arsh} e^x}{2} + \frac{e^x \cdot \sqrt{1 + e^{2x}}}{2} \right]_0^2,$$

that is,

$$S = \pi \cdot \left(\operatorname{arsh} e^2 + e^2 \cdot \sqrt{1 + e^4} - \operatorname{arsh} 1 - \sqrt{2} \right) \approx 174.35.$$

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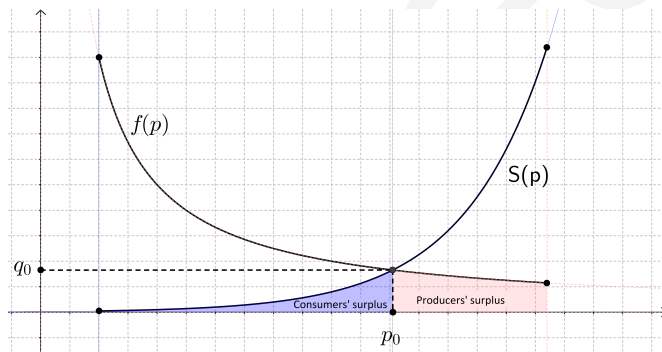
2.5. Application of Riemann integral for economic problems

Theoretical summary. If $f(p)$ is a demand function, $S(p)$ is a supply function and $(p_0; q_0)$ is the equilibrium point then *producers' surplus* is

$$\int_{p_{min}}^{p_0} f(p) dp,$$

consumers' surplus is

$$\int_{p_0}^{p_{max}} S(p) dp.$$

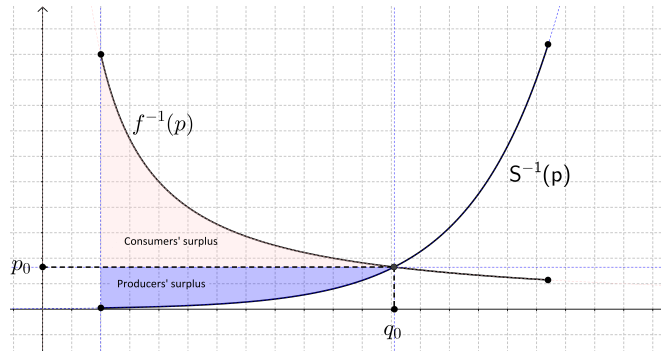


If $f^{-1}(p)$ is an inverse demand function and $S^{-1}(p)$ is an inverse supply function, and $(p_0; q_0)$ is the equilibrium point, then producers' surplus is

$$p_0 \cdot q_0 - \int_{q_{min}}^{q_0} f^{-1}(q) dq,$$

the consumers' surplus is

$$\int_{q_{min}}^{q_0} S^{-1}(q) dq - p_0 \cdot q_0.$$



If $MC(q)$ is a marginal cost function and $q_1 < q_2$ then the cost is increasing production from q_1 to q_2 units is

$$\int_{q_1}^{q_2} MC(q) \, dq.$$

Applied mathematical knowledge. Newton-Leibniz theorem, primitive functions.

Sample problem. The demand function for a product is

$$f(p) = \frac{90}{p} - 2 \quad (1 \leq p \leq 45)$$

the supply function is

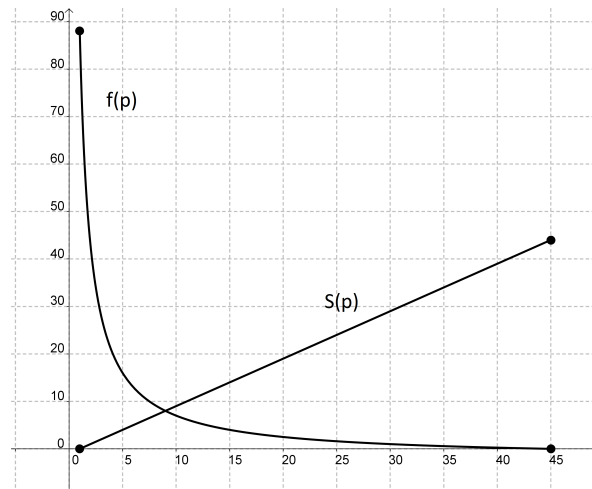
$$S(p) = p - 1 \quad (1 \leq p \leq 45).$$

Variable p is given in dollars.

- Sketch the graphs of the demand and supply functions.
- Calculate the equilibrium price and quantity.
- Mark the equilibrium point in the coordinate system that you have used in question a).
- Calculate consumers' surplus.
- Calculate producers' surplus.

Solution:

- The graphs of the demand and supply functions are



b) The equilibrium price is the solution of the equation $f(p) = S(p)$, that is,

$$\frac{90}{p} - 2 = p - 1.$$

Applying algebraic transformations, we get that

$$p^2 + p - 90 = 0.$$

Applying the quadratic formula, we have

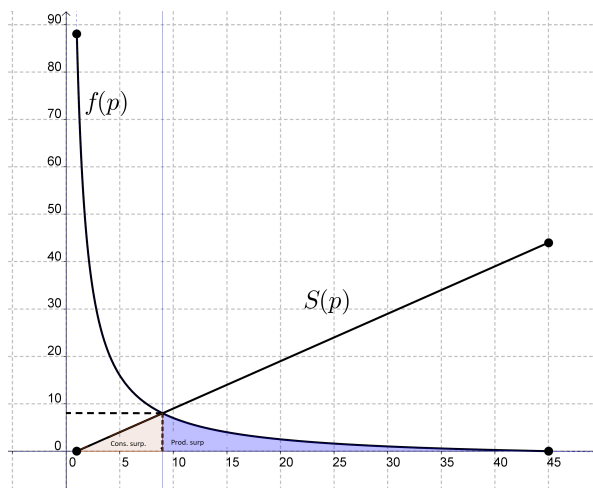
$$p_{1,2} = \frac{-1 \pm \sqrt{1 + 360}}{2} = \frac{-1 \pm 19}{2}.$$

Since $p > 0$, we get that $p = 9$.

The equilibrium quantity is

$$f(9) = \frac{90}{9} - 2 = 8 \text{ units.}$$

c) The figure below shows the equilibrium point, consumers' surplus and producers' surplus:



d) Since

$$\int_1^9 p - 1 \, dp = \left[\frac{1}{2} \cdot p^2 - p \right]_1^9 =$$

$$= \frac{81}{2} - 9 - \left(\frac{1}{2} - 1 \right) = 32,$$

the consumers' surplus is 32\$.

e) Since

$$\int_9^{45} \frac{90}{p} - 2 \, dp = [90 \ln p - 2p]_9^{45} =$$

$$= 90 \ln 45 - 90 - (90 \ln 9 - 18) \approx 72.85,$$

producers' surplus is 72.85\$.

Problems

Problem 2.5.1. Suppose the demand function for a product is

$$f(p) = 2\,000 - 20p \quad (10 \leq p \leq 100),$$

the supply function is

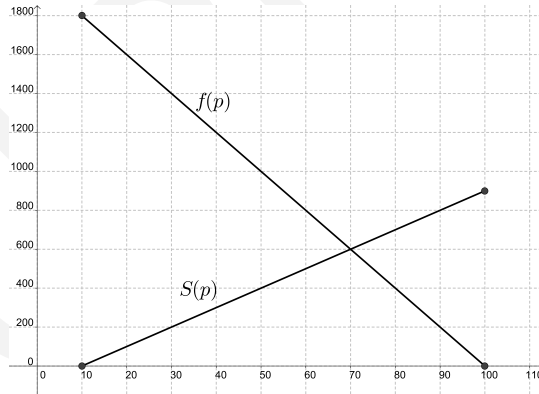
$$S(p) = 10p - 100 \quad (10 \leq p \leq 100).$$

Variable p is given in dollars.

- Sketch the graphs of the demand and supply functions.
- Calculate the equilibrium price and quantity.
- Mark the equilibrium point in the previous coordinate system.
- Calculate consumers' surplus.
- Calculate producers' surplus.

Solution:

- The graphs of the demand and supply functions:



- The equilibrium price is the solution of the equation $f(p) = S(p)$.

Since

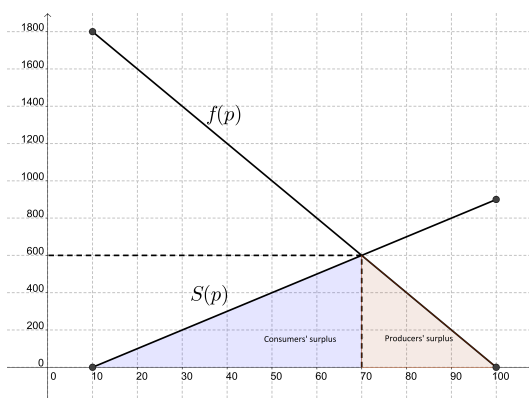
$$10p - 100 = 2\,000 - 20p \quad \Rightarrow \quad p = 70,$$

then the equilibrium price is 70\$.

The equilibrium quantity is

$$f(70) = 10 \cdot 70 - 100 = 600 \text{ units.}$$

- c) The figure shows equilibrium point, consumers' surplus and producers' surplus:



- d) Since

$$\int_{10}^{70} 10p - 100 \, dp = [5p^2 - 100p]_{10}^{70} =$$

$$= 5 \cdot 70^2 - 100 \cdot 70 - (5 \cdot 10^2 - 100 \cdot 10) = 18\,000,$$

the consumers' surplus is 18 000 \$.

- e) Since

$$\int_{70}^{100} 2\,000 - 20p \, dp = [2\,000p - 10p^2]_{70}^{100} =$$

$$= 2\,000 \cdot 100 - 10 \cdot 100^2 - (2\,000 \cdot 70 - 10 \cdot 70^2) = 9\,000,$$

the producers' surplus is 9 000 \$.

Problem 2.5.2. A manufacturer's marginal cost function is

$$MC(q) = 0.6q + 2.$$

If the production is presently set at $q = 80$ units per week, how much more would it cost to increase production to 100 units per week?

Solution:

We have to calculate the value of the difference

$$C(100) - C(80).$$

According to Newton-Leibniz theorem, we get that

$$\begin{aligned} C(100) - C(80) &= \int_{80}^{100} C'(q) \, dq = \int_{80}^{100} MC(q) \, dq = \\ &= \int_{80}^{100} 0.6q + 2 \, dq = [0.3q^2 + 2q]_{80}^{100} = \\ &= (0.3 \cdot 100^2 + 2 \cdot 100) - (0.3 \cdot 80^2 + 2 \cdot 80) = \\ &= 3\,200 - 2\,080 = 1\,120. \end{aligned}$$

The cost to increase production from 80 to 100 units is 1 120\$.

2.6. Applications of Lorenz-functions

Theoretical summary. Continuous function $L(x)$ is a *Lorenz-function*, if the following properties are fulfilled

- the domain of the function is interval $[0; 1]$, because this interval means percent $0 \rightarrow 100$;
- $L(0) = 0$ since no money is earned by 0 households;
- $L(1) = 1$ because all of the income is earned by the entire population;
- $L(x) \leq x$ for all $x \in [0; 1]$;
- range of function $L(x)$ is interval $[0; 1]$;
- function $L(x)$ is a strictly monotonic increasing function. (It means that more of the total income is earned by more of the households.)

The graph of a Lorenz function lies inside the unit square in the first quadrant.

The Lorenz function $L(x)$ gives the proportion of the total income earned by the lowest proportion x of the population. It can also be used to show distribution of assets (total wealth, rather than income).

Economists consider it to be a measure of social inequality. It was developed by Max O. Lorenz in 1905 for representing inequality of the wealth distribution.

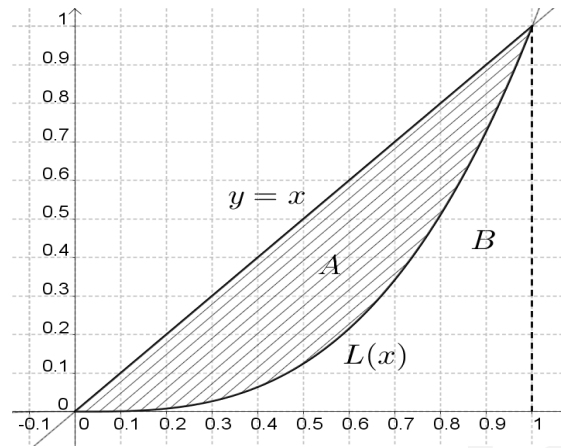
For example if $L(x)$ is a Lorenz-function, $L(0.25) = 0.10$ would mean that the poorest 25% of households earns 10% of the total income. $L(0.90) = 0.55$ would mean that the poorest 90% earns 55% of the total income. Equivalently, the richest 10% households earn 45% of the total income.

Two extreme cases that help us understand the Lorenz-function.

- a) *Absolute Equality of Income.* Everyone earns exactly the same amount of money. In this situation $L(x) = x$, that is, $x\%$ of the people earn $x\%$ of the income.
- b) *Absolute Inequality of Income.* Nobody earns any income except one person (who earns it all). In this situation

$$L(x) = \begin{cases} 0, & \text{for } 0 \leq x < 1 \\ 1, & \text{for } x = 1. \end{cases}$$

Let A be the area between the line $y = x$ representing perfect income equality and the Lorenz function $L(x)$. This is the shaded area in the figure



Let B denote the region under the Lorenz curve, then the Gini index is

$$G = \frac{A}{A + B}.$$

Since $A + B = \frac{1}{2}$, we get that

$$G = \frac{A}{\frac{1}{2}} = 2A.$$

Calculating shaded area A in the figure below, using definite integrals, we get that

$$G = 2A = 2 \cdot \int_0^1 x - L(x) \, dx.$$

Applying the properties of the definite integral, we get that

$$\begin{aligned} G &= 2 \cdot \int_0^1 x \, dx - 2 \cdot \int_0^1 L(x) \, dx = \\ &= 2 \cdot \left[\frac{x^2}{2} \right]_0^1 - 2 \cdot \int_0^1 L(x) \, dx = \\ &= 1 - 2 \cdot \int_0^1 L(x) \, dx. \end{aligned}$$

Notice that when there is perfect income equality, then $L(x) = x$ is the diagonal and we have $A = 0$, thus $G = 2A = 0$. When there is absolute income

inequality, then $B = 0$ and $A = \frac{1}{2}$, thus $G = 1$. G always falls in interval $[0; 1]$ with values closer to 0 representing more equally distributed income.

