A high-order non-conforming finite element family

doktori (PhD) értekezés

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Dr. Stoyan Gisbert
témavezető

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Contents

List of Figures

List of Notations

$B_{c,\Delta}^{(k)}$	kth-order conforming bubble function defined over a triangle Δ
$B_{n,\Delta}^{(k)}$	kth-order non-conforming bubble function defined over a triangle Δ
$C^{\infty}(\Omega)$	space of infinitely differentiable functions over Ω
$C_0^{\infty}(\Omega)$	space of infinitely differentiable functions having a compact support in Ω
$\boldsymbol{\Delta}$	Laplace operator
Δ	a triangle
Δ_0	the reference triangle
diam	diameter
div	the divergence operator
$\Gamma = \partial(\Omega)$	the boundary of Ω
grad	the gradient operator
h	the discretization parameter
$H_0^1(\Omega)$	the Sobolev space of functions with square integrable gradient and zero boundary values (in the sense of traces)
$H^1(\Omega)$	the Sobolev space of functions with square integrable gradient
$L^2(\Omega)$	space of square integrable functions over Ω
$L_0^2(\Omega)$	space of square integrable functions with zero integral over Ω
L_k	the kth-order Legendre polynomial defined on the interval $[0,1]$
λ_i	barycentric coordinates, $i = 1, 2, 3$
Ω	a domain in $\mathbb{R}^2$
P_h	the discrete pressure space
$\mathsf{P}_k^{(\alpha,\beta)}$	the kth-order Jacobi polynomial on the interval $[-1,1]$ with parameters (α,β) and leading coefficient 1.
$\mathbb{P}_k$	the space of the polynomials with maximal degree k
$\mathbb{R}$	the set of the real numbers
rot	the rotation operator
$\mathcal{T}_h$	a triangulation of Ω
V_h	the discrete velocity space

Preface

Numerous numerical experiments show that the use of high-order finite elements is advantageous in solving flow problems [16], [22]. It is well known that the motion of an incompressible viscous fluid can be described by the Stokes equations. In the present dissertation we will consider triangular finite elements for the two-dimensional Stokes problem.

An example of a finite element family that is defined for arbitrary order are the Taylor-Hood finite elements where the velocity and the pressure are approximated trianglewise by polynomials of order k and $k-1$, respectively, and the polynomials are assumed to be continuous on the common sides of two adjacent triangles. The element is inf-sup stable [5].

In the case of the pressure it is common to use discontinuous approximations to ensure the elementwise mass conservation property.

In [25] Scott and Vogelius investigated the $\mathbb{P}_k/\mathbb{P}_{k-1}$ finite element pair (polynomials of order k for the velocity and $(k-1)$ for the pressure) where there is no continuity requirement for the discrete pressures. The element is inf-sup stable for $k \geq 4$ if the triangulation does not contain near-singular points, i.e. points which are close to the situation where the edges meeting in this point lie on two straight lines.

An other problem caused by the singular points is that in the presence of these points the nullspace of the discrete gradient operator is larger than the nullspace of the gradient operator in the original problem (where it contains only constant functions). This means that while in the continuous case the pressure can be determined uniquely up to an additive constant, after the finite element discretization of the problem we have a system of linear equations which has a higher dimensional nullspace.

To ensure the grid independent stability one may consider the non-conforming elements. In [11] Crouzeix and Raviart defined a first-order non-conforming finite element pair, the second-, and third-order elements were described by Fortin and Soulie [13], and by Crouzeix and Falk [10], respectively. In all the three cases the kth-order ($k = 1, 2, 3$) polynomials which approximate the velocity are continuous on the common side of two neighboring triangles only in the kth-order Gauss-Legendre points. These points are the roots of the kth-order Legendre polynomial defined over the given side of the triangle. In [11] the authors showed that the continuity requirements in these points ensure the optimal order of convergence.

The construction of the second-order element differs from the cases $k = 1, 3$. If $k = 1$ or $k = 3$ then for the degrees of freedom one can choose the $3k$ Gauss-Legendre points on the sides of the triangle and $\frac{(k-2)(k-1)}{2}$ points distributed uniformly inside the triangle.

If $k = 2$ then there exists a second-order polynomial which disappears in all the six second-order Gauss-Legendre points on the sides of any triangle. This polynomial is called a second-order non-conforming bubble function. In this case one gets the velocity part of the finite element by adding trianglewise the bubble function to the velocity space of the second-order Scott-Vogelius elements.

In the past few years several authors have dealt with the study of non-conforming finite elements.

In [7] the elements of order $k = 4$ and $k = 6$ are investigated. Similarly to the second-order case the authors define bubble functions of order k and the non-conforming elements are given by enriching the corresponding conforming velocity spaces by these bubbles.

In [18] a non-conforming finite element family is described that is defined for arbitrary order. The authors also use bubble functions, but the construction differs from the earlier one: here a polynomial of order $k + 1$ is added trianglewise to the kth-order velocity space.

In the present work we deal with the description of a non-conforming finite element family which generalizes the low-order ($k = 1, 2, 3$) cases. The family is defined for arbitrary order and for even k it is inf-sup stable without a restriction on the grid. We show that for even k the non-conforming bubble removes from the kernel of the discrete gradient the non-constant pressures arising for the Scott-Vogelius elements if singular points are present.

This dissertation consists of four chapters. In Chapter 1 we give a short review of the basic notions and theorems about the finite element methods.

Since in the even order cases we will define the non-conforming element based on the conforming Scott-Vogelius elements in Chapter 2 we examine the algebraic properties of this family.

In Section 2.1.2 we investigate the nullspace of the discrete gradient operator when the triangulation contains singular points. Using the result about its dimension [25] we give a basis of this nullspace. Here we use orthogonal polynomials defined over triangles [17].

In Section 2.2 we study the second-order non-conforming element defined by Fortin and Soulie - for even k we will construct the new non-conforming finite element family (called Gauss-Legendre elements) similarly to this case.

In Section 2.3 we shortly describe the fourth-, and sixth-order elements defined by Y.Cha, M. Lee and S. Lee [7].

The detailed examination of the Gauss-Legendre elements is given in Chapter 3. In Section 3.1 we define the elements: velocity and pressure are approximated trianglewise by polynomials of order k and $k-1$, respectively, and the discrete velocities are continuous on the common side of two neighboring triangles in the kth-order Gauss-Legendre points. For odd k we specify suitable degrees of freedom for the velocity part, and connected with the interpolation problem arising here we determine a general formula for the even-order non-conforming bubble function. For $k = 2$ this gives the bubble used in [13] and for $k = 4, 6$, apart from a conforming part, the bubbles defined in [7].

In Section 3.2 we show that in the case of Gauss-Legendre elements for even k the nullspace of the discrete gradient operator is one-dimensional: it consists of constant functions. Here we use the results of Section 2.1.2 and we prove that by adding trianglewise the non-conforming bubble function to the conforming velocity space the nullspace described in Section 2.1.2 becomes one-dimensional.

For even k the proof of stability is based on the macroelement technique which was described for the conforming cases by Stenberg [26]. In Section 3.3 a slight modification of this method for the non-conforming case is given. For the proof we need the result of Section 3.2 about the kernel of the discrete gradient.

The non-conforming bubble function has an important role in the stability. However, for odd k there does not exist a kth-order polynomial defined over a triangle which disappears on the sides of the triangle only in the kth-order Gauss-Legendre points. In Section 3.1 we have described functions that can be considered as non-conforming bubbles over two adjacent triangles: they are equal to zero in the Gauss-Legendre points on the boundary sides of the quadrilateral formed by the triangles. In Section 3.4 we investigate whether the proof of stability given in the even order cases can be modified using these bubbles.

In Chapter 4 we present some numerical results using Matlab computations. For different values of k and for various triangulations in the case of Scott-Vogelius and Gauss-Legendre elements we give the value of the discrete inf-sup stability constant and we solve a test problem using fourth-order Gauss-Legendre elements.

Chapter 1

Introduction and preliminary results

In the present chapter we introduce the basic notions and propositions that are needed in the presentation of the results of the author. Most of the definitions and theorems presented here can be found in the monographs of Atkinson and Han [1], Braess [3], Brezzi and Fortin [6] and Ciarlet [8]. Besides this we provide a wide overview of the historical background of this research.

1.1 The Stokes equations

In the present work we will deal with the finite element solution of the two-dimensional Stokes equations. These equations describe the motion of an incompressible viscous fluid in a 2-dimensional domain. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz continuous boundary. The Stokes problem is the following: find $\vec{u} : \Omega \to \mathbb{R}^2$ and $p : \Omega \to \mathbb{R}$ such that

$$-\vec{\boldsymbol{\Delta}}\vec{u} + \operatorname{grad} p = \vec{f}, \tag{1.1.1}$$

$$-\operatorname{div}\vec{u} = 0, \tag{1.1.2}$$

$$\vec{u}|_\Gamma = \vec{u}_0, \tag{1.1.3}$$

where $\Gamma = \partial\Omega$ is the boundary of the domain, and $\vec{f}$ is a given external force field. Here $\vec{u} = (u_1, u_2)^T$ is the velocity vector and p is the pressure.

Applying Green's formula we obtain that for the solvability of the equations we must have

$$\int_\Gamma \vec{u}_0 \vec{n} \,\mathrm{d}s = \int_\Omega \operatorname{div}\vec{u} \,\mathrm{d}x\,\mathrm{d}y = 0,$$

where $\vec{n}$ is the outward-pointing normal unit vector to Γ.

In (1.1.1)–(1.1.3) the pressure is determined only up to an additive constant, but for a fixed p the velocity vector is uniquely given, since the operator $\vec{\mathbf{\Delta}}$ is positive definite.

A solution $\vec{u}$ and p of (1.1.1)–(1.1.3) is called a classical solution if $\vec{u} \in (C^2(\Omega) \cap C(\overline{\Omega}))^2$ and $p \in C^1(\Omega)$.

1.2 Saddle point problem and the inf-sup condition

Before formulating the weak version of the Stokes equations let us consider the solvability conditions of a saddle point problem. Let V and M be two Hilbert spaces with the norms $||\cdot||_V$ and $||\cdot||_M$, respectively, and let $a : V \times V \to \mathbb{R}$, $b : V \times M \to \mathbb{R}$ be continuous bilinear functionals,

$$|a(u,v)| \leq M_a ||u||_V ||v||_V, \quad |b(u,q)| \leq M_b ||u||_V ||q||_M.$$

Suppose that $f \in V'$ and $g \in M'$, where V' and M' are the dual spaces of V and M, respectively. Consider the following saddle point problem: find $(u,p) \in V \times M$ such that

$$a(u,v) + b(v,p) = \langle f, v\rangle \quad \forall v \in V, \tag{1.2.1}$$

$$b(u,q) = \langle g, q\rangle \quad \forall q \in M. \tag{1.2.2}$$

Theorem 1.2.1 *(Brezzi's theorem) Suppose that the bilinear forms a and b are continuous and let*

$$\begin{aligned} \mathcal{L} : V \times M &\to V' \times M' \\ (u,p) &\mapsto (f,g) \end{aligned}$$

be the linear mapping defined by (1.2.1)–(1.2.2). *Then $\mathcal{L}$ is an isomorphism if and only if*

1. *for the bilinear form a with some $\alpha > 0$*

   $$a(v,v) \geq \alpha ||v||_V^2$$

   holds for all $v \in V_0 := \{v \in V \ : \ b(v,q) = 0 \quad \forall q \in M\}$ (that means a is V_0-elliptic), and

2. *there exists a constant β such that*

   $$\inf_{p \in M} \sup_{v \in V} \frac{b(v,p)}{||v||_V \cdot ||p||_M} \geq \beta > 0 \tag{1.2.3}$$

   holds.

Moreover, if the conditions above are satisfied then the solution of the problem (1.2.1)–(1.2.2) *is stable in the sense*

$$
\begin{aligned}
||u||_V &\le \frac{||f||}{\alpha} + \left(a + \frac{M_a}{\alpha}\right)\frac{||g||}{\beta},\\
||p||_M &\le \left(a + \frac{M_a}{\alpha}\right)\frac{1}{\beta}\left(||f|| + \frac{M_a}{\beta}||g||\right).
\end{aligned}
$$

The inequality (1.2.3) is called inf-sup condition.

1.3 Variational formulation

In this section we give the weak formulation of the problem (1.1.1)–(1.1.3) which is a starting point to get the finite element solution of the Stokes equations.

Let $C^\infty(\Omega)$ denote the space of infinitely differentiable functions, and let $C_0^\infty(\Omega)$ be the subspace of $C^\infty(\Omega)$ which consists of functions having a compact support in Ω. In order to ensure the uniqueness of the solution p we will assume that

$$\int_\Omega p \, \mathrm{d}x = 0,$$

and we define the space $L_0^2(\Omega)$ in the usual way:

$$L_0^2(\Omega) := \left\{ q \in L^2(\Omega) \ \ : \ \ \int_\Omega q \, \mathrm{d}x = 0 \right\}.$$

Let

$$H^m(\Omega) := \left\{ u \in L^2(\Omega) \ \ : \ \ D^\alpha u \in L^2(\Omega), \ \forall \ |\alpha| \le m \right\}$$

be the Sobolev space of functions which have square integrable αth weak derivatives for all $|\alpha| \le m$. Denote by $(\cdot,\cdot)_0$ the usually inner product in L^2. The space $H^m(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_m := \sum_{|\alpha| \le m} (D^\alpha u, D^\alpha v)_0.$$

The corresponding norm is

$$||u||_m := \sqrt{\sum_{|\alpha| \le m} ||D^\alpha u||^2_{L^2(\Omega)}}, \tag{1.3.1}$$

and the function

$$|u|_m := \sqrt{\sum_{|\alpha| = m} ||D^\alpha u||^2_{L^2(\Omega)}} \tag{1.3.2}$$

defines a semi-norm on $H^m(\Omega)$.

First we will deal with the equations (1.1.1)–(1.1.3) in the case of homogeneous boundary conditions that is when $\vec{u}_0 \equiv 0$.

Denote by $H_0^m(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm $||\cdot||_m$. If Ω is bounded, then $|\cdot|_m$ is a norm on $H_0^m(\Omega)$ which is equivalent to $||\cdot||_m$ (see [3, Ch. II. §1.]).

Multiplying (1.1.1) with a test-function $\vec{v} \in (H_0^1(\Omega))^2$ and integrating over Ω, from the Green-formula we obtain

$$\int_\Omega \operatorname{grad}\vec{u} : \operatorname{grad}\vec{v}\,\mathrm{d}x - \int_\Omega p \operatorname{div}\vec{v}\,\mathrm{d}x = (\vec{f}, \vec{v})_0,$$

where

$$\operatorname{grad}\vec{u} : \operatorname{grad}\vec{v} := \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j},$$
$$(\vec{f}, \vec{v})_0 := \int_\Omega (f_1 v_1 + f_2 v_2)\,\mathrm{d}x.$$

Similarly, from (1.1.2) with a test-function $q \in L_0^2(\Omega)$ we have

$$-\int_\Omega q \operatorname{div}\vec{u}\,\mathrm{d}x = 0.$$

Denote by $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ the following bilinear functionals:

$$a(\vec{u}, \vec{v}) := \int_\Omega \operatorname{grad}\vec{u} : \operatorname{grad}\vec{v}\,\mathrm{d}x, \tag{1.3.3}$$
$$b(\vec{v}, p) := -\int_\Omega p \operatorname{div}\vec{v}\,\mathrm{d}x. \tag{1.3.4}$$

Using these notations in the case of homogeneous boundary conditions the variational formulation of the equations (1.1.1)–(1.1.3) is the following: find $\vec{u} \in (H_0^1(\Omega)^2$ and $p \in L_0^2(\Omega)$ such that

$$a(\vec{u}, \vec{v}) + b(\vec{v}, p) = (\vec{f}, \vec{v})_0 \quad \forall \vec{v} \in (H_0^1(\Omega))^2, \tag{1.3.5}$$
$$b(\vec{u}, q) = 0 \quad \forall q \in L_0^2(\Omega). \tag{1.3.6}$$

It can be shown (see e.g. [3, Ch. III, §5.]) that if for a solution $(\vec{u}, p)$ of (1.3.5)–(1.3.6) $\vec{u} \in (C^2(\Omega) \cap C^0(\overline{\Omega}))^2$ and $p \in C^1(\Omega)$ hold, then $(\vec{u}, p)$ is a solution of (1.1.1)–(1.1.3).

To prove the existence and uniqueness of the solution of (1.3.5)–(1.3.6), we will use the Nečas theorem [20]:

Theorem 1.3.1 *Let* Ω *be a domain with Lipschitz continuous boundary. For all* $q \in L_0^2(\Omega)$ *there exists a* $\vec{u} \in (H_0^1(\Omega))^2$ *such that*

$$q = \operatorname{div} \vec{u} \quad \textit{and} \quad |\vec{u}|_1 \leq c_0 ||q||_{L_2(\Omega)} \tag{1.3.7}$$

hold, and the constant c_0 *is independent of* q *and* $\vec{u}$*.*

Theorem 1.3.2 *If* Ω *is a domain with Lipschitz continuous boundary then the variational problem* (1.3.5)–(1.3.6) *has a unique solution.*

Proof. We have to check the conditions of Theorem 1.2.1. The bilinear forms a and b are continuous and since $|\vec{u}|_1 = a(\vec{u}, \vec{u})^{1/2}$ is a norm on $(H_0^1(\Omega))^2$, the bilinear form a is elliptic on $(H_0^1(\Omega))^2$. To verify the inf-sup condition let $q \in L_0^2(\Omega)$ be an arbitrary function and let $\vec{u} \in (H_0^1(\Omega))^2$ be a function for which (1.3.7) holds. Then the inf-sup inequality follows from

$$b(-\vec{u}, q) = \int_\Omega q^2 \, \mathrm{d}x = ||q||_{L_2(\Omega)}^2 \geq \frac{1}{c_0} ||q||_{L_2(\Omega)} \cdot |\vec{u}|_1 .$$

□

1.4 Finite elements

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain. We divide Ω into finitely many subdomains and we will approximate the solution of the variational problem with functions which are polynomials on each subdomain.

Definition 1.4.1 *Let* $\mathcal{T}_h = \{\Delta_i, \ i = 1, \dots, n\}$ *be a partition of* Ω *into triangles, where* $\operatorname{diam} \Delta_i \leq h$ *for all* $i = 1, \dots, n$*. We call* $\mathcal{T}_h$ *a triangulation of* Ω *if the following conditions are satisfied*

1. $\overline{\Omega} = \bigcup_{i=1}^n \Delta_i$,
2. *if* $i \neq j$ *and* $\Delta_i \cap \Delta_j \neq \emptyset$ *then exactly one of the following two conditions is satisfied*
    - a. $\Delta_i \cap \Delta_j$ *consists of exactly one point, and this point is a common vertex of* Δ_i *and* Δ_j,
    - b. $\Delta_i \cap \Delta_j$ *is a common edge of* Δ_i *and* Δ_j.

Definition 1.4.2 *(See [3]) A finite element is a triple* $(T, \Pi_T, \sum_T)$ *with the following properties:*

1. T *is a polyhedron in* $\mathbb{R}^2$,

2. *Π_T is a subspace of $C(T)$ with finite dimension s,*

3. *$\sum_T$ is a set of s linearly independent functionals on Π_T. Each $p \in \Pi_T$ is uniquely defined by the values of the s functionals in $\sum_T$. Since the functionals usually involve point evaluation of a function or of its derivatives at points in T, we call these functionals interpolation conditions.*

Remark 1.4.3 Usually the set Π_T itself is called a finite element.

Definition 1.4.4 *If there is a set of points which uniquely determines any function in the finite element space by its values at the given points, these points are called nodal points and the functionals that map the nodal points on the function values are called (Lagrange type) degrees of freedom.*

Remark 1.4.5 In Definition 1.4.2 polyhedrons could be both triangles and quadrilaterals. However, in the present work we use only triangular finite elements.

Remark 1.4.6 (See [3]) Let $t \geq 0$ and Δ a given triangle. Suppose that $z_1, \ldots, z_s$ are $s = 1 + 2 + \cdots + (t+1)$ points in Δ which lie on $t+1$ lines, see Figure 1.1. Then for each $f \in C(\Delta)$ there exists a unique polynomial p of degree t satisfying the interpolation conditions

$$p(z_i) = f(z_i), \quad i = 1, \ldots, s.$$

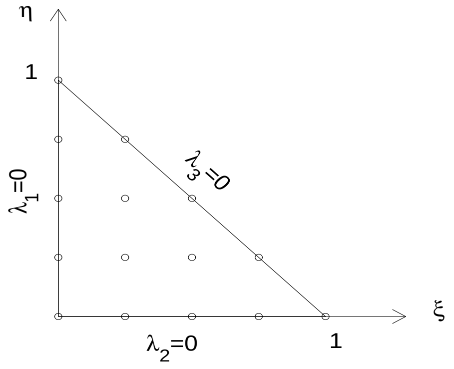


Figure 1.1: The nodal points for the standard Lagrangian basis on Δ_0 if $k = 4$.

Barycentric coordinates. To describe the polynomials given on a triangle we will use barycentric coordinates. Let Δ be a triangle with vertices $S_i = (x_i, y_i)$, $i = 1, 2, 3$, and let $P = (x, y)$ be a point in Δ. If for $\lambda_1, \lambda_2, \lambda_3$

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{1.4.1}$$

holds, then we call $\lambda_1, \lambda_2, \lambda_3$ the barycentric coordinates of P in Δ. The determinant of the matrix in (1.4.1) is equal to $2|\Delta|$, where

$$|\Delta| = \frac{1}{2}(x_1y_2 - y_1x_2 + x_2y_3 - y_2x_3 + x_3y_1 - y_3x_1)$$

is the area of the triangle Δ. Thus λ_1, λ_2, λ_3 are simply the ratios of the areas of triangles S_2S_3P, S_1S_3P and S_1S_2P, respectively, to $|\Delta|$.

The following lemma will be useful if we want to calculate integrals of polynomials over the triangle Δ (see [15]).

Lemma 1.4.7 *Let $\Delta \in \mathcal{T}_h$ be an arbitrary triangle and denote by $\lambda_1, \lambda_2, \lambda_3$ the barycentric coordinates in Δ. In this case*

$$\int_\Delta \lambda_1^k \lambda_2^\ell \lambda_3^m \, dx \, dy = 2|\Delta| \frac{k!\ell!m!}{(k + \ell + m + 2)!}.$$

Denote by $\mathbb{P}_k(\Delta)$ the space of polynomials given on Δ with maximal degree k. In what follows we will need the standard Lagrangian basis of $\mathbb{P}_k(\Delta)$, where $\Delta \in \mathcal{T}_h$. Consider the points $\gamma_i^{(k)}$, $i = 1, \dots (k + 1)(k + 2)/2$, in Δ with the barycentric coordinates (see Figure 1.1)

$$\lambda_1 = \frac{j}{k}, \quad \lambda_2 = \frac{\ell}{k}, \quad j = 0, \dots, k, \;\; \ell = 0, \dots, k - j.$$

Then $\{u_i \in \mathbb{P}_k(\Delta), \;\; i = 1, \dots, (k + 1)(k + 2)/2\}$ is the standard Lagrangian basis of $\mathbb{P}_k(\Delta)$, if

$$u_i(\gamma_j^{(k)}) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

holds for all $i, j = 1, \dots, (k + 1)(k + 2)/2$.

Remark 1.4.8 Let $\mathcal{T}_h$ be a triangulation of the polygonal domain Ω. Consider on each triangle the nodal points corresponding to the standard Lagrangian basis of order k. In this case there are $(k + 1)$ points on the edges of the triangles and the restriction of an arbitrary polynomial of order k to an edge is uniquely determined by its values at these points. This means that if $f : \Omega \to \mathbb{R}$ is a function for which $f|_\Delta \in \mathbb{P}_k(\Delta)$ holds for all $\Delta \in \mathcal{T}_h$, and f is continuous on the common side of two adjacent triangles in these nodal points then f is globally continuous on Ω.

Definition 1.4.9 *(See [3]) Let* $\mathcal{T}_h$ *be a triangulation of* Ω*, and let* V_h *be a family of finite elements for the partition* $\mathcal{T}_h$*. This family is called affine family if there exists a reference element* $(\Delta_0, \Pi_0, \sum_0)$ *such that for every* $\Delta \in \mathcal{T}_h$ *there exists an affine mapping* $F_\Delta(x) = B_\Delta x + b_\Delta$, $B_\Delta \in \mathbb{R}^{2\times 2}$, $b_\Delta \in \mathbb{R}^2$*, which has the following properties:*

(i) $F_\Delta : \Delta_0 \to \Delta$ *and* $F_\Delta(\Delta_0) = \Delta$*,*

(ii) for every $v \in V_h$

$$v|_\Delta(x) = p(F_\Delta^{-1}(x))$$

holds where $p \in \Pi_0$*.*

In what follows we will deal with affine finite element families. For simplicity sometimes we will call affine finite element families just finite element spaces.

1.5 The discrete Stokes problem

Let $\mathcal{T}_h$ be a triangulation of a polygonal domain Ω and let $V_h \subset (H_0^1(\Omega))^2$ and $P_h \subset L^2(\Omega)$ be finite element spaces. We will approximate the solution of (1.3.5)–(1.3.6) in the finite dimensional spaces V_h and P_h. Since the spaces V_h and P_h are subspaces of $(H_0^1(\Omega))^2$ and $L^2(\Omega)$, respectively, the bilinear forms a and b are well defined on $V_h \times V_h$ and on $V_h \times P_h$, and the discrete form of the equations (1.3.5)–(1.3.6) is the following: find $\vec{u}_h \in V_h$ and $p_h \in P_h$ such that

$$a(\vec{u}_h, \vec{v}_h) + b(\vec{v}_h, p_h) = (\vec{f}, \vec{v}_h)_0 \quad \forall \vec{v}_h \in V_h, \tag{1.5.1}$$

$$b(\vec{u}_h, q_h) = 0 \quad \forall q_h \in P_h \cap L_0^2(\Omega). \tag{1.5.2}$$

In this case we call the approximation **conforming approximation** and the finite element spaces V_h and P_h conforming finite element spaces.

Let $\{\vec{v}_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^M$ be bases of the spaces V_h and P_h, respectively. The matrix form of the equations (1.5.1)–(1.5.2) is

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} f_h \\ 0 \end{pmatrix}, \tag{1.5.3}$$

where

$$\begin{aligned} A_h &:= (a_{ij})_{i,j=1}^N, & a_{ij} &:= a(\vec{v}_i, \vec{v}_j), \\ B_h &:= (b_{ij})_{i,j=1}^{N,M}, & b_{ij} &:= b(\vec{v}_j, q_i), \\ f_h &:= (f_i)_{i=1}^N, & f_i &:= (\vec{f}, \vec{v}_i)_0, \\ \vec{u}_h &:= \sum_{i=1}^N y_i \vec{v}_i, & p_h &:= \sum_{i=1}^M z_i q_i, \\ \underline{u} &:= (y_1, \ldots, y_N)^T, & \underline{p} &:= (z_1, \ldots, z_M)^T. \end{aligned}$$

In the original problem (1.1.1)–(1.1.3) the pressure is determined only up to an additive constant, since the nullspace of the gradient operator is one-dimensional and it contains only constant functions. Similarly, considering the problem (1.3.5)–(1.3.6), the space

$$\{q \in L^2(\Omega) \;\;:\;\; b(\vec{v}, q) = 0 \quad \forall \vec{v} \in (H_0^1(\Omega))^2\}$$

contains only the constant functions, so in the solution of (1.3.5)–(1.3.6) the function p is uniquely determined in $L_0^2(\Omega)$. Using the notations B and B^T for the operators $B : \vec{v} \mapsto b(\vec{v}, \cdot)$ and $B^T : q \mapsto b(\cdot, q)$, we obtain that $\ker B^T$ is one-dimensional.

