

**Comparison of renormalization group schemes for sine-Gordon-type models**I. Nándori,<sup>1</sup> S. Nagy,<sup>2</sup> K. Sailer,<sup>2</sup> and A. Trombettoni<sup>3</sup><sup>1</sup>*Institute of Nuclear Research of the Hungarian Academy of Sciences, H-4001 Debrecen, P.O.Box 51, Hungary*<sup>2</sup>*Department of Theoretical Physics, University of Debrecen, Debrecen, Hungary*<sup>3</sup>*SISSA and INFN, Sezione di Trieste, via Beirut 2/4, I-34151, Trieste, Italy*

(Received 1 April 2009; published 17 July 2009)

The scheme dependence of the renormalization group (RG) flow has been investigated in the local potential approximation for two-dimensional periodic, sine-Gordon type field-theoretic models discussing the applicability of various functional RG methods in detail. It was shown that scheme-independent determination of such physical parameters is possible as the critical frequency (temperature) at which Kosterlitz-Thouless-Berezinskii type phase transition takes place in the sine-Gordon and the layered sine-Gordon models, and the critical ratio characterizing the Ising-type phase transition of the massive sine-Gordon model. For the latter case, the Maxwell construction represents a strong constraint on the RG flow, which results in a scheme-independent infrared value for the critical ratio. For the massive sine-Gordon model also the shrinking of the domain of the phase with spontaneously broken periodicity is shown to take place due to the quantum fluctuations.

DOI: [10.1103/PhysRevD.80.025008](https://doi.org/10.1103/PhysRevD.80.025008)

PACS numbers: 11.10.Gh, 11.10.Hi, 11.10.Kk

**I. INTRODUCTION**

Since the invention of the renormalization group (RG) method [1], it has been the main goal of functional RG to describe systems where the usual approximations (e.g. perturbation theory) failed. Strongly correlated electrons, critical phenomena, quark confinement are examples where a nonperturbative treatment is required. The RG equations are functional integro-differential equations, consequently, they can only be handled by using truncations (for reviews see [2]). The truncated RG flow depends on the particular choice of the so-called regulator function of the RG method, i.e. on the renormalization scheme, hence, the predicting power of the RG method is weakened. Indeed, once approximations are used a dependence of physical results on the choice of the regulator function is observed, consequently, any RG scheme makes sense if this dependence is weak. Therefore, it is of relevance to clarify how far the results obtained are independent (or at least weakly dependent) of the particular choice of the renormalization scheme used.

The purpose of this paper is to investigate the RG scheme dependence of two-dimensional (2D) periodic scalar field theories with possible inclusion of explicit symmetry breaking mass terms, i.e. the sine-Gordon (SG) [3], the massive sine-Gordon (MSG) [4] and the multicomponent layered sine-Gordon (LSG) [5] models. Our motivation is twofold: (i) from the *methodological* point of view, 2D SG-type models represent a new platform to consider RG scheme dependence since the comparison of results obtained by various RG schemes has been investigated previously for the O(N) symmetric polynomial scalar field theory mostly in three dimensions [6–10] (ii) from the *phenomenological* point of view, 2D SG-type models have important direct realizations in high-energy and con-

densed matter physics, consequently, their physical parameters should be determined independently of the particular choice of the RG scheme used.

In particular, our goal is to consider under which conditions it is possible to determine scheme independently three physical parameters, the critical frequency  $\beta_c^2$  of the SG theory, the critical ratio  $u/M^2$  of the MSG model, and the layer-dependent critical value  $\beta_c^2(N_L)$  of the LSG model. The SG scalar field theory (the Bose form of the massive Thirring fermionic model) represents the simplest nontrivial quantum field theory, which is used to study the confinement mechanism. It has two phases separated by the critical value of the frequency  $\beta_c^2 = 8\pi$ . The 2D SG model belongs to the universality class of the 2D Coulomb gas and the 2D-XY spin model, which have been applied to describe the Kosterlitz-Thouless-Berezinskii (KTB) [11] phase transition of many condensed matter systems. Because of the exact mapping of the generating functional of the SG model onto the partition function of the Coulomb gas,  $\beta_c^2$  is related to the critical temperature of XY-like condensed matter systems [12,13], which can be measured directly. Moreover, the MSG model describes the vortex dynamics of 2D superconducting thin films [14,15] and it is the bosonized version of the one-flavor massive Schwinger model, i.e. the 2D quantum electrodynamics (QED<sub>2</sub>) [4,16–18], which possesses confinement properties. The Ising-type phase transition of the MSG model is controlled by the dimensionless quantity  $u/M^2$ , where  $u$  is the Fourier amplitude and  $M$  is the scalar mass. It is related to the critical ratio of QED<sub>2</sub>, which separates the confining and the half-asymptotic phases of the fermionic model, which has been calculated by semiclassical and lattice methods [19–21]. The LSG model is used to describe the vortex dynamics of magnetically coupled layered superconductors [15,22–25] and considered to be the bosonized form of

the multiflavor Schwinger model (i.e. multiflavor QED<sub>2</sub>) [16,25,26]. It is known that the critical frequency of the LSG model (and consequently the critical temperature of the corresponding condensed matter system) depends on the number of layers  $N_L$  in a particular way  $\beta_c^2(N_L) = 8\pi N_L / (N_L - 1)$  [15]. This layer number dependence can be observed experimentally [27]. Therefore, the critical frequency  $\beta_c^2$  of the SG theory, the critical ratio  $u/M^2$  of the MSG model and the layer-dependent critical value  $\beta_c^2(N_L)$  of the LSG model are physical parameters, consequently, it is an important issue to consider under which conditions it is possible to determine them scheme independently, which is the main goal of the present work.

The structure of our paper is the following: Some general remarks on RG scheme dependence is given in Sec. II. The applicability of various RG schemes to the 2D SG model and the scheme dependence of the critical frequency  $\beta_c^2$  is discussed in Sec. III. In Sec. IV, we investigate the scheme dependence of the critical ratio of the MSG model, which characterizes the boundary of the phases of the model. In Sec. V, we consider the dependence of the critical frequency  $\beta_c^2(N_L)$  of the LSG model on the RG scheme used. Section VI presents the summary and our concluding remarks. A brief overview of the frequently used RG methods is given in Appendix A in order to remind the reader of their advantages and drawbacks and settling the notations used throughout the paper. Finally, in Appendix B the scheme dependence of the renormalization of the one-component polynomial scalar field theory is reviewed.