Applied mathematical knowledge. Properties of definit integral.

Sample problem. Consider function $L(x) = x^p$, where $p \geq 1$ is a parameter.

- Show that, function $L(x)$ is a Lorenz-function.
- Prove that Lorenz-function $L(x)$ has a Gini-index

$$G = \frac{p-1}{p+1}.$$

- Determine the value of p , such that the Gini index is equal to 0.45, if the Lorenz-function has the form of a power function $L(x) = x^p$.
- According to this model, how much of the family income is earned by the top 5% of families?

Solution:

a) Since

- $L(0) = 0^p = 0$;
- $L(1) = 1^p = 1$;
- $L(x) = x^p \leq x$ for all $x \in [0; 1]$;
- range of function $L(x)$ is interval $[0; 1]$;
- if $x_1 < x_2$, then $x_1^p < x_2^p$, thus $L(x)$ is strictly monotonic increasing function,

$L(x)$ is a Lorenz-function.

b) Since

$$\int_0^1 L(x) dx = \int_0^1 x^p dx = \left[\frac{x^{p+1}}{p+1} \right]_0^1 = \frac{1}{p+1},$$

the Gini-coefficient is

$$G = 1 - 2 \cdot \frac{1}{p+1} = \frac{p+1-2}{p+1} = \frac{p-1}{p+1}.$$

c) Since

$$G = \frac{p-1}{p+1},$$

we have to solve the equation

$$\frac{p-1}{p+1} = 0.45.$$

Multiplying both sides by $p+1$, we get that

$$p-1 = 0.45 \cdot (p+1).$$

Expand the expression, we get that

$$p-1 = 0.45p + 0.45.$$

Transforming the equation into the form

$$0.55p = 1.45,$$

thus $p = \frac{29}{11} \approx 2.64$.

d) Since

$$L(0.95) = 0.95^{2.64} = 0.8734,$$

the poorest 95% of households earns 87.34% of the total income, equivalently, the richest 5% of households earn 12.68% of the total income.

Problems

Problem 2.6.1. Check that function

$$L(x) = x^2$$

is a Lorenz function. What is the Gini-coefficient?

Solution:

Since

- $L(0) = 0^2 = 0$;
- $L(1) = 1^2 = 1$;
- $L(x) = x^2 \leq x$ for all $x \in [0; 1]$,
- range of function $L(x)$ is interval $[0; 1]$;
- if $x_1 < x_2$, then $x_1^2 < x_2^2$, thus $L(x)$ is strictly monotonic increasing function,

then $L(x)$ is a Lorenz function.

Since

$$\int_0^1 x^2 dx = \left[\frac{x^3}{3} \right]_0^1 = \frac{1}{3},$$

then the Gini-coefficient is

$$G = 1 - 2 \cdot \frac{1}{3} = \frac{1}{3}.$$

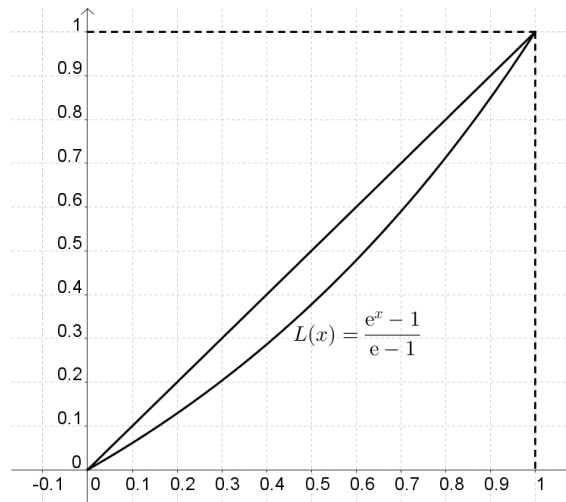
Problem 2.6.2. A European country has a Lorenz-function for household incomes given by the function

$$L(x) = \frac{e^x - 1}{e - 1} \quad (0 \leq x \leq 1).$$

- a) Sketch the graph of the function $L(x)$.
- b) Show that this function satisfies the properties of Lorenz-function.
- c) Determine $L(0.5)$ and interpret what it means.
- d) What is the Gini coefficient for this country? Give your interpretation and commentary.

Solution:

- a) The graph of function $L(x)$ is



b) Since

- $L(0) = \frac{e^0 - 1}{e - 1} = 0$;
- $L(1) = \frac{e^1 - 1}{e - 1} = 1$;
- $L(x) = \frac{e^x - 1}{e - 1} \leq x$ for all $x \in [0; 1]$;
- range of function $L(x)$ is interval $[0; 1]$;
- if $x_1 < x_2$, then $L(x_1) < L(x_2)$, thus $L(x)$ is strictly monotonic increasing function,

then $L(x)$ is a Lorenz function.

c) Since

$$L(0.5) = \frac{e^{0.5} - 1}{e - 1} = \frac{\sqrt{e} - 1}{e - 1} = 0.3775,$$

it would mean that the poorest 50% of households earns 37.75% of the total income.

d) Since

$$\begin{aligned}\int_0^1 L(x) dx &= \int_0^1 \frac{e^x - 1}{e - 1} dx = \left[\frac{e^x - x}{e - 1} \right]_0^1 = \\ &= \frac{1}{e - 1} \cdot (e - 1 - 1) = \frac{1}{e - 1} \cdot (e - 2) \approx 0.418,\end{aligned}$$

then the Gini-coefficient is

$$G = 1 - 2 \cdot 0.418 = 0.164.$$

Problem 2.6.3. Find the Gini-index corresponding to the Lorenz function

$$L(x) = x^3.$$

Solution:

Since

$$\int_0^1 L(x) dx = \int_0^1 x^3 dx = \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{4}$$

then the Gini-coefficient is

$$G = 1 - 2 \cdot \frac{1}{4} = \frac{1}{2}.$$

Problem 2.6.4. One type of function, which is often used to model Lorenz curves is

$$L(x) = a \cdot x + (1 - a) \cdot x^p.$$

Suppose that $a = \frac{1}{4}$ and that the Gini index for the distribution of wealth in a country is known to be $\frac{9}{16}$. Find the value of p that fits this situation.

Solution:

Since $a = \frac{1}{4}$, we get that

$$L(x) = \frac{1}{4} \cdot x + \frac{3}{4} \cdot x^p.$$

The integral of function $L(x)$ from 0 to 1 is

$$\int_0^1 \frac{1}{4} \cdot x + \frac{3}{4} \cdot x^p dx = \left[\frac{x^2}{8} + \frac{3}{4} \cdot \frac{x^{p+1}}{p+1} \right]_0^1 = \frac{1}{8} + \frac{3}{4} \cdot \frac{1}{p+1}.$$

The Gini index is $\frac{9}{16}$, thus we get that

$$\frac{9}{16} = 1 - 2 \cdot \left(\frac{1}{8} + \frac{3}{4} \cdot \frac{1}{p+1} \right).$$

Applying algebraic transformation, we get that

$$\frac{9}{16} = \frac{3}{4} - \frac{3}{2} \cdot \frac{1}{p+1}.$$

Thus

$$\frac{3}{16} = \frac{3}{2} \cdot \frac{1}{p+1},$$

that is $8 = p + 1$, consequently $p = 7$. Thus function $L(x)$ is

$$L(x) = \frac{1}{4} \cdot x + \frac{3}{4} \cdot x^7.$$

2.7. Kinematics of a particle

Theoretical summary. In the section about the „differential calculus of functions with one variable” we got the formula

$$a(t) = \dot{v}(t) = \ddot{s}(t).$$

Applying the above formula we can calculate the $v(t)$ and $a(t)$ functions from the $s(t)$ function. Let's deal with the inverse direction, and derive formulas for the calculation of the $v(t)$ and $s(t)$ functions from the $a(t)$ function. The connection between the $a(t)$ and $v(t)$ functions:

$$a(t) = \dot{v}(t).$$

Let's integrate both sides of the above equation between values t_0 and t_1

$$\int_{t_0}^{t_1} a(t) dt = \int_{t_0}^{t_1} \dot{v}(t) dt = [v(t)]_{t_0}^{t_1} = v(t_1) - v(t_0).$$

From the above equation we get that

$$v(t_1) = v(t_0) + \int_{t_0}^{t_1} a(t) dt.$$

Thus, from the velocity at moment t_0 and the function $a(t)$, we can calculate the velocity at any moment t_1 .

Similarly, the connection between functions $v(t)$ and $s(t)$ is

$$v(t) = \dot{s}(t).$$

From the above equation, we get that

$$s(t_1) = s(t_0) + \int_{t_0}^{t_1} v(t) dt.$$

Thus, from the position at moment t_0 and function $v(t)$, we can calculate the position at any moment t_1 . Replacing variable t_1 by t and t by τ , we can

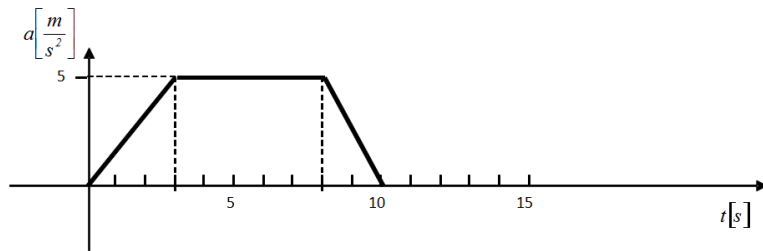
determine functions $v(t)$ and $s(t)$ with the following formulas:

$$v(t) = v(t_0) + \int_{t_0}^t a(\tau) d\tau$$

$$s(t) = s(t_0) + \int_{t_0}^t v(\tau) d\tau.$$

Applied mathematical knowledge. Primitive functions, Newton-Leibniz formula.

Sample problem. The figure below shows the acceleration-time function of a car:



- Find the acceleration-time function.
- Calculate the velocity-time function and sketch its graph if $v(0) = 0$.
- Calculate the position-time function of the car if $s(0) = 0$.

Solution:

- The acceleration-time function consists of three cases. The first one is linear, the second one is constant functions and the third one is linear also. The slopes of the functions are denoted as m_i ($i = 0, 1, 2$). The acceleration of the car at moment t_i is $a(t_i)$.

The acceleration-time function in the i th part is

$$a(t) = a(t_i) + m_i \cdot (t - t_i).$$

If $t \in [0; 3]$ then

$$a(t) = a(t_0) + m_0 \cdot (t - t_0) = 0 + \frac{5}{3} \cdot (t - 0) = \frac{5}{3}t \left[\frac{\text{m}}{\text{s}^2} \right].$$

If $t \in [3; 8]$ then

$$a(t) = a(t_1) + m_1 \cdot (t - t_1) = 5 + 0 \cdot (t - 3) = 5 \left[\frac{\text{m}}{\text{s}^2} \right].$$

If $t \in [8; 10]$ then

$$a(t) = a(t_2) + m_2 \cdot (t - t_2) = 5 - \frac{5}{2} \cdot (t - 8) = -\frac{5}{2}t + 25 \left[\frac{\text{m}}{\text{s}^2} \right].$$

That is the acceleration-time function is

$$a(t) = \begin{cases} \frac{5}{3}t, & \text{if } 0 \leq t \leq 3 \\ 5, & \text{if } 3 \leq t \leq 8 \\ -\frac{5}{2}t + 25, & \text{if } 8 \leq t \leq 10. \end{cases}$$

b) If $t \in [0; 3]$ then

$$v(t) = v(t_0) + \int_{t_0}^t a(\tau) \, d\tau = 0 + \int_0^t \frac{5}{3}\tau \, d\tau = \left[\frac{5}{6}\tau^2 \right]_0^t = \frac{5}{6}t^2 \left[\frac{\text{m}}{\text{s}} \right].$$

Thus

$$v(3) = \frac{15}{2} \left[\frac{\text{m}}{\text{s}} \right].$$

If $t \in [3; 8]$ then

$$\begin{aligned} v(t) &= v(t_1) + \int_{t_1}^t a(\tau) \, d\tau = \frac{15}{2} + \int_3^t 5 \, d\tau = \\ &= \frac{15}{2} + [5\tau]_3^t = \frac{15}{2} + 5t - 15 = 5t - \frac{15}{2} \left[\frac{\text{m}}{\text{s}} \right]. \end{aligned}$$

Thus

$$v(8) = 5 \cdot 8 - \frac{15}{2} = \frac{65}{2} \left[\frac{\text{m}}{\text{s}} \right].$$

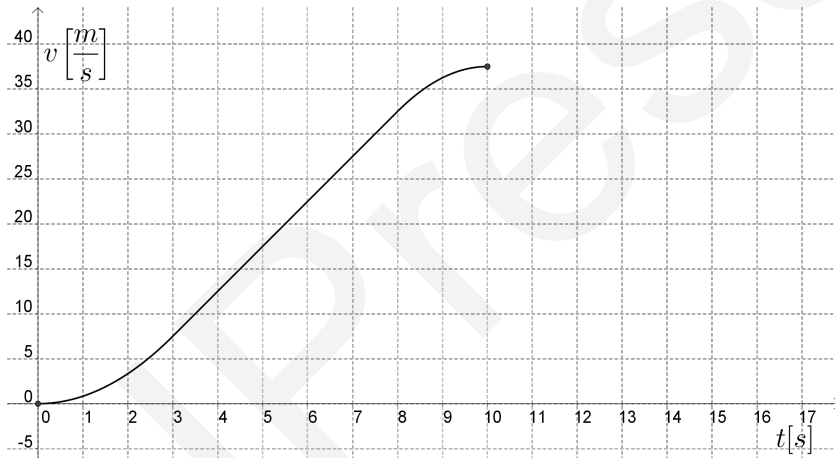
If $t \in [8; 10]$ then

$$\begin{aligned} v(t) &= v(t_2) + \int_{t_2}^t a(\tau) \, d\tau = \frac{65}{2} + \int_8^t -\frac{5}{2}\tau + 25 \, d\tau = \\ &= \frac{65}{2} + \left[-\frac{5}{4}\tau^2 + 25\tau \right]_8^t = \frac{65}{2} - \frac{5}{4}t^2 + 25t + \frac{5}{2} \cdot 32 - 25 \cdot 8 = \\ &= -\frac{5}{4}t^2 + 25t - \frac{175}{2} \left[\frac{\text{m}}{\text{s}} \right]. \end{aligned}$$

