



**Neighbourhood sequences and character
recognition by Walsh transformation**

doktori (PhD) értekezés

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Tanúsítom, hogy Hajdu András doktorjelölt 1997 – 2000 között a fent megnevezett Doktori Iskola Informatika programjának keretében irányítással végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységgel meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

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Dr. Kormos János

To my wife and son

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Abstract

This PhD dissertation contains new results in the fields of digital image processing and digital geometry. We made an effort to present results that can be interesting from both theoretical and practical points of view to the reader. The dissertation consists of two parts corresponding to two areas of digital image processing. We present some new results in the theory of neighbourhood sequences, and a character recognition algorithm based on Walsh transformation. The first part was composed mainly in theoretical, while the second mainly in practical respect which is also reflected in their structures. Thus in the first part we use the conventional mathematical outline and notation (numbering, typeface, definition, theorem, proof, etc.), while the second part was composed in a more informal style. The structures of the parts are similar: we review the literature of the given topic, aim at the results we want to reach, present the results of our investigations, and finally finish by discussing possible further research. Throughout the dissertation we always indicate those results which are already known from literature.

In the first part we examine the properties of neighbourhood sequences which provide a tool to generalise the classical neighbourhood relations that are used in many places in digital geometry and topology. The mapping of the continuous (Euclidean) space to a digital grid frequently arises in digital image processing. An important research area within this topic is the approximation of the Euclidean metric by digital ones. Since neighbourhood sequences generate digital distance functions (sometimes metrics), they can be applied here well. The theory of neighbourhood sequences can

be handled nicely and easily, so beside the most usual digital grids it can be simply extended to other ones which improves potential applicability, as well. We study neighbourhood sequences from algebraic, metrical and geometric points of view. As an illustration of the applicability of our theoretical results, we show some possibilities of approximating the Euclidean metric using neighbourhood sequences. The new results include the generalisation of the concept of neighbourhood sequences, the extension of the structural analysis to arbitrary dimension, the introduction of velocity and metric on the set of neighbourhood sequences, the extension of the 2-dimensional (2D) geometric investigations to higher dimensions and the approximation of the 2D Euclidean metric by distance functions generated by neighbourhood sequences. These results are published in the papers [16], [21], [19] and [20], respectively.

In the second part of the dissertation we describe an own character recognition algorithm that is based on the well-known Walsh transformation which is frequently applied in several fields of digital image processing. Since this transformation is invariant for magnification and preserves geometry nicely, it is also well applicable for character recognition. To prove the reliability and efficiency of our method, we perform a noise tolerance test and compare recognition accuracy against other well-known character recognition procedures. To illustrate the applicability of our character recognition algorithm, we give the example of compression typeset documents. Finally, we discuss some possibilities to increase the accuracy rate of character recognition algorithms. Our algorithm and its application is published in the papers [15] and [14], respectively.

Part I

Neighbourhood sequences

Chapter 1

Introduction

We give a broad overview of results from the fields of digital image processing and digital topology that can be considered as the basis and precedents of our investigations on neighbourhood sequences. Furthermore, we present more specific known results about neighbourhood sequences, and embed our corresponding results into the literature.

In digital applications it is often very useful to have an appropriate (digital) distance function on the digital grid the application is based on. Thus the investigation of digital distance functions and metrics becomes more and more important. See the survey paper [28] of MELTER for an account. According to practical applicability many studies are restricted to two dimensions. MELTER and TOMESCU [29] investigated path generated digital metrics, while HARARY et al. [22] graph generated digital distances. For three and higher dimensional analysis we can mention the paper of OKABE et al. [34] about the properties of paths, and the paper of BORGEFORS [1] with certain generalizations of former two dimensional results. YAMASHITA and HONDA [43] found a condition for a sequence of neighbourhoods to generate digital metrics. YAMASHITA and IBARAKI [44] extended and generalized the results of [43] using a general neighbourhood definition. Investigating a special case of the results in [44], DAS, CHAKRABARTI, CHATTERJI and MUKHERJEE published several papers [3] – [10]

in this topic.

One of the most essential tasks of this area is to give a convenient digital metric which approximates the Euclidean metric well. Many of the above mentioned papers also contain such results, like [3], [7], [8], [9], [10], and [44]. A detailed description about the basic concepts of digital topology can be found in [24] and [42].

The theory of neighbourhood sequences is an important part of this topic. By the help of such sequences, we can describe planar (or higher dimensional) movements. The classical digital – cityblock and chessboard – motions were introduced by ROSENFELD and PFALTZ [36]. The cityblock motion allows movements only in horizontal and vertical directions, while the chessboard motion in diagonal directions, as well. Based on these two types of motions, the authors in [36] defined two distances. The d_4 or d_8 distance of two points is the number of steps required to reach one point from the other, where only cityblock or chessboard motions can be used, respectively. To obtain a better approximation of the Euclidean distance, ROSENFELD and PFALTZ recommended the alternate use of the cityblock and chessboard motions, which defines the distance d_{oct} . It was also pointed out that other combinations of the cityblock and chessboard motions may give better distance functions.

By allowing arbitrary periodic mixture of the cityblock and chessboard motions, DAS et al. [5] introduced the concept of periodic (cyclic) neighbourhood sequences, and generalised it to arbitrary dimension. We note that the concept of periodic neighbourhood sequences is actually a specialization of the N -sequences introduced by YAMASHITA and IBARAKI in [44]. Moreover, the authors in [5] established a formula for calculating the distance of two points in a finite dimensional digital space, determined by such a periodic neighbourhood sequence. Using this formula, DAS in [4] showed that on the set of periodic 2D-neighbourhood sequences a “natural” partial ordering relation can be introduced. Furthermore, he investigated the structure of this set and some of its subsets with respect to this relation. More precisely, he proved that under this ordering, the set of l -periodic 2D-neighbourhood sequences forms a distributive lattice. Recently, FAZEKAS in [13] has proved that a similar partial ordering can also be introduced for

neighbourhood sequences in 3D.

In the first part of the dissertation we add new results to these investigations. We generalise the concept of neighbourhood sequences by allowing not periodic sequences only. We show that the results of DAS [4] and FAZEKAS [13], about ordering the set of periodic neighbourhood sequences, can be extended to arbitrary finite dimension, even using these general sequences. Furthermore, we investigate the structure of the set and some subsets of these general neighbourhood sequences under this ordering. We involve into our investigations all the sets studied by DAS [4] in the periodic case. Unfortunately, in several cases we obtain negative results: some of the structures considered do not have nice properties. However, for the sake of completeness we provide these results, too. Instead of this “natural” partial ordering we propose another relation which is in close connection with the original one. More precisely, the “natural” ordering is a refinement of the relation introduced here. We show that under this new ordering, the examined sets usually form lattices with nice properties.

We extend all these results to ∞D , which is the most interesting case theoretically. Moreover, the lattice obtained in ∞D under the new ordering relation is the closure of the union of the finite dimensional lattices, in some sense. These results are also presented in [16], joint with A. FAZEKAS and L. HAJDU.

The “natural” partial ordering has some unpleasant structural properties (it fails to be a complete ordering) on the set of neighbourhood sequences. However, in certain applications it can be useful to compare any two neighbourhood sequences, i.e. to decide which one spreads “faster”. For this purpose we introduce a norm-like concept for neighbourhood sequences, called velocity, and investigate its properties. The concept is introduced in a way to fit the relation “faster”. Furthermore, we define a metric for neighbourhood sequences. We also work out a possible application scheme for distributing information. The obtained results are also contained in [21] which is joint with L. HAJDU.

The main advantage of neighbourhood sequences over the classical distances d_4 , d_8 and d_{oct} is that they provide more flexibility in moving on the plane. Geometrically, the corresponding “disks” are diamonds for the

distance d_4 , squares for d_8 , and octagons for d_{oct} . DAS and CHATTERJI [9] investigated the quality of the approximation of the Euclidean distance by a periodic neighbourhood sequence in a geometric sense. To this purpose they analysed the geometric properties of the octagons occupied by a periodic neighbourhood sequence during “spreading” on the 2D plane.

As we mentioned above, the approximation of the Euclidean metric with digital distances is an important and extensively investigated research area. Using general neighbourhood sequences (which are not required to be periodic) Euclidean distance can be approximated more precisely. For this reason, we extend the geometric description of 2D periodic neighbourhood sequences presented by DAS and CHATTERJI in [9] to general neighbourhood sequences, and embed the former results into this more general structure. Moreover, we perform our analysis in arbitrary finite dimension instead of $\mathbb{Z}^2$, where $\mathbb{Z}$ stands for the set of the integer numbers. We give the coordinates of vertices of polyhedra occupied by neighbourhood sequences, and perform the symmetry and convexity analysis of these bodies. Since 2D and 3D square grids play a very important role in digital image processing, we investigate the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$ in detail. These results are also presented in [19].

DAS in [3] determined distance functions induced by periodic neighbourhood sequences that provide good approximations of the Euclidean distance in a certain sense. This analysis is a special case of the one presented in [44], moreover, DAS restricted his investigations to the so called simple metrics.

We perform approximations of the 2D Euclidean distance by distance functions based on general neighbourhood sequences. To approximate the Euclidean distance on $\mathbb{Z}^2$, we compare the disks of positive integer radii of the Euclidean distance and the distance functions corresponding to the neighbourhood sequences. In contrast with [3], we involve the whole set of neighbourhood sequences into our investigations. Interestingly, the sequences we obtain are (mostly) Beatty sequences, thus they can be constructed very easily. For every type of approximation under consideration we give neighbourhood sequences such that the corresponding distance functions are metrics on $\mathbb{Z}^2$. Thus we get good approximations of the Euclidean dis-

tance by digital metrics. In particular, we determine the digital metric that best approximates the Euclidean one from below “uniformly”, i.e. independently of the sense of approximation considered. Our results concerning the approximation of the Euclidean distance by digital metrics are joint with L. HAJDU and also can be found in [20].

The structure of the first part of the dissertation is as follows. In Chapter 2 we give our notation, and provide some properties of the concepts introduced. From Chapter 3 on we present our new results. In Chapter 3 we examine the lattice structure of the set of neighbourhood sequences with respect to the “natural” partial ordering in finite dimension, and perform the same analysis in ∞D , as well (see also [16]). In Chapter 4 we introduce the concept of velocity and define a metric on the set of neighbourhood sequences, and present an application scheme of using velocity for distributing information in a general network model (c.f. [21], too). In Chapter 5 we extend the former 2D geometric results of DAS and CHATTERJI [9] to arbitrary finite dimension, and give a detailed and illustrated description of the geometric behaviour of neighbourhood sequences in $\mathbb{Z}^2$ and $\mathbb{Z}^3$ (see also [19]). In Chapter 6 we formulate three problems about approximating the 2D Euclidean metric with digital distance functions coming from neighbourhood sequences, then we solve these problems by giving the best approximating neighbourhood sequences. The sequence we suggest to approximate the Euclidean distance is compared with that recommended by DAS in [3], see also [20]. Finally, in Chapter 7 we close our investigations with some concluding remarks, and discuss the possible extensions and applicability of our results.

Chapter 2

Basic concepts and notation

We give the basic definitions and notation in this chapter. From now on, n will denote an arbitrary positive integer, while $\mathbb{N}$ and $\mathbb{R}$ will stand for the set of natural and real numbers, respectively.

Definition 2.1. Let p be a point in $\mathbb{Z}^n$. The i -th coordinate of p is indicated by $\text{Pr}_i(p)$ ($i = 1, \dots, n$). Let M be an integer with $0 \leq M \leq n$. The points $p, q \in \mathbb{Z}^n$ are M -neighbours, if the following two conditions hold:

- $|\text{Pr}_i(p) - \text{Pr}_i(q)| \leq 1 \quad (1 \leq i \leq n),$
- $\sum_{i=1}^n |\text{Pr}_i(p) - \text{Pr}_i(q)| \leq M.$

Definition 2.2. The sequence $A = (A(i))_{i=1}^{\infty}$, where $A(i) \in \{1, \dots, n\}$ for all $i \in \mathbb{N}$, is called a general n -dimensional (shortly nD) neighbourhood sequence. If for some $l \in \mathbb{N}$, $A(i+l) = A(i)$ ($i \in \mathbb{N}$), then A is periodic with period l . In this case we briefly write $A = (A(1), A(2), \dots, A(l))$. The set of the general nD -neighbourhood sequences will be denoted by S_n .

Remark 2.3. We note that the above concept of the general nD -neighbourhood sequences is actually a generalization of the notion of neighbourhood sequences introduced in [5]. The authors in [3], [4], [5], [6], [9] and [13] dealt only with periodic sequences. For the sake of simplicity, from now on we omit the attribute “general”.

Definition 2.4. Let $p, q \in \mathbb{Z}^n$ and $A \in S_n$. The point sequence $p = p_0, p_1, \dots, p_m = q$, where p_{i-1} and p_i are $A(i)$ -neighbours in $\mathbb{Z}^n$ ($1 \leq i \leq m$), is called an A -path from p to q of length m . The A -distance $d(p, q; A)$ of p and q is defined as the length of the shortest A -path(s) between them. As a brief notation, we also use $d(A)$ for the A -distance.

Remark 2.5. Clearly, there always exist A -paths from p to q . If more “shortest” paths exist, the distance between p and q is defined as the common length of these paths.

Using the above distance we cannot obtain a metric in $\mathbb{Z}^n$ for every n D-neighbourhood sequence. In order to prove this, consider the following simple example.

Example 2.6. Let $A = (2, 1) \in S_2$, and $p = (0, 0)$, $q = (1, 1)$ and $r = (2, 2) \in \mathbb{Z}^2$. Then $d(p, q; A) = 1$, $d(q, r; A) = 1$, but $d(p, r; A) = 3$. So $d(p, q; A)$ fails to be a metric, since the triangular inequality does not hold.

We have the following natural question: knowing A , how can we decide whether the distance function related to A is a metric on the n -dimensional digital plane, or not? The answer in the periodic case can be found in [5], while NAGY extended this result to general neighbourhood sequences in [30]. Here we formulate NAGY’s result only in 2D, as we will use only this in our investigations.

Theorem 2.7. (see [30]) Let $A \in S_2$. $d(A)$ is a metric if and only if for any $s, t \in \mathbb{N}$

$$\sum_{i=1}^s A(i) \leq \sum_{i=t}^{s+t-1} A(i).$$

Later on we need an explicit formula to calculate the A -distance of two points of $\mathbb{Z}^n$. The following lemma provides such a tool. Before formulating the lemma, we need to introduce some further notation.

Notation 2.8. Let $p, q \in \mathbb{Z}^n$, and $A = (A(i))_{i=1}^\infty \in S_n$. Let

$$||p - q|| = \sum_{h=1}^n |\text{Pr}_h(p) - \text{Pr}_h(q)|,$$

and for every $i \in \mathbb{N}$ and $j \in \{1, \dots, n\}$ put

$$A^{(j)}(i) = \min(A(i), j), \quad \text{and} \quad f_j^A(i) = \sum_{k=1}^i A^{(j)}(k).$$

Let

$$x = (x(1), x(2), \dots, x(n))$$

be the nonincreasing ordering of the numbers $|\text{Pr}_i(p) - \text{Pr}_i(q)|$ with $i = 1, \dots, n$, that is, $x(i_1) \geq x(i_2)$ if $i_1 < i_2$. For $j = 1, \dots, n$ put

$$a_j = \sum_{k=1}^{n-j+1} x(k).$$

Finally, for any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer which is less than or equal to x , and $\lceil x \rceil$ the smallest integer which is greater than or equal to x .

DAS et al. in [5] provided an algorithm to calculate $d(A)$, where A is a periodic nD -neighbourhood sequence. By the following lemma we extend this result to any neighbourhood sequence belonging to S_n .

Lemma 2.9. *Let $p, q \in \mathbb{Z}^n$, and $A \in S_n$. Write $c = \|p - q\|$, and let*

$$g_j^A(i) = f_j^A(c) - f_j^A(i-1) - 1, \quad i = 1, \dots, c.$$

Then the A -distance of p and q can be calculated as

$$d(p, q; A) = \max_{j=1}^n d_j^A(p, q),$$

where

$$d_j^A(p, q) = \sum_{i=1}^c \left\lfloor \frac{a_j + g_j^A(i)}{f_j^A(c)} \right\rfloor.$$

Proof. As we mentioned, a similar statement was formulated by DAS et al. in [5]. They showed that in case of a periodic neighbourhood sequence with period l the statement of the lemma holds with

$$g_j^A(i) = f_j^A(l) - f_j^A(i-1) - 1, \quad i = 1, \dots, l$$

and

$$d_j^A(p, q) = \sum_{i=1}^l \left\lfloor \frac{a_j + g_j^A(i)}{f_j^A(l)} \right\rfloor.$$

Suppose that A is any n D-neighbourhood sequence. It is obvious that to calculate the A -distance of the points $p, q \in \mathbb{Z}^n$, it is enough to know the first c elements of A , since $d(p, q; A) \leq c$. In other words, the elements of A after the c -th element are not involved in the calculation of $d(p, q; A)$. So if we consider the neighbourhood sequence A as if it was a neighbourhood sequence with period c (changing its elements $A(i)$ with $i > c$ appropriately), then applying the above result from [5], the lemma follows. $\square$

It is natural to ask what kind of relation exists between the distance functions generated by two given neighbourhood sequences A and B . The complexity of the problem can be characterized by the following 2D periodic example from [4]. Let $A = (1, 1, 2)$, $B = (1, 1, 1, 2, 2, 2)$. Consider the points $\mathbf{0} = (0, 0)$, $p = (3, 1)$ and $q = (6, 3)$. We have $d(\mathbf{0}, p; A) = 3 < 4 = d(\mathbf{0}, p; B)$, but $d(\mathbf{0}, q; A) = 7 > 6 = d(\mathbf{0}, q; B)$. So the distances generated by A and B cannot be compared.

With the following definition we introduce a natural ordering relation on S_n which will be investigated in detail in this dissertation.

Definition 2.10. For any $A, B \in S_n$ we define the relation $\sqsubseteq^*$ in the following way:

$$A \sqsubseteq^* B \iff d(p, q; A) \leq d(p, q; B), \text{ for all } p, q \in \mathbb{Z}^n.$$

Beside S_n , we investigate the structure of all those sets under the ordering relation $\sqsubseteq^*$ which were studied by DAS [4] in the periodic case. These subsets of S_n are given below.

Notation 2.11. Let S'_n , $S'_n(l_{\geq})$ and $S'_n(l)$ be the sets of periodic, at most l -periodic, and l -periodic ($l \in \mathbb{N}$) n D-neighbourhood sequences, respectively.

Remark 2.12. It is evident that $\sqsubseteq^*$ is a partial ordering relation in S_n , hence in all of its subsets. Moreover, this relation in 2D and 3D, is identical to those introduced by DAS [4] and FAZEKAS [13], respectively.

In our investigations we also need the well-known concept of L_p (Euclidean) metrics.

Definition 2.13. Let $q, r \in \mathbb{R}^n$. The L_p ($p > 0$) distance of q and r is

$$L_p(q, r) = \left(\sum_{i=1}^n |\text{Pr}_i(q) - \text{Pr}_i(r)|^p \right)^{\frac{1}{p}},$$

and

$$L_{\infty}(q, r) = \max_{i=1}^n (|\text{Pr}_i(q) - \text{Pr}_i(r)|).$$

Remark 2.14. We have $L_{p_1}(q, r) \geq L_{p_2}(q, r)$ for every $q, r \in \mathbb{R}^n$, provided that $p_1 \leq p_2$.

We also investigate the way a neighbourhood sequence spreads in the digital space starting from a point of $\mathbb{Z}^n$. This spreading is translation-invariant, so for simplicity we may choose the origin $\mathbf{0}$ of $\mathbb{Z}^n$ as the starting point.

Definition 2.15. Let $A \in S_n$. For every $k \in \mathbb{N}$, let

$$\widehat{A}_k = \{q \in \mathbb{Z}^n : d(\mathbf{0}, q; A) \leq k\}.$$

So $\widehat{A}_k$ is the region occupied by A after k steps. Let $H(\widehat{A}_k)$ be the convex hull of $\widehat{A}_k$ in $\mathbb{R}^n$.

In 2D, we will compare the regions occupied by neighbourhood sequences with the corresponding Euclidean disks. For that purpose, we introduce the following notation.

Definition 2.16. For every $k \in \mathbb{N}$, let

$$O_k = \{q \in \mathbb{Z}^2 : L_2(\mathbf{0}, q) \leq k\},$$

and

$$G_k = \{q \in \mathbb{R}^2 : L_2(\mathbf{0}, q) \leq k\}$$

be the Euclidean disks of radius k in $\mathbb{Z}^2$ and $\mathbb{R}^2$, respectively. The sets $\widehat{A}_k$ and O_k will be called the k -disks of the distances $d(A)$ and L_2 , respectively.

The following remarks summarize some simple observations about the geometric behaviour of neighbourhood sequences.

Remark 2.17.

- The set $\widehat{A}_k$ is digitally convex in the usual sense (see, e.g. [42] p. 171, Definition 4.3.4.).
- For any $k \in \mathbb{N}$, the region $H(\widehat{A}_k)$ does not depend on the order of the first k elements of A .
- For any $A \in S_n$, the sequence of regions $(H(\widehat{A}_k))_{k=1}^\infty$ is a strictly monotone increasing sequence. That is, $l > k$ implies $H(\widehat{A}_l) \supsetneq H(\widehat{A}_k)$.

We will often use in our investigations the number of the values i ($i = 1, \dots, n$) occurring among the first k elements of an n D-neighbourhood sequence A .

Definition 2.18. Let $A = (A(i))_{i=1}^\infty \in S_n$. For every $i = 1, \dots, n$ and $k \in \mathbb{N}$ put

$$A(i, k) = \#\{A(j) : A(j) = i, 1 \leq j \leq k\}.$$

For convenience, write $A(i, 0) = 0$.

We use the densities of the elements of the neighbourhood sequences to analyse the asymptotic behaviour of the occupied regions. Since these density values do not always exist, we consider the lower and upper densities, as well.

Definition 2.19. Let $A \in S_n$. For $i = 1, \dots, n$ let

$$s^A(i) = \liminf_{k \rightarrow \infty} \frac{A(i, k)}{k}, \quad \text{and} \quad S^A(i) = \limsup_{k \rightarrow \infty} \frac{A(i, k)}{k}.$$

Definition 2.20. Let $A \in S_n$. The density of the value i ($i = 1, \dots, n$) in A is defined as

$$D^A(i) = \lim_{k \rightarrow \infty} \frac{A(i, k)}{k},$$

if this limit exists.

Remark 2.21.

- If $D^A(i)$ exists for every $i = 1, \dots, n$, then $\sum_{i=1}^n D^A(i) = 1$.
- If $A \in S_n$ is periodic, then $D^A(i)$ exists for every $i = 1, \dots, n$, and it is a rational number. More precisely, $D^A(i)$ is the relative frequency of the value i inside a period.
- If $s^A(i) = S^A(i)$, then $D^A(i)$ exists, and $D^A(i) = s^A(i) = S^A(i)$, $1 \leq i \leq n$.

Chapter 3

Algebraic properties of neighbourhood sequences

In this chapter we investigate the lattice structure of neighbourhood sequences. The structural analyses are arranged into two sections within this chapter. In Section 3.1 we examine the n D digital space, while in Section 3.2 we extend our investigations to $\mathbb{Z}^\infty$. We give the introduction to these cases separately, at the beginning of the actual section.

We recall some definitions from lattice theory that will be needed later on. These basic concepts also can be found e.g. in [2].

Definition 3.1. Let $(P, \leq)$ be a partially ordered set, and $S \subseteq P$. An element $a \in P$ is the least upper bound (or greatest lower bound) of S , if for all $x \in S$, $a \geq x$ (or $a \leq x$), and $b \geq a$ (or $b \leq a$) for every upper bound (or lower bound) b of S . Moreover, if every pair of elements (x, y) with $x, y \in P$ has a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$, then $(P, \leq)$ is called a lattice.

Definition 3.2. The lattice $(P, \leq)$ is distributive if for all $x, y, z \in P$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Remark 3.3. Clearly, $(P, \leq)$ is distributive if and only if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Definition 3.4. The lattice $(P, \leq)$ is complete if its every subset $S \subseteq P$ has a least upper bound $\bigvee S$ and a greatest lower bound $\bigwedge S$.

Remark 3.5. It is well known that $(P, \leq)$ is complete if its every subset has a least upper bound.

Definition 3.6. Let $(P, \leq)$ be a complete lattice and $S \subseteq P$. The set $S^c = \{x \in P : x \leq \bigvee S\}$ is called the closure of S .

3.1 Lattice of neighbourhood sequences in nD

In [4] DAS introduced the “natural” partial ordering relation $\sqsubseteq^*$ on the set of periodic 2D-neighbourhood sequences. Furthermore, he investigated the structure of this set and some of its subsets with respect to this ordering. More precisely, he proved that under this ordering, the set of l -periodic 2D-neighbourhood sequences forms a distributive lattice. Recently, FAZEKAS in [13] has proved that a similar partial ordering can also be introduced for neighbourhood sequences in 3D.

In this section we extend these structural investigations to arbitrary finite dimension. Unfortunately, in most cases, the set of nD -neighbourhood sequences S_n and its subsets defined in Notation 2.11, do not form a nice structure with respect to $\sqsubseteq^*$. Although some of the forthcoming results are very easy to prove, they are of certain importance because provide some information about the differences of the sets involved.

In the proofs of our results a fundamental connection between the functions $f_j^A(i)$ and the relation $\sqsubseteq^*$ will be well applicable. Such a connection was shown by DAS [4], in case of periodic 2D-neighbourhood sequences. A similar result was proved by FAZEKAS in 3D (see [13]). Together with A. FAZEKAS and L. HAJDU in [16] we extended these results to nD with arbitrary $n \in \mathbb{N}$, to general nD -neighbourhood sequences. Clearly, this case also includes the periodic one. In particular our result is new even

for $n = 2$ and 3 . As we mentioned, this theorem will be very useful in our investigations of the lattice structure of S_n (and its subsets) under the relation $\sqsubseteq^*$.

Theorem 3.7. *For any nD -neighbourhood sequences $A = (A(i))_{i=1}^\infty$ and $B = (B(i))_{i=1}^\infty$*

$$d(p, q; A) \leq d(p, q; B), \quad \text{for all } p, q \in \mathbb{Z}^n$$

if and only if

$$f_j^A(i) \geq f_j^B(i), \quad \text{for all } i \in \mathbb{N}, j \in \{1, \dots, n\}.$$

Proof. First we prove that if $d(p, q; A) \leq d(p, q; B)$ for any p, q , then $f_j^A(i) \geq f_j^B(i)$ for every $i \in \mathbb{N}$, $j \in \{1, \dots, n\}$. The proof is indirect. Assume that there are $i \in \mathbb{N}$ and $j \in \{1, \dots, n\}$ for which $f_j^A(i) < f_j^B(i)$ holds. Put

$$u_k = \begin{cases} \#\{B(t) : 1 \leq t \leq i, B(t) = k\}, & \text{for } 1 \leq k < j, \\ \#\{B(t) : 1 \leq t \leq i, B(t) \geq k\}, & \text{for } k = j. \end{cases}$$

Let $p = (0, 0, \dots, 0)$ and

$$\text{Pr}_h(q) = \begin{cases} \sum_{k=h}^j u_k, & \text{for } 1 \leq h \leq j, \\ 0, & \text{for } j < h \leq n. \end{cases}$$

It is clear that $d(p, q; B)$ is equal to i . On the other hand, by the assumption $f_j^B(i) > f_j^A(i)$, and by the definition of p and q , we have $d(p, q; A) > i$ which is a contradiction.