## II. MOTIVATION AND GENERAL REMARKS ON SCHEME DEPENDENCE

In the framework of the Kadanoff-Wilson [1] RG approach the differential RG transformations are realized via a blocking construction, the successive elimination of the degrees of freedom that lie above the running ultraviolet (UV) momentum cutoff  $k$ . Consequently, the effective theory defined by the (bare or effective) blocked action contains quantum fluctuations whose frequencies are smaller than the momentum cutoff. This procedure generates, e.g. the functional RG flow equation

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2} \text{Tr}(\Gamma_k^{(2)}[\phi] + R_k)^{-1} k\partial_k R_k$$

for the effective action  $\Gamma_k[\phi]$  when various types of regulator functions  $R_k$  are used, where  $\Gamma_k^{(2)}[\phi]$  denotes the second functional derivative of the effective action (see e.g. [2]). Here,  $R_k$  is a properly chosen infrared (IR) regulator function that fulfills a few basic constraints to ensure that  $\Gamma_k$  approaches the bare action in the UV limit ( $k \rightarrow \Lambda$ ) and the full quantum effective action in the IR limit ( $k \rightarrow 0$ ). Indeed, various renormalization schemes are constructed in such a manner that the RG flow starts at the bare action and provides the effective action in the IR limit,

so that the physical predictions (e.g. fixed points, critical exponents) are independent of the renormalization scheme particularly used.

Since RG equations are functional partial differential equations it is not possible to solve them in general; hence, approximations are required. One of the commonly used systematic approximation is the truncated derivative expansion where the (bare or effective) action is expanded in powers of the derivative of the field

$$\Gamma_k[\phi] = \int_x \left[ V_k(\phi) + Z_k(\phi) \frac{1}{2} (\partial_\mu \phi)^2 + \dots \right].$$

In the local potential approximation (LPA) higher derivative terms are neglected and the wave-function renormalization is set equal to constant, i.e.  $Z_k \equiv 1$ . The solution of the RG equations sometimes requires further approximations, e.g. the potential can be expanded in powers of the field variable (with a truncation at the power  $N$ )

$$V_k(\phi) = \sum_{n=1}^N c_n(k) \phi^n,$$

where the scale dependence is encoded in the coupling constants  $c_n(k)$ . Since the approximated RG flow depends on the choice of the regulator function the physical results could become scheme dependent.

A general issue is the comparison of results obtained by various RG schemes [6–10], which has been investigated for the O(N) symmetric polynomial scalar field theory in three dimensions in great detail. Some of the main results are as follows: (i) for a given RG equation the explicit dependence on the regulator function disappears in the LPA, i.e. the results should not depend on the form of the regulator function if no further approximations (e.g. truncation in powers of the field) are used, (ii) RG equations linearized around the trivial UV Gaussian fixed point ( $V_* = 0$ ) provide the same UV scaling laws in various RG schemes, consequently, critical exponents of the Gaussian fixed point are scheme independent in the LPA, (iii) nontrivial fixed points (like the Wilson-Fisher fixed point) and the critical exponents characterizing the scaling in their neighborhood obtained in the LPA become scheme dependent, i.e. the estimates of these physical quantities depend on the particular choice of the RG scheme. Therefore, it is generally assumed that once approximations are used nontrivial fixed points and their critical behavior become scheme dependent. However, in this paper we show that one can find models that have important physical realizations, and their physical parameters (fixed points, critical ratio) can be obtained scheme-independently even in the LPA.

Our goal here is to consider the scheme dependence of the low-energy behavior of various RG methods by considering the renormalization of the 2D periodic scalar field theory, i.e. the 2D SG model [3] with possible inclusion of explicit symmetry breaking mass terms, i.e. the MSG [4]

and the multicomponent LSG model [5]. The functional RG approaches are investigated in the LPA. Four types of RG methods are compared: the Wegner-Houghton (WH-RG) [28], the Polchinski (P-RG) [29], the Callan-Symanzik type (CS-RG) [30], and the effective average action (EAA-RG) [31–33] approaches. In the latter case, we use two types of regulator functions: the optimized [6] and the powerlike (quartic) [33] regulators. The phase structure of various SG-type models obtained at quantum and classical level are also compared.

As a rule, scheme dependence is expected even in the exact (not truncated) RG flow at intermediate scales between the UV and IR scales since various schemes realize the elimination of quantum fluctuations in a different manner. However, the disappearance of scheme dependence of the exact RG flow is expected in the deep IR limit if no approximations are involved. In the UV limit even the LPA of the RG flow is able to produce results, which are independent of the particular choice of the renormalization scheme since various RG equations linearized at the trivial Gaussian fixed point are the same in the LPA. Nevertheless, RG flows obtained in the LPA show spurious scheme dependence at the nontrivial Wilson-Fisher fixed point. Therefore, it is a natural question to ask whether it is possible to obtain scheme-independent results at nontrivial fixed points. The answer can be affirmative if an additional constraint influences the RG flow. For example, theories exhibiting spontaneous symmetry breaking have superuniversal effective IR behavior due to the Maxwell cut in the symmetry broken phase. Superuniversality represents so strong constraints on the RG flow that the scheme dependence disappears if the effects of the truncation of the functional subspace are under control by using a sufficiently large functional subspace. In this paper, we will show that this general view holds for the MSG model where scheme-independent results are obtained for the critical ratio  $(u/M^2)_c$  being one of the typical IR characteristics of the Ising-type phase transition of the MSG model. It will, however, also be shown that RG flow equations linearized at the UV Gaussian fixed point provide us scheme-dependent results for the critical ratio  $(u/M^2)_c$  of the MSG model, which characterizes the phase structure far beyond the validity of the UV scaling laws, in the deep IR region.

Consequently, one expects that UV scaling laws cannot be used to determine the critical behavior of any nontrivial fixed point situated in the IR or in the crossover regions. At crossover scales between the UV and IR scaling regions even “exact” RG equations (where LPA is the only approximation used) produce results being influenced by the choice of renormalization scheme. Nevertheless, we shall show that the critical frequency  $\beta_c^2 = 8\pi$  at which the SG model and the layer-dependent critical value  $\beta_c^2(N_L) = 8\pi N_L/(N_L - 1)$  at which the multicomponent ( $N_L > 1$ ) LSG models undergo a KTB-type phase transition can be

obtained exactly (independently of the invented RG scheme) even by the UV linearized-RG flow. This rather surprising result is due to the following circumstances: (i) although UV linearized flow equations obtained in various RG schemes are different for the LSG model (where the linearization is performed in the periodic piece of the potential and not in the full potential), they provide us the same scheme-independent critical frequency, (ii) the KTB-type phase transition of SG and LSG models is governed by the fundamental Fourier mode, which has the same scaling law in the UV and IR regions, (iii) higher harmonics are generated by RG transformations, and they have different UV and IR scalings, but they do not influence the critical behavior.

### III. SINE-GORDON MODEL

In this section we compare the applicability of the frequently used functional RG methods to two-dimensional one-component Euclidean scalar fields with periodic self-interaction. In Appendix A, we review the notation and the main properties of various RG schemes used by us which are the following: the WH-RG, the P-RG, the CS-RG, and the EAA-RG approaches. In the latter case, we use the optimized and the quartic regulator functions. The WH-RG, the functional CS-RG, and the EAA-RG with powerlike regulator with  $b = 1$  are equivalent methods in the LPA for  $d = 2$  even if spinodal instability (SI) [34] occurs (see Appendix A). Therefore, we shall compare the WH-RG, the EAA-RG with optimized and quartic regulators and the P-RG. It was shown [9] that the P-RG and the EAA-RG with optimized regulator can be transformed into each other by a suitable Legendre transformation (in LPA); however, their singularity structures are different. We shall demonstrate this for the SG model since a SI, i.e. an IR singularity appears in the RG flow in the weak-coupling phase but the P-RG method is not able to indicate this. Therefore, we shall conclude that any of the discussed RG schemes are expected to be applicable to the investigation of the IR behavior of the MSG model except the P-RG method, which is inappropriate for quantitative analysis in that case.