Returning to the discrete problem (1.5.1)–(1.5.2) it may happen that the space

$$\{q_h \in P_h \;\;:\;\; b(\vec{v}_h, q_h) = 0 \quad \forall \vec{v}_h \in V_h\},$$

or with matrix notations the space $\ker B_h^T$ contains also non-constant functions. In this case p can not be determined uniquely in $P_h/\mathbb{R}$, there are present “energy-free” pressures.

In the formulation of the theorem about the solvability and uniqueness we take into account these cases too (see [6]).

Theorem 1.5.1 *Suppose that*

(1) the bilinear form a is elliptic on the space

$$V_{h,0} := \{\vec{v}_h \in V_h \;\;:\;\; b(\vec{v}_h, q_h) = 0 \quad \forall q_h \in P_h\},$$

i.e. there exists $\alpha_h > 0$ *such that*

$$a(\vec{v}_h, \vec{v}_h) \geq \alpha_h ||\vec{v}_h||_1 \quad \forall \vec{v}_h \in V_{h,0},$$

(2) and with a constant $\beta_h > 0$

$$\sup_{\vec{v}_h \in V_h} \frac{b(\vec{v}_h, q_h)}{|\vec{v}|_1} \geq \beta_h \inf_{q_{0h} \in \ker B_h^T} ||q_h + q_{0h}||_{L^2(\Omega)} = \beta_h ||q_h||_{L^2(\Omega)/\ker B_h^T}$$

holds for all $q_h \in P_h$,

then the problem (1.5.1)–(1.5.2) *is uniquely solvable in* $V_h \times (P_h/\ker B_h^T)$. *Moreover, if* $\beta_h \geq \beta_0 > 0$ *holds with a constant* β_0 *independent of* h, *then the solution is stable and*

$$||\vec{u} - \vec{u}_h||_1 \leq c_1 (\inf_{\vec{v}_h \in V_h} ||\vec{u} - \vec{v}_h||_1 + \inf_{q_h \in P_h} ||p - q_h||_{L^2(\Omega)}),$$

$$||p - p_h||_{L^2(\Omega)/\ker B_h^T} \leq c_2 (\inf_{\vec{v}_h \in V_h} ||\vec{u} - \vec{v}_h||_1 + \inf_{q_h \in P_h} ||p - q_h||_{L^2(\Omega)}),$$

where the constants c_1 *and* c_2 *are independent of* h.

The condition (2) is called discrete inf-sup condition or Babuška-Brezzi condition.

A conforming finite element pair for the approximation of the Stokes problem was described by Scott and Vogelius [24]. Velocity and pressure are approximated trianglewise by polynomials of degrees k and $k-1$, respectively, and the discrete velocities are continuous on the common side of two adjacent triangles. The element is defined for arbitrary order k.

The Scott-Vogelius element pair is an example for the phenomenon mentioned above: in the case of certain triangulations $\dim\ker B_h^T > 1$, see [25]. The dimension of $\ker B_h^T$ depends on the number of singular points of the triangulation. Scott and Vogelius [25] give the dimension of $\ker B_h^T$ for $k \geq 4$ without describing the space itself. For $k \geq 4$ they prove the stability of the element under a condition which is also connected with the singular points. We will examine the Scott-Vogelius elements in a detailed form in Chapter 2.

Several authors deal with the problem of avoiding the unpleasant grid condition. In [21] macroelements are used to obtain inf-sup stability for $k = 2, 3$: each triangle of the triangulation is divided into 3 triangles by connecting its centroid with the three vertices.

An other possibility to ensure the stability is the enlargement the discrete velocity space: we do not assume the continuity of the discrete velocities on the common side of two neighboring triangles, but in this case $V_h \not\subset (H^1(\Omega))^2$.

1.6 Non-conforming approximations

If the finite element spaces V_h and P_h where we want to approximate the solution of the variational problem do not lie in the spaces $(H^1(\Omega))^2$ and in $L^2(\Omega)$, respectively, then we call the finite elements non-conforming finite elements.

An example for non-conforming finite elements is the first order Crouzeix-Raviart element [11], where

$$\begin{aligned}
V_h :=& \big\{\vec{v} \;:\; \vec{v}|_\Delta \in (\mathbb{P}_1(\Delta))^2, \quad \forall \Delta \in \mathcal{T}_h,\ \vec{v} \text{ is continuous at the midpoints} \\
&\quad \text{of the edges and } \vec{v} = 0 \text{ at the midpoints of the edges on } \partial\Omega\big\}, \\
P_h :=& \{q \;:\; q|_\Delta \in \mathbb{P}_0(\Delta), \quad \forall \Delta \in \mathcal{T}_h\}.
\end{aligned}$$

Since the elements of V_h (the discrete velocities) are not continuous on the common side of two adjacent triangles (the continuity is required only in one point) the space V_h is not a subspace of $(H^1(\Omega))^2$ and we can not define the bilinear forms a and b as in (1.3.3)–(1.3.4). Let

$$a(\vec{u}, \vec{v}) := \sum_{\Delta \in \mathcal{T}_h} \int_\Delta \operatorname{grad} \vec{u} : \operatorname{grad} \vec{v} \,\mathrm{d}x, \tag{1.6.1}$$

$$b(\vec{v}, p) := -\sum_{\Delta \in \mathcal{T}_h} \int_\Delta p \operatorname{div} \vec{v} \,\mathrm{d}x, \tag{1.6.2}$$

and we define the semi-norm $|\cdot|_{1,h}$ as

$$|\vec{v}|_{1,h} := \sqrt{a(\vec{v},\vec{v})}. \tag{1.6.3}$$

In the remaining part of this work we will deal with approximations where the coordinate functions of the discrete velocities are trianglewise polynomials of a given order with or without continuity assumptions between the triangles. If $\vec{u},\vec{v} \in (H^1(\Omega))^2$ then (1.6.1)–(1.6.2) define the same bilinear forms as (1.3.3)–(1.3.4), and $|\vec{v}|_{1,h} = |\vec{v}|_1$, so in what follows we consider (1.6.1)–(1.6.2) as the definition of a and b.

It is easy to see that for the space V_h the so-called patch-test is fulfilled (with $k = 1$):

1. if Δ_1 and Δ_2 are two adjacent triangles with the common edge E then for all $\vec{v}_h \in V_h$
$$\int_E q(\vec{v}_{h,1} - \vec{v}_{h,2})\,\mathrm{d}s = 0 \qquad \forall q \in \mathbb{P}_{k-1}(E)$$
holds, where $\vec{v}_{h,1} = \vec{v}_h|_{\Delta_1}$, $\vec{v}_{h,2} = \vec{v}_h|_{\Delta_2}$,
2. for all edges which lies on $\partial\Omega$ and for all $\vec{v}_h \in V_h$
$$\int_E q\vec{v}_h\,\mathrm{d}s = 0 \qquad \forall q \in \mathbb{P}_{k-1}(E)$$
holds.

Conditions 1. and 2. imply that the semi-norm $|\cdot|_{1,h}$ defines a norm on V_h. In general: if the space V_h consists of functions that are polynomials of order k on each triangle and the patch test is fulfilled for the space V_h then (1.6.1) and (1.6.3) define a norm on V_h (see [11]). From this we obtain the usual requirement for the discrete velocities in the kth-order case: the velocities are continuous between the triangles in the kth-order Gauss-Legendre points (these are the roots of the kth-order Legendre polynomial defined on the sides of the triangles), and they are equal to 0 in the kth-order Gauss-Legendre points of the edges on $\partial\Omega$. In this work we will consider discretisations where in the case of kth-order discrete velocities the discrete pressures (the elements of P_h) are trianglewise polynomials of order $k-1$.

A second order non-conforming finite element is the Fortin–Soulie element (see [13] and (2.2.1)–(2.2.2) in the present work), while the element corresponding to the case $k = 3$ (Crouzeix–Falk element) is investigated in [10]. In [7] the authors construct non-conforming finite elements of orders 4 and 6 (see in Section 2.3).

In the present dissertation we will deal with the generalization of the above cases. The detailed description of the higher order elements can be found in Chapter 3, where we prove the inf-sup stability of these non-conforming elements for all even k. Since in the case of even k the nullspace of the discrete gradient contains only the constant functions (see Theorem 3.2.1) we consider the conditions for the solvability and stability only in this case (a general theorem for the cases where $\dim\ker B^T > 1$ can be found in [6]).

If the patch-test is satisfied then the bilinear form a is elliptic on V_h and for the unique solvability of the discrete variational problem (1.5.1)–(1.5.2) we need only the discrete inf-sup condition:

there exists a constant $\beta_h > 0$ such that

$$\sup_{\vec{v}_h \in V_h} \frac{b(\vec{v}_h, q_h)}{|\vec{v}_h|_{1,h}} \geq \beta_h ||q_h||_{L^2(\Omega)} \qquad \forall q_h \in P_h \cap L_0^2(\Omega).$$

If $\beta_h \geq \beta_0 > 0$ for all h, where β_0 is independent of h then the solution is stable.

1.7 Error estimates

In [11] the authors investigate the error bounds in the case when the velocity and the pressure are approximated trianglewise by polynomials of order k and $k-1$, respectively. They prove that if the solutions of (1.5.1)–(1.5.2) are smooth enough, i.e.

$$\vec{u} \in \{\vec{v} \ : \ \vec{v} \in (H_0^1(\Omega))^2, \ \operatorname{div}\vec{v} = 0\} \cap (H^{k+1}(\Omega))^2, \qquad p \in H^k(\Omega),$$

and the patch-test is fulfilled then under suitable conditions

$$\begin{aligned} ||\vec{u}_h - \vec{u}||_{1,h} &\leq C_1 \cdot h^k(|\vec{u}|_{k+1} + |p|_k), \\ ||\vec{u}_h - \vec{u}||_{L^2(\Omega)} &\leq C_2 \cdot h^{k+1}(|\vec{u}|_{k+1} + |p|_k), \\ ||p_h - p||_{L^2(\Omega)/\mathbb{R}} &\leq C_3 \cdot h^k(|\vec{u}|_{k+1} + |p|_k), \end{aligned}$$

where the constants C_1, C_2 and C_3 depend on the triangulation.

1.8 Crouzeix-Velte decomposition

The Crouzeix-Velte decomposition of the Sobolev space $(H_0^1(\Omega))^d$, $d = 2,3$, of vector functions defined over a Lipschitz continuous domain $\Omega \subset \mathbb{R}^d$ is a decomposition into three orthogonal subspaces which was described first in [9] and later, independently, in [32]. In [32] and [12] the decomposition is used to get more information about the inf-sup constant of the Stokes problem.

By partial integration one can show that

$$a(\vec{u}, \vec{v}) = (\operatorname{div}\vec{u}, \operatorname{div}\vec{v})_0 + (\operatorname{rot}\vec{u}, \operatorname{rot}\vec{v})_0$$

holds for all $\vec{u}, \vec{v} \in (H_0^1(\Omega))^d$, $d = 2,3$. From this representation of the $|\cdot|_1$ norm we obtain the following orthogonal decomposition of the space $(H_0^1(\Omega))^d$:

$$(H_0^1(\Omega))^d = V_0 \oplus V_1 \oplus V_\beta,$$

where

$$V_0 := \{\vec{v} \in (H_0^1(\Omega))^d \quad : \quad \operatorname{div} \vec{v} = 0\},$$
$$V_1 := \{\vec{v} \in (H_0^1(\Omega))^d \quad : \quad \operatorname{rot} \vec{v} = 0\},$$

and the orthogonality is understood in the sense of $a(\cdot,\cdot)$. A similar decomposition exists for the space $L^2(\Omega)$:

$$L^2(\Omega) = P_0 \oplus P_1 \oplus P_\beta,$$

where

$$P_0 := \ker \operatorname{grad}, \qquad P_1 := \operatorname{div} V_1, \qquad P_\beta := \operatorname{div} V_\beta.$$

The inf-sup constant is determined by the space V_β only (see [27]), that is

$$\inf_{0 \neq q \in L_0^2(\Omega)} \sup_{\vec{v} \in (H_0^1(\Omega))^2} \frac{b^2(\vec{v}, q)}{|\vec{v}|_1^2 ||q||_{L^2(\Omega)}^2} = \frac{1}{1+\kappa^2},$$

where

$$\kappa = \sup_{\vec{v} \in V_\beta} \frac{||\operatorname{rot} \vec{v}||_{L_2(\Omega)}}{||\operatorname{div} \vec{v}||_{L_2(\Omega)}}.$$

Let V_h and P_h be finite element spaces and consider the matrix form (1.5.3) of the discrete Stokes problem. Denote by M_h the mass matrix of the pressure basis:

$$M_h = (m_{ij})_{i,j=1}^M, \qquad m_{ij} := (q_i, q_j)_0,$$

and let $\operatorname{div}_h : V_h \rightarrow P_h$ and $\operatorname{rot}_h : V_h \rightarrow P_h$ be the discrete divergence and rotation operators that are defined as projections into the pressure space:

$$(\operatorname{div}_h \vec{v}_h, q)_0 = (\operatorname{div} \vec{v}, q)_0 \qquad \forall q \in P_h,$$
$$(\operatorname{rot}_h \vec{v}_h, q)_0 = (\operatorname{rot} \vec{v}, q)_0 \qquad \forall q \in P_h.$$

The discrete Crouzeix-Velte decomposition is the orthogonal decomposition of the discrete velocity space V_h into three subspaces,

$$V_h = V_{h,0} \oplus V_{h,1} \oplus V_{h,\beta}, \tag{1.8.1}$$

where $V_{h,0} := \ker \operatorname{div}_h$, $V_{h,1} := \ker \operatorname{rot}_h$, and the third subspace $V_{h,\beta}$ might be empty. We call (1.8.1) proper if $V_{h,\beta} \neq \emptyset$ holds. The decomposition (1.8.1) can be characterized by the generalized eigenvalue problem

$$B_h^T M_h^{-1} B_h \underline{u} = \mu_h A_h \underline{u}. \tag{1.8.2}$$

If the decomposition (1.8.1) exists, then the eigenvalues of (1.8.2) are in $[0, 1]$, and the eigenvectors corresponding to the eigenvalues $\mu_h = 0$ and $\mu_h = 1$ span the spaces

$V_{h,0}$ and $V_{h,1}$, respectively, and the eigenvectors corresponding to $\mu_h \in (0,1)$ span the space $V_{h,\beta}$, see [27]. Using transformation $\underline{p} = M_h^{-1} B_h \underline{u}$ from (1.8.2) we obtain the eigenproblem

$$S_h \underline{p} = \lambda_h M_h \underline{p}, \tag{1.8.3}$$

where

$$S_h := B_h A_h^{-1} B_h^T \tag{1.8.4}$$

is the so-called Schur complement operator. The eigenvalues of the problems (1.8.2) and (1.8.3) coincide not counting the multiplicity of the zero eigenvalue. The zero eigenvalues of (1.8.3) correspond to the eigenvectors $p_h \in \ker B_h^T$ and the discrete inf-sup constant is the square root of the smallest nonzero eigenvalue of (1.8.3), see [6].

For some finite elements and triangulations the spectrum of the Schur operator S_h can be seen in Chapter 4.

The discrete inf-sup constant can be used to optimize iterative methods for the solution of the discrete problem (1.5.3) (see [29]).

Chapter 2

Some finite element families

In this chapter we deal with a conforming finite element family for the two-dimensional Stokes problem, namely with the Scott-Vogelius elements. The family is defined for arbitrary order k and the elements are inf-sup stable for $k \geq 4$ under a grid condition. In the case of certain triangulations (if there are present so-called singular points on the grid) we find a phenomenon mentioned in the previous chapter: the nullspace of the discrete gradient operator also contains non-constant functions. In Section 2.1.2 we describe a basis of this nullspace which will be useful in the calculations in Chapter 3.

In Section 2.2 we study a second order non-conforming finite element that is known inf-sup stable without assumptions on the grid, and we show that in this case the nullspace of the discrete gradient - on analogy of the continuous case - is one-dimensional.

In Section 2.3 we review two methods to construct higher order non-conforming elements that are inf-sup stable on arbitrary grids.

Throughout this chapter if it does not lead to confusion in the notations of the discrete velocity and pressure functions we will omit the index h.

2.1 Scott-Vogelius elements

2.1.1 Description and properties

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and $\mathcal{T}_h$ be a triangulation of Ω.

In [24] and [25] Scott and Vogelius examined a higher order conforming finite element family: velocities and pressures are approximated trianglewise by polynomials of order k and $(k-1)$, respectively. Moreover, the discrete velocities are assumed to be continuous, but there is no continuity requirement on the pressure:

$$P_h(\Omega) = \big\{ p \in L^2(\Omega) \ : \ p|_\Delta \in \mathbb{P}_{k-1}(\Delta), \ \Delta \in \mathcal{T}_h \big\}, \tag{2.1.1}$$

$$V_{h,k}(\Omega) = \big\{ \vec{v} \in (H_0^1(\Omega))^2 \ : \ \vec{v}|_\Delta \in (\mathbb{P}_k(\Delta))^2, \ \Delta \in \mathcal{T}_h \big\}. \tag{2.1.2}$$

For $k \geq 4$ Scott and Vogelius in [24] proved the stability of the finite element pair (2.1.1)–(2.1.2) under a condition connected with the grid singularity.

Definition 2.1.1 *(See [25]) A vertex of the triangulation is called singular point if the edges meeting at this vertex lie on two straight lines.*

In the case of an inner singular point four triangles meet around a common vertex and the common sides of the triangles fall into two straight lines (see Figure 2.1). The four possible cases of the boundary singular points are showed on Figure 2.2.

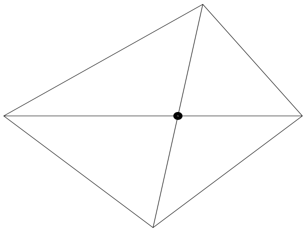


Figure 2.1: Inner singular point.

Scott and Vogelius [25] introduced a function which measures how close a non-singular vertex is to being singular:

Definition 2.1.2 *Let x_0 be a non-singular vertex of $\mathcal{T}_h$ and let θ_i, $i = 1, \dots, n$, be the angles of the triangles Δ_i, $i = 1, \dots, n$, meeting at x_0 (the triangles are numbered sequentially). Then we define the function $R(x_0)$ as*

$$R(x_0) := \max\{|\theta_i + \theta_j - \pi|, \quad \textit{where} \quad 1 \leq i, j \leq n, \quad i - j = 1 \mod n\}.$$

Definition 2.1.3 *(See [25]) We call the family of triangulations $\{\mathcal{T}_h\}$, $0 < h \leq 1$, quasiuniform if there exists a constant $\kappa > 0$ such that*

$$\kappa \cdot h \geq \rho_\Delta \qquad \forall \Delta \in \mathcal{T}_h, \qquad 0 < h \leq 1,$$

where ρ_Δ denotes the supremum of diameters of discs contained in Δ.

The stability of the element (2.1.1)–(2.1.2) holds only in the case if the triangulation does not contain near-singular points.

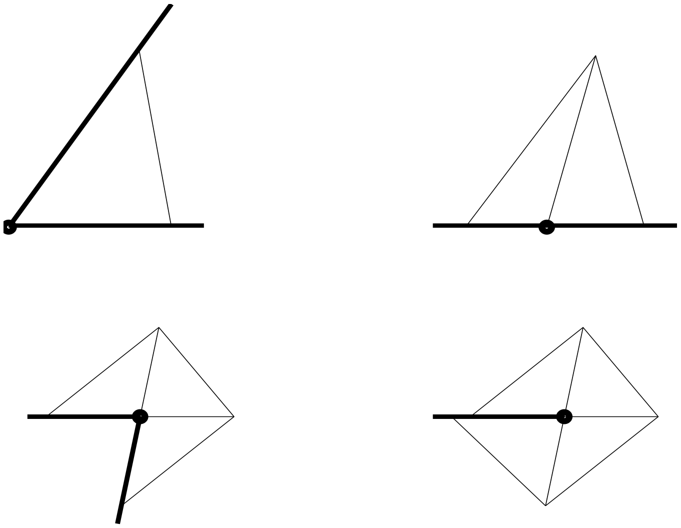


Figure 2.2: The four types of boundary singular point. The thick line is the boundary of Ω.

Theorem 2.1.4 *(See [25]) Let $\mathcal{T}_h$, $0 < h \leq 1$, be a quasiuniform family of triangulations. Assume that*

$$\min\{R(x_0) \ : \ x_0 \text{ is a non-singular inner vertex of } \mathcal{T}_h\} \geq \delta > 0,$$

where the constant δ is independent of h. Then for any $k \geq 4$ the finite element family (2.1.1)–(2.1.2) *is inf-sup stable.*

It follows from the results described in Section 1.7 that – under a condition with respect to the grid – for arbitrary $k \geq 4$ (2.1.1)–(2.1.2) define a finite element pair such that the corresponding discrete solution converges to the solution of the variational problem (1.3.5)–(1.3.6), the convergence of the velocity and pressure part is of order $k+1$ and k in the norms $||\cdot||_{1,h}$ and $||\cdot||_{L^2(\Omega)}$, respectively.

In what follows we examine the nullspace of the discrete gradient operator. Let σ be the number of the singular points in the triangulation, and consider the matrix form (1.5.3) of the discrete Stokes equations corresponding to the finite element (2.1.1)–(2.1.2). Then (see [25]) in the case of $k \geq 4$ the range of B_h is equal to

$$\frac{1}{2}(k+1)kT - \sigma - 1,$$

where T is the number of the triangles in $\mathcal{T}_h$. Since $\dim P_h = T \cdot k(k+1)/2$, we have

$$\dim \ker B_h^T = \sigma + 1.$$

Lemma 2.1.5 *In the case of the finite element* (2.1.1)–(2.1.2) *for* $k \geq 4$ *the nullspace of the discrete gradient operator*

$$N_{V_{h,k}(\Omega)} := \{p \in P_h(\Omega) \ : \ b(\vec{v}, p) = 0, \quad \textit{for all} \quad \vec{v} \in V_{h,k}(\Omega)\}, \tag{2.1.3}$$

is $\sigma + 1$ *dimensional, where* σ *is the number of the singular points in* $\mathcal{T}_h$*.*

Definition 2.1.6 *Let* Ω *be the unit square. We call the triangulation* $\mathcal{T}_h$ *criss-cross grid when the sides of* Ω *are divided into* n *equal parts and all the small squares are divided into four triangles by their diagonals. Then* $\mathcal{T}_h$ *has* n^2 *singular points.*

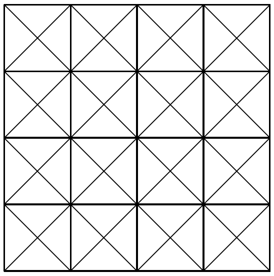


Figure 2.3: Criss-cross triangulation of the unit square in the case of $n = 4$

The criss-cross grid contains many singular points, and on grids produced by standard triangulation programs, near-singular and singular points can often be observed [19], [23]. The grids on Figure 2.4 and 2.5 were generated by the triangulation program of the Matlab PDE Toolbox, the second grid is a refinement of the first one.

In the case when near-singular points approach singular points, the right inverse of the divergence operator is blowing up (see [25]).

2.1.2 The nullspace of the discrete gradient

To describe the nullspace we will use orthogonal polynomials given on a triangle. For this aim, we denote by $\mathsf{P}_n^{(\alpha,\beta)}$ the Jacobi polynomial of order n on the interval $[-1, 1]$ with parameters α, β and with leading coefficient 1 defined as

$$\mathsf{P}_n^{(\alpha,\beta)}(x) = \frac{1}{2^n} \sum_{m=0}^{n} \binom{n+\alpha}{m} \binom{n+\beta}{n-m} (x-1)^{n-m} (x+1)^m. \tag{2.1.4}$$

Let us denote by Δ_0 the reference triangle,

$$\Delta_0 = \{(\xi, \eta) \in \mathbb{R}^2 \quad | \quad 0 \leq \xi \leq 1, \quad 0 \leq \eta \leq 1 - \xi\},$$

and by $\lambda_1, \lambda_2, \lambda_3$ the barycentric coordinates in the reference triangle Δ_0: $\lambda_1 = \xi$, $\lambda_2 = \eta$, $\lambda_3 = 1 - \xi - \eta$.

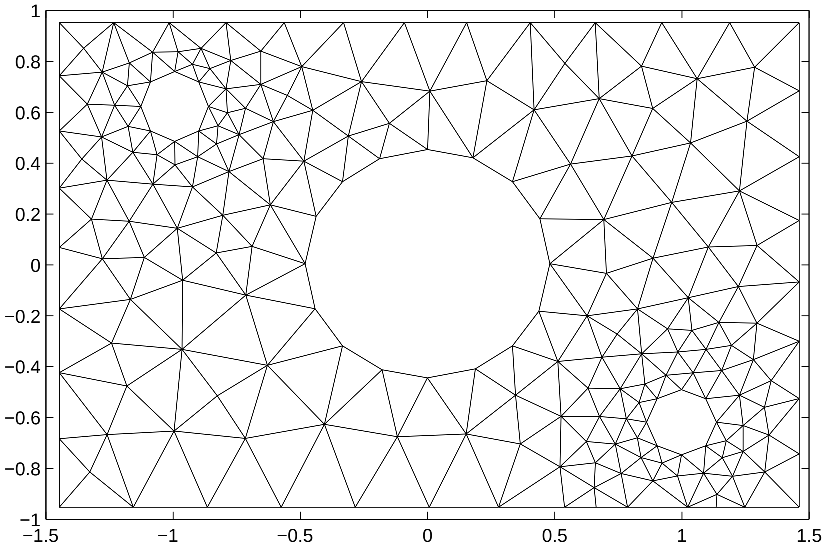


Figure 2.4: A triangulation with 208 nodes and 347 triangles.

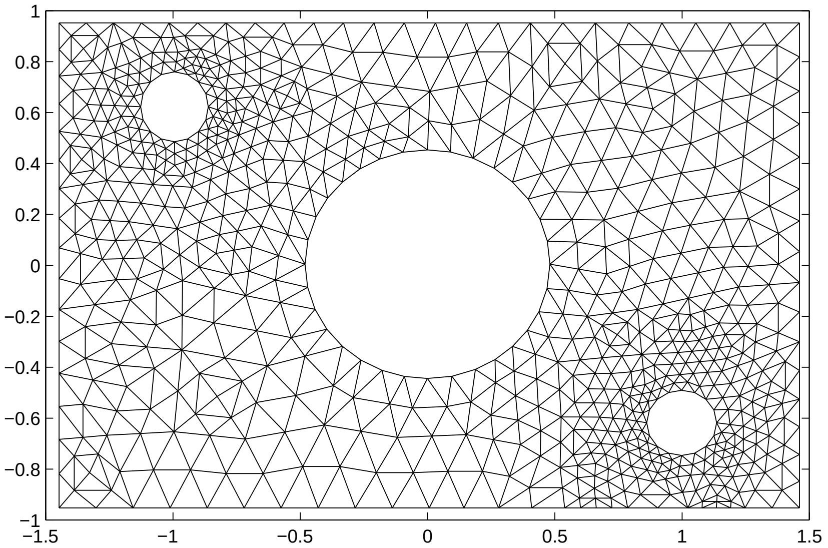


Figure 2.5: A triangulation with 765 nodes and 1388 triangles.