Thus the velocity-time function is

$$v(t) = \begin{cases} \frac{5}{6}t^2, & \text{if } 0 \leq t \leq 3 \\ 5t - \frac{15}{2}, & \text{if } 3 \leq t \leq 8 \\ -\frac{5}{4}t^2 + 25t - \frac{175}{2}, & \text{if } 8 \leq t \leq 10. \end{cases}$$

The graph of the velocity-time function is



c) If $t \in [0; 3]$ then

$$\begin{aligned} s(t) &= s(t_0) + \int_{t_0}^t v(\tau) d\tau = 0 + \int_0^t \frac{5}{6}\tau^2 d\tau = \\ &= \left[\frac{5}{18}\tau^3 \right]_0^t = \frac{5}{18}t^3 \left[\frac{m}{s} \right]. \end{aligned}$$

Thus

$$s(3) = \frac{5}{18} \cdot 27 = \frac{15}{2} \left[\frac{m}{s} \right].$$

If $t \in [3; 8]$ then

$$\begin{aligned}
 s(t) &= s(t_1) + \int_{t_1}^t v(\tau) \, d\tau = \\
 &= \frac{15}{2} + \int_3^t 5\tau - \frac{15}{2} \, d\tau = \\
 &= \frac{15}{2} + \left[\frac{5}{2}\tau^2 - \frac{15}{2}\tau \right]_3^t = \\
 &= \frac{15}{2} + \frac{5}{2}t^2 - \frac{15}{2}t - \frac{45}{2} + \frac{45}{2} = \\
 &= \frac{5}{2}t^2 - \frac{15}{2}t + \frac{15}{2} \left[\frac{\text{m}}{\text{s}} \right].
 \end{aligned}$$

Thus

$$s(8) = \frac{5}{2} \cdot 64 - \frac{15}{2} \cdot 8 + \frac{15}{2} = \frac{215}{2} \left[\frac{\text{m}}{\text{s}} \right].$$

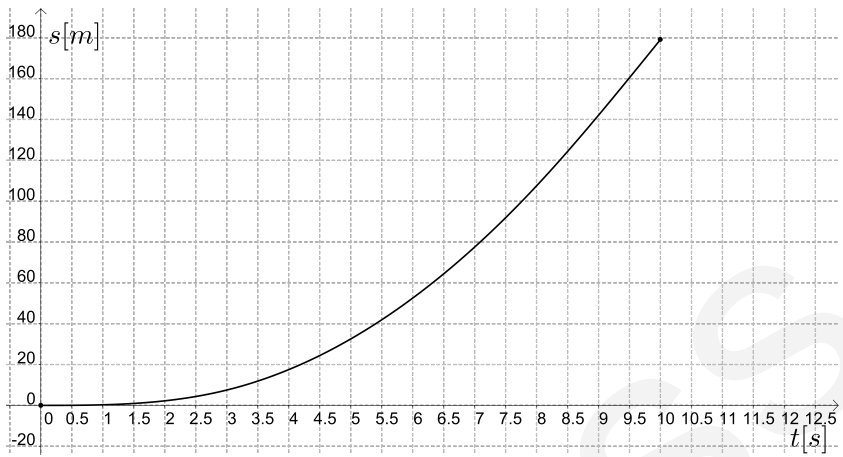
If $t \in [8; 10]$ then

$$\begin{aligned}
 s(t) &= s(t_2) + \int_{t_2}^t v(\tau) \, d\tau = \frac{215}{2} + \int_8^t -\frac{5}{4}\tau^2 + 25\tau - \frac{175}{2} \, d\tau = \\
 &= \frac{215}{2} + \left[-\frac{5}{12}\tau^3 + \frac{25}{2}\tau^2 - \frac{175}{2}\tau \right]_8^t = \\
 &= \frac{215}{2} - \frac{5}{12}t^3 + \frac{25}{2}t^2 - \frac{175}{2}t + \frac{5}{12} \cdot 512 - \frac{25}{2} \cdot 64 + \frac{175}{2} \cdot 8 = \\
 &= -\frac{5}{12}t^3 + \frac{25}{2}t^2 - \frac{175}{2}t + \frac{1325}{6} \left[\frac{\text{m}}{\text{s}} \right].
 \end{aligned}$$

That is, the position-time function is

$$s(t) = \begin{cases} \frac{5}{18}t^3, & \text{if } 0 \leq t \leq 3 \\ \frac{5}{2}t^2 - \frac{15}{2}t + \frac{15}{2}, & \text{if } 3 \leq t \leq 8 \\ -\frac{5}{12}t^3 + \frac{25}{2}t^2 - \frac{175}{2}t + \frac{1325}{6}, & \text{if } 8 \leq t \leq 10. \end{cases}$$

The graph of the position-time function is



Problems**Problem 2.7.1.** The velocity-time function of a car is

$$v(t) = A \cdot \frac{1 - e^{-a(t+b)}}{1 + e^{-a(t+b)}} \left[\frac{\text{m}}{\text{s}} \right] \quad (t \geq 0).$$

- a) Calculate the position-time function of the car, if $s(0) = 0$ [m].
 b) If

$$A = 36.25 \left[\frac{\text{m}}{\text{s}} \right]; \quad a = 0.145 \left[\frac{1}{\text{s}} \right]; \quad b = 6.494 \text{ [s]},$$

then find its position-time function!

- c) Calculate the covered distance by the car in 10 [s].

Solution:

- a) The position-time function of the car is calculated as

$$s(t) = s(0) + \int_0^t v(\tau) \, d\tau = \int_0^t A \cdot \frac{1 - e^{-a(\tau+b)}}{1 + e^{-a(\tau+b)}} \, d\tau.$$

We have to find a primitive function of the function

$$f(\tau) = \frac{1 - e^{-a(\tau+b)}}{1 + e^{-a(\tau+b)}}.$$

Introducing the notation

$$e^{-a(\tau+b)} = y$$

and τ , we get that

$$\begin{aligned} e^{-a(\tau+b)} &= y \\ -a \cdot (\tau + b) &= \ln y \\ \tau &= -\frac{1}{a} \cdot \ln y - b. \end{aligned}$$

Thus

$$\frac{d\tau}{dy} = -\frac{1}{a} \cdot \frac{1}{y}.$$

Integrating by substitution, we get that

$$- \int \frac{1-y}{1+y} \cdot \frac{1}{a} \cdot \frac{1}{y} \, dy = \frac{1}{a} \cdot \int \frac{y-1}{y \cdot (1+y)} \, dy.$$

Decomposing the

$$\frac{y-1}{y \cdot (1+y)}$$

fraction into partial fractions, we get that

$$\frac{y-1}{y \cdot (1+y)} = \frac{A}{y} + \frac{B}{1+y}.$$

We have to find the values of the constants A and B :

$$\begin{aligned} \frac{y-1}{y \cdot (1+y)} &= \frac{A}{y} + \frac{B}{1+y} \\ y-1 &= A \cdot (1+y) + B \cdot y \\ y-1 &= A + A \cdot y + B \cdot y \\ y-1 &= (A+B) \cdot y + A. \end{aligned}$$

We have to solve the system of equations below:

$$\begin{aligned} A+B &= 1 \\ A &= -1. \end{aligned}$$

The value of A is -1 . The value of B is 2 .

Thus we get that

$$\frac{y-1}{y \cdot (1+y)} = \frac{-1}{y} + \frac{2}{1+y}.$$

Applying the elementary properties of the primitive functions, by simple calculations, we get that

$$\begin{aligned} \frac{1}{a} \cdot \int \frac{y-1}{y \cdot (1+y)} dy &= \frac{1}{a} \cdot \int \frac{-1}{y} + \frac{2}{1+y} dy = \\ &= \frac{1}{a} \cdot (-\ln|y| + 2 \cdot \ln|1+y|) = \\ &= \frac{1}{a} \cdot (-\ln|e^{-a(\tau+b)}| + 2 \cdot \ln|1+e^{-a(\tau+b)}|) = \\ &= \frac{1}{a} \cdot (a \cdot (\tau+b) + 2 \cdot \ln|1+e^{-a(\tau+b)}|) = \\ &= \frac{2 \cdot \ln|1+e^{-a(\tau+b)}|}{a} + \tau + b. \end{aligned}$$

Thus the position-time function of the car is

$$\begin{aligned}
 s(t) &= \int_0^t A \cdot \frac{1 - e^{-a(\tau+b)}}{1 + e^{-a(\tau+b)}} + B \, d\tau = \\
 &= A \cdot \left[\frac{2 \cdot \ln |1 + e^{-a(\tau+b)}|}{a} + \tau + b + B \cdot \tau \right]_0^t = \\
 &= A \cdot \left(\frac{2 \cdot \ln |1 + e^{-a(t+b)}|}{a} + t + b + B \cdot t - \right. \\
 &\quad \left. - \frac{2 \cdot \ln |1 + e^{-a \cdot b}|}{a} - b \right) = \\
 &= A \cdot \left(\frac{2 \cdot \ln |1 + e^{-a(t+b)}|}{a} + t - \frac{2 \cdot \ln |1 + e^{-a \cdot b}|}{a} + B \cdot t \right).
 \end{aligned}$$

b) If $A = 36.25$, $a = 0.145$ and $b = 6.494$, then the position-time function of the car is

$$\begin{aligned}
 s(t) &= 36.25 \cdot \left(\frac{2 \cdot \ln |1 + e^{-0.145 \cdot (t+6.494)}|}{0.145} + t - \right. \\
 &\quad \left. - \frac{2 \cdot \ln |1 + e^{-0.145 \cdot 6.494}|}{0.145} - 1.5t \right).
 \end{aligned}$$

c) The covered distance is

$$\begin{aligned}
 s(10) &= 36.25 \cdot \left(\frac{2 \cdot \ln |1 + e^{-0.145 \cdot (10+6.494)}|}{0.145} + 10 - \right. \\
 &\quad \left. - \frac{2 \cdot \ln |1 + e^{-0.145 \cdot 6.494}|}{0.145} - 1.5 \cdot 10 \right) \approx 27.19 \text{ [m]}.
 \end{aligned}$$

Problem 2.7.2. The velocity-time function of a ship is

$$v(t) = A \cdot (1 - e^{-Bt}).$$

a) Find the position-time function of the ship, if $s(0) = 0$ [m].

b) If

$$A = 6.412 \left[\frac{\text{m}}{\text{s}} \right]; \quad B = 0.459 \left[\frac{1}{\text{s}} \right],$$

then calculate the position-time function of the ship.

c) What is the position of the ship at $t = 15$ [s] and $t = 20$ [s]?

Solution:

a) The position-time function is

$$\begin{aligned} s(t) &= s(t_0) + \int_0^t v(\tau) \, d\tau = s(t_0) + \int_0^t A \cdot (1 - e^{-B\tau}) \, d\tau = \\ &= \int_0^t A \cdot (1 - e^{-B\tau}) \, d\tau = A \cdot \left[\tau + \frac{e^{-B\tau}}{B} \right]_0^t = \\ &= A \cdot \left(t + \frac{e^{-Bt}}{B} - \frac{1}{B} \right). \end{aligned}$$

b) Substituting A and B , we get that the position-time function is

$$\begin{aligned} s(t) &= 6.412 \cdot \left(t + \frac{e^{-0.459t}}{0.459} - \frac{1}{0.459} \right) = \\ &= 6.412t - 13.97e^{-0.459t} - 13.97 \text{ [m]}. \end{aligned}$$

c) Since

$$s(15) = 6.412 \cdot 15 - 13.97e^{-0.459 \cdot 15} - 13.97 \approx 82.22 \text{ [m]},$$

then the position of the ship at $t = 15$ is

$$s(15) = 82.22 \text{ [m]}.$$

Similarly,

$$s(20) = 6.412 \cdot 20 - 13.97e^{-0.459 \cdot 20} - 13.97 \approx 114.27$$

thus the position of the ship at $t = 20$ is

$$s(20) = 114.27 \text{ [s]}.$$

Problem 2.7.3. Suppose that acceleration-time function of a sports car is

$$a(t) = 112 \cdot \left(\frac{2 - \ln(t+1)}{(t+1)^{\frac{3}{2}}} \right).$$

- Calculate the velocity-time function of the sports car, if $v(0) = 0$.
- Find the top speed of the car.
- Find the convexity of the velocity-time function. Calculate its inflection point(s).
- Sketch the graph of the velocity-time function.

