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Effective results for polynomial values of (alternating) power sums of arithmetic progressions

András Bazsó[1,2]

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Abstract
We prove effective finiteness results concerning polynomial values of the sums

$$b^k + (a+b)^k + \dots + (a\,(x-1)+b)^k$$

and

$$b^k - (a+b)^k + (2a+b)^k - \ldots + (-1)^{x-1}\,(a\,(x-1)+b)^k\,,$$

where $a \neq 0, b, k$ are given integers with $\gcd(a, b) = 1$ and $k \geq 2$.

Keywords Diophantine equations · Bernoulli polynomials · Euler polynomials

Mathematics Subject Classification 11D41 · 11B68

1 Introduction

Problems concerning power sums and alternating power sums of consecutive integers has a long history in the literature of combinatorics and number theory. It is well known that the sum

$$S_k(n) = 1^k + 2^k + \ldots + (n-1)^k \tag{1.1}$$

can be expressed by the Bernoulli polynomials $B_k(x)$ as

$$S_k(n) = \frac{1}{k+1}\left(B_{k+1}(n) - B_{k+1}\right), \tag{1.2}$$

where the polynomials $B_k(x)$ are defined by the generating series

$$\frac{t\,e^{tx}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}$$

✉ András Bazsó
bazsoa@science.unideb.hu

[1] Institute of Mathematics, University of Debrecen, P.O. Box 400, Debrecen 4002, Hungary

[2] HUN-REN-UD Equations, Functions, Curves and their Applications Research Group, Debrecen, Hungary

and $B_{k+1} = B_{k+1}(0)$. Hence S_k can be extended to real values x, i.e., to the polynomial

$$\mathrm{S}_k(x) = \frac{1}{k+1}\left(B_{k+1}(x) - B_{k+1}\right). \tag{1.3}$$

It is also well known that the alternating power sum

$$T_k\,(n) := -1^k + 2^k - \cdots + (-1)^{n-1}(n-1)^k$$

can be expressed by means of the classical Euler polynomials $E_k(x)$ via:

$$T_k\,(n) = \frac{E_k(0) + (-1)^{n-1}E_k(n)}{2},$$

where the classical Euler polynomials $E_k(x)$ are usually defined by the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{m=0}^{\infty} E_m(x)\frac{t^m}{m!} \quad (|t| < \pi).$$

For the properties of Bernoulli and Euler polynomials which will be often used in this paper, sometimes without special reference, we refer to the paper of Brillhart [1] and the book of Abramowitz and Stegun [2].

A classical problem of Lucas [3], from 1875, was the study of square values of $\mathrm{S}_k(x)$. Later, in 1956, Schäffer [4] investigated n-th power values, that is, the Diophantine equation

$$\mathrm{S}_k(x) = y^n \quad \text{in integers } x, y. \tag{1.4}$$

For $k \geq 1$, $n \geq 2$ he proved an ineffective finiteness result on the solutions x, y of (1.4) provided that $(k, n) \notin \{(1, 2), (3, 2), (3, 4), (5, 2)\}$. In the exceptional cases (k, n) he proved the existence of infinitely many solutions. Moreover, Schäffer proposed a still unproven conjecture which says that if (k, n) is not in the above exceptional set, then the only nontrivial solution of Eq. (1.4) is $(k, n, x, y) = (2, 2, 24, 70)$. In 1980, Győry, Tijdeman and Voorhoeve [5] proved effective finiteness for the solutions of (1.4) in the general case when, in (1.4), n is also unknown. Several generalizations of (1.4) have been considered, e.g. in the papers of Voorhoeve, Győry and Tijdeman [6, 7], Brindza [8], Dilcher [9] and Urbanowicz [10–12]. Schäffer's conjecture has been confirmed only in a few cases: for $n = 2$ and $k \leq 58$ by Jacobson, Pintér and Walsh [13]; and for $n \geq 2$ and $k \leq 11$ by Bennett, Győry and Pintér [14]. For further generalizations of (1.4) and related results see the survey paper of Győry and Pintér [15] and the references given there.