Conversely, suppose that $f_j^A(i) \geq f_j^B(i)$, for every $i \in \mathbb{N}$, $j \in \{1, \dots, n\}$. Let $p, q \in \mathbb{Z}^n$, and put $c = \sum_{h=1}^n |\text{Pr}_h(p) - \text{Pr}_h(q)|$. Without loss of generality we may assume that $c \geq 1$. To derive $d(p, q; A) \leq d(p, q; B)$, by Lemma 2.9, it is sufficient to show that for every $j \in \{1, \dots, n\}$

$$d_j^A(p, q) = \sum_{k=1}^c \left\lfloor \frac{a_j + g_j^A(k)}{f_j^A(c)} \right\rfloor \leq \sum_{k=1}^c \left\lfloor \frac{a_j + g_j^B(k)}{f_j^B(c)} \right\rfloor = d_j^B(p, q)$$

holds. For this we prove that for any fixed j with $j \in \{1, \dots, n\}$

$$\left\lfloor \frac{a_j + g_j^A(k)}{f_j^A(c)} \right\rfloor \leq \left\lfloor \frac{a_j + g_j^B(k)}{f_j^B(c)} \right\rfloor, \quad \text{for } 1 \leq k \leq c.$$

Using the definition of $g_j(k)$ given in Lemma 2.9, the above inequalities are equivalent to the following ones:

$$\left\lfloor \frac{a_j + f_j^A(c) - f_j^A(k-1) - 1}{f_j^A(c)} \right\rfloor \leq \left\lfloor \frac{a_j + f_j^B(c) - f_j^B(k-1) - 1}{f_j^B(c)} \right\rfloor$$

with $1 \leq k \leq c$ which are the same as

$$1 + \left\lfloor \frac{(a_j - 1) - f_j^A(k-1)}{f_j^A(c)} \right\rfloor \leq 1 + \left\lfloor \frac{(a_j - 1) - f_j^B(k-1)}{f_j^B(c)} \right\rfloor, \quad 1 \leq k \leq c.$$

If $(a_j - 1) - f_j^B(k-1) \geq 0$, then we even have

$$\frac{(a_j - 1) - f_j^A(k-1)}{f_j^A(c)} \leq \frac{(a_j - 1) - f_j^B(k-1)}{f_j^B(c)}.$$

Indeed, this inequality is equivalent to

$$f_j^B(c)(a_j - 1 - f_j^A(k-1)) \leq f_j^A(c)(a_j - 1 - f_j^B(k-1)),$$

which clearly holds because of our assumption $f_j^B(i) \leq f_j^A(i)$ ($i \in \mathbb{N}$).

In case of $(a_j - 1) - f_j^B(k-1) < 0$, by the definitions of f_j and a_j , we have

$$\left\lfloor \frac{(a_j - 1) - f_j^B(k-1)}{f_j^B(c)} \right\rfloor = -1.$$

However, using again $f_j^B(i) \leq f_j^A(i)$ ($i \in \mathbb{N}$), now the equality

$$\left\lfloor \frac{(a_j - 1) - f_j^A(k-1)}{f_j^A(c)} \right\rfloor = -1$$

is also valid which completes the proof of the theorem. $\square$

Now we formulate our results about the lattice structure of S_n (and its subsets) under $\sqsubseteq^*$ (see also A. FAZEKAS, A. HAJDU and L. HAJDU [16]).

Theorem 3.8. $(S_2, \sqsubseteq^*)$ is a complete distributive lattice.

Proof. Let S be any non-empty subset of S_2 . Put $B(1) = \max_{A \in S} f_2^A(1)$. Inductively, for $i \geq 2$ let

$$B(i) = \begin{cases} 1, & \text{if } 1 + \sum_{h=1}^{i-1} B(h) \geq f_2^A(i), \text{ for all } A \in S, \\ 2, & \text{otherwise.} \end{cases}$$

Let $B = (B(i))_{i=1}^\infty$. Using Theorem 3.7, by the definition of B we have $B \sqsubseteq^* A$ for every $A \in S$. (Clearly, $f_1^B(i) = f_1^A(i) = i$ for every $i \in \mathbb{N}$.) Let B' be an arbitrary upper bound of S , and let $k \in \mathbb{N}$ be also arbitrary, but fixed. The definition of B implies that $f_2^B(k) = f_2^A(k)$ for some $A \in S$. Hence, as B' is an upper bound of S , $f_2^{B'}(k) \geq f_2^A(k) = f_2^B(k)$ for this A which proves the minimality of B . Now, by Remark 3.5 we have that $(S_2, \sqsubseteq^*)$ is a complete lattice.

Let A_1, A_2, A_3 be arbitrary elements of S_2 and let $i \in \mathbb{N}$. Now

$$\begin{aligned} f_2^{A_1 \wedge (A_2 \vee A_3)}(i) &= \min \left(f_2^{A_1}(i), \max \left(f_2^{A_2}(i), f_2^{A_3}(i) \right) \right) = \\ \max \left(\min \left(f_2^{A_1}(i), f_2^{A_2}(i) \right), \min \left(f_2^{A_1}(i), f_2^{A_3}(i) \right) \right) &= f_2^{(A_1 \wedge A_2) \vee (A_1 \wedge A_3)}(i), \end{aligned}$$

which proves the distributive property of the lattice $(S_2, \sqsubseteq^*)$. The proof of the theorem is complete. $\square$

Now we show that the above theorem does not hold in higher dimensions. Roughly speaking, the reason of this phenomenon is that from 3D on, we have to deal with $n - 1 \geq 2$ “non-trivial” $f_j(i)$.

Proposition 3.9. $(S_n, \sqsubseteq^*)$ is not a lattice for $n \geq 3$.

Proof. Fix $n \geq 3$. Put $A_1 = (3, 1)$, $A_2 = (2)$, $B_1 = (2, 1, 3, 1, 3, 1, 3, 1, 3, \dots)$ and $B_2 = (1, 3, 1, 1, 1, 1, \dots)$. By Theorem 3.7, $A_1 \sqsubseteq^* B_1$ and $A_2 \sqsubseteq^* B_1$. On the other hand, if $B_3 \in S_n$, $B_3 \neq B_1$ then $A_1 \sqsubseteq^* B_3$ and $A_2 \sqsubseteq^* B_3$ implies $B_3 \not\sqsubseteq^* B_1$. Indeed, the first element of B_3 can be at most 2. Suppose that the first i elements of B_1 and B_3 are identical, but the $(i+1)$ -th elements are different. Now $B_3 \sqsubseteq^* B_1$ would yield $B_3(i+1) > B_1(i+1)$. If $B_1(i+1) = 1$ then $B_3(i+1) \geq 2$ which implies $f_2^{B_3}(i+1) > f_2^{A_1}(i+1)$, contradicting $A_1 \sqsubseteq^* B_3$. Otherwise, when $B_1(i+1) = 3$ then $B_3(i+1) \geq 4$, whence $f_4^{B_3}(i+1) > f_4^{A_2}(i+1)$ which contradicts $A_2 \sqsubseteq^* B_3$. (The latter case can occur only when $n \geq 4$.) This means that if $A_1 \wedge A_2$ exists in S_n , then it must be B_1 . However, it is clear that $A_1 \sqsubseteq^* B_2$, $A_2 \sqsubseteq^* B_2$, but $B_1 \not\sqsubseteq^* B_2$. Hence, A_1 and A_2 have no greatest lower bound in S_n , and the statement follows. $\square$

Concerning some special sets of periodic sequences, we show that similar unpleasant properties of $\sqsubseteq^*$ also occur. In what follows we list these “negative” results. We note that Propositions 3.10 (case $n = 2$) and 3.11 disprove Corollary 1 and Theorem 4 of DAS [4], respectively.

Proposition 3.10. $(S'_n, \sqsubseteq^*)$ is not a lattice for $n \geq 2$.

Proof. Let $A_1 = (2, 1, 1)$ and $A_2 = (1, 2, 2)$. The least upper bound of these sequences in $(S_n, \sqsubseteq^*)$ is $B = (2, 1, 2, 1, 2, 2, 1, 2, 2, 1, 2, 2, \dots)$ which is clearly not in S'_n . Suppose that $B' = (B'(1), \dots, B'(l))$, and $B' = A_1 \vee A_2$ in S'_n . In this case $B' \sqsubseteq^* B$ in S_n , but $B' \neq B$, hence for some $i \in \mathbb{N}$, $B'(i) > B(i)$ must hold; suppose that i is the smallest number with this property. Putting $B''(j) = B(j)$ for $1 \leq j \leq 3i$ and $B'' = (B''(1), \dots, B''(3i))$, we have $B'' \sqsubseteq^* A_1, A_2$, but $B'' \not\sqsubseteq^* B'$ which contradicts $B' = A_1 \vee A_2$ in S'_n . The proof of the proposition is complete. $\square$

In [4] DAS proved that the set $S'_2(l)$ for any $l \geq 1$ forms a distributive lattice with respect to $\sqsubseteq^*$. However, this is not true for $S'_2(l_{\geq})$ in general.

Proposition 3.11. $(S'_2(l_{\geq}), \sqsubseteq^*)$ is not a lattice for any $l \geq 5$.

Proof. First let $A_1 = (1, 2, 2)$ and $A_2 = (1, 2, 2, 2, 1)$. One can readily verify that A_1 and A_2 have no least upper bound in $S'_2(5_{\geq})$, thus the statement holds for $l = 5$.

Let now $l \geq 6$, and choose an arbitrary even number k with $\frac{l}{6} < k \leq \frac{l}{3}$. Put $X = (X_1, \dots, X_k) = (1, 2, 2, 1, 2, 1, 2, 1, \dots, 2, 1)$ and $Y = (Y_1, \dots, Y_k) = (2, 1, 1, 2, 2, 1, 2, 1, 2, 1, \dots, 2, 1)$, i.e. $X_1 = 1, X_2 = 2$ and $X_i = (i \bmod 2) + 1$ for $3 \leq i \leq k$ while $Y_1 = Y_4 = 2, Y_2 = Y_3 = 1$ and $Y_i = (i \bmod 2) + 1$ for $5 \leq i \leq k$. Let $A_1 = (X, Y)$ be the sequence of period $2k$, obtained by writing Y after X , i.e. the elements of the $2k$ long period of A_1 with odd indices equal to 2 and with even indices equal to 1, except for $A_1(1) = A_1(k+3) = 1, A_1(2) = A_1(k+4) = 2$. Similarly, let $A_2 = (X, Y, X)$ be a sequence of period $3k$. Furthermore, set $B_1 = (2, 1)$ and $B_2 = (1, 2, 2)$. Observe that $A_1, A_2, B_1, B_2 \in S'_2(l_{\geq})$, and $B_i \sqsubseteq^* A_j$ for $i, j = 1, 2$. In S_2 we have $A_1 \vee A_2 = A = (X, Y, X, 2, 1, 2, 1, \dots, 2, 1)$ of length $6k$ and $B_1 \wedge B_2 = B = (1, 2, 2, 1, 2, 1, 2, 1, \dots)$ which is not periodic. We claim that $A_1 \vee A_2$ does not exist in $S'_2(l_{\geq})$. Indeed, if such a $C = (C(i))_{i=1}^{\infty} \in S'_2(l_{\geq})$ exists, then in S_2 , $A \sqsubseteq^* C \sqsubseteq^* B$ must hold. However, every period of C should be even, moreover, $C(2i-1) + C(2i) = 3$ should hold for every $i \in \mathbb{N}$. Furthermore, $C(1) = 1$ should also be valid which in view of $l < 6k$ and $A \sqsubseteq^* C$ is impossible, and the proposition is proved. $\square$

Remark 3.12. It is easy to check that $(S'_2(l_{\geq}), \sqsubseteq^*)$ is a distributive lattice if $1 \leq l \leq 4$. Hence the statement of Theorem 4 of DAS [4] is valid in these special cases only.

Proposition 3.13. $(S'_n(l_{\geq}), \sqsubseteq^*)$ is not a lattice for any $l \geq 2, n \geq 3$.

Proof. Let $l \geq 5$. It is clear that the sequences A_1 and A_2 , defined in the proof of the previous proposition, have no least upper bound in $S'_n(l_{\geq})$.

Let now $2 \leq l \leq 4$, and let $B_1 = (B_1(1), \dots, B_1(l))$ and $B_2 = (2)$, where B_1 is defined in the following way: $B_1(1), \dots, B_1(l)$ are the first l elements of $(3, 1, 3, 1)$. It is easy to check that these sequences have no greatest lower bound in $S'_n(l_{\geq})$ which completes the proof. $\square$

As we mentioned above, $S'_2(l)$ is a distributive lattice for every $l \in \mathbb{N}$ (see [4]). The following result shows that this statement cannot be generalised to nD with $n \geq 3$.

Proposition 3.14. *$(S'_n(l), \sqsubseteq^*)$ is not a lattice for any $l \geq 2, n \geq 3$.*

Proof. Let $l \geq 2$ and $n \geq 3$ be arbitrary, but fixed integers. Choose the sequences $A_1 = (3, 1)$, $A_2 = (2)$, $B_1 = (2, 1, 3, 1, 3, 1, 3, \dots)$ and $B_2 = (1, 3, 1, 1, 1, \dots)$ in S_n . Define $A'_j = (A'_j(1), \dots, A'_j(l))$, and $B'_j = (B'_j(1), \dots, B'_j(l))$ for $j = 1, 2$ in $S'_n(l)$ in the following way: $A'_j(i) = A_j(i)$, $B'_j(i) = B_j(i)$, $i = 1, \dots, l$. Just as in the proof of Proposition 3.9, it is easy to show that if $A'_1 \wedge A'_2$ exists in $S'_n(l)$, then it must be B'_1 . However, $B'_2 \sqsubseteq^* A'_1, A'_2$, but $B'_2 \not\sqsubseteq^* B'_1$ which completes the proof. $\square$

The above results show that under the relation $\sqsubseteq^*$ we can obtain a nice structure neither in S_n , nor in various subsets of it. Now we introduce another ordering relation which is in close connection with $\sqsubseteq^*$. Moreover, S_n and its subsets considered above, form much nicer structures under this new relation.

Definition 3.15. For any $A_1 = (A_1(i))_{i=1}^\infty$, $A_2 = (A_2(i))_{i=1}^\infty \in S_n$ we define the relation $\sqsubseteq$ in the following way:

$$A_1 \sqsubseteq A_2 \iff A_1(i) \geq A_2(i) \text{ for every } i \in \mathbb{N}.$$

Remark 3.16. It is clear that $\sqsubseteq^*$ is a proper refinement of $\sqsubseteq$ in S_n , S'_n , $S'_n(l_{\geq})$ and $S'_n(l)$.

We examine the structure of S_n , S'_n , $S'_n(l_{\geq})$ and $S'_n(l)$ with respect to $\sqsubseteq$. As we will see, the structures we get will be much nicer than in case of $\sqsubseteq^*$.

Proposition 3.17. *$(S_n, \sqsubseteq)$ is a complete distributive lattice with greatest lower bound $\bigwedge S_n = (1)$ and least upper bound $\bigvee S_n = (n)$.*

Proof. From the definition of $\sqsubseteq$ it follows that this relation is reflexive, antisymmetric and transitive on S_n . Thus $(S_n, \sqsubseteq)$ is a partially ordered set.

It is clear that for every $A_1, A_2 \in S_n$, $A_1 \wedge A_2$ and $A_1 \vee A_2$ exist in S_n , and we have

$$A_1 \wedge A_2 = (\min(A_1(i), A_2(i)))_{i=1}^{\infty},$$

$$A_1 \vee A_2 = (\max(A_1(i), A_2(i)))_{i=1}^{\infty}.$$

Thus $(S_n, \sqsubseteq)$ is a lattice. Let now $S = \{A_\gamma : A_\gamma \in S_n, \gamma \in \Gamma\}$ with some index set Γ , and put $A(i) = \max_{\gamma \in \Gamma} A_\gamma(i)$, where $A_\gamma(i)$ is the i -th element of A_γ . Clearly, $A = (A(i))_{i=1}^{\infty}$ is the least upper bound of S , hence by Remark 3.5, $(S_n, \sqsubseteq)$ is a complete lattice.

The statements that this lattice is distributive and $\bigwedge S_n = (1)$, $\bigvee S_n = (n)$ are trivial. $\square$

Proposition 3.18. *$(S'_n, \sqsubseteq)$ is a distributive lattice with greatest lower bound $\bigwedge S'_n = (1)$ and least upper bound $\bigvee S'_n = (n)$.*

Proof. It is clear that if $A_1, A_2 \in S'_n$, then $A_1 \wedge A_2$ and $A_1 \vee A_2$ – given in the proof of the previous proposition – are also in S'_n . Thus, as $S'_n \subseteq S_n$ and $(1), (n) \in S'_n$, we immediately obtain the statement. $\square$

However, the ordering relation $\sqsubseteq$ has worse properties in S'_n than in S_n . This is shown by the following “negative” result.

Proposition 3.19. *For $n \geq 2$, $(S'_n, \sqsubseteq)$ is not a complete lattice.*

Proof. To prove the statement, we give a counterexample. To find such an example, we construct a monotonously increasing and a monotonously decreasing sequence in S'_n , such that their “limit” sequence is the same, however, this “limit” is in $S_n \setminus S'_n$. This will show that $(S'_n, \sqsubseteq)$ is not complete.

For the precise formulation of the above idea we need some notation. If $A \in S'_n$ is l -periodic for some $l \in \mathbb{N}$, i.e. $A = (A(1), \dots, A(l))$, then put $A^2(l) = (A(1), \dots, A(l), A(1), \dots, A(l))$. (That is, we write A into a $2l$ -periodic form). Moreover, if $A \in S'_2(l)$ with $A = (A(1), \dots, A(l))$, then for $i \in \{1, \dots, l\}$ let $A^{(i)}(l) = (A'(1), \dots, A'(l))$ such that $A'(j) = A(j)$ for

$j \neq i$ and $A'(i) = 3 - A(i)$. (That is, all the elements of A remains the same, except for the i -th which is changed.) Finally, put

$$u_t = \frac{t(t+1)}{2}, \quad \text{and} \quad v_t = \frac{t(t+3)}{2} \quad (t = 0, 1, 2, \dots).$$

Clearly, the length of the closed interval $I_t = [u_t, v_t]$ is t , and these intervals for $t \geq 0$ provide a partition of $\mathbb{N} \cup \{0\}$.

Now let $A_{-1} = (1)$, $B_{-1} = (2)$, and define the sequences A_k and B_k in S'_n in the following way. If $k \in I_t$ where t is even, then let $A_k = A_{k-1}^2(2^k)$, $B_k = (B_{k-1}^2(2^k))^{\langle 2^k \rangle}(2^{k+1})$, and if t is odd then put $A_k = (A_{k-1}^2(2^k))^{\langle 2^k \rangle}(2^{k+1})$, $B_k = B_{k-1}^2(2^k)$. That is, to obtain A_k and B_k , we write the 2^k long periods of A_{k-1} and B_{k-1} after A_{k-1} and B_{k-1} , respectively, and then we change exactly one of the 2^{k+1} -th long periods obtained, at the 2^k -th place; the parity of t determines which period is to be modified. For the first few values of k we get $A_{-1} = (1)$, $A_0 = (1, 1)$, $A_1 = (1, 2, 1, 1)$, $A_2 = (1, 2, 1, 2, 1, 2, 1, 1)$, $A_3 = (1, 2, 1, 2, 1, 2, 1, 1, 1, 2, 1, 2, 1, 1)$ and $B_{-1} = (2)$, $B_0 = (1, 2)$, $B_1 = (1, 2, 1, 2)$, $B_2 = (1, 2, 1, 2, 1, 2, 1, 2)$, $B_3 = (1, 2, 1, 2, 1, 2, 1, 1, 1, 2, 1, 2, 1, 2)$. From the definition of the sequences A_k and B_k immediately follows that $A_{k-1} \sqsubseteq A_k$ and $B_{k-1} \supseteq B_k$ for every $k \in \mathbb{N} \cup \{0\}$, with strict inequalities in both sequences infinitely often. Moreover, observe that for every $k \in \mathbb{N} \cup \{0, -1\}$ for the 2^{k+1} long periods $(A_k(1), \dots, A_k(2^{k+1}))$ and $(B_k(1), \dots, B_k(2^{k+1}))$ of A_k and B_k , respectively, we have $A_k(i) = B_k(i)$ for $1 \leq i < 2^{k+1}$ and $A_k(2^{k+1}) < B_k(2^{k+1})$. This fact is just a straightforward consequence of the shape of A_{-1} and B_{-1} , and of the definition of the sequences. Now for every $i \in \mathbb{N}$ choose a $k \in \mathbb{N}$ with $2^{k+1} > i$, and put $C(i) = A_k(i)$ and $C = (C(i))_{i=1}^\infty$. By the above mentioned property of A_k and B_k , C is well-defined. We claim that $C \in S_n \setminus S'_n$. Indeed, from the definition of A_k and B_k it follows that for every $t \in \mathbb{N} \cup \{0\}$ we have $C(i) = C(i + ms)$ with $1 \leq i \leq 2^{u_t}$, $s = 2^{u_t}$ and $0 \leq m \leq 2^{v_t - u_t + 1} - 1$, but $C(2^{u_t}) \neq C(2^{v_t + 1}) = C(2^{u_t + 1})$. However, it is impossible for any periodic sequence to have this property.

Let now $S = \{A_k : k \in \mathbb{N} \cup \{0, -1\}\}$, and suppose that $B = \bigvee S$ in S'_n for some $B = (B(1), \dots, B(l))$. Then in S_n we have $B \supseteq C$, but $B \neq C$,

hence for some $i \in \mathbb{N}$, $B(i) > C(i)$. However, this implies $B_k \not\sqsubseteq B$ in S'_n , if $2^{k+1} > i$ which by $B_k \sqsubseteq A_j$, $j \in \mathbb{N} \cup \{0, -1\}$, contradicts $B = \bigvee S$ in S'_n . Hence S has no least upper bound in S'_n which proves that $(S'_n, \sqsubseteq)$ is not complete. $\square$

The next proposition shows that $S'_n(l_{\geq})$ is not a “natural” subset of S_n , in the sense that it does not form a nice structure even under $\sqsubseteq$. Of course, it is not surprising in view of the following observation: if A_1 and A_2 are in $S'_n(l_{\geq})$, then $A_1 \vee A_2$ and $A_1 \wedge A_2$ defined in S_n , does not belong to $S'_n(l_{\geq})$ in general.

Proposition 3.20. *$(S'_n(l_{\geq}), \sqsubseteq)$ is not a lattice for any $n \geq 2$ and $l \geq 6$.*

Proof. Let $n \geq 2$ be arbitrary. Observe that $(1, 1, 2, 1)$ and $(1, 2, 1, 2, 1, 1)$ for $6 \leq l \leq 11$, and $(1, 1, 1, 1, 1, 1, 2, 1)$ and $(1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 1)$ for $12 \leq l \leq 14$ have no least upper bound in $S'_n(l_{\geq})$.

Let now $l \geq 15$, and choose a prime p with $\max(3, \frac{l}{6}) < p \leq \frac{l}{3}$. (By Bertrand's postulate, for $l \geq 18$ such a prime always exists, and for $l < 18$ we may take $p = 5$.) Put $A_1 = (1, 2)$ and $A_2 = (1, 1, 1, \dots, 1, 1, 1, 2)$, where the period of A_2 is $3p$ (i.e. the first $3p - 1$ elements of the $3p$ long period of A_2 equal to 1, and the last element is 2). Moreover, let $B_1 = (1, 2, 2, 2, 1, 2)$ and $B_2 = (B_2(1), \dots, B_2(2p))$ with $B_2(i) = 1$ if i is odd but $i \neq p$ and $B_2(i) = 2$ otherwise (that is when i is even or $i = p$). Now it is easy to check that $A_1, A_2, B_1, B_2 \in S'_n(l_{\geq})$, and $A_1, A_2 \sqsubseteq B_1$, $A_1, A_2 \sqsubseteq B_2$. Moreover, in S_n we have $A_1 \vee A_2 = B_1 \wedge B_2$. However, as $A_1 \vee A_2$ is of period $6p$ (but not of shorter period), it is not in $S'_n(l_{\geq})$, hence A_1 and A_2 cannot have a least upper bound in $S'_n(l_{\geq})$. This completes the proof of the proposition. $\square$

Remark 3.21. We note that $(S'_n(l_{\geq}), \sqsubseteq)$ is a distributive lattice if $1 \leq l \leq 5$, for every $n \in \mathbb{N}$. We omit the trivial proof of this statement.

Proposition 3.22. *$(S'_n(l), \sqsubseteq)$ is a distributive lattice for every $n, l \in \mathbb{N}$.*

Proof. As for any $A_1, A_2 \in S'_n(l)$ the sequences $A_1 \vee A_2$ and $A_1 \wedge A_2$ defined in S_n are also in $S'_n(l)$, the statement follows from Proposition 3.17. $\square$

3.2 Lattice of neighbourhood sequences in ∞D

Throughout this section we denote the set of infinite integer sequences by $\mathbb{Z}^\infty$, i.e. $\mathbb{Z}^\infty = \{(z_i)_{i=1}^\infty : z_i \in \mathbb{Z}\}$. We shall refer to the elements of $\mathbb{Z}^\infty$ as points.

Our purpose is to extend the result of the previous section, concerning $\mathbb{Z}^n$, to this general case (see also A. FAZEKAS, A. HAJDU and L. HAJDU [16]). Moreover, we will extend Lemma 2.9 to $\mathbb{Z}^\infty$, too. First we give some definitions that are natural generalizations of the concepts in Chapter 2.

Definition 3.23. Let p and q be two points in $\mathbb{Z}^\infty$. The i -th coordinate of the point p is indicated by $\text{Pr}_i(p)$. The points $p, q \in \mathbb{Z}^\infty$ are called M -neighbours for some $M \in \mathbb{N} \cup \{\infty\}$, if

- $|\text{Pr}_i(p) - \text{Pr}_i(q)| \leq 1 \quad (i \in \mathbb{N}),$
- $\sum_{i=1}^\infty |\text{Pr}_i(p) - \text{Pr}_i(q)| \leq M.$

Definition 3.24. An infinite sequence $A = (A(i))_{i=1}^\infty$, where $A(i) \in \mathbb{N} \cup \{\infty\}$ is called an ∞D -neighbourhood sequence. If for some $l \in \mathbb{N}$, $A(i) = A(i + l)$ holds for every $i \in \mathbb{N}$, then A is called periodic, with period l , or simply l -periodic. In this case we will use the abbreviation $A = (A(1), \dots, A(l))$. The set of the ∞D -neighbourhood sequences will be denoted by S_∞ .

Definition 3.25. Let $p, q \in \mathbb{Z}^\infty$ and $A \in S_\infty$. The point sequence $p = p_0, p_1, \dots, p_m = q$, where p_{i-1} and p_i are $A(i)$ -neighbours in $\mathbb{Z}^\infty$ ($1 \leq i \leq m$), is called an A -path from p to q of length m . If such a path exists, then the A -distance of p and q is defined as the common length of the shortest A -paths from p to q . It will be denoted by $d(p, q; A)$ or briefly $d(A)$. If there is no A -path from p to q , then we put $d(p, q; A) = \infty$.

Remark 3.26. Observe that the following two statements are equivalent:

- $d(p, q; A) = \infty$ for every $A \in S_\infty$,
- the set $\{|\text{Pr}_i(p) - \text{Pr}_i(q)| : i \in \mathbb{N}\}$ is unbounded.