For the generalized SG model characterized by the local potential

$$\tilde{V}_k(\phi) = \sum_{n=1}^N \tilde{u}_n(k) \cos(n\beta\phi), \quad (1)$$

exhibiting periodicity in the internal space the dimensionless couplings are represented by the Fourier amplitudes  $\tilde{u}_n(k)$  and the “frequency”  $\beta$  is a scale-independent, dimensionless parameter in the LPA. If the higher harmonics are neglected, i.e. for  $N = 1$  the generalized SG model reduces to the well-known SG model [3]. We restrict ourselves to the RG analysis of bare models with  $\tilde{u}_1(\Lambda) > 0$  (positive fugacity for “charges”  $\pm 1$  of the equivalent



Coulomb gas). Another interesting generalization of the periodic model is the multicomponent LSG model where the two-dimensional SG interaction terms are coupled by a particular mass matrix. This model receives important applications in high-energy and in low-temperature physics, and its KTB-type phase transition has been discussed in [15,16,22–25]. In Sec. V, we consider the scheme dependence of the critical frequency of the multicomponent LSG model.

Let us first briefly summarize the results of the WH-RG analysis of the generalized SG model obtained previously in the LPA [12,13,35,36]. The phase structure (in LPA) is sketched in Fig. 1 and has also been discussed in [35]. The  $\beta^2$  axis represents the line of Gaussian fixed points, which are IR ones in the strong-coupling phase with  $\beta^2 > 8\pi$  and UV ones in the weak-coupling phase with  $\beta^2 < 8\pi$ . The two phases are separated by the vertical line at  $\beta^2 = 8\pi$  where the KTB-type [11] phase transition takes place. In the figure only the single parameter axis  $\tilde{u}_1$  is depicted. In order to get a more reliable picture of the phase diagram, one has to imagine a continuous sequence of nonintersecting infinite-dimensional hypersurfaces, which are intersected by the  $\beta^2$  axis at  $\tilde{u}_n = 0$ , as well as, by the line of the IR fixed points in the weak-coupling phase at some  $\tilde{u}_n(0) \neq 0$ . Then the hypersurface at  $\beta^2 = 8\pi$  separates the strong- and weak-coupling phases. In the UV limit ( $k \leq \Lambda$ ) the linearized-RG equations provide the UV scaling laws

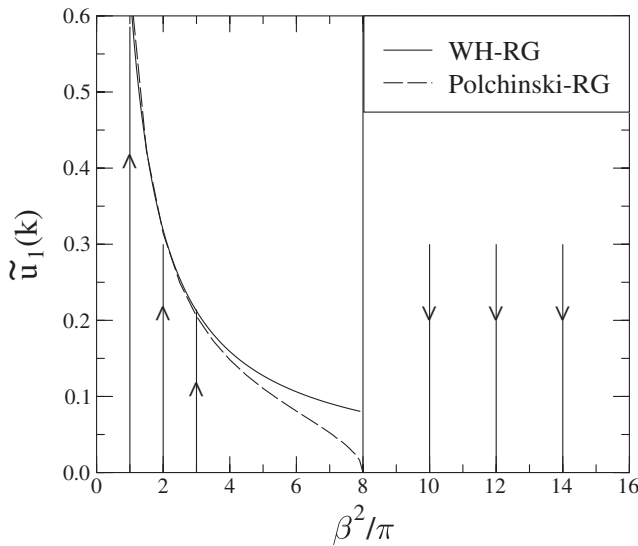


FIG. 1. The phase structure of the SG model obtained in LPA ( $\beta^2$  is scale independent). The arrows indicate the direction of the RG flow. The nontrivial IR fixed points for  $\beta^2 < 8\pi$  are obtained by the WH-RG (full line) and by the P-RG (dashed line) methods. The line of IR fixed points is given by the analytic expression  $\tilde{u}_1(0) = 2/\beta^2$  for the WH-RG method for  $\beta^2 < 8\pi$  and  $\tilde{u}_1(0) \sim 0.03(8\pi - \beta^2)^{1/2}$  for the P-RG approach for  $\beta^2 \lesssim 8\pi$ .

$$\tilde{u}_n(k) = \tilde{u}_n(\Lambda) \left( \frac{k}{\Lambda} \right)^{(n^2 \beta^2 / 4\pi) - 2}. \quad (2)$$

In the strong-coupling phase ( $\beta^2 > 8\pi$ ) all the Fourier amplitudes are UV irrelevant, whereas in the weak-coupling one ( $\beta^2 < 8\pi$ ) the first few Fourier amplitudes become relevant, depending on the value of  $\beta^2$ . In both the strong- and weak-coupling phases the dimensionful blocked potential tends to a constant effective potential for  $k \rightarrow 0$ , being the only function that is simultaneously convex and periodic, but there is a significant difference in the IR scaling of the dimensionless couplings. The insertion of the ansatz (1) into the WH-RG equation Eq. (A2) yields the RG flow equations

$$(2 + k\partial_k)n\tilde{u}_n = \frac{\beta^2}{4\pi}n^3\tilde{u}_n + \frac{\beta^2}{2} \sum_{s=1}^{\infty} sA_{n,s}(2 + k\partial_k)\tilde{u}_s, \quad (3)$$

for the couplings  $\tilde{u}_n$  where  $A_{n,s}(k) = (n-s)^2\tilde{u}_{|n-s|} - (n+s)^2\tilde{u}_{n+s}$ . Equation (A2), as well as Eq. (A1) are valid unless SI arises. In the strong-coupling phase  $\beta^2 > 8\pi$  no SI occurs, Eq. (3) holds at any scale and every Fourier amplitude is irrelevant. The IR scaling laws are given by

$$\tilde{u}_n(k) = c_n \left( \frac{k}{\Lambda} \right)^{n\eta}, \quad (4)$$

with  $\eta = \beta^2/4\pi - 2 > 0$  and the constants  $c_n$  depending on  $\beta^2$  and the single bare parameter  $c_1 = \tilde{u}_1(\Lambda)$  via the recursion relation (for  $n > 1$ ) [35]

$$c_n = \frac{\frac{1}{2}\beta^2 \sum_{s=1}^{n-1} (2 + s\eta)s(n-s)^2 c_{n-s} c_s}{n(2 + n\eta - n^2 \frac{\beta^2}{4\pi})}. \quad (5)$$