In [16], Bilu et al. considered the Diophantine equations

$$\mathrm{S}_k(x) = \mathrm{S}_\ell(y), \tag{1.5}$$

and

$$\mathrm{S}_k(x) = y\,(y+1)\,(y+2)\ldots(y+(\ell-1)). \tag{1.6}$$

They proved ineffective finiteness results on the integer solutions x, y of these equations for $k < \ell$, moreover, they established effective statements for certain small values of k and ℓ. E.g., they proved that, for $\ell \in \{2, 4\}$ and $k \geq 2$, all integer solutions x, y of Eq. (1.6) satisfy $\max\{|x|, |y|\} < C(k)$, where $C(k)$ is an effectively computable constant depending only on k.

For a positive integer $n \geq 2$ and for $a \neq 0$, b coprime integers, let

$$S_{a,b}^k\,(n) = b^k + (a+b)^k + (2a+b)^k + \ldots + (a\,(n-1)+b)^k\,. \tag{1.7}$$

Further, let

$$T_{a,b}^{k}\left(n\right)=b^{k}-\left(a+b\right)^{k}+\left(2a+b\right)^{k}-\ldots+\left(-1\right)^{n-1}\left(a\left(n-1\right)+b\right)^{k}. \tag{1.8}$$

Howard [17] showed, by means of generating functions, that the above power sums are related to the Bernoulli and Euler polynomials in the following ways

$$S_{a,b}^{k}\left(n\right)=\frac{a^{k}}{k+1}\left(B_{k+1}\left(n+\frac{b}{a}\right)-B_{k+1}\left(\frac{b}{a}\right)\right), \tag{1.9}$$

$$T_{a,b}^{k}\left(n\right)=\frac{a^{k}}{2}\left(E_{k}\left(\frac{b}{a}\right)+\left(-1\right)^{n-1}E_{k}\left(n+\frac{b}{a}\right)\right), \tag{1.10}$$

respectively.

Thus we can extend $S_{a,b}^{k}$ for every real value x as

$$\mathtt{S}_{a,b}^{k}\left(x\right)=\frac{a^{k}}{k+1}\left(B_{k+1}\left(x+\frac{b}{a}\right)-B_{k+1}\left(\frac{b}{a}\right)\right), \tag{1.11}$$

and, depending on the power of -1 in (1.10), the following polynomial extensions arise for $T_{a,b}^{k}\left(n\right)$:

$$\mathtt{T}_{a,b}^{k+}(x)=\frac{a^{k}}{2}\left(E_{k}\left(\frac{b}{a}\right)+E_{k}\left(x+\frac{b}{a}\right)\right), \tag{1.12}$$

$$\mathtt{T}_{a,b}^{k-}(x)=\frac{a^{k}}{2}\left(E_{k}\left(\frac{b}{a}\right)-E_{k}\left(x+\frac{b}{a}\right)\right). \tag{1.13}$$

Clearly, for positive integer values x, we have $\mathtt{T}_{a,b}^{k+}(x)=T_{a,b}^{k}\left(x\right)$ if x is odd, and $\mathtt{T}_{a,b}^{k-}(x)=T_{a,b}^{k}\left(x\right)$ if x is even.

In [18], Kreso, Luca, Pintér and the present author generalized the results of Bilu et al. [16] on Eq. (1.5) to the equation

$$\mathtt{S}_{a,b}^{k}\left(x\right)=\mathtt{S}_{c,d}^{\ell}\left(y\right) \tag{1.14}$$

where x, y are unknown integers, and k, ℓ, a, b, c, d are given positive integers with $k<\ell$, $\gcd(a,b)=\gcd(c,d)=1$. Recently, in [19], Kreso, Luca, Pintér, Rakaczki and the author gave further generalizations of the results of [16] to the equation

$$\mathtt{S}_{a,b}^{k}(x)=y\left(y+c\right)\left(y+2c\right)\ldots\left(y+\left(\ell-1\right)c\right) \tag{1.15}$$

where x, y are unknown integers, and k, ℓ, a, b, c are given positive integers with $\gcd(a,b)=1$.