To prove our main results concerning ∞ D-neighbourhood sequences, we need three lemmas. The first lemma shows that for any $A \in S_\infty$, the function $d(p, q; A)$ has some “symmetry” properties. We note that by Lemma 2.9, the same is also true in n D for every $n \in \mathbb{N}$.

Lemma 3.27. *Let $p, q \in \mathbb{Z}^\infty$ and $A = (A(i))_{i=1}^\infty \in S_\infty$. The distance value $d(p, q; A)$ depends only on the differences of the coordinates of the points, i.e. on the numbers $|\text{Pr}_i(p) - \text{Pr}_i(q)|$, $i \in \mathbb{N}$. In particular, for any $p, q \in \mathbb{Z}^\infty$ and $A \in S_\infty$, we have $d(p, q; A) = d(q, p; A)$.*

Proof. First, it is clear that for every $a, b, x \in \mathbb{Z}^\infty$ and $A \in S_\infty$, $d(a, b; A) = d(a - x, b - x; A)$ holds. Thus we may suppose that $p = \mathbf{0} = (0, 0, 0, \dots)$. Let $q, q' \in \mathbb{Z}^\infty$ with $|\text{Pr}_i(q)| = |\text{Pr}_i(q')|$ for every $i \in \mathbb{N}$. To prove the first part of the lemma, it is sufficient to show that $d(\mathbf{0}, q; A) = d(\mathbf{0}, q'; A)$. For this purpose, first put $d(\mathbf{0}, q; A) = k < \infty$ and let $\mathbf{0} = q_0, q_1, \dots, q_k = q$ be a path from $\mathbf{0}$ to q . For $i = 0, \dots, k$ define the points q'_i in the following way:

$$\text{Pr}_j(q'_i) = \begin{cases} \text{Pr}_j(q_i), & \text{if } \text{Pr}_j(q) = \text{Pr}_j(q'), \\ -\text{Pr}_j(q_i), & \text{if } \text{Pr}_j(q) = -\text{Pr}_j(q'). \end{cases}$$

Now we have $q'_0 = \mathbf{0}$ and $q'_k = q'$. Moreover, for $i = 1, \dots, k$ and $j \in \mathbb{N}$, $\text{Pr}_j(q'_{i-1}) \neq \text{Pr}_j(q'_i)$ implies $\text{Pr}_j(q_{i-1}) \neq \text{Pr}_j(q_i)$. Indeed, suppose that $\text{Pr}_j(q_{i-1}) = \text{Pr}_j(q_i)$ and let $\varepsilon \text{Pr}_j(q) = \text{Pr}_j(q')$ with $\varepsilon \in \{1, -1\}$. However, by the definition of $\text{Pr}_j(q'_{i-1})$ and $\text{Pr}_j(q'_i)$, in this case we would have $\text{Pr}_j(q'_{i-1}) = \varepsilon \text{Pr}_j(q_{i-1}) = \varepsilon \text{Pr}_j(q_i) = \text{Pr}_j(q'_i)$ which would be a contradiction. Hence $\sum_{i=1}^\infty |\text{Pr}_j(q'_i) - \text{Pr}_j(q'_{i-1})| \leq \sum_{i=1}^\infty |\text{Pr}_j(q_i) - \text{Pr}_j(q_{i-1})|$ which shows that q'_{i-1} and q'_i are $A(i)$ -neighbours. This proves $d(\mathbf{0}, q'; A) \leq k$. Exchanging q and q' we have $d(\mathbf{0}, q; A) = d(\mathbf{0}, q'; A)$ in the case $k < \infty$. However, if $d(\mathbf{0}, q; A) = \infty$, just as above we must have $d(\mathbf{0}, q'; A) = \infty$, as well. Indeed, $d(\mathbf{0}, q'; A) = k < \infty$ would imply $d(\mathbf{0}, q; A) \leq k$ which would be a contradiction.

The second statement of the lemma is an immediate consequence of the first one. $\square$

If the points $p, q \in \mathbb{Z}^\infty$ differ only at finitely many coordinates, then their distance is certainly finite (regardless of A), and the points of a short-

est path connecting them belongs to an n D subspace of $\mathbb{Z}^\infty$ for some $n \in \mathbb{N}$. By this observation, the following two lemmas are obvious. However, since these lemmas play important roles in the proof of Theorem 3.30, and for the convenience of the reader, we provide the easy proofs of these statements.

Lemma 3.28. *Let $p, q \in \mathbb{Z}^\infty$ such that $\#\{i : \text{Pr}_i(p) \neq \text{Pr}_i(q)\} = c < \infty$. Let $A = (A(i))_{i=1}^\infty \in S_\infty$, and let $B = (B(i))_{i=1}^\infty \in S_\infty$ defined by $B(i) = \min(A(i), c)$ ($i \in \mathbb{N}$). Then $d(p, q; B) = d(p, q; A)$.*

Proof. It is clear that $d(p, q; A) \leq d(p, q; B)$, so we have only to show the opposite relation. Put $H = \{i : \text{Pr}_i(p) \neq \text{Pr}_i(q)\}$, and $m = d(p, q; A)$. Since $c < \infty$, we have $m < \infty$. Hence there exists an A -path $p = p_0, p_1, \dots, p_m = q$ from p to q in $\mathbb{Z}^\infty$. For $1 \leq i \leq m - 1$ let

$$\text{Pr}_j(q_i) = \begin{cases} \text{Pr}_j(p_i), & \text{for } j \in H, \\ \text{Pr}_j(p), & \text{for } j \in \mathbb{N} \setminus H, \end{cases}$$

and set $q_0 = p$ and $q_m = q$. Now by the definitions of B and the points q_i , it is clear that $q_0, q_1, \dots, q_m$ is a B -path from p to q which yields $d(p, q; B) \leq m = d(p, q; A)$, and the lemma is proved. $\square$

Lemma 3.29. *Let $p, q \in \mathbb{Z}^\infty$ such that the set $H = \{i : \text{Pr}_i(p) \neq \text{Pr}_i(q)\}$ is finite. Let $A \in S_\infty$ with $d(p, q; A) = m$. Then there exists an A -path $p = q_0, q_1, \dots, q_m = q$ from p to q in $\mathbb{Z}^\infty$ such that for every $i = 1, \dots, m$ and $j \in \mathbb{N} \setminus H$ we have $\text{Pr}_j(q_{i-1}) = \text{Pr}_j(q_i)$.*

Proof. The points $q_0, \dots, q_m$ defined in the proof of Lemma 3.28 clearly have the desired properties. $\square$

In Theorem 3.30 we describe how $d(p, q; A)$ can be calculated. This result is of independent interest, but it will be useful in the proof of Theorem 3.32, too.

Theorem 3.30. *Let p and q be two distinct points in $\mathbb{Z}^\infty$ such that the set $\{|\text{Pr}_i(p) - \text{Pr}_i(q)| : i \in \mathbb{N}\}$ is bounded and let $A = (A(i))_{i=1}^\infty \in S_\infty$. For $c \geq 1$ let $H_c = \{i : |\text{Pr}_i(p) - \text{Pr}_i(q)| \geq c\}$, and put $k = \min\{c : \#H_c < \infty\}$ and $h = \#H_k$. Let $B(i) = \min(h, A(i))$, and $B = (B(i))_{i=1}^\infty$ ($i \in \mathbb{N}$).*

Moreover, put $r = (\text{Pr}_i(p))_{i \in H_k}$ and $s = (\text{Pr}_i(q))_{i \in H_k}$. Let t be defined by the following properties:

- $A(t) = \infty$,
- $\#\{i : i \leq t \text{ and } A(i) = \infty\} = k - 1$.

If such t does not exist, then put

$$t = \begin{cases} 0, & \text{if } k = 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Now the following equality holds:

$$d(p, q; A) = \max(d_h(r, s; B), t),$$

where for $h \geq 1$, $d_h(r, s; B)$ is the h -dimensional B -distance of r and s , and $d_0(r, s; B) = 0$.

Proof. By Lemma 3.27, without loss of generality we may suppose that $\text{Pr}_i(p) \leq \text{Pr}_i(q)$ for every $i \in \mathbb{N}$. Furthermore, it is clear that if $h = 0$, then $d(p, q; A) = t$, and if $t = 0$, then $d(p, q; A) = d_h(r, s; B)$.

First we prove that $d(p, q; A) \leq \max(d_h(r, s; B), t)$. Assume that $h \geq 1$, and put $d_h(r, s; B) = x$. Let $r = r_0, r_1, \dots, r_x = s$ be a B -path from r to s in $\mathbb{Z}^h$. First suppose that $x \geq t$, and let $1 \leq m_1 < m_2 < \dots < m_t = t$ be the $k - 1$ indices with $A(m_i) = \infty$, $i = 1, \dots, t$. Set $M = \{m_1, \dots, m_t\}$. Define the points q_i for $i = 0, \dots, x$ in $\mathbb{Z}^\infty$ in the following way: $q_0 = p$,

$$\text{Pr}_j(q_i) = \begin{cases} \text{Pr}_j(r_i), & \text{for } j \in H_k, i \in \{1, \dots, x\}, \\ \text{Pr}_j(q_{i-1}), & \text{for } j \in \mathbb{N} \setminus H_k, \\ & i \in \{1, \dots, x\} \setminus M, \\ \min(\text{Pr}_j(q_{i-1}) + 1, \text{Pr}_j(q)), & \text{for } j \in \mathbb{N} \setminus H_k, i \in M. \end{cases}$$

By this definition, the points q_{i-1}, q_i are $A(i)$ -neighbours for $i = 1, \dots, x$. Moreover, we have $q_x = q$. Indeed, for $j \in H_k$ clearly $\text{Pr}_j(q_x) = \text{Pr}_j(r_x) = \text{Pr}_j(q)$ holds. On the other hand, as $x \geq t$, for $j \in \mathbb{N} \setminus H_k$ the j -th coordinate is increased $k - 1$ times (if necessary), and we obtain $\text{Pr}_j(q_x) = \text{Pr}_j(q)$ for these indices, too. Hence in this case we have $d(p, q; A) \leq x =$

$\max(d_h(r, s; B), t)$. Now suppose that $t > x$. If $t = \infty$, then there is nothing to prove, so suppose that $t < \infty$. Let the points $r_0, \dots, r_x$ be as before, and for $0 \leq i \leq x$ define the points q_i in the same way. For $i = x + 1, \dots, t$ put

$$\Pr_j(q_i) = \begin{cases} \min(\Pr_j(q_{i-1}) + 1, \Pr_j(q)), & \text{if } A(i) = \infty \text{ and } j \in \mathbb{N} \setminus H_k, \\ \Pr_j(q_{i-1}), & \text{otherwise.} \end{cases}$$

It is clear that for $i = 1, \dots, t$ the points q_{i-1}, q_i are $A(i)$ -neighbours. Moreover, we have $q_t = q$. Indeed, for $j \in H_k$ we even have $\Pr_j(q_x) = \Pr_j(q)$. On the other hand, if $j \in \mathbb{N} \setminus H_k$, then the j -th coordinate is increased (at most) $k - 1$ times which yields $\Pr_j(q_t) = \Pr_j(q)$ for such indices, too. Hence, in this case we have again

$$d(p, q; A) \leq t = \max(d_h(r, s; B), t).$$

Now we prove that $d(p, q; A) \geq \max(d_h(r, s; B), t)$. If $m = \infty$, then we are ready. So put $d(p, q; A) = m < \infty$. Let $p = p_0, p_1, \dots, p_m = q$ be an A -path from p to q in $\mathbb{Z}^\infty$. For $i = 0, \dots, m$ define q_i in the following way:

$$\Pr_j(q_i) = \begin{cases} \Pr_j(p_i), & \text{for } j \in H_k, \\ 0, & \text{for } j \in \mathbb{N} \setminus H_k. \end{cases}$$

Clearly, for $i = 1, \dots, m$, q_{i-1} and q_i are $A(i)$ -neighbours. Hence, by Lemma 3.28 and Lemma 3.29, and by the definitions of r, s and B , we have

$$d(p, q; A) \geq d(q_0, q_m; A) = d(q_0, q_m; B) \geq d_h(r, s; B).$$

Let now u, v , and w be any three points in $\mathbb{Z}^\infty$ such that u and v are $A(i)$ -neighbours for some $i \in \mathbb{N}$. For $g \in \mathbb{N}$ let $U_g = \{i : |\Pr_i(u) - \Pr_i(w)| \geq g\}$, $V_g = \{i : |\Pr_i(v) - \Pr_i(w)| \geq g\}$, and put $k_u = \min\{g : \#U_g < \infty\}$, $k_v = \min\{g : \#V_g < \infty\}$. If $\#U_g = \infty$ or $\#V_g = \infty$ for every $g \in \mathbb{N}$, then put $k_u = \infty$ or $k_v = \infty$, respectively. Now, if $A(i) < \infty$, then clearly $k_u = k_v$, and even if $A(i) = \infty$, we have $|k_u - k_v| \leq 1$ (with the agreement $|\infty - \infty| = 0$). Hence, by the definition of k and t , we obtain $d(p, q; A) \geq t$ which implies $d(p, q; A) \geq \max(d_h(r, s; B), t)$. The proof of the theorem is now complete. $\square$

Remark 3.31. It is interesting to note that combining the above result with the formula provided for $d(p, q; A)$ in Lemma 2.9, it is possible to calculate explicitly the distance of two points in $\mathbb{Z}^\infty$, determined by an ∞ D-neighbourhood sequence. On the other hand, if we take $p, q \in \mathbb{Z}^\infty$ such that they differ only at finitely many places, then we have $k = 1$ whence $t = 0$ in Theorem 3.30. This shows that the distance defined in $\mathbb{Z}^\infty$ is in fact a generalization of the distances introduced in the finite dimensional cases.

The following result is the extension of Theorem 3.7 to $\mathbb{Z}^\infty$.

Theorem 3.32. *Let $A_c = (A_c(i))_{i=1}^\infty \in S_\infty$ with $c = 1, 2$. For $i, j \in \mathbb{N}$, $c = 1, 2$, put $f_j^{A_c}(i) = \sum_{k=1}^i \min(A_c(k), j)$. Then*

$$d(p, q; A_1) \leq d(p, q; A_2) \quad \text{for all } p, q \in \mathbb{Z}^\infty$$

if and only if

$$f_j^{A_1}(i) \geq f_j^{A_2}(i) \quad \text{for all } i, j \in \mathbb{N}.$$

Proof. First we derive the second property from the first one. Contrary to the second statement, suppose that for some $h, j \in \mathbb{N}$, $f_j^{A_1}(h) < f_j^{A_2}(h)$ holds. Let

$$u_k = \begin{cases} \#\{A_2(t) : A_2(t) = k, 1 \leq t \leq h\}, & \text{for } 1 \leq k < j, \\ \#\{A_2(t) : A_2(t) \geq k, 1 \leq t \leq h\}, & \text{for } k = j, \end{cases}$$

and put $p = (0, 0, 0, \dots)$, $q = (u_j, u_{j-1} + u_j, \dots, u_1 + u_2 + \dots + u_j, 0, 0, \dots)$. For $c = 1, 2$ and $i \in \mathbb{N}$ set $B_c(i) = \min(A_c(i), j)$ and $B_c = (B_c(i))_{i=1}^\infty$. Using Lemma 3.28, by the constructions of p, q and B_c , $c = 1, 2$, and by $f_j^{A_1}(h) < f_j^{A_2}(h)$, we have

$$d(p, q; A_1) = d(p, q; B_1) > d(p, q; B_2) = d(p, q; A_2),$$

which contradicts the first statement, and the first part of the theorem is proved.

Now we prove that the second statement implies the first one. Fix two arbitrary points, p and q in $\mathbb{Z}^\infty$. Without loss of generality we may

suppose that the set $\{|\text{Pr}_i(p) - \text{Pr}_i(q)| : i \in \mathbb{N}\}$ is bounded, otherwise we have $d(p, q; A_1) = d(p, q; A_2) = \infty$. Let h, r, s, t_c and B_c denote the parameters, points and sequences corresponding to p, q , and A_c , $c = 1, 2$, defined in Theorem 3.30. Using this theorem, it is sufficient to show that $t_1 \leq t_2$ and $d_h(r, s; B_1) \leq d_h(r, s; B_2)$ in $\mathbb{Z}^h$.

First suppose that $t_1 > t_2$. This implies that for some $n \in \mathbb{N}$, $\#\{i : i \leq n, \text{ and } A_1(i) = \infty\} < \#\{i : i \leq n, \text{ and } A_2(i) = \infty\}$. However, in this case for some m we clearly have $f_m^{A_1}(n) < f_m^{A_2}(n)$ which contradicts the second statement. Now we prove that $d_h(r, s; B_1) \leq d_h(r, s; B_2)$. To do this, observe that B_c for $c = 1, 2$ is an h D-neighbourhood sequence. Moreover, for $1 \leq j \leq h$ the functions $f_j^{A_c}$ are just the same as those corresponding to B_c , $c = 1, 2$. Hence, by Theorem 3.7 we have $d_h(r, s; B_1) \leq d_h(r, s; B_2)$ which completes the proof of the theorem. $\square$

Now we study the structure of the ∞ D-neighbourhood sequences. First we define two ordering relations on them which are the extensions of the finite dimensional orderings to this general case.

Definition 3.33. Let $A_1, A_2 \in S_\infty$. We write $A_1 \sqsupseteq^* A_2$ if and only if for every $p, q \in \mathbb{Z}^\infty$, $d(p, q; A_1) \leq d(p, q; A_2)$.

Remark 3.34. By Theorem 3.32, $A_1 \sqsupseteq^* A_2$ if and only if for every $i, j \in \mathbb{N}$, $f_j^{A_1}(i) \geq f_j^{A_2}(i)$ holds.

Remark 3.35. Let S'_∞ , $S'_\infty(l_\geq)$ and $S'_\infty(l)$ be the sets of periodic, at most l -periodic and l -periodic ($l \in \mathbb{N}$) ∞ D-neighbourhood sequences, respectively. It is clear that $\sqsupseteq^*$ is an antisymmetric, transitive relation, i.e. a partial ordering on S_∞ and on all its above subsets. However, just as in the finite dimensional case, $(S_\infty, \sqsupseteq^*)$, $(S'_\infty, \sqsupseteq^*)$, $(S'_\infty(l_\geq), \sqsupseteq^*)$ and $(S'_\infty(l), \sqsupseteq^*)$ with $l \geq 2$ are not lattices. These statements are simple consequences of Propositions 3.9 through 3.14.

Definition 3.36. For $A_1 = (A_1(i))_{i=1}^\infty$ and $A_2 = (A_2(i))_{i=1}^\infty$ in S_∞ we write $A_1 \sqsupseteq A_2$ if and only if $A_1(i) \geq A_2(i)$ for every $i \in \mathbb{N}$.

It turns out that $\sqsupseteq$ has much more pleasant properties than $\sqsupseteq^*$ in case of ∞ D-neighbourhood sequences, too.

Proposition 3.37. *For any $l \in \mathbb{N}$, $(S_\infty, \sqsubseteq)$, $(S'_\infty, \sqsubseteq)$ and $(S'_\infty(l), \sqsubseteq)$ are distributive lattices, with greatest lower bound (1) and least upper bound (∞) . Moreover, the first and third lattices are complete, while the second one is not.*

Proof. The statement is a simple consequence of Propositions 3.17, 3.18, 3.19 and 3.22. We omit the details. $\square$

Proposition 3.38. *$(S'_\infty(l_\geq), \sqsubseteq)$ is a complete distributive lattice for $1 \leq l \leq 5$. If $l \geq 6$, then $(S'_\infty(l_\geq), \sqsubseteq)$ is not a lattice.*

Proof. The statements are straightforward consequences of Proposition 3.20 and Remark 3.21. $\square$

The following result provides some information about the structures of several other subsets of S_∞ under $\sqsubseteq$.

Proposition 3.39. *Let $S_\infty^* = \{A : A = (A(i))_{i=1}^\infty \text{ with } A(i) \in \mathbb{N}\}$, $S_\infty^+ = \bigcup_{n=1}^\infty S_n$ and $S_\infty^- = \bigcup_{n=1}^\infty S'_n$. Then $(S_\infty^*, \sqsubseteq)$, $(S_\infty^+, \sqsubseteq)$ and $(S_\infty^-, \sqsubseteq)$ are non-complete distributive lattices (sublattices of S_∞) with greatest lower bound (1), but having no least upper bounds.*

Proof. Let S be any of the sets S_∞^* , S_∞^+ , S_∞^- . It is evident that for each $A_1, A_2 \in S$, $A_1 \vee A_2$ and $A_1 \wedge A_2$ defined in S_∞ , also belong to S . Hence these three sets clearly form distributive lattices under $\sqsubseteq$. It is obvious that (1) is the greatest lower bound of each lattice, and that they have no least upper bounds. The last observation immediately implies the non-completeness of the lattices. $\square$

Finally, we show that the lattice $(S_\infty, \sqsubseteq)$ (and in some special cases $(S'_\infty(l_\geq), \sqsubseteq)$ or $(S'_\infty(l), \sqsubseteq)$) can be considered as the closure of the union of the finite dimensional lattices studied in the previous section. We also include the lattices $(S'_\infty, \sqsubseteq)$ and $(S_\infty^*, \sqsubseteq)$ into this consideration.

Proposition 3.40. *Let $S_\infty^-(l_\geq) = \bigcup_{n=1}^\infty S'_n(l_\geq)$ and $S_\infty^-(l) = \bigcup_{n=1}^\infty S'_n(l)$ ($l \in \mathbb{N}$). For any $l \in \mathbb{N}$, in S_∞*

$$(S_\infty, \sqsubseteq) = (S'_\infty, \sqsubseteq)^c = (S_\infty^*, \sqsubseteq)^c = (S_\infty^+, \sqsubseteq)^c =$$

$$(S_{\infty}^-, \sqsupseteq)^c = (S_{\infty}^-(l_{\geq}), \sqsupseteq)^c = (S_{\infty}^-(l), \sqsupseteq)^c$$

holds. Moreover, for any $l \in \mathbb{N}$, in $S'_{\infty}(l)$ we have

$$(S'_{\infty}(l), \sqsupseteq) = (S_{\infty}^-(l), \sqsupseteq)^c.$$

Finally, for $1 \leq l \leq 5$, in $S'_{\infty}(l_{\geq})$

$$(S'_{\infty}(l_{\geq}), \sqsupseteq) = (S_{\infty}^-(l_{\geq}), \sqsupseteq)^c.$$

Proof. The statements are trivial by noting that S_{∞} , $S'_{\infty}(l)$ ($l \in \mathbb{N}$) and $S'_{\infty}(l_{\geq})$ ($1 \leq l \leq 5$) are complete lattices with respect to $\sqsupseteq$, and that all the examined sets have the least upper bound (∞) in S_{∞} (or $S'_{\infty}(l)$ or $S'_{\infty}(l_{\geq})$, respectively). $\square$

Chapter 4

Metrical properties of neighbourhood sequences

Suppose that $A \sqsubseteq^* B$ holds for some $A, B \in S_n$. In this case we can say that A is “faster” than B , since the A -distance is less than or equal to the B -distance of any two points in $\mathbb{Z}^n$. In other words, A spreads faster in the digital space, and reaches not less points from the origin within the same number of steps than B .

In the previous chapter we could see that $\sqsubseteq^*$ has some unpleasant properties. It fails to be a complete ordering on the set of neighbourhood sequences. Moreover, the structure obtained is not even a lattice in higher dimensions. However, in certain applications it can be useful to compare any two neighbourhood sequences, i.e. to decide which one spreads “faster”. For this purpose, we introduce a norm-like concept, called velocity, on the set of neighbourhood sequences, and investigate its properties. This concept has to be introduced in a way to fit the relation $\sqsubseteq^*$. So we need some preliminaries before defining velocity. Furthermore, we introduce a metric for neighbourhood sequences. The results presented here can also be found in [21].

The structure of this chapter is as follows. In Section 4.1 we clarify which conditions should be met by the notion of velocity. In Section 4.2 the

concept of velocity is introduced, and some basic properties are proved. In Section 4.3 we give some theoretical examples to illustrate the behaviour of velocity, and in Section 4.4 we show how this concept can be applied to distributing information in a general network model. In Section 4.5 we define a metric on the set of neighbourhood sequences, and study its properties.

4.1 Preliminaries

By defining velocity, we assign a positive real number to every neighbourhood sequence. In this section we give some natural conditions which should be met by this concept.

- (I) *Velocity must be sensitive for the behaviour of the sequences in different dimensions.*

It can happen that a sequence spreads “faster” than another one in higher dimensions, but they have the same “speed” in lower dimensional subspaces. For example, in 3D the constant sequences (3) and (2) have the same velocity on the planes $[x, y]$, $[x, z]$, $[y, z]$ defined by the coordinate axes; or the sequences (1,3) and (2) behave differently in the 2D subspaces of $\mathbb{Z}^3$. These features should be reflected in the definition of velocity.

- (II) *The elements of a neighbourhood sequence should contribute to its velocity by different weights.*

There are two reasons to establish this condition. First, it is natural to consider the initial elements of the sequences more important than the elements which occur later. The second reason comes from theoretical necessity. Namely, if we want to take into consideration all elements of the sequences, then we have to guarantee the convergence of certain sums or series of the (weighted) elements of the sequences.

- (III) *Velocity must be defined in a way to fit the natural ordering.*

This condition is evident: velocity should preserve the ordering relation $\sqsubseteq^*$. If a neighbourhood sequence is faster than another one, its

velocity should be larger as well. As $\sqsubseteq^*$ is only a partial ordering, the opposite statement cannot be true. However, our velocity concept, introduced in the next section, will have the nice property that in a certain sense this opposite statement also holds (cf. Theorem 4.10).

4.2 Assigning velocity to neighbourhood sequences

According to condition (II), first we give the concept of a weight system which will be appropriate in our further investigations.

Definition 4.1. Let $n \in \mathbb{N}$. The set of functions $\delta_j : \mathbb{N} \rightarrow \mathbb{R}$ ($j = 1, \dots, n$) is called a weight system, if the following three conditions hold:

- $\delta_j(i) \geq 0$ ($i \in \mathbb{N}$, $j = 1, \dots, n$),
- $\sum_{i=1}^{\infty} \delta_j(i) < \infty$ ($j = 1, \dots, n$),
- δ_j is monotone decreasing ($j = 1, \dots, n$).

In order to meet condition (I), we introduce the concept of velocity in two steps. First, we assign an n -tuple to every neighbourhood sequence. The elements of this n -tuple reflect the “velocity” of the given neighbourhood sequence in the subspaces of $\mathbb{Z}^n$ of dimensions from 1 to n . Then, we define one descriptive velocity value.

Definition 4.2. Let $A \in S_n$, and δ_j ($j = 1, \dots, n$) be a weight system. The j -dimensional velocity of A is defined as

$$v_j^A = \sum_{i=1}^{\infty} A^{(j)}(i) \delta_j(i).$$

The velocity of A is given by

$$v^A = \frac{1}{n} \sum_{j=1}^n v_j^A.$$

Remark 4.3. Let T be the linear space of bounded real sequences over $\mathbb{R}$, and let δ_j ($j = 1, \dots, n$) be a weight system. It is well-known (see e.g. [23]) that for every j , with the norm

$$\|(x_i)_{i=1}^\infty\| = \sum_{i=1}^\infty |x_i| \delta_j(i) \quad ((x_i)_{i=1}^\infty \in T),$$

T becomes a Banach space. Thus, for any $A = (A(i))_{i=1}^\infty$, v_j^A could be defined as $v_j^A = \|(A^{(j)}(i))_{i=1}^\infty\|$.