Here, the well-justified approximation  $A_{n,s}(k) \approx (n-s)^2\tilde{u}_{|n-s|}$  has been used. The dimensionless blocked potential becomes flat, i.e. all couplings  $\tilde{u}_n$  vanish in the IR limit  $k \rightarrow 0$ . For  $\beta^2 < 8\pi$  the SI occurs in the RG flow when the propagator diverges,  $k_{\text{SI}}^2 + V''_{k_{\text{SI}}}(\phi) = 0$ . The scaling laws just above the scale  $k_{\text{SI}}$  of the SI are given by (4), but now with  $\eta < 0$ , which modifies the scaling essentially. Namely, all the Fourier amplitudes  $\tilde{u}_n$  become relevant at the scale  $k_{\text{SI}}$ . Even more radical change of the IR scaling laws has been observed. Making use of the tree-level blocking relation (A3) one finds that the SI results in the building up of a condensate of increasing amplitude  $\rho$  for decreasing scale  $k < k_{\text{SI}}$ . In the weak-coupling phase the dimensionless effective potential remains a nonvanishing periodic one (the line of nontrivial IR fixed points in Fig. 1), graphically obtained by setting forth along the  $\phi$  axis the section of the parabola

$$\tilde{V}_{k \rightarrow 0}(\phi) = \frac{2}{\beta^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(n\beta\phi) = -\frac{1}{2}\phi^2, \quad (6)$$

with  $\phi \in [-\pi/\beta, \pi/\beta]$  periodically. Each parabola section is the one of just the same parabola (A4), one would

find as the nontrivial fixed point of the polynomial theory. The periodic dimensionless effective potential is the continuous, sectionally differentiable, periodic solution of Eq. (A5). Let us note that Eq. (3) yields the fixed-point equation

$$2n\tilde{u}_n = \frac{\beta^2}{4\pi} n^3 \tilde{u}_n + \frac{1}{2} \beta^2 \sum_{s=1}^{\infty} s A_{n,s} 2\tilde{u}_s \quad (7)$$

if no SI occurs in the RG flow.

Using the same machinery in the framework of the P-RG, one obtains from Eq. (A9) the flow equations

$$(2 + k\partial_k) n\tilde{u}_n = \frac{\beta^2}{4\pi} n^3 \tilde{u}_n - \frac{1}{2} \beta^2 \sum_{s=1}^{\infty} s A_{n,s} 2\tilde{u}_s. \quad (8)$$

Inserting the ansatz (4) into Eq. (8) we get

$$(2 + n\eta) n c_n k^{n\eta} = \frac{\beta^2}{4\pi} n^3 c_n k^{n\eta} - \frac{\beta^2}{2} \sum_{s=1}^{\infty} s A_{n,s} 2 c_s k^{s\eta},$$

which gives  $\eta = \beta^2/4\pi - 2$  again. For  $\beta^2 > 8\pi$  ansatz (4) works with the neglect of the term proportional to  $\tilde{u}_{n+s}(k)$  in the expression of  $A_{n,s}(k)$  similarly to the WH-RG and yields the recursion relation

$$c_n = -\frac{\frac{1}{2} \beta^2 \sum_{s=1}^{n-1} 2s(n-s)^2 c_{n-s} c_s}{n(2 + n\eta - n^2 \frac{\beta^2}{4\pi})}, \quad (9)$$

with  $c_1 = \tilde{u}_1(\Lambda)$ ,  $c_n = \tilde{u}_1^n(\Lambda) R_n$  for  $n > 1$ ,  $R_1 = 1$  and the constants  $R_n$  for  $n > 1$ , satisfying the recursion relation (9), become independent of the bare couplings. In Fig. 2, the flow of the first few couplings can be followed. All dimensionless couplings tend to zero in the IR limit  $k \rightarrow 0$  again.

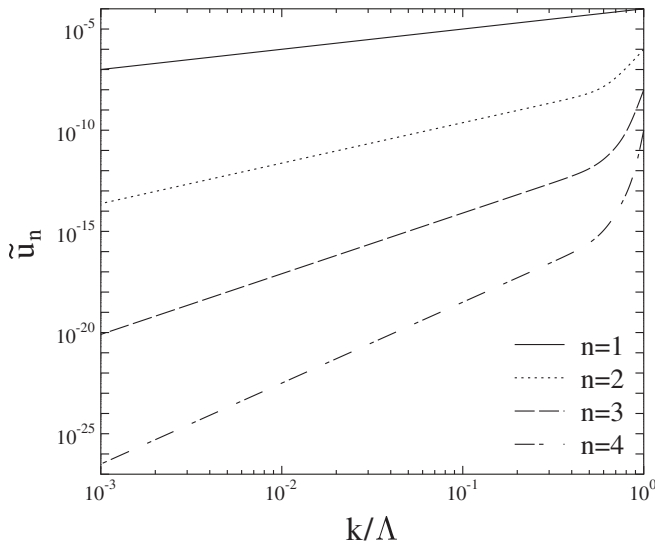


FIG. 2. The P-RG flow of the first few dimensionless couplings of the generalized SG model for  $\beta^2 = 12\pi$ .

For  $\beta^2 < 8\pi$  the numerical solution of the flow Eqs. (8) for the dimensionless couplings  $\tilde{u}_n(k)$  follows the power-law behavior of Eq. (4), now with  $\eta < 0$ , in a wide range above a certain scale  $k_c$ , but the couplings  $\tilde{u}_n(k)$  go to constant values in the deep IR region, as it is demonstrated in Fig. 3. Let us note that the inverse propagator vanishes around the scale  $\sim k_c$ , which signals the appearance of the SI (i.e.  $k_c \sim k_{SI}$ ). However, the P-RG flow equations do not exhibit any singularity at that scale  $k_c$ . Equation (8) provides a scale-independent, IR fixed-point solution now satisfying Eq. (7). Our numerical results show in Fig. 4 for  $\beta^2$  close to but below  $8\pi$  ( $\beta^2 \leq 8\pi$ ) that the  $\beta$  dependence of  $\tilde{u}_n(0)$  can be factored out as

$$\tilde{u}_n(0) = (8\pi - \beta^2)^{n/2} a_n, \quad (10)$$

where the numbers  $a_n$  are independent of any parameter of the model. Therefore, the nonvanishing, universal dimensionless IR effective potential obtained by the P-RG method reminds one of the similar result of the WH-RG analysis, but the parabolic shape of the dimensionless effective potential in its periods cannot be recovered. As a consequence, the line of IR fixed points of the SG model in the weak-coupling phase has been modified, see the dashed line in Fig. 1.