Solution:

- a) The velocity-time function is the Riemann-integral of the acceleration-time function

$$\begin{aligned} v(t) &= v(0) + \int_0^t 112 \cdot \left(\frac{2 - \ln(\tau + 1)}{(\tau + 1)^{\frac{3}{2}}} \right) d\tau \\ &= \int_0^t 112 \cdot \left(\frac{2 - \ln(\tau + 1)}{(\tau + 1)^{\frac{3}{2}}} \right) d\tau. \end{aligned}$$

We have to find the primitive function of the function

$$f(\tau) = \frac{2 - \ln(\tau + 1)}{(\tau + 1)^{\frac{3}{2}}}.$$

In alternative form, function $f(\tau)$ is

$$f(\tau) = \frac{2 - \ln(\tau + 1)}{(\tau + 1)^{\frac{3}{2}}} = \frac{2}{(\tau + 1)^{\frac{3}{2}}} - \frac{\ln(\tau + 1)}{(\tau + 1)^{\frac{3}{2}}}.$$

The first component of the indefinite integral is

$$\begin{aligned} \int \frac{2}{(\tau + 1)^{\frac{3}{2}}} d\tau &= \int 2 \cdot (\tau + 1)^{-\frac{3}{2}} d\tau = \\ &= -4 \cdot (\tau + 1)^{-\frac{1}{2}} + c = -\frac{4}{\sqrt{\tau + 1}} + c. \end{aligned}$$

The second component of the indefinite integral is

$$\begin{aligned} \int \frac{\ln(\tau + 1)}{(\tau + 1)^{\frac{3}{2}}} d\tau &= \int \ln(\tau + 1) \cdot (\tau + 1)^{-\frac{3}{2}} d\tau = \\ &= \ln(\tau + 1) \cdot \frac{(\tau + 1)^{-\frac{1}{2}}}{-\frac{1}{2}} - \int \frac{1}{\tau + 1} \cdot \frac{(\tau + 1)^{-\frac{1}{2}}}{-\frac{1}{2}} d\tau = \\ &= -2 \cdot \ln(\tau + 1) \cdot (\tau + 1)^{-\frac{1}{2}} + 2 \cdot \int (\tau + 1)^{-\frac{3}{2}} d\tau = \\ &= -2 \cdot \ln(\tau + 1) \cdot (\tau + 1)^{-\frac{1}{2}} - 4 \cdot (\tau + 1)^{-\frac{1}{2}} + c = \\ &= \frac{-2 \cdot \ln(\tau + 1) - 4}{\sqrt{\tau + 1}} + c. \end{aligned}$$

Consequently, a primitive function of $f(\tau)$ is

$$F(\tau) = -\frac{4}{\sqrt{\tau + 1}} - \frac{-2 \cdot \ln(\tau + 1) - 4}{\sqrt{\tau + 1}} = \frac{2 \cdot \ln(\tau + 1)}{\sqrt{\tau + 1}}.$$

Thus the velocity-time function is

$$\begin{aligned} v(t) &= \int_0^t 112 \cdot \left(\frac{2 - \ln(\tau + 1)}{(\tau + 1)^{\frac{3}{2}}} \right) d\tau = 112 \cdot \int_0^t \left(\frac{2 - \ln(\tau + 1)}{(\tau + 1)^{\frac{3}{2}}} \right) d\tau = \\ &= \left[112 \cdot \frac{2 \cdot \ln(\tau + 1)}{\sqrt{\tau + 1}} \right]_0^t = 112 \cdot \frac{2 \cdot \ln(t + 1)}{\sqrt{t + 1}} = \\ &= \frac{224 \cdot \ln(t + 1)}{\sqrt{t + 1}}. \end{aligned}$$

- b) The speed has a maximum value when the acceleration of the car is zero, thus we have to solve the equation $a(t) = 0$, that is,

$$112 \cdot \left(\frac{2 - \ln(t + 1)}{(t + 1)^{\frac{3}{2}}} \right) = 0.$$

This equation holds, if and only is,

$$2 = \ln(t + 1),$$

that is,

$$e^2 - 1 = t.$$

Thus the car reaches its maximum speed at $t \approx 6.39$ seconds.

The maximum speed is

$$v(6.39) \approx 154.67 \left[\frac{\text{m}}{\text{s}} \right].$$

- c) The zero of function $v(t)$ is the solution of the equation $v(t) = 0$, that is,

$$\frac{224 \cdot \ln(t + 1)}{\sqrt{t + 1}} = 0.$$

This equation holds, if and only is,

$$\ln(t + 1) = 0,$$

that is,

$$t + 1 = 1 \quad \Rightarrow \quad t = 0.$$

The inflection point of function $v(t)$ is a solution of the equation $\dot{a}(t) = 0$, that is,

$$\begin{aligned}\dot{a}(t) &= 112 \cdot \frac{-\frac{1}{t+1} \cdot (t+1)^{\frac{3}{2}} - (2 - \ln(t+1)) \cdot \frac{3}{2} \cdot (t+1)^{\frac{1}{2}}}{(t+1)^3} = \\ &= 112 \cdot \frac{-(t+1)^{\frac{1}{2}} - \frac{3}{2} \cdot (t+1)^{\frac{1}{2}} \cdot (2 - \ln(t+1))}{(t+1)^3} = \\ &= 112 \cdot \frac{\sqrt{t+1} \cdot \left(-1 - 3 + \frac{3}{2} \cdot \ln(t+1)\right)}{(t+1)^3}.\end{aligned}$$

The equation $\dot{a}(t) = 0$ holds, if and only if,

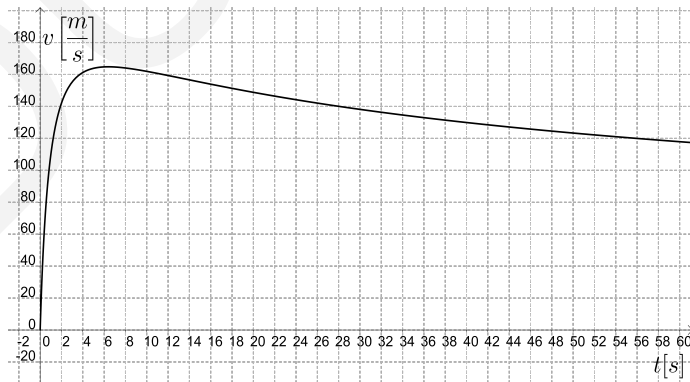
$$\begin{aligned}-4 + \frac{3}{2} \cdot \ln(t+1) &= 0 \\ \ln(t+1) &= \frac{8}{3} \\ t+1 &= e^{\frac{8}{3}},\end{aligned}$$

thus $t \approx 13.39$ [s].

Second-derivative sign table of function $v(t)$ is

	$t < 13.39$	$t = 13.39$	$t > 13.39$
$\ddot{v}(t)$	-	0	+
$v(t)$	concave	inflection point	convex

d) The graph of the velocity-time function is



Problem 2.7.4. The velocity-time function of a car is

$$v(t) = 100 \cdot \left(\frac{e^t}{1 + e^t} - \frac{e^{-t}}{1 + e^{-t}} \right) \left[\frac{\text{m}}{\text{s}} \right],$$

furthermore $s(0) = 200 \cdot \ln 2$ [m].

- a) Give the initial velocity of the car.
- b) Calculate the velocity at $t = 2$.
- c) Calculate the average acceleration at time interval $[0; 2]$ [s].
- d) Calculate the top speed of the car.
- e) Sketch the graph of the velocity-time function.
- f) Calculate the position-time function of the car.
- g) Give the initial position of the car.
- h) Calculate the position at $t = 2$.
- i) Calculate the average velocity at time interval $[0; 2]$ [s].
- j) Sketch the graph of function $s(t)$.
- k) Determine the acceleration-time function.
- l) Give the initial acceleration of the car.
- m) Calculate the acceleration at $t = 2$.
- n) Determine the limits of acceleration-time function at infinity.
- o) Sketch the graph of the acceleration-time function.

Solution:

- a) The initial velocity of the sports car is

$$v(0) = 100 \cdot \frac{e^0 - 1}{e^0 + 1} = 0 \left[\frac{\text{m}}{\text{s}} \right].$$

- b) The value of velocity-time function at $t = 2$ is

$$v(2) = 100 \cdot \frac{e^2 - 1}{e^2 + 1} = 76.16 \left[\frac{\text{m}}{\text{s}} \right].$$

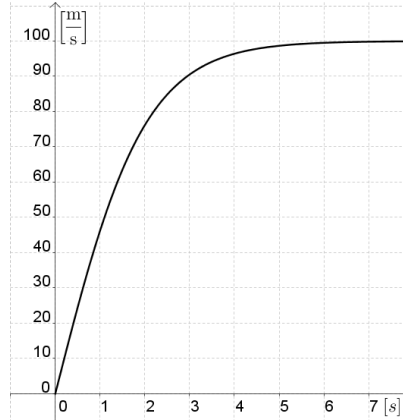
- c) The average acceleration is

$$\frac{v(2) - v(0)}{2 - 0} = \frac{76.16 - 0}{2} = 38.08 \left[\frac{\text{m}}{\text{s}^2} \right].$$

d) Applying the L'Hospital Rule, we have the top speed is

$$v_{\max} = \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} 100 \cdot \frac{e^t - 1}{e^t + 1} = 100 \cdot \lim_{t \rightarrow \infty} \frac{e^t}{e^t} = 100 \left[\frac{\text{m}}{\text{s}} \right].$$

e) The graph of the velocity-time function is



f) The position-time function is

$$\begin{aligned} s(t) &= \int \left(\frac{e^t}{1+e^t} - \frac{e^{-t}}{1+e^{-t}} \right) dt = \\ &= 100 \cdot (\ln(1+e^t) + \ln(1+e^{-t})) + c. \end{aligned}$$

Since $s(0) = 200 \ln 2$, then

$$200 \cdot \ln 2 = 100 \cdot (\ln 2 + \ln 2) + c \quad \Rightarrow \quad c = 0,$$

therefore the position-time function is

$$s(t) = 100 \cdot (\ln(1+e^t) + \ln(1+e^{-t})).$$

g) The initial position is

$$s(0) = 100 \cdot (\ln 2 + \ln 2) = 138.6 \text{ [m]}.$$

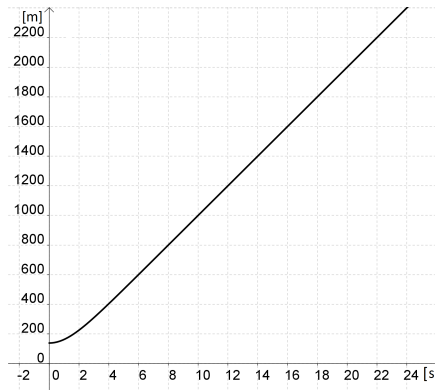
h) The position at $t = 2$ is

$$s(2) = 100 \cdot (\ln(1+e^2) + \ln(1+e^{-2})) = 225.4 \text{ [m]}.$$

i) The average velocity at time interval $[0; 2]$ is

$$\frac{s(2) - s(0)}{2 - 0} = \frac{225.4 - 138.6}{2} = 43.4 \left[\frac{\text{m}}{\text{s}} \right].$$

j) The graph of the position-time function is



k) The acceleration-time function is

$$\begin{aligned} a(t) = \dot{v}(t) &= 100 \cdot \frac{e^t \cdot (e^t + 1) - (e^t - 1) \cdot e^t}{(e^t + 1)^2} = \\ &= 100 \cdot \frac{e^{2t} + e^t - e^{2t} + e^t}{(e^t + 1)^2} = 100 \cdot \frac{2 \cdot e^t}{(e^t + 1)^2} = \frac{200 \cdot e^t}{(e^t + 1)^2} \left[\frac{\text{m}}{\text{s}^2} \right]. \end{aligned}$$

l) The initial acceleration is

$$a(0) = \frac{200 \cdot e^0}{(e^0 + 1)^2} = \frac{200}{4} = 50 \left[\frac{\text{m}}{\text{s}^2} \right].$$

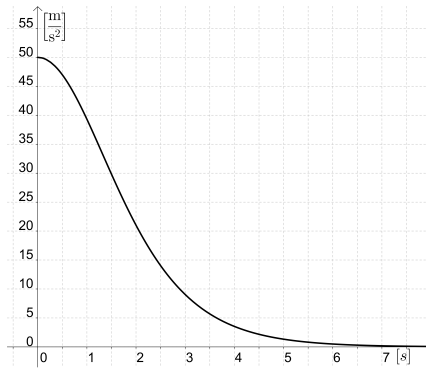
m) The acceleration at $t = 2$ is

$$a(2) = \frac{200 \cdot e^2}{(e^2 + 1)^2} = \frac{200}{4} = 21 \left[\frac{\text{m}}{\text{s}^2} \right].$$

n) Limit of acceleration-time function at infinity is

$$\begin{aligned} \lim_{t \rightarrow \infty} a(t) &= \lim_{t \rightarrow \infty} \frac{200 \cdot e^t}{(e^t + 1)^2} = \lim_{t \rightarrow \infty} \frac{200 \cdot e^t}{2 \cdot (e^t + 1) \cdot e^t} = \\ &= \lim_{t \rightarrow \infty} \frac{100}{e^t + 1} = 0. \end{aligned}$$

o) The graph of the acceleration-time function is



2.8. Mechanical work

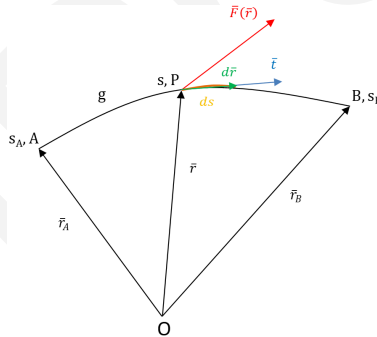
Theoretical summary. The term work is used in everyday language to mean the total amount of effort required to perform a task. In physics it has a technical meaning that depends on the idea of a force.

Intuitively, you can think of a force as describing a push or pull on an object - for example, a horizontal push of a book across a table or the downward pull of the earth's gravity on a ball. In general, the work of a force $\vec{F}(\vec{r})$ between point A and B along curve g is defined by the following line integral:

$$W_{\vec{F}(\vec{r}),g}^{A \rightarrow B} = \int_{\vec{r}_A,g}^{\vec{r}_B} \vec{F}(\vec{r}) \, d\vec{r}.$$

Let's introduce tangential unit vector \vec{t} by the following definition:

- the magnitude of \vec{t} is unit, that is, $|\vec{t}| = 1$;
- the direction of \vec{t} is parallel to the tangent line of the track at the given point;
- the sense of \vec{t} is the same as that of the track.



We can express the small displacement $d\vec{r}$ of the particle with tangential unit vector \vec{t} and the small covered distance ds of the particle

$$d\vec{r} = \vec{t} \, ds.$$

The work of force $\vec{F}(\vec{r})$ can be written as

$$W_{\vec{F}(\vec{r}),g}^{A \rightarrow B} = \int_{\vec{r}_A,g}^{\vec{r}_B} \vec{F}(\vec{r}) \, d\vec{r} = \int_{\vec{r}_A,g}^{\vec{r}_B} \vec{F}(\vec{r}) \cdot \vec{t} \, ds = \int_{s_A}^{s_B} F_t(s) \, ds.$$

In the formula above $F_t(s)$ is the tangential component of force $\vec{F}(\vec{r})$.

Thus for the calculation of work we get

$$W_{F_t(s)}^{A \rightarrow B} = \int_{s_A}^{s_B} F_t(s) ds.$$

Specially, in case of a linear track along the x axis $F_t(s) = F(x)$ and $ds = dx$ therefore

$$W = W_{F(x)}^{x_A \rightarrow x_B} = \int_{x_A}^{x_B} F(x) dx.$$

In the formula above $F(x)$ is the x component of the force. For example, if we compress or extend a spring with Δx , then the magnitude of the spring force will be linearly proportional with Δx . Thus the above magnitude of force can be calculated as

$$|F(x)| = D \cdot \Delta x,$$

where D is the *stiffness* of the spring and

$$\Delta x = |x - x_0|$$

is its deformation and x_0 is its *unloaded length*.