Remark 4.4. For every $A \in S_n$ we have

$$\sum_{i=1}^\infty \delta_j(i) \leq v_j^A \leq n \sum_{i=1}^\infty \delta_j(i).$$

Remark 4.5. By the definition of v_j^A ($j = 1, \dots, n$) and v^A , we have that for every $\varepsilon > 0$ there exists some $k_0 \in \mathbb{N}$ such that for any $k > k_0$,

$$v_j^A - \sum_{i=1}^k A^{(j)}(i) \delta_j(i) < \varepsilon \quad (j = 1, \dots, n),$$

and also

$$v^A - \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^k A^{(j)}(i) \delta_j(i) < \varepsilon.$$

This shows that regardless of the system δ_j , the j -dimensional velocities and the velocity of A is “determined” by the first “few” terms of A .

In the next section we study the behaviour of the velocity with respect to various weight systems. Now we show how conditions (I), (II) and (III) are met by this velocity concept.

The velocity vector $(v_1^A, v_2^A, \dots, v_n^A)$, thus also v^A is obviously sensitive for the behaviour of the sequence A in subspaces of $\mathbb{Z}^n$ of dimensions from 1 to n . Thus, (I) is satisfied. As we use a weight system to define $(v_1^A, v_2^A, \dots, v_n^A)$ and v^A , the requirements of (II) are also met. The following theorem verifies that our velocity concept satisfies condition (III) as well.

Theorem 4.6. *Let $A, B \in S_n$ with $A \sqsupseteq^* B$, and let δ_j ($j = 1, \dots, n$) be a weight system. Then, $v_j^A \geq v_j^B$ for every $j = 1, \dots, n$.*

Proof. Put $A = (A(i))_{i=1}^\infty$ and $B = (B(i))_{i=1}^\infty$, and fix some j with $j = 1, \dots, n$. Let $k \in \mathbb{N}$ be arbitrary. Since δ_j is monotone decreasing, we can write

$$\begin{aligned}\delta_j(k-1) &= \delta_j(k) + \varepsilon_j(k-1), \\ \delta_j(k-2) &= \delta_j(k) + \varepsilon_j(k-1) + \varepsilon_j(k-2), \\ &\vdots \\ \delta_j(1) &= \delta_j(k) + \varepsilon_j(k-1) + \varepsilon_j(k-2) + \dots + \varepsilon_j(1),\end{aligned}$$

with $\varepsilon_j(m) \geq 0$ ($m = 1, \dots, k-1$). Put $\varepsilon_j(k) = \delta_j(k)$. Using these relations, by a simple calculation we get

$$\begin{aligned}\sum_{i=1}^k A^{(j)}(i)\delta_j(i) - \sum_{i=1}^k B^{(j)}(i)\delta_j(i) &= \\ \sum_{m=1}^k \varepsilon_j(m) \left(\sum_{h=1}^m A^{(j)}(h) - \sum_{h=1}^m B^{(j)}(h) \right).\end{aligned}\tag{4.1}$$

Observe that as $\sum_{h=1}^m A^{(j)}(h) = f_j^A(m)$ and $\sum_{h=1}^m B^{(j)}(h) = f_j^B(m)$, by Theorem 3.7 and Definition 2.10, $A \sqsupseteq^* B$ implies

$$\sum_{h=1}^m A^{(j)}(h) \geq \sum_{h=1}^m B^{(j)}(h)$$

for every m with $1 \leq m \leq k$. As $\varepsilon_j(m) \geq 0$ ($m = 1, \dots, k$), (4.1) yields

$$\sum_{i=1}^k A^{(j)}(i)\delta_j(i) \geq \sum_{i=1}^k B^{(j)}(i)\delta_j(i).$$

By letting $k \rightarrow \infty$, we obtain $v_j^A \geq v_j^B$, and the proof is complete. $\square$

Remark 4.7. By the definition of velocity, the above theorem implies that if $A \sqsupseteq^* B$, then $v^A \geq v^B$.

In the following two remarks we explain why some alternative ways of introducing velocity would not be appropriate.

Remark 4.8. The monotonicity of δ_j is necessary to have Theorem 4.6. Indeed, let $A, B \in S_2$ be defined by

$$A = (2, 1, 1, 1, 1, \dots), \quad \text{and} \quad B = (1, 2, 1, 1, 1, \dots).$$

Moreover, let δ_1 be arbitrary, and put

$$\delta_2(i) = \begin{cases} \frac{1}{4}, & \text{if } i = 1, \\ \frac{1}{2^{i-1}}, & \text{otherwise.} \end{cases}$$

Clearly, $A \sqsupseteq^* B$ holds, but $v_2^A = \frac{6}{4}$ and $v_2^B = \frac{7}{4}$. Thus, $v_2^B \geq v_2^A$, and also $v^B \geq v^A$ in this case.

Remark 4.9. It would be possible to define v_j^A in a more general way. Namely, for any $m > 0$ we could put

$$v_{j,m}^A = \left(\sum_{i=1}^{\infty} (A^{(j)}(i))^m \delta_j(i) \right)^{\frac{1}{m}}.$$

However, on one hand the case $m < 1$ does not seem to be interesting. On the other hand, in case of $m > 1$ it is easy to find sequences $A, B \in S_n$ and a weight system δ_j ($j = 1, \dots, n$) such that Theorem 4.6, hence condition (III) fails for them.

As one can easily see, it can happen that with some weight system δ_j , $v_j^A \geq v_j^B$ for every $j = 1, \dots, n$, but $A \sqsupseteq^* B$ does not hold. However, in some sense we can reverse Theorem 4.6. More precisely, we have

Theorem 4.10. *Let $A, B \in S_n$. If for any weight system δ_j ($j = 1, \dots, n$), $v_j^A \geq v_j^B$ holds for all $j = 1, \dots, n$, then $A \sqsupseteq^* B$.*

Proof. Let $k \in \mathbb{N}$ be arbitrary, and for every j with $j \in \{1, \dots, n\}$ set

$$\delta_j^{(k)}(i) = \begin{cases} 1, & \text{if } i \leq k, \\ \frac{1}{2^{i-k}n}, & \text{otherwise.} \end{cases}$$

Clearly, the system $\delta_j^{(k)}$ ($j = 1, \dots, n$) is a weight system. Thus, by our assumptions we have

$$\sum_{i=1}^{\infty} A^{(j)}(i) \delta_j^{(k)}(i) = v_j^A \geq v_j^B = \sum_{i=1}^{\infty} B^{(j)}(i) \delta_j^{(k)}(i).$$

Hence, for every $j = 1, \dots, n$

$$\sum_{i=1}^k A^{(j)}(i) + \sum_{h=k+1}^{\infty} \frac{A^{(j)}(h)}{2^{h-k}n} \geq \sum_{i=1}^k B^{(j)}(i) + \sum_{h=k+1}^{\infty} \frac{B^{(j)}(h)}{2^{h-k}n}$$

holds. Replacing $\sum_{i=1}^k A^{(j)}(i)$ and $\sum_{i=1}^k B^{(j)}(i)$ by $f_j^A(k)$ and $f_j^B(k)$, respectively, we get

$$f_j^A(k) - f_j^B(k) \geq \sum_{h=k+1}^{\infty} \frac{B^{(j)}(h)}{2^{h-k}n} - \sum_{h=k+1}^{\infty} \frac{A^{(j)}(h)}{2^{h-k}n} \geq \frac{1}{n} - 1.$$

Since $f_j^A(k) - f_j^B(k)$ is an integer, we may infer that

$$f_j^A(k) - f_j^B(k) \geq 0.$$

By Theorem 3.7 the proof is complete. $\square$

Remark 4.11. It can be easily verified that the condition $v_j^A \geq v_j^B$ for all $j = 1, \dots, n$ cannot be replaced by $v^A \geq v^B$.

4.3 Examples of weight systems

In this section we give examples of weight systems, and analyse the behaviour of the velocity concept. We investigate exponentially decreasing systems, and calculate the velocity of some concrete sequences with respect to different weight systems.

Let $c > 1$, and put

$$\delta_j(i) = \frac{1}{c^{i-1}}, \text{ for every } j = 1, \dots, n \text{ and } i \in \mathbb{N}.$$

Obviously, δ_j is a weight system with

$$\sum_{i=1}^{\infty} \delta_j(i) = \frac{c}{c-1} \quad (j = 1, \dots, n).$$

Consider the n D-neighbourhood sequences

$A = (h, 1, 1, 1, 1, \dots)$, and $B = (1, n, n, n, n, \dots)$, where $2 \leq h \leq n$.

Then

$$v^A = v_j^A = \frac{1}{c-1} + h, \text{ and } v^B = v_j^B = \frac{n}{c-1} + 1 \quad (j = 1, \dots, n).$$

Clearly, the sequences A and B cannot be compared by the ordering $\sqsubseteq^*$. We show how the relation between the velocity values of A and B change according to the choice of the parameter c .

First, suppose that $c > n$. Then, we have

$$v^A = v_j^A = \frac{1}{c-1} + h \geq \frac{1}{c-1} + 2 = \frac{c}{c-1} + 1 > \frac{n}{c-1} + 1 = v_j^B = v^B.$$

Using this weight system, we obtain a very strong condition, namely that $v^A > v^B$ if and only if A precedes B lexicographically.

Now, let $c = 2$. In this case we have

$$v^A = v_j^A = 1 + h \leq 1 + n = v_j^B = v^B,$$

with equality only for $h = n$.

Finally, set $c < 2$. By a simple calculation, we get $v^B = v_j^B > v_j^A = v^A$.

Summarizing, using such a weight system, we can get rid of the (sometimes excessive) importance of the first “few” elements of a neighbourhood sequence. In particular, for every $k \in \mathbb{N}$, by choosing a suitable c , we can have $v_j^B > v_j^A$ for the nD -sequences

$$A = (\underbrace{n, n, \dots, n}_k, 1, 1, 1, \dots), \text{ and } B = (\underbrace{1, 1, \dots, 1}_k, n, n, n, \dots).$$

On the other hand, by the appropriate choice of c we can give large significance of the first “few” elements of the neighbourhood sequences, ignoring their later elements.

By choosing other (e.g. polinomially decreasing) types of weight systems, we can have different properties. The weight system should be chosen appropriately for the actual application, as we can see from a practical example given in the following section.

4.4 An application for distributing information

In this section we give an application scheme of neighbourhood sequences and velocity in a network model, where the members of the network are the points of $\mathbb{Z}^2$. As we will discuss it in Chapter 7, neighbourhood sequences and velocity can be introduced also in other types of grids. Hence this application scheme could be used in such cases, too.

The network model shown in Figure 4.1 has an information source at the center (origin) of the system which distributes information to the other members (clients) of the network. The system is based on priority, that is if a client is “closer” to the origin than another one, it has greater priority, and receives the information earlier. We can think of subscription systems for instance, where clients pay different fees according to their position with respect to the information source.

It is worth indexing the clients according to their “reachability” from the origin. For this purpose, if a client sits on the point $(x, y) \in \mathbb{Z}^2$, then

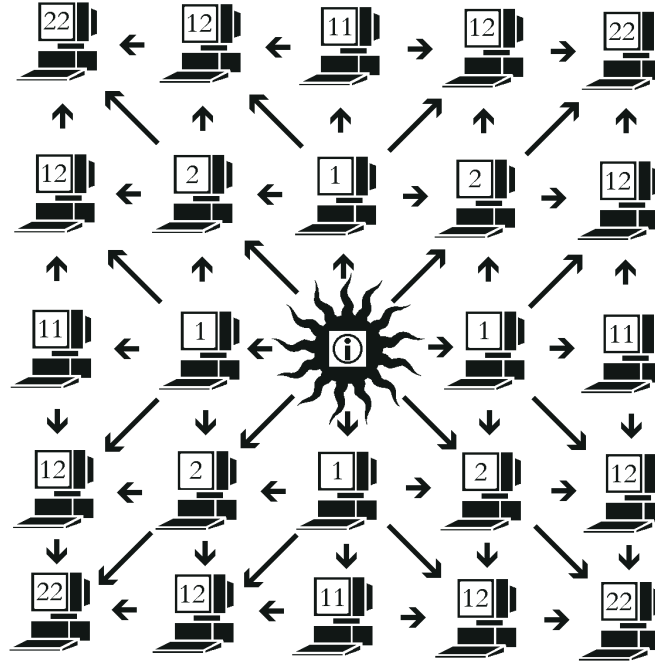


Figure 4.1: 2D priority-based model for distributing information.

its index will be given by the first few (significant) elements of the slowest neighbourhood sequence A , for which $d((x, y), (0, 0); A)$ is minimal. The clients with the same index have equal priority, so they should pay the same fee. In particular, clients indexed by “1” have the greatest priority.

In this model we use 2D-neighbourhood sequences to deliver the information to the clients. Suppose that the cost of distributing information decreases with the number of 2-s in the chosen neighbourhood sequence. Knowing the importance of the information, we have to choose one of the cheapest sequences which is still “fast” enough. That is, we take a neighbourhood sequence, whose velocity fits the importance of the information to be sent.

By choosing an appropriate weight system, we can increase and decrease the initial priority of the clients in the network. If we do not take much care of the clients residing far from the source, we need to choose a weight system which decreases rapidly. In the opposite case we can take a very slowly decreasing weight system.

This network model can be easily extended to $\mathbb{Z}^3$. In this case, we can take more advantage of the behaviour of neighbourhood sequences in lower dimensional subspaces. If we know in advance that a special type of information is important only for a group of clients, we can place these clients onto or close to the $[x, y]$, $[y, z]$ and $[x, z]$ planes. Thus for the distribution of this special kind of information we can choose quite a cheap neighbourhood sequence which consists of mainly 1 and 2 values. To have a similar possibility in 2D, we have to put such clients near the coordinate axes, and use sequences containing mostly 1-s.

4.5 Metric spaces of neighbourhood sequences

We introduce a metric on the set of neighbourhood sequences in a similar fashion as we did it for velocity.

Definition 4.12. Let $\Delta = \{\delta_j : j = 1, \dots, n\}$ be a weight system and $A, B \in S_n$. The distance ϱ_Δ of these sequences is defined by

$$\varrho_\Delta(A, B) = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^{\infty} |A^{(j)}(i) - B^{(j)}(i)| \delta_j(i).$$

Remark 4.13. One can easily verify that for any weight system Δ , the function ϱ_Δ is a metric on S_n .

Remark 4.14. The metric space (S_n, ϱ_Δ) is bounded. Its diameter is

$$\text{diam}(S_n, \varrho_\Delta) = \varrho_\Delta((1), (n)) = \frac{n-1}{n} \sum_{j=1}^n \sum_{i=1}^{\infty} \delta_j(i).$$

In what follows, we establish some useful and interesting properties of these metric spaces.

Theorem 4.15. *For any weight system Δ , (S_n, ϱ_Δ) is a complete metric space.*

Proof. Let Δ be any weight system. We prove the theorem by showing that every Cauchy sequence in S_n has a limit. We actually construct this limit sequence in the proof. Let $(A_k)_{k=1}^\infty$ be a Cauchy sequence in (S_n, ϱ_Δ) , and let $m \in \mathbb{N}$. By the definition of ϱ_Δ , there exists some $\varepsilon_m > 0$, such that for any $B, C \in S_n$, $\varrho_\Delta(B, C) < \varepsilon_m$ implies that the first m elements of B and C coincide. By the Cauchy property of $(A_k)_{k=1}^\infty$, there exists some $k_0 \in \mathbb{N}$ such that for every $k_1, k_2 > k_0$, $\varrho_\Delta(A_{k_1}, A_{k_2}) < \varepsilon_m$, whence the first m elements of the neighbourhood sequences A_{k_1} and A_{k_2} are identical. Define the sequence A in the following way. For every $m \in \mathbb{N}$ choose a $k_0 \in \mathbb{N}$, such that the m -th elements of A_{k_1} and A_{k_2} with $k_1, k_2 \geq k_0$ are equal. Let $A(m)$ be this element, and put $A = (A(m))_{m=1}^\infty$. Clearly, A is well defined. By the construction of A we immediately get that $\lim_{k \rightarrow \infty} A_k = A$. $\square$

Definition 4.16. A sequence $(A_k)_{k=1}^\infty$ is monotone increasing (resp. decreasing), if $A_{i+1} \supseteq^* A_i$ (resp. $A_i \supseteq^* A_{i+1}$) holds for every $i \in \mathbb{N}$.

Theorem 4.17. *Every monotone increasing or decreasing sequence $(A_k)_{k=1}^\infty$, with $A_k \in S_n$ ($k \in \mathbb{N}$) is convergent.*

Proof. As in the previous proof, we construct the limit of $(A_k)_{k=1}^\infty$. We may assume that $(A_k)_{k=1}^\infty$ is monotone increasing, the proof in the other case is similar.

Put $A_k = (A_k(i))_{i=1}^\infty$. As $(A_k)_{k=1}^\infty$ is increasing, so is $(A_k(1))_{k=1}^\infty$. As $n \geq A_k(1)$ ($k \in \mathbb{N}$), there exists some $k_0 \in \mathbb{N}$ such that for any $k_1, k_2 \geq k_0$ we have $A_{k_1}(1) = A_{k_2}(1)$. Put $A(1) = A_{k_0}(1)$. Suppose that $A(i)$ is already given for $i \leq m$, and define $A(m+1)$ in the following way. Choose $t_0 \in \mathbb{N}$ such that for $t_1, t_2 \geq t_0$ and $1 \leq i \leq m$, $A_{t_1}(i) = A_{t_2}(i)$ holds. Since $(A_k)_{k=t_0}^\infty$ is increasing, so is the sequence $(A_k(m+1))_{k=t_0}^\infty$. As $n \geq A_k(m+1)$ for every $k \in \mathbb{N}$, there exists some $s_0 \in \mathbb{N}$ such that for any $s_1, s_2 \geq s_0$ we have $A_{s_1}(m+1) = A_{s_2}(m+1)$. Put $A(m+1) = A_{s_0}(m+1)$.

From the construction of $A = (A(m))_{m=1}^{\infty}$ it is clear that for every $m \in \mathbb{N}$ there exists some $k_0 \in \mathbb{N}$, such that if $k \geq k_0$, the first m elements of A_k and A coincide. Thus, $\lim_{k \rightarrow \infty} A_k = A$, and the theorem follows. $\square$

The next result shows that the Bolzano-Weierstrass theorem is true in the constructed metric spaces.

Proposition 4.18. *For any weight system Δ , every subset of (S_n, ϱ_{Δ}) of infinite cardinality has an accumulation point.*

Proof. Let H be an infinite subset of S_n . We construct an accumulation point of H . Let $A(1)$ be a number which is the first element of infinitely many sequences in H . Suppose that $A(i)$ with $i \leq m$ is already defined. Let $A(m+1)$ be a number which is the $(m+1)$ -th element of infinitely many such sequences in H , whose first m elements are $A(1), A(2), \dots, A(m)$. Put $A = (A(m))_{m=1}^{\infty}$. Clearly, A is an accumulation point of H . $\square$

Periodic neighbourhood sequences may play important role in certain applications. The following result shows that they form a dense subset of (S_n, ϱ_{Δ}) . As the set of periodic neighbourhood sequences is countable, this also yields that (S_n, ϱ_{Δ}) is a separable metric space.

Theorem 4.19. *For any weight system Δ , the set of periodic neighbourhood sequences is dense in (S_n, ϱ_{Δ}) .*

Proof. Let $A = (A(i))_{i=1}^{\infty} \in S_n$ and $\varepsilon > 0$. By the definition of ϱ_{Δ} there exists some $k_0 \in \mathbb{N}$, such that if the first k_0 elements of $B \in S_n$ is the same as those of A , then $\varrho_{\Delta}(A, B) < \varepsilon$ holds. So put $B(i) = A(i \bmod k_0)$ ($i \in \mathbb{N}$), and $B = (B(i))_{i=1}^{\infty}$. Clearly, B is periodic and $\varrho_{\Delta}(A, B) < \varepsilon$, thus the proof is complete. $\square$

Chapter 5

Geometric properties of neighbourhood sequences

DAS and CHATTERJI [9] investigated some geometric behaviour of the regions occupied by 2D periodic neighbourhood sequences. They analysed the geometric properties of these octagons, and performed an approximation of the Euclidean distance by digital metrics coming from neighbourhood sequences. In this chapter we extend these geometric investigations to arbitrary finite dimensions. Moreover, we use general neighbourhood sequences in our analysis. Our achievements are also presented in [19]. The structure of this chapter is as follows. In Section 5.1 we extend the former results of [9] to the n D digital space $\mathbb{Z}^n$. We perform the geometric analysis of regions occupied by n D-neighbourhood sequences, with respect to convexity, symmetry, etc. In Section 5.2 we give a detailed and illustrated description of the geometric behaviour of (general) neighbourhood sequences in $\mathbb{Z}^2$ and $\mathbb{Z}^3$.

5.1 Geometric properties of n D-neighbourhood sequences

We start our geometric investigations in the general digital space $\mathbb{Z}^n$. Since neighbourhood sequences spread in an “isotropic” way, the occupied regions are symmetric objects. More precisely, we have the following theorem.

Theorem 5.1. *Let $A \in S_n$ and $k \in \mathbb{N}$. If a point $p \in \mathbb{Z}^n$ with coordinates $(p_1, p_2, \dots, p_n)$ belongs to $\widehat{A_k}$, then the points with coordinates*

$$(\lambda_1 p_{i_1}, \lambda_2 p_{i_2}, \dots, \lambda_n p_{i_n})$$

also belong to $\widehat{A_k}$. Here $\lambda_j = \pm 1$ ($j = 1, \dots, n$), and $(i_1, i_2, \dots, i_n)$ is an arbitrary permutation of $(1, 2, \dots, n)$.

Proof. Suppose that $p \in \widehat{A_k}$ with coordinates $(p_1, p_2, \dots, p_n)$. Let $p' \in \mathbb{Z}^n$ with coordinates $(\lambda_1 p_{i_1}, \lambda_2 p_{i_2}, \dots, \lambda_n p_{i_n})$, where $\lambda_i = \pm 1$ ($i = 1, \dots, n$), and $(i_1, i_2, \dots, i_n)$ is an arbitrary permutation of $(1, 2, \dots, n)$. Observe that for every $j \in \{1, \dots, n\}$, the same a_j are obtained in Lemma 2.9 for both $(\mathbf{0}, p)$ and $(\mathbf{0}, p')$, and $\|\mathbf{0}, p\| = \|\mathbf{0}, p'\|$. Using Lemma 2.9 we get $d(\mathbf{0}, p; A) = d(\mathbf{0}, p'; A)$ which completes the proof. $\square$

Remark 5.2. It is easy to verify that the theorem also holds for the points of $H(\widehat{A_k})$.

Using the above results we can find hyperplanes, for which the regions occupied by neighbourhood sequences are symmetric.

Remark 5.3. Let $A \in S_n$ and $k \in \mathbb{N}$. Then $H(\widehat{A_k})$ is symmetric to those $(n-1)$ D hyperplanes that contain $(n-1)$ coordinate axes (this implies that the coordinate values can change sign), and to their rotations by 45° around any axis they contain (this implies that coordinates can be permuted).

Now we calculate the coordinates of vertices of polyhedra occupied by neighbourhood sequences.

Theorem 5.4. *Let $A \in S_n$ and $k \in \mathbb{N}$. The vertices of $H(\widehat{A}_k)$ are exactly those points, whose coordinates are the permutation of the values*

$$\left(\lambda_1 \sum_{i=1}^n A(i, k), \lambda_2 \sum_{i=2}^n A(i, k), \dots, \lambda_n A(n, k) \right),$$

where $\lambda_j = \pm 1$, for every $j = 1, \dots, n$, and λ_j can be different in every permutation.

Proof. We prove the theorem by induction. DAS and CHATTERJI in [9] showed that in 2D the coordinates of the vertices of the occupied regions are the permutation of the following ones

$$(\lambda_1(A(1, k) + A(2, k)), \lambda_2 A(2, k)),$$

where $\lambda_j = \pm 1$, ($j = 1, 2$).

Now suppose that for any $H(\widehat{A}_k)$ in $\mathbb{Z}^n$ the coordinates of the vertices are the permutations of

$$\left(\lambda_1 \sum_{i=1}^n A(i, k), \lambda_2 \sum_{i=2}^n A(i, k), \dots, \lambda_n A(n, k) \right),$$

with $\lambda_j = \pm 1$ ($j = 1, \dots, n$). Let $B \in S_{n+1}$ and $k \in \mathbb{N}$. The projection of $H(\widehat{B}_k) \subset \mathbb{Z}^{n+1}$ onto $\mathbb{Z}^n$ can be obtained by changing the $(n+1)$ -th coordinate to 0. Moreover, since in $\mathbb{Z}^n$ there are only n coordinates, in determining the projection, the values $(n+1)$ in B act as the values n . Thus, by the induction hypothesis, the coordinates of the vertices of the projection of $H(\widehat{B}_k)$ are

$$\left(\lambda_1 \sum_{i=1}^{n+1} A(i, k), \lambda_2 \sum_{i=2}^{n+1} A(i, k), \dots, \lambda_n \sum_{i=n}^{n+1} A(i, k), 0 \right), \quad (5.1)$$

where the first n coordinates can be permuted arbitrarily, and $\lambda_j = \pm 1$ ($j = 1, \dots, n$).

Now to get the vertices of $H(\widehat{B}_k)$ in $\mathbb{Z}^{n+1}$ we should choose the “maximal” value for the $(n+1)$ -th coordinate in (5.1). It means that the absolute

value of the $(n+1)$ -th coordinate must be $A(n+1, k)$. The symmetric behaviour of $H(\widehat{B}_k)$ implies that the coordinates of the vertices of $H(\widehat{B}_k)$ are the permutations of the coordinates

$$\left(\lambda_1 \sum_{i=1}^{n+1} A(i, k), \lambda_2 \sum_{i=2}^{n+1} A(i, k), \dots, \lambda_{n+1} A(n+1, k) \right),$$

with $\lambda_j = \pm 1$ ($j = 1, \dots, n+1$), and the proof is complete. $\square$

Remark 5.5. Using the above theorem we obtain that the maximal number of vertices of $H(\widehat{A}_k)$ is $n! \cdot 2^n$. The polyhedron $H(\widehat{A}_k)$ can be degenerate if some of the elements $(1, \dots, n)$ do not occur in A , causing the decrease of the number of the vertices.

We investigate the existence of some convergence limits related to the sequence of polyhedra $(H(\widehat{A}_k))_{k=1}^\infty$. We use the densities of the $1, \dots, n$ elements in A to calculate the corresponding limits.

In the general n -dimensional case only the asymptotic behaviour of the diameter of $H(\widehat{A}_k)$ is considered. We work with the usual definition of diameter, that is $\text{diam}(H(\widehat{A}_k)) = \sup_{p, q \in H(\widehat{A}_k)} L_2(p, q)$.