Theorem 2.1.7 *The polynomial*

$$\mathsf{P}_n^{(\alpha,\beta+\gamma+1)}(1-2\lambda_3)$$

is orthogonal to $\mathbb{P}_{n-1}(\Delta_0)$ *with respect to the weight function* $\lambda_3^\alpha \lambda_2^\beta \lambda_1^\gamma$.

Proof. For $\alpha, \beta, \gamma > -1$ the polynomials

$$P_{n,k}^{\alpha,\beta,\gamma}(x,y) := \mathsf{P}_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(2x-1)\cdot x^k \cdot \mathsf{P}_k^{(\beta,\gamma)}(2x^{-1}y-1), \quad n \ge k \ge 0,$$

are orthogonal with respect to the weight function $(1-x)^\alpha(x-y)^\beta y^\gamma$ on the triangular region $\{(x,y) : 0 < y < x < 1\}$ (see [17]).

Let $k = 0$. If we transform the triangle given above onto our reference triangle (taking $x = 1-\eta = 1-\lambda_2$, $y = \xi = \lambda_1$), and interchange λ_2 and λ_3 (see Lemma 1.4.7), we obtain the statement of the theorem. □

Integral transformations. Let Δ be a triangle with vertices $(0,0)$, (a_1,b_1), (a_2,b_2), and

$$B = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

be the affine transformation which maps Δ_0 onto Δ. Then

$$\int_\Delta \frac{\partial v(x,y)}{\partial x} q(x,y)dxdy = \int_{\Delta_0}\left(\frac{\partial u(\xi,\eta)}{\partial \xi}b_2 - \frac{\partial u(\xi,\eta)}{\partial \eta}b_1\right)p(\xi,\eta)d\xi d\eta, \tag{2.1.5}$$

$$\int_\Delta \frac{\partial v(x,y)}{\partial y} q(x,y)dxdy = \int_{\Delta_0}\left(-\frac{\partial u(\xi,\eta)}{\partial \xi}a_2 + \frac{\partial u(\xi,\eta)}{\partial \eta}a_1\right)p(\xi,\eta)d\xi d\eta,$$

where

$$u(\xi,\eta) = v(x(\xi,\eta), y(\xi,\eta)),$$
$$p(\xi,\eta) = q(x(\xi,\eta), y(\xi,\eta)).$$

Theorem 2.1.8 *If the triangulation of a polygonal domain* Ω *contains* σ *singular points, then for* $k \ge 4$ *there exists a basis of* $N_{V_{h,k}(\Omega)}$, *which can be described as follows. Besides the constant function, to each singular point corresponds a function which is zero everywhere–except on the triangles around the given point.*

Lemma 2.1.9 *Let* S_1 *be a boundary singular point of type I. (see Figure 2.6). Denote by* Δ_1 *the triangle which contains the point* S_1, *and let* S_2 *and* S_3 *be the other two vertices of* Δ_1 *(then* S_1S_2 *and* S_1S_3 *lie on the boundary of* Ω*). Denote by* $\lambda_3^{(1)}$ *the barycentric coordinate on* Δ_1, *which is equal to 1 in the point* S_1. *Then the function*

$$q|_{\Delta_1} = \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3^{(1)}), \qquad q|_{\Omega\setminus\Delta_1} \equiv 0 \tag{2.1.6}$$

is an element of $N_{V_{h,k}(\Omega)}$.

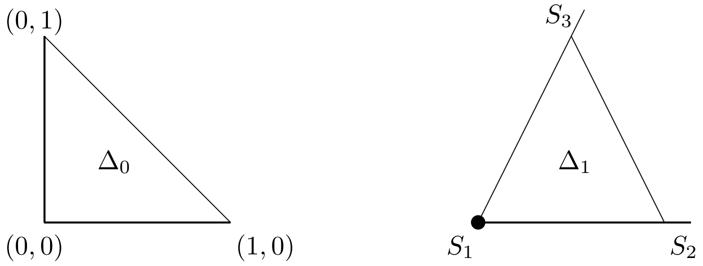


Figure 2.6: The reference triangular and a boundary singular point of type I.

Proof. Without loss of generality we may assume that the coordinates of the vertices of Δ_1 are $S_1 = (0,0)$, $S_2 = (a_2, b_2)$ and $S_3 = (a_3, b_3)$. Denote by Φ the following basis of $(\mathbb{P}_k(\Delta_0))^2$

$$\Phi = \left\{ \begin{pmatrix} u_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_i \end{pmatrix} \quad i = 1, \ldots, (k+1)(k+2)/2 \right\},$$

where $\{u_i\}$ is the standard Lagrangian basis for the scalar case, $u_i \in \mathbb{P}_k(\Delta_0)$.

Let B_1 be the affine transformation which maps the reference triangle Δ_0 onto Δ_1 ($x = x(\xi, \eta)$, $y = y(\xi, \eta)$, $(1,0) \to (a_2, b_2)$, $(0,1) \to (a_3, b_3)$). Since the vectors (a_2, b_2), (a_3, b_3) are linearly independent and $q|_{\Omega \setminus \Delta_1} \equiv 0$, using the corresponding integral transformation we obtain that for the function q

$$b(\vec{v}, q) = 0, \qquad \forall \vec{v} \in V_{h,k}(\Omega),$$

holds if and only if

$$\int_{\Delta_0} \frac{\partial u_i(\xi,\eta)}{\partial \xi} p(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta = 0, \quad \int_{\Delta_0} \frac{\partial u_i(\xi,\eta)}{\partial \eta} p(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta = 0, \tag{2.1.7}$$

where $p(\xi, \eta) = q(x(\xi, \eta), y(\xi, \eta))$ and u_i, $i = 1, \ldots, (k-1)k/2$, are those elements of the Lagrangian basis on Δ_0 which can be written in the form $u_i = \lambda_1 \lambda_2 g_i^{(k-2)}$ for some $g_i^{(k-2)} \in \mathbb{P}_{k-2}(\Delta_0)$. Due to the boundary conditions $\vec{v}|_{\partial\Omega} = 0$ we do not have to deal with the remaining elements of the Lagrangian basis. Thus

$$\int_{\Delta_0} \frac{\partial u_i(\xi,\eta)}{\partial \xi} p(\xi,\eta)\, \mathrm{d}\xi\, \mathrm{d}\eta = \int_{\Delta_0} \left(g_i^{(k-2)} + \lambda_1 \frac{\partial g_i^{(k-2)}}{\partial \xi} \right) \lambda_2 \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3)\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{2.1.8}$$

and Theorem 2.1.7 implies (taking there $n = k-1$, $\alpha = 0$, $\beta = 1$, $\gamma = 0$) that the last integral is equal to 0. Similarly, the integral

$$\int_{\Delta_0} \frac{\partial u_i(\xi,\eta)}{\partial \eta} p(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_{\Delta_0} \left(g_i^{(k-2)} + \lambda_2 \frac{\partial g_i^{(k-2)}}{\partial \eta} \right) \lambda_1 \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{2.1.9}$$

is equal to 0 (Theorem 2.1.7, $n = k-1$, $\alpha = 0$, $\beta = 0$, $\gamma = 1$). □

Lemma 2.1.10 *Let S_0 be a boundary singular point of type II and Δ_1, Δ_2 be the triangles containing the point S_0. Denote by S_1, S_2 and S_2, S_3 the remaining vertices of Δ_1 and Δ_2, respectively, where S_1S_3 lie on the boundary of Ω (see Figure 2.7). We may assume that the point S_0 is at the origin, i.e. $S_0 = (0,0)$, and let $S_1 = (a_1, b_1)$, $S_3 = (a_3, b_3) = -t_0(a_1, b_1)$ with some $t_0 > 0$. If*

$$q|_{\Delta_1} = \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3^{(1)}), \qquad q|_{\Delta_2} = -\frac{1}{t_0}\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3^{(2)}), \qquad q|_{\Omega\setminus(\Delta_1\cup\Delta_2)} \equiv 0,$$

where $\lambda_2^{(1)}$ and $\lambda_1^{(2)}$ are those barycentric coordinates in Δ_1 resp. Δ_2 which are equal to 1 in the point S_0, then $q \in N_{V_{h,k}(\Omega)}$.

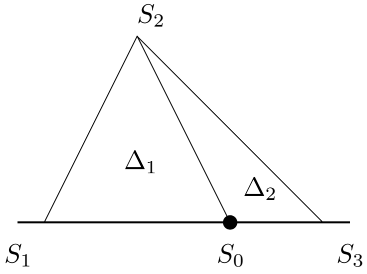


Figure 2.7: A boundary singular point of type II.

Proof. Let B_1 and B_2 be the affine transformations from Δ_0 onto Δ_1 and Δ_2, respectively:

$$B_1 = \begin{pmatrix} a_2 & a_1 \\ b_2 & b_1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} a_3 & a_2 \\ b_3 & b_2 \end{pmatrix}. \tag{2.1.10}$$

We may assume that the basis functions $\vec{v} \in V_{h,k}(\Omega)$ have the form $\vec{v} = \binom{v}{0}$ or $\vec{v} = \binom{0}{v}$, so we have to examine the equation $b(\vec{v}, q) = 0$ in two cases. In the first case

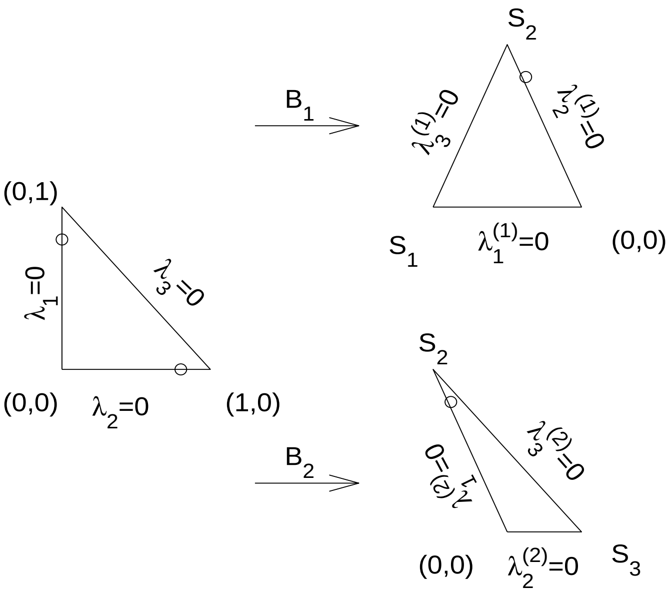


Figure 2.8: The transformation B_1 and B_2

using (2.1.10) and integral transformations, from $b(\vec{v}, q) = 0$ we obtain

$$\begin{aligned}\int_\Omega q\frac{\partial v}{\partial x}\,\mathrm{d}x\,\mathrm{d}y &= \int_{\Delta_1} q\frac{\partial v}{\partial x}\,\mathrm{d}x\,\mathrm{d}y + \int_{\Delta_2} q\frac{\partial v}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\Delta_0}\left(\frac{\partial u^{(1)}(\xi,\eta)}{\partial\xi}b_1 - \frac{\partial u^{(1)}(\xi,\eta)}{\partial\eta}b_2\right)\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)\,\mathrm{d}\xi\,\mathrm{d}\eta \qquad (2.1.11)\\ &\quad - \frac{1}{t_0}\int_{\Delta_0}\left(\frac{\partial u^{(2)}(\xi,\eta)}{\partial\xi}b_2 - \frac{\partial u^{(2)}(\xi,\eta)}{\partial\eta}b_3\right)\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)\,\mathrm{d}\xi\,\mathrm{d}\eta,\end{aligned}$$

where $v = u^{(1)} \circ B_1^{-1}$ on Δ_1 and $v = u^{(2)} \circ B_2^{-1}$ on Δ_2. First we will assume that v takes nonzero values only on one of the triangles Δ_1 and Δ_2, e.g. on Δ_1. Then $u^{(2)} \equiv 0$ and $u^{(1)} = \lambda_1\lambda_2\lambda_3 g^{(k-3)}$ holds for some $g^{(k-3)} \in \mathbb{P}_{k-3}(\Delta_0)$. In this case the last integral is equal to 0, and for the first integral we can apply equations (2.1.8) and (2.1.9) implying $b(\vec{v}, q) = 0$.

If v takes nonzero values on the common side of Δ_1 and Δ_2 then $u^{(1)}(\xi,\eta) = u^{(2)}(\eta,\xi)$ (see Fig. 2.8). Now, observe that the polynomial $\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)$ is symmetric in ξ and η that implies

$$\int_{\Delta_0} \frac{\partial u^{(1)}(\xi,\eta)}{\partial \xi} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_{\Delta_0} \frac{\partial u^{(2)}(\xi,\eta)}{\partial \eta} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)\,\mathrm{d}\xi\,\mathrm{d}\eta,$$
$$\int_{\Delta_0} \frac{\partial u^{(1)}(\xi,\eta)}{\partial \eta} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)\,\mathrm{d}\xi\,\mathrm{d}\eta = \int_{\Delta_0} \frac{\partial u^{(2)}(\xi,\eta)}{\partial \xi} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)\,\mathrm{d}\xi\,\mathrm{d}\eta.$$

Since $b_3 = -t_0 b_1$ and since the function $u^{(1)}$ can be written in the form $\lambda_1 \lambda_3 g_1^{(k-2)}$, (see Fig. 2.8) where $g^{(k-2)} \in \mathbb{P}_{k-2}(\Delta_0)$, equation (2.1.11) takes the form

$$\int_\Omega q \frac{\partial v}{\partial x}\,\mathrm{d}x\,\mathrm{d}y = -b_2\left(1+\frac{1}{t_0}\right)\int_{\Delta_0} \frac{\partial u^{(1)}(\xi,\eta)}{\partial \eta} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)\,\mathrm{d}\xi\,\mathrm{d}\eta$$
$$= -b_2\left(1+\frac{1}{t_0}\right)\int_{\Delta_0}\left(-g^{(k-2)} + \lambda_3 \frac{\partial g^{(k-2)}}{\partial \xi}\right)\lambda_1 \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)\,\mathrm{d}\xi\,\mathrm{d}\eta = 0.$$

(see Theorem 2.1.7 taking $n = k-1$, $\alpha = 0$, $\beta = 0$, $\gamma = 1$).

In the case $\vec{v} = \binom{0}{v}$, the equation $b(\vec{v}, q) = 0$ for the piecewise polynomial q can be proved in the same way. $\square$

Lemma 2.1.11 *Let S_0 be a boundary singular point of type III, and Δ_1, Δ_2, Δ_3 be the triangles which contain the point S_0. Denote by S_i, $i = 1,2,3,4$, the remaining vertices of the triangles Δ_i, $i = 1,2,3$ (see Figure 2.9). Here S_0S_1 and S_0S_4 are parts of the boundary of Ω. Let $S_0 = (0,0)$ and $S_i = (a_i, b_i)$, $i = 1,2,3,4$, where $(a_3,b_3) = -t_0(a_1,b_1)$ and $(a_4,b_4) = -t_1(a_2,b_2)$ with some $t_0, t_1 > 0$. Then the function*

$$q|_{\Delta_1} = \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3^{(1)}), \qquad q|_{\Delta_2} = -\frac{1}{t_0}\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3^{(2)}),$$
$$q|_{\Delta_3} = \frac{1}{t_0 t_1}\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3^{(3)}), \qquad q|_{\Omega\setminus(\Delta_1\cup\Delta_2\cup\Delta_3} \equiv 0,$$

where $\lambda_3^{(1)}$, $\lambda_3^{(2)}$ and $\lambda_3^{(3)}$ are the coordinates in Δ_1, Δ_2 and Δ_3, respectively, which are equal to 1 in the point S_0, is an element of $N_{V_{h,k}(\Omega)}$.

Proof. To obtain the statement of the lemma we apply the argumentation of Lemma 2.1.10 to Δ_1, Δ_2 and to Δ_2, Δ_3. $\square$

Lemma 2.1.12 *Let S_0 be a boundary singular point of type IV, and Δ_i, $i = 1,2,3,4$, be the triangles which contain the point S_0. Denote by S_i, $i = 1,\ldots,5$, the remaining vertices of the triangles Δ_i, $i = 1,2,3,4$ (see Figure 2.9). Here $S_0S_4S_5$ is a part of the boundary of Ω. Let $S_0 = (0,0)$ and $S_i = (a_i,b_i)$, $i = 1,\ldots,5$, where $(a_3,b_3) =$*

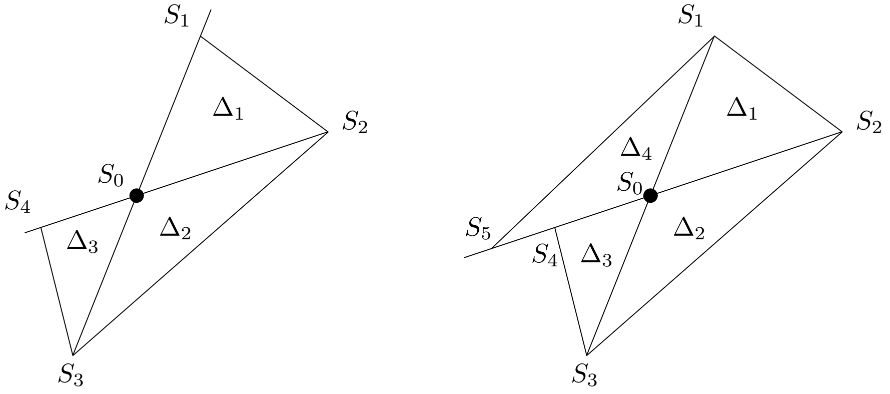


Figure 2.9: A boundary singular point of type III. and IV.

$-t_0(a_1, b_1)$, $(a_4, b_4) = -t_1(a_2, b_2)$ and $(a_5, b_5) = -t_2(a_2, b_2)$ with some $t_0, t_1, t_2 > 0$. Then the function q defined as

$$\begin{aligned}
&q|_{\Delta_1} = \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(1)}), \qquad q|_{\Delta_2} = -\frac{1}{t_0}\mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(2)}),\\
&q|_{\Delta_3} = \frac{1}{t_0 t_1}\mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(3)}), \qquad q|_{\Delta_4} = -\frac{1}{t_2}\mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(4)}),\\
&q|_{\Omega\setminus(\Delta_1\cup\Delta_2\cup\Delta_3} \equiv 0,
\end{aligned}$$

where $\lambda_3^{(i)}$, $i = 1, \ldots, 4$, are the coordinates in Δ_i, $i = 1, \ldots, 4$, which are equal to 1 in the point S_0, is an element of $N_{V_{h,k}(\Omega)}$.

Proof. Apply the argumentation of the proof of Lemma 2.1.10 to $\Delta_1 \cup \Delta_2$, $\Delta_2 \cup \Delta_3$ and $\Delta_4 \cup \Delta_1$. □

Lemma 2.1.13 *Let S_0 be an inner singular point, and Δ_i, $i = 1, 2, 3, 4$, be the triangles around S_0. Let $S_0 = (0, 0)$ and $S_i = (a_i, b_i)$, $i = 1, 2, 3, 4$, the remaining vertices of Δ_i, $i = 1, 2, 3, 4$, where $(a_3, b_3) = -t_0(a_1, b_1)$, $(a_4, b_4) = -t_1(a_2, b_2)$ with some $t_0, t_1 > 0$. Then for the piecewise polynomial defined as*

$$\begin{aligned}
&q|_{\Delta_1} = \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(1)}), \qquad q|_{\Delta_2} = -\frac{1}{t_0}\mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(2)}),\\
&q|_{\Delta_3} = \frac{1}{t_0 t_1}\mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(3)}), \qquad q|_{\Delta_4} = -\frac{1}{t_1}\mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(4)}),\\
&q|_{\Omega\setminus(\Delta_1\cup\Delta_2\cup\Delta_3\cup\Delta_4} \equiv 0,
\end{aligned}$$

$q \in N_{V_{h,k}(\Omega)}$ holds. Here $\lambda_3^{(i)}$, $i = 1, \ldots, 4$, are the barycentric coordinates in Δ_i, $i = 1, 2, 3, 4$, which are equal to 1 in the point S_0.

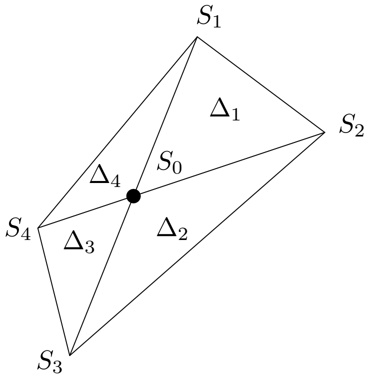


Figure 2.10: An inner singular point.

Proof. We have to prove that if $\vec{v}$ is the element of the basis of $V_{h,k}(\Omega)$ which is nonzero in the origin, then $b(\vec{v}, q) = 0$ holds. We may assume that $\vec{v}$ has the form $\vec{v} = \binom{v}{0}$ or $\vec{v} = \binom{0}{v}$. In the first case

$$\begin{aligned} b(\vec{v}, q) = \int_\Omega \frac{\partial v}{\partial x} q \, \mathrm{d}x \, \mathrm{d}y &= \int_{\Delta_1} \frac{\partial v}{\partial x} q|_{\Delta_1} \, \mathrm{d}x \, \mathrm{d}y + \int_{\Delta_2} \frac{\partial v}{\partial x} q|_{\Delta_2} \, \mathrm{d}x \, \mathrm{d}y \\ &+ \int_{\Delta_3} \frac{\partial v}{\partial x} q|_{\Delta_3} \, \mathrm{d}x \, \mathrm{d}y + \int_{\Delta_4} \frac{\partial v}{\partial x} q|_{\Delta_4} \, \mathrm{d}x \, \mathrm{d}y. \end{aligned}$$

Let u be the element of the standard Lagrangian basis on Δ_0 which is equal to 1 in the point $\lambda_3 = 1$, and let $p_0 = \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3)$. After the corresponding integral transformations we obtain

$$\begin{aligned} b(\vec{v}, q) &= \int_{\Delta_0} \left(\frac{\partial u}{\partial \xi} b_1 - \frac{\partial u}{\partial \eta} b_2 \right) p_0 \, \mathrm{d}\xi \, \mathrm{d}\eta - \frac{1}{t_0} \int_{\Delta_0} \left(\frac{\partial u}{\partial \xi} b_2 - \frac{\partial u}{\partial \eta} b_3 \right) p_0 \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &+ \frac{1}{t_0 t_1} \int_{\Delta_0} \left(\frac{\partial u}{\partial \xi} b_3 - \frac{\partial u}{\partial \eta} b_4 \right) p_0 \, \mathrm{d}\xi \, \mathrm{d}\eta - \frac{1}{t_1} \int_{\Delta_0} \left(\frac{\partial u}{\partial \xi} b_4 - \frac{\partial u}{\partial \eta} b_1 \right) p_0 \, \mathrm{d}\xi \, \mathrm{d}\eta \\ &= \frac{t_1 - 1}{t_1} \int_{\Delta_0} \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) b_1 p_0 \, \mathrm{d}\xi \, \mathrm{d}\eta + \frac{t_0 - 1}{t_0} \int_{\Delta_0} \left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta} \right) b_2 p_0 \, \mathrm{d}\xi \, \mathrm{d}\eta, \end{aligned}$$

where we have used that $(a_3, b_3) = -t_0(a_1, b_1)$ and $(a_4, b_4) = -t_1(a_2, b_2)$. Similarly, if $\vec{v} = \binom{0}{v}$ holds then

$$\begin{aligned} b(\vec{v}, q) = \int_\Omega \frac{\partial v}{\partial y} q \,\mathrm{d}x \,\mathrm{d}y = & \frac{t_1 - 1}{t_1} \int_{\Delta_0} \left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi} \right) a_1 p_0 \,\mathrm{d}\xi \,\mathrm{d}\eta + \\ & \frac{t_0 - 1}{t_0} \int_{\Delta_0} \left(\frac{\partial u}{\partial \eta} - \frac{\partial u}{\partial \xi} \right) a_2 p_0 \,\mathrm{d}\xi \,\mathrm{d}\eta. \end{aligned}$$

As u and p_0 are functions of $\lambda_3 = 1 - \xi - \eta$, we have

$$\int_{\Delta_0} \frac{\partial u}{\partial \xi} p_0 \,\mathrm{d}\xi \,\mathrm{d}\eta = \int_{\Delta_0} \frac{\partial u}{\partial \eta} p_0 \,\mathrm{d}\xi \,\mathrm{d}\eta$$

which implies $b(\vec{v}, q) = 0$. □

Proof of Theorem 2.1.8. We will prove that the constant function and the functions from $N_{V_{h,k}(\Omega)}$ described in Lemmas 2.1.6–2.1.13 (corresponding to the singular points of $\mathcal{T}_h$) are linearly independent. Since the functions in Lemmas 2.1.6–2.1.13 have nonzero values only in the triangles around the considered singular point we have to examine only the cases when a triangle contains more than 1 singular point.

We give two versions of the proof: the first is valid only in the case when all the triangles in $\mathcal{T}_h$ contain at most two singular points, while the second one is valid in all cases.

1. version. Let $\Delta \in \mathcal{T}_h$ be a triangle with vertices S_i, $i = 1, 2, 3$. Denote by λ_i that barycentric coordinate in Δ which is equal to 1 in S_i, $i = 1, 2, 3$, respectively. We will assume that S_1 and S_2 are singular points of $\mathcal{T}_h$. Denote by q_1 and q_2 the corresponding element of $N_{V_{h,k}(\Omega)}$ and let $q_3 \equiv 1$. In this case we may assume

$$q_1|_\Delta = \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_1), \qquad q_2|_\Delta = \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_2).$$

Let $A \in \mathbb{R}^{3\times 3}$ be a matrix such that a_{ij} is equal to the value of q_i in the point S_j, $i, j = 1, 2, 3$. Then using (2.1.4) we obtain

$$A = \begin{pmatrix} \mathsf{P}_{k-1}^{(0,2)}(-1) & \mathsf{P}_{k-1}^{(0,2)}(1) & \mathsf{P}_{k-1}^{(0,2)}(1) \\ \mathsf{P}_{k-1}^{(0,2)}(1) & \mathsf{P}_{k-1}^{(0,2)}(-1) & \mathsf{P}_{k-1}^{(0,2)}(1) \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

where $a = (-1)^{k-1} k(k+1)/2$. It follows from $\det(A) = (a-1)^2$ that the functions q_i, $i = 1, 2, 3$, are linearly independent if $k > 1$. □

2. version. Let $\Delta \in \mathcal{T}_h$ be a triangle with vertices S_i, $i = 1, 2, 3$. Denote by λ_i that barycentric coordinate in Δ which is equal to 1 in S_i, $i = 1, 2, 3$, respectively. We will assume that all the vertices of Δ are singular points in $\mathcal{T}_h$. Denote by q_i, $i = 1, 2, 3$, the corresponding element of $N_{V_{h,k}(\Omega)}$, and let $q_4 \equiv 1$. In this case we may assume

$$q_i|_\Delta = \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_i), \qquad i = 1, 2, 3.$$

If there exist constants α_i, $i = 1, 2, 3$, such that

$$\sum_{i=1}^{3} \alpha_i q_i|_\Delta = q_4|_\Delta$$

holds, then

$$\sum_{i=1}^{3} \alpha_i \int_\Delta q_i|_\Delta \lambda_1\lambda_2\lambda_3 \,\mathrm{d}x\,\mathrm{d}y = \int_\Delta \lambda_1\lambda_2\lambda_3 \,\mathrm{d}x\,\mathrm{d}y.$$

After the corresponding integral transformations we obtain

$$\sum_{i=1}^{3} \alpha_i \int_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_i^{(0)})\lambda_1^{(0)}\lambda_2^{(0)}\lambda_3^{(0)} \,\mathrm{d}\xi\,\mathrm{d}\eta = \int_{\Delta_0} \lambda_1^{(0)}\lambda_2^{(0)}\lambda_3^{(0)} \,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{2.1.12}$$

where we used notations $\lambda_i^{(0)}$, $i = 1, 2, 3$, to denote the barycentric coordinates in Δ_0 (this notation differs from the usual one). It follows from Lemma 1.4.7 that the last integral is equal to $\frac{1}{5!}$ and

$$\int_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_i^{(0)})\lambda_1^{(0)}\lambda_2^{(0)}\lambda_3^{(0)} \,\mathrm{d}\xi\,\mathrm{d}\eta = \int_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(0)})\lambda_1^{(0)}\lambda_2^{(0)}\lambda_3^{(0)} \,\mathrm{d}\xi\,\mathrm{d}\eta$$

holds for $i = 1, 2, 3$. Since $k \geq 4$, Theorem 2.1.7 implies (taking there $n = k - 1$, $\alpha = \beta = 0$, $\gamma = 1$) that

$$\int_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3^{(0)})\lambda_1^{(0)}\lambda_2^{(0)}\lambda_3^{(0)} \,\mathrm{d}\xi\,\mathrm{d}\eta = 0,$$

which contradicts to (2.1.12). That means the functions q_i, $i = 1, 2, 3, 4$ are linearly independent. □

The lower order cases. In Lemmas 2.1.6–2.1.13 we do not use the condition $k \geq 4$, the results remain valid for the lower order cases, too. In both proofs of Theorem 2.1.8 to prove that the functions described in Lemmas 2.1.6–2.1.13 are linearly independent, it suffices to consider a weaker condition for k. In the first proof we used only the condition $k > 1$. In the second proof we need a lower bound for k only in order to apply Theorem 2.1.7, however for our purposes it is sufficient to require $k > 2$. This means that the constant function and the functions defined in Lemmas 2.1.6–2.1.13 are linearly independent for all $k > 2$, and for all $k > 1$ in the case when all the triangles contain at most two singular points. The condition $k \geq 4$ is important when we apply Lemma 2.1.5 in the proof of Theorem 2.1.8: in the lower order cases the dimension of the nullspace can be greater than $\sigma + 1$.