Since the sign of $x - x_0$ is opposite, $F(x)$ can be written as

$$F(x) = -D \cdot (x - x_0).$$

Specifically, if the unloaded state belongs to $x_0 = 0$, then the formula above can be written as

$$F(x) = -D \cdot x.$$

Thus, if x is positive then $F(x)$ is negative and vice versa. Thus the work of $F(x)$ is

$$W = - \int_a^b D \cdot x dx.$$

Applied mathematical knowledge. Methods for finding the primitive function of one-variable functions, Newton-Leibniz formula.

Sample problem. A spring is 1 meter long in unloaded state. We extend it with a force of 24 [N] to length 1.8 meter.

- Determine the spring stiffness.
- How much work does the spring do while we extend it to 2 [cm]?
- What is the length of the spring at 45 [N] load?

Solution:

a) We can determine the spring stiffness from formula

$$|F(x)| = D \cdot \Delta x,$$

which was described in the theoretical summary. In this case the extension of the spring is 0.8 meter, thus we get the formula

$$24 = D \cdot 0.8,$$

which results in

$$D = 30 \left[\frac{\text{N}}{\text{m}} \right].$$

b) If the unloaded state belongs to $x = 0$, then

$$F(x) = -30x \text{ [N]}.$$

The work done by the spring force between points with coordinates $x = 0$ and $x = 2$ is

$$W = - \int_0^2 30x \, dx = - [15x^2]_0^2 = -60 \text{ [J]}.$$

c) As $F(x) = -30x$, then

$$45 = -30x \quad \Rightarrow \quad x = -1.5 \text{ [m]},$$

thus the length of the spring is 1.5 [m].

Problems

Problem 2.8.1. A spring is 10 [cm] long in unloaded state. We extend it with a force of 800 [N] to length 14 [cm].

- Determine the spring stiffness.
- How much work does the spring do while we extend it to 3 meters?
- What is the length of the spring at 45 [N] load?

Solution:

- We can determine the spring stiffness from formula

$$|F| = D \cdot \Delta x,$$

which was described in the theoretical summary. In this case the extension of the spring is 0.04 meter, thus we get the formula

$$800 = D \cdot 0.04,$$

which results in

$$D = 20\,000 \left[\frac{\text{N}}{\text{m}} \right].$$

- If the unloaded state belongs to $x = 0$, then

$$F(x) = -20\,000x \text{ [N]}.$$

The work done by the spring force between points with coordinates $x = 0$ and $x = 2$ is

$$W = - \int_0^{0.02} 20\,000x \, dx = - [10\,000x^2]_0^{0.02} = -4 \text{ [J]}.$$

- As $F(x) = -20\,000x$, then

$$1\,600 = -20\,000x \quad \Rightarrow \quad x = -0.08 \text{ [m]},$$

thus the length of the spring will be 8 [cm].

Problem 2.8.2. A particle is located a distance from the origin, a force of

$$F(x) = x^2 + 2x \text{ [N]}$$

acts on it. How much work is done in moving it from $x = 1$ [m] to $x = 3$ [m]?

Solution:

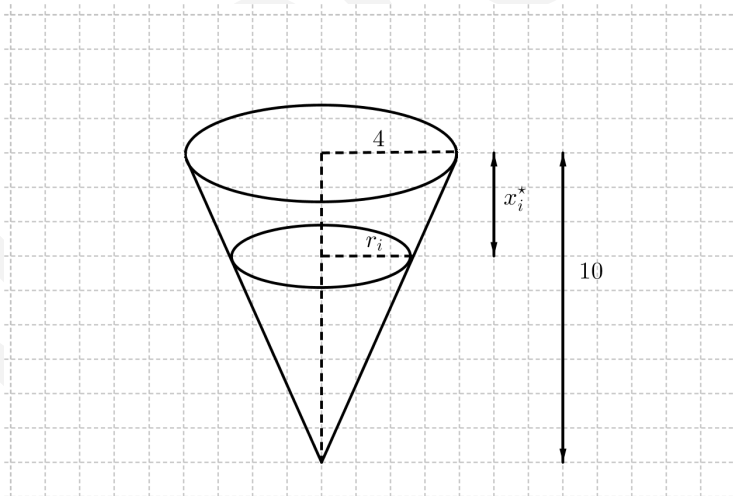
Work is

$$\int_1^3 x^2 + 2x \, dx = \left[\frac{x^3}{3} + x^2 \right]_1^3 = \left(\frac{3^3}{3} + 3^2 \right) - \left(\frac{1^3}{3} + 1^2 \right) = \frac{50}{3} \text{ [J]}.$$

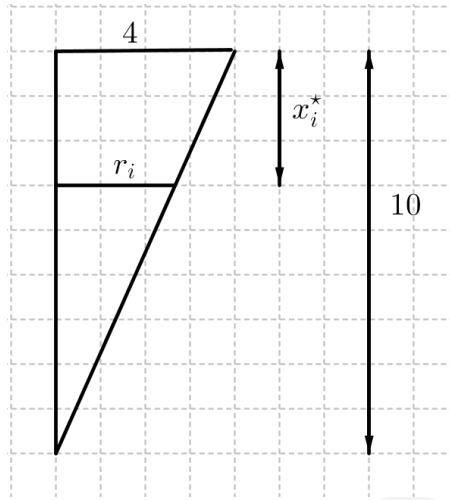
Problem 2.8.3. A tank has the shape of an inverted circular cone with height 10 [m] and base radius 4 [m]. It is filled with water to a height of 8 [m]. Find the work required to empty the tank by pumping all of the water to the top of the tank. The density of water is $1\,000 \left[\frac{\text{kg}}{\text{m}^3} \right]$.

Solution:

Let's measure depths from the top of the tank by introducing a vertical coordinate line as in the figure:



The water extends from a depth of 2 [m] to a depth of 10 [m] and we divide the interval $[2; 10]$ into n subintervals with endpoints $x_0; x_1; \dots; x_n$ and choose x_i^* in the i^{th} subinterval. This divides the water into n layers. The i^{th} layer is approximated by a circular cylinder with radius r_i and height Δx . We can compute r_i from similar triangles, using below:



Since

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10},$$

then

$$r_i = 4 - 0.4x_i^*.$$

An approximation to the volume of the i^{th} layer of water is

$$V_i = r_i^2 \cdot \pi \cdot \Delta x = (4 - 0.4x_i^*)^2 \cdot \pi \cdot \Delta x,$$

and thus its mass is

$$m_i = \rho \cdot V_i \approx 1000 \cdot (4 - 0.4x_i^*)^2 \cdot \pi \cdot \Delta x.$$

The force required to raise this layer must overcome the force of gravity and thus

$$F_i = g \cdot m_i \approx 9.81 \cdot 1000 \cdot (4 - 0.4x_i^*)^2 \cdot \pi \cdot \Delta x = 9810 \cdot (4 - 0.4x_i^*)^2 \cdot \pi \cdot \Delta x.$$

Each particle in the layer must travel a distance of approximately x_i^* . Thus the work W_i is

$$W_i = F_i \cdot x_i^* \approx 9810 \cdot (4 - 0.4x_i^*)^2 \cdot \pi \cdot \Delta x \cdot x_i^*.$$

To find the total work done in emptying the entire tank, we add the contributions of each of the n layers and then take the limit as $n \rightarrow \infty$

$$\begin{aligned} W &= \lim_{n \rightarrow \infty} \sum_{i=1}^n 9810 \cdot (4 - 0.4x_i^*)^2 \cdot \pi \cdot \Delta x \cdot x_i^* = \\ &= \int_2^{10} 9810 \cdot (4 - 0.4x)^2 \cdot \pi \, dx = 9810 \cdot \pi \cdot \left[\frac{(4 - 0.4x)^3}{3 \cdot (-0.4)} \right]_2^{10} = \\ &= 9810 \cdot \pi \cdot \left(0 - \frac{3 \cdot 2^3}{-1.2} \right) = 1\,009\,878 \text{ [J]}. \end{aligned}$$

Problem 2.8.4. If 6 [J] of work is needed to stretch a spring from 10 [cm] to 12 [cm] and another 10 [J] is needed to stretch it from 12 [cm] to 14 [cm]. What is the natural length of the spring?

Solution:

Let L be the natural length of the spring in meters. Then

$$\begin{aligned} 6 &= \int_{0.10-L}^{0.12-L} -D \cdot x \, dx = -D \cdot \left[\frac{x^2}{2} \right]_{0.10-L}^{0.12-L} = \\ &= -\frac{D}{2} \cdot [(0.12 - L)^2 - (0.10 - L)^2] = \\ &= -\frac{D}{2} \cdot (0.0144 - 0.24L + L^2 - 0.01 + 0.2L - L^2) = \\ &= -\frac{D}{2} \cdot (0.0044 - 0.04L). \end{aligned}$$

On the other hand

$$\begin{aligned} 10 &= \int_{0.12-L}^{0.14-L} -D \cdot x \, dx = -D \cdot \left[\frac{x^2}{2} \right]_{0.12-L}^{0.14-L} = \\ &= -\frac{D}{2} \cdot [(0.14 - L)^2 - (0.12 - L)^2] = \\ &= -\frac{D}{2} \cdot (0.0196 - 0.28L + L^2 - 0.0144 + 0.24L - L^2) = \\ &= -\frac{D}{2} \cdot (0.0052 - 0.04L). \end{aligned}$$

Thus we get the system of equations below:

$$-12 = D \cdot (0.0044 - 0.04L)$$

$$-20 = D \cdot (0.0052 - 0.04L).$$

If we divide the second equation by first equation, we get that:

$$\frac{0.0052 - 0.04L}{0.0044 - 0.04L} = \frac{5}{3}.$$

Transforming equation another equivalent form, we get that:

$$3 \cdot (0.0052 - 0.04L) = 5 \cdot (0.0044 - 0.04L).$$

Expand the expression, we get that:

$$0.0156 - 0.12L = 0.0220 - 0.2L,$$

thus

$$0.08L = 0.0064,$$

that is $L = 0.08$ [m]. Since

$$-D \cdot (0.0044 - 0.0032) = 12,$$

then

$$-0.0012D = 12 \quad \Rightarrow \quad D = -10\,000.$$

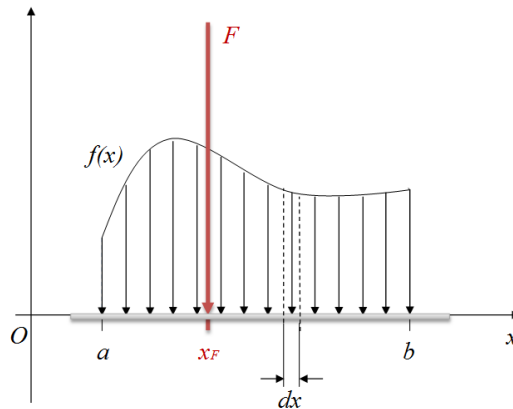
2.9. Parallel force system distributed along a line

Theoretical summary. A beam can be loaded with a *distributed force*. The distributed force, contrary to the concentrated force, does not act on a given point of a beam but is distributed along a section $[a; b]$ of it. A distributed force is always perpendicular to the beam and it is described by its intensity $f(x)$ at a point with coordinate x . If we multiply $f(x)$ by length dx of a small section of the beam, we get the magnitude of the force acting on the section:

$$dF = f(x) dx.$$

As a result, the magnitude of the force acting on the whole section $[a; b]$ is

$$F = \int_a^b f(x) dx.$$



The moment of the distributed force can be calculated relative to any point of the beam. The scalar moment of the distributed force - acting on line segment with length dx - can be calculated relative to point O by the formula below:

$$dM_O = -dF \cdot x = -x \cdot f(x) dx.$$

Thus the moment of the distributed force - acting on section $[a; b]$ - relative to point O is calculated as

$$M_O = - \int_a^b x \cdot f(x) dx.$$

We must mention here that the counter clockwise direction of rotation is considered as positive.

Concentrated force F is called as the *resultant of the distributed force* if its moment is the same relative to any point of the beam (so specifically relative to point O) as that of the distributed force. That is

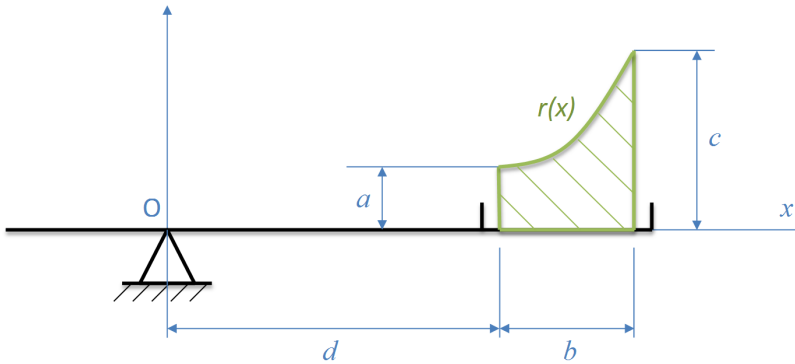
$$-F \cdot x_F = M_O = - \int_a^b x \cdot f(x) dx.$$

The coordinate x_F of the point of application of the resultant F is

$$x_F = \frac{- \int_a^b x \cdot f(x) dx}{-F} = \frac{\int_a^b x \cdot f(x) dx}{\int_a^b f(x) dx}.$$

Applied mathematical knowledge. Methods for finding the primitive function of a one-variable function, Newton-Leibniz formula.

Sample problem. The figure shows a two-armed scale in static equilibrium. In one of its pans there is a prism made of steel.



The longitudinal axis of the prism is perpendicular to the plane of the figure, so we can see the base of the prism. A part from the geometrical data represented in the figure we also know the length l of the prism, its mass density ρ and the magnitude of gravitational acceleration g . It is also known that the connection between intensity $f(x)$ of the distributed force, exerted by the prism on the pan of the scale, and function $r(x)$, describing the cross section of the prism, can be given as

$$f(x) = l \cdot \rho \cdot g \cdot r(x).$$

Data:

$$a = 0.05 \text{ [m]}; \quad b = 0.1 \text{ [m]}; \quad c = 0.15 \text{ [m]}; \quad l = 0.1 \text{ [m]};$$

$$d = 0.25 \text{ [m]}; \quad \rho = 7800 \left[\frac{\text{kg}}{\text{m}^3} \right]; \quad g = 9.81 \left[\frac{\text{m}}{\text{s}^2} \right].$$

- Give the formula of function $r(x)$ if we know, that its graph is a parabola the peak of which is the point with coordinates $(d; a)$.
- Determine the magnitude of the resultant of the distributed force which is exerted on the scale pan by the prism.
- Determine the scalar moment of the distributed force, the prism exerts on the scale pan, relative to point O .
- Calculate the distance between the point of application of the resultant force and point O .