Theorem 5.6. *Let $A \in S_n$, and suppose that $D^A(i)$ exists for every $i = 1, \dots, n$. Then*

$$\lim_{k \rightarrow \infty} \frac{\text{diam}(H(\widehat{A}_k))}{k} = 2 \left(\sum_{i=1}^n i D^A(i)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n i D^A(i) D^A(j) \right)^{\frac{1}{2}}.$$

Proof. By Remark 5.3 and Theorem 5.4, $\text{diam}(H(\widehat{A}_k))$ is the L_2 distance of the points

$$\left(\sum_{i=1}^n A(i, k), \sum_{i=2}^n A(i, k), \dots, A(n, k) \right),$$

and

$$\left(-\sum_{i=1}^n A(i, k), -\sum_{i=2}^n A(i, k), \dots, -A(n, k) \right).$$

We obtain

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \frac{\text{diam}(H(\widehat{A}_k))}{k} = \\
& \lim_{k \rightarrow \infty} \frac{2 \left(\sum_{i=1}^n i A(i, k)^2 + 2 \sum_{i=1}^{n-1} i \sum_{j=i+1}^n A(i, k) A(j, k) \right)^{\frac{1}{2}}}{k} = \\
& = 2 \left(\sum_{i=1}^n i D^A(i)^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n i D^A(i) D^A(j) \right)^{\frac{1}{2}}
\end{aligned}$$

which completes the proof. $\square$

5.2 Geometric properties of 2D- and 3D-neighbourhood sequences

In this section we present a detailed investigation of the geometry of neighbourhood sequences in $\mathbb{Z}^2$ and $\mathbb{Z}^3$. We note that our analysis could be performed in higher dimensions and also in other kind of digital spaces (e.g. that are based on hexagonal or triangular neighbouring relations).

5.2.1 The 2D case

In [9] DAS and CHATTERJI showed that for every 2D periodic neighbourhood sequence A , $H(\widehat{A}_k)$ is always an octagon. They calculated the coordinates of the vertices of the octagon, and also the length of its sides based on the relative frequency of the 1 and 2 values in a period of A . Using general neighbourhood sequences we obtain similar results, but in this case we have to work with the densities of the 1 and 2 values in the neighbourhood sequence.

The well-known cityblock and chessboard motions can be represented by the constant neighbourhood sequences (1) and (2), respectively. In these cases we also obtain “octagons” after every step, though they become

degenerate as it can be seen in Figure 5.1. In this figure we show a “typical” octagon occupied by the periodic neighbourhood sequence $(1, 2)$, as well.

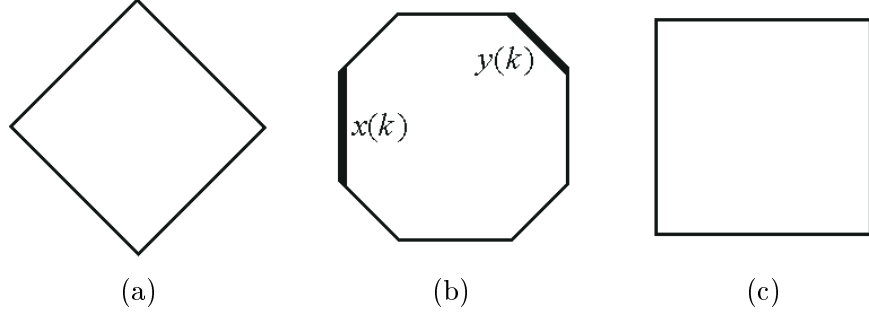


Figure 5.1: Octagons occupied by the 2D-neighbourhood sequence A , when (a) $A = (1)$, (b) $A = (1, 2)$, (c) $A = (2)$.

Remark 5.7. If we start any 2D-neighbourhood sequence A from the origin, then for every $k \in \mathbb{N}$ the octagon $H(\widehat{A}_k)$ is symmetric (by Remark 5.3) to both coordinate axes, and to their 45° rotations.

In what follows, we calculate some geometric parameters of these octagons, namely the length of their sides, their perimeters and areas. We give these data in terms of $A(1, k)$ and $A(2, k)$.

Definition 5.8. Let A be a 2D-neighbourhood sequence. Let $x^A(k)$ be the length of the horizontal and $y^A(k)$ be the length of the inclined sides of the octagon $H(\widehat{A}_k)$, illustrated in Figure 5.1. Moreover, let $P_{2D}^A(k)$ be the perimeter and $V_{2D}^A(k)$ be the area of this octagon.

Remark 5.9. Because of symmetry, the lengths of the horizontal and vertical sides of these octagons are equal, and the same holds for the inclined sides.

Proposition 5.10. *Using the above notation, the following relations hold:*

- $x^A(k) = 2A(2, k)$,

- $y^A(k) = \sqrt{2}A(1, k)$,
- $P_{2D}^A(k) = 4(\sqrt{2}A(1, k) + 2A(2, k))$,
- $V_{2D}^A(k) = 2(A(1, k)^2 + 4A(1, k)A(2, k) + 2A(2, k)^2)$.

Proof. From Theorem 5.4 we know the coordinates of the vertices of $H(\widehat{A_k})$. So the statement follows by a simple calculation. $\square$

Now we describe the asymptotic behaviour of the sequences $x^A(k)/k$, $y^A(k)/k$, $P_{2D}^A(k)/k$, and $V_{2D}^A(k)/k^2$. We will do this using the lower and upper densities of the 1 and 2 values in the neighbourhood sequence.

Theorem 5.11. *Let $A \in S_2$. Then we have*

- $\liminf_{k \rightarrow \infty} \frac{x^A(k)}{k} = 2(1 - S^A(1)), \quad \limsup_{k \rightarrow \infty} \frac{x^A(k)}{k} = 2(1 - s^A(1));$
- $\liminf_{k \rightarrow \infty} \frac{y^A(k)}{k} = \sqrt{2}s^A(1), \quad \limsup_{k \rightarrow \infty} \frac{y^A(k)}{k} = \sqrt{2}S^A(1);$
- $\liminf_{k \rightarrow \infty} \frac{P_{2D}^A(k)}{k} = 8 + (4\sqrt{2} - 8)S^A(1), \quad \limsup_{k \rightarrow \infty} \frac{P_{2D}^A(k)}{k} = 8 + (4\sqrt{2} - 8)s^A(1);$
- $\liminf_{k \rightarrow \infty} \frac{V_{2D}^A(k)}{k^2} = 2(2 - S^A(1)^2), \quad \limsup_{k \rightarrow \infty} \frac{V_{2D}^A(k)}{k^2} = 2(2 - s^A(1)^2).$

Proof. Using the identity $A(1, k) + A(2, k) = k$ and the well-known properties of $\liminf$ and $\limsup$, the statements are immediate consequences of Proposition 5.10. $\square$

We note that as $s^A(1) + S^A(2) = s^A(2) + S^A(1) = 1$, the above $\liminf$ and $\limsup$ values can be expressed using $s^A(2)$ and $S^A(2)$. If the densities $D^A(1)$ and $D^A(2)$ exist, we can obtain the above limits by their help.

Corollary 5.12. *Let $A \in S_2$. If $D^A(1)$ exists, then we have*

- $\lim_{k \rightarrow \infty} \frac{x^A(k)}{k} = 2(1 - D^A(1)),$
- $\lim_{k \rightarrow \infty} \frac{y^A(k)}{k} = \sqrt{2}D^A(1),$
- $\lim_{k \rightarrow \infty} \frac{P_{2D}^A(k)}{k} = 8 + (4\sqrt{2} - 8)D^A(1),$

- $\lim_{k \rightarrow \infty} \frac{V_{2D}^A(k)}{k^2} = 2(2 - D^A(1)^2).$

Remark 5.13. We can formulate the opposite statement as well. Namely, for any $A \in S_2$ the density $D^A(1)$ exists, if and only if any of the convergence limits formulated in Corollary 5.12 exist.

5.2.2 The 3D case

We continue our analysis by studying the geometry of neighbourhood sequences in $\mathbb{Z}^3$. As a consequence of Theorem 5.4 we obtain that if $A \in S_3$ and $k \in \mathbb{N}$, then the region $H(\widehat{A}_k)$ is a polyhedron with at most $3! \cdot 2^3 = 48$ vertices, 72 edges and 26 faces. The vertices can be obtained by permuting the coordinates $(\lambda_1(A(1, k) + A(2, k) + A(3, k)), \lambda_2(A(2, k) + A(3, k)), \lambda_3 A(3, k))$, where $\lambda_i = \pm 1$, $(i = 1, 2, 3)$. Moreover, the polyhedron $H(\widehat{A}_k)$ is symmetric to the planes $[x, y]$, $[y, z]$ and $[x, z]$, and to their 45° rotations around each of the coordinate axes they contain. The symmetry of such a polyhedron to these planes can be observed in Figures 5.2, 5.3 and 5.4.

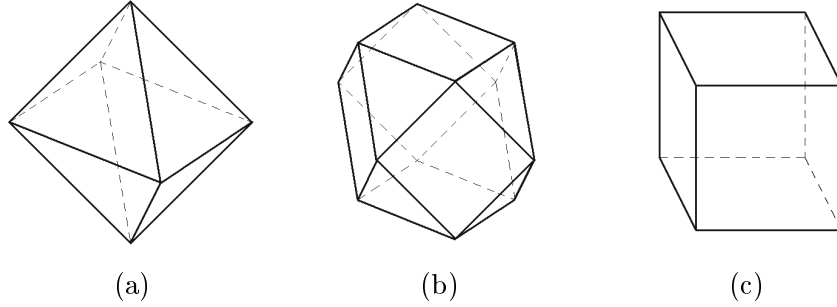


Figure 5.2: Polyhedra occupied by the constant 3D-neighbourhood sequence A , when (a) $A = (1)$, (b) $A = (2)$, (c) $A = (3)$.

We give some examples to illustrate how 3D-neighbourhood sequences spread in $\mathbb{Z}^3$. Beside the general case, we show degenerate polyhedra that belong to special types of neighbourhood sequences. In Figure 5.2, we can

see polyhedra occupied by constant neighbourhood sequences. In our next example (see Figure 5.3) we illustrate the occupied polyhedra of neighbourhood sequences containing only two different values.

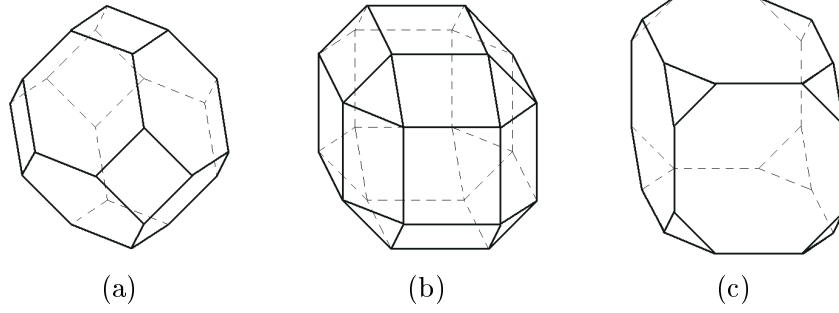


Figure 5.3: Polyhedra occupied by the 3D-neighbourhood sequence A containing two elements, when (a) $A = (1, 2)$, (b) $A = (1, 3)$, (c) $A = (2, 3)$.

Finally, in Figure 5.4 we show the occupied polyhedron of the 3D-neighbourhood sequence $(1, 2, 3)$ after k steps. We also indicate the edges which can be of different lengths in the general case.

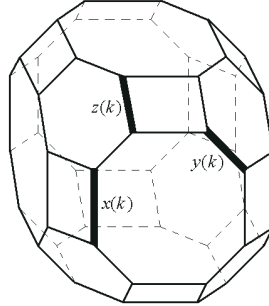


Figure 5.4: Polyhedron occupied by the 3D-neighbourhood sequence $A = (1, 2, 3)$.

Similarly to the 2D case, we calculate some geometric parameters of these polyhedra, namely the length of their sides, their surfaces and vol-

umes. Again, we investigate the existence of some convergence limits related to the sequence of polyhedra $(H(\widehat{A_k}))_{k=1}^{\infty}$. As in 2D, we express these values by the densities of the elements in the neighbourhood sequence A . The side lengths, surfaces and volumes of these polyhedra can be given in terms of $A(1, k)$, $A(2, k)$ and $A(3, k)$.

Definition 5.14. Let $A \in S_3$. By symmetry, the sides of $H(\widehat{A_k})$ are of (at most) three different lengths. They are indicated by $x^A(k)$, $y^A(k)$ and $z^A(k)$, as it is shown in Figure 5.4. Moreover, let $P_{3D}^A(k)$ be the surface and $V_{3D}^A(k)$ be the volume of this polyhedron.

Proposition 5.15. *Using the above notation, the following relations hold:*

- $x^A(k) = 2A(3, k)$;
- $y^A(k) = \sqrt{2}A(2, k)$;
- $z^A(k) = \sqrt{2}A(1, k)$;
- $P_{3D}^A(k) = 6P_{oct} + 12P_{rec} + 8P_{hex}$, where

$$P_{oct} = 2(A(2, k)^2 + 4A(2, k)A(3, k) + 2A(3, k)^2),$$

$$P_{rec} = 2\sqrt{2}A(1, k)A(3, k),$$

$$P_{hex} = \frac{\sqrt{3}}{2}(A(1, k)^2 + 4A(1, k)A(2, k) + A(2, k)^2);$$
- $V_{3D}^A(k) = 2P_{oct}H_{oct} + 4P_{rec}H_{rec} + \frac{8}{3}P_{hex}H_{hex}$, where

$$H_{oct} = A(1, k) + A(2, k) + A(3, k),$$

$$H_{rec} = \frac{\sqrt{2}}{2}(A(1, k) + 2A(2, k) + 2A(3, k)),$$

$$H_{hex} = \frac{\sqrt{3}}{3}(A(1, k) + 2A(2, k) + 3A(3, k)).$$

Proof. Similarly to Proposition 5.10 in the 2D case, the length of the sides can be obtained by calculating the L_2 distance of certain vertices, whose coordinates are given by Theorem 5.4. The surface consists of 6 octagons, 12 rectangles and 8 hexagons. The volume can be derived by adding the volumes of certain pyramids, with octagonal, rectangular and hexagonal bases, respectively. $\square$

Now we study the convergence limits of the sequences $x^A(k)/k$, $y^A(k)/k$, $z^A(k)/k$, $P_{3D}^A(k)/k^2$, and $V_{3D}^A(k)/k^3$. As in the 2D case, we use the lower and upper density values of the elements in A to estimate these limits.

Theorem 5.16. *Let $A \in S_3$. Then we have*

$$\begin{aligned}
\liminf_{k \rightarrow \infty} \frac{x^A(k)}{k} &= 2s^A(3), & \limsup_{k \rightarrow \infty} \frac{x^A(k)}{k} &= 2S^A(3); \\
\liminf_{k \rightarrow \infty} \frac{y^A(k)}{k} &= \sqrt{2}s^A(2), & \limsup_{k \rightarrow \infty} \frac{y^A(k)}{k} &= \sqrt{2}S^A(2); \\
\liminf_{k \rightarrow \infty} \frac{z^A(k)}{k} &= \sqrt{2}s^A(1), & \limsup_{k \rightarrow \infty} \frac{z^A(k)}{k} &= \sqrt{2}S^A(1); \\
\liminf_{k \rightarrow \infty} \frac{P_{3D}^A(k)}{k^2} &\geq 12(s^A(2)^2 + 4s^A(2)s^A(3) + 2s^A(3)^2) + 24\sqrt{2}s^A(1)s^A(3) + \\
&\quad 4\sqrt{3}(s^A(1)^2 + 4s^A(1)s^A(2) + s^A(2)^2); \\
\limsup_{k \rightarrow \infty} \frac{P_{3D}^A(k)}{k^2} &\leq 12(S^A(2)^2 + 4S^A(2)S^A(3) + 2S^A(3)^2) + 24\sqrt{2}S^A(1)S^A(3) + \\
&\quad 4\sqrt{3}(S^A(1)^2 + 4S^A(1)S^A(2) + S^A(2)^2); \\
\liminf_{k \rightarrow \infty} \frac{V_{3D}^A(k)}{k^3} &\geq 4(s^A(2)^2 + 4s^A(2)s^A(3) + 2s^A(3)^2) + \\
&\quad 8s^A(1)s^A(3)(1 + s^A(2) + s^A(3)) + \\
&\quad \frac{4}{3}(s^A(1)^2 + 4s^A(1)s^A(2) + s^A(2)^2)(1 + s^A(2) + 2s^A(3)); \\
\limsup_{k \rightarrow \infty} \frac{V_{3D}^A(k)}{k^3} &\leq 4(S^A(2)^2 + 4S^A(2)S^A(3) + 2S^A(3)^2) + \\
&\quad 8S^A(1)S^A(3)(1 + S^A(2) + S^A(3)) + \\
&\quad \frac{4}{3}(S^A(1)^2 + 4S^A(1)S^A(2) + S^A(2)^2)(1 + S^A(2) + 2S^A(3)).
\end{aligned}$$

Proof. The statements are immediate consequences of Proposition 5.15 and the well-known properties of $\liminf$ and $\limsup$. $\square$

Contrary to 2D (Theorem 5.11), in some cases we can have only inequalities for $\liminf$ and $\limsup$ in 3D and also in higher dimensions. If the densities of the elements of a neighbourhood sequence exist we get equalities.

Corollary 5.17. *Let $A \in S_3$. If $D^A(1)$, $D^A(2)$ and $D^A(3)$ exist, then we have*

$$\begin{aligned}
\lim_{k \rightarrow \infty} \frac{x^A(k)}{k} &= 2D^A(3); \\
\lim_{k \rightarrow \infty} \frac{y^A(k)}{k} &= \sqrt{2}D^A(2); \\
\lim_{k \rightarrow \infty} \frac{z^A(k)}{k} &= \sqrt{2}D^A(1); \\
\lim_{k \rightarrow \infty} \frac{P_3^A(k)}{k^2} &= 12(D^A(2)^2 + 4D^A(2)D^A(3) + 2D^A(3)^2) + 24\sqrt{2}D^A(1)D^A(3) + \\
&\quad 4\sqrt{3}(D^A(1)^2 + 4D^A(1)D^A(2) + D^A(2)^2); \\
\lim_{k \rightarrow \infty} \frac{V_{3D}^A(k)}{k^3} &= 4(D^A(2)^2 + 4D^A(2)D^A(3) + 2D^A(3)^2) + \\
&\quad 8D^A(1)D^A(3)(1 + D^A(2) + D^A(3)) + \\
&\quad \frac{4}{3}(D^A(1)^2 + 4D^A(1)D^A(2) + D^A(2)^2)(1 + D^A(2) + 2D^A(3)).
\end{aligned}$$

Remark 5.18. Similarly to the 2D case, an opposite statement also can be formulated. Namely, for any 3D-neighbourhood sequence the densities $D^A(1)$, $D^A(2)$ and $D^A(3)$ exist if and only if the convergence limit(s) concerning any two sides, or the surface or the volume in Corollary 5.17 exist(s). This claim easily follows from Theorem 5.16.

Chapter 6

Approximating the Euclidean distance by digital metrics

In this chapter we discuss some possibilities of approximating the Euclidean metric L_2 in $\mathbb{Z}^2$ by the help of digital metrics induced by neighbourhood sequences. The main advantage of neighbourhood sequences over the classical distances d_4 , d_8 and d_{oct} is that they provide more flexibility in moving on the plane.

As $L_1(q, r) \leq L_2(q, r) \leq L_\infty(q, r)$ for any $q, r \in \mathbb{Z}^2$, it is a natural problem to find the neighbourhood sequences, whose distance functions approximate L_2 best, since clearly,

$$L_1(q, r) = d_4(q, r) = d(q, r; (1)), \quad \text{and} \quad L_\infty(q, r) = d_8(q, r) = d(q, r; (2)).$$

Thus, for every $A \in S_2$, we have $L_1(q, r) \leq d(q, r; A) \leq L_\infty(q, r)$.

Using neighbourhood sequences, Das [3] determined distance functions that provide good approximations of the L_2 distance in a certain sense. However, he used periodic neighbourhood sequences only, and restricted his investigations to the so called simple metrics. In contrast with the periodic case, using general neighbourhood sequences we can give the actually best

approximating sequence, instead of a finite part of it. In our approximation procedures, we will compare the regions occupied by a 2D-sequence with the Euclidean disks. For illustration, Figure 6.1 shows that the metric d_{oct} (generated by $(1, 2)$) is “closer” to L_2 than d_4 or d_8 .

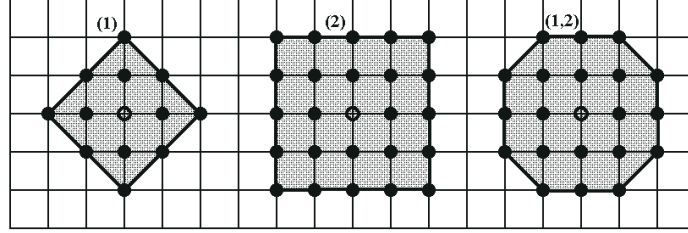


Figure 6.1: The regions occupied by the neighbourhood sequences (1) , (2) and $(1, 2)$ after two steps.

Since the best approximating sequences we obtain are (mostly) Beatty sequences, we give a brief overview on their most important properties. Let $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$, and let $A = (A(i))_{i=1}^{\infty}$, $B = (B(i))_{i=1}^{\infty}$ be sequences of 1-s and 2-s, defined by

$$A(i) = \lfloor i\alpha \rfloor - \lfloor (i-1)\alpha \rfloor + 1, \quad B(i) = \lceil i\alpha \rceil - \lceil (i-1)\alpha \rceil + 1 \quad (i \in \mathbb{N}).$$

The sequences A, B are called Beatty sequences on the letters 1, 2. Clearly, for every $k \in \mathbb{N}$ we have

$$A(2, k) = \lfloor k\alpha \rfloor \quad \text{and} \quad B(2, k) = \lceil k\alpha \rceil.$$

Conversely, these equalities define Beatty sequences which are uniquely determined. We refer to [26] for the basic properties of Beatty sequences and their generalizations.

The structure of this chapter is as follows. In Section 6.1 we formulate three problems about approximating the Euclidean distance with digital ones coming from neighbourhood sequences. In Section 6.2 we solve these

problems, by giving the best approximating neighbourhood sequences. Finally, in Section 6.3 we compare the sequences obtained with those recommended by DAS in [3]. As we use a different method to measure the error of the approximation than the author in [3], we choose a third type of error function for this purpose. Our results are also published in [20].

6.1 Three approximation problems

To decide how a digital distance $d(A)$ approximates the Euclidean distance L_2 on $\mathbb{Z}^2$, we will compare their k -disks $\widehat{A}_k$ and O_k , respectively. A natural approach could be to choose the number of integer points in the symmetric difference $\widehat{A}_k \nabla O_k$ as an error function. However, there is no exact formula for the number of integer points inside O_k . So we will follow a slightly different method. Namely, we will compare the sets $H(\widehat{A}_k)$ and G_k , and we will choose A to minimize the area of $H(\widehat{A}_k) \nabla G_k$. Of course, it can be done only separately for each $k \in \mathbb{N}$. However, surprisingly it turns out that for every k the very same A can be chosen to minimize this area. So this neighbourhood sequence A can be regarded as the one that approximates L_2 best (in the above sense).

According to these principles, we will investigate the function

$$TE_A(k) = \text{Area}(H(\widehat{A}_k) \nabla G_k),$$

called the total error of the approximation at the k -th step. We will also use the relative error at the k -th step, defined as

$$RE_A(k) = \frac{TE_A(k)}{k^2 \pi},$$

and the limit relative error (if it exists)

$$RE_A = \lim_{k \rightarrow \infty} RE_A(k).$$

We perform several types of approximation. Our aims can be summarized by the following problems. The first problem concerns the general case.

Problem 6.1. Find a neighbourhood sequence $\Lambda \in S_2$ (if exists) such that for every $A \in S_2$ and $k \in \mathbb{N}$

$$\text{Area}(H(\widehat{\Lambda}_k) \nabla G_k) \leq \text{Area}(H(\widehat{A}_k) \nabla G_k).$$

We consider separately the case when the octagons $H(\widehat{A}_k)$ cover G_k for every $k \in \mathbb{N}$, that is the corresponding function $d(A)$ minorates L_2 .

Problem 6.2. Find a neighbourhood sequence $\Phi \in S_2$ (if exists) such that $H(\widehat{\Phi}_k) \supseteq G_k$ for every $k \in \mathbb{N}$, and for every $A \in S_2$, $H(\widehat{A}_k) \supseteq G_k$ implies that

$$\text{Area}(H(\widehat{\Phi}_k) \setminus G_k) \leq \text{Area}(H(\widehat{A}_k) \setminus G_k).$$

Note that it does not make sense to consider a problem with $H(\widehat{A}_k) \subseteq G_k$. Indeed, observe that $H(\widehat{A}_k)$ is contained in G_k if and only if the first k elements of A are all 1-s. This is the reason why we do not take up the problem of majorating L_2 by digital metrics $d(A)$. Figure 6.2 illustrates Problems 6.1 and 6.2.

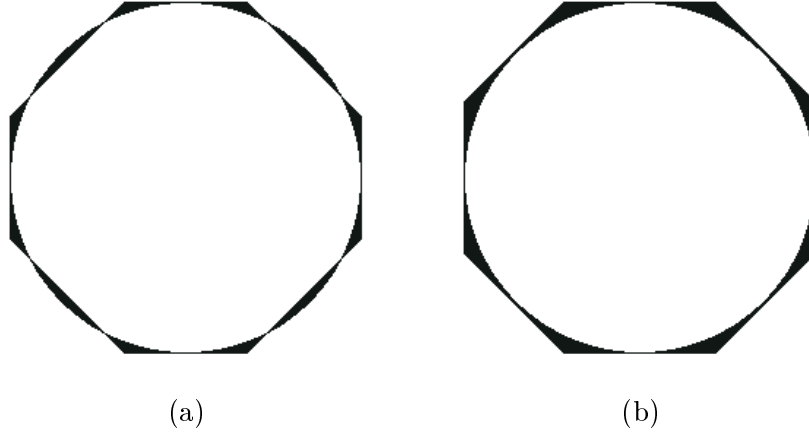


Figure 6.2: The error of approximation (a) in the general case (b) when $H(\widehat{A}_k) \supseteq G_k$.

We will construct two neighbourhood sequences, satisfying the requirements of Problems 6.1 and 6.2, respectively. Moreover, we will give a sequence such that the corresponding distance function $d(A)$ is a *metric*, and it can be considered as the digital metric which approximates L_2 best in the sense of Problem 6.1.

We also investigate the following “discrete” version of Problem 6.2. Note that Problem 6.1 does not have a similar variant.

Problem 6.3. *Find a neighbourhood sequence $\Psi \in S_2$ (if exists) such that $\widehat{\Psi}_k \supseteq O_k$ for every $k \in \mathbb{N}$, and if $A \in S_2$ with $\widehat{A}_k \supseteq O_k$, then $\widehat{A}_k \supseteq \widehat{\Psi}_k$.*

Observe that the sequence Ψ has the nice property that the corresponding distance function $d(\Psi)$ is “uniformly” the best one to approximate L_2 from below. That is, for any $A \in S_2$, if

$$d(q, r; A) \leq L_2(q, r), \quad \text{for any } q, r \in \mathbb{Z}^2,$$

then

$$d(q, r; A) \leq d(q, r; \Psi), \quad \text{for any } q, r \in \mathbb{Z}^2.$$

In Theorem 6.8 we will solve Problem 6.3, by constructing the sequence Ψ having the desired property. Interestingly, it will turn out that $d(\Psi)$ is a metric on $\mathbb{Z}^2$. To show this, the following lemma will be useful.

