

ON THE SUM OF DIGITS OF NUMERATORS OF BERNOULLI NUMBERS

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ABSTRACT. Let $b > 1$ be an integer. We prove that for most n , the sum of the digits in base b of the numerator of the Bernoulli number B_{2n} exceeds $c \log n$, where $c := c(b) > 0$ is some constant depending on b .

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1. INTRODUCTION

Let $\{B_n\}_{n \geq 0}$ be the sequence of Bernoulli numbers given by $B_0 = 1$ and

$$B_n = 1 - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k}{n-k+1} \quad \text{for all } n \geq 1.$$

Then $B_1 = -1/2$ and $B_{2n+1} = 0$ for all $n \geq 0$. Furthermore, we have $(-1)^{n+1} B_{2n} > 0$. Write $B_{2n} =: (-1)^{n+1} C_n / D_n$ with coprime positive integers C_n and D_n . The denominator D_n is well-understood by the von Staudt–Clausen theorem which asserts that

$$D_n = \prod_{p-1|2n} p. \tag{1}$$

Let $b > 1$ be an integer. For a positive integer m , put $s_b(m)$ for the sum of digits of m in base b . In [3], it was shown that there exists a positive constant c_0 depending on b such that the inequality $s_b(n!) > c_0 \log n$ holds for all positive integers n . Here, we prove that a similar inequality holds with $n!$ replaced by C_n on a set of n of asymptotic density 1.

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Theorem 1. *The inequality*

$$s_b(C_n) > \frac{\log n}{6 \log b}$$

holds on a set of positive integers n of asymptotic density 1.

In what follows, we use the Landau symbols O and o and the Vinogradov symbols \gg and \ll with their usual meaning. We also use c_0, c_1, \dots for computable positive constants and x_0 for a large real number, not necessarily the same from one occurrence to the next.

Proof. Consider the following set of positive integers:

$$\mathcal{M}_b(x) := \left\{ n \in [x/2, x) : s_b(C_n) < \frac{\log x}{6 \log b} \right\}. \quad (2)$$

We need to show that $\#\mathcal{M}_b(x) = o(x)$ as $x \rightarrow \infty$, for after this the conclusion of Theorem 1 will follow by replacing x by $x/2$, then by $x/4$, and so on, and summing up the resulting estimates.

Put $y := \log x$ and consider the set

$$\mathcal{L}(x) := \{n \in [x/2, x) : p-1 \mid 2n \text{ for some prime } p \geq y\}. \quad (3)$$

It follows from Theorem 2 in [2] that

$$\#\mathcal{L}(x) = o(x) \quad \text{as } x \rightarrow \infty. \quad (4)$$

We now put

$$\mathcal{N}_b(x) := \mathcal{M}_b(x) \setminus \mathcal{L}(x). \quad (5)$$

In light of (4), it suffices to show that $\#\mathcal{N}_b(x) = o(x)$ as $x \rightarrow \infty$.

Let

$$\mathcal{D}(x) := \{D_n : n \in \mathcal{N}_b(x)\}. \quad (6)$$

Since $n \notin \mathcal{L}(x)$, it follows that if $p \mid C_n$, then $p < y$. Thus,

$$\#\mathcal{D}(x) \leq 2^{\pi(y)} = x^{o(1)} \quad \text{as } x \rightarrow \infty. \quad (7)$$

For $n \in \mathcal{N}_b(x)$, we write

$$C_n = d_1 b^{n_1} + d_2 b^{n_2} + \dots + d_s b^{n_s}, \quad (8)$$

where $d_1, \dots, d_s \in \{1, \dots, b-1\}$ and $n_1 > n_2 > \dots > n_s$. We put $t := t(n)$ for the smallest index $i \in \{1, 2, \dots, s-1\}$ such that $b^{n_1 - n_{i+1}} > n^2$ if it exists and set $t := s$ otherwise. From the definition of $t(n)$, we see immediately that

$$\begin{aligned} C_n &= (d_1 b^{n_1} + \dots + d_t b^{n_t}) (1 + O(n^{-2})) \\ &:= b^{m_n} E_n (1 + O(x^{-2})), \end{aligned} \quad (9)$$

where $m = m_n := n_t$ and $E_n := d_1 b^{n_1 - n_t} + d_2 b^{n_2 - n_t} + \dots + d_t$.