Solution:

- As the graph of function $r(x)$ is parabolic and the peak of the parabola is point $(d; a) = (0.25; 0.05)$, the function that describes the graph of the parabola is

$$r(x) = a \cdot (x - 0.25)^2 + 0.05.$$

It can also be seen in the figure above that point

$$(d + b; c) = (0.35; 0.15)$$

is on the graph of function $r(x)$. Applying all of these we get the following formula:

$$a \cdot (0.35 - 0.25)^2 + 0.05 = 0.15.$$

The solution of the previous equation is

$$0.01a + 0.05 = 0.15 \quad \Rightarrow \quad 0.01a = 0.1 \quad \Rightarrow \quad a = 10.$$

The function that describes the graph of the parabola is

$$r(x) = 10 \cdot (x - 0.25)^2 + 0.05 = 10x^2 - 5x + 0.675 \left[\frac{\text{N}}{\text{m}} \right].$$

- As it is given in the problem, the connection between intensity $f(x)$ of the distributed force, that is exerted on the scale pan, and function $r(x)$ which describes the cross-section of the prism is

$$\begin{aligned} f(x) &= l \cdot \rho \cdot g \cdot r(x) = 0.1 \cdot 7800 \cdot 9.81 \cdot (10x^2 - 5x + 0.675) = \\ &= 7651.8 \cdot (10x^2 - 5x + 0.675). \end{aligned}$$

The magnitude of the force, exerted by the prism on the pan is

$$\begin{aligned}
 F &= \int_{0.25}^{0.35} f(x) \, dx = \int_{0.25}^{0.35} 7651.8 \cdot (10x^2 - 5x + 0.675) \, dx = \\
 &= 7651.8 \cdot \int_{0.25}^{0.35} 10x^2 - 5x + 0.675 \, dx = \\
 &= 7651.8 \cdot \left[\frac{10}{3}x^3 - \frac{5}{2}x^2 + 0.675x \right]_{0.25}^{0.35} = \\
 &= 7651.8 \cdot \left(\frac{10}{3} \cdot 0.35^3 - \frac{5}{2} \cdot 0.35^2 + 0.675 \cdot 0.35 \right) - \\
 &\quad - 7651.8 \cdot \left(\frac{10}{3} \cdot 0.25^3 - \frac{5}{2} \cdot 0.25^2 + 0.675 \cdot 0.25 \right) = \\
 &= 7651.8 \cdot \frac{1}{120} = 63.765 \text{ [N]}.
 \end{aligned}$$

c) The scalar moment is

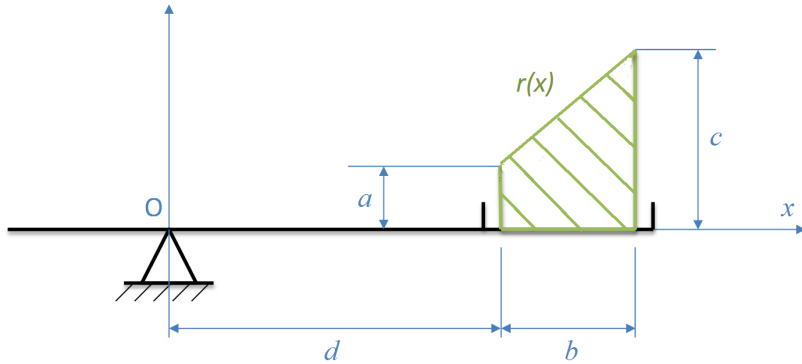
$$\begin{aligned}
 M_O &= - \int_{0.25}^{0.35} x \cdot f(x) \, dx = -7651.8 \cdot \int_{0.25}^{0.35} 10x^3 - 5x^2 + 0.675x \, dx = \\
 &= -7651.8 \cdot \left[\frac{5}{2}x^4 - \frac{5}{3}x^3 + \frac{27}{80}x^2 \right]_{0.25}^{0.35} = -19.77 \text{ [Nm]}.
 \end{aligned}$$

d) Coordinate x_F of the point of application of the resultant force is

$$x_F = \frac{- \int_{0.25}^{0.35} x \cdot f(x) \, dx}{- \int_{0.25}^{0.35} f(x) \, dx} \approx \frac{-19.77}{-63.77} \approx 0.31.$$

Problems

Problem 2.9.1. The figure shows a two-armed scale in static equilibrium. In one of its pans there is a prism made of steel.



The longitudinal axis of the prism is perpendicular to the plane of the figure, so we can see the base of the prism. The function $f(x)$ is given by

$$f(x) = l \cdot \rho \cdot g \cdot r(x),$$

where l , ρ and g are in sample problem.

Data:

$$a = 2 \text{ [m]}; \quad b = 3 \text{ [m]}; \quad c = 4 \text{ [m]}; \quad d = 8 \text{ [m]}$$

$$\rho = 2800 \left[\frac{\text{kg}}{\text{m}^3} \right]; \quad g = 9.81 \left[\frac{\text{m}}{\text{s}^2} \right].$$

- Give the formula of function $r(x)$.
- Determine the magnitude of the resultant of the distributed force which is exerted on the scale pan by the prism.
- Determine the scalar moment of the distributed force, the prism exerts on the scale pan, relative to point O .
- Calculate the distance between the point of application of the resultant force and point O .

Solution:

- The graph of function $r(x)$ is a line which can be described by the following function:

$$r(x) = m \cdot x + k.$$

The points (8; 2) and (11; 4) are on the straight line. Thus we get the system of equations below

$$\begin{aligned} 2 &= 8m + b \\ 4 &= 11m + b. \end{aligned}$$

The solution of the system of equations $m = \frac{2}{3}$ and $k = -10$. Function $r(x)$ is

$$r(x) = \frac{2}{3}x - 10.$$

- b) As it is given in the problem, the connection between intensity $f(x)$ of the distributed force, that is exerted on the scale pan by the prism, and function $r(x)$ which describes the cross-section of the prism is

$$\begin{aligned} f(x) &= -l \cdot \rho \cdot g \cdot r(x) = -3 \cdot 2800 \cdot 9.81 \cdot \left(\frac{2}{3}x - \frac{10}{3}\right) = \\ &= -82404 \cdot \left(\frac{2}{3}x - \frac{10}{3}\right). \end{aligned}$$

The magnitude of the force, exerted by the prism on the pan is

$$\begin{aligned} F &= \int_8^{11} f(x) \, dx = -82404 \cdot \int_8^{11} \left(\frac{2}{3}x - \frac{10}{3}\right) dx = \\ &= -82404 \left[\frac{1}{3} \cdot x^2 - \frac{10}{3}x \right]_8^{11} = -82404 \left(\frac{1}{3} \cdot 11^2 - \frac{10}{3} \cdot 11 \right) - \\ &- 82404 \left(\frac{1}{3} \cdot 8^2 - \frac{10}{3} \cdot 8 \right) \approx -741636 \text{ [N]}. \end{aligned}$$

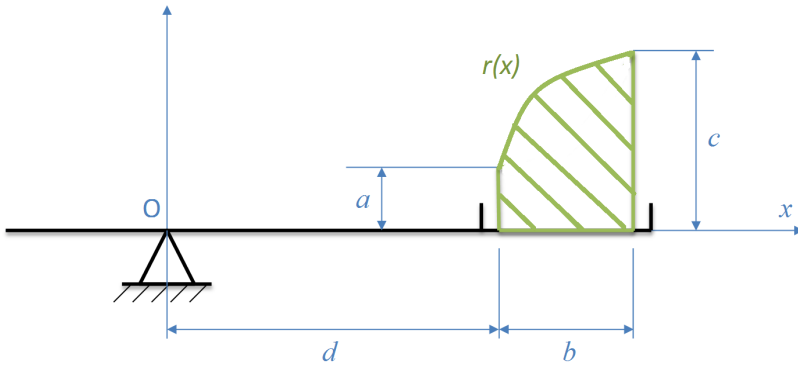
- c) The scalar moment is

$$\begin{aligned} M_O &= - \int_0^2 x \cdot f(x) \, dx = -82.404 \cdot \int_0^2 \left(0.21x^{\frac{3}{2}} - 0.5x\right) dx = \\ &= -82.404 \cdot \left[0.21 \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 0.5 \frac{x^2}{2} \right]_0^2 \approx \\ &\approx -82.404 \cdot (-0.52) \approx 42.850 \text{ [Nm]}. \end{aligned}$$

d) Coordinate x_F of the point of application of the resultant force is

$$x_F = \frac{-\int_0^2 x \cdot f(x) dx}{\int_0^2 f(x) dx} = \frac{42.850}{49.773} = 0.86.$$

Problem 2.9.2. The figure shows a two-armed scale in static equilibrium. In one of the pans of the scale there is a prism made of steel



The connection between the intensity $f(x)$ of the distributed force, exerted by the prism on the pan of the scale, and function $r(x)$, describing the cross section of the prism, can be given as

$$f(x) = -l \cdot \rho \cdot g \cdot r(x).$$

Data:

$$a = 0.2 \text{ [m]}; \quad b = 2 \text{ [m]}; \quad c = 0.5 \text{ [m]}; \quad l = 3 \text{ [m]}$$

$$\rho = 2800 \left[\frac{\text{kg}}{\text{m}^3} \right]; \quad g = 9.81 \left[\frac{\text{m}}{\text{s}^2} \right].$$

a) The formula of function $r(x)$ can be given as

$$r(x) = A \cdot \sqrt{x} - B.$$

Calculate the values of constants A and B .

- b) Determine the magnitude of the resultant of the distributed force which is exerted on the scale pan by the prism.
- c) Determine the scalar moment of the distributed force, the prism exerts on the scale pan, relative to point O .

- d) Calculate the distance between the point of application of the resultant force and point O .

Solution:

- a) The points $(0; -0.5)$ and $(2; -0.2)$ are on the graph of function $r(x)$, thus the next system of equations holds

$$\begin{aligned} -0.5 &= -B \\ -0.2 &= A \cdot \sqrt{2} - B. \end{aligned}$$

From the first equation, we get that $B = 0.5$. From the second equation it follows that $A \approx 0.21$. The function $r(x)$ is

$$r(x) = 0.21 \cdot \sqrt{x} - 0.5.$$

- b) As it is given in the problem, the connection between intensity $f(x)$ of the distributed force, that is exerted on the scale pan by the prism, and function $r(x)$ which describes the cross-section of the prism is

$$\begin{aligned} f(x) &= -l \cdot \rho \cdot g \cdot r(x) = -3 \cdot 2800 \cdot 9.81 \cdot (0.21 \cdot \sqrt{x} - 0.5) = \\ &= -82404 \cdot (0.21 \cdot \sqrt{x} - 0.5). \end{aligned}$$

The magnitude of the force, exerted by the prism on the pan is

$$\begin{aligned} F &= \int_0^2 f(x) \, dx = \\ &= -82404 \cdot \int_0^2 0.21 \cdot \sqrt{x} - 0.5 \, dx = \\ &= -82404 \left[0.21 \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 0.5x \right]_0^2 = \\ &= -82404 \cdot \left(0.21 \cdot \frac{2^{\frac{3}{2}}}{\frac{3}{2}} - 0.5 \cdot 2 \right) - 0 = \\ &= -82404 \cdot (-0.604) \approx 49773 \text{ [N]}. \end{aligned}$$

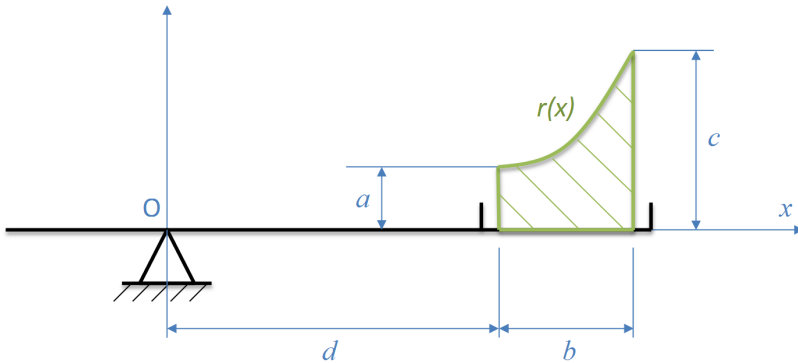
c) The scalar moment is

$$\begin{aligned} M_O &= - \int_0^2 x \cdot f(x) \, dx = -82\,404 \cdot \int_0^2 0.21x^{\frac{3}{2}} - 0.5x \, dx = \\ &= -82\,404 \cdot \left[0.21 \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 0.5 \frac{x^2}{2} \right]_0^2 \approx \\ &\approx -82\,404 \cdot (-0.52) \approx 42\,850 \text{ [Nm]}. \end{aligned}$$

d) Coordinate x_F of the point of application of the resultant force is

$$x_F = \frac{- \int_0^2 x \cdot f(x) \, dx}{\int_0^2 f(x) \, dx} = \frac{42\,850}{49\,773} = 0.86.$$

Problem 2.9.3. The figure shows a two-armed scale in static equilibrium. In one of its pans there is a prism made of steel



The longitudinal axis of the prism is perpendicular to the plane of the figure, so we can see the base of the prism. A part from the geometrical data represented in the figure we also know the length l of the prism, its mass density ρ and the magnitude of gravitational acceleration g . It is also known that the connection between intensity $f(x)$ of the distributed force, exerted by the prism on the pan of the scale, and function $r(x)$, describing the cross section of the prism, can be given as

$$f(x) = l \cdot \rho \cdot g \cdot r(x).$$

Data:

$$a = 0.5 \text{ [m]}; \quad b = 1 \text{ [m]}; \quad c = 1.5 \text{ [m]}; \quad l = 0.5 \text{ [m]};$$

$$d = 2 \text{ [m]}; \quad \rho = 5\,000 \left[\frac{\text{kg}}{\text{m}^3} \right]; \quad g = 9.81 \left[\frac{\text{m}}{\text{s}^2} \right].$$

a) Calculate the values of parameters α and β in the function

$$r(x) = \alpha \cdot 2^x + \beta.$$

- b) Determine the magnitude of the resultant of the distributed force which is exerted on the scale pan by the prism.
- c) Determine the scalar moment of the distributed force, the prism exerts on the scale pan, relative to point O .
- d) Calculate the distance between the point of application of the resultant force and point O .