Lemma 6.4. *Let $\alpha \in \mathbb{R}$ with $0 \leq \alpha \leq 1$, and let $A = (A(i))_{i=1}^{\infty} \in S_2$ be the unique sequence with $A(2, k) = \lfloor k\alpha \rfloor$ for every $k \in \mathbb{N}$. Then $d(A)$ is a metric.*

Proof. Suppose to the contrary that $d(A)$ is not a metric. Then by using Theorem 2.7, for some $n, N \in \mathbb{N}$ with $n < N$

$$\sum_{i=1}^n A(i) > \sum_{i=N-n+1}^N A(i) \tag{6.1}$$

holds. Clearly, we may suppose that $N - n + 1 > n$. We rewrite (6.1) as

$$A(2, n) > A(2, N) - A(2, N - n). \tag{6.2}$$

Since all the numbers in (6.2) are integers, we obtain

$$A(2, N - n) \geq A(2, N) + 1 - A(2, n). \quad (6.3)$$

Observe that by the definition of A , for every $k \in \mathbb{N}$ we have

$$\frac{A(2, k)}{k} \leq \alpha < \frac{A(2, k) + 1}{k}. \quad (6.4)$$

Combining (6.3) and (6.4), we get

$$(N - n)\alpha \geq A(2, N - n) \geq A(2, N) + 1 - A(2, n) > N\alpha - n\alpha \quad (6.5)$$

which is a contradiction. Hence the lemma follows. $\square$

6.2 The solution of the approximation problems

In this section we construct “extremal” sequences described in Problems 6.1, 6.2, and 6.3. We start with Problem 6.2, being the simplest to handle.

6.2.1 Approximating L_2 from below

In this subsection we consider only neighbourhood sequences A with G_k contained in $H(\widehat{A_k})$ for all $k \in \mathbb{N}$. As we have already mentioned, it means that the corresponding distance function $d(A)$ minorates L_2 . The next result gives a solution to Problem 6.2.

Theorem 6.5. *Let $\Phi = (\Phi(i))_{i=1}^{\infty}$ be the unique 2D-neighbourhood sequence defined by $\Phi(2, k) = \lceil k(\sqrt{2} - 1) \rceil$ ($k \in \mathbb{N}$), that is*

$$\Phi(i) = \lceil i(\sqrt{2} - 1) \rceil - \lceil (i - 1)(\sqrt{2} - 1) \rceil + 1 \quad (i \in \mathbb{N}).$$

Then $H(\widehat{\Phi_k}) \supseteq G_k$ for any $k \in \mathbb{N}$. Moreover, $A \in S_2$, $H(\widehat{A_k}) \supseteq G_k$ implies that

$$\text{Area}(H(\widehat{\Phi_k}) \setminus G_k) \leq \text{Area}(H(\widehat{A_k}) \setminus G_k).$$

Proof. Let k be a fixed positive integer, and let $A \in S_2$ be arbitrary such that $H(\widehat{A_k})$ contains G_k . Clearly, the vertices of $H(\widehat{A_k})$ with positive coordinates are $(k, A(2, k))$ and $(A(2, k), k)$. Hence $H(\widehat{A_k}) \supseteq G_k$ implies that the line $x + y = k + A(2, k)$ has at most one point in common with the circle $x^2 + y^2 = k^2$ in the plane $[x, y]$. A simple calculation gives that it is equivalent to

$$(k - A(2, k))^2 - 2A(2, k)^2 \leq 0, \quad (6.6)$$

that is $A(2, k)/k \geq \sqrt{2} - 1$.

One can easily check that $\text{Area}(H(\widehat{A_k})) = 4k^2 - 2(k - A(2, k))^2$, whence the total error of the approximation at the k -th step is

$$TE_A(k) = 4k^2 - 2(k - A(2, k))^2 - \pi k^2. \quad (6.7)$$

Clearly, $TE_A(k)$ is minimal, when $A(2, k)$ is maximal. Taking into consideration that $A(2, k)/k \geq \sqrt{2} - 1$ and $A(2, k)$ is an integer, we obtain that $TE_A(k)$ takes its minimum when $A(2, k) = \lceil k(\sqrt{2} - 1) \rceil$. Hence the theorem follows. $\square$

Figure 6.3 shows how the octagons $H(\widehat{\Phi_k})$ approximate G_k for $k = 2, 5, 7, 9, 12$. The dark regions show the error of the approximation.

Remark 6.6. The octagons $H(\widehat{\Phi_k})$ are almost regular. (For regularity we should have $A(2, k) = k(\sqrt{2} - 1)$ which is impossible.) Obviously, the ratio of the lengths of the inclined and horizontal (or vertical) sides of $H(\widehat{\Phi_k})$ tends to 1 as $k \rightarrow \infty$.

Remark 6.7. For the k -th total error of the approximation of L_2 with $d(\Phi)$ we get

$$TE_\Phi(k) = (4 - \pi)k^2 - 2(k - \Phi(2, k))^2.$$

Thus for the k -th relative error and for the limit relative error we obtain

$$RE_\Phi(k) = \frac{4 - \pi}{\pi} - \frac{2}{\pi} \left(1 - \frac{\Phi(2, k)}{k} \right)^2$$

and

$$RE_\Phi = \frac{8(\sqrt{2} - 1) - \pi}{\pi} = 0.054786175\dots$$

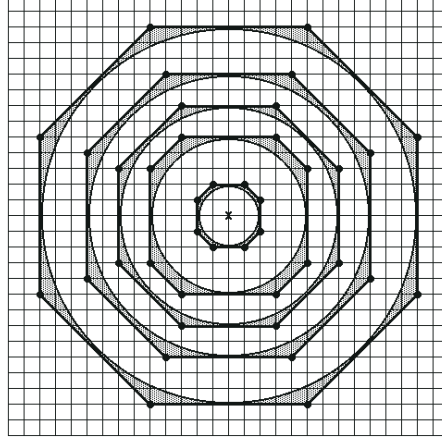


Figure 6.3: Approximating G_k by $H(\widehat{\Phi}_k)$ for $k = 2, 5, 7, 9, 12$.

By the following theorem we solve Problem 6.3.

Theorem 6.8. *Let $\Psi = (\Psi(i))_{i=1}^{\infty} \in S_2$ be the unique sequence defined by $\Psi(2, k) = \lfloor k(\sqrt{2} - 1) \rfloor$ ($k \in \mathbb{N}$), that is*

$$\Psi(i) = \lfloor i(\sqrt{2} - 1) \rfloor - \lfloor (i-1)(\sqrt{2} - 1) \rfloor + 1 \quad (i \in \mathbb{N}).$$

Then for every $k \in \mathbb{N}$, $O_k \subseteq \widehat{\Psi}_k$. Moreover, if $A \in S_2$ and $O_k \subseteq \widehat{A}_k$ for some $k \in \mathbb{N}$, then $\widehat{\Psi}_k \subseteq \widehat{A}_k$.

Proof. Let $k \in \mathbb{N}$ be fixed, and suppose that $O_k \not\subseteq \widehat{\Psi}_k$. Let Φ be the neighbourhood sequence defined in Theorem 6.5. Then, as $\widehat{\Phi}_k \supseteq O_k$, there exists an integer point q of O_k which is also in $\widehat{\Phi}_k \setminus \widehat{\Psi}_k$. Since $\Phi(2, k) = \Psi(2, k) + 1$, q must be on the border of $\widehat{\Phi}_k$.

Moreover, by $H(\widehat{\Phi}_k) \supseteq G_k$, the only possibility is that q is the tangent point of G_k and one of the inclined sides of $H(\widehat{\Phi}_k)$. Using symmetry, this implies that q belongs to one of the lines $y = x$ or $y = -x$. However, the points being both on the perimeter of G_k and on one of these lines, do not

have integer coordinates. This contradiction shows that $O_k \subseteq \widehat{\Psi}_k$ for every $k \in \mathbb{N}$.

To prove the second statement, we show that for any $k \in \mathbb{N}$, every inclined side of $\widehat{\Psi}_k$ contains an integer point from O_k . For this purpose, observe that as $\Psi(2, k)/k < \sqrt{2} - 1$, we have

$$(k - \Psi(2, k))^2 - 2\Psi(2, k)^2 > 0. \quad (6.8)$$

That is, by formula (6.6) in the proof of Theorem 6.5, every inclined side of $H(\widehat{\Psi}_k)$ has two points in common with the perimeter of G_k . A simple calculation yields that the intersection points with positive coordinates are

$$\left(\frac{k + \Psi(2, k) + Q}{2}, \frac{k + \Psi(2, k) - Q}{2} \right)$$

and

$$\left(\frac{k + \Psi(2, k) - Q}{2}, \frac{k + \Psi(2, k) + Q}{2} \right),$$

where

$$Q = \sqrt{(k - \Psi(2, k))^2 - 2\Psi(2, k)^2}.$$

For the Euclidean distance D of these points we obtain

$$D = \sqrt{2[(k - \Psi(2, k))^2 - 2\Psi(2, k)^2]}.$$

As k and $\Psi(2, k)$ are integers, by (6.8) we infer that

$$(k - \Psi(2, k))^2 - 2\Psi(2, k)^2 \geq 1$$

which yields $D \geq \sqrt{2}$. As the line containing the above intersection points is given by $x + y = k + \Psi(2, k)$ in the plane $[x, y]$, $D \geq \sqrt{2}$ implies that there is at least one integer point on the corresponding inclined side of $H(\widehat{\Psi}_k)$ which also belongs to G_k . However, this shows that if $A \in S_2$ with $\widehat{A}_k \supseteq O_k$, then $A(2, k) \geq \Psi(2, k)$ must be valid. Thus $\widehat{A}_k \supseteq \widehat{\Psi}_k$, and the theorem follows. $\square$

Remark 6.9. By Lemma 6.4, $d(\Psi)$ is a metric on $\mathbb{Z}^2$. That is, among the digital metrics corresponding to neighbourhood sequences, $d(\Psi)$ is the best one to approximate L_2 from below in $\mathbb{Z}^2$.

6.2.2 The general case

In this subsection we solve Problem 6.1. For this purpose the following lemma will be useful.

Lemma 6.10. *Define the function $E : [0, 1] \rightarrow \mathbb{R}$ by*

$$E(y) = \begin{cases} -y^2 + 2y, & \text{if } y \geq \sqrt{2} - 1, \\ 2 \arccos(y(y+2)) - 2(y+1)\sqrt{1-2y-y^2} - y^2 + 2y, & \text{otherwise.} \end{cases}$$

Then $E(y)$ has a global minimum at $y_0 = \frac{2\sqrt{6}-3}{5}$. Moreover, $E(y)$ is strictly monotone decreasing in $[0, y_0]$ and strictly monotone increasing in $[y_0, 1]$.

Proof. Observe that $E(y)$ is continuous on $[0, 1]$. Taking the first two derivatives of E on the intervals $[0, \sqrt{2} - 1]$ and $[\sqrt{2} - 1, 1]$ separately, by a simple calculation the lemma follows from elementary calculus. $\square$

Theorem 6.11. *Let the neighbourhood sequence $\Lambda = (\Lambda(i))_{i=1}^\infty$ be defined by*

$$\Lambda(i) = \begin{cases} 1, & \text{if } E\left(\frac{\Lambda(2,i-1)}{i}\right) < E\left(\frac{\Lambda(2,i-1)+1}{i}\right), \\ 2, & \text{otherwise,} \end{cases}$$

where E is introduced in the previous lemma. Then for any $A \in S_2$ and $k \in \mathbb{N}$,

$$\text{Area}(H(\widehat{\Lambda}_k) \nabla G_k) \leq \text{Area}(H(\widehat{A}_k) \nabla G_k).$$

Proof. Let k be a fixed positive integer and $A \in S_2$. Using (6.7) when the inclined sides of $H(\widehat{A}_k)$ do not intersect G_k , and by a simple calculation in the opposite case, we obtain that the k -th total error of the approximation of L_2 by $d(A)$ is

$$TE_A(k) = \begin{cases} 2k^2 - \pi k^2 + 2k^2(2y - y^2), & \text{if } y \geq \sqrt{2} - 1, \\ 2k^2 - \pi k^2 + 2k^2(2 \arccos(y(y+2)) - 2(y+1)\sqrt{1-2y-y^2} - y^2 + 2y), & \text{otherwise,} \end{cases}$$

where $y = A(2, k)/k$.

Clearly, $TE_A(k)$ is minimal if and only if $E(y)$ is minimal for A . By Lemma 6.10

$$\min_{\substack{t \in \mathbb{Z} \\ 0 \leq t \leq k}} E\left(\frac{t}{k}\right) = \min\left(E\left(\frac{\lfloor ky_0 \rfloor}{k}\right), E\left(\frac{\lceil ky_0 \rceil}{k}\right)\right),$$

where $y_0 = \frac{2\sqrt{6}-3}{5}$.

We prove that for every $k \in \mathbb{N}$, $\Lambda(2, k) = \lfloor ky_0 \rfloor$ or $\lceil ky_0 \rceil$ according to whether

$$E\left(\frac{\lfloor ky_0 \rfloor}{k}\right) < E\left(\frac{\lceil ky_0 \rceil}{k}\right),$$

or not. This will imply that $E(y)$ is minimal for Λ , whence $TE_\Lambda(k) \leq TE_A(k)$ for every $A \in S_2$ and $k \in \mathbb{N}$.

We proceed by induction on k . For $k = 1$ the statement is obvious: $\Lambda(1) = 2$ and $E(1) < E(0)$. Suppose that for some k we have $\Lambda(2, k) = \lfloor ky_0 \rfloor$; the case when $\Lambda(2, k) = \lceil ky_0 \rceil$ is similar. Then, by the induction hypothesis,

$$E\left(\frac{\lfloor ky_0 \rfloor}{k}\right) < E\left(\frac{\lceil ky_0 \rceil}{k}\right) = E\left(\frac{\lfloor ky_0 \rfloor + 1}{k}\right). \quad (6.9)$$

We distinguish two cases.

(i) Assume first that $\lfloor ky_0 \rfloor = \lfloor (k+1)y_0 \rfloor < \lceil (k+1)y_0 \rceil = \lceil ky_0 \rceil$.

Now if

$$E\left(\frac{\lfloor ky_0 \rfloor}{k+1}\right) < E\left(\frac{\lfloor ky_0 \rfloor + 1}{k+1}\right), \quad (6.10)$$

then by definition $\Lambda(k+1) = 1$, whence

$$\Lambda(2, k+1) = \Lambda(2, k) = \lfloor ky_0 \rfloor = \lfloor (k+1)y_0 \rfloor.$$

Since we can write (6.10) as

$$E\left(\frac{\lfloor (k+1)y_0 \rfloor}{k+1}\right) < E\left(\frac{\lceil (k+1)y_0 \rceil}{k+1}\right),$$

the statement is also true for $k + 1$. On the other hand, if

$$E\left(\frac{\lfloor ky_0 \rfloor}{k+1}\right) \geq E\left(\frac{\lfloor ky_0 \rfloor + 1}{k+1}\right), \quad (6.11)$$

then we obtain $\Lambda(k+1) = 2$. This yields

$$\Lambda(2, k+1) = \Lambda(2, k) + 1 = \lceil ky_0 \rceil = \lceil (k+1)y_0 \rceil.$$

Now the statement for $k+1$ follows from rewriting (6.11) as

$$E\left(\frac{\lceil (k+1)y_0 \rceil}{k+1}\right) \leq E\left(\frac{\lfloor (k+1)y_0 \rfloor}{k+1}\right).$$

(ii) Assume now that $\lfloor ky_0 \rfloor < \lceil ky_0 \rceil = \lfloor (k+1)y_0 \rfloor < \lceil (k+1)y_0 \rceil$.

In this case we have

$$\frac{\lfloor ky_0 \rfloor}{k+1} < \frac{\lfloor ky_0 \rfloor}{k} \leq \frac{\lfloor (k+1)y_0 \rfloor}{k+1} < y_0 < \frac{\lceil ky_0 \rceil}{k} \leq \frac{\lceil (k+1)y_0 \rceil}{k+1}. \quad (6.12)$$

Combining (6.9) with Lemma 6.10, we immediately obtain

$$E\left(\frac{\lfloor (k+1)y_0 \rfloor}{k+1}\right) < E\left(\frac{\lceil (k+1)y_0 \rceil}{k+1}\right).$$

Moreover, (6.12) yields

$$E\left(\frac{\lfloor ky_0 \rfloor}{k+1}\right) > E\left(\frac{\lfloor (k+1)y_0 \rfloor}{k+1}\right) = E\left(\frac{\lfloor ky_0 \rfloor + 1}{k+1}\right).$$

Hence, by definition, $\Lambda(k+1) = 2$ and

$$\Lambda(2, k+1) = \Lambda(2, k) + 1 = \lfloor ky_0 \rfloor + 1 = \lfloor (k+1)y_0 \rfloor.$$

Thus the statement follows for $k+1$ also in this case.

Observe that as $0 < y_0 < 1/2$, the above two cases cover all the possibilities. Thus the sequence Λ minimizes $E(y)$. This implies that $TE_\Lambda(k)$ is minimal for every $k \in \mathbb{N}$, and the theorem follows. $\square$

Remark 6.12. We have $\Lambda = (2, 1, 1, 1, 2, 1, 2, 1, 1, 2, 1, 1, \dots)$. For the k -th total and relative errors of Λ we obtain

$$TE_{\Lambda}(k) = k^2(2 - \pi + 2E(y)),$$

and

$$RE_{\Lambda}(k) = \frac{2 - \pi + 2E(y)}{\pi},$$

where y and $E(y)$ are as in Theorem 6.11. By $\lim_{k \rightarrow \infty} y = \frac{2\sqrt{6}-3}{5}$ we get

$$RE_{\Lambda} = \frac{2 - \pi}{\pi} + \frac{2}{\pi} \left(2 \arccos \left(\frac{3 + 8\sqrt{6}}{25} \right) + \frac{4\sqrt{6} - 11}{5} \right) = 0.046525347 \dots$$

Figure 6.4 illustrates how $H(\widehat{\Lambda}_k)$ approximates G_k for $k = 2, 5, 7, 9, 12$. The dark regions show the error of the approximation.

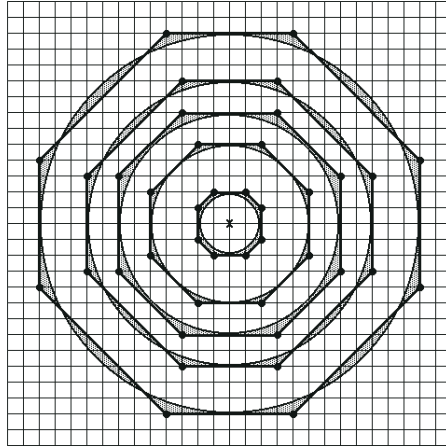


Figure 6.4: Approximating G_k by $H(\widehat{\Lambda}_k)$ for $k = 2, 5, 7, 9, 12$.

Clearly, $d(\Lambda)$ is not a metric on $\mathbb{Z}^2$. Another unpleasant feature of Λ is that it is not easy to generate: to obtain its k -th element, we have to

calculate the first $k - 1$ elements previously. Now we give two sequences which are easy to construct, and for every $k \in \mathbb{N}$, one of them is also the “best” to approximate G_k .

Corollary 6.13. *For $j = 1, 2$ and $i \in \mathbb{N}$ put*

$$C^{[j]}(i) = \begin{cases} j, & \text{if } i = 1, \\ \lfloor i \frac{2\sqrt{6}-3}{5} \rfloor - \lfloor (i-1) \frac{2\sqrt{6}-3}{5} \rfloor + 1, & \text{if } i > 1, \end{cases}$$

and write $C^{[1]} = (C^{[1]}(i))_{i=1}^\infty$ and $C^{[2]} = (C^{[2]}(i))_{i=1}^\infty$. Then for every $A \in S_2$ and $k \in \mathbb{N}$,

$$\min(TE_{C^{[1]}}(k), TE_{C^{[2]}}(k)) \leq TE_A(k).$$

Proof. Observe that for every $k \in \mathbb{N}$, $C^{[1]}(2, k) = \lfloor ky_0 \rfloor$ and $C^{[2]}(2, k) = \lceil ky_0 \rceil$, where $y_0 = \frac{2\sqrt{6}-3}{5}$. Hence, $\widehat{\Lambda}_k = \widehat{C_k^{[1]}}$ or $\widehat{C_k^{[2]}}$ for every $k \in \mathbb{N}$, and the statement follows from Theorem 6.11. $\square$

Remark 6.14. As $C^{[1]}(2, k) = \lfloor ky_0 \rfloor$ for every $k \in \mathbb{N}$, by Lemma 6.4, $d(C^{[1]})$ is a metric on $\mathbb{Z}^2$. Thus in a sense, $d(C^{[1]})$ can be considered to be the best metric (coming from a neighbourhood sequence) to approximate the Euclidean distance on $\mathbb{Z}^2$. Note that $RE_{C^{[1]}} = RE_\Lambda$.

6.3 Comparing approximation results

In this section we compare our results with those of DAS in [3]. He used an error function which measures the average difference between the Euclidean distance and the “simple metric value” generated by a neighbourhood sequence. DAS concluded that the periodic neighbourhood sequence $S = (1, 1, 2, 1, 2)$ which generates a “simple metric”, should be used to approximate L_2 . Note that for every $k \in \mathbb{N}$, $H(\widehat{S}_k) \not\subseteq G_k$. We propose to use the sequence $C^{[1]}$ defined in Corollary 6.13 to approximate L_2 , so we compare S and $C^{[1]}$ here.

Since we used a different error function than Das in [3], we choose a third one for the comparison. We examine how the k -disks $\widehat{A}_k$ approximate

the k -disks G_k in digital sense. That is, we count the k -th discrete total error

$$DTE_A(k) = \#(\widehat{A}_k \nabla O_k),$$

being the number of grid points in the symmetric difference of $\widehat{A}_k$ and O_k , where $A \in S_2$, $k \in \mathbb{N}$. The k -th discrete relative error of the approximation is defined as

$$DRE_A(k) = \frac{DTE_A(k)}{\#O_k}.$$

The following table shows the discrete relative errors of the neighbourhood sequences $S = (1, 1, 2, 1, 2)$ given in [3] and $C^{[1]}$ given in Corollary 6.13 which provide metrics on $\mathbb{Z}^2$.

Table 6.1: Discrete relative errors of metrics generated by S and $C^{[1]}$.

k	$DRE(k)$	
	$S = (1, 1, 2, 1, 2)$	$C^{[1]} = \left(\lfloor i \frac{2\sqrt{6}-3}{5} \rfloor - \lfloor (i-1) \frac{2\sqrt{6}-3}{5} \rfloor + 1 \right)_{i=1}^{\infty}$
10	0.12618297...	0.11356467...
50	0.06322498...	0.05863607...
100	0.05551135...	0.05169176...
200	0.05218540...	0.04909694...
500	0.05023499...	0.04755593...
1000	0.04955390...	0.04704176...

From Table 6.1 we can see that $C^{[1]}$ behaves better in this “digital” sense.

Chapter 7

Conclusion and discussion

In our investigations we dealt with neighbourhood sequences defined on $\mathbb{Z}^n$ and $\mathbb{Z}^\infty$. However, there can be applications, where the grid points form another kind of structure (e.g. triangular or hexagonal); for a survey on planar grids see [35]. The concept and structural analysis of neighbourhood sequences presented in Chapter 3 (and also in [16]) can be easily generalised to these grids, see [31, 32] for the case of triangular and hexagonal grids. We note that all other results of the dissertation could be extended to these grids.

In some of our investigations (Chapter 5 and 6) we focused our attention on the most frequently used spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$. Beside considering other types of digital grids, it also seems to be interesting to extend these results to arbitrary dimension. For example, we note that neighbourhood sequences may have many applications not only in 2D, but also from 3D on. In 3D one can think of three-dimensional pictures, for instance in medical applications (see e.g. [12] and the references given there). Interestingly, even in 2D, higher dimensional neighbourhood sequences can be useful. For example, in case of colour images (when three independent colour parameters appear), or during tracking motion (when time can be considered as a third dimension). The motivation and theoretical background of such investigations can be found e.g. in [18].

In Chapter 4 (see also [21]) we have shown an application scheme of the presented results. As other possible applicable results, we mention the geometric results of Chapter 5 (and [19], as well) which can be used in approximating convex and symmetric bodies in 2D or 3D. In Chapter 6 we found neighbourhood sequences which generate digital metrics, and approximate the Euclidean distance on $\mathbb{Z}^2$ well (see also [20]). These results can be used in any applications, where well approximating digital metrics are needed.

Part II

Character recognition by Walsh transformation

Chapter 8

Introduction

The history of character recognition by tools of digital image processing dates back to the 1950's (see e.g., [41]). In that time new problems and challenges occurred in practical applications and since then several character recognition methods were developed to solve them. Most of these algorithms can be applied to typeset texts, and there exist procedures to process handwritten characters. The first step one should take during a character recognition process is to select the most suitable method for the specific application. We note here that nowadays there exist reliable and cheap software (e.g. Recognita (see www.recognita.hu), Abbyy FineReader (see www.abbyy.com)) for general character recognition purposes, so theoretical research should focus on those special areas (such as the recognition of handwritten text), where the application of these programs is not suitable.

A character recognition process usually contains a scanning step, pre-processing (binarization – segmentation), feature extraction (skeletonization – contour extraction), the actual recognition and classification, sometimes postprocessing, and verification. For a comprehensive survey on feature extraction methods see [41], where several algorithms are presented and compared. A commonly used method produces a thinning on the characters to obtain their skeletons so the recognition is based on some kind of

skeleton analysis. An overview on character recognition algorithms based on thinning can be found in [39]. Skeletonization is applied to replace the original image with a smaller data structure and has the advantage of saving memory.

Algorithms for recognizing both typeset and handwritten characters can be based on e.g. projection histograms [40], zoning [11], or as a more theoretical analysis, the recognition can be executed according to the Fourier descriptors [41].

The recognition of handwritten characters means a more sophisticated problem than the recognition of machine printed text. Nowadays, OCR packages are able to recognize only neatly written text but there are investigations in progress to recognize common handwritten documents as well. One of these algorithms contains a thinning step, and an additional stroke segmentation part is inserted to make the algorithm able to recognize Chinese characters (see [17], [25] and [33]). The recognition of Arabic text is also a popular research area with growing literature, see e.g. [27].

In the second part of the dissertation we present a character recognition algorithm which basically supports the recognition of typeset documents. The algorithm performs best for alphabets with small number of symbols and without accented characters. We focused on segmentation, classification, and made some experiments to verify the reliability of our character recognition method. To classify a character, usually a feature vector is composed and the actual recognition is achieved by searching for the closest prototype feature vector. The dimension of the feature vector can vary in different applications. We also have to choose a suitable metrics to measure the difference between two feature vectors. In our investigations we tried to find a method which generates easily separable feature vectors, that is the prototype vectors have large distance from each other. We found that the well-known Walsh transformation has this property. The feature vectors generated by this transformation can be separated more effectively, than in other popular character recognition methods like zoning or projection histograms. Using Walsh transformation with underdetermined feature vectors, we also obtain a noise filtering effect which is very useful in character recognition processes. Feature extraction is not needed, since

the classification step of our algorithm is based on the Walsh transforms of the image, and they can be calculated without any modifications. We built our character recognition algorithm into an application by which we can recognize and compress typeset documents for further transmission. The results of the above studies are also presented in [14, 15] which are joint with A. FAZEKAS.

The structure of the second part is as follows. In Chapter 9 we describe our character recognition system and an application based on it. In Chapter 10 we summarize our current work in character recognition, and discuss some possibilities of improving the accuracy of character recognizers.

Chapter 9

Description of the recognition system

In the first step of our algorithm, a segmentation procedure divides the original binary image into smaller ones. These smaller segments are stored in a chained list. The segments are actually rectangles, and beside the image information, the coordinates of the upper left pixel and the size of the rectangle are also stored. The detailed description of this segmentation algorithm can be found in Section 9.1.

Our character recognition method is based on the Walsh transformation which is described in Section 9.2. For every segment a 64-dimensional feature vector is composed according to the first 64 Walsh transforms of the segment.