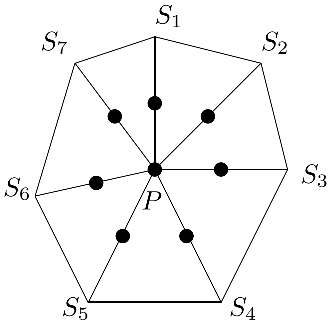


Figure 2.11: A domain Ω and a triangulation where $N_{V_{h.2}(\Omega)} \geq N-2=5$

Example 2.1.14 Let $k=2$ and let Ω be a convex polyhedron with N vertices. We consider the triangulation of Ω which contains N triangles: an arbitrary inner point P is connected with the vertices S_i, $i=1,\ldots,N$, of Ω (here $\sigma=0$). It follows from the homogeneous boundary conditions that the space $V_{h,k}(\Omega)$ is $2(N+1)$-dimensional (we can write the basis functions of $V_{h,k}(\Omega)$ in the form

$$\vec{v}^{(i)} = \begin{pmatrix} v_i \\ 0 \end{pmatrix} \qquad \text{or} \qquad \vec{v}^{(i+N+1)} = \begin{pmatrix} 0 \\ v_i \end{pmatrix}, \qquad i=1,\ldots,N+1,$$

where v_i, $i=1,\ldots,N+1$, are the basis functions in the scalar case corresponding to the midpoint of the section PS_i, $i=1,\ldots,N$, and to the point P, see Figure 2.11). Since the space $P_h(\Omega)$ contains polynomials which are trianglewise linear functions, the unknown function q in the equation

$$b(\vec{v},q)=0, \qquad \forall \vec{v} \in V_{h,k}(\Omega),$$

has $3N$ free parameters. Thus the nullspace of the discrete gradient is at least $N-2$ dimensional.

Example 2.1.15 Let $k=2$, $\Omega=[0,1]^2$ and consider the standard triangulation of Ω (the sides of Ω are divided into n equal parts, and all the small squares are divided into two triangles by their southwest-northeast diagonals, see Figure 2.12). In this case $\mathcal{T}_h$ has 2 singular points (northwest and southeast corner). Denote by Δ_1 and Δ_2 the triangles containing the singular points. It follows from Lemma 2.1.6 that besides the constant functions the functions

$$q_1|_{\Delta_1} = \mathsf{P}_1^{(0,2)}(1-2\lambda_3^{(1)}) = 1-4\lambda_3^{(1)}, \qquad q_1|_{\Omega\setminus\Delta_1} \equiv 0,$$
$$q_2|_{\Delta_2} = \mathsf{P}_1^{(0,2)}(1-2\lambda_3^{(2)}) = 1-4\lambda_3^{(2)}, \qquad q_2|_{\Omega\setminus\Delta_2} \equiv 0$$

are in the nullspace of the discrete gradient operator. Here $\lambda_3^{(1)}$ and $\lambda_3^{(2)}$ are the barycentric coordinates in Δ_1 and Δ_2, which are equal to 1 in the corresponding singular points.

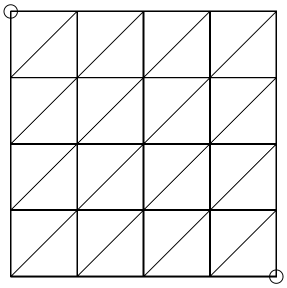


Figure 2.12: Standard triangulation of the unit square in the case of $n = 4$

We say that a triangle $\Delta \in \mathcal{T}_h$ is of type I. or II., if it is an upper, or a lower triangle in one of the small squares. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = B \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

be the affine transformation which maps the reference triangle Δ_0 onto Δ, where (a, b) is the coordinate of the vertex of Δ at the rectangle.
If the triangle is of type I., then with $h = 1/N$

$$B = h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

while for a triangle of type II. we have

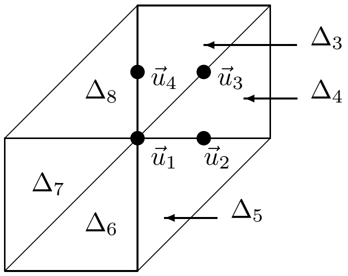
$$B = h \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Next, we will prove that the functions

$$\begin{aligned} q_3|_{\Delta_i} &= \lambda_1^{(i)}, \qquad & \Delta_i \in \mathcal{T}_h, \\ q_4|_{\Delta_i} &= \lambda_2^{(i)}, \qquad & \Delta_i \in \mathcal{T}_h \end{aligned}$$

are also in the nullspace of the discrete gradient. Here $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ are those barycentric coordinates in Δ_i which correspond (after the affine transformation of Δ_0 onto Δ_i) to λ_1 and λ_2.

Consider an inner point of the triangulation and the 6 triangles around the point.



It suffices to prove that

$$\int_\Omega q_j \operatorname{div} \vec{u}_i \, \mathrm{d}x \, \mathrm{d}y = 0, \qquad j = 3,4, \quad i = 1,2,3,4, \tag{2.1.13}$$

where $\vec{u}_i \in V_{h,k}$, $i = 1,2,3,4$, are the basis functions belonging to the points drawn on the figure and $\vec{u}_i = \binom{u}{0}$ or $\vec{u}_i = \binom{0}{u}$, where u is the corresponding element of the local basis.

In the scalar case the standard Lagrangian basis on Δ_0 is:

$$\begin{aligned} v_1 &= \lambda_3(2\lambda_3 - 1), \\ v_2 &= 4\lambda_3\lambda_2, \\ v_3 &= \lambda_2(2\lambda_2 - 1), \\ v_4 &= 4\lambda_3\lambda_1, \\ v_5 &= 4\lambda_1\lambda_2, \\ v_6 &= \lambda_1(2\lambda_1 - 1). \end{aligned}$$

For vectors $\vec{u}_2$, $\vec{u}_3$ and $\vec{u}_4$ equality (2.1.13) holds since these functions have nonzero values only in two adjacent triangles, where one of the triangles is of type I. and the other is of type II. Further, transforming these two triangles onto Δ_0 the velocity functions in the two triangles belonging to the nodal point on the common side correspond to the same element of the local basis in Δ_0.

E.g. if $i = 3$, $j = 3$ using that the triangles Δ_3 and Δ_4 is of type I. and II., respectively, we obtain

$$\begin{aligned} \int_\Omega q_3 \frac{\partial \vec{u}_3}{\partial x} \, \mathrm{d}x \, \mathrm{d}y &= \int_{\Delta_3} \lambda_1^{(3)} \frac{\partial \vec{u}_3}{\partial x} \, \mathrm{d}x \, \mathrm{d}y + \int_{\Delta_4} \lambda_1^{(4)} \frac{\partial \vec{u}_3}{\partial x} \, \mathrm{d}x \, \mathrm{d}y \\ &= h \int_{\Delta_0} \lambda_1 \frac{\partial v_5}{\partial \eta} \, \mathrm{d}\xi \, \mathrm{d}\eta - h \int_{\Delta_0} \lambda_1 \frac{\partial v_5}{\partial \eta} \, \mathrm{d}\xi \, \mathrm{d}\eta = 0. \end{aligned}$$

Here $v_5 = 4\lambda_1\lambda_2$ is the corresponding element of the local basis on Δ_0. Similarly

$$\begin{aligned} \int_\Omega q_3 \frac{\partial \vec{u}_3}{\partial y} \, \mathrm{d}x \, \mathrm{d}y &= \int_{\Delta_3} \lambda_1^{(3)} \frac{\partial \vec{u}_3}{\partial y} \, \mathrm{d}x \, \mathrm{d}y + \int_{\Delta_4} \lambda_1^{(4)} \frac{\partial \vec{u}_3}{\partial y} \, \mathrm{d}x \, \mathrm{d}y \\ &= h \int_{\Delta_0} \lambda_1 \frac{\partial v_5}{\partial \xi} \, \mathrm{d}\xi \, \mathrm{d}\eta - h \int_{\Delta_0} \lambda_1 \frac{\partial v_5}{\partial \xi} \, \mathrm{d}\xi \, \mathrm{d}\eta = 0. \end{aligned}$$

In the case of $\vec{u}_1$ in a similar way we obtain that for $j = 3, 4$

$$\int\limits_{\Delta_3\cup\Delta_6} q_j \operatorname{div} \vec{u}_1 \,\mathrm{d}x\,\mathrm{d}y = 0, \qquad \int\limits_{\Delta_4\cup\Delta_7} q_j \operatorname{div} \vec{u}_1 \,\mathrm{d}x\,\mathrm{d}y = 0, \qquad \int\limits_{\Delta_5\cup\Delta_8} q_j \operatorname{div} \vec{u}_1 \,\mathrm{d}x\,\mathrm{d}y = 0.$$

From the reasoning above we obtain that the nullspace is at least $\sigma + 3$ dimensional.

Example 2.1.16 If $k = 2$ and $\Omega = [0, 1]^2$ then in the case of the criss-cross grid, similarly to the cases $k \geq 4$, the nullspace is $\sigma + 1$ dimensional.

Example 2.1.17 If $k = 3$, Ω is a convex polyhedron with N vertices and we consider the same triangulation of Ω as in Example 2.1.14, then we obtain that the nullspace is at least 2 dimensional (here $\sigma = 0$).

2.2 Fortin-Soulie element

A possibility to avoid the unpleasant grid condition mentioned in Section 2.1 is enlarging the velocity space $V_h(\Omega)$.

The idea is coming from [13], where a second order non-conforming finite element pair was investigated. In that paper the continuity of the discrete velocities between the triangles is not assumed, the continuity is required only in the second order Gauss-Legendre points. The corresponding discrete spaces are

$$P_h(\Omega) = \{p \in L_2(\Omega),\ \ p|_\Delta \in \mathbb{P}_1(\Delta),\ \ \Delta \in \mathcal{T}_h\} \tag{2.2.1}$$

$$\begin{aligned} V_{h,2}^{nc}(\Omega) = \big\{\vec{v} \in (L_2(\Omega))^2,\ \ &\vec{v}|_\Delta \in (\mathbb{P}_2(\Delta))^2,\ \text{ and } \vec{v} \text{ is continuous in all} \\ &\text{2nd-order Gauss-Legendre points of all sides of } \Delta,\ \Delta \in \mathcal{T}_h\big\}, \end{aligned} \tag{2.2.2}$$

$$\text{the norm in } V_{h,2} \text{ being defined as } |\vec{v}_h|_{1,h} := \left(\sum_{\Delta\in\mathcal{T}_h} |\vec{v}_h|^2_{1,\Delta}\right)^{1/2}.$$

It is understood that in this case instead of homogeneous boundary conditions the velocity components v_ℓ, $\ell = 1, 2$, satisfy

$$\int_E q v_\ell \,\mathrm{d}s = 0, \quad q \in \mathbb{P}_1(E)$$

for all edges $E \subset \partial\Delta \cap \partial\Omega$ and for all $\Delta \in \mathcal{T}_h$.

Let $\Delta \in \mathcal{T}_h$ be a given triangle, and denote by λ_1, λ_2, λ_3 the barycentric coordinates in Δ. Then on the sides of Δ the function

$$b_\Delta := 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 2 \tag{2.2.3}$$

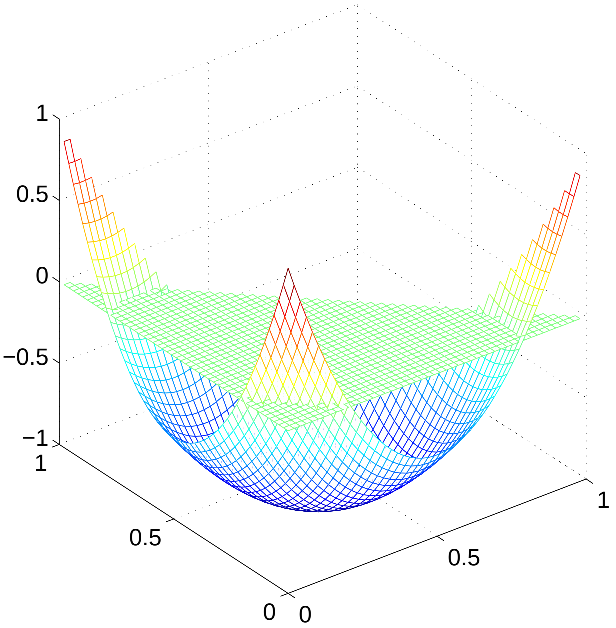


Figure 2.13: The function b_Δ on the reference triangle.

is equal to the second order Legendre polynomial; b_Δ disappears in the second order Gauss-Legendre points on the sides of Δ. From this follows that one gets the velocity space (2.2.2) by adding trianglewise this function to the conforming velocity space $V_{h,2}(\Omega)$:

$$V_{h,2}^{nc}(\Omega) = V_{h,2}(\Omega) + \left\{ \vec{v},\ \ \vec{v}|_\Delta = \begin{pmatrix} \alpha_\Delta \\ \beta_\Delta \end{pmatrix} b_\Delta,\ \ \alpha_\Delta, \beta_\Delta \in \mathbb{R},\ \ \Delta \in \mathcal{T}_h \right\}.$$

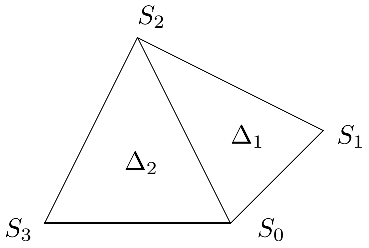
The authors announced that the element pair (2.2.1)–(2.2.2) is inf-sup stable without a restriction on the grid.

Lemma 2.2.1 *In the case of the finite element* (2.2.1)–(2.2.2) *the nullspace of the discrete gradient is one-dimensional, it contains only constant functions.*

Proof. Let $q \in P_h$ be a function for which $b(\vec{v}, q) = 0$ holds for all $\vec{v} \in V_{h,2}^{nc}$, and let $\Delta \in \mathcal{T}_h$ be a triangle from the support of q. Let $\vec{v}|_\Delta = \binom{b_\Delta}{0}$, $\vec{v}|_{\Omega \setminus \Delta} \equiv 0$, then using that $\frac{\partial q}{\partial x_1} \equiv \text{const}$ we obtain

$$0 = b(\vec{v}, q) = -\int_\Delta \frac{\partial q}{\partial x_1} b_\Delta \, \mathrm{d}x = -\frac{\partial q}{\partial x_1} \int_\Delta b_\Delta \, \mathrm{d}x = \frac{\partial q}{\partial x_1} \cdot \frac{1}{4},$$

which means that $\frac{\partial q}{\partial x_1} \equiv 0$. Similarly, we have $\frac{\partial q}{\partial x_2} \equiv 0$, so the function q is constant on Δ. To prove that q is constant on Ω let Δ_1 and Δ_2 be two adjacent triangles with vertices S_0, S_1, S_2 and S_0, S_2, S_3, respectively (see the figure). We may assume that $S_0 = (0,0)$ and let $S_i = (a_i, b_i)$, $i = 1, 2, 3$.



Denote by v the element of the Lagrangian basis of the scalar case which has nonzero value in the midpoint of the side S_0S_2. Let $\vec{v} = \binom{v}{0}$ and $q|_{\Delta_i} \equiv c_i$, $i = 1, 2$. Then, using $\vec{v}|_{\Omega \setminus (\Delta_1 \cup \Delta_2)} \equiv 0$ after the corresponding integral transformations from $b(\vec{v}, q) = 0$ we obtain

$$0 = \int\limits_{\Delta_1 \cup \Delta_2} \frac{\partial v}{\partial x_1} q \, \mathrm{d}x = c_1 \int\limits_{\Delta_0} \left(\frac{\partial u^{(1)}}{\partial \xi} b_2 - \frac{\partial u^{(1)}}{\partial \eta} b_1 \right) \mathrm{d}x + c_2 \int\limits_{\Delta_0} \left(\frac{\partial u^{(2)}}{\partial \xi} b_3 - \frac{\partial u^{(2)}}{\partial \eta} b_2 \right) \mathrm{d}x,$$

where $u^{(1)}$ and $u^{(2)}$ are the corresponding elements of the standard Lagrangian basis on Δ_0: $u^{(1)} = \lambda_2\lambda_3$ and $u^{(2)} = \lambda_1\lambda_3$. Then

$$0 = c_1 \int\limits_{\Delta_0} -\lambda_2 b_2 - (\lambda_3 - \lambda_2) b_1 \, \mathrm{d}x + c_2 \int\limits_{\Delta_0} (\lambda_3 - \lambda_1) b_3 + \lambda_1 b_2 \, \mathrm{d}x = \frac{1}{6} b_2 (c_2 - c_1)$$

holds. If $\vec{v} = \binom{0}{v}$ in a similar way we obtain

$$0 = \frac{1}{6} a_2 (c_2 - c_1).$$

Since $(a_2, b_2) \neq (0, 0)$ holds we have $c_1 = c_2$ that completes the proof. □

2.3 Some higher order non-conforming finite elements

Several authors tried to generalize the results of Fortin and Soulie to higher order cases. In [7] the cases $k = 4$ and $k = 6$ were investigated. The construction of the elements was similar to the case $k = 2$ of Fortin and Soulie, the authors extended the space $V_{h,4}(\Omega)$ and $V_{h,6}(\Omega)$ by adding to the local basis trianglewise functions

$$\begin{aligned}\Psi_4 =&\lambda_1^4 + \lambda_2^4 + \lambda_3^4 + 36(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2)\\ &- 16(\lambda_1^3\lambda_2 + \lambda_2^3\lambda_1 + \lambda_2^3\lambda_3 + \lambda_3^3\lambda_2 + \lambda_3^3\lambda_1 + \lambda_1^3\lambda_3)\end{aligned} \tag{2.3.1}$$

and

$$\begin{aligned}\Psi_6 =&\lambda_1^6 + \lambda_2^6 + \lambda_3^6 - 400(\lambda_1^3\lambda_2^3 + \lambda_2^3\lambda_3^3 + \lambda_3^3\lambda_1^3)\\ &- 36(\lambda_1^5\lambda_2 + \lambda_1^5\lambda_3 + \lambda_2^5\lambda_1 + \lambda_2^5\lambda_3 + \lambda_3^5\lambda_1 + \lambda_3^5\lambda_2)\\ &+ 225(\lambda_1^4\lambda_2^2 + \lambda_1^4\lambda_3^2 + \lambda_2^4\lambda_1^2 + \lambda_2^4\lambda_3^2 + \lambda_3^4\lambda_1^2 + \lambda_3^4\lambda_2^2),\end{aligned} \tag{2.3.2}$$

respectively. These functions have the same properties as the function (2.2.3), that is Ψ_k disappears in the kth-order Gauss-Legendre points on the triangle sides, $k = 4, 6$, which means that the discrete velocities are assumed to be continuous between the triangles only in the Gauss-Legendre points. The authors proved the inf-sup condition for these elements, but the usual bilinear form corresponding to the divergence was augmented by a stabilizing term containing jumps across triangle sides and a free parameter which was chosen suitably during the proof of the stability.

In [18] the authors described a family of non-conforming finite elements of arbitrary order k by adding to the kth-order conforming velocity space a polynomial of degree $k+1$ on each triangle. They proved the stability of this element without a restriction on the grid.

Chapter 3

Gauss-Legendre elements

In this chapter we define a non-conforming finite element family of arbitrary order k, and we prove its inf-sup stability for any even k without assumptions on the grid. This family generalizes the well-known low-order cases, $k = 1$ (Crouzeix-Raviart [11]), $k = 2$ (Fortin-Soulie [13]), $k = 3$ (Crouzeix-Falk [10]), where non-conforming approximations for the velocities are used.

3.1 Definitions and properties

Definition 3.1.1 *(See in [30]) We define the non-conforming kth-order Gauss-Legendre element on Ω as*

$$\begin{aligned} P_h(\Omega) &= \{p \in L^2(\Omega),\ p|_\Delta \in \mathbb{P}_{k-1}(\Delta),\ \Delta \in \mathcal{T}_h\} && (3.1.1)\\ V^{nc}_{h,k}(\Omega) &= \{\vec{v} \in (L^2(\Omega))^2,\ \vec{v}|_\Delta \in (\mathbb{P}_k(\Delta))^2,\ \textit{and } \vec{v} \textit{ is continuous in all} \\ &\qquad \textit{kth-order Gauss-Legendre points of all sides of } \Delta,\ \Delta \in \mathcal{T}_h,\ \vec{v} = 0 \textit{ in} \\ &\qquad \textit{all kth-order Gauss-Leg. points on the triangle sides } E \subset \Gamma\}, && (3.1.2)\\ &\qquad \textit{the norm in } V^{nc}_{h,k} \textit{ being defined as } |\vec{v}_h|_{1,h,\Omega} := \left(\sum_{\Delta\in\mathcal{T}_h} |\vec{v}_h|^2_{1,\Delta}\right)^{1/2}. \end{aligned}$$

As we mentioned in Section 1.6 the continuity in the Gauss-Legendre points indicates that the patch-test is fulfilled, i.e. the seminorm $|.|_{1,h,\Omega}$ is a norm on $V^{nc}_{h,k}(\Omega)$.

The question we consider next is to specify of suitable degrees of freedom for the velocity parts of these elements.

Consider a fixed triangle Δ with barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$ and sides s_1, s_2, s_3, see Figure 3.1. We will examine the following interpolation problem:

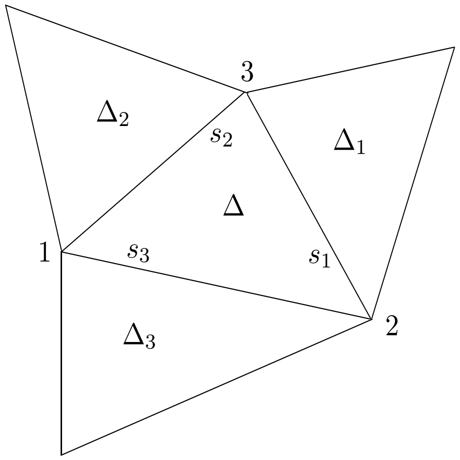


Figure 3.1: A triangle and its three neighbours

select a kth-degree polynomial p which is equal to given arbitrary values g_i, $i = 1,\ldots,\frac{(k-2)(k-1)}{2}$, in the inner points of Δ with barycentric coordinates

$$\lambda_1 = \frac{j}{k}, \lambda_2 = \frac{\ell}{k}, \lambda_3 = \frac{m}{k}, \;\; 1 \le j, \ell, m \le k-2, \;\; j+\ell+m = k, \tag{3.1.3}$$

and

$$p(\gamma_{i,j}^{(k)}) = g_{i,j} \quad i = 1,2,3, \quad j = 1,\ldots,k,$$

where $\gamma_{i,j}^{(k)}$, $j = 1,\ldots,k$, are the kth-order Gauss-Legendre points on the side s_i of Δ, and $g_{i,j}$, $i = 1,2,3$, $j = 1,\ldots,k$, are arbitrary prescribed values.

Theorem 3.1.2 *The nullspace of the above interpolation problem is trivial for odd k and one-dimensional for even k.*

Proof. Let $v_{0,k}(\lambda_1,\lambda_2,\lambda_3)$ be an element of the nullspace. Since $v_{0,k} \in \mathbb{P}_k$, it can be uniquely described by the standard Lagrangian basis $\{w_i\}_{i=1}^{d(k)}$, where $d(k) = \frac{(k+1)(k+2)}{2}$. We assume that the w_i corresponding to the (Lagrangian) boundary points have been ordered first. Considering that $v_{0,k} = 0$ in the inner points we obtain

$$v_{0,k} = \sum_{i=1}^{3k} \alpha_i w_i, \tag{3.1.4}$$

where $3k$ gives the number of boundary points. Let $w_j(\lambda_1,\lambda_2,\lambda_3)$, $j = 1,2,3$, be the basis functions which correspond to the vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, respectively. Using that on each of the sides s_1, s_2, s_3 the nullspace is one-dimensional – it is spanned by the kth-degree Legendre-polynomial $L_k(s)$ – and $L_k(1) =: c \neq 0$, $L_k(0) = (-1)^k c$, on the side s_1 (where $\lambda_1 = 0, \lambda_2 = 1 - s, \lambda_3 = s$, $s \in [0,1]$) we have $v_{0,k}|_{s_1} = \alpha L_k(s)$ for some constant α and can write

$$\begin{aligned} v_{0,k}(0,0,1) &= \alpha L_k(1) = \alpha c = \alpha_3 w_3(0,0,1), \\ v_{0,k}(0,1,0) &= \alpha L_k(0) = \alpha(-1)^k c = \alpha_2 w_2(0,1,0). \end{aligned}$$

Using the same reasoning on s_2 and remarking that in the starting point $(0,0,1)$, where $s = 0$, the value αc is already known, we have $v_{0,k}|_{s_2} = \alpha(-1)^k L_k(s)$ and

$$v_{0,k}(1,0,0) = \alpha(-1)^k c.$$

Finally, continuing on s_3, we find that $v_{0,k}$ takes the same values in the two endpoints of s_3 which means $\alpha = 0$ for odd values of k. Hence, $v_{0,k}$ disappears on all sides of the triangle. This together with (3.1.4) implies $v_{0,k} \equiv 0$. For even k, we have $v_{0,k} = \alpha L_k(s)$ on all three sides (which determines it uniquely in the form (3.1.4)), and hence the nullspace is one-dimensional. $\square$

Based on the theorem above we can described a basis of $V_{h,k}^{nc}(\Omega)$ as follows.