Solution:

a) Points $(d; a) = (2; 0.5)$ and $(d+b; c) = (3; 1.5)$ are on the graph of function $r(x)$ thus the next system of equations holds

$$0.5 = \alpha \cdot 2^2 + \beta$$

$$1.5 = \alpha \cdot 2^3 + \beta.$$

The solution of the equation $\alpha = \frac{1}{4}$ and $\beta = -\frac{1}{2}$, thus we get that

$$r(x) = \frac{1}{4} \cdot 2^x - \frac{1}{2}.$$

b) As it is given in the problem, the connection between intensity $f(x)$ of the distributed force, that is exerted on the scale pan, and function $r(x)$ which describes the cross-section of the prism is

$$\begin{aligned} f(x) &= l \cdot \rho \cdot g \cdot r(x) = 0.5 \cdot 5\,000 \cdot 9.81 \cdot (0.25 \cdot 2^x - 0.5) = \\ &= 6\,131.25 \cdot 2^x - 12\,262.5. \end{aligned}$$

The magnitude of the force, exerted by the prism on the pan is

$$\begin{aligned} F &= \int_2^3 f(x) \, dx = \int_2^3 6\,131.25 \cdot 2^x - 12\,262.5 \, dx = \\ &= \left[\frac{6\,131.25}{\ln 2} \cdot 2^x - 12\,262.5x \right]_2^3 = 23\,120 \text{ [N]}. \end{aligned}$$

c) The scalar moment is

$$\begin{aligned} M_O &= - \int_2^3 x \cdot f(x) \, dx = \int_2^3 6\,131.25 \cdot x \cdot 2^x - 12\,262.5x \, dx = \\ &= \int_2^3 6\,131.25 \cdot (x \cdot 2^x - 2x) \, dx. \end{aligned}$$

A primitive function of function $x \cdot 2^x$ can be calculated as

$$\int x \cdot 2^x \, dx = x \cdot \frac{2^x}{\ln 2} - \int \frac{2^x}{\ln 2} \, dx = x \cdot \frac{2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2}.$$

Consequently

$$M_O = 6\,131.25 \cdot \left[x \cdot \frac{2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} - x^2 \right]_2^3 \approx 59\,827 \text{ [J]}.$$

d) Coordinate x_F of the point of application of the resultant force is

$$x_F = \frac{- \int_2^3 x \cdot f(x) \, dx}{\int_2^3 f(x) \, dx} \approx \frac{-59\,827}{23\,120} \approx 2.59.$$

2.10. The speed distribution of ideal gases

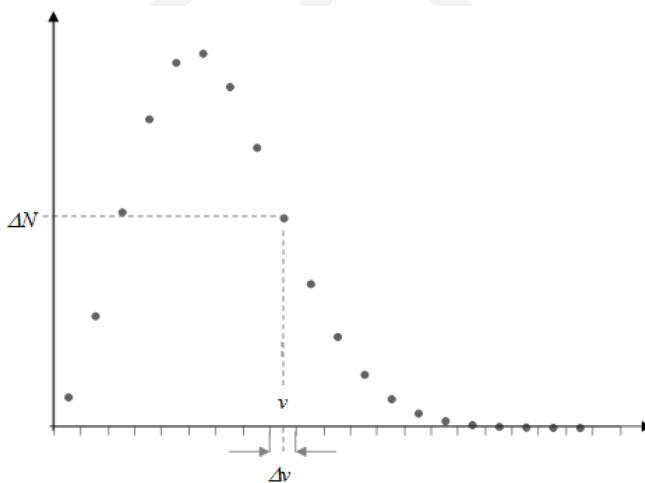
Theoretical summary. Any gas molecule has a diameter of approximately 10^{-10} [m].

The number of particles in 22.41 [dm³] normal state ($P = 1.013 \cdot 10^5$ [Pa], $T = 273.15$ [K]) in ideal gas is $6 \cdot 10^{23}$ [piece]. In spite of the great number of particles ideal gases are very rare.

If they were of the size of a tennis ball then there would be 10-20 of them in an average size room in normal state. As a result (considering that there are short distance interactions between the particles) the interaction between them is negligible.

The gas particles are in continuous movement. There are slower and faster particles among them in every moment. Let's divide the full speed range $[0; \infty[$ [$\frac{m}{s}$] to intervals with length Δv .

Let's plot the average number of particles ΔN falling into the respective intervals as a function of their average (mean) velocity. This way we get the approximate velocity magnitude distribution of the gas particles



Decreasing length Δv of the intervals the above approximation will be more and more accurate. Instead of ΔN , it is advised to plot the ratio $\frac{N}{v}$ because it is independent of length Δv . Decreasing Δv down to zero we get the derivative

of the speed distribution function

$$n(v) = \lim_{\Delta v \rightarrow 0} \frac{\Delta N}{\Delta v} = \frac{dN}{dv}.$$

The formula of the function above can be found in literature, that is

$$n(v) = A \cdot v^2 \cdot e^{-B \cdot v^2},$$

where

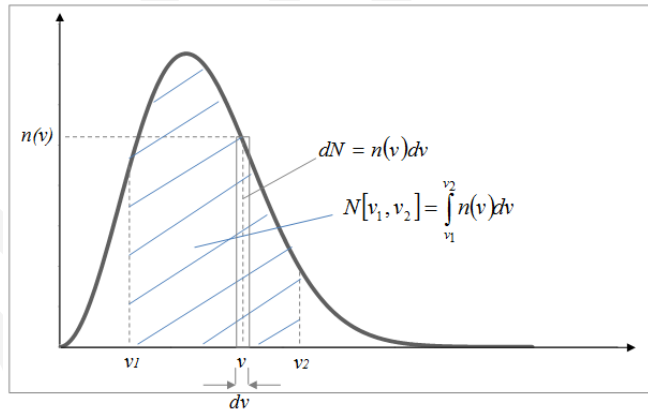
$$A = N \cdot \sqrt{\frac{2}{\pi}} \cdot \left(\frac{m_0}{k \cdot T}\right)^3;$$

$$B = \frac{m_0}{2 \cdot k \cdot T},$$

where m_0 , v and N denote the mass, speed and total number of gas particles, and T is the absolute temperature of the gas. The value of Boltzmann constant k is

$$k = 1.38 \cdot 10^{-23} \left[\frac{\text{J}}{\text{K}} \right].$$

The graph of function $n(v)$ is presented in the figure below



From function $n(v)$ the number of gas particles in the speed range $[v_1; v_2]$ can be calculated as

$$N[v_1; v_2] = \int_{v_1}^{v_2} n(v) dv.$$

The average speed of the particles can be calculated as

$$v_{\text{average}} = \frac{\int_0^{\infty} v \cdot n(v) dv}{\int_0^{\infty} n(v) dv} = \int_0^{\infty} v \cdot n(v) dv.$$

The average kinetic energy of the particles can be calculated as

$$\varepsilon_{\text{average}} = \frac{\int_0^{\infty} \varepsilon \cdot n(v) dv}{\int_0^{\infty} n(v) dv} = \int_0^{\infty} \varepsilon \cdot n(v) dv.$$

In the above equation kinetic energy ε can be calculated as

$$\varepsilon = \frac{1}{2} \cdot m_0 \cdot v^2$$

Since the speed of each particle is in the range $[0; \infty[$, the total number of gas particles can be calculated as

$$n = \int_0^{\infty} n(v) dv.$$

Applied mathematical knowledge. Methods for the finding of the primitive function. Newton-Leibnitz formula. Calculation of the approximate value of the Riemann integral (trapezoid formula). Application of different mathematical software for numerical integration.

Sample problem. Semiconducting materials are experimentally investigated in the Institute of Nuclear Research of the Hungarian Academy of Sciences in Debrecen. Inside the vacuum chamber - that is applied for the measurements - there is high vacuum. The vacuum is generated by a turbo-molecular pump. At the beginning of the experiment the number of nitrogen molecules in the chamber is $N = 10^{12}$. Other data of the remaining nitrogen gas are as follows

$$m_0 = 4.652 \cdot 10^{-25} \text{ [kg]}; \quad T = 273.15 \text{ [K]}.$$

- Determine the function $n(v)$ of the gas molecules and its first and second derivatives.
- Determine speed v^* at which most of the particles move (speed v^* is the maximum of the function $n(v)$).

- c) Analyse the convexity of function $n(v)$ and determine its inflection points.
 d) Sketch the graph of function $n(v)$.
 e) What is the average number of particles in the speed interval $[100; 200][m/s]$?
 f) Determine the average speed of the particles.
 g) Determine the average kinetic energy of the particles.

Solution:

a) The value of parameter A can be calculated as

$$A = N \cdot \sqrt{\frac{2}{\pi} \cdot \left(\frac{m_0}{k \cdot T}\right)^3} = 10^{12} \cdot \sqrt{\frac{2}{\pi} \cdot \left(\frac{4.652 \cdot 10^{-26}}{1.38 \cdot 10^{-23} \cdot 273.15}\right)^3} \approx 34.592 \left[\frac{\text{s}^3}{\text{m}^3}\right],$$

while parameter B is

$$B = \frac{m_0}{2k \cdot T} = \frac{4.652 \cdot 10^{-26}}{2 \cdot 1.38 \cdot 10^{-23} \cdot 273.15} \approx 6.17 \cdot 10^{-5} \left[\frac{\text{s}^2}{\text{m}^2}\right].$$

On the basis of the above results function $n(v)$ is

$$n(v) = A \cdot v^2 \cdot e^{-B \cdot v^2} = 34.592 \cdot v^2 \cdot e^{-6.17 \cdot 10^{-5} \cdot v^2} \left[\frac{\text{s}}{\text{m}}\right].$$

The first derivative of function $n(v)$ is

$$\begin{aligned} n'(v) &= 2Av \cdot e^{-B \cdot v^2} - A \cdot v^2 \cdot e^{-B \cdot v^2} \cdot 2Bv = \\ &= e^{-B \cdot v^2} \cdot (2Av - 2ABv^3). \end{aligned}$$

The second derivative of function $n(v)$ is

$$\begin{aligned} n''(v) &= -2Bv \cdot e^{-B \cdot v^2} \cdot (2Av - 2ABv^3) + \\ &+ e^{-B \cdot v^2} \cdot (2A - 6ABv^2) = \\ &= e^{-B \cdot v^2} \cdot (4AB^2v^4 - 10ABv^2 + 2A). \end{aligned}$$

- b) For the determination of the maximum of function $n(v)$, first we have to solve equation $n'(v) = 0$. Thus we have to solve the equation

$$e^{-B \cdot v^2} \cdot (2Av - 2ABv^3) = 0.$$

Since function $x \mapsto e^x$ is positive for all $x \in \mathbb{R}$ and $A \neq 0$ furthermore $v > 0$, the solution of the equation above is

$$v = \frac{1}{\sqrt{B}}.$$

Since

$$\begin{aligned} n''\left(\frac{1}{\sqrt{B}}\right) &= e^{-1} \cdot \left(4AB^2 \cdot \frac{1}{B^2} - 10AB \cdot \frac{1}{B} + 2A\right) \\ &= e^{-1} \cdot (-4A) = -\frac{4A}{e} < 0, \end{aligned}$$

the function has a local maximum at $v = \frac{1}{\sqrt{B}}$. Since this is the only one local maximum, this is also the global maximum too. Consequently we get that

$$v^* = \frac{1}{\sqrt{B}} = 127.3 \left[\frac{\text{m}}{\text{s}} \right].$$

- c) It is known that there is a global maximum of $n(v)$ at $v = \frac{1}{\sqrt{B}}$. Since $e^{-B \cdot v^2} \neq 0$, zeros of the $n''(v)$ function are the solutions of the equation below

$$4AB^2v^4 - 10ABv^2 + 2A = 0.$$

Let's introduce the notation $v^2 = u$. Then we have to solve the equation

$$4AB^2u^2 - 10ABu + 2A = 0.$$

Since $A \neq 0$, we get that

$$4B^2u^2 - 10Bu + 2 = 0.$$

Applying the quadratic formula, we get that

$$u_{1,2} = \frac{10B \pm \sqrt{100b^2 - 32B^2}}{8B^2} = \frac{10B \pm \sqrt{68B}}{8B^2} = \frac{5B \pm \sqrt{17B}}{4B^2},$$

that is,

$$u_1 = \frac{5 + \sqrt{17}}{4B} \approx \frac{2.28}{B}; \quad u_2 = \frac{5 - \sqrt{17}}{4B} \approx \frac{0.47}{B}.$$

Since $v > 0$, thus

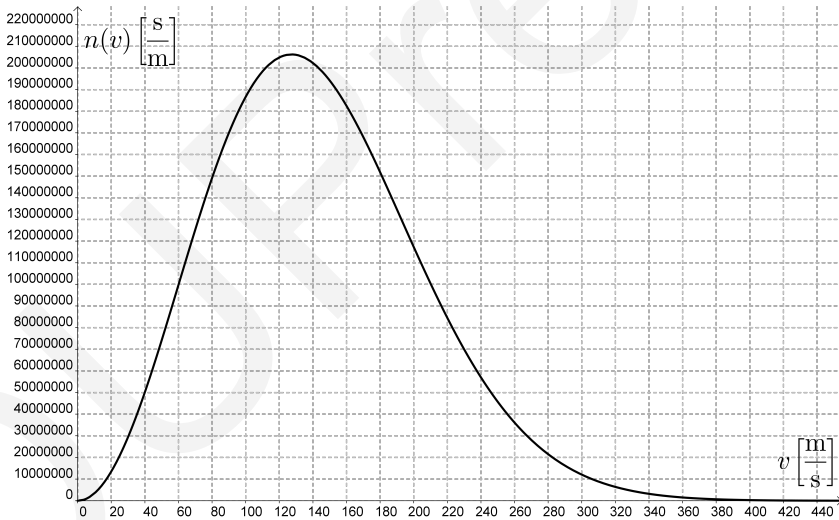
$$v_1 = \frac{1.51}{\sqrt{B}}; \quad v_2 = \frac{0.47}{\sqrt{B}}.$$