The Diophantine equations

$$\mathtt{S}_{a,b}^{k}\left(x\right)=g(y), \tag{1.16}$$

$$\mathtt{T}_{a,b}^{k+}\left(x\right)=g(y) \tag{1.17}$$

and

$$\mathtt{T}_{a,b}^{k-}\left(x\right)=g(y), \tag{1.18}$$

where $g(y)\in\mathbb{Q}[y]$ with $\deg g\geq 3$ have been investigated in the literature first in the case $(a,b)=(1,0)$. Rakaczki [20] and independently Kulkarni and Sury [21] characterized those pairs $(k, g(y))$ for which Eq. (1.16) has infinitely many integer solutions. Kreso and Rakaczki

[22] proved an analogous result for Eqs. (1.17) and (1.18). For further related results we refer to the papers of Kulkarni and Sury [23, 24], and of Bennett [25].

The results of [20–22] have been extended to the general case, i.e, to Eqs. (1.16)–(1.18) by the present author in [26]. We note that all the mentioned results on Eqs. (1.16)–(1.18) were ineffective as their proofs mainly relied on the finiteness criterion of Bilu and Tichy [27], and on the decomposition properties of the polynomials involved (which were described in the papers [16, 28, 29]). For a detailed discussion on the theory of polynomial decomposition we refer to the monograph of Schinzel [30].

In the present paper we consider Eqs. (1.16)–(1.18) in the cases when $g(y) \in \mathbb{Q}[y]$ with $\deg g = 2$, and for $g(y) = cy^{\ell} + d$ with $c, d \in \mathbb{Q}$, $c \neq 0$ and $\ell \geq 2$. In the quadratic case we provide effective upper bounds on the solutions (x, y), while in the latter case we also let ℓ be unknown, and prove effective finiteness for the triple (x, y, ℓ) as well. Our results have important special cases. They imply, e.g., that there are only finitely many power values of $\mathrm{S}_{a,b}^{k}(x)$ which can be considered as a generalization of the results of [5]. Further, our Theorems 2.1, 2.2, 2.3 extend the main results of [26] in an effective way to the quadratic case.

2 Main results

Let $a \neq 0, b, k \in \mathbb{Z}$, $\gcd(a, b) = 1$, and let $g \in \mathbb{Q}[x]$ be an arbitrary polynomial. Consider the Diophantine equations

$$\mathrm{S}_{a,b}^{k}(x) = g(y), \tag{1.19}$$

$$\mathrm{T}_{a,b}^{k+}(x) = g(y) \tag{1.20}$$

and

$$\mathrm{T}_{a,b}^{k-}(x) = g(y), \tag{1.21}$$

in $x, y \in \mathbb{Z}$.

First we consider the case when g is a quadratic polynomial. On Eq. (1.16) we prove the following.

Theorem 2.1 *Let a, b, k and g be as above. Then for $k \geq 2$, $k \notin \{3, 5\}$, and $\deg g = 2$, there exists an effectively computable constant $C_1(a, b, k, g)$ depending only on a, b, k and g such that $\max(|x|, |y|) < C_1(a, b, k, g)$ for each integer solutions of Eq. (1.16).*

As we mentioned above, this result extends Theorem 2.1 of [26] in an effective way.