Proposition 3.1.3 *For even k we can start from a standard Lagrangian basis in every Δ including its boundary. The resulting finite element velocities are continuous everywhere. Moreover, on each triangle there exists a nontrivial polynomial $v_{0,k}$ of order k such that on the sides of the corresponding triangle it is equal to a multiple of the kth-order Legendre polynomial defined on the given side. By adding this polynomial trianglewise to the Lagrangian basis we obtain trianglewise k-th order polynomials that are continuous on the common sides of two adjacent triangles in all the kth-order Gauss-Legendre points.*

Proof. The first part of the proposition follows from Remark 1.4.8. The existence of the nontrivial polynomial $v_{0,k}$ follows from Theorem 3.1.2. $\square$

Proposition 3.1.4 *For odd k, based on the above theorem, inside the triangles we can use a standard Lagrangian basis whereas on the triangle sides the Gauss-Legendre points can be taken as degrees of freedom.*

In what follows we establish a form of the basis for the (components of the) velocity space for any k which is suitable for our subsequent considerations, and we look once more on the above interpolation problem with the aim to show that the problem are not specially the Gauss-Legendre points.

On a fixed triangle Δ, such a basis can be defined as follows:

a) The basis functions corresponding to degrees of freedom connected with inner points of Δ appear for $k \geq 3$ and can be represented as

$$B_{c,\Delta}^{(3)} q_i^{(k-3)}, \quad i = 1, \ldots, \frac{(k-2)(k-1)}{2}, \tag{3.1.5}$$

where $B^{(3)}_{c,\Delta} := \lambda_1\lambda_2\lambda_3$ is the conforming bubble function of degree 3 and $q_i^{(k-3)} = q_i^{(k-3)}(\lambda_1, \lambda_2, \lambda_3)$ are polynomials of degree $k-3$ disappearing in all but one of the $\frac{(k-2)(k-1)}{2}$ inner points of Δ with barycentric coordinates (3.1.3). Observe that $\frac{(k-2)(k-1)}{2}$ is also the number of free parameters of a polynomial of degree $k-3$. Therefore the basis functions corresponding to inner points are well defined by (3.1.5).

b) There remain $\frac{(k+2)(k+1)}{2} - \frac{(k-2)(k-1)}{2} = 3k$ degrees of freedom on the sides of Δ, on each of which they are connected (through the corresponding barycentric coordinate) to k knots $\{\gamma_j\}_{j=1}^k$ in $(0,1)$ which we assume to be distributed symmetrically with respect to $\frac{1}{2}$. These knots define the polynomials $\omega_k(s) := \prod_{j=1}^k (s-\gamma_j)$ and $\omega_{2,k-1} := \prod_{j=2}^k (s-\gamma_j)$.

For the basis functions of Δ corresponding to any side s_i we can write

$$w = w(\lambda_1, \lambda_2, \lambda_3) = \alpha\omega_k(\lambda_i) + \beta\omega_k(\lambda_{i+1}) + \gamma\omega_k(\lambda_{i+2}) + \lambda_{i+1}\lambda_{i+2}q^{(k-2)}(\lambda_1, \lambda_2, \lambda_3), \tag{3.1.6}$$

where the barycentric coordinates are numbered cyclically ($\lambda_4 = \lambda_1$, $\lambda_5 = \lambda_2$), and $q^{(k-2)}$ are suitable polynomials of degree $k-2$. Further, restricting w to, say, the side s_1, i.e. the side on which $\lambda_1 = 0, \lambda_2 = 1-s, \lambda_3 = s$, we get

$$w = w(s) = \alpha\omega_k(0) + \beta\omega_k(1-s) + \gamma\omega_k(s) + s(1-s)q^{(k-2)}(s). \tag{3.1.7}$$

We shall clarify under which conditions the polynomials given by (3.1.7) can take every prescribed value in the knots $\{\gamma_j\}_{j=1}^k$ on side s_1. In these points, due to their symmetry with respect to $\frac{1}{2}$, both polynomials $\omega_k(1-s)$ and $\omega_k(s)$ are zero, and therefore the degrees of freedom β, γ obviously belong to the other sides of Δ.

For the remaining part $a + s(1-s)q_i^{(k-2)}(s)$ (where $a := \alpha\omega_k(0)$, and here $\omega_k(0) \neq 0$) we have the following result.

Lemma 3.1.5 *The interpolation problem of selecting a constant a and a polynomial $q_{(k-2)}$ of order $(k-2)$ to arbitrary values $g_1, \ldots, g_k$ prescribed in the knots $\{\gamma_j\}_{j=1}^k$ which are symmetrically distributed in $(0,1)$:*

$$(a + s(1-s)q_{(k-2)}(s))_{|s=\gamma_j} = g_j, \quad j = 1, \ldots, k, \tag{3.1.8}$$

is uniquely solvable iff the following conditions are satisfied:

1) the knots γ_j are pairwise different from each other;

2) k is odd.

Proof. We denote the determinant of the problem by $W(\gamma_1, \ldots, \gamma_k) = W_{1,\ldots,k}$, that is

$$W_{1,\ldots,k} := \begin{vmatrix} 1 & \gamma_1(1-\gamma_1) & \ldots & \gamma_1^{k-1}(1-\gamma_1) \\ \ldots & \ldots & \ldots & \ldots \\ 1 & \gamma_k(1-\gamma_k) & \ldots & \gamma_k^{k-1}(1-\gamma_k) \end{vmatrix}.$$

By subtracting the first row from the other rows and by subsequent simplifications we obtain

$$\begin{aligned} W_{1,\dots,k} &= \prod_{\ell=2}^{k}(\gamma_\ell-\gamma_1)\begin{vmatrix} 1-\gamma_1-\gamma_2 & \gamma_2(1-\gamma_2) & \dots & \gamma_2^{k-2}(1-\gamma_2) \\ \dots & \dots & \dots & \dots \\ 1-\gamma_1-\gamma_k & \gamma_k(1-\gamma_k) & \dots & \gamma_k^{k-2}(1-\gamma_k) \end{vmatrix} \\ &= \prod_{\ell=2}^{k}(\gamma_\ell-\gamma_1)\left\{(1-\gamma_1)W_{2,\dots,k} - \right. \\ &\qquad \left. -\prod_{\ell=2}^{k}\gamma_\ell \begin{vmatrix} 1 & 1-\gamma_2 & \gamma_2(1-\gamma_2) & \dots & \gamma_2^{k-3}(1-\gamma_2) \\ \dots & \dots & \dots & \dots & \\ 1 & 1-\gamma_k & \gamma_k(1-\gamma_k) & \dots & \gamma_k^{k-3}(1-\gamma_k) \end{vmatrix}\right\} \\ &= \prod_{\ell=2}^{k}(\gamma_\ell-\gamma_1)\left\{(1-\gamma_1)W_{2,\dots,k} + (-1)^{k-1}\prod_{\ell=2}^{k}\gamma_\ell \prod_{2\le m<\ell\le k}(\gamma_\ell-\gamma_m)\right\}, \end{aligned}$$

where in the last step the formula for Vandermonde determinants has been used.

This recursion for $W_{1,\dots,k}$, in the special case $k=2$, gives already

$$W_{1,2} = (\gamma_2-\gamma_1)(1-\gamma_1-\gamma_2) = (\gamma_2-\gamma_1)\left\{\omega_2(1)-\omega_2(0)\right\}.$$

The general case then follows by induction remarking that

$$(1-\gamma_1)\left\{\omega_{2,k-1}(1)-\omega_{2,k-1}(0)\right\} + (-1)^{k-1}\prod_{\ell=2}^{k}\gamma_\ell = \omega_k(1)-\omega_k(0).$$

Finally,

$$W_{1,\dots,k} = \prod_{1\le m<\ell\le k}(\gamma_\ell-\gamma_m)\left\{\omega_k(1)-\omega_k(0)\right\},$$

that completes the proof of the lemma, since, due to the symmetry we have $\omega_k(1) = \prod_{j=1}^{k}\gamma_j = (-1)^k\omega_k(0)$. □

In what follows for odd k we give an explicit form of the basis functions corresponding to the boundary points of Δ. Similarly to Lemma 3.1.5 let γ_i, $i=1,\dots,k$ be pairwise different real numbers which are symmetrically distributed in $(0,1)$. Since k is odd we have

$$\gamma_{\frac{k+1}{2}} = \frac{1}{2}, \qquad \gamma_{k-j} = 1-\gamma_j, \qquad j=1,\dots,\frac{k-1}{2}.$$

Consider the following points on the sides of the triangle:

$$\begin{aligned} \gamma_{1,j}^{(k)} &= (0,1-\gamma_j,\gamma_j), \quad j=1,\dots,k, \qquad \text{on the side } s_1, \\ \gamma_{2,j}^{(k)} &= (\gamma_j,0,1-\gamma_j), \quad j=1,\dots,k, \qquad \text{on the side } s_2, \\ \gamma_{3,j}^{(k)} &= (1-\gamma_j,\gamma_j,0), \quad j=1,\dots,k, \qquad \text{on the side } s_3. \end{aligned}$$

Theorem 3.1.6 *Let* $1 \leq j \leq \frac{k-1}{2}$*, so we have* $\gamma_j = \frac{1}{2} - a_j$ *and* $\gamma_{k-j} = \frac{1}{2} + a_j$ *with some* $a_j \in (0, 1/2)$*. For* $i = 1, 2, 3,$ *and* $j = 1, \dots, \frac{k-1}{2}$ *let the piecewise polynomials* $q_{i,j}^{(k)}$ *and* $q_{i,k-j}^{(k)}$ *be defined as*

$$q_{i,j}^{(k)}(\lambda_1, \lambda_2, \lambda_3) := \tag{3.1.9}$$
$$\left[(\lambda_{i+1}\lambda_{i+2}(\lambda_{i+2} + A_j) + B_j) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{(k-1)/2} (\lambda_{i+1}\lambda_{i+2} - \gamma_\ell \gamma_{k-\ell}) - C_j \right] + \frac{C_j}{\omega_k(0)} \omega_k(\lambda_i),$$

$$q_{i,k-j}^{(k)}(\lambda_1, \lambda_2, \lambda_3) := \tag{3.1.10}$$
$$\left[(\lambda_{i+1}\lambda_{i+2}(\lambda_{i+1} + A_j) + B_j) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{(k-1)/2} (\lambda_{i+1}\lambda_{i+2} - \gamma_\ell \gamma_{k-\ell}) - C_j \right] + \frac{C_j}{\omega_k(0)} \omega_k(\lambda_i),$$

where the barycentric coordinates are numbered cyclically and

$$A_j := \frac{1 - 2a_j - 4a_j^2}{4a_j}, \quad B_j := \frac{4a_j^2 - 1}{16a_j}, \quad C_j := B_j \prod_{\substack{\ell=1 \\ \ell \neq j}}^{(k-1)/2} (-\gamma_\ell \gamma_{k-\ell}) \,.$$

If $j = \frac{k+1}{2}$ *then let*

$$q_{i,\frac{k+1}{2}}^{(k)}(\lambda_1, \lambda_2, \lambda_3) := \frac{1}{C} \left[\prod_{j=1}^{(k-1)/2} (\lambda_{i+1}\lambda_{i+2} - \gamma_j(1 - \gamma_j)) - D \right] + \frac{D}{C\omega_k(0)} \omega_k(\lambda_i), \tag{3.1.11}$$

where

$$D := \prod_{j=1}^{(k-1)/2} \gamma_j(\gamma_j - 1) \quad \text{and} \quad C := \prod_{j=1}^{(k-1)/2} \left(\frac{1}{4} - \gamma_j(1 - \gamma_j) \right).$$

Then

$$\begin{aligned} & q_{i,j}^{(k)}(\gamma_{m,n}^{(k)}) \neq 0 \qquad \text{if} \quad (m,n) = (i,j), \\ \text{and} \quad & q_{i,j}^{(k)}(\gamma_{m,n}^{(k)}) = 0 \qquad \text{if} \quad (m,n) \neq (i,j) \end{aligned}$$

hold for $i = 1, 2, 3,$ $j = 1, \dots, k$*. Moreover, the functions* $q_{i,j}^{(k)}$ *can be written in the form*

$$q_{i,j}^{(k)}(\lambda_1, \lambda_2, \lambda_3) = \lambda_{i+1}\lambda_{i+2} g_{i,j}^{(k-2)}(\lambda_1, \lambda_2, \lambda_3) + c_{0,j}\omega_k(\lambda_i),$$

where $c_{0,j} \in \mathbb{R}$*, and* $g_{i,j}^{(k-2)}$ *are suitable polynomials of order* $k-2$*.*

Proof. First we prove the second part of the theorem. Since the constants C_j, $j = 1, \ldots, (k-1)/2$, are equal to the constant terms of the polynomials

$$(\lambda_{i+1}\lambda_{i+2}(\lambda_{i+2} + A_j) + B_j) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{(k-1)/2} (\lambda_{i+1}\lambda_{i+2} - \gamma_\ell \gamma_{k-\ell})$$

and

$$(\lambda_{i+1}\lambda_{i+2}(\lambda_{i+1} + A_j) + B_j) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{(k-1)/2} (\lambda_{i+1}\lambda_{i+2} - \gamma_\ell \gamma_{k-\ell}) \,,$$

and D is equal to the constant term of the polynomial

$$\prod_{j=1}^{(k-1)/2} (\lambda_{i+1}\lambda_{i+2} - \gamma_j(1-\gamma_j)) \,,$$

moreover, the non-constant terms of the polynomials contain the multiplier $\lambda_{i+1}\lambda_{i+2}$ we obtain that in (3.1.9), (3.1.10) and (3.1.11) the expressions in the brackets can be written in the form $\lambda_{i+1}\lambda_{i+2} g_{i,j}^{(k-2)}$.

We prove the first part of the theorem for $i = 3$, that is in the case of the functions $q_{3,j}^{(k)}$, $j = 1, \ldots, k$, which correspond to the knots on the triangle side s_3.

Since the knots belonging to the sides s_1 and s_2 have the coordinates $(0, 1-\gamma_n, \gamma_n)$ and $(\gamma_n, 0, 1-\gamma_n)$, respectively, from (3.1.6) we obtain

$$q_{3,j}^{(k)}(\gamma_{1,n}^{(k)}) = c_{0,j}\omega_k(\gamma_n) = 0, \qquad q_{3,n}^{(k)}(\gamma_{2,n}^{(k)}) = c_{0,j}\omega_k(1-\gamma_n) = 0,$$

where we have used that γ_n and $1-\gamma_n$ are roots of the polynomial ω_k.

Next, consider the points $\gamma_{3,j}^{(k)} = (1-\gamma_j, \gamma_j, 0)$, $j = 1, \ldots, k$, on the side s_3. If $j = n = \frac{k+1}{2}$ then we obtain

$$q_{3,\frac{k+1}{2}}^{(k)}(\gamma_{3,n}^{(k)}) = q_{3,\frac{k+1}{2}}^{(k)}\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \frac{1}{C}\left[\prod_{j=1}^{(k-1)/2}\left(\frac{1}{4} - \gamma_j(1-\gamma_j)\right) - D\right] + \frac{D}{C} = 1.$$

If $j = \frac{k+1}{2}$, $n \neq \frac{k+1}{2}$ then

$$\prod_{j=1}^{(k-1)/2} (\gamma_n(1-\gamma_n) - \gamma_j(1-\gamma_j)) = 0,$$

that implies $q_{3,\frac{k+1}{2}}^{(k)}(\gamma_{3,n}^{(k)}) = 0$.

If $j < \frac{k+1}{2}$ then the constants A_j and B_j are determined in such a way that $\gamma^{(k)}_{3,k-j} = (1-\gamma_{k-j}, \gamma_{k-j}, 0) = (\frac{1}{2} - a_j, \frac{1}{2} + a_j, 0)$ and $\gamma^{(k)}_{3,\frac{k+1}{2}} = (\frac{1}{2}, \frac{1}{2}, 0)$ are roots of the polynomial

$$\lambda_1\lambda_2(\lambda_2 + A_j) + B_j,$$

i.e. A_j and B_j are the solutions of the system of linear equations

$$\begin{aligned} \left(\frac{1}{4} - a_j^2\right)\left(\frac{1}{2} + a_j\right) + \left(\frac{1}{4} - a_j^2\right) A_j + B_j &= 0, \\ \frac{1}{8} + \frac{1}{4} A_j + B_j &= 0. \end{aligned}$$

Thus all values $\gamma^{(k)}_{3,n}$, $n = 1, \ldots, k$, $n \neq j$ are roots of the polynomial

$$(\lambda_1\lambda_2(\lambda_2 + A_j) + B_j) \prod_{\substack{\ell=1 \\ \ell \neq j}}^{(k-1)/2} (\lambda_1\lambda_2 - \gamma_\ell\gamma_{k-\ell}), \tag{3.1.12}$$

which implies $q_{3,j}(\gamma^{(k)}_{3,n}) = 0$, $n = 1, \ldots, k$, $n \neq j$. Using the same A_j and B_j we obtain that $\gamma^{(k)}_{3,j} = (1-\gamma_j, \gamma_j, 0) = (\frac{1}{2} + a_j, \frac{1}{2} - a_j, 0)$ and $\gamma^{(k)}_{3,\frac{k+1}{2}} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ are roots of the polynomial

$$\lambda_1\lambda_2(\lambda_1 + A_j) + B_j,$$

and then $q_{3,k-j}(\gamma^{(k)}_{3,n}) = 0$, $n = 1, \ldots, k$, $n \neq k-j$.

Finally, we have to prove that $q_{3,j}(\gamma^{(k)}_{3,j}) \neq 0$ and $q_{3,k-j}(\gamma^{(k)}_{3,k-j}) \neq 0$ hold. Assume that $\gamma^{(k)}_{3,j} = (\frac{1}{2} + a_j, \frac{1}{2} - a_j, 0)$ is also a root of the polynomial (3.1.12). Then

$$\left(\frac{1}{4} - a_j^2\right)\left(\frac{1}{2} - a_j\right) + \left(\frac{1}{4} - a_j^2\right) A_j + B_j = 0,$$

and this – together with the first equation of the system of equations above – means that $a_j = 0$, which contradicts $j \neq \frac{k+1}{2}$. In a similar way we obtain $q_{3,k-j}(\gamma^{(k)}_{3,k-j}) \neq 0$. □

Remark 3.1.7 For $j = 1, \ldots, k$, $j \neq (k+1)/2$ let

$$S_j := q^{(k)}_{i,j}(\gamma^{(k)}_{i,j}) = \frac{a_j(4a_j^2 - 1)}{2} \prod_{\substack{\ell=1 \\ \ell \neq j}}^{(k-1)/2} \left(\left(\frac{1}{4} - a^2\right) - \gamma_\ell\gamma_{k-\ell}\right).$$

Then $S_{k-j} = S_j$, and for the polynomials

$$\overline{q}^{(k)}_{i,j}(\lambda_1, \lambda_2, \lambda_3) := \begin{cases} \frac{1}{S_j} q^{(k)}_{i,j}(\lambda_1, \lambda_2, \lambda_3), & \text{if } j \neq \frac{k+1}{2}, \\ q^{(k)}_{i,j}(\lambda_1, \lambda_2, \lambda_3), & \text{if } j = \frac{k+1}{2}, \end{cases}$$

we have

$$\overline{q}_{i,j}^{(k)}(\gamma_{m,n}) = \begin{cases} 1 & \text{if} \quad (m,n) = (i,j), \\ 0 & \text{if} \quad (m,n) \neq (i,j), \end{cases}$$

and the coefficients of $\omega_k(\lambda_i)$ in $\overline{q}_{i,j}^{(k)}$ and in $\overline{q}_{i,k-j}^{(k)}$ are equal.

Remark 3.1.8 If γ_j, $j = 1, \dots, k$ are the kth-order Gauss-Legendre points given on the interval $[0,1]$, then

$$\omega_k(\lambda_i) = \mathsf{P}_k^{(0,0)}(1 - 2\lambda_i).$$

Returning to the even order cases we define a function given on a triangle which is equal to the Legendre polynomial on each sides of the triangle:

Definition 3.1.9 *Let $\Delta \in \mathcal{T}_h$ be a given triangle. Then the function*

$$B_{n,\Delta}^{(k)} := \frac{1}{2} \left\{ \sum_{i=1}^{3} \mathsf{P}_k^{(0,0)}(1 - 2\lambda_i) - 1 \right\} \tag{3.1.13}$$

is called kth-order non-conforming bubble function.

On the sides of Δ (e.g. $\lambda_1 = 0$, $\lambda_2 = s$, $\lambda_3 = 1 - s$) we have

$$B_{n,\Delta}^{(k)} = \frac{1}{2} \left\{ \mathsf{P}_k^{(0,0)}(1 - 2s) + \mathsf{P}_k^{(0,0)}(2s - 1) \right\} = \begin{cases} L_k(s) & \text{for even } k, \\ 0 \text{ for odd } k, \end{cases}$$

where L_k denotes the kth-degree Legendre polynomial on the interval $[0,1]$. That is, (3.1.13) gives the general formula for the polynomials causing the singularity in Theorem 3.1.2 for even k and called there $v_{0,k}$.

Remark 3.1.10 For even k one gets the non-conforming velocity space $V_{h,k}^{nc}(\Omega)$ by adding trianglewise a kth order non-conforming bubble function to the conforming velocity space $V_{h,k}(\Omega)$ (defined by (2.1.2)):

$$V_{h,k}^{nc}(\Omega) = V_{h,k}(\Omega) + \left\{ \vec{v}, \ \vec{v}|_\Delta = \begin{pmatrix} \alpha_\Delta \\ \beta_\Delta \end{pmatrix} B_{n,\Delta}^{(k)}, \ \alpha_\Delta, \beta_\Delta \in \mathbb{R}, \ \Delta \in \mathcal{T}_h \right\}. \tag{3.1.14}$$

Example 3.1.11 For $k = 2$ we have

$$B_{n,\Delta}^{(2)} = \frac{1}{2} \left\{ \sum_{i=1}^{3} \mathsf{P}_2^{(0,0)}(1 - 2\lambda_i) - 1 \right\} = 3 \sum_{i=1}^{3} \lambda_i^2 - 2,$$

which is the “neutral” function (2.2.3) used by Fortin and Soulie (see [13]).

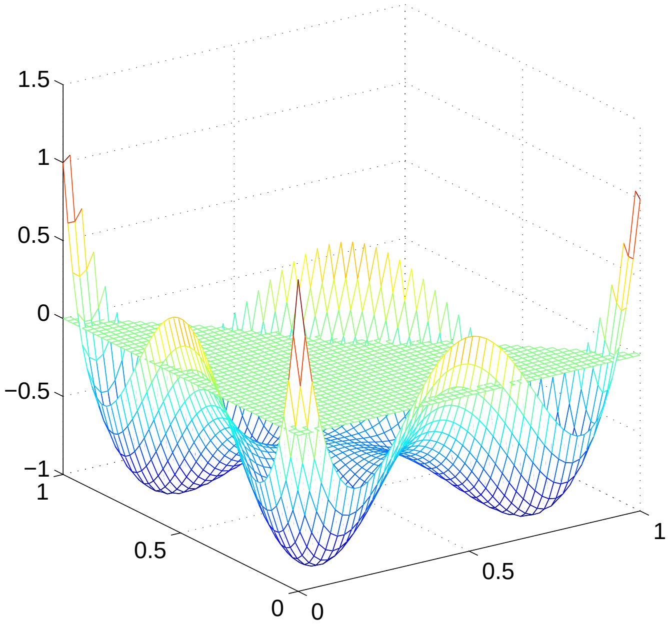


Figure 3.2: The 4th-order non-conforming bubble function on the reference triangle.

Example 3.1.12 For $k = 4$ the non-conforming bubble function has the form

$$B^{(4)}_{n,\Delta} := \frac{1}{2}\left\{\sum_{i=1}^{3} \mathsf{P}_4^{(0,0)}(1-2\lambda_i) - 1\right\} = 35\sum_{i=1}^{3}\lambda_i^4 - 60\sum_{i=1}^{3}\lambda_i^2 + 26 - 210\lambda_1\lambda_2\lambda_3.$$

Denote by Ψ_4 the 4th-order non-conforming bubble function used in [7] (see the formula (2.3.1) in Section 2.3).
Using that

$$\lambda_1^3\lambda_2 + \lambda_2^3\lambda_1 + \lambda_2^3\lambda_3 + \lambda_3^3\lambda_2 + \lambda_3^3\lambda_1 + \lambda_1^3\lambda_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 - \lambda_1^4 - \lambda_2^4 - \lambda_3^4,$$

and $\lambda_3 = 1 - \lambda_1 - \lambda_2$ we obtain

$$\begin{aligned}\Psi_4 = {}& \lambda_1^4 + \lambda_2^4 + \lambda_3^4 + 36(\lambda_1^4 + \lambda_2^4 + 3\lambda_1^2\lambda_2^2 + 2\lambda_1^3\lambda_2 + 2\lambda_1\lambda_2^3 - 2\lambda_1^3 - 2\lambda_2^3 \\ & - 2\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^2 + \lambda_1^2 + \lambda_2^2) - 16(1 - 3\lambda_1 - 3\lambda_2 + 3\lambda_1^2 + 3\lambda_2^2 + 6\lambda_1\lambda_2 \\ & - 3\lambda_1^2\lambda_2 - 3\lambda_1\lambda_2^2 - \lambda_1^4 - \lambda_2^4 - \lambda_3^4).\end{aligned}$$

By reordering the above equation we have

$$\begin{aligned}\Psi_4 = {} & 17(\lambda_1^4+\lambda_2^4+\lambda_3^4)+18(2\lambda_1^4+2\lambda_2^4+6\lambda_1^2\lambda_2^2+4\lambda_1^3\lambda_2+4\lambda_1\lambda_2^3-4\lambda_1^3-4\lambda_2^3\\ & -12\lambda_1^2\lambda_2-12\lambda_1\lambda_2^2+6\lambda_1^2+6\lambda_2^2+12\lambda_1\lambda_2-4\lambda_1-4\lambda_2+1)\\ & -60(1+2\lambda_1^2+2\lambda_2^2-2\lambda_1-2\lambda_2+2\lambda_1\lambda_2)+26-192(\lambda_1\lambda_2-\lambda_1^2\lambda_2-\lambda_1\lambda_2^2).\end{aligned}$$

The expressions in the second, third and fourth brackets are equal to

$$\begin{aligned}&\lambda_1^4+\lambda_2^4+(1-\lambda_1-\lambda_2)^4=\lambda_1^4+\lambda_2^4+\lambda_3^4,\\ &\lambda_1^2+\lambda_2^2+(1-\lambda_1-\lambda_2)^2=\lambda_1^2+\lambda_2^2+\lambda_3^2,\\ &\lambda_1\lambda_2(1-\lambda_1-\lambda_2)=\lambda_1\lambda_2\lambda_3,\end{aligned}$$

respectively, so

$$\Psi_4 = B_{n,\Delta}^{(4)} + 18\lambda_1\lambda_2\lambda_3,$$

which means that the functions Ψ and $B_{n,\Delta}^{(4)}$ differ only in a conforming bubble function.

Example 3.1.13 For k=6 we have

$$\begin{aligned}B_{n,\Delta}^{(6)} :={} & \frac{1}{2}\left\{\sum_{i=1}^{3}\mathsf{P}_6^{(0,0)}(1-2\lambda_i)-1\right\} = 462\sum_{i=1}^{3}\lambda_i^6-1890\sum_{i=1}^{3}\lambda_i^4+2415\sum_{i=1}^{3}\lambda_i^2\\ & -986-315\lambda_1\lambda_2\lambda_3\left[11\sum_{i=1}^{3}\lambda_i^2-25\right],\end{aligned}$$

and for the 6th-order non-conforming bubble function Ψ_6 mentioned in [7] (see (2.3.2) in Section 2.3)

$$\Psi_6 = B_{n,\Delta}^{(6)} - 15\lambda_1\lambda_2\lambda_3\left[210\lambda_1\lambda_2\lambda_3+15\sum_{i=1}^{3}\lambda_i^2-13\right]$$

holds.