Let $\mathcal{E}_b(x)$ be the subset of all possible values for E_n for $n \in \mathcal{N}_b(x)$. Let us find an upper bound for the cardinality of this set. Observe first that

$$E_n < b^{n_1 - n_t + 1} \leq b^{2(\log x)/(\log b) + 1}. \quad (10)$$

The positive integers $E := E_n$ bounded by the quantity appearing on the right-hand side of inequality (10) have at most $K := \lfloor 2(\log x)/(\log b) + 2 \rfloor$ digits in base b . As $n \in \mathcal{N}_b(x)$, the number of nonzero digits of E_n is bounded by $S := \lfloor (\log x)/(6 \log b) \rfloor$, and the number of possible values for E is at most

$$\begin{aligned} \sum_{i=0}^S \binom{K}{i} (b-1)^i &\leq (S+1) \binom{K}{S} (b-1)^S \leq (S+1) \left(\frac{(b-1)eK}{S} \right)^S \\ &\leq \left(\frac{\log x}{6 \log b} + 1 \right) \left(\frac{(b-1)e}{3} + o(1) \right)^{(\log x)/(6 \log b)} = x^{\delta + o(1)} \end{aligned}$$

as $x \rightarrow \infty$, where

$$\delta := \frac{\log((b-1)e/3)}{6 \log b} < \frac{1}{6}.$$

Thus, we get that

$$\#\mathcal{E}_b(x) = x^{\delta + o(1)} \quad \text{as } x \rightarrow \infty. \quad (11)$$

We next use the formula

$$\zeta(2n) = (-1)^{n+1} B_{2n} \frac{(2\pi)^{2n}}{2(2n)!} = \frac{C_n (2\pi)^{2n}}{D_n 2(2n)!},$$

as well as the approximation

$$\zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = 1 + O\left(\frac{1}{2^{2n}}\right),$$

to get that

$$C_n = D_n \frac{2(2n)!}{(2\pi)^{2n}} \zeta(2n) = D_n \frac{2(2n)!}{(2\pi)^{2n}} \left(1 + O\left(\frac{1}{2^{2n}}\right) \right). \quad (12)$$

We take logarithms in (12) above to arrive at

$$\log C_n - \log D_n - \log(2(2n)!) + 2n \log(2\pi) = \log \left(1 + O\left(\frac{1}{2^{2n}}\right) \right) = O\left(\frac{1}{2^n}\right). \quad (13)$$

Taking logarithms in (9) and comparing the result with (13), we get

$$\log(2(2n)!) - 2n \log(2\pi) = \log(E_n/D_n) + m_n \log b + O\left(\frac{1}{x^2}\right). \quad (14)$$

Now fix a pair of numbers $(D, E) \in \mathcal{D}_b(x) \times \mathcal{E}_b(x)$ and look at the set

$$\mathcal{N}_{b,D,E}(x) := \{n \in \mathcal{N}_b(x) : (D_n, E_n) = (D, E)\}. \quad (15)$$

We let $z := x^{1/5}$ and show that if $x > x_0$, then every subinterval I of $[x/2, x]$ of length z contains at most two elements of $\mathcal{N}_{b,D,E}(x)$. Well, assume that this is not so and let $n_0 < n_1 < n_2$ be all three in $\mathcal{N}_{b,D,E}(x)$, where $n_2 - n_0 \leq z$. Write $n_i = n_0 + k_i$ for $i = 0, 1, 2$, with $0 = k_0 < k_1 < k_2 \leq z$. We evaluate relation (14) in $n := n_i := n_0 + k_i$ for $i = 0, 1, 2$ getting

$$\log(2(2n_i)!) + 2n_i \log(2\pi) = \log(E/D) + m_{n_i} \log b + O\left(\frac{1}{x^2}\right), \quad (i = 0, 1, 2). \quad (16)$$

We let $\Lambda := (\lambda_0, \lambda_1, \lambda_2)$ be a nonzero vector of integers such that

$$\begin{cases} \lambda_0 + \lambda_1 + \lambda_2 = 0, \\ n_0 \lambda_0 + n_1 \lambda_1 + n_2 \lambda_2 = 0. \end{cases} \quad (17)$$

For example, we can put $\Lambda := (k_1 + k_2, -k_2, -k_1)$. With this choice,

$$\max\{|\lambda_i| : i = 0, 1, 2\} \leq 2z. \quad (18)$$

Note also that system (17) is equivalent to

$$\begin{cases} \lambda_0 + \lambda_1 + \lambda_2 = 0, \\ k_1 \lambda_1 + k_2 \lambda_2 = 0. \end{cases} \quad (19)$$