After calculating the feature vector of a given segment, we judge whether the segment contains text information (letter, number, etc.) or an unrecognizable symbol. Section 9.3 describes the way how the decision is made and the possibilities of training the algorithm. Figure 9.1 shows the scheme of the algorithm.

We used commonly applied character sets (OCR-A, OCR-B, CMR, TIMES NEW ROMAN and license plate characters) to test our algorithm from several points of view, like computation speed, noise sensitivity, recognition

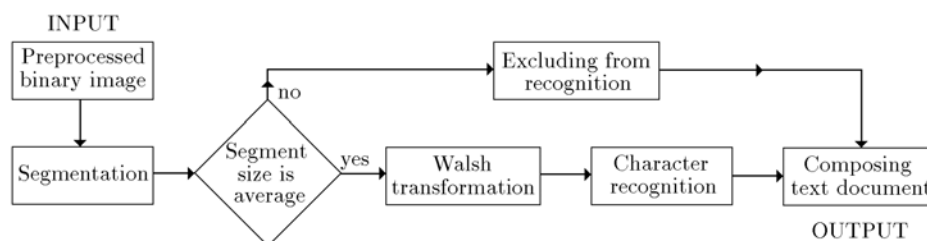


Figure 9.1: Theoretical scheme of the algorithm.

failure. We made other statistical investigations as well, such as correlation analysis. Our experimental results are summarized in Section 9.4. The above described studies about our character recognition algorithm are also published in [14].

To prove the practical applicability of our character recognition algorithm, we built our method into an application which compress typeset documents. The description of this application and our experimental results are presented in Section 9.5 and also in [15].

9.1 Segmentation

The first step of the algorithm performs a segmentation procedure on the digital image. We try to determine minimal storing rectangles for those subsets of the image foreground which can be separated by horizontal and vertical lines. These rectangles can be determined by calculating their size and the coordinates of their upper left pixel. These storing rectangles are sometimes referred to as segments. Using storing rectangles, our segmentation procedure does not extract the connected foreground components in the case of a ligature. This segmentation method is able to handle ligature appearances in the text which depend on the applied character set. In case of a ligature the recognition is obviously unsuccessful, since the pictures of

the letters change drastically, see Figure 9.2, where a ligature “fi” appears in the Hungarian word “figyelő” (observer). Our algorithm is preconditioned for this phenomenon, and can be trained to recognize ligatures. In our experiments we also used CMR character set which allows ligatures beside the ligature-free sets OCR-A and OCR-B (which were composed for optical character recognition).

The segmentation can be made parallel easily, and the whole procedure can be performed as an alternating recursive sequence of the following horizontal and vertical segmentation steps.

Horizontal segmentation step: We have a pixel running from left to right, starting from the upper left corner of a segment we have already extracted. If we find an object (foreground) point in the given row then we go down one pixel and start running a pixel from the beginning of this row. The procedure continues till the running pixel reaches the right side of the segment (we find a row which does not contain object points). In this case we obtain a new segment. The top row of the new segment will be the uppermost row of the original segment which contains an object point. The bottom row of the new segment will be the lowermost row of the original segment which has been already processed and contains an object point. After defining the new segment, we go on with processing the original segment, starting from that row which did not contain object points.

Vertical segmentation step: The vertical segmentation procedure is analogous to the horizontal one, but here the pixel runs from top to bottom, starting from the upper left corner of a segment. We go right one pixel till a column is found which does not contain object points. In this case a new segment is defined.

Horizontal segmentation steps are followed by vertical segmentation steps, and vice versa. Alternating these two steps, each of the existing segments is divided into smaller segments. If the number of the segments does not change during a segmentation step, the algorithm stops. The following description explains briefly – in meta language format – how the horizontal segmentation works. The vertical segmentation works similarly.

```

Type SP=~Segment;
Segment=Record Of
    X,Y:Word; {*Upper left pixel coordinates*}
    LX,LX:Word; {*Size of the segment*}
    Code:Byte; {*Code of the recognized character*}
    Picture:Pointer; {*The address of the segment*}
    Next:SP; {*Next element of the chained list*}
End;
.
Function HSegmentation(Var Head:SP):Boolean; {*True for a new segment*}
Var NewHead:SP; {*New segment*}
Var SY,EY:Word; {*Beginning and end of the segment*}
Begin
    SY:=NextEmptyLine;
    EY:=NextEmptyLine;
    If (SY=Head^.Y) And (EY=Head^.Y+Head^.LY-1) Then
        HSegmentation:=False; {*Image cannot be segmented any more*}
        Exit
    End;
    While (EY-SY<>1) Do Begin {*Ignoring pairs of empty lines*}
        SY:=EY;
        EY:=NextEmptyLine
    End;
    New(NewHead); {*Creating a new element in the list*}
    CutPicture(Head,NewHead,SY,EY); {*Current segment is defined by SY,EY*}
    NewHead^.Next:=Head^.Next;
    Head^.Next:=NewHead;
    Head:=NewHead;
    HSegmentation:=HSegmentation(Head) Or True {*Processing the remained part*}
End;

```

In the first step of the segmentation procedure the whole original binary image is considered as the initial segment, and a horizontal segmentation step is performed. The segmentation algorithm stops if during a horizontal or vertical segmentation step we cannot find a row or a column which does not contain an object point – in other words, new segments cannot be defined. The steps of the segmentation algorithm can be seen in Figure 9.2, where the segments are represented by rectangles. The result of the first (horizontal) step is a line of text if the original binary image is a text document.

Notice that the top of the line is determined by the highest character (“f”), and the bottom is determined by the lowest character (“g”). The second (vertical) segmentation step divides the text line into characters, but the rectangles that store the characters contain relatively large background components which can be eliminated with the following (horizontal) segmentation step. If the document contains some graphic parts then the

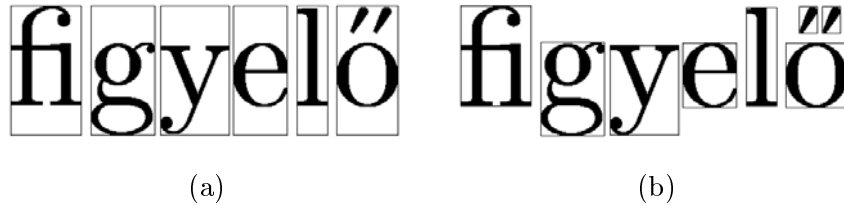


Figure 9.2: Result of segmentation (a) after two (b) after four steps.

number of the segmentation steps that are necessary to segment such a graphic part depends on the complexity of the graphics.

The segmentation of accented characters is an interesting and difficult problem. In the case of English text after three segmentation steps (horizontal – vertical – horizontal) the storing rectangles cannot be reduced any more, while in the case of Hungarian text (which contains accented characters), we have the same situation only after the fourth segmentation step. Figure 9.2 illustrates this case as well, where the segmentation of the Hungarian accented character “ő” is performed in four steps. It is rather difficult to recognize accented characters, since the accent is segmented separately. Analysing the placement of the segments of small size can help to recognize these characters. For example, it can be useful to examine if a vowel takes place below a segment with small size. Our system was not trained to recognize accented characters.

The input image (a text document) can be distorted in many ways: it can be rotated, corrupted by noise, etc. The image might be rotated if e.g. the document was inserted into the scanner improperly. Our method tolerates rotation with small degrees. If the degree of the rotation is too large, the horizontal segmentation is not able to separate the document lines in the first step, since the lowest pixel of a line is “under” the highest pixel of the next line. The degree of the maximal tolerable rotation α can be obtained as

$$\alpha = \arctan \frac{\text{line-space}}{\text{paper width} - \text{horizontal margins}} \quad .$$

For example, if we assume the line-space to be 4 mm and the page size to be A4 with small margins, we approximately have 1.5° as the degree of the maximal tolerable rotation.

9.2 Character recognition

According to the average size of the segments, we can decide whether an element stores text information (character) or something else, like a graphic or noise. Now we show the role of Walsh transformation in our recognition algorithm.

9.2.1 The Walsh transformation

Walsh transformation was applied for several purposes in digital image processing, but never to character recognition. We found that this transformation gives a verbose description of the image, like symmetrical relations, placement of the foreground and background pixels, and so on. That is why it is useful to apply it for character recognition. We got good results for simple documents.

The Walsh transformation $W(u, v)$ in 2D is a transformation of the form

$$W(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y)g(x, y, u, v).$$

Here f is the binary image of size $N \times N$, where N is supposed to be a power of 2. The intensity of the pixel (x, y) is $f(x, y)$, and $u, v = 0, \dots, N - 1$. The kernel function g of the transformation is defined by

$$g(x, y, u, v) = \frac{1}{N} \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-i-1}(u) + b_i(y)b_{n-i-1}(v)},$$

where $b_i(w)$ is the i -th bit in the binary expansion of w , and $n = \log_2 N$.

We compute N^2 Walsh transforms altogether which can be organized into the N^2 -dimensional feature vector

$$(W(0, 0), W(0, 1), \dots, W(0, N-1), W(1, 0), W(1, 1), \dots, W(N-1, N-1)).$$

The Walsh transform is injective in the sense that if we consider two different binary images, then the corresponding feature vectors are also different. If we compose a feature vector which does not contain all the Walsh transform values, then this vector can be the same for two different images (see Subsection 9.2.2). The Walsh transformation is separable, as its kernel function can be separated:

$$g(x, y, u, v) = g_1(x, u) g_2(y, v).$$

Moreover, the equality

$$g(x, y, u, v) = g_1(x, u) g_1(y, v)$$

also holds (the factors are functionally equivalent), thus the Walsh transformation is symmetric, as well. With these two properties the computation of the 2D transforms can be made considerably faster, since it can be simplified to the computation of two 1D Walsh transformations, and the symmetry makes the computation even faster. All of these properties of the Walsh transformation are well-known from the literature, see e.g. [18]. Other analytical properties of Walsh functions also can be found in [38].

9.2.2 Application of the Walsh transformation

To perform the Walsh transformation, first we have to magnify the original image to the size of $2^n \times 2^n$ for some n . This does not mean a considerable modification, since the Walsh transformation is invariant for magnification. In our algorithm the image size was fixed as 32×32 and we used linear transformations to magnify the segments. However, we compute only the following 64 Walsh transforms instead of the $32 \times 32 = 1024$ ones:

$$(W(0, 0), W(0, 1), W(0, 2), \dots, W(0, 7), W(1, 0), W(1, 1), \dots, W(7, 7)).$$

There are three reasons for reducing the number of Walsh transforms:

- We can save computation time;
- These 64 values describe the global features and symmetric relations of the binary image well;
- We can filter out some noise corruption from the image, since the computation of less Walsh transforms results in a blurring effect. For a detailed description about the noise sensitivity of the algorithm, see Section 9.4 which summarizes our experimental results.

An example for the latter property is shown in Figure 9.3, where the Walsh transforms $W(i, j)$ with $0 \leq i, j \leq 3$, coincide for two 8×8 binary images.

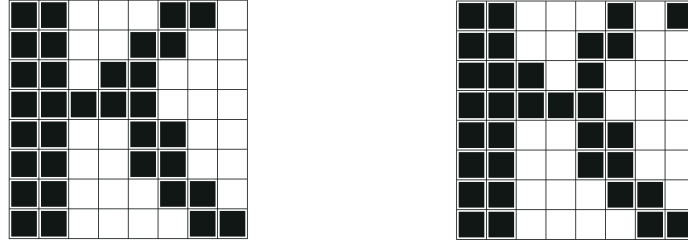


Figure 9.3: Different images with the same feature vector.

Magnifying the segments to the same size 32×32 can cause a problem in character recognition. Though the lower and upper cases of the characters usually look different, some characters have similar lower and upper cases (e.g. “w” and “W”) which become identical during magnification. We can get rid of this problem by storing the ratio of the side lengths of the storing rectangle for every segment. The proper case can be restored by comparing the ratio of the side lengths of the original storing rectangle.

In our investigations we used the metric L_1 for measuring the distance of the feature vectors. The difference (distance) of the 64D feature vectors of different characters is significant, so the recognition of a given character is quite reliable. To verify this statement check Table 9.1 which contains the L_1 distance values between the feature vectors of prototype OCR digits. The values show strong differences which fortifies our conception to calculate only a 64 dimensional feature vector for each character.

Table 9.1: Distance values between feature vectors of prototype OCR digits.

	0	1	2	3	4	5	6	7	8
1	2248								
2	1460	2184							
3	1524	1836	1216						
4	2504	2440	2424	2592					
5	1900	1972	1610	1458	2660				
6	1664	2020	1926	2110	2062	2072			
7	2324	2304	2044	1948	2580	2428	2630		
8	1552	2190	1536	1864	2404	1510	1934	2584	
9	1892	2252	2052	1952	2244	1980	1888	2026	2088

In the previous subsection we described how the 2D Walsh transformation can be performed by two 1D transformations. In our algorithm we used a faster method and computed the transforms directly from tables. The value

$$N \cdot g(x, y, u, v) = \prod_{i=0}^{n-1} (-1)^{b_i(x)b_{n-i-1}(u) + b_i(y)b_{n-i-1}(v)} \quad (9.1)$$

in the kernel function can be ± 1 . For example, let us consider the case $n = 1$, $N = 2$. The corresponding Table 9.2 has a traditional form, where the cells of the table contain $+$ or $-$ signs according to the value of (9.1).

Table 9.2: The kernel values of the Walsh transformation for $N=2$.

$\begin{smallmatrix} (x, y) \\ (u, v) \end{smallmatrix}$	(0, 0)	(0, 1)	(1, 0)	(1, 1)
(0, 0)	+	+	+	+
(0, 1)	+	-	+	-
(1, 0)	+	+	-	-
(1, 1)	+	-	-	+

9.2.3 Comparing our method with other algorithms

Our algorithm was compared with two classical character recognition methods: one of them is based on projection histograms, the other one is based on zoning. The reason why we involved these character recognition algorithms into our analysis is that both of them use feature vectors to classify the recognizable characters and assign a 64D feature vector to every recognizable segment, similarly to our algorithm. We worked with several font types: OCR-A, OCR-B, CMR, and TIMES NEW ROMAN. By fixing a prototype alphabet (letters, digits, punctuation marks, and other symbols), first we computed the average distance values for every symbol from the rest of the alphabet. Table 9.3 contains our results according to the different character recognition algorithms. We present only the first 10 lower case letters of the alphabet here.

Table 9.3: Average distance values of the first 10 letters from the rest of the prototype alphabet.

Letter	Walsh		Projection		Zoning	
	CMR	OCR	CMR	OCR	CMR	OCR
a	2375	2795	478	615	316	406
b	2011	2323	335	432	238	310
c	2281	2613	356	470	280	373
d	1974	2396	333	439	244	316
e	2288	2742	390	642	294	417
f	1983	2478	334	477	236	321
g	1930	2377	330	416	233	306
h	1965	2326	338	462	241	314
i	1628	2272	344	487	205	311
j	1655	2328	338	508	194	314

The entries of Table 9.3 indicate more significant difference values in the case of Walsh transformation which results in better recognition performance. The detailed description and results of the tests we made to

check the recognition accuracy of our method can be found in Section 9.4. Another comparative analysis is discussed in Chapter 10.

9.2.4 Character recognition – feature and prototype vectors

As it is described in the previous subsection, we obtain a 64 dimensional feature vector by computing some of the Walsh transforms for a segment. This feature vector is compared with prototype vectors which contain the same 64 Walsh transforms of prototype characters. For a given feature vector we find the closest prototype vector by using a suitable distance function. If we assume that the size of the character segments lie in an interval, we can exclude the segments with too small (noise) or too large (graphic parts) size from the recognition process before the decision step. The recognition is based on the distance between the feature vector of the analysed character and the prototype vector.

9.3 Decision and training

As we mentioned before, the document to be processed should contain only text and some special symbols. For these kind of documents the algorithm works in a fairly reliable way. To make the algorithm more flexible we inserted a decision step into the recognition process to handle unrecognizable segments. We have the following two possibilities to make a decision about the recognizability of a segment.

2-level decision: With this step the given segment is always recognized which means that the algorithm finds the prototype character whose feature vector has the minimal distance from the feature vector of the given segment.

3-level decision: With this step we classify the segments as recognizable, or uncertain ones. In the uncertain cases the algorithm has “doubts” about the recognizability of the given segments. It happens when the minimal distance is larger than a threshold value.

During character recognition we create a 64D feature vector for every segment, then calculate the minimal distance between this vector and the

prototype vectors. The decision steps above are based on this distance value. In the 3-level decision step we use one critical value. The decision can be made according to the relation between the minimal distance value and the critical value. In the 3-level case, if the minimal distance value is larger than the critical value the segment is considered unrecognizable.

The critical values can be given in different ways. For example, in our experiments we used $\frac{1}{3} \times \text{the minimal distance between any two of the prototype vectors}$ as the critical value for the 3-level decision. Another possibility is to give the critical value for every prototype vector separately.

There is a training part inserted into the algorithm which is independent of the recognition step. Adaptive recognition would be questionable, since the false recognition of a given character would ruin the corresponding prototype vector. If we would like to apply the algorithm to a document using an unknown font type, first we have to train our process to recognize the new symbols. We have two possibilities to do this. If we have the font in electronic form, we can easily compose an artificial document containing the whole alphabet without any noise. Scanning through this document we can train the algorithm for every symbol of the alphabet. If we cannot compose such an artificial document (e.g. we do not have the font type in electronic form), then the algorithm must be trained by using 3 to 10 samples for every symbol from scanned material.

9.4 Experimental results

We calculated the average computation times the algorithm needed to process one page of printed text which contained 27 rows. It took 3.5 seconds to segment the document and additional 4.5 seconds to perform the recognition step. The test was executed on a Personal Computer at a moderate performance level (Celeron 433 processor). The segmentation step of the algorithm takes approximately the same time to finish as the actual recognition step. The performance of the algorithm can be improved by making the procedure parallel. As we investigated other features, we did not implement parallel segmentation and character recognition which would reduce

the computation time drastically. In the case of parallel processing the segmentation of the whole document and the segmentation of one character would take approximately the same time, and the same holds for character recognition, too.

We inserted a noise generator step into the character recognition process after the segmentation. This way we corrupted the image with different noise types (global, contour) at different levels before executing the recognition step. We applied uniformly distributed additive noise corruption and the level of the corruption is defined as the percentage of the pixels which are affected by the noise corruption. Global noise corruption means that all the points of the binary image are involved in the noise corruption, while in the case of contour noise corruption only the contour points are affected.

To perform a comparative analysis, we made some tests for our recognition algorithm, and for the methods based on projection histograms, and zoning. Synthetic images were generated with the character sets CMR, OCR-A, OCR-B which contained the most regular elements of these sets. These prototype documents were used to train the algorithms, so the prototype feature vectors were calculated. We composed some one page test documents containing regular text, and generated 100 noisy images for every input document. For noise corruption we used global and contour noise separately and parallelly, as well. Moreover, tests were made on actually scanned printed material which is equivalent to a small (approximately 1%) global and a moderate (approximately 20%) contour noise corruption. Using these samples, we made an analysis according to the recognition accuracy of our method for each symbol of the alphabets. The result of this test for CMR can be found in Appendix A, where “Rate” describes the efficiency of the recognition, while “Failure” shows the resulted false characters in case of misrecognition. We also calculated an average accuracy value to measure the recognition efficiency of the above algorithms. Appendix B summarizes these results according to several types and levels of noise corruption. The “—” signs in the table indicate those cases, when the recognition accuracy fell drastically.

Testing our method, experiments indicated strong correlation among the levels of the noise corruption, the feature vector of the segment, and its

distance from the corresponding prototype vector. At the significance level 0.001 the correlation coefficient is $r = 0.7875$. Table 9.4 summarizes the type and the frequency of some typical recognition failures that occurred according to the different types of noise corruption. We applied a 2-level decision during the recognition.

Table 9.4: Recognition failures according to different noise corruption.

	Global noise				Contour noise			
Failure	20%	25%	30%	35%	30%	40%	50%	60%
i→I	1%	5%	28%	37%			2%	16%
l→1	2%	3%	5%	5%	2%	2%	13%	4%
l→I		6%	17%	23%		18%	47%	82%
1→l		2%	6%	8%				1%
1→I			2%	4%	2%	4%	28%	38%
6→0			3%	6%				2%
t→l			1%	8%			1%	1%

If the recognition is restricted only to digits then the dimension of the feature space can be reduced. According to a factor analysis the dimension of the feature space can be reduced from 64 to 48 without violating the accuracy of the recognition. We can have a moderate recognition accuracy (at the level of 90%) if we use only a 16-dimensional feature space.

9.5 An application – compressing typeset documents

As a practical application we built our character recognition method into an information loss compressive algorithm, see [14]. Recognizable characters are letters, numbers, separators, punctuation marks, and so on. This compressive algorithm can be used, when the main goal is to transmit the compressed document through some telecommunication channel. The original digital images are supposed to contain basically text information – with a small number (actually 84) of symbols – recorded in a typographically

fixed form, and optionally some graphic parts. During the compression our main purpose is to preserve the visual information of the document which is the most important for a human reader. For example, fax documents usually have these properties. The characters can be magnified as the algorithm is invariant for magnification. The experimental analysis indicated that the algorithm tolerates noise corruption quite well. The noise sensitivity of the method can be reduced further by lexical analysis. The algorithm was tested in several cases, and proved to be pretty efficient and reliable for simple documents.

In our algorithm we use two basically different compression methods. The graphic parts of the document are compressed by an information preserving compression program, while the text information (letters, numbers, etc.) is recognized and the characters are encoded with their character code from a character map which obviously means a more efficient, but information loss compression. All this information is organized into a chained list. Processing an element of the chained list, according to the average size of the stored binary images it can be decided whether the given element stores text information (character) or a graphic part. If it is considered as a character according to its size, but the character recognition is not successful, then we consider this segment as a graphic part and compress it in that way.

If we use well-chosen documents and character set we can achieve very good results. For example, if the same character set is used for composing the document and for restoring it then the restored document perfectly fits (bit by bit) the original one without any loss of information.

The segmentation procedure also means an efficient compression, since it eliminates (at least when the original image is not affected by noise corruption) the large background components from the compressed data structure. We used an LZW based (see, e.g. [37], Chapter 5, p. 127-132) compression program and investigated the efficiency of the compression, when our method was also applied. We tested our algorithm for a document which contains only text information and for a document with graphic parts. For text information we experienced approximately 4-, while for graphic parts 2-times better compression performance.

Chapter 10

Conclusion and discussion

In the second part of the dissertation we present a character recognition algorithm which is based on Walsh transformation, see also [15]. Walsh transformation is invariant under magnification, and nicely preserves the geometric relations of an image. To recognize a segmented character, we compose a feature vector from its Walsh transforms and compare that with prototype vectors. We can improve noise tolerance by reducing the dimension of the feature vector space. We inserted our character recognition algorithm into an application for compressing typeset text and got good results for simple documents, see also [14].

We have some investigations in progress about improving the accuracy rate of character recognition algorithms. This is a significant problem, since there are applications of great importance based on character recognition. Using our analyses below, any character recognition algorithm can be examined in this way which uses feature and prototype vectors.

To improve accuracy, we can investigate the vector space the prototype vectors belong to. The recognition is more reliable, if these vectors can be well separated in the feature space. In this way, we can compare different alphabets, and try to improve the accuracy rate by modifying some elements of the alphabet. (For example, on some EU license plates the symbol 0 may be changed to $\emptyset$.) A possible improvement is to determine the “worst”

elements of the alphabet, and modify their appearance. In this procedure cluster analysis also can be useful to find the similar characters.

We examined the possibilities of performing the above analyses for the Hungarian license plate alphabet. Since the corresponding official standard was available only in printed form, we also had to compose the digital version of the Hungarian prototype characters. The modifications of the characters of an alphabet is a sophisticated typographical problem, so we discuss some theoretical possibilities here. Depending on the applied character recognition we might determine the theoretically best alphabet. It is a bit like using bar code, but this case the best alphabet could be used as a supplementary one which is used only by the character recognizer. Another possibility is to modify the worst characters which can be done for example, by defining modification rules that preserve main homotopy, or by constructing several possible versions of every character, and find the best alphabet.

Summary

This PhD dissertation consists of two main parts, containing new results in digital image processing. The first part is on neighbourhood sequences, while in the second part we present a character recognition algorithm based on Walsh transformation. The structure of the two parts are similar. First we overview the existing literature, then present the new theoretical results according to the given topic. Finally we show possible application models and areas or actual applications which are not restricted to digital image processing only.

The first part of the dissertation contains investigations about neighbourhood sequences which can be used in digital geometry and digital topology. By using such sequences, we can define more general digital metrics than the classical ones. We define the notion of general neighbourhood sequences for arbitrary finite and infinite digital spaces. We introduce ordering relations on the set of neighbourhood sequences, and investigate the algebraic structures obtained under these orderings. These relations do not make it possible to compare any two neighbourhood sequences, so we introduce a norm-like velocity concept on the set of neighbourhood sequences. Using velocity, we also define a metric, and analyse the topological properties of the metric space obtained. We show an application model, too, which is based on the velocity of neighbourhood sequences. We continue with the geometric analysis of neighbourhood sequences, and extend some former results in this topic. We finish the analysis of neighbourhood sequences with investigating possible approximation methods of the Euclidean metric in $\mathbb{Z}^2$. We show that using digital metrics based on neighbourhood sequen-

ces we can give better approximation of the Euclidean distance than by classical digital metrics.

In the second part of the dissertation we outline a character recognition procedure based on Walsh transformation. Walsh transformation is frequently applied in several fields of digital image processing. Since this transformation is invariant for magnification and preserves geometry nicely, it is well applicable for character recognition, as well. Taking advantage of the properties of Walsh transformation, our algorithm is quite effective and economical. We also perform a noise analysis which shows a nice noise-tolerance property. To illustrate the actual applicability of our character recognition algorithm, we give the example of compressing typeset documents. We perform a test on the reliability of our method, and compare its recognition accuracy with classical character recognition procedures. Finally, we discuss some possibilities to increase the accuracy rate of the character recognition algorithms.

Összefoglaló

Ez a PhD értekezés a digitális képfeldolgozás területére eső új eredményeket tartalmaz. A dolgozat két részre tagolódik. Az első részben a szomszédsági szekvenciák elméletével kapcsolatos eredményeinket ismertetjük, míg a második részben egy Walsh transzformáción alapuló karakterfelismerő eljárást mutatunk be.

I. Szomszédsági szekvenciák

A klasszikus digitális – 4-szomszédsos (cityblock) és 8-szomszédsos (chess-board) – mozgásokat $\mathbb{Z}^2$ -ben ROSENFELD és PFALTZ [36] vezette be. Egy 4-szomszédsos lépéssel csak vízszintesen vagy függőlegesen, míg 8-szomszédsos lépéssel átlósan is mozoghatunk. Ezekhez a lépésekhez kötődően ROSENFELD és PFALTZ két távolságfüggvényt definiált. Két pont d_4 , illetve d_8 távolsága az ahhoz szükséges 4-, illetve 8-szomszédsos lépések minimális száma, hogy az egyik pontból a másikba jussunk. Az euklideszi metrika egy jobb közelítéséhez ROSENFELD és PFALTZ a 4- és 8-szomszédsos lépések felváltva történő alkalmazását javasolta, amely a d_{oct} távolságot adja. A szerzők [36]-ben azt is megemlítették, hogy a közelítés tovább javítható a 4- és 8-szomszédsos lépések további kombinációival.