Remark 3.1.14 In the case of odd k, if a kth-order polynomial defined over a triangle Δ is equal to 0 in the kth-order Gauss-Legendre points of the sides of Δ, then it takes 0 on the whole boundary of the triangle (see Theorem 3.1.2). This means that for odd k there does not exist a non-conforming bubble function defined over a single triangle. However, it follows from Theorem 3.1.6 that there exist k linearly independent functions which behave similarly to a non-conforming bubble over two adjacent triangles. These are those elements of the Lagrangian basis which correspond to the Gauss-Legendre points of the common side of the triangles. If Δ_1 and Δ_2 are two adjacent triangles, then these bubbles have the form

$$u|_{\Delta_1} = \lambda_1^{(1)}\lambda_2^{(1)}g_1^{(k-2)} + \alpha \mathsf{P}_k^{(0,0)}(1-2\lambda_3^{(1)}), \quad u|_{\Delta_2} = \lambda_1^{(2)}\lambda_2^{(2)}g_2^{(k-2)} + \alpha \mathsf{P}_k^{(0,0)}(1-2\lambda_3^{(2)}),$$

where $\alpha \in \mathbb{R}$, $\lambda_i^{(1)}$ and $\lambda_i^{(2)}$, $i = 1,2,3$, are the barycentric coordinates in Δ_1 and Δ_2, respectively, $\lambda_3^{(1)}$ and $\lambda_3^{(2)}$ are those coordinates which are equal to zero on the common side of the triangles, and $g_1^{(k-2)}$, $g_2^{(k-2)}$ are polynomials of degree $k-2$. The equality of the coefficients of $\mathsf{P}_k^{(0,0)}$ in $u|_{\Delta_1}$ and $u|_{\Delta_2}$ follows from Remark 3.1.7.

3.2 The nullspace of the discrete gradient in the case of even order Gauss-Legendre elements

In this section we prove that for even k the non-conforming bubbles remove the algebraic singularity of Scott-Vogelius elements. Using (3.1.14) it is sufficient to show that for all non-constant $p \in N_{V_{h,k}(\Omega)}$ (see (2.1.3)) there exist constants $\alpha_\Delta, \beta_\Delta \in \mathbb{R}$, $\Delta \in \mathcal{T}_h$, such that

$$b(\vec{v}, p) \neq 0$$

holds, where $\vec{v}|_\Delta = \binom{\alpha_\Delta}{\beta_\Delta} B_{n,\Delta}^{(k)}$ for all $\Delta \in \mathcal{T}_h$.

Let us denote by $N_{V_{h,k}^{nc}}(\Omega)$ the nullspace of the discrete gradient operator in the case of the finite element pair (3.1.1)–(3.1.2), i.e.

$$N_{V_{h,k}^{nc}}(\Omega) := \left\{ q \in P_h(\Omega) \ : \ b(\vec{v}, q) = 0 \quad \forall \vec{v} \in V_{h,k}^{nc}(\Omega) \right\}.$$

Theorem 3.2.1 *For even k the space $N_{V_{h,k}^{nc}}(\Omega)$ is one-dimensional, it contains only constant functions.*

Proof. For $k = 2$ see Lemma 2.2.1. Let $k \geq 4$ and let $q \in P_h$ be a non-constant function for which $b(\vec{v}, q) = 0$ holds for all $\vec{v} \in V_{h,k}(\Omega)$ (also for all velocity function from the conforming velocity space). Denote by Δ one of the triangles from the support of q. We may suppose that the vertices of Δ are $(0,0)$, (a_1, b_1) and (a_2, b_2). Then, let $\vec{u}, \vec{v} \in V_{h,k}^{nc}(\Omega)$ be the following functions:

$$\vec{u}|_\Delta = \begin{pmatrix} B_{n,\Delta}^{(k)} \\ 0 \end{pmatrix}, \qquad \vec{u}|_{\Omega\setminus\Delta} = 0,$$

and

$$\vec{v}|_\Delta = \begin{pmatrix} 0 \\ B_{n,\Delta}^{(k)} \end{pmatrix}, \qquad \vec{v}|_{\Omega\setminus\Delta} = 0.$$

We may assume that

$$q|_\Delta = \sum_{j=1}^{3} \alpha_j \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_j) \circ B^{-1} + \alpha_4, \tag{3.2.1}$$

where $\alpha_j \in \mathbb{R}$, $j = 1,2,3,4$, and

$$B = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

is the affine transformation which maps the reference triangle Δ_0 onto Δ. Then using

$$\frac{d}{dx}\mathsf{P}_k^{(0,0)}(x) = \frac{k+1}{2}\mathsf{P}_{k-1}^{(1,1)}(x), \tag{3.2.2}$$

(see [14]) and

$$\int_\Delta \operatorname{div}\vec{u}\,\mathrm{d}x\,\mathrm{d}y = \int_\Delta \operatorname{div}\vec{v}\,\mathrm{d}x\,\mathrm{d}y = 0$$

we obtain

$$\begin{aligned}
&\int_\Omega q\operatorname{div}\vec{u}\,\mathrm{d}x\,\mathrm{d}y = \int_\Delta q\frac{\partial B_{n,\Delta}^{(k)}}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\sum_{j=1}^{3}\alpha_j \int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_j)\left(b_2\frac{\partial}{\partial\xi} - b_1\frac{\partial}{\partial\eta}\right)\left(\sum_{i=1}^{3}\mathsf{P}_k^{(0,0)}(1-2\lambda_i) - 1\right)\mathrm{d}\xi\,\mathrm{d}\eta \\
&= b_2\frac{k+1}{2}\sum_{j=1}^{3}\alpha_j\int\limits_{\Delta_0}\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_j)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_1))\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&- b_1\frac{k+1}{2}\sum_{j=1}^{3}\alpha_j\int\limits_{\Delta_0}\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_j)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_2))\,\mathrm{d}\xi\,\mathrm{d}\eta,
\end{aligned}$$

and

$$
\begin{aligned}
&\int_\Omega q \operatorname{div} \vec{v} \,\mathrm{d}x\,\mathrm{d}y = \int_\Delta q \frac{\partial B_{n,\Delta}^{(k)}}{\partial y} \,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2} \sum_{j=1}^{3} \alpha_j \int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_j) \left(-a_2 \frac{\partial}{\partial \xi} + a_1 \frac{\partial}{\partial \eta}\right) \left(\sum_{i=1}^{3} \mathsf{P}_k^{(0,0)}(1-2\lambda_i) - 1\right) \mathrm{d}\xi\,\mathrm{d}\eta \\
&= -a_2 \frac{k+1}{2} \sum_{j=1}^{3} \alpha_j \int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_j)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_1))\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&+ a_1 \frac{k+1}{2} \sum_{j=1}^{3} \alpha_j \int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_j)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_2))\,\mathrm{d}\xi\,\mathrm{d}\eta.
\end{aligned}
$$

If $j = 1$ in the sums above, then using that for even k the constant terms of the polynomials

$$\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_1) = \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(2(\lambda_2+\lambda_3)-1)$$

and

$$\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_2)$$

are equal to $2k$ and 0, respectively (see (2.1.4)), and that the polynomial $\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_1)$ is orthogonal to $\mathbb{P}_{k-2}(\Delta_0)$ with respect to the weight functions λ_2 and λ_3, we obtain

$$\int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_1)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_1))\,\mathrm{d}\xi\,\mathrm{d}\eta = 2k \cdot C, \tag{3.2.3}$$

$$\int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_1)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_2))\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \tag{3.2.4}$$

where $C = \int_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_1)\,\mathrm{d}\xi\,\mathrm{d}\eta$. Similarly, using the same constant C for $j = 2$ we have

$$\int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_2)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_1))\,\mathrm{d}\xi\,\mathrm{d}\eta = 0, \tag{3.2.5}$$

$$\int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_2)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_2))\,\mathrm{d}\xi\,\mathrm{d}\eta = 2k \cdot C. \tag{3.2.6}$$

Finally, if $j = 3$ then

$$\int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_1))\,\mathrm{d}\xi\,\mathrm{d}\eta = -2k \cdot C, \tag{3.2.7}$$

$$\int\limits_{\Delta_0} \mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_3)(\mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_3) - \mathsf{P}_{k-1}^{(1,1)}(1-2\lambda_2))\,\mathrm{d}\xi\,\mathrm{d}\eta = -2k \cdot C. \tag{3.2.8}$$

Equations (3.2.3)–(3.2.8) imply that

$$\int\limits_{\Omega} q \operatorname{div} \vec{u}\,\mathrm{d}x\,\mathrm{d}y = k(k+1)C \cdot \left[b_2(\alpha_1 - \alpha_3) - b_1(\alpha_2 - \alpha_3)\right],$$

and

$$\int\limits_{\Omega} q \operatorname{div} \vec{v}\,\mathrm{d}x\,\mathrm{d}y = -k(k+1)C \cdot \left[a_2(\alpha_1 - \alpha_3) - a_1(\alpha_2 - \alpha_3)\right].$$

$C \neq 0$, otherwise $\mathsf{P}_{k-1}^{(0,2)}(1-2\lambda_i)$ would be orthogonal to $\mathbb{P}_{k-1}(\Delta_0)$, which is a contradiction. Since the vectors (a_1, b_1) and (a_2, b_2) are linearly independent we have that one of the integrals above differs from zero. ☐

3.3 The macroelement technique and the stability of the even order Gauss-Legendre elements

We will prove for even $k \geq 2$ the stability of (3.1.1)–(3.1.2) using a modification of the macroelement technique for conforming elements described in [26]. We suppose that the triangulation $\mathcal{T}_h$ is regular in the usual sense that there exists a constant $\kappa > 1$ independent of h such that

$$h_\Delta \leq \kappa \rho_\Delta \quad \forall \Delta \in \mathcal{T}_h,$$

where h_Δ is the diameter of Δ and ρ_Δ is the maximum diameter of all circles contained in Δ.

Definition 3.3.1 *(see [26]) A macroelement M is a collection of adjacent triangles of $\mathcal{T}_h$. It is said to be equivalent to a reference macroelement $\hat{M}$ if there is a mapping $F_M : \hat{M} \to M$ satisfying the following conditions:*

(i) *F_M is continuous and one-to-one,*

(ii) *$F_M(\hat{M}) = M$,*

(iii) *if $\hat{M} = \bigcup_{j=1}^m \hat{\Delta}_j$, where $\hat{\Delta}_j$, $j = 1, \ldots, m$, are the triangles of which $\hat{M}$ is composed, then $\Delta_j = F_M(\hat{\Delta}_j)$, $j = 1, \ldots, m$, are the triangles forming M,*

(iv) $F_{M|_{\hat{\Delta}_j}} = F_{\Delta_j} \circ F_{\hat{\Delta}_j}^{-1}$, *$j = 1, \ldots, m$, where $F_{\hat{\Delta}_j}$ and F_{Δ_j} are the affine mappings from the reference triangle onto $\hat{\Delta}_j$ and Δ_j, respectively.*

We will denote by $\mathcal{E}_{\hat{M}}$ the family of macroelements that are equivalent to the reference element $\hat{M}$.

For a given macroelement M let us define the following spaces

$$\begin{aligned}
V_{0,M}^{nc} =& \big\{\vec{v} \ : \ \vec{v}|_\Delta \in (\mathbb{P}_k(\Delta))^2 \ \ \forall \Delta \in M, \ \text{ and } \vec{v} \text{ is continuous in all} \\
& \quad k\text{th-order Gauss-Legendre points of all sides of } \Delta, \ \Delta \subset M, \ \vec{v} = 0 \ \text{ in all} \\
& \quad k\text{th-order Gauss-Legendre points of all edges on } \partial M\big\}, \\
P_M =& \big\{p \in L^2(M) \ : \ p|_\Delta \in \mathbb{P}_{k-1}(\Delta) \ \ \forall \Delta \subset M\big\}, \\
P_{M,0} =& P_M \cap L_0^2(M), \\
N_M^{nc} =& \left\{ p \in P_M \ : \ \int_M p \operatorname{div} \vec{v} \,\mathrm{d}x = 0 \ \ \forall \vec{v} \in V_{0,M}^{nc} \right\}.
\end{aligned}$$

In the proof of stability the dimension of the space N_M^{nc} plays a key role.

The following lemma is a slight modification of the Lemma 3.1. of [26].

Lemma 3.3.2 *Let $\mathcal{E}_{\hat{M}}$ be a class of macroelements. Suppose that for every $M \in \mathcal{E}_{\hat{M}}$ the space N_M^{nc} is one-dimensional, consisting of functions that are constant on M. Then there exists a positive constant $\beta_{\hat{M}} = \beta(\hat{M}, \kappa)$ such that the condition*

$$\sup_{\vec{v} \in V_{0,M}^{nc}, \ \vec{v} \neq 0} \frac{(\operatorname{div} \vec{v}, p)_M}{|\vec{v}|_{1,h,M}} \geq \beta_{\hat{M}} ||p||_{0,M}, \quad \forall p \in P_{M,0},$$

holds for every $M \in \mathcal{E}_{\hat{M}}$.

In Lemma 3.1 of [26] the author uses $V_{0,M}$ instead of $V_{0,M}^{nc}$, where

$$V_{0,M} = \big\{\vec{v} \in (H_0^1(M))^2 \ : \ \vec{v} \in (\mathbb{P}_k(\Delta))^2, \quad \forall \Delta \subset M\big\},$$

and – corresponding to the conforming case – the norm is $|.|_{1,M}$. Since in the proof of the original lemma [26, Lemma 3.1] only the finiteness of the dimension of $V_{0,M}$ is applied the proof remains valid in the non-conforming case too.

Proof. Let $M \in \mathcal{E}_{\hat{M}}$ be an arbitrary fixed macroelement, and let

$$\beta_M := \inf_{\substack{p \in P_{M,0} \\ ||p||_{0,M}=1}} \ \sup_{\substack{\vec{v} \in V_{0,M}^{nc} \\ |\vec{v}|_{1,h,M}=1}} (\operatorname{div} \vec{v}, p)_M.$$

Since N_M^{nc} consists of functions that are constant on M, and $P_{M,0}$ and $V_{0,M}^{nc}$ are finite dimensional, we obtain $\beta_M > 0$. We have to prove that there exists a constant $\beta_{\hat{M}}$ such that

$$\beta_M \geq \beta_{\hat{M}} > 0 \qquad \forall M \in \mathcal{E}_{\hat{M}}.$$

Let $\hat{x}^1, \ldots, \hat{x}^d$ be the vertices of the triangles in $\hat{M}$. Every $M \in \mathcal{E}_{\hat{M}}$ is uniquely defined by its vertices $x^i = F_M(\hat{x}^i)$, $i = 1, \ldots, d$ and so $\beta_M = \beta(x^1, \ldots, x^d)$. Consider the vertices as a point $X = (x^1, \ldots, x^d)$ in $\mathbb{R}^{2d}$, and $\beta_m = \beta(X)$ as a function of X. Let $h_M = \max_{\Delta \subset M}\{h_\Delta\}$. We may assume that $h_M = 1$ and that x^1 coincides with the origin in $\mathbb{R}^2$, since the general case can be handled by a scaling argument using the mapping $G(x) = h_M^{-1}(x - x^1)$. Since x^1 is chosen as the origin, every vertex $x^1, \ldots, x^d$ lies within a given distance from the origin. Every $\Delta \subset M$ has a diameter less than or equal to unity and satisfies the regularity assumption $h_\Delta \leq \kappa\rho_\Delta$. This means that the point X belongs to a compact set $D \subset \mathbb{R}^{2d}$. The function β is continuous, and since $\beta(X) > 0$ for every $X \in D$, we conclude that there is a constant $\beta_{\hat{M}} > 0$ such that $\beta(X) \geq \beta_{\hat{M}}$ for every $X \in D$. $\square$

We will prove the stability of (3.1.1)–(3.1.2) only for even $k \geq 2$, so we need only two of the three stability conditions of Stenberg [26]:

Definition 3.3.3 *We say that the macroelement conditions are satisfied if there is a fixed set of equivalence classes $\mathcal{E}_{\hat{M}_i}$, $i = 1, \ldots, n$, $n \geq 1$, such that*

(i) for each $M \in \mathcal{E}_{\hat{M}_i}$, $i = 1, \ldots, n$, the space N_M^{nc} is one-dimensional, consisting of functions that are constant on M,

(ii) for each h, the triangles in $\mathcal{T}_h$ can be grouped together to form macroelements such that for the so obtained macroelement partitioning $\mathcal{M}_h$ of $\overline{\Omega}$, each $M \in \mathcal{M}_h$ belongs to some of the classes $\mathcal{E}_{\hat{M}_i}$, $i = 1, \ldots, n$.

Theorem 3.3.4 *If the above macroelement conditions are satisfied then for $k \geq 2$ the inf-sup condition holds for the finite element family defined in* (3.1.1)–(3.1.2).

To prove the theorem for all $p \in P_h$ we consider the decomposition

$$p = \Pi_h p + (I - \Pi_h)p,$$

where Π_h is the L^2-projection from P_h onto the space

$$Q_h = \left\{\mu \in L_0^2(\Omega) \;\; : \;\; \mu|_M \;\text{ is constant for all }\; M \in \mathcal{M}_h\right\}.$$

Lemma 3.3.5 *Suppose the macroelement conditions are valid. Then there exists a constant $C_1 > 0$ such that for every $p \in P_h$ there is a $\vec{v} \in V_{h,k}^{nc}(\Omega)$ satisfying*

$$(\operatorname{div} \vec{v}, p) = (\operatorname{div} \vec{v}, (I - \Pi_h)p) \geq C_1 ||(I - \Pi_h)p||_0^2$$

and

$$|\vec{v}|_{1,h} \leq ||(I - \Pi_h)p||_0.$$

Proof. The proof is similar to the proof of Lemma 3.2. of [26]. It follows from Lemma 3.3.2 that for every $M \in \mathcal{M}_h$ there exists $\vec{v}_M \in V_{0,M}^{nc}$ such that

$$(\operatorname{div} \vec{v}_M, (I-\Pi_h)p)_M \geq C_1 ||(I-\Pi_h)p||_{0,M}^2,$$

and

$$|\vec{v}_M|_{1,h,M} \leq ||(I-\Pi_h)p||_{0,M},$$

where $C_1 = \min_{1\leq i\leq n} \beta_{\hat{M}_i}$ and the constants $\beta_{\hat{M}_i} = \beta(\hat{M}_i, \kappa) > 0$ are the same as in Lemma 3.3.2. Define $\vec{v} \in V_{0,M}^{nc}$ by

$$\vec{v}|_M = \vec{v}_M \in V_{0,M}^{nc} \quad \forall M \in \mathcal{M}_h.$$

Then $\vec{v} = 0$ in all kth-order Gauss-Legendre points of the edges of ∂M for all $M \in \mathcal{M}_h$, and we obtain that $\vec{v} \in V_{h,k}^{nc}(\Omega)$ and that

$$(\operatorname{div} \vec{v}, \Pi_h p) = 0 \quad \forall p \in P_h. \qquad \Box$$

Since we want to prove stability for even $k \geq 2$, we are going to use Lemma 3.3 of [26] where we omit the condition referring to the linear case.

Lemma 3.3.6 *(see [26]) If the macroelement conditions are satisfied there exists a constant $C_2 > 0$ such that for every $p \in P_h$ there is a $\vec{g} \in V_{h,k}(\Omega)$ satisfying*

$$(\operatorname{div} \vec{g}, \Pi_h p) = ||\Pi_h p||_0^2 \quad \textit{and} \quad |\vec{g}|_1 \leq C_2 ||\Pi_h p||_0.$$

Proof of Theorem 3.3.4. Let $p \in P_h$ be arbitrary, let $\vec{v} \in V_{h,k}^{nc}(\Omega)$ and C_1 be as in Lemma 3.3.5, and let $\vec{g} \in V_{h,k}(\Omega)$, C_2 be as in Lemma 3.3.6. Set $\vec{z} = \vec{v} + \delta\vec{g}$ where $\delta = 2C_1(1+C_2^2)^{-1}$. Then the proof of the theorem is the same as in [26]:

$$\begin{aligned}
(\operatorname{div} \vec{z}, p) &= (\operatorname{div} \vec{v}, p) + \delta(\operatorname{div} \vec{g}, p) \\
&\geq C_1 ||(I-\Pi_h)p||_0^2 + \delta(\operatorname{div} \vec{g}, \Pi_h p) + \delta(\operatorname{div} \vec{g}, (I-\Pi_h)p) \\
&\geq C_1 ||(I-\Pi_h)p||_0^2 + \delta||\Pi_h p||_0^2 - \delta|g|_1 ||(I-\Pi_h)p||_0 \\
&\geq \left(C_1 - \frac{\delta C_2^2}{2}\right) ||(I-\Pi_h)p||_0^2 + \frac{\delta}{2}||\Pi_h p||_0^2 \\
&= C_1(1+C_2^2)^{-1}||p||_0^2
\end{aligned}$$

and

$$|z|_{1,h} \leq ||(I-\Pi_h)p||_0 + \delta C_2 ||\Pi_h p||_0 \leq C||p||_0. \qquad \Box$$

Theorem 3.3.7 *For even $k \geq 2$ the finite element family defined in* (3.1.1)–(3.1.2) *is inf-sup stable.*

Proof. It follows from Theorem 3.3.4 that it is enough to define finitely many classes of macroelements for which the macroelement conditions are satisfied. Since in the case of even k the nullspace $N_{V_{h,k}^{nc}}(\Omega)$ is one-dimensional for arbitrary polygonal domain Ω (it consists of functions which are constant on Ω, see Theorem 3.2.1) we obtain that for an arbitrary chosen macroelement class $\mathcal{E}_{\hat{M}}$ the space N_M^{nc} is one-dimensional for all $M \in \mathcal{E}_{\hat{M}}$. This means that the only restriction in the choice of the macroelement classes is condition (ii) of Definition 3.3.3.

The simplest possibility is to define only one class, such that each macroelement from this class consists of one triangle and let the reference macroelement be the reference triangle Δ_0. □

Besides the stability an another advantage of the even-order Gauss-Legendre elements is given in the following theorem, [30].

Theorem 3.3.8 *For even order Gauss-Legendre elements the discrete Crouzeix-Velte decomposition does exists.*

If the Crozeix-Velte decomposition exists the eigenvalues of the discrete Schur complement operator lie in $[0, 1]$, moreover the spectrum of the operator gives information about the dimension of the nullspace of the discrete gradient and about the discrete inf-sup constant (see in Section 1.8).

The spectrum on the unit square in the case of some elements and triangulations can be seen in Chapter 4.

3.4 Some remarks on the odd order Gauss-Legendre elements

The key objects of the stability proof of the previous section are the equivalence classes of macroelements (which are equivalent under trianglewise affine mappings, as long as regularity of the triangulation is preserved) and the property that the corresponding nullspaces in the discrete pressure space are one-dimensional.

For even k the presence of the non-conforming bubble function ensures that the nullspace is one-dimensional, but for odd k there does not exists a non-conforming bubble defined over a single triangle (see Remark 3.1.14).

Let k be an arbitrary odd number, $q \in N_{V_{h,k}^{nc}}(\Omega)$ and let $\Delta \in \mathcal{T}_h$ be a triangle. Consider the velocities $\vec{v}_{\Delta,0} \in V_{h,k}^{nc}(\Omega)$ which take nonzero values only inside of Δ. Then $\vec{v}_{\Delta,0}$ has the form

$$\begin{pmatrix} B_{c,\Delta}^{(3)} q_i^{(k-3)} \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ B_{c,\Delta}^{(3)} q_i^{(k-3)} \end{pmatrix}, \quad i = 1, \ldots, \frac{(k-2)(k-1)}{2}, \tag{3.4.1}$$

where $B^{(3)}_{c,\Delta}q_i^{(k-3)}$ are the functions described by (3.1.5). Since the functions (3.4.1) are the same as the functions taking nonzero values only inside of Δ in the case of the conforming velocity space, from Theorem 2.1.8 we obtain that for $k \geq 5$ the function $q|_\Delta$ is in the space spanned by

$$q|_{\Delta,i} = \mathsf{P}^{(0,2)}_{k-1}(1-2\lambda_i), \quad i = 1,2,3, \quad q|_{\Delta,4} \equiv 1.$$

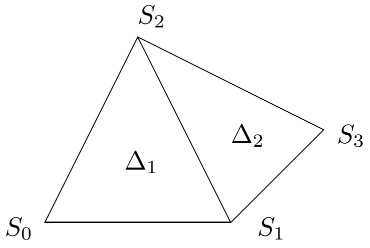
Here λ_i, $i = 1,2,3$, are the barycentric coordinates in Δ.

For odd k the boundary conditions and Theorem 3.1.2 imply that if $\Omega = \Delta$ and the triangulation $\mathcal{T}_h$ contains only one triangle, then the space $V^{nc}_{h,k}(\Omega)$ coincides with the conforming velocity space $V_{h,k}(\Omega)$. Then the nullspace of the discrete gradient is 4 dimensional.

If $k = 3$ (Theorem 2.1.8 is valid only for $k \geq 4$) and $\Omega = \Delta$, then it follows from (3.4.1) that the space $V^{nc}_{h,k}(\Omega)$ is 2 dimensional, while $\dim P_h = \dim \mathbb{P}_2(\Delta) = 6$, which means that the dimension of the nullspace is at least 4.

Since for odd k there exist bubble functions defined over two adjacent triangles (see Remark 3.1.14) it is natural to ask whether in the proof of stability the macroelements can be chosen as two adjacent triangles.

Let $\Omega = \Delta_1 \cup \Delta_2$, where Δ_1 and Δ_2 are two adjacent triangles (see the figure). We may assume that $S_0 = (0,0)$, and denote by (a_i, b_i) the coordinates of S_i, $i = 1,2,3$.