Taking the linear combination of the three relations (16) with the coefficients given by Λ , we get

$$\sum_{i=0}^2 \lambda_i \log(2(2n_i)!) = \Gamma \log b + O\left(\frac{z}{x^2}\right), \quad (20)$$

where $\Gamma := \sum_{i=0}^2 \lambda_i m_{n_i} \in \mathbb{Z}$. Write

$$2(2n_i)! = 2(2n_0)!(2n_0 + 1) \cdots (2n_i) =: 2(2n_0)!X_i, \quad (i = 0, 1, 2).$$

Hence,

$$\begin{aligned} \sum_{i=0}^2 \lambda_i \log(2(2n_i)!) &= \sum_{i=0}^2 \lambda_i (\log(2(2n_0)!) + \log X_i) \\ &= \sum_{i=0}^2 \lambda_i \log(2(2n_0)!) + \sum_{i=0}^2 \lambda_i \log X_i \\ &= \sum_{i=0}^2 \lambda_i \log X_i = \lambda_1 \log X_1 + \lambda_2 \log X_2 \end{aligned}$$

where in the above equalities we used the first equation of (17) as well as the fact that $X_0 = 1$. Writing

$$\begin{aligned}
\log X_i &= \sum_{j=1}^{2k_i} \log(2n_0 + j) = 2k_i \log(2n_0) + \sum_{j=1}^{2k_i} \log \left(1 + \frac{j}{n_0} \right) \\
&= 2k_i \log(2n_0) + \sum_{j=1}^{2k_i} \left(\frac{j}{n_0} + O \left(\frac{j^2}{n_0^2} \right) \right) \\
&= 2k_i \log(2n_0) + \sum_{j=1}^{2k_i} \frac{j}{n_0} + O \left(\sum_{j=0}^{2k_i} \frac{j^2}{n_0^2} \right) \\
&= 2k_i \log(2n_0) + \frac{2k_i(2k_i + 1)}{n_0} + O \left(\frac{k_i^3}{x^2} \right),
\end{aligned}$$

for $i = 1, 2$, we get that

$$\begin{aligned}
\lambda_1 \log X_1 + \lambda_2 \log X_2 &= 2(\lambda_1 k_1 + \lambda_2 k_2) \log(2n_0) + \frac{4\lambda_1 k_1^2 + 4\lambda_2 k_2^2}{n_0} \\
&\quad + \frac{2\lambda_1 k_1 + 2\lambda_2 k_2}{n_0} + O \left(\frac{k_2^4}{n_0^2} \right) \\
&= 4 \left(\frac{\lambda_1 k_1^2 + \lambda_2 k_2^2}{n_0} \right) + O \left(\frac{z^4}{x^2} \right).
\end{aligned}$$

Inserting the above estimate into the left-hand side of (20), we get

$$4 \left(\frac{\lambda_1 k_1^2 + \lambda_2 k_2^2}{n_0} \right) = \Gamma \log b + O \left(\frac{z}{x^2} + \frac{z^4}{x^2} \right) = \Gamma \log b + O \left(\frac{1}{x^{6/5}} \right). \tag{21}$$

If $\Gamma = 0$, then the above estimate (21) implies that

$$|\lambda_1 k_1^2 + \lambda_2 k_2^2| = O \left(\frac{n_0}{x^{6/5}} \right) = O \left(\frac{1}{x^{1/5}} \right) = o(1)$$

as $x \rightarrow \infty$, showing that $\lambda_1 k_1^2 + \lambda_2 k_2^2 = 0$ for $x > x_0$. This equation together with equations (19) shows that $\lambda_1 = \lambda_2 = \lambda_3 = 0$, which is false. Thus, $\Gamma \neq 0$, showing the right-hand side in estimate (21) is $\gg 1$, and since the left-hand side of estimate (21) is $O(k_2^3/x)$, we get that $k_2 \gg x^{1/3}$. This is not possible for large x because $k_2 \leq z$. We conclude that indeed for large x , I cannot contain three numbers from $\mathcal{N}_{b,D,E}(x)$. This shows that

$$\#\mathcal{N}_{b,D,E}(x) \leq \left\lfloor \frac{x - x/2}{z} \right\rfloor + 2 \ll x^{4/5}.$$

Hence, by estimates (7) and (11), we get that

$$\begin{aligned}\mathcal{N}_b(x) &= \sum_{(D,E) \in \mathcal{D}_b(x) \times \mathcal{E}_b(x)} \#\mathcal{N}_{b,D,E} \ll x^{4/5} \times \#\mathcal{D}_b(x) \times \#\mathcal{E}_b(x) \\ &\leq x^{4/5+\delta+o(1)} < x^{29/30+o(1)} = o(x)\end{aligned}$$

as $x \rightarrow \infty$, which is what we wanted to prove. \square

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