In the alternating case, i.e., for Eqs. (1.17),(1.18), we have

Theorem 2.2 *Let a, b, k and g be as above. Then for $k \geq 7$ and $\deg g = 2$, there exists an effectively computable constant $C_2(a, b, k, g)$ depending only on a, b, k and g such that $\max(|x|, |y|) < C_2(a, b, k, g)$ for each integer solutions of Eq. (1.17).*

Theorem 2.3 *Let a, b, k and g be as above. Then for $k \geq 7$ and $\deg g = 2$, there exists an effectively computable constant $C_3(a, b, k, g)$ depending only on a, b, k and g such that $\max(|x|, |y|) < C_3(a, b, k, g)$ for each integer solutions of Eq. (1.18).*

Analogously to Theorem 2.1, the above two theorems extend Theorem 2.2 and 2.3 of [26], respectively.

Now we turn our attention to the case when, in Eqs. (1.16)–(1.18), the polynomial g is linear in some power of y. More precisely, we consider the equations

$$\mathtt{S}_{a,b}^{k}(x) = cy^{\ell} + d, \tag{2.1}$$

$$\mathtt{T}_{a,b}^{k+}(x) = cy^{\ell} + d \tag{2.2}$$

and

$$\mathtt{T}_{a,b}^{k-}(x) = cy^{\ell} + d, \tag{2.3}$$

in integers x, y and $\ell \geq 2$, where $c, d \in \mathbb{Q}$ with $c \neq 0$.

In this case we can also bound the exponent ℓ from above together with x and y.

Theorem 2.4 *Let a, b, c, d, k and ℓ be as above. Then for $k \geq 2$, $k \notin \{3, 5\}$, there exists an effectively computable constant $C_4(a, b, c, d, k)$ depending only on a, b, c, d and k such that $\max(|x|, |y|, \ell) < C_4(a, b, c, d, k)$ for each integer solutions of Eq. (2.1) with $|y| > 1$.*

In the special case $c = 1, d = 0$, our Theorem 2.4 implies effective finiteness for the power values of $\mathtt{S}_{a,b}^{k}(x)$, i.e. for a generalization of Schäffer's Eq. (1.4). It can be considered as an extension of Theorem 1 of [5] as well. We note that, in the above mentioned paper [19], we also proved effective finiteness for the power values of $\mathtt{S}_{a,b}^{k}(x)$ in a slightly more general case $k \geq 1, \ell \geq 2$ (cf. Theorem 1.1 in [19]). The reader can find a detailed analysis of the exceptional cases $k \in \{1, 3, 5\}$ there.

We prove analogous results in the alternating case.

Theorem 2.5 *Let a, b, c, d, k and ℓ be as above. Then for $k \geq 7$, there exists an effectively computable constant $C_5(a, b, c, d, k)$ depending only on a, b, c, d and k such that $\max(|x|, |y|, \ell) < C_5(a, b, c, d, k)$ for each integer solutions of Eq. (2.2) with $|y| > 1$.*

Theorem 2.6 *Let a, b, c, d, k and ℓ be as above. Then for $k \geq 7$, there exists an effectively computable constant $C_6(a, b, c, d, k)$ depending only on a, b, c, d and k such that $\max(|x|, |y|, \ell) < C_6(a, b, c, d, k)$ for each integer solutions of Eq. (2.3) with $|y| > 1$.*

3 Proofs of the Theorems

In this section, we prove our theorems. In our arguments we need some lemmas and notation. First we recall two classical effective results on hyper- and superelliptic equations.

Let $f(x) \in \mathbb{Z}[x]$ be a nonzero polynomial of degree d. Write H for the naive height (i.e., the maximum of the absolute values of the coefficients) of f. Further, let α be a nonzero rational number. Consider the Diophantine equation

$$f(x) = \alpha y^{N}. \tag{3.1}$$

The following result is a special case of a result of Bérczes, Brindza and Hajdu [31]. For the first results of this type, we refer to Schinzel and Tijdeman [32] and Tijdeman [33].