Lemma 3.4.1 *The function* q *defined by*

$$q|_{\Delta_i} = \mathsf{P}^{(0,2)}_{k-1}(1-2\lambda_3^{(i)}), \qquad i = 1,2,$$

lies in the nullspace $N_{V^{nc}_{h,k}}(\Omega)$*. Here* $\lambda_3^{(i)}$ *is that barycentric coordinate in* Δ_i *which is equal to 0 on the common side of the triangles.*

Proof. From the discussion above follows that we have to prove $b(\vec{v}, q) = 0$ only for the functions $\vec{v}$ which take nonzero values on the edge S_1S_2. Let $\vec{v}$ be an element of the basis of $V^{nc}_{h,k}(\Omega)$, which corresponds to one of the Gauss points on the common side of the triangles (see Theorem 3.1.6), i.e. on the side S_1S_2 it takes zero in all Gauss points but one. Then $\vec{v} = \binom{v}{0}$ or $\vec{v} = \binom{0}{v}$, where (see Remark 3.1.14)

$$v|_{\Delta_1} = \alpha \mathsf{P}_k^{(0,0)}(1 - 2\lambda_3^{(1)}) + \lambda_1^{(1)}\lambda_2^{(1)} g_1^{(k-2)}, \quad v|_{\Delta_2} = \alpha \mathsf{P}_k^{(0,0)}(1 - 2\lambda_3^{(2)}) + \lambda_1^{(2)}\lambda_2^{(2)} g_2^{(k-2)},$$

and functions $g_1^{(k-2)}$ and $g_2^{(k-2)}$ are appropriate polynomials of order $k-2$. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_2 - a_3 & a_1 - a_3 \\ b_2 - b_3 & b_1 - b_3 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}, \tag{3.4.2}$$

be the affine transformations which maps the reference triangle Δ_0 onto Δ_1 and Δ_2, respectively. If $\vec{v}$ has the form $\binom{v}{0}$, then using (3.2.2) we obtain

$$\begin{aligned}
b(\vec{v}, q) &= \int\limits_{\Delta_1 \cup \Delta_2} \frac{\partial v}{\partial x} q \, \mathrm{d}x \, \mathrm{d}y \\
&= b_2 \int\limits_{\Delta_0} \left(\alpha(k+1)\mathsf{P}_{k-1}^{(1,1)}(1 - 2\lambda_3) + \lambda_2 g_{0,1}^{(k-2)} + \lambda_1\lambda_2 \frac{\partial g_{0,1}^{(k-2)}}{\partial \xi} \right) \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3) \, \mathrm{d}\xi \, \mathrm{d}\eta \\
&- b_1 \int\limits_{\Delta_0} \left(\alpha(k+1)\mathsf{P}_{k-1}^{(1,1)}(1 - 2\lambda_3) + \lambda_1 g_{0,1}^{(k-2)} + \lambda_1\lambda_2 \frac{\partial g_{0,1}^{(k-2)}}{\partial \eta} \right) \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3) \, \mathrm{d}\xi \, \mathrm{d}\eta \\
&+ (b_1 - b_3) \int\limits_{\Delta_0} \left(\alpha(k+1)\mathsf{P}_{k-1}^{(1,1)}(1 - 2\lambda_3) + \lambda_2 g_{0,2}^{(k-2)} + \lambda_1\lambda_2 \frac{\partial g_{0,2}^{(k-2)}}{\partial \xi} \right) \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3) \, \mathrm{d}\xi \, \mathrm{d}\eta \\
&- (b_2 - b_3) \int\limits_{\Delta_0} \left(\alpha(k+1)\mathsf{P}_{k-1}^{(1,1)}(1 - 2\lambda_3) + \lambda_1 g_{0,2}^{(k-2)} + \lambda_1\lambda_2 \frac{\partial g_{0,2}^{(k-2)}}{\partial \eta} \right) \mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3) \, \mathrm{d}\xi \, \mathrm{d}\eta,
\end{aligned}$$

where $g_{0,1}^{(k-2)} = g_1^{(k-2)} \circ F_1^{-1}$ and $g_{0,2}^{(k-2)} = g_2^{(k-2)} \circ F_2^{-1}$ with the affine mappings F_1 and F_2 described by (3.4.2). Since the polynomial $\mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3)$ is orthogonal to $\mathbb{P}_{k-2}(\Delta_0)$ with respect to the weight functions λ_1 and λ_2 we obtain that $b(\vec{v}, q)$ is equal to

$$\alpha(k+1)\left(b_2 - b_1 + (b_1 - b_3) - (b_2 - b_3)\right) \int_{\Delta_0} \mathsf{P}_{k-1}^{(1,1)}(1 - 2\lambda_3)\mathsf{P}_{k-1}^{(0,2)}(1 - 2\lambda_3) \, \mathrm{d}\xi \, \mathrm{d}\eta = 0.$$

Similarly, $b(\vec{v}, q) = 0$ holds for $\vec{v} = \binom{0}{v}$. □

Lemma 3.4.1 implies that we can not choose the macroelement as two adjacent triangles.

In the case of $k = 3$ Crouzeix and Falk proved (see [10]) the stability for certain triangulations. They used macroelements consist of four triangles and at the end of their paper they announced that they think the stability of the element for any triangulation of a convex polygon.

Theorem 3.4.2 *(See [30, 31]) For odd-order Gauss-Legendre elements there is no Crouzeix-Velte decomposition.*

The spectrum of the discrete Schur complement operator in the case of the first-order Crouzeix-Raviart element (see in Section 1.6) on the unit square using a standard triangulation is shown in Chapter 4. In this case the eigenvalues are in $[0, 2)$: the Crouzeix-Velte decomposition does not exist.

Chapter 4

Numerical results

4.1 Eigenvalues of the discrete Schur complement operator

In this section we show some results of Matlab computations where the discrete Schur complement of the Stokes problem (defined by (1.8.4)) is computed. The zero eigenvalues of the Schur complement operator S_h correspond to the eigenvectors $p_h \in \ker B^T$, i.e. the number of the zero eigenvalues gives the dimension of the nullspace of the discrete gradient operator. The square root of the smallest nonzero eigenvalue of S_h is the discrete inf-sup constant (see Section 1.8).
Let $\Omega = [0,1]^2$ and consider homogeneous boundary conditions. In what follows we use the notation below

- k for the degree of the velocity elements,
- n for the number of intervals into which $[0,1]$ is subdivided,
- v for the dimension of the velocity space V_h,
- p for the dimension of the pressure space P_h,
- σ for the number of the singular points of the triangulation,
- λ_i, $i = 1, \ldots, v$, for the eigenvalues of S_h (in increasing order).

On Figure 4.1 one can see the eigenvalues of S_h in the case of fourth-order Scott-Vogelius elements and fourth-order Gauss-Legendre elements using the standard triangulation with $n = 6$. Here $\sigma = 2$ singular points are present and $v = 1250$, $p = 720$, $\lambda_1 = \lambda_2 = \lambda_3 = 0,\ \lambda_4 = 0.025975$ for the Scott-Vogelius elements and $v = 1394$, $p = 720$, $\lambda_1 = 0, \lambda_2 = 0.056153$ for the Gauss-Legendre elements.

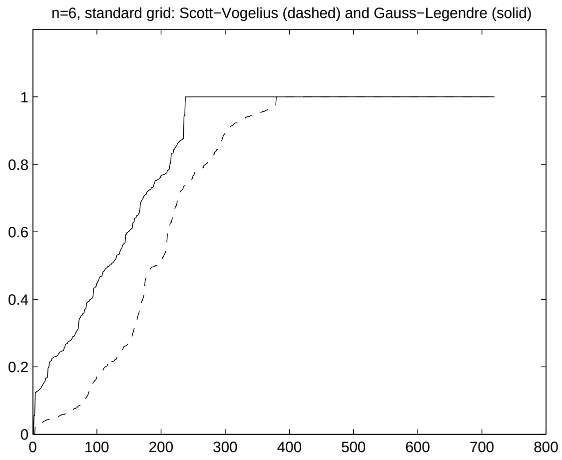


Figure 4.1: Eigenvalues of S_h on the standard triangulation of the unit square, n=6. Lower curve: $\mathbb{P}_4^2 - \mathbb{P}_3$ (Scott-Vogelius element), here $\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 0.025975$, upper curve: $(\mathbb{P}_4 + B_n^{(4)})^2 - P_3$ (Gauss-Legendre element): $\lambda_2 = 0.056153$.

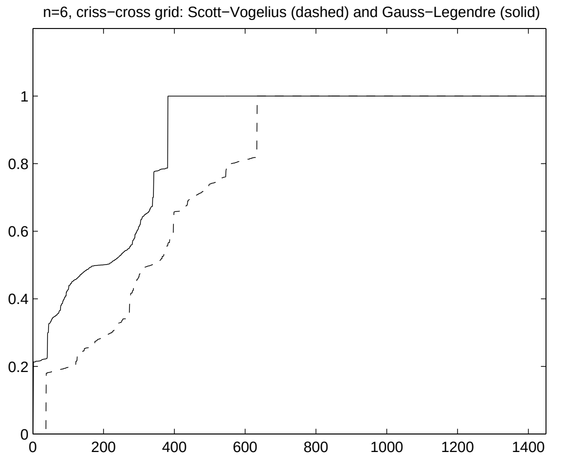


Figure 4.2: Eigenvalues of S_h on the criss-cross triangulation of the unit square, n=6. Lower curve: $\mathbb{P}_4^2 - \mathbb{P}_3$ (Scott-Vogelius element), here $\lambda_1 = \ldots = \lambda_{37} = 0$, $\lambda_{38} = 0.179739$, upper curve: $(\mathbb{P}_4 + B_n^{(4)})^2 - \mathbb{P}_3$ (Gauss-Legendre element): $\lambda_2 = 0.212876$

	Gauss-Legendre		Scott-Vogelius		
n	v	λ_2	v	$\lambda_{\sigma+2}$	$\sigma+2$
2	258	0.212708	226	0.178406	6
3	602	0.213040	530	0.179234	11
4	1090	0.213073	962	0.179562	18
5	1722	0.213091	1522	0.179690	27
6	2498	0.212876	2210	0.179739	38
7	3418	0.211434	3026	0.179757	51
8	4482	0.210255	3970	0.179764	66
9	5690	0.209270	5042	0.179766	83
10	7042	0.208431	6242	0.179767	102

Table 4.1: The first positive eigenvalue of S_h

Moreover, in Table 4.1 we present some values (rounded to 6 digits) of the first positive eigenvalue of S_h for the same elements on the unit square using the criss-cross grid. Here $v = 64n^2 - 16n + 2$ and $v = 72n^2 - 16n + 2$ in the case of Scott-Vogelius and Gauss-Legendre elements, respectively. For the whole spectrum of S_h for $n = 6$, see Figure 4.2.

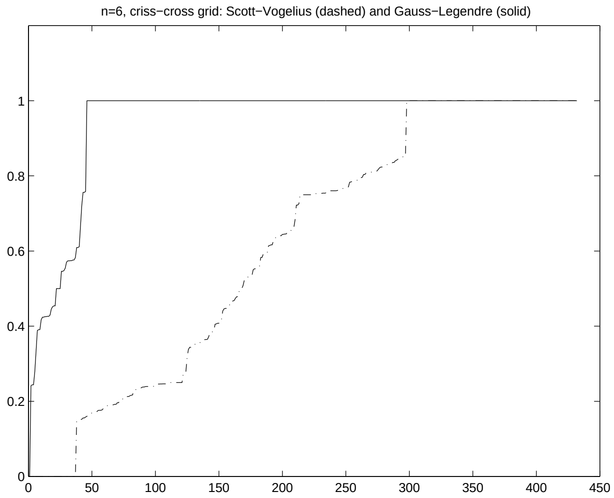


Figure 4.3: Eigenvalues of S_h on the criss-cross triangulation of the unit square, n=6. Lower curve: $\mathbb{P}_2^2 - \mathbb{P}_1$ (Scott-Vogelius element), here $\lambda_1 = \ldots = \lambda_{37} = 0$, $\lambda_{38} = 0.1478315$, upper curve: $(\mathbb{P}_2 + B_n^{(2)})^2 - \mathbb{P}_1$ (Gauss-Legendre element): $\lambda_2 = 0.2414476$

In the case of the second order Scott-Vogelius and Gauss-Legendre (i.e. Fortin-Soulie) elements the eigenvalues of S_h can be seen on Figure 4.3. Here we used the criss-cross triangulation of the unit square for $n = 6$. Then $\sigma = 36$ and $v = 530$, $p = 432$ in the conforming case, while $v = 818$, $p = 432$ in the non-conforming case. In all of the previous cases the eigenvalues lie in the $[0, 1]$ interval which is a consequence of the existence of the Crouzeix-Velte decomposition (see Section 1.8).

Figure 4.4 shows the spectrum of the first-order Crouzeix-Raviart element (see Section 1.6) on the unit square using the standard triangulation with $n = 20$. In this case $v = 2n(3n - 2) = 2320$, while $p = 2n^2 = 800$. The first nonzero eigenvalue is equal to 0.270649 and the eigenvalues are in $[0, 2)$, so the element does not have Crouzeix-Velte decomposition (see [31]).

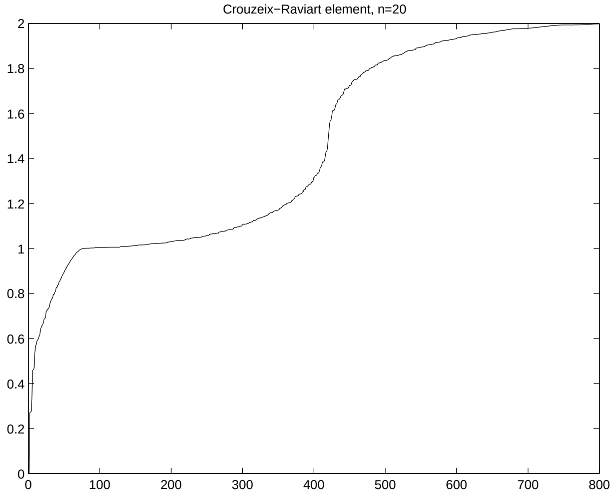


Figure 4.4: Eigenvalues of S_h in the case of first order Crouzeix-Raviart element.

4.2 Solution of a test problem

Let $\Omega = [0, 1]^2$ and consider the Stokes equations in the case of

$$\begin{aligned} u(x, y) &= (\sin x \sin y, \cos x \cos y)^T, \\ p(x, y) &= 2 \cos x \sin x - 2 \sin(1)(1 - \cos(1)), \\ f(x, y) &= (0, 4 \cos x \cos y)^T. \end{aligned}$$

This problem is described in [4] and also investigated in [18].

For the solution of the Stokes equations we used the criss-cross triangulation of the unit square and fourth-order Gauss-Legendre and Scott-Vogelius elements. Numerical calculations were performed using Matlab.

We solved the linear equation by Uzawa algorithm with conjugate directions (see [3, Chapter IV]).

In Tables 4.2 and 4.3 we present the number of iteration steps and the errors of the discrete velocity and pressure, where the norm $||\cdot||_{(0,h)}$ is defined on $\mathbb{R}^N$ as

$$||x||^2_{(0,h)} := h^2 \sum_{i=1}^{N} x_i^2, \qquad x = (x_1, \ldots, x_N)^T,$$

and $h = 1/n$.

n	$\|\|u_h - u\|\|_{(0,h)}$	$\|\|p_h - p\|\|_{(0,h)}$	steps
2	$1.82630613 \cdot 10^{-6}$	$2.459486 \cdot 10^{-2}$	76
4	$7.37690783 \cdot 10^{-8}$	$5.930648 \cdot 10^{-3}$	97
8	$3.83938455 \cdot 10^{-9}$	$1.145399 \cdot 10^{-3}$	98
16	$2.26807489 \cdot 10^{-10}$	$3.598438 \cdot 10^{-4}$	98
32	$1.38766900 \cdot 10^{-11}$	$8.950061 \cdot 10^{-5}$	99

Table 4.2: Gauss-Legendre elements.

n	$\|\|u_h - u\|\|_{(0,h)}$	$\|\|p_h - p\|\|_{(0,h)}$	steps
2	$1.30900265 \cdot 10^{-6}$	$2.459450 \cdot 10^{-2}$	92
4	$4.14507706 \cdot 10^{-8}$	$5.930621 \cdot 10^{-3}$	110
8	$1.30255577 \cdot 10^{-9}$	$1.145398 \cdot 10^{-3}$	119
16	$4.08131908 \cdot 10^{-11}$	$3.598437 \cdot 10^{-4}$	121
32	$1.28732502 \cdot 10^{-12}$	$8.950061 \cdot 10^{-5}$	122

Table 4.3: Scott-Vogelius elements.

Summary

In the present work we deal with a high-order non-conforming finite element family for the two dimensional Stokes equations.

Let $\{\vec{v}_i\}_{i=1}^N$ and $\{q_i\}_{i=1}^M$ be bases of the given finite element spaces V_h (velocities) and P_h (pressures), respectively. The matrix form of the discrete Stokes equations is

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} f_h \\ 0 \end{pmatrix},$$

where

$$\begin{aligned} A_h &:= (a_{ij})_{i,j=1}^N, & a_{ij} &:= a(\vec{v}_i, \vec{v}_j), \\ B_h &:= (b_{ij})_{i,j=1}^{N,M}, & b_{ij} &:= b(\vec{v}_j, q_i), \\ f_h &:= (f_i)_{i=1}^N, & f_i &:= (\vec{f}, \vec{v}_i)_0, \\ \vec{u}_h &:= \sum_{i=1}^N y_i \vec{v}_i, & p_h &:= \sum_{i=1}^M z_i q_i, \\ \underline{u} &:= (y_1, \ldots, y_N)^T, & \underline{p} &:= (z_1, \ldots, z_M)^T, \end{aligned}$$

and the bilinear functionals $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined as

$$\begin{aligned} a(\vec{u}, \vec{v}) &:= \sum_{\Delta \in \mathcal{T}_h} \int_\Delta \sum_{i=1}^2 \sum_{j=1}^2 \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x, \\ b(\vec{v}, p) &:= - \sum_{\Delta \in \mathcal{T}_h} \int_\Delta p \operatorname{div} \vec{v} \, \mathrm{d}x. \end{aligned}$$

To prove stability of the finite element pair (V_h, P_h) we have to check the inf-sup condition

$$\sup_{\vec{v}_h \in V_h} \frac{b(\vec{v}_h, q_h)}{|\vec{v}|_1} \geq \beta_h \inf_{q_{0h} \in \ker B_h^T} ||q_h + q_{0h}||_{L^2(\Omega)} = \beta_h ||q_h||_{L^2(\Omega)/\ker B_h^T} \quad \forall q_h \in P_h,$$

where $\beta_h \geq \beta_0 > 0$ with a constant β_0 independent of h.

First we investigate the Scott-Vogelius finite element family where velocities and pressures are approximated trianglewise by polynomials of order k and $(k-1)$, respectively. Moreover, the discrete velocities are assumed to be continuous, but there is no continuity requirement on the pressure:

$$P_h(\Omega) := \left\{ p \in L^2(\Omega) \;\;:\;\; p|_\Delta \in \mathbb{P}_{k-1}(\Delta),\;\; \Delta \in \mathcal{T}_h \right\}, \tag{5.1.1}$$

$$V_{h,k}(\Omega) := \left\{ \vec{v} \in (H_0^1(\Omega))^2 \;\;:\;\; \vec{v}|_\Delta \in (\mathbb{P}_k(\Delta))^2,\;\; \Delta \in \mathcal{T}_h \right\}. \tag{5.1.2}$$

For $k \geq 4$ Scott and Vogelius [24] proved the stability of the finite element pair (2.1.1)–(2.1.2) under a condition connected with the grid singularity. A vertex of the triangulation is called a singular point if the edges meeting at this vertex lie on two straight lines. In [24] the authors showed that in the case of $k \geq 4$ if the number of the singular points in the triangulation is equal to σ then the nullspace of the discrete gradient operator is $\sigma + 1$ dimensional. Moreover, for $k \geq 4$ the finite element pair (2.1.1)–(2.1.2) is inf-sup stable if the triangulation does not contain near-singular points. In the case when near-singular points approach singular points, the right inverse of the divergence operator is blowing up.

The aim of the present dissertation is to describe a finite element family that is – similarly to the Scott-Vogelius elements – defined for arbitrary order k, moreover it is inf-sup stable without a restriction on the grid.

In Chapter 2 we investigate the nullspace of the discrete gradient operator in the case of the Scott-Vogelius elements (in [25] just the dimension of this space is given). We show that if the triangulation of a polygonal domain Ω contains σ singular points, then for $k \geq 4$ there exists a basis of the nullspace which can be given in the following way. Besides the constant function, to each singular point corresponds a function which is zero everywhere except on the triangles around the given point.

In the description of the basis we use orthogonal polynomials defined on triangles (see in [17]).

At the end of this chapter we give some examples where in the case of $k = 2, 3$ the dimension of the nullspace can be greater as $\sigma + 1$.

In Chapter 3 we define the Gauss-Legendre finite element family for arbitrary order k. In this case the velocities are assumed to be continuous on the common side of two adjacent triangles only in certain points that are the roots of the kth-order Legendre polynomial given on this side. The discrete spaces are

$$P_h(\Omega) := \left\{ p \in L^2(\Omega),\;\; p|_\Delta \in \mathbb{P}_{k-1}(\Delta),\;\; \Delta \in \mathcal{T}_h \right\}, \tag{5.1.3}$$

$$\begin{aligned} V_{h,k}^{nc}(\Omega) := \big\{ &\vec{v} \in (L^2(\Omega))^2,\;\; \vec{v}|_\Delta \in (\mathbb{P}_k(\Delta))^2,\;\; \text{and } \vec{v} \text{ is continuous in all} \\ &k\text{th-order Gauss-Legendre points of all sides of } \Delta,\;\; \Delta \in \mathcal{T}_h,\;\; \vec{v} = 0 \;\text{ in} \\ &\text{all } k\text{th-order Gauss-Leg. points on the triangle sides } \; E \subset \Gamma \big\}, \end{aligned} \tag{5.1.4}$$

while the norm in $V_{h,k}^{nc}(\Omega)$ is defined as

$$|\vec{v}_h|_{1,h,\Omega} := \left(\sum_{\Delta \in \mathcal{T}_h} |\vec{v}_h|_{1,\Delta}^2 \right)^{1/2} .$$

Since $V_{h,k}^{nc}(\Omega) \not\subset (H^1(\Omega))^2$, the approximation is non-conforming.

In Subsection 3.1 we specify suitable degrees of freedom for the velocity parts of these elements. Let Δ be a given triangle and consider the kth-order Gauss-Legendre points on the sides of Δ together with the $\frac{(k-2)(k-1)}{2}$ inner points having barycentric coordinates

$$\lambda_1 = \frac{j}{k}, \lambda_2 = \frac{\ell}{k}, \lambda_3 = \frac{m}{k},\ \ 1 \le j, \ell, m \le k-2,\ \ j + \ell + m = k.$$

The nullspace of the interpolation problem of finding a kth-order polynomial which takes prescribed values in these points is trivial if k is odd and one-dimensional if k is even. For odd values of k we can use the points above as degrees of freedom, while for even k there exist ("bubble"-) functions which take zero values in the kth-order Gauss-Legendre points.

On the triangle Δ we define the kth-order non-conforming bubble function as

$$B_{n,\Delta}^{(k)} := \frac{1}{2} \left\{ \sum_{i=1}^{3} \mathsf{P}_k^{(0,0)}(1 - 2\lambda_i) - 1 \right\} .$$

For odd k it is equal to zero on the sides of Δ, while for even k one gets the non-conforming velocity space $V_{h,k}^{nc}(\Omega)$ by adding trianglewise a kth-order non-conforming bubble function to the conforming velocity space $V_{h,k}(\Omega)$, that is

$$V_{h,k}^{nc}(\Omega) = V_{h,k}(\Omega) + \left\{ \vec{v},\ \ \vec{v}|_\Delta = \begin{pmatrix} \alpha_\Delta \\ \beta_\Delta \end{pmatrix} B_{n,\Delta}^{(k)},\ \ \alpha_\Delta, \beta_\Delta \in \mathbb{R},\ \ \Delta \in \mathcal{T}_h \right\} .$$

In Chapter 3 for even k we prove the stability of (5.1.3)–(5.1.4) using a modification of the macroelement technique of Stenberg [26]. We define finitely many macroelement classes satisfying

(i) for each macroelement M from the classes the nullspace of the discrete gradient operator defined over M is one-dimensional, consisting of functions that are constant on M, i.e. $\dim N_M^{nc} = 1$, where

$$N_M^{nc} = \left\{ p \in P_M \ \ : \ \ \int_M p \operatorname{div} \vec{v} \, \mathrm{d}x = 0 \ \ \forall \vec{v} \in V_{0,M}^{nc} \right\} ,$$

$$\begin{aligned} V_{0,M}^{nc} = \{ \vec{v} \ \ : \ \ & \vec{v}|_\Delta \in (\mathbb{P}_k(\Delta))^2 \ \ \forall \Delta \in M, \ \text{ and } \vec{v} \text{ is continuous in all} \\ & k\text{th-order Gauss-Legendre points of all sides of } \Delta,\ \Delta \subset M,\ \ \vec{v} = 0 \ \text{ in all} \\ & k\text{th-order Gauss-Legendre points of all edges on } \partial M \} , \end{aligned}$$

(ii) for each h the triangles in $\mathcal{T}_h$ can be grouped to form macroelements such that for the so obtained macroelement partitioning each macroelement belongs to one of the macroelement classes.

If the above macroelement conditions are satisfied then for $k \geq 2$ the inf-sup condition holds for the finite element family defined in (5.1.3)–(5.1.4).

An important step in the proof of stability is the study of the nullspace. In Section 3.2 we prove that for even k by adding trianglewise a non-conforming bubble function to the space $V_{h,k}(\Omega)$ the nullspace of the discrete gradient operator becomes one-dimensional. In the proof we use that the basis of the nullspace in the conforming case has already been described in Subsection 2.1.2.

Since in the case of even k the nullspace $N_{V_{h,k}^{nc}}(\Omega)$ is one-dimensional for arbitrary polygonal domains Ω the only restriction in the choice of the macroelement classes is the macroelement condition (ii). The simplest possibility is to define only one class, such that each macroelement from this class consists of a single triangle.

In Chapter 4 we give some numerical results where for different values of k and for various triangulations in the case of Scott-Vogelius and Gauss-Legendre elements we compute the discrete inf-sup constants. Further, using fourth-order Gauss-Legendre elements we solve a test equation described in [4].

Összefoglaló

A dolgozat a két-dimenziós Stokes feladat numerikus megoldása kapcsán egy magas rendű nemkonform végeselem család vizsgálatával foglalkozik.

A Stokes feladat egy összenyomhatatlan, viszkózus folyadék lassú áramlását leíró egyenletrendszer:

$$-\boldsymbol{\vec{\Delta}}\vec{u} + \operatorname{grad} p = \vec{f}, \tag{6.1.1}$$

$$-\operatorname{div}\vec{u} = 0, \tag{6.1.2}$$

$$\vec{u}|_\Gamma = \vec{u}_0, \tag{6.1.3}$$

ahol Ω egy két-dimenziós tartomány Lipschitz folytonos határral (Ω peremét jelöli Γ), $\vec{u} : \Omega \to \mathbb{R}^2$ és $p : \Omega \to \mathbb{R}$ a keresett sebesség, illetve nyomás, $\vec{f}$ adott külső erő.

Az egyenletekből a nyomás csak egy additív konstantól eltekintve határozható meg egyértelműen, adott p esetén a sebesség vektor egyértelmű.

A feladatnak homogén peremfeltételek esetén a numerikus megoldáshoz hasznos variácós megfogalmazása a következő. Olyan $\vec{u} \in (H_0^1(\Omega)^2$ és $p \in L_0^2(\Omega)$ függvényeket keresünk, melyek kielégítik az alábbi egyenleteket:

$$a(\vec{u},\vec{v}) + b(\vec{v},p) = (\vec{f},\vec{v})_0 \quad \forall \vec{v} \in (H_0^1(\Omega))^2, \tag{6.1.4}$$

$$b(\vec{u},q) = 0 \quad \forall q \in L_0^2(\Omega). \tag{6.1.5}$$

Az $a(\cdot,\cdot)$ és $b(\cdot,\cdot)$ bilineáris funkcionálok értelmezése

$$a(\vec{u},\vec{v}) := \int_\Omega \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x, \tag{6.1.6}$$

$$b(\vec{v},p) := -\int_\Omega p \operatorname{div}\vec{v}\,\mathrm{d}x. \tag{6.1.7}$$

Hasonlóan (6.1.4)–(6.1.5) típusú egyenletekre vezet például a stacionárius hővezetést leíró egyenletrendszer.

A (6.1.4)–(6.1.5) feladat egyértelműen megoldható, mert az $a(\cdot,\cdot)$ funkcionál elliptikus és teljesül az úgy nevezett inf-sup feltétel: létezik olyan β konstans, hogy

$$\inf_{p \in L_0^2(\Omega)} \sup_{\vec{v} \in (H_0^1(\Omega))^2} \frac{b(v,p)}{|\vec{v}|_1 \cdot ||p||_0} \geq \beta > 0,$$

ahol $|\vec{v}|_1^2 = a(\vec{v},\vec{v})$.