The signs of the second derivative of function $n(v)$ is shown in the table below:

	$v < \frac{0.47}{\sqrt{B}}$	$v = \frac{0.47}{\sqrt{B}}$	$\frac{0.47}{\sqrt{B}} < v < \frac{2.28}{\sqrt{B}}$	$v = \frac{2.28}{\sqrt{B}}$	$v > \frac{2.28}{\sqrt{B}}$
$n''(v)$	+	0	-	0	+
$n(v)$	convex	ip	concave	ip	convex

In the table above, "ip" means the inflection point.

d) The graph of the function $n(v)$ is



e) The average number of particles in the speed interval $[100; 200]$:

$$N[100; 200] = \int_{100}^{200} n(v) dv = \int_{100}^{200} A \cdot v^2 \cdot e^{-B \cdot v^2} dv.$$

Calculating the approximate value of integral by trapezoid formula with $n = 5$:

$$\int_{100}^{200} A \cdot v^2 \cdot e^{-B \cdot v^2} dv \approx \frac{n(100) + n(120)}{2} \cdot 20 + \frac{n(120) + n(140)}{2} \cdot 20 +$$

$$+ \frac{n(140) + n(160)}{2} \cdot 20 + \frac{n(160) + n(180)}{2} \cdot 20 +$$

$$+ \frac{n(180) + n(200)}{2} \cdot 20 = 10 \cdot (n(100) + 2 \cdot n(120) +$$

$$+ 2 \cdot n(140) + 2 \cdot n(160) + 2 \cdot n(180) + n(200)).$$

Thus the average number of particles in the speed interval $[100; 200]$ approximately $4.73 \cdot 10^{11}$.

f) The average speed of the particles is

$$v_{\text{average}} = \frac{\int_0^{\infty} v \cdot n(v) dv}{\int_0^{\infty} n(v) dv} \approx 143.65 \left[\frac{m}{s} \right].$$

The calculations were done with a mathematical software (Wolfram Alpha).

g) The average kinetic energy of the particles is

$$\varepsilon_{\text{average}} = \frac{\int_0^{\infty} \varepsilon \cdot n(v) dv}{\int_0^{\infty} n(v) dv} \approx 5.66 \cdot 10^{-21} \text{ [J]}.$$

The calculations were done with a mathematical software (Wolfram Alpha).

Problems

Problem 2.10.1. Semiconducting materials are experimentally investigated in the Institute of Nuclear Research of the Hungarian Academy of Sciences in Debrecen. Inside the vacuum chamber - that is applied for the measurements - there is high vacuum. The vacuum is generated by a turbo-molecular pump. At the beginning of the experiment the number of nitrogen molecules in the chamber is $N = 10^4$. Other data of the remaining nitrogen gas are as follows

$$m_0 = 4.652 \cdot 10^{-26} \text{ [kg]}; \quad T = 293.15 \text{ [K]}; \quad k = 1.38 \cdot 10^{-23} \left[\frac{\text{J}}{\text{K}} \right].$$

- Determine the function $n(v)$ of the gas molecules and its first and second derivatives.
- Determine speed v^* at which most of the particles move (speed v^* is the maximum of the function $n(v)$).

Solution:

- The value of parameter A can be calculated as

$$\begin{aligned} A &= N \cdot \sqrt{\frac{2}{\pi} \cdot \left(\frac{m_0}{k \cdot T} \right)^3} = 2 \cdot 10^4 \cdot \sqrt{\frac{2}{\pi} \cdot \left(\frac{4.652 \cdot 10^{-26}}{1.38 \cdot 10^{-23} \cdot 273.15} \right)^3} \approx \\ &\approx 6.22 \cdot 10^{-4} \left[\frac{\text{s}^3}{\text{m}^3} \right]. \end{aligned}$$

The value of parameter A can be calculated as

$$B = \frac{m_0}{2k \cdot T} = \frac{4.652 \cdot 10^{-26}}{2 \cdot 1.38 \cdot 10^{-23} \cdot 273.15} \approx 5.74 \cdot 10^{-6} \left[\frac{\text{s}^2}{\text{m}^2} \right].$$

Thus, we get that, the function $n(v)$ is

$$n(v) = A \cdot v^2 \cdot e^{-B \cdot v^2} = 6.22 \cdot 10^{-4} \cdot v^2 \cdot e^{-5.74 \cdot 10^{-6} \cdot v^2} \left[\frac{\text{S}}{\text{m}} \right].$$

The first derivative of function $n(v)$ is

$$\begin{aligned} n'(v) &= 2Av \cdot e^{-B \cdot v^2} - A \cdot v^2 \cdot e^{-B \cdot v^2} \cdot 2Bv = \\ &= e^{-B \cdot v^2} \cdot (2Av - 2ABv^3). \end{aligned}$$

The second derivative of function $n(v)$ is

$$\begin{aligned} n''(v) &= -2Bv \cdot e^{-B \cdot v^2} \cdot (2Av - 2ABv^3) + \\ &+ e^{-B \cdot v^2} \cdot (2A - 6ABv^2) = \\ &= e^{-B \cdot v^2} \cdot (4AB^2v^4 - 10ABv^2 + 2A). \end{aligned}$$

- b) For the determination of the maximum of function $n(v)$, first we have to solve equation $n'(v) = 0$. Thus we have to solve the equation

$$e^{-B \cdot v^2} \cdot (2Av - 2ABv^3) = 0.$$

Since function $x \mapsto e^x$ is positive for all $x \in \mathbb{R}$ and $A \neq 0$ furthermore $v > 0$, the solution of the equation above is

$$v = \frac{1}{\sqrt{B}}.$$

Since

$$\begin{aligned} n''\left(\frac{1}{\sqrt{B}}\right) &= e^{-1} \cdot \left(4AB^2 \cdot \frac{1}{B^2} - 10AB \cdot \frac{1}{B} + 2A\right) \\ &= e^{-1} \cdot (-4A) = -\frac{4A}{e} < 0, \end{aligned}$$

the function has a local maximum at $v = \frac{1}{\sqrt{B}}$. Since this is the only one local maximum, this is also the global maximum too. Consequently, we have

$$v^* = \frac{1}{\sqrt{B}} = 417.04 \left[\frac{\text{m}}{\text{s}}\right].$$

Problem 2.10.2. How many pieces of gas particle is in a mol of normal nitrogen gas speed exceeds the normal air condition sound barrier? $v_{\text{sound}} = 331.5 \left[\frac{\text{m}}{\text{s}}\right]$

Data:

$$N = 6.022 \cdot 10^{23} [\text{piece}]; \quad p = 1.013 \cdot 10^5 [\text{Pa}];$$

$$T = 273.15 [\text{K}]; \quad m_0 = 4.652 \cdot 10^{-26} [\text{kg}].$$

Solution:

The value of parameter A can be calculated as

$$\begin{aligned} A &= N \cdot \sqrt{\frac{2}{\pi} \cdot \left(\frac{m_0}{k \cdot T}\right)^3} = \\ &= 6.022 \cdot 10^{23} \cdot \sqrt{\frac{2}{\pi} \cdot \left(\frac{4.652 \cdot 10^{-26}}{1.38 \cdot 10^{-23} \cdot 273.15}\right)^3} \approx \\ &\approx 1.87 \cdot 10^{16} \left[\frac{\text{s}^3}{\text{m}^3}\right]. \end{aligned}$$

The value of parameter B can be calculated as

$$B = \frac{m_0}{2k \cdot T} = \frac{4.652 \cdot 10^{-26}}{2 \cdot 1.38 \cdot 10^{-23} \cdot 273.15} \approx 5.74 \cdot 10^{-6} \left[\frac{\text{s}^2}{\text{m}^2}\right].$$

The function $n(v)$ is

$$n(v) = A \cdot v^2 \cdot e^{-B \cdot v^2} = 1.87 \cdot 10^{16} \cdot v^2 \cdot e^{-5.74 \cdot 10^{-6} \cdot v^2} \left[\frac{\text{s}}{\text{m}}\right].$$

The number of particles is

$$\int_{331.5}^{\infty} A \cdot v^2 \cdot e^{-B \cdot v^2} dv \approx 4.45 \cdot 10^{23}.$$

The calculations were done with a mathematical software (Wolfram Alpha).

References

- [1] Árki Tamás – Konfárné Nagy Klára – Kovács István – Trembeczki Csaba – Urbán János, *Sokszínű matematika feladatgyűjtemény 11-12*, Mozaik Kiadó, Szeged, 2010.
- [2] Babcsányi István – Gyurmánczi János – Szabó Lajos – Wettl Ferenc, *Matematika feladatgyűjtemény I.*, Műegyetemi Kiadó, 2009.
- [3] Bartha Gábor – Bogdán Zoltán – Duró Lajosné dr. – Gyapjas Ferencné – Hack Frigyes – dr. Kántor Sándorné – dr. Korányi Erzsébet, *Matematika feladatgyűjtemény II.*, Nemzeti Tankönyvkiadó, Budapest, 2000.
- [4] Bárczy Barnabás, *Differenciálszámítás*, Műszaki Könyvkiadó, Budapest, 1994.
- [5] Benkő Pálné – Diószegi Ferencné – Serény György, *Matematika feladattár II.*, Műegyetemi Kiadó, 2002.
- [6] Császár Ákosné, *Matematika I/1*, Műegyetemi Kiadó, 2003.
- [7] Csikós Pajor Gizella – Péics Hajnalka, *Analízis elméleti összefoglaló és példatár*, Bolyai Farkas Alapítvány, Zenta, 2010.
- [8] Farkas István, *Differenciálszámítás gyakorlati jegyzet*, Debreceni Egyetem, 2005.
- [9] Dr. Geröcs László–Juhász István–Orosz Gyula– Paróczay József–Számadó László–Szászné Dr. Simon Judit, *Matematika emelt szintű tananyag*, Nemzedékek Tudása Tankönyvkiadó, 2013.
- [10] Gilbert János – Sólyom András – Kocsányi László, *Fizika mérnököknek I-II*, Egyetemi Tankönyv, Műegyetemi Kiadó, 1999.
- [11] J. Harcet – L. Heinrichs – P. M. Seiler – M. T. Skoumal, *Mathematics Higher Level*, Oxford University Press, 2012.
- [12] Horváth Eszter – Inges János – Nagyné Pálmai Piroska – Róka Sándor – Tassy Gergely, *Tehetség gondozás a matematikában*, <http://users.itk.ppke.hu/adorjan/matematika/list.html>, 2011.
- [13] Jakus G. – Kis M. – Magyar T. – Zombori N., *Analízis példatár*, Budapest, 2014.
- [14] Kézi Csaba Gábor, *Differenciálszámítás és alkalmazásai*, DUPress, 2016.
- [15] Kézi Csaba Gábor, *Differenciálszámítás és alkalmazásai feladatgyűjtemény*, DUPress, 2016.
- [16] Király Balázs, *Analízis (gyakorlat támogató jegyzet)*, elektronikus oktatási segédanyag, <http://tamop412a.ttk.pte.hu/files/analizis.pdf>, 2011.
- [17] Középiskolai Matematikai és Fizikai Lapok, 1996. január, 51. oldal, 2869. fizika feladat, <http://db.komal.hu/KomalHU/felhivatoz.phtml?id=41005>

- [18] Nagyné Kondor Rita – Szíki Gusztáv Áron, *Matematika eszközök mérnöki alkalmazásokban*, Egyetemi jegyzet, Debreceni Egyetem, 2011.
- [19] Kovács József – Takács Gábor – Takács Miklós, *Analízis*, Nemzeti Tankönyvkiadó, Budapest, 1986.
- [20] Kovács István – Trembeczki Csaba, *Sokszínű matematika feladatgyűjtemény, az analízis elemei 11–12 emelt szint*, Mozaik Kiadó, Szeged, 2011.
- [21] Lengyel Csilla Mária, *Szélsőérték-feladatok különböző megoldási módszerei*, ELTE, szakdolgozat, 2012.
- [22] Lial M. L. – Greenwell R. N. – Ritchey N. P., *Calculus with applications*, Pearson, 2012.
- [23] Mendelson E., *3000 solved problems in calculus*, McGraw-Hill Companies, 1988.
- [24] Nándori Frigyes – Szirbik Sándor, *Statika oktatási segédlet a Gépészmérnöki és Informatikai Kar Bsc levelezős hallgatói részére*, Mechanikai Tanszék, Miskolc-Egyetemváros, 2008.
- [25] Pintér Lajos, *Analízis I.*, Typotex, 1998.
- [26] Rosser M., *Basic mathematics for economists*, Routledge, 2003.
- [27] Sikolya Eszter, *Analízis jegyzet Matematikatanári Szakosok részére*, elektronikus jegyzet, tankonyvtar.ttk.bme.hu, 2013.
- [28] Simon Anita, *Az analízis néhány közgazdaságtani alkalmazása*, szakdolgozat, Eötvös Loránd Tudományegyetem, Természettudományi Kar, 2009.
- [29] Szentelekiné Dr. Páles Ilona, *Analízis példatár*, Nemzeti Tankönyvkiadó, Budapest, 2011.
- [30] Stewart J., *Calculus*, Brooks/Cole, 2012.
- [31] Tan S. T., *Applied Calculus for the Managerial*, Life and Social Sciences, Brooks/Cole, 1999.
- [32] Thomas G. B. – Weir M. D. – Hass J. – Giordano F. R., *Thomas féle kalkulus I. kötet*, Typotex, Budapest, 2008.

Contents

Chapter 1. Differential calculus of one variable functions	3
1.1. Kinematics of a particle: description of the motion	4
1.2. Kinetics of a particle: constrained motion on a 2D or 3D curve	37
1.3. Application of the least squares method	67
1.4. Extremum problems in engineering practice	85
1.5. Differential calculus in economics	105
Chapter 2. Integral calculus of one variable functions	129
2.1. Application of indefinite integral for economic problems	130
2.2. Area between the graphs of two functions	141
2.3. Volume of solids of revolution	148
2.4. Surface area of revolutions	154
2.5. Application of Riemann integral for economic problems	162
2.6. Applications of Lorenz-functions	169
2.7. Kinematics of a particle	177
2.8. Mechanical work	194
2.9. Parallel force system distributed along a line	202
2.10. The speed distribution of ideal gases	213
References	223

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