Lemma 3.1 *If $f(x)$ has at least two distinct roots and $|y| > 1$, then, in (3.1), we have $N < C_7(d, H, \alpha)$, where $C_7(d, H, \alpha)$ is an effectively computable constant depending only on d, H and α.*

The next result is a special case of an effective theorem of Brindza [34]. To formulate it, we need further notation. For a finite set of rational primes S, let $\mathbb{Z}_S$ denote the set of rational numbers whose denominator (in reduced form) has no prime divisor outside S. By the *height* $h(s)$ of a rational number $s = u/v$ with $u, v \in \mathbb{Z}$, $\gcd(u, v) = 1$, we mean $h(s) = \max\{|u|, |v|\}$.

Lemma 3.2 *Let S be a finite set of rational primes. If, in (3.1), either $N = 2$ and $f(x)$ has at least three roots of odd multiplicity, or $N \geq 3$ and $f(x)$ has at least two roots of multiplicities coprime to N, then for each solutions $x, y \in \mathbb{Z}_S$ of (3.1) we have* $\max(h(x), h(y)) < C_8(d, H, S, \alpha, N)$*, where $C_8(d, H, S, \alpha, N)$ is an effectively computable constant depending only on d, H, S, α and N.*

We recall the following result concerning Bernoulli polynomials $B_k(x)$ which is due to Brillhart [1].

Lemma 3.3 *If k is odd, then $B_k(x)$ has no multiple roots. For even k, the only polynomial which can be a multiple factor of $B_k(x)$ over $\mathbb{Q}$ is $x^2 - x - \beta$, where β is an odd, positive integer. Further, the multiplicity of any multiple root of B_k is* 2*.*

The following result, which provides information on the root structure of shifted Bernoulli polynomials, will be crucial in applying Lemma 3.2 in the proofs of Theorems 2.1 and 2.4.

Lemma 3.4 (Lemma 2.2 in [18]) *For every $s \in \mathbb{Q}$ and rational integer $k \geq 3$ with $k \notin \{4, 6\}$ the polynomial $B_k(x) + s$ has at least three roots of odd multiplicity.*

Now we are in position to prove our Theorems 2.1 and 2.4.

Proof of Theorem 2.1 Let $k \geq 2$, $k \notin \{3, 5\}$, and let $g \in \mathbb{Q}[x]$ be an arbitrary polynomial with $\deg g = 2$. Assume that Eq. (1.16) holds. Then there exist rational numbers A, B, C with $A \neq 0$ such that

$$g(x) = Ax^2 + Bx + C. \tag{3.2}$$

Obviously, we can rewrite (1.16) as

$$S_{a,b}^k(x) + \nu = A(y + \mu)^2, \tag{3.3}$$

where $\mu = \frac{B}{2A}$ and $\nu = \frac{B^2-4AC}{4A}$. Thus, to bound x, y, in view of Lemma 3.2, it is sufficient to show that the polynomial $S_{a,b}^k(x) + \nu$ has at least three roots of odd multiplicity. By (1.11), we have

$$S_{a,b}^k(x) + \nu = \frac{a^k}{k+1}\left(B_{k+1}\left(x + \frac{b}{a}\right) + s\right) \tag{3.4}$$

with $s = -B_{k+1}\left(\frac{b}{a}\right) + \frac{(k+1)\nu}{a^k} \in \mathbb{Q}$, and, since the number of roots as well as the multiplicities of the roots of a polynomial remain unchanged if we replace its variable by a linear polynomial of that, or if we multiply the polynomial by a nonzero constant, Lemma 3.4 implies the existence of three roots of odd multiplicity for $S_{a,b}^k(x) + \nu$. This completes the proof. □