As to the ambiguity of the decision based on the numerics whether the SI does or does not occur during the flow, it is in order to make here an important remark. At the first glance one expects that the accuracy of such a decision will be increased by increasing the number of Fourier modes, that of the couplings  $\tilde{u}_n$  taken into account. With decreasing scale  $k$  the dimensionful periodic potential becomes rather flat in each period. According to our numerical experience it approaches rather smoothly the potential (6) for  $\beta^2 \leq 4\pi$  and then the occurrence of the SI

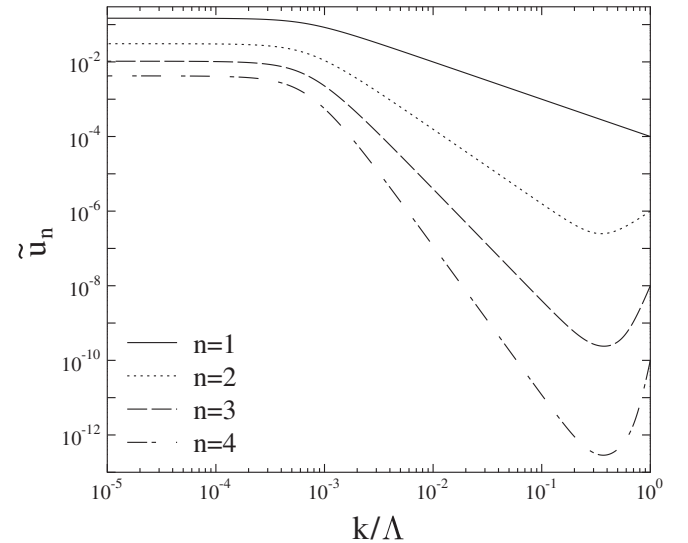


FIG. 3. The flow of the first few dimensionless couplings of the SG model for  $\beta^2 = 4\pi$  obtained by the P-RG method.

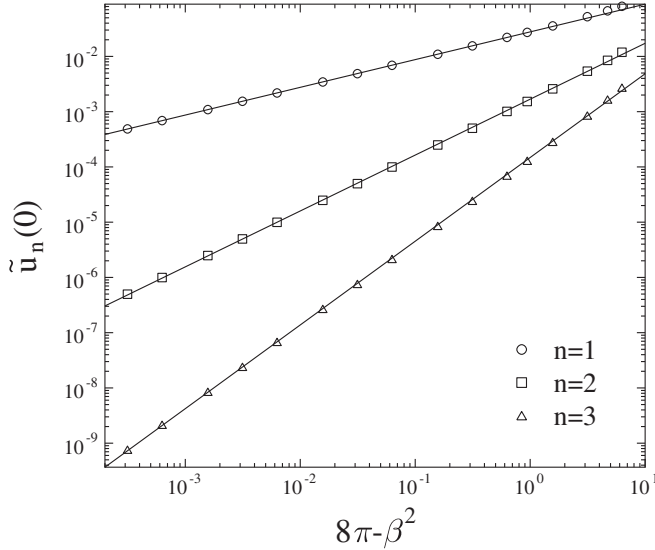


FIG. 4. The IR values  $\tilde{u}_n(0)$  of the dimensionless couplings show power-law behavior in the vicinity of the critical value of  $\beta^2$ . The markers are obtained by solving the nonlinear system of equations in Eq. (7), and the lines are plotted by a fit giving 0.5, 1.0, 1.5 for the slopes, respectively, for  $n = 1, 2, 3$ .

can be detected without any doubt. However, the matter of

things becomes much worse for  $0 < 8\pi - \beta^2 \ll 8\pi$  when there occur numerical instabilities if one reconstructs the potential. In such a case one cannot decide unambiguously with the numerical method based on the Fourier expansion of the potential whether the SI does occur indeed. Solving the RG equation derived in the LPA for the potential without using any further expansion seems to be a reliable way to settle this point, but it lies out of the scope of the present paper.

As to the next, let us turn now to the discussion of the EAA-RG flow for the generalized SG model (1). Using the optimized regulator, and deriving Eq. (A18) with respect to the field and multiplying its both sides by  $(1 + \tilde{V}_k'')^2$ , one obtains the evolution equations

$$(2 + k\partial_k)n\tilde{u}_n = \frac{\beta^2}{4\pi}n^3\tilde{u}_n + \beta^2 \sum_{p=1}^{\infty} pA_{n,p}(2 + k\partial_k)\tilde{u}_p - \frac{1}{4}\beta^4 \sum_{q=1}^{\infty} \sum_{p=1}^{\infty} pA_{n,q}A_{q,p}(2 + k\partial_k)\tilde{u}_p \quad (11)$$

for the Fourier amplitudes  $\tilde{u}_n$ . For  $\beta^2 > 8\pi$  the IR scaling laws are given again by Eq. (4) with  $\eta = \beta^2/(4\pi) - 2 > 0$  and the recursion relation for  $c_n$ 's

$$c_n = \frac{\beta^2 \sum_{p=1}^{n-1} (2 + p\eta)p(n-p)^2 c_{n-p}c_p}{n(2 + n\eta - n^2 \frac{\beta^2}{4\pi})} - \frac{\frac{1}{4}\beta^4 \sum_{q=1}^{n-1} \sum_{p=1}^{q-1} (2 + p\eta)p(n-q)^2(q-p)^2 c_{n-q}c_{q-p}c_p}{n(2 + n\eta - n^2 \frac{\beta^2}{4\pi})}. \quad (12)$$

For  $\beta^2 < 8\pi$  the numerical solution of the system (11) of the coupled flow equations exhibits the following features: If one takes into account a sufficiently large number of Fourier modes, one finds that there exists a nonvanishing scale  $k_{\text{SI}}$  at which the inverse propagator vanishes that signals the presence of the SI. The numerical problem of deciding whether the SI occurs or does not, remains just the same as in the WH-RG framework. For  $\beta^2$  in the vicinity of the critical value  $8\pi$  and restricting oneself to the first few Fourier amplitudes, one finds that the inverse propagator does not vanish in the IR region. In this case, the IR effective potential is similar to that obtained by the P-RG, with the scaling property (10) [but with different numbers  $a_n$ ].

Let us now discuss the flow in the framework of the EAA-RG with quartic regulator. For  $\beta^2 > 8\pi$  the SI does not occur. On the basis of the result of the WH-RG analysis, for  $\beta^2 < 8\pi$  one would expect the occurrence of the SI, i.e. vanishing of  $1 + 1/2\tilde{V}_k''(\phi)$  at some scale  $k_{\text{SI}} \neq 0$ . For the single mode potential  $\tilde{V}_k(\phi) = \tilde{u}(k)\cos(\beta\phi)$  this happens for  $\phi = 0$  if  $1 - 1/2\tilde{u}(k_{\text{SI}})\beta^2 = 0$  and then  $1 - 1/2\tilde{V}_{k_{\text{SI}}}''(\pi/\beta) = 0$  holds as well. Therefore, the momentum scale  $k_+$  where the quartic regulator RG is nonanalytic

and the scale of the SI  $k_- = k_{\text{SI}}$  coincide for the periodic model. The decision based on numerics on the existence of the SI suffers from the same problems as for the other RG schemes, and the IR scaling laws are also expected to be similar to those obtained on the basis of Eqs. (3), (8), and (11). Since the SI occurs when the propagator in the right-hand side of Eq. (A13) develops a pole, and periodicity should not be violated, the IR fixed-point potential should be obtained by setting forth periodically the parabola section in  $[-\pi/\beta, \pi/\beta]$  given by Eqs. (A14) and (A15).

In conclusion, the IR scaling laws determined by various RG methods are qualitatively the same in the strong-coupling phase of the generalized SG model. In the framework of WH-RG and the EAA-RG with the power-law regulator, the SI has been treated explicitly in the weak-coupling phase, which provides a reliable determination of the IR scaling laws even in that phase. In this case, the functional form of the low-energy effective potential is found to be the same but its exact value is scheme dependent [see e.g. Eqs. (A14) and (A15)]. However, the potential can always be rescaled by a constant, which leaves the physical results unchanged; consequently, only the functional form of the IR effective potential is of physical significance.