A (6.1.4)–(6.1.5) végeselem diszkretizációjához az Ω (poligonális) tartományt háromszögekre osztjuk, a keresett sebességet és nyomást háromszögenként adott fokszámú polinomokkal közelítjük.

Egy – a dolgozatban részletesebben is vizsgált – végeselem család a Scott-Vogelius elemek. Itt a sebességet háromszögenként k-ad, a nyomást háromszögenként $k-1$-edrendű polinomokkal közelítjük, a sebesség esetén feltételezzük, hogy a polinomok a szomszédos háromszögek közös oldalán folytonosak, míg a nyomást közelítő polinomok esetén nincs ilyen feltétel. A végeselem pár tetszőleges $k \geq 1$ egész esetén definiált. A legfeljebb k-adfokú polinomok terét $\mathbb{P}_k$-val jelölve az Ω tartomány egy $\mathcal{T}_h$ triangularizációja esetén a $(H_0^1(\Omega))^2$ és az $L^2(\Omega)$ tereket approximáló V_h és P_h terek a következők:

$$V_h = V_{h,k}(\Omega) = \left\{\vec{v} \in (H_0^1(\Omega))^2 \ : \ \vec{v}|_\Delta \in (\mathbb{P}_k(\Delta))^2, \ \ \Delta \in \mathcal{T}_h\right\}, \tag{6.1.8}$$

$$P_h(\Omega) = \left\{p \in L^2(\Omega) \ : \ p|_\Delta \in \mathbb{P}_{k-1}(\Delta), \ \ \Delta \in \mathcal{T}_h\right\}. \tag{6.1.9}$$

Ha – mint a fenti esetben – a V_h, ill. P_h tér része annak a térnek, ahol az eredeti (6.1.4)–(6.1.5) problémát megfogalmaztuk (jelen esetben a $(H_0^1(\Omega))^2$, ill. $L^2(\Omega)$ térnek), akkor konform approximációról beszélünk. Ekkor a diszkrét variációs feladat: olyan $\vec{u}_h \in V_h$ és $p_h \in P_h$ függvényeket keresünk, melyekre

$$a(\vec{u}_h,\vec{v}_h)+b(\vec{v}_h,p_h) = (\vec{f},\vec{v}_h)_0 \quad \forall \vec{v}_h \in V_h, \tag{6.1.10}$$

$$b(\vec{u}_h,q_h) = 0 \quad \forall q_h \in P_h \cap L_0^2(\Omega) \tag{6.1.11}$$

teljesül. Ha $\{\vec{v}_i\}_{i=1}^N$ és $\{q_i\}_{i=1}^M$ a V_h, illetve P_h terek bázisai, akkor a diszkrét feladat mátrix alakja

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{p} \end{pmatrix} = \begin{pmatrix} f_h \\ 0 \end{pmatrix},$$

ahol

$$\begin{aligned}
A_h &:= (a_{ij})_{i,j=1}^N, & a_{ij} &:= a(\vec{v}_i,\vec{v}_j),\\
B_h &:= (b_{ij})_{i,j=1}^{N,M}, & b_{ij} &:= b(\vec{v}_j,q_i),\\
f_h &:= (f_i)_{i=1}^N, & f_i &:= (\vec{f},\vec{v}_i)_0,\\
\vec{u}_h &:= \sum_{i=1}^N y_i\vec{v}_i, & p_h &:= \sum_{i=1}^M z_i q_i,\\
\underline{u} &:= (y_1,\dots,y_N)^T, & \underline{p} &:= (z_1,\dots,z_M)^T.
\end{aligned}$$

A (6.1.10)–(6.1.11) feladat egyértelműen megoldható és a megoldás stabil, ha létezik olyan $\beta_h > 0$ konstans, hogy

$$\sup_{\vec{v}_h \in V_h} \frac{b(\vec{v}_h,q_h)}{|\vec{v}|_1} \geq \beta_h \inf_{q_{0h} \in \ker B_h^T} ||q_h + q_{0h}||_{L^2(\Omega)} = \beta_h ||q_h||_{L^2(\Omega)/\ker B_h^T}$$

minden $q_h \in P_h$ esetén, és $\beta_h \geq \beta_0 > 0$, ahol β_0 nem függ h-tól.

Amíg folytonos esetben az inf-sup feltétel mindig teljesül, ha az Ω tartomány pereme Lipschitz-folytonos [20], addig a diszkrét inf-sup feltétel teljesülését minden végeselem pár esetén meg kell vizsgálni.

Egy másik, a diszkretizáció során felmerülő esetleges probléma, hogy a diszkrét feladat nem feltétlenül örökli a (6.1.1)–(6.1.3) feladatnak azt a tulajdonságát, hogy a nyomás egy additív konstanstól eltekintve egyértelműen meghatározott. Diszkrét esetben a gradiens operátornak megfelelő B^T operátor nulltere akár több dimenziós is lehet, ami ,,energiamentes" nyomás függvények megjelenését jelenti. Tipikus példa erre a (6.1.8)–(6.1.9) véges elem pár, ahol a B^T nullterének dimenziója függ a tartomány diszkretizációjától és bizonyos triangularizációk esetén egészen magas is lehet. Ugyanis a nulltér dimenziója a rácsban jelenlévő úgy nevezett szinguláris pontok számától függ.

A triangularizáció egy csúcsát akkor nevezzük szingulárisnak, ha az ott találkozó élek két egyenesen fekszenek (belső szinguláris pontra lásd a 2.1 ábrát, a perem szinguláris pont négy típusa a 2.2 ábrán látható).

Scott és Vogelius [25] belátták, hogy ha $k \geq 4$ és a triangularizáció σ darab szinguláris pontot tartalmaz, akkor a B^T diszkrét gradiens operátor nulltere $\sigma + 1$ dimenziós. Az általuk adott bizonyítás kombinatorikus, csak a nulltér dimenziójáról ad információt, a nulltér leírását nem tartalmazza.

A szerzők bebizonyították (lásd [25]), hogy a (6.1.8)–(6.1.9) elem $k \geq 4$ esetén stabil egy – szintén a szinguláris pontokkal kapcsolatos – rácsfeltétel teljesülése esetén. A stabilitás feltétele, hogy a triangularizáció ne tartalmazzon közel szinguláris pontokat, azaz ha x_0 egy nem szinguláris pont és Δ_i, $i = 1, \ldots, n \leq 4$ az x_0 pontot tartalmazó háromszögek, melyeknek az x_0-nál lévő szögei Θ_i, $i = 1, \ldots, n$ (a háromszögeket sorban számozva), akkor az

$$R(x_0) := \max\{|\theta_i + \theta_j - \pi|, \quad \text{ahol} \quad 1 \leq i, j \leq n, \quad i - j = 1 \mod n\}$$

függvénnyel teljesüljön

$$\min\{R(x_0) \ : \ x_0 \text{ egy nem szinguláris belső pont } \} \geq \delta > 0,$$

ahol δ nem függ h-tól.

Ha a h diszkretizációs paraméter csökkenésével a közel szinguláris pont tart a szinguláris helyzethez, akkor a stabilitás nem teljesül.

Szinguláris, vagy közel szinguláris pontokkal találkozhatunk akár standard rácsgeneráló programok által készített rácsok esetén is, lásd például a 2.4 és 2.5 ábrákat, melyek a Matlab egy automata triangularizációs függvényével készültek (a második rács az első finomítása), de az úgynevezett criss-cross rács is számos szinguláris pontot tartalmaz (lásd a 2.3 ábrát).

Sokan próbálkoztak ezt a szinguláris pontokkal kapcsolatos rácsfeltételt kiküszöbölni. Qin [21] a triangularizáció minden háromszögét a súlypont felhasználásával három kisebb háromszögre bontotta és az így kapott makroelemeken definiált polinomokkal közelítette a megoldást.

Több munka kiindulópontja a Fortin és Soulie [13] által definiált másodrendű végeselem pár volt. A szerzők ott a $V_h = V_{h,2}$ tér bővítésével biztosították a szinguláris pontoktól független stabilitást, az általuk definiált végeselem pár:

$$\begin{aligned} P_h(\Omega) &= \{p \in L_2(\Omega),\ \ p|_\Delta \in \mathbb{P}_1(\Delta),\ \ \Delta \in \mathcal{T}_h\} \qquad (6.1.12)\\ V_{h,2}^{nc}(\Omega) &= \big\{\vec{v} \in (L_2(\Omega))^2,\ \ \vec{v}|_\Delta \in (\mathbb{P}_2(\Delta))^2,\ \ \text{és } \vec{v} \text{ folytonos két szomszédos} \\ &\qquad \text{háromszög közös oldalán a másodrendű Gauss-Legendre pontokban,} \\ &\qquad \vec{v} = 0 \text{ a } \Gamma\text{-n fekvő oldalak mśodrendű Gauss-Legendre pontjaiban}\big\}. \qquad (6.1.13)\end{aligned}$$

Itt tehát a diszkrét sebességek folytonosságát két szomszédos háromszög közös oldalán csak bizonyos pontokban követeljük meg: a másodfokú Legendre polinom zérushelyein. Ekkor a $V_{h,2}^{nc}$ tér nem része a $(H^1(\Omega))^2$ térnek, a véges elem nem konform. Ebben az esetben az $a(\cdot,\cdot)$ és $b(\cdot,\cdot)$ funkcionálokat a következő módon definiáljuk:

$$a(\vec{u},\vec{v}) := \sum_{\Delta\in\mathcal{T}_h} \int_\Delta \sum_{i=1}^{2}\sum_{j=1}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x, \qquad (6.1.14)$$

$$b(\vec{v},p) := -\sum_{\Delta\in\mathcal{T}_h} \int_\Delta p \operatorname{div}\vec{v}\,\mathrm{d}x, \qquad (6.1.15)$$

míg a $|\cdot|_{1,h}$ szemi-norma

$$|\vec{v}|_{1,h} := \sqrt{a(\vec{v},\vec{v})}. \qquad (6.1.16)$$

A $V_{h,2}^{nc}(\Omega)$ tér a $V_{h,2}$ konform térből származtatható oly módon, hogy a $V_{h,2}$ tér lokális bázisához háromszögenként hozzáadunk egy másodfokú polinomot (egy úgy nevezett buborék függvényt), amely a háromszög oldalain a másodrendű Gauss-Legendre pontokban eltűnik:

$$V_{h,2}^{nc}(\Omega) = V_{h,2}(\Omega) + \left\{\vec{v},\ \ \vec{v}|_\Delta = \begin{pmatrix}\alpha_\Delta\\ \beta_\Delta\end{pmatrix} b_\Delta,\ \ \alpha_\Delta,\beta_\Delta \in \mathbb{R},\ \ \Delta\in\mathcal{T}_h\right\},$$

ahol λ_i-vel, $i = 1,2,3$, jelölve a Δ háromszögben a baricentrikus koordinátákat

$$b_\Delta := 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 2.$$

Matthies és Tobiska [18] egy tetszőleges k rend esetén definiált nemkonform elem családot írt le, de a szerzők a k-adrendű konform teret háromszögenként egy $k+1$-edfokú buborék függvénnyel bővítették. Az így kapott véges elem a rács szingularitásától függetlenül stabil.

Y. Cha, M. Lee és S. Lee [7] a $k = 4, 6$ esetet vizsgálta, itt a konform térhez adott buborék függvény fokszáma 4, ill. 6, de a stabilitás bizonyításához a $b(\cdot,\cdot)$ bilineáris formát egy stabilizáló taggal egészítették ki, amely egy alkalmasan megválasztott paramétert tartalmaz.

Jelen dolgozat témája egy nemkonform végeselem család leírása, mely tetszőleges k rend esetén definiált és páros k esetén a rács szingularitásától függetlenül stabil.

A (6.1.12)–(6.1.13) elemhez hasonlóan a k-adrendű Gauss-Legendre elemet a következő módon definiáljuk:

$$P_h(\Omega) = \{p \in L_2(\Omega),\ \ p|_\Delta \in \mathbb{P}_{k-1}(\Delta),\ \ \Delta \in \mathcal{T}_h\} \tag{6.1.17}$$

$$\begin{aligned} V_{h,k}^{nc}(\Omega) = \big\{ & \vec{v} \in (L_2(\Omega))^2,\ \ \vec{v}|_\Delta \in (\mathbb{P}_k(\Delta))^2,\ \ \text{és } \vec{v} \text{ folytonos két szomszédos} \\ & \text{háromszög közös oldalán a } k\text{-adrendű Gauss-Legendre pontokban,} \\ & \vec{v} = 0 \text{ a } \Gamma\text{-n fekvő oldalak } k\text{-adrendű Gauss-Legendre pontjaiban} \big\}. \end{aligned} \tag{6.1.18}$$

Ez a család jól ismert alacsony rendű véges elemek általánosítása, $k = 1$, $k = 2$, $k = 3$ esetén (6.1.17)–(6.1.18) rendre a Crouzeix-Raviart [11], Fortin-Soulie [13] és Crouzeix-Falk [10] elemeket definiálja.

Az $a(\cdot,\cdot)$ és $b(\cdot,\cdot)$ bilineáris formákat és a $|\cdot|_{1,h}$ szemi-normát itt is az (6.1.14)–(6.1.16) képletekkel értelmezzük. Mivel a sebességek folytonosságát két szomszédos háromszög közös oldalán a k-adrendű Gauss-Legendre pontokban követeljük meg, az úgy nevezett ,,patch-feltétel" teljesül és (6.1.16) normát definiál $V_{h,k}^{nc}(\Omega)$-n [11].

A Gauss-Legendre elemek vizsgálatával a dolgozat 3. fejezete foglalkozik. A 3.1 alfejezetben az elem részletes leírása szerepel, itt először a $V_{h,k}^{nc}$ szabadsági fokait vizsgáljuk. Egy adott Δ háromszög esetén tekintsük a háromszög oldalain a k-adrendű Gauss-Legendre pontokat, a háromszög belsejében pedig a

$$\lambda_1 = \frac{j}{k}, \lambda_2 = \frac{\ell}{k}, \lambda_3 = \frac{m}{k},\ \ 1 \le j, \ell, m \le k-2,\ \ j + \ell + m = k,$$

baricetrikus koordinátákkal leírt pontokat. Ha olyan $p \in \mathbb{P}_k(\Delta)$ polinomot keresünk, mely az így definiált $3k + \frac{(k-2)(k-1)}{2} = \frac{(k+1)(k+2)}{2}$ darab pontban megadott értékeket vesz fel, a keletkező interpolációs feladat nulltere triviális, ha k páratlan és egydimenziós, ha k páros. Ezek alapján páratlan k esetén a fenti pontokat választhatjuk szabadsági fokoknak, páros k esetén viszont létezik olyan, az adott háromszög fölött definiált k-adfokú polinom, amely a háromszög peremén a k-adrendű Legendre pontokban eltűnik. A k-adrendű nemkonform buborék függvényt tetszőleges páros k esetére a következő alakban definiáltuk:

$$B_{n,\Delta}^{(k)} := \frac{1}{2}\left\{ \sum_{i=1}^{3} \mathsf{P}_k^{(0,0)}(1 - 2\lambda_i) - 1 \right\}, \tag{6.1.19}$$

ahol $\mathsf{P}_k^{(0,0)}$ jelöli a k-adfokú, $[-1, 1]$-en definiált 1 főegyütthatójú Legendre polinomot. $k = 2$ esetén ez a Fortin és Soulie által definiált másodrendű b_Δ buborék függvénnyel egyenlő, míg $k = 4$ és $k = 6$ esetén a Y. Cha, M. Lee és S. Lee [7] által használt buborék függvényektől csak egy konform (a háromszög minden oldalán eltűnő) buborék tagban különbözik.

Beláttuk, hogy páratlan k esetén ha egy polinom értéke a k-adrendű Gauss-Legendre pontokban nulla, akkor a polinom a háromszög teljes peremén eltűnik.

Páros rendű elemek esetén (hasonlóan a másodrendű elemhez) a $V_{h,k}^{nc}(\Omega)$ teret a konform $V_{h,k}$ tér alábbi bővítéseként is értelmezhetjük:

$$V_{h,k}^{nc}(\Omega) = V_{h,k}(\Omega) + \left\{ \vec{v},\ \ \vec{v}|_\Delta = \begin{pmatrix} \alpha_\Delta \\ \beta_\Delta \end{pmatrix} B_{n,\Delta}^{(k)},\ \ \alpha_\Delta, \beta_\Delta \in \mathbb{R},\ \ \Delta \in \mathcal{T}_h \right\}.$$

A (6.1.17)–(6.1.18) elem stabilitását páros k esetén bizonyítottuk, ehhez a 3.3 alfejezetben a konform elemekre alkalmazható Stenberg-féle makroelem módszert (lásd [26]) módosítottuk nemkonform elemek esetére. A módszer lényege, hogy véges sok olyan makroelem osztályt definiálunk, hogy

(i) az osztályok mindegyikére teljesül, hogy az abba tartozó M makroelemek fölött a diszkrét gradiens operátor nulltere egy-dimenziós (csak a konstans függvényeket tartalmazza), azaz $\dim N_M^{nc} = 1$, ahol

$$\begin{aligned} N_M^{nc} &= \left\{ p \in P_h \ \ : \ \ \int_M p \operatorname{div} \vec{v} \,\mathrm{d}x = 0 \ \ \forall \vec{v} \in V_{0,M}^{nc} \right\}, \\ V_{0,M}^{nc} &= \big\{ \vec{v} \ \ : \ \ \vec{v}|_\Delta \in (\mathbb{P}_k(\Delta))^2 \ \ \forall \Delta \in M, \ \ \text{és } \vec{v} \text{ folytonos minden } k\text{-adrendű} \\ &\qquad \text{Gauss-Legendre pontban két szomszédos háromszög közös oldalán,} \\ &\qquad \vec{v} = 0 \ \text{ a } k\text{-adrendű Gauss-Leg. pontokban a } \partial M\text{-hez tartozó oldalakon} \big\}, \end{aligned}$$

(ii) tetszőleges h esetén a triangularizáció háromszögei összecsoportosíthatóak makroelemekké úgy, hogy a makroelemek egyértelműen osztályba sorolhatóak.

Ha a fenti két makroelem feltétel teljesül, akkor a véges elem inf-sup stabil.

A stabilitási bizonyítás egy lényeges eleme a nulltér vizsgálata, amihez előbb a 2.1.2 alfejezetben leírtuk a nullteret a konform Scott-Vogelius elem esetén minden $k \geq 4$-re. Itt felhasználva Scott és Vogelius állítását a nulltér dimenziójáról tetszőleges $k \geq 4$ esetén megadtuk a nulltér egy bázisát. Ha σ jelöli a szinguláris pontok számát, akkor a bázis a konstans függvényen kívül σ darab olyan függvényt tartalmaz, melyek mindegyike megfeleltethető valamelyik szinguláris pontnak: csak a pontot tartalmazó háromszögekben vesz fel nullától különböző értékeket.

A bázis leírásához háromszögek fölött definiált ortogonális polinomokat használtunk (lásd [17]), alapvető szerepet játszott a $(0,2)$ paraméterű $(k-1)$-edrendű Jacobi-polinom.

A 2.1.2 alfejezet végén néhány példával szemléltettük, hogy a Scott-Vogelius elemek esetén ha $k = 2, 3$ a nulltér dimenziója nagyobb is lehet, mint $\sigma + 1$.

A 3.2 alfejezetben – felhasználva a 2.1.2 alfejezetben leírt bázist – megmutattuk, hogy páros $k \geq 4$ esetén a nemkonform buborék függvénnyel bővítve a konform teret a nulltér egy-dimenzióssá válik. A $k = 2$ esetre a bizonyítás (mivel itt a konform elemhez tartozó nulltér leírása nem ismert) más módszerrel a 2.2 alfejezetben található. Így beláttuk, hogy tetszőleges páros k-ra a diszkrét gradiens operátor nulltere egy-dimenziós a triangularizációtól és lényegében az Ω tartománytól függetlenül, ezért a stabilitási bizonyításban a makroelem osztályok definiálásánál csak az (ii) makroelem feltétel teljesülésére kell figyelnünk. A legegyszerűbb eset, ha csupán egyetlen osztályt definiálunk –az ebbe az osztályba tartozó makroelemek egyetlen háromszögből állnak, így az (ii) feltétel triviálisan teljesül.

A páros rendű Gauss-Legendre elemek stabilitásából $k = 2$ esetén következik a (6.1.12)–(6.1.13) elem stabilitása is. Fortin és Soulie [13] munkájukban ugyan állítják a másodrendű elem stabilitását, ennek bizonyítását azonban nem közölték.

A Gauss-Legendre elemek stabilitását csak páros rend esetén sikerült bizonyítani. Páratlan k esetén nem létezik olyan nemkonform buborék függvény, amely a háromszög oldalain csak a Gauss-Legendre pontokban egyenlő nullával – ennek a függvénynek lényeges szerepe van abban, hogy páros k esetén a diszkrét gradiens operátor nulltere egy-dimenziós. A 3.1 alfejezetben a páratlan k rendű Gauss-Legendre elemek leírásánál két szomszédos (közös oldallal rendelkező) háromszög fölött olyan k-adfokú polinomokat definiáltunk, melyek a két háromszög alkotta négyszög külső oldalán a k-adrendű Gauss-Legendre pontokban eltűnnek. Természetesen adódik a kérdés, hogy nem lehet-e páratlan rend esetén a stabilitást úgy bizonyítani, hogy makroelemként két szomszédos háromszög alkotta négyszögeket választunk? A 3.4 alfejezetben megmutattuk, hogy így választva a makroelemeket az (i) makroelem feltétel nem teljesül, a stabilitás így nem bizonyítható.

A $k = 3$ esetben bizonyos triangularizációk esetén a stabilitást Crouzeix és Falk [10] bizonyította, a szerzők a cikkük végén megjegyzik, hogy sejtésük szerint a stabilitás tetszőleges rács esetén igaz.

A 4. fejezetben a nemkonform véges elemekkel kapcsolatos numerikus eredményeket közlünk. Matlab programok segítségével a Scott-Vogelius és a Gauss-Legendre elemek esetén az egységnégyzet különböző triangularizációira kiszámítottuk a diszkrét inf-sup konstanst néhány k értékre (lásd [31]). A futási eredmények alátámasztják a diszkrét gradiens nullterének dimenziójára vonatkozó elméleti eredményeket is.

Szintén Matlab segítségével az egységnégyzet criss-cross triangularizációja esetén a negyedrendű Gauss-Legendre elemet használva megoldottunk egy Braess és Sarazin által [4] ismertetett tesztfeladatot.

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Appendix A

List of Publications

Papers

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- Zhang, Y., Fogg, G. E., Simulation of multi-scale heterogeneity of porous media and parameter sensitivity analysis. *Sci. China Ser. E* **46** (2003), no. 5, 459–474.
- Li, W. D., Zhang, C. R., Burt, J. E., Zhu, A. X., Feyen, J., Two-dimensional Markov chain simulation of soil type spatial distribution. *Soil Sci. Soc. Am. J.* **68** (2004), no. 5, 1479–1490.
- Zhang, C. R., Li, W. D., Markov chain modeling of multinomial land-cover classes, *GIScience and Remote Sensing* **42** (2005), no. 1, 1–18.
- Zhang, C. R., Li, W. D., Application of Transiograms to Markov Chain Simulation and Spatial Uncertainty Assessment of Land-Cover Classes, *GIScience and Remote Sensing* **42** (2005), no. 4, 297–319.
- Li, W. D., Zhang, C. R., Burt, J. E., Zhu, A. X., A Markov chain-based probability vector approach for modeling spatial uncertainties of soil classes. *Soil Sci. Soc. Am. J.* **69** (2005), no. 6, 1931–1942.
- Li, W. D., Zhang, C. R., A generalized Markov chain approach for conditional simulation of categorical variables from grid samples. *Transactions in GIS* **10** (2006), no. 4, 651–669.
- Brown, J. D., Heuvelink, G. B. M., The Data Uncertainty Engine (DUE): A software tool for assessing and simulating uncertain environmental variables. *Computers & Geosciences* **33** (2007), no. 2, 172–190.

  - Li, W. D., A fixed-path Markov chain algorithm for conditional simulation of discrete spatial variables. *Math. Geol* **39** (2007), no. 2 159–176.

[2] Stoyan, G., Strauber, G., Baran, Á., Generalizations to discrete and analytical Crouzeix-Velte decompositions. *Numerical Linear Algebra with Applications* **11** (2004), no. 5-6, 565–590. (Impact faktor: 0.727)

[3] Stoyan, G., Baran, Á., Crouzeix-Velte decompositions for higher order finite elements. *Computers and Mathematics with Applications* **51** (2006), no. 6-7, 967–986. (Impact faktor: 0.430)

[4] Baran, Á., Stoyan, G., Gauss-Legendre elements: a stable higher order non-conforming finite element family. *Computing* **79** (2007), no. 1, 1–21. (Impact faktor: 0.949)

Proceedings

[1] Baran, S., Szabó, Á., An application of simulated annealing to ML-estimation of a partially observed Markov chain. *Proceedings of the 3rd International Conference on Applied Informatics*, Eger–Noszvaj, Hungary, August 25–28, 1997, 85–95.

  - Verdes, E., *The π^* index: computation, characterization and application of a new goodness of fit measure.* Ph.D. thesis, University of Debrecen, 2002.
  - Goldstein, P., Karaga, M., Kosor, M., Nizetić, I., Tadić, M., Vlah, D., Hidden Markov models and multiple alignments of protein sequences. In Drmać, Z. et al. (eds.), *Proceedings of the Conference on Applied Mathematics and Scientific Computing*, Springer, Dordrecht, 2005, pp. 187–196.

[2] Baran, Á., Baran, S., On the weak convergence of a continuous state space simulated annealing. *Proceedings of the 4th International Conference on Applied Informatics*, Eger–Noszvaj, Hungary, August 30–September 3, 1999, 231–240.

Appendix B

Conference talks

[1] *Crouzeix-Velte decompositions for higher order finite element families.* Applied Mathematics and Scietific Computing, Brijuni, Croatia, June 23–27, 2003.

[2] *Crouzeix-Velte spectrum for some finite elements.* 6^{th} International Conference on Applied Informatics, Eger, Hungary, January 27–31, 2004.

[3] *Some special properties of Scott-Vogelius elements.* Third Croatian Congress of Mathematics, Split, Croatia, June 16–18, 2004.

[4] *A higher order non-conforming finite element family.* Applied Mathematics and Scietific Computing, Brijuni, Croatia, June 19–24, 2005.

[5] *Egy stabil, magas rendű nemkonform végeselem család.* Áramlástan Numerikus Módszerei: Elmélet és Alkalmazások, Győr, Magyarország, 2006. november 13.

A high-order non-conforming finite element family

Értekezés a doktori (PhD) fokozat megszerzése érdekében a
matematika tudományában

Írta: Baran Ágnes Éva okleveles matematikus és matematika tanár

Készült a Debreceni Egyetem Matematika és Számítástudományok Doktori Iskolája
(Valószínűségelmélet, matematikai statisztika és alkalmazott matematika
alprogramja) keretében

Témavezető: Dr. Stoyan Gisbert

A doktori szigorlati bizottság:

elnök: Dr. ....
tagok: Dr. ....
Dr. ....

A doktori szigorlat időpontja: 2007. ....

Az értekezés bírálói:

Dr. ....
Dr. ....
Dr. ....

A bírálóbizottság:

elnök: Dr. ....
tagok: Dr. ....
Dr. ....
Dr. ....
Dr. ....

Az értekezés védésének időpontja: 2007. ....