Proof of Theorem 2.4 We rewrite Eq. (2.1) as

$$S_{a,b}^k(x) - d = cy^\ell. \tag{3.5}$$

To prove that ℓ is bounded, by Lemma 3.1, we only need to show that the polynomial $\mathrm{S}_{a,b}^{k}(x) - d$ has two distinct roots for every $d \in \mathbb{Q}$. Assuming to the contrary, for some $d \in \mathbb{Q}$, we have

$$\mathrm{S}_{a,b}^{k}(x) - d = R(Ux + V)^{k+1} \tag{3.6}$$

with some $R, U, V \in \mathbb{Q}$. However, as $k \geq 2$, this implies that the derivative

$$(\mathrm{S}_{a,b}^{k}(x) - d)' = \frac{a^k}{k+1} B'_{k+1}\left(x + \frac{b}{a}\right) = a^k B_k\left(x + \frac{b}{a}\right) \tag{3.7}$$

has a rational root of multiplicity k. This contradicts Lemma 3.3, whence ℓ is bounded as required. (Indeed, for $k \leq 3$, then the Bernoulli polynomial B_k would have a rational root of multiplicity at least 3, and for $k = 2$, B_2 would have a double rational root which are both impossible.)

Now we give a bound for $\max(|x|, |y|)$. For $\ell = 2$, the statement is a straightforward consequence of Theorem 2.1. In the sequel, let $\ell \geq 3$. By what we have already proved, we may assume that ℓ is fixed. Further, in view of Theorem 2.1, we may suppose that ℓ is odd. Clearly, without loss of generality we may assume that in fact ℓ is an odd prime. By (3.5) and Lemma 3.2, it suffices to show that the polynomial on the left hand side of (3.5) has at least two roots of multiplicities coprime to ℓ. Suppose to the contrary, i.e., that we have

$$\mathrm{S}_{a,b}^{k}(x) - d = (Kx + L)(w(x))^{\ell}, \tag{3.8}$$

with some $K, L \in \mathbb{Q}$, $w \in \mathbb{Q}[x]$. Taking derivatives in (3.8), by (3.7), we obtain

$$a^k B_k\left(x + \frac{b}{a}\right) = w(x)^{\ell-1}\left(Kw(x) + \ell(Kx + L)w'(x)\right). \tag{3.9}$$

Thus, every root of $w(x)$ is a root of $a^k B_k\left(x + \frac{b}{a}\right)$ of multiplicity at least $\ell - 1 \geq 2$. This immediately contradicts Lemma 3.3 if $\ell \geq 4$. In the case $\ell = 3$, Lemma 3.3 implies $w(x) = (x^2 - x - \beta)$ with an odd positive integer β. Then, we obtain from (3.9) that $k = 6$ and that $B_6\left(x + \frac{b}{a}\right)$ should have a multiple root, which is a contradiction. This completes the proof. □

In the sequel, we turn to the alternating case, i.e., to the proofs of Theorems 2.2,2.3 and 2.5,2.6. Before proving our results, besides the abovementioned auxiliary results, we need the following two lemmas. The first is a deep result of Rakaczki [35] concerning the root structure of shifted Euler polynomials.

Lemma 3.5 *Let $k \geq 7$ be an integer. Then the shifted Euler polynomial $E_k(x) + z$ has at least three simple zeros for arbitrary complex number z.*

The following result, which describes the multiple roots of Euler polynomials $E_k(x)$ is due to Brillhart [1].

Lemma 3.6 *If k is even, then $E_k(x)$ has no multiple roots. For odd k, the only polynomial which can be a multiple factor of $E_k(x)$ over $\mathbb{Q}$ is $x^2 - x - 1$. Further, the multiplicity of any multiple root of E_k is 2.*

For simplicity we prove Theorems 2.2 and 2.3 jointly. We introduce the following notation. Let $\mathrm{T}_{a,b}^{k\pm}(x) \in \left\{\mathrm{T}_{a,b}^{k+}(x), \mathrm{T}_{a,b}^{k-}(x)\right\}$.