#### IV. MASSIVE SINE-GORDON MODEL

The MSG model for the number of dimensions  $d = 2$ , characterized by the dimensionless bare potential

$$\tilde{V}_{k=\Lambda}(\phi) = \frac{1}{2}\tilde{M}_\Lambda^2 \phi^2 + \tilde{u}(\Lambda) \cos(\beta\phi) \quad (13)$$

exhibits two phases. For  $\beta^2 > 8\pi$  only the phase with explicitly broken periodicity is present, and that phase extends to  $\beta^2 < 8\pi$  if the bare dimensionless ratio  $\tilde{u}(\Lambda)/\tilde{M}_\Lambda^2$  is smaller than a critical upper bound  $[\tilde{u}(\Lambda)/\tilde{M}_\Lambda^2]_c$  depending on the parameters  $\beta^2$  and  $\tilde{M}_\Lambda^2$ . For  $\beta^2 < 8\pi$  and  $\tilde{u}(\Lambda)/\tilde{M}_\Lambda^2 > [\tilde{u}(\Lambda)/\tilde{M}_\Lambda^2]_c$  the periodicity is spontaneously broken, and the RG trajectories in that phase merge into a single trajectory in the deep IR region, which is characterized by the unique ratio  $\tilde{u}(k \rightarrow 0)/\tilde{M}_{k \rightarrow 0}^2 = [\tilde{u}/\tilde{M}^2]_c = [u/M^2]_c$  depending on  $\beta^2$  only. Our main goal is to discuss whether the determination of that unique ratio does depend on the choice of the previously discussed renormalization schemes. Let us note that the  $Z_2$  symmetry  $\phi \rightarrow -\phi$  can be used to distinguish the phases of the MSG model, it suffers a spontaneous breakdown in the phase with spontaneously broken periodicity when a condensate appears.

The phase structure of the MSG model has been discussed in the framework of the WH-RG in the LPA in [17] making use of the more general ansatz

$$\tilde{V}_k(\phi) = \frac{1}{2}\tilde{M}_k^2 \phi^2 + \sum_{n=1}^N \tilde{u}_n(k) \cos(n\beta\phi) \quad (14)$$

for the local potential. [The parameter  $\tilde{u}$  in the expression (13) corresponds to  $\tilde{u}_1(\Lambda)$ , and we are looking for the unique value of  $\tilde{u}_1(k \rightarrow 0)/\tilde{M}_{k \rightarrow 0}^2 = [u/M^2]_c$  in the deep IR region for the phase with spontaneously broken periodicity.] The periodic piece of the potential (14) possesses the discrete symmetry under the shift  $\phi \rightarrow \phi + 2\pi/\beta$  of the field variable. This is the symmetry the generalized SG model with the potential (1) exhibits, and—as discussed in the previous section—in the weak-coupling phase ( $\beta^2 < 8\pi$ ) of the generalized SG model a condensate appears in the IR limit, due to which the periodicity is broken spontaneously. Because of the explicit mass term the MSG model offers the opportunity to investigate the interplay between the spontaneous and explicit breaking of periodicity. The following features of the phase structure have been determined in the framework of the WH-RG method in the LPA in [17,22]:

- (1) The dimensionful mass  $M^2$  remains constant during the blocking, which provides the trivial scaling  $\tilde{M}_k^2 = k^{-2}M^2$  for the dimensionless mass. In the LPA (when no wave-function renormalization is incorporated) also the parameter  $\beta^2$  is constant during the flow.

- (2) The linearized WH-RG flow equation

$$(2 + k\partial_k)\tilde{u}_1(k) = \frac{\beta^2}{4\pi}\tilde{u}_1(k)\frac{k^2}{k^2 + M^2} \quad (15)$$

for the coupling  $\tilde{u}_1(k)$  of the fundamental mode of the local potential provides the UV scaling law

$$\tilde{u}_1(k) = \tilde{u}_1(\Lambda)\left(\frac{k}{\Lambda}\right)^{-2}\left(\frac{k^2 + M^2}{\Lambda^2 + M^2}\right)^{(\beta^2/8\pi)}. \quad (16)$$

The coupling  $\tilde{u}_1(k)$  is relevant in the UV scaling region, so that the scaling law (16) loses its validity for IR scales, irrespectively of  $\beta^2$ . Nevertheless, it was observed numerically that Eq. (16) provides a rather good description of the RG flow for scales  $k \gtrsim M$  for  $\beta^2 > 8\pi$ . The critical ratio

$$\left[\frac{\tilde{u}_1(\Lambda)}{\tilde{M}_\Lambda^2}\right]_c = \frac{2}{\beta^2}\left(\frac{1 + M_\Lambda^2}{2M_\Lambda^2}\right)^{(\beta^2/8\pi)} \quad (17)$$

for the bare parameters of the MSG model has also been discussed in the framework of the linearized-RG [see Eq. (9) of [17]].

- (3) Inserting the ansatz (14) into the WH-RG equation Eq. (A2) yields the RG flow equations

$$\begin{aligned} \left(1 + \frac{M^2}{k^2}\right)(2 + k\partial_k)n\tilde{u}_n &= \frac{\beta^2}{4\pi}n^3\tilde{u}_n + \frac{\beta^2}{2} \\ &\times \sum_{s=1}^{\infty} sA_{n,s}(2 + k\partial_k)\tilde{u}_s \end{aligned} \quad (18)$$

for the couplings  $\tilde{u}_n(k)$ . For  $k^2 \gg M^2$  the RG flow is close to that of the SG model. For strong coupling  $\beta^2 > 8\pi$  no SI occurs during the flow, the inequality  $1 + \tilde{V}_k'' > 0$  holds for all scales  $k$ . The numerical solution of Eq. (18) provides scaling laws for  $M < k \ll \Lambda$  not differing significantly from those of the SG model, and below the scale  $k \sim \mathcal{O}(M)$  the trivial scaling  $\tilde{u}_n(k) \sim k^{-2}$  occurs. The parameter region with  $\beta^2 > 8\pi$  belongs to the phase with explicit breaking of periodicity.