A 4- és 8-szomszédsos lépések tetszőleges periodikus sorozatba való rendezésével DAS és szerzőtársai [5]-ben bevezették a $\mathbb{Z}^2$ -beli periodikus szomszédsági szekvencia fogalmát, majd kiterjesztették azt tetszőleges véges dimenzióra. Megjegyezzük, hogy a periodikus szomszédsági szekvencia a YAMASHITA és IBARAKI [44] által bevezetett ún. N -szekvencia speciális

esete. A témakör részletes irodalmi áttekintését a disszertáció 1. fejezete tartalmazza.

1. Szomszédsági szekvenciák struktúrális vizsgálata

Legyen n egy pozitív egész szám. Egy $p \in \mathbb{Z}^n$ pont i -edik ($i = 1, \dots, n$) koordinátáját jelölje $\text{Pr}_i(p)$. Legyen M egy olyan egész, hogy $0 \leq M \leq n$. A $p, q \in \mathbb{Z}^n$ pontokat M -szomszédoknak mondjuk, ha kielégítik a következő két feltételt:

- $|\text{Pr}_i(p) - \text{Pr}_i(q)| \leq 1 \quad (1 \leq i \leq n),$
- $\sum_{i=1}^n |\text{Pr}_i(p) - \text{Pr}_i(q)| \leq M.$

Az $A = (A(i))_{i=1}^\infty$ sorozatot, ahol $A(i) \in \{1, \dots, n\}$ minden $i \in \mathbb{N}$ -re, n -dimenziós (röviden nD) általános szomszédsági szekvenciának nevezzük. Ha létezik olyan $l \in \mathbb{N}$, melyre $A(i+l) = A(i)$ ($i \in \mathbb{N}$), akkor A periodikus l periódussal. Az nD általános szomszédsági szekvenciák halmazát S_n -nel jelöljük. Megjegyezzük, hogy az általános szomszédsági szekvencia az [5]-ben bevezetett periodikus szomszédsági szekvencia általánosítása.

Legyen $p, q \in \mathbb{Z}^n$ és $A \in S_n$. A $p = p_0, p_1, \dots, p_m = q$ pontsorozatot, ahol p_{i-1} és p_i $A(i)$ -szomszédok $\mathbb{Z}^n$ -ben ($1 \leq i \leq m$), egy p -t q -val összekötő m hosszú A -útnak nevezzük. A p és q pont $d(p, q; A)$ -val jelölt A -távolságán a pontok között vezető legrövidebb A -út hosszát értjük. Az A -távolságot röviden $d(A)$ -val jelöljük. Könnyen látható, hogy nem minden A szekvencia indukál metrikát. Minden $i \in \mathbb{N}$ és $j \in \{1, \dots, n\}$ esetén vezessük be az alábbi jelöléseket:

$$A^{(j)}(i) = \min(A(i), j) \quad \text{és} \quad f_j^A(i) = \sum_{k=1}^i A^{(j)}(k).$$

Sokszor fontos két pont A -távolságának meghatározása. DAS szerzőtársaival ezt [5]-ben megtette periodikus szekvenciákra nézve. A disszertációban (lásd 2. fejezet, 2.9. Lemma) ezt kiterjesztjük az általános esetre.

Legyen $A, B \in S_n$ tetszőleges. Defináljuk a $\sqsubseteq^*$ relációt az alábbi módon:

$$A \sqsubseteq^* B \iff d(p, q; A) \leq d(p, q; B), \text{ minden } p, q \in \mathbb{Z}^n \text{ esetén.}$$

Ez a reláció a DAS [4] által periodikus esetben bevezetett rendezés általánosítása. A disszertáció 3. fejezetében (lásd 3.7. Tétel) megmutatjuk, hogy

$$d(p, q; A) \leq d(p, q; B), \quad \forall p, q \in \mathbb{Z}^n\text{-re}$$

pontosan akkor, ha

$$f_j^A(i) \geq f_j^B(i), \quad \forall i \in \mathbb{N}\text{-re}, j \in \{1, \dots, n\}\text{-re.}$$

Az értekezés 3.1. szakaszában azt is megvizsgáljuk, hogy a szomszédsági szekvenciák halmazai milyen hálóelméleti struktúrát alkotnak a fenti rendezésre nézve. S_n mellett annak ugyanazon típusú részhalmazait elemezzük, mint DAS [4]-ben a periodikus esetben. Mivel a $\sqsubseteq^*$ rendezés ritkán indukál szép hálóstruktúrákat, ezért bevezetjük annak egy durvább változatát is, amelyet használva szebb strukturális eredményekhez jutunk. Minden $A = (A(i))_{i=1}^\infty, B = (B(i))_{i=1}^\infty \in S_n$ esetén definiáljuk a $\sqsubseteq$ relációt az alábbi módon:

$$A \sqsubseteq B \iff A(i) \geq B(i), \text{ minden } i \in \mathbb{N} \text{ esetén.}$$

A fenti strukturális eredményeinket a ∞ dimenziós digitális térre is kiterjesztjük a disszertáció 3.2. szakaszában. Az ott kapott hálóstruktúrák bizonyos értelemben a véges dimenziós hálók uniói lezártjának tekinthetők. Az értekezés 2. és 3. fejezete mellett ezek az eredmények [16]-ben is közlésre kerültek FAZEKAS ATTILA és HAJDU LAJOS társszerzőkkel.

2. Szomszédsági szekvenciák metrikus vizsgálata

A $\sqsubseteq^*$ rendezés csak féligrendezés a szomszédsági szekvenciák halmazán, ám elképzelhetők olyan alkalmazások, ahol szükség lehet bármely két szekvencia „gyorsaságának” összehasonlítására. Ezen probléma megoldására a

disszertáció 4. fejezetében bevezetünk egy norma jellegű sebességfüggvényt a szekvenciák halmazán, és megvizsgáljuk annak tulajdonságait. A sebesség definiálásához a következő feltételeket vesszük alapul:

- (i) *A sebességnek érzékenynek kell lennie a szekvencia egyes alterekben való viselkedésére.*
- (ii) *A szekvencia elemei különböző mértékben járuljanak hozzá a sebességhez.*
- (iii) *A sebességnek összhangban kell lennie a $\sqsubseteq^*$ rendezéssel.*

A fentieknek megfelelően a sebességet a következőképpen adjuk meg. A $\delta_j : \mathbb{N} \rightarrow \mathbb{R}$ ($j = 1, \dots, n$) függvényhalmazt súlyrendszernek nevezzük, ha teljesül az alábbi három feltétel:

- $\delta_j(i) \geq 0$ ($i \in \mathbb{N}$, $j = 1, \dots, n$),
- $\sum_{i=1}^{\infty} \delta_j(i) < \infty$ ($j = 1, \dots, n$),
- δ_j monoton csökkenő ($j = 1, \dots, n$).

Legyen $A \in S_n$ és δ_j ($j = 1, \dots, n$) egy súlyrendszer. Az A szekvencia j -dimenziós sebessége

$$v_j^A = \sum_{i=1}^{\infty} A^{(j)}(i) \delta_j(i),$$

az A sebessége pedig

$$v^A = \frac{1}{n} \sum_{j=1}^n v_j^A.$$

Mivel $\sqsubseteq^*$ csak féligrendezés, ezért (iii) csak az egyik irányban teljesülhet. A másik irány megvalósításához még egy további feltétel szükséges:

$$v_j^A \geq v_j^B, \quad \forall \delta_j\text{-re } (j = 1, \dots, n) \implies A \sqsubseteq^* B, \quad \forall j = 1, \dots, n\text{-re.}$$

A sebesség bevezetéséhez hasonlóan a disszertáció 4.5. szakaszában metrikát is definiálunk a szomszédsági szekvenciák halmazán, és elvégezzük a szokásos vizsgálatokat. Legyen $\Delta = \{\delta_j : j = 1, \dots, n\}$ egy súlyrendszer és $A, B \in S_n$. Ezen szekvenciák ϱ_Δ távolságát a következőképpen definiáljuk:

$$\varrho_\Delta(A, B) = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^{\infty} |A^{(j)}(i) - B^{(j)}(i)| \delta_j(i).$$

Az értekezés 4.5. szakaszában megmutatjuk azt is, hogy minden Δ súlyrendszer esetén (S_n, ϱ_Δ) teljes metrikus tér. Továbbá, a metrikus tér minden korlátos monoton sorozata konvergens, és minden korlátos végtelen részhalmazának létezik torlódási pontja. A periodikus sorozatok halmaza a metrikus tér egy sűrű részhalmazát adja. A sebesség és metrika bevezetésével kapcsolatos eredményeket és gyakorlati példákat a disszertáció 4. fejezete és a HAJDU LAJOSSAL közös [21] tartalmazza.

3. Szomszédsági szekvenciák geometriai vizsgálata

A disszertáció 5. fejezetében a szomszédsági szekvenciák által elfoglalt tartományokat vizsgáljuk meg. A szekvenciák „terjedése” eltolásinvariáns, ezért az egyszerűség kedvéért a $\mathbb{Z}^n$ tér $\mathbf{0}$ origójából indulunk. Legyen $A \in S_n$ és minden $k \in \mathbb{N}$ -re

$$\widehat{A}_k = \{q \in \mathbb{Z}^n : d(\mathbf{0}, q; A) \leq k\}$$

az A által az első k lépésben elfoglalt tartomány. Jelölje $H(\widehat{A}_k)$ az $\widehat{A}_k$ konvex burkát $\mathbb{R}^n$ -ben.

Approximációs módszereinkhez a szomszédsági szekvenciákhoz és a klasszikus euklideszi metrikához tartozó „körök” összehasonlítása szolgál alapul. Minden $k \in \mathbb{N}$ esetén legyen

$$O_k = \{q \in \mathbb{Z}^2 : L_2(\mathbf{0}, q) \leq k\},$$

illetve

$$G_k = \{q \in \mathbb{R}^2 : L_2(\mathbf{0}, q) \leq k\}$$

a k sugarú euklideszi körlap $\mathbb{Z}^2$ -ben, illetve $\mathbb{R}^2$ -ben.

Számításainkban gyakran használjuk egy $A \in S_n$ szomszédsági szekvencia első k elemében előforduló azonos elemek számát. Legyen $A = (A(i))_{i=1}^\infty \in S_n$ és minden $i = 1, \dots, n$ és $k \in \mathbb{N}$ esetén

$$A(i, k) = \#\{A(j) : A(j) = i, 1 \leq j \leq k\}.$$

Az elfoglalt tartományok aszimptotikus tanulmányozásához a szomszédsági szekvencia egyes elemeinek sűrűségét (amennyiben ez nem létezik, alsó és felső sűrűségét) használjuk a disszertáció 5. fejezetében.

DAS és CHATTERJI [9] az euklideszi metrika szomszédsági szekvenciákkal való közelítésének hatékonyságát vizsgálta. Nevezetesen, azon nyolcszögek geometriai tulajdonságait elemezték, amelyeket egy szomszédsági szekvencia az origóból kiindulva a síkban való terjedésekor elfoglal. Általánosított szomszédsági szekvenciákat használva az euklideszi metrika jobban közelíthető. Ennek megfelelően a disszertáció 5. fejezetében kiterjesztjük DAS és CHATTERJI [9] 2D-ben elvégzett geometriai vizsgálatait általános szomszédsági szekvenciákra, és beágyazzuk a korábbi eredményeket az új környezetbe. $\mathbb{Z}^2$ helyett ezeket a vizsgálatainkat tetszőleges véges dimenzióra elvégezzük. Egy sok helyen jól felhasználható eredményként (lásd disszertáció 5. fejezet, 5.4. Tétel) először megadjuk az elfoglalt poliéderek csúcspontjait. Legyen $A \in S_n$ és $k \in \mathbb{N}$. A $H(\widehat{A_k})$ poliéder csúcsai pontosan azok a pontok, melyeknek koordinátái az alábbiak egy permutációja

$$\left(\lambda_1 \sum_{i=1}^n A(i, k), \lambda_2 \sum_{i=2}^n A(i, k), \dots, \lambda_n A(n, k) \right),$$

ahol $\lambda_j = \pm 1$ ($j = 1, \dots, n$), és λ_j eltérhet az egyes permutációkban.

Az n -dimenziós esetben, a disszertáció 5.1. szakaszában meghatározzuk az elfoglalt poliéderek átmérőjét, és vizsgáljuk az átmérő és a lépésszám hányadosának aszimptotikus viselkedését. Mivel a 2D-s és 3D-s eset kiemelt jelentőséggel bír a képfeldolgozásban, ezért a $\mathbb{Z}^2$ és $\mathbb{Z}^3$ tereket behatóan vizsgáljuk a disszertáció 5.2. szakaszában. A szomszédsági szekvencia elemeinek sűrűségével (illetve alsó/felső sűrűségével) meghatározzuk az elfoglalt tartományok oldalhosszának, területének és térfogatának aszimptotikus viselkedését. Geometriai eredményeink a disszertáció 5. fejezete mellett [19]-ben is közlésre kerültek.

4. Az euklideszi metrika közelítése digitális metrikákkal

Az euklideszi metrika digitális távolságokkal való közelítése széleskörben kutatott terület. DAS [3]-ban olyan szomszédsági szekvenciák által in-

dukált távolságfüggvényeket adott meg, amelyek egy bizonyos értelemben az euklideszi metrika jó közelítéseit adják. Mi is végrehajtottunk egy hasonló, de általánosabb vizsgálatot az euklideszi metrika közelítésére szomszédsági szekvenciák segítségével a disszertáció 6. fejezetében. Approximációs modellünk geometriai megközelítésen alapszik, nevezetesen az euklideszi metrika és a szomszédsági szekvenciák távolságfüggvényeihez tartozó különböző sugarú köröket hasonlítjuk össze. Szemben DAS [3] eredményeivel, periodikus helyett általános szekvenciákkal dolgozunk, és lehetőség nyílik a legjobban közelítő szomszédsági szekvencia explicit, zárt alakban való megadására is. Vizsgálatunk a szomszédsági szekvenciák egész halmazára kiterjed, és megadjuk a legjobban közelítő, metrikát generáló szekvenciákat is.

Három approximációs problémát fogalmazunk meg, amelyek megoldása a legjobban közelítő szomszédsági szekvenciákat szolgáltatja. Az alábbiakban $T(H)$ a H síkbeli tartomány területét jelöli.

1. probléma. Keressük meg azt a $\Lambda \in S_2$ szomszédsági szekvenciát, amelyre minden $A \in S_2$ és $k \in \mathbb{N}$ esetén

$$T(H(\widehat{\Lambda}_k) \nabla G_k) \leq T(H(\widehat{A}_k) \nabla G_k).$$

2. probléma. Keressük meg azt a $\Phi \in S_2$ szomszédsági szekvenciát, amelyre minden $k \in \mathbb{N}$ esetén $H(\widehat{\Phi}_k) \supseteq G_k$, továbbá ha $A \in S_2$ és $H(\widehat{A}_k) \supseteq G_k$, akkor

$$T(H(\widehat{\Phi}_k) \setminus G_k) \leq T(H(\widehat{A}_k) \setminus G_k).$$

A disszertáció 6.2. szakaszában megadunk két, a fenti problémákat kielégítő szomszédsági szekvenciát. Ezen kívül meghatározunk egy szekvenciát, amely olyan digitális metrikát indukál, amely az L_2 -t az első probléma értelmében legjobban közelíti. A következő kérdés a második probléma diszkrét változatára vonatkozik.

3. probléma. Keressük meg azt a $\Psi \in S_2$ szomszédsági szekvenciát, amelyre $\widehat{\Psi}_k \supseteq O_k$ minden $k \in \mathbb{N}$ esetén, és ha $A \in S_2$ úgy, hogy $\widehat{A}_k \supseteq O_k$, akkor $\widehat{A}_k \supseteq \widehat{\Psi}_k$.

A probléma megoldásához ugyancsak egy olyan szomszédsági szekvenciát tudunk megadni a disszertáció 6.2. szakaszában, ami metrikát generál. A megoldás során ú.n. Beatty-sorozatokot kapunk, így a legjobban közelítő

szomszédsági szekvenciákat zárt alakban meg tudjuk adni. Approximációs eredményeink az értekezés 6. fejezetében és a HAJDU LAJOS társszerzővel közös [20] cikkben is megtalálhatók.

II. Karakterfelismerés Walsh transzformációval

A disszertáció második részében egy Walsh transzformáción alapuló, saját fejlesztésű karakterfelismerőt mutatunk be, amely elsősorban nyomda-technikailag rögzített szövegek felismeréséhez használható. Az algoritmus elkészítése során a szegmentálásra és osztályozásra fektettük a hangsúlyt, és kísérletekkel ellenőriztük rendszerünk megbízhatóságát. A digitális képfeldolgozáson belül gyakran használt Walsh transzformációt nagyításinvarianciája és geometriát megőrző tulajdonsága teszi alkalmassá karakterfelismerésre. A felismerni kívánt karakterekhez Walsh transzformáltakból álló sajátosságvektorokat rendelünk, amelyeket etalonvektorok alapján osztályozunk. A Walsh etalonvektorok meglehetősen jól szeparálhatók, ezért a tévesztés esélye kisebb, mint más klasszikus (például projekciós histogramon vagy zónázáson alapuló) módszerek esetében. Algoritmusunkban nem számolunk ki minden transzformáltat, ami csökkenti a számítási időt és fokozza a zajjal szembeni toleranciát. Módszerünk hatékonyságának és megbízhatóságának bizonyításául bemutatjuk eljárásunk zajtűrő képességét, továbbá más felismerőkkel való összehasonlításának eredményeit.

Az algoritmusunk első lépésként a kiindulási bináris képet kisebb szegmensekre bontja, amelyeket egy láncolt listában tárol el. A szegmensek téglalapok, melyek bal felső pontjainak koordinátái és méretei kerülnek tárolásra a képi információval együtt. A szegmentációs algoritmus részletes leírása az értekezés 9.1. szakaszában található meg. Karakterfelismerési eljárásunk a disszertáció 9.2. szakaszában leírt Walsh transzformáción alapul. Minden szegmenshez egy 64D-s sajátosságvektort készítünk a szegmens első 64 Walsh transzformáltjából. A sajátosságvektor elkészítése után eldöntjük, hogy a szegmens felismerhető karaktert vagy egyéb képadatot

tartalmaz-e. Az értekezés 9.3. szakasza tartalmazza a különböző szinteken végrehajtható döntési folyamatot és az algoritmus taníthatóságának feltételeit. A hatékonyság ellenőrzésére, a zajérzékenységre és a felismerés pontosságára vonatkozó tesztjeinkbe olyan gyakran használt betűtípusokat vontunk be, mint az OCR-A, OCR-B, CMR, TIMES NEW ROMAN és a magyar rendszámtáblákon szereplő karakterek. Ezeken kívül más statisztikai teszteket, például korrelációs analízist is végeztünk. Kísérleti tapasztalatainkat a disszertáció 9.4. szakasza rögzíti. Az itt leírt eredmények a FAZEKAS ATTILA társszerzővel közös [15] publikációban ugyancsak megtalálhatók.

Karakterfelismerő eljárásunknak egy gyakorlati alkalmazását is adtuk: az eljárást beépítettük egy géppel írt dokumentumok tömörítésére használható, karakterfelismerésen alapuló rendszerbe. Az alkalmazás leírását és a végrehajtott teszteket az értekezés 9.5. szakasza és a FAZEKAS ATTILÁVAL közös [14] is tartalmazza.

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APPENDIX A

Recognition accuracy and failures for the CMR alphabet at 40% global and contour noise using Walsh transformation

	Rate	Failure		Rate	Failure		Rate	Failure
a	98%	@ H	A	100%		?	100%	
b	99%	h	B	100%		!	63%	l
c	95%	e 0 6 E	C	86%	0 6 c	\	55%	I 1
d	100%		D	100%		#	100%	
e	100%		E	100%		&	100%	
f	100%		F	100%		~	17%	-
g	100%		G	100%		{	57%	I
h	100%		H	100%		}	61%	I)
i	54%	I	I	100%		0	100%	
j	51%	I }]	J	100%		1	96%	I
k	98%	h	K	100%		2	100%	
l	36%	I 1	L	100%		3	100%	
m	100%		M	100%		4	100%	
n	96%	u	N	100%		5	100%	
o	95%	. O	O	100%		6	95%	0
p	100%		P	100%		7	93%	I 1
q	97%	H	Q	98%	0	8	100%	
r	99%	1	R	99%	H	9	100%	
s	83%	H S 8	S	98%	8	+	99%	;
t	81%	1 I [1 (	T	100%		%	100%	
u	99%	0	U	100%		/	73%	I 1
v	90%	V M	V	98%	v	=	63%	-
w	100%		W	50%	w	(	51%	I 1 l t
x	99%	\	X	100%		)	48%	I }] l
y	99%	I	Y	100%		-	100%	
z	98%	E Z	Z	100%		.	84%	=
						,	54%	I ! 1 l ;
						:	42%	I 1
						;	67%	I i l {
						@	100%	
						[	66%	l
						]	57%	l i ;

APPENDIX B

Recognition accuracy according to different types and levels of noise corruption

Noise level		Walsh		Projection hist.		Zoning	
Global	Contour	CMR	OCR	CMR	OCR	CMR	OCR
5%	0%	100%	100%	100%	100%	100%	100%
10%	0%	100%	100%	99.95%	99.69%	99.96%	100%
15%	0%	99.94%	100%	87.02%	94.65%	99.33%	100%
20%	0%	99.69%	100%	70.46%	80.94%	95.60%	100%
25%	0%	99.15%	100%	57.80%	65.60%	89.89%	99.80%
30%	0%	97.27%	99.95%	–	–	82.06%	98.49%
35%	0%	94.85%	99.91%	–	–	72.48%	95.66%
40%	0%	90.58%	99.55%	–	–	59.12%	90.51%
45%	0%	84.10%	98.12%	–	–	–	78.32%
50%	0%	75.08%	94.08%	–	–	–	–
55%	0%	–	86.39%	–	–	–	–
60%	0%	–	72.22%	–	–	–	–
0%	20%	100%	100%	100%	100%	100%	100%
0%	25%	99.93%	100%	99.99%	100%	99.99%	100%
0%	30%	99.86%	100%	99.93%	100%	99.86%	100%
0%	35%	99.63%	100%	99.37%	100%	99.51%	100%
0%	40%	99.14%	100%	96.08%	99.88%	98.11%	100%
0%	45%	98.26%	100%	89.33%	98.94%	96.43%	100%
0%	50%	96.96%	100%	78.93%	95.80%	94.17%	99.95%
0%	55%	95.39%	99.84%	–	85.64%	90.82%	99.80%
0%	60%	93.36%	99.32%	–	69.04%	84.19%	98.34%
0%	65%	88.92%	97.40%	–	–	72.60%	93.04%
0%	70%	80.63%	91.26%	–	–	–	78.71%
0%	75%	64.67%	77.60%	–	–	–	55.94%
5%	5%	100%	100%	99.96%	100%	100%	100%
10%	10%	99.96%	100%	96.58%	99.31%	99.77%	100%
15%	15%	99.75%	100%	76.70%	90.51%	97.10%	100%
20%	20%	99.58%	100%	–	73.54%	91.31%	100%
25%	25%	98.45%	100%	–	55.89%	81.52%	99.68%
30%	30%	96.45%	99.98%	–	–	70.48%	97.34%
35%	35%	93.65%	99.89%	–	–	–	91.65%
40%	40%	88.54%	99.04%	–	–	–	79.20%
45%	45%	79.95%	94.48%	–	–	–	56.54%
50%	50%	–	83.26%	–	–	–	–
scan	scan	98.86%	100%	95.45%	100%	95.45%	100%

APPENDIX C Publications of András Hajdu

Publications in reviewed journals

1. A. Fazekas and A. Hajdu: An algorithm using Walsh transformation for compressing typeset documents, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **15** (1999), 61-68.
2. A. Fazekas and A. Hajdu: Recognizing typeset documents using Walsh transformation, *J. Comput. Inf. Technol.* **9** (2001), 101-112.
3. A. Fazekas, A. Hajdu and L. Hajdu: Lattice of generalized neighbourhood sequences in nD and ∞D , *Publ. Math. Debrecen* **60** (2002), 405-427.
4. A. Hajdu and L. Hajdu: Velocity and distance of neighbourhood sequences, *Acta Cybernet.*, submitted.
5. A. Hajdu: Geometry of Neighbourhood Sequences, *Pattern Recognition Lett.*, submitted.
6. A. Hajdu and L. Hajdu: Approximating the Euclidean distance by digital metrics, *J. Approx. Theory*, submitted.

Other refereed publications

7. A. Fazekas and A. Hajdu: Analysing the noise sensitivity of skeletonization algorithms, *Studia Universitatis Babes-Bolyai* **2** (1997), 55-70.
8. Á. Baráz, A. Fazekas, A. Hajdu and R. Jónás: A gyakorlati feladatok szerepe a digitális képfeldolgozás oktatásában és megoldásukat támogató szoftverek, *Informatika a Felsőoktatásban'99* (1999), 116-121.
9. A. Fazekas, A. Hajdu and L. Hajdu: Véges dimenziós szekvenciák strukturális vizsgálata, *KÉPAF'02* (2000), 29-32.
10. A. Fazekas, A. Hajdu and L. Hajdu: Végtelen dimenziós szekvenciák strukturális vizsgálata, *KÉPAF'02* (2000), 33-36.

11. A. Hajdu: Geometric and topological properties of generalized neighbourhood sequences, *5th International Conference of Applied Informatics*, Eger (2001), to appear.
12. A. Fazekas and A. Hajdu: Extracting feature vectors for character recognition by Walsh transformation, *KÉPAF'03* (2002), 272-278.
13. A. Hajdu and B. Nagy: Approximating the Euclidean circle using neighbourhood sequences, *KÉPAF'03* (2002), 260-271.
14. A. Hajdu and Z. Zörgő: Orvosi szoftver keretrendszer műtéti tervezéshez, *KÉPAF'03* (2002), 140-151.
15. A. Fazekas, G. Fazekas, A. Hajdu and J. Kormos: A digitális képfeldolgozás oktatásának eredményei és néhány aktuális kérdése a Debreceni Egyetemen, *Informatika a Felsőoktatásban 2002* (2002), 758-765.
16. A. Hajdu and J. Kormos: A Debreceni egyetem intézeteinek együttműködése különböző képfeldolgozási projekteken, *Informatika a Felsőoktatásban 2002* (2002), 773-780.

Presentations

17. A. Fazekas and A. Hajdu: A character recognition algorithm based on Walsh transformation, 2nd conference of PhD students in Computer Science, Miskolc, Hungary, 1999.
18. A. Fazekas and A. Hajdu: Properties of generalized neighbourhood sequences in finite dimension, 4th International Conference on Applied Informatics, Eger-Noszvaj, Hungary, 1999.
19. A. Hajdu: Image processing applications at the University of Debrecen, Summer School on Image Processing, Szeged, Hungary, 2001.
20. A. Hajdu, as a member of the IKTA-4 6/2001 consortium: MEDIP – Platformfüggetlen szoftver keretrendszer orvosi képfeldolgozáshoz (projektindító előadás), Budapest, Hungary, 2001.

**NEIGHBOURHOOD SEQUENCES AND
CHARACTER RECOGNITION BY WALSH TRANSFORMATION**

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a Matematika tudományágban

Írta: Hajdu András okleveles matematikus

Készült a Debreceni Egyetem Matematika doktori iskolája
(Informatika programja) keretében

Témavezető: Dr. Kormos János

A doktori szigorlati bizottság:

elnök:	Dr.	
tagok:	Dr.	
	Dr.	

A doktori szigorlat időpontja: 200...

Az értekezés bírálói:

Dr.	
Dr.	
Dr.	

A bírálóbizottság:

elnök:	Dr.	
tagok:	Dr.	
	Dr.	
	Dr.	
	Dr.	

Az értekezés védésének időpontja: 200...