Proof of Theorems 2.2 and 2.3 Let $g \in \mathbb{Q}[x]$ be an arbitrary polynomial with $\deg g = 2$, and assume that Eqs. (1.17) or (1.18) holds. We adapt the argument of the proof of Theorem 2.1 to our case. There exist rational numbers A, B, C with $A \neq 0$ such that $g(x) = Ax^2 + Bx + C$, and one can rewrite (1.17) or (1.18) as

$$\mathrm{T}_{a,b}^{k\pm}(x) + \nu = A(y+\mu)^2, \tag{3.10}$$

with $\mu = \frac{B}{2A}$ and $\nu = \frac{B^2-4AC}{4A}$. Thus, to bound x, y, in view of Lemma 3.2, it is sufficient to show that the polynomial on the left hand side of (3.10) has at least three roots of odd multiplicity. By (1.12), we have

$$\mathrm{T}_{a,b}^{k\pm}(x) + \nu = \pm\frac{a^k}{2}\left(E_k\left(x+\frac{b}{a}\right)+s\right) \tag{3.11}$$

with $s = E_k\left(\frac{b}{a}\right) + \frac{2\nu}{a^k} \in \mathbb{Q}$, whence, as $k \geq 7$, Lemma 3.5 implies the existence of three simple roots for $\mathrm{T}_{a,b}^{k\pm}(x) + \nu$, which completes the proof. □

Similarly, we prove Theorems 2.5 and 2.6 jointly by using the above notation.

Proof of Theorems 2.5 and 2.6 We adapt the argument of the proof of Theorem 2.4. Rewriting Eqs. (2.2) or (2.3) as

$$\mathrm{T}_{a,b}^{k\pm}(x) - d = cy^{\ell}, \tag{3.12}$$

in view of Lemma 3.1, we can prove that ℓ is bounded if we can show that the polynomial $\mathrm{T}_{a,b}^{k\pm}(x) - d$ has two distinct roots for every $d \in \mathbb{Q}$. Let us assume to the contrary. Then, for some $d \in \mathbb{Q}$, we have

$$\mathrm{T}_{a,b}^{k\pm}(x) - d = R(Ux+V)^k \tag{3.13}$$

with some $R, U, V \in \mathbb{Q}$. However, as $k \geq 7$, this implies that the derivative

$$(\mathrm{T}_{a,b}^{k\pm}(x) - d)' = \pm\frac{a^k}{2}E_k'\left(x+\frac{b}{a}\right) = \pm\frac{ka^k}{2}E_{k-1}\left(x+\frac{b}{a}\right) \tag{3.14}$$

has a rational root of multiplicity at least 6, which contradicts Lemma 3.6. Hence ℓ is bounded as required.

If $\ell = 2$, the effective upper bound on $\max(|x|, |y|)$, in Eqs. (2.2) or (2.3), follows directly from Theorems 2.2 and 2.3, respectively. Let $\ell \geq 3$, and by the first part of the proof, we may assume that ℓ is fixed. Further, in view of Theorems 2.2 and 2.3, we may suppose, without loss of generality, that ℓ is an odd prime. By (3.12) and Lemma 3.2, it suffices to show that the polynomial on the left hand side of (3.12) has at least two roots of multiplicities coprime to ℓ. Assuming to the contrary, we have

$$\mathrm{T}_{a,b}^{k\pm}(x) - d = (Kx+L)(w(x))^{\ell}, \tag{3.15}$$

with some $K, L \in \mathbb{Q}$, $w \in \mathbb{Q}[x]$. Taking derivatives in (3.15), by (3.14), we obtain

$$\pm\frac{ka^k}{2}E_{k-1}\left(x+\frac{b}{a}\right) = w(x)^{\ell-1}\left(Kw(x) + \ell(Kx+L)w'(x)\right). \tag{3.16}$$

Thus, every root of $w(x)$ is a root of $\pm\frac{ka^k}{2}E_{k-1}\left(x+\frac{b}{a}\right)$ of multiplicity at least $\ell - 1 \geq 2$. This immediately contradicts Lemma 3.6 if $\ell \geq 4$. In the case $\ell = 3$, Lemma 3.6 implies $w(x) = x^2 - x - 1$. Then, we infer from (3.16) that $k = 7$ and that $E_6\left(x+\frac{b}{a}\right)$ should have a multiple root, which is a contradiction again by Lemma 3.6. This completes the proof. □

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