- (4) For weak coupling  $\beta^2 < 8\pi$  two phases appear. On the one hand, the RG flow on the trajectories started at  $\beta^2 < 8\pi$  and  $\tilde{u}_1(\Lambda)/\tilde{M}_\Lambda^2 > [u_1(\Lambda)/M_\Lambda^2]_c$  develops SI at some finite scale  $k_{\text{SI}} > M$ , so that one has to evaluate the flow using the tree-level blocking relation (A3). In this case, the parameters  $\tilde{u}_n(k)$  become superuniversal (i.e. independent of the bare parameters) for  $k < k_{\text{SI}}$  signaling the presence of the condensate and the spontaneous breaking of periodicity, similarly to the behavior of the weak-coupling phase of the SG model. It should be noticed that the detection of occurring SI in the evaluation of the trajectories suffers the same ambiguity for  $\beta^2$  close to but smaller than  $8\pi$  as in the case of the generalized SG model. For  $k < k_{\text{SI}}$  the tree-level



evolution yields the periodic piece of the potential with parabola sections

$$U_k(\phi) = (k^2 + M^2) \frac{2}{\beta^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos(n\beta\phi) \\ = -\frac{1}{2}(k^2 + M^2)\phi^2 \quad (19)$$

for  $\phi \in [-\pi/\beta, \pi/\beta]$ . Consequently, the first dimensionful Fourier amplitude tends to the constant  $u_1(k \rightarrow 0) = \frac{2}{\beta^2} M^2$ , i.e. the IR value of the critical ratio is  $[u/M^2]_c = 2/\beta^2$ . The parabolic potential (19) represents the nontrivial solution of the differential equation  $k^2 + V_k''(\phi) = k^2 + M^2 + U_k''(\phi) = 0$ . The effective potential  $V_{k \rightarrow 0} = U_{k \rightarrow 0} + 1/2 M^2 \phi^2 = -1/2 k^2 \phi^2$  is superuniversal, i.e. independent of the bare parameters  $\tilde{u}_n(\Lambda)$ .

On the other hand, the phase with explicit breaking of periodicity extends to the region with  $\beta^2 < 8\pi$  bounded from above by  $[u_1(\Lambda)/M_\Lambda^2]_c$ . In this phase no SI occurs along the RG trajectories and below the scale  $k = M$  the Fourier amplitudes start to scale as  $\tilde{u}_n(k) \sim k^{-2}$ . In this case, the effective potential  $U_{k \rightarrow 0}$  turns out to depend on the single bare parameter  $\tilde{u}_1(\Lambda)$ .

The phase diagram of the MSG model for  $\beta^2 = 4\pi$  is depicted in Fig. 5 where one can see that the RG trajectories in the phase with spontaneously broken periodicity merge into a single trajectory in the deep IR limit (full lines). It is worthwhile noticing that the KTB phase tran-

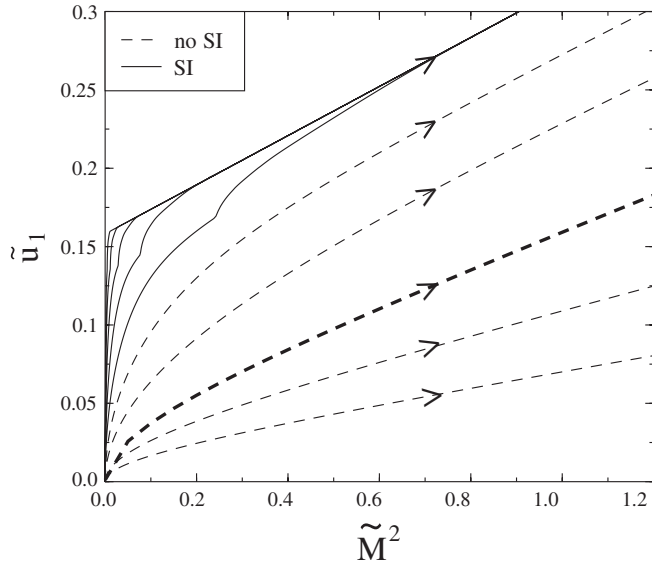


FIG. 5. Phase structure of the MSG model in LPA for  $\beta^2 = 4\pi$  obtained by the full WH-RG method. The arrows indicate the direction of the flow. The numerical solution of the exact RG equation shows that the separatrix obtained by the linearized flow (21) [wide dashed line] is modified by higher order corrections.

sition exhibited by the massless SG model disappears in the MSG model due to the presence of the explicit mass term, as demonstrated in [17,22] using WH-RG and CS-RG methods.

Let us consider the critical RG trajectory on the  $\tilde{u}_1 - \tilde{M}^2$  plane, which separates the phases of the MSG model. At linearized level this is constructed from Eq. (16) as

$$\tilde{u}_{1c}(\tilde{M}^2) = \tilde{u}_{1c}(\Lambda) \left( \frac{\tilde{M}^2}{\tilde{M}_\Lambda^2} \right)^{1-(\beta^2/8\pi)} \left( \frac{1 + \tilde{M}^2}{1 + \tilde{M}_\Lambda^2} \right)^{\beta^2/8\pi}, \quad (20)$$

where  $\tilde{M}^2 = \tilde{M}_k^2 = k^{-2} M^2$  is the running coupling and  $\tilde{M}_\Lambda^2 = \Lambda^{-2} M^2$  is constant. By inserting the critical ratio (17) obtained for the initial bare values into Eq. (20) one finds

$$\tilde{u}_{1c}(\tilde{M}^2) = \frac{2}{\beta^2} \tilde{M}^2 \left( \frac{1 + \tilde{M}^2}{2\tilde{M}^2} \right)^{\beta^2/8\pi}. \quad (21)$$

At the mass scale it gives  $\tilde{u}_{1c}(\tilde{M}^2 = 1) = 2/\beta^2 \approx 0.159$  (see the wide dashed line in Fig. 5). From Eq. (21) one can read off the IR value of the critical ratio

$$\left[ \frac{u}{M^2} \right]_c = \lim_{\tilde{M}^2 \rightarrow \infty} \left( \frac{\tilde{u}_{1c}}{\tilde{M}^2} \right) = \frac{2}{\beta^2} \left( \frac{1}{2} \right)^{\beta^2/8\pi}, \quad (22)$$

which is smaller than the superuniversal ratio  $[u/M^2]_c = 2/\beta^2$  obtained by the exact WH-RG method (for  $\beta^2 \neq 0$ ) and coincides with it only in the limit  $\beta^2 \rightarrow 0$ , see the dashed line in Fig. 6. Indeed, the numerical solution of the exact WH-RG Eq. (18) shows that one finds RG trajectories (see, Fig. 5), which lie above the critical one obtained in the framework of the linearized WH-RG.

In Fig. 6 we plotted the superuniversal ratio  $[u/M^2]_c = u_1(k \rightarrow 0)/M^2$  against the parameter  $\beta^2$  obtained in the framework of the WH-RG. According to our numerical results the effective potential (19) implies  $[u/M^2]_c = 2/\beta^2$ . According to lattice results [19] the phase transition of the MSG model is assumed to belong to the same universality class as the two-dimensional Ising model (with the critical exponents  $\nu = 1$ ,  $\beta = 1/8$  [37]). Since the MSG model for  $\beta^2 = 4\pi$  represents the bosonized version of QED<sub>2</sub>, the ratio  $[u/M^2]_c = 1/(2\pi) \approx 0.1591$  obtained by the WH-RG method can also be used to determine the phase transition point of QED<sub>2</sub> given as  $[m/g]_c = 2\sqrt{\pi} e^{-\gamma} [u/M^2]_c \approx 0.3168$ , where  $m$  is the fermion mass,  $g$  is the coupling between the fermionic and the gauge field and  $\gamma \approx 0.5774$  is the Euler's constant. It is shown in Fig. 6 that our WH-RG result is in good agreement with the result  $[m/g]_c = 0.31-0.33$  obtained by density matrix and lattice methods [19-21].

It is illustrative to compare the critical ratio  $[u/M^2]_c = 2/\beta^2$  obtained by the WH-RG method with the estimation based on the classical potential (13) by solving  $V_\Lambda''(\phi = 0) = 0$ , which gives  $[u/M^2]_c = 1/\beta^2$ , i.e.  $[m/g]_c = 2\sqrt{\pi} e^{-\gamma} / \beta^2 \approx 0.1584$ , a value being one-half of the one obtained by taking the quantum fluctuations into account.



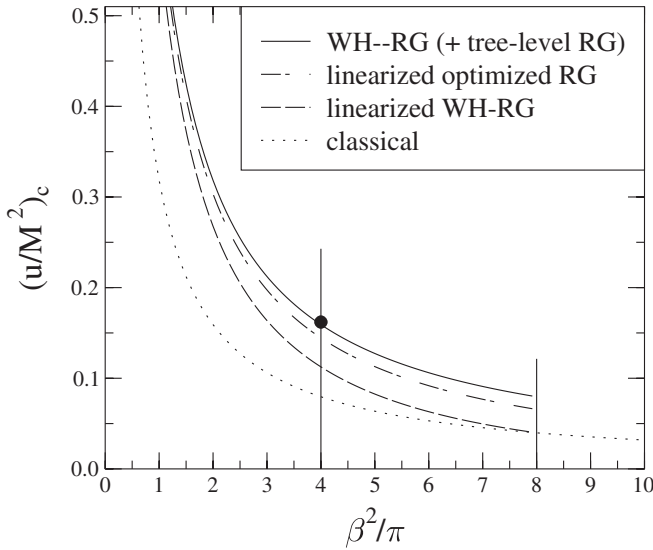


FIG. 6. The critical ratio that separates the phases of the MSG model obtained at classical level  $(u/M^2)_c = 1/\beta^2$  is compared to that of determined by WH-RG and EAA-RG methods. The full circle represents the results  $(u/M^2)_c = 0.156\text{--}0.168$  calculated by lattice methods [19–21] at  $\beta^2 = 4\pi$ . The solid line stands for  $(u/M^2)_c = 2/\beta^2$ , which is obtained by the full WH-RG (+ tree-level RG) method. The full EAA-RG (+ tree-level RG) method provides the same critical ratio (solid line). The dashed line is determined by the linearized WH-RG equation, and it is given by (22). The dashed-dotted line is obtained by the linearized optimized EAA-RG approach, see Eq. (27), which gives better result than the linearized WH-RG. For  $\beta^2 = 4\pi$  it is shown that only the usage of the full-RG equation gives a reliable result for the critical ratio, i.e.  $(u/M^2)_c = 0.159$ , which coincides to the results of the lattice calculations.

This means that the slope of the critical RG trajectory that separates the phases of the MSG model is steeper when the effect of the quantum fluctuations is taken into account, i.e. the latter enlarges the parameter region with explicitly broken symmetry, see Fig. 7. In general, the effect of quantum fluctuations leads to the shrinking of the region of spontaneous breakdown of symmetry. For example, the polynomial model defined by Eq. (B1) exhibits a double-well bare potential for the initial conditions with  $g_2(\Lambda) < 0$

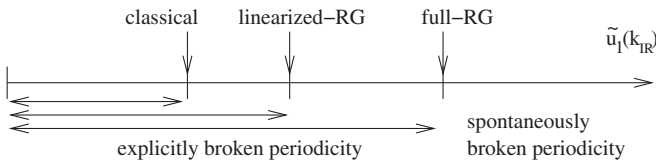


FIG. 7. Regions of explicitly and spontaneously broken periodicity of the MSG model obtained by various approximations. The critical value of the fundamental Fourier amplitude  $\tilde{u}_1(k_{\text{IR}})$  is calculated by classical, linearized-RG, and full-RG methods at some IR momentum scale  $k_{\text{IR}} \ll M \ll \Lambda$ . The effect of quantum fluctuations leads to the shrinking of the region of spontaneous breakdown of symmetry.

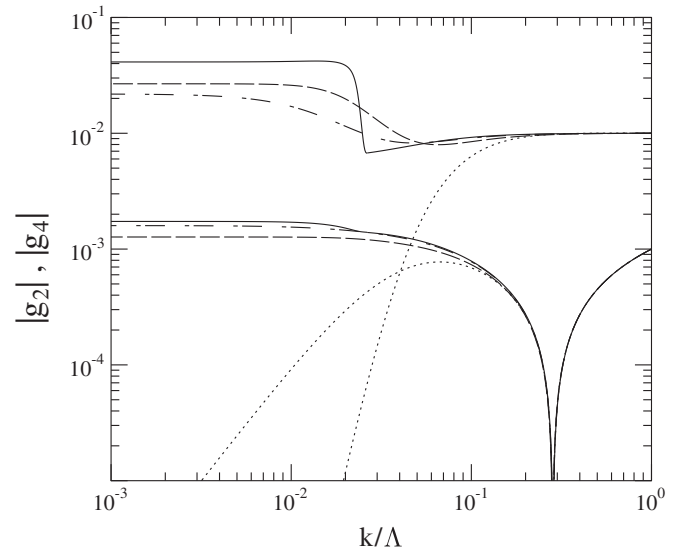


FIG. 8. The RG flow of the dimensionful couplings  $g_2(k)$  and  $g_4(k)$  obtained by various RG methods for the same initial condition  $g_2(\Lambda) = -0.001$ ,  $g_4(\Lambda) = 0.01$ ,  $g_{2n>4}(\Lambda) = 0$ . The full, dashed, and dashed-dotted lines correspond to the EAA-RG with quartic regulator, the EAA-RG with optimized regulator, and the WH-RG flow, respectively, and the dotted lines represent the flow obtained by the P-RG method. The mass  $g_2(k)$  changes its sign in any cases at  $(k/\Lambda) \sim 3 \cdot 10^{-1}$  for the given initial conditions. If the flow is determined by the quartic regulator RG,  $g_4(k)$  is nonanalytic at the momentum scale  $k_+ \approx 2.64 \cdot 10^{-2}$  for which  $k_+^2 = g_2(k_+)$ .

chosen in Appendix B, but the running mass  $g_2(k)$  changes its sign due to quantum effects (see Fig. 8), and the minimum of the effective potential remains at  $\phi = 0$ . Let us note that the classical analysis is not able to distinguish between the weak ( $\beta^2 < 8\pi$ ) and the strong ( $\beta^2 > 8\pi$ ) coupling regimes, it determines the critical ratio  $[u/M^2]_c = 1/\beta^2$  independently of the actual choice of the frequency  $\beta^2$ . Only the detailed RG study of the MSG model can show that for  $\beta^2 > 8\pi$  the SI, i.e. the condensate does not appear; hence the critical value cannot be extended for  $\beta^2 > 8\pi$ , see the vertical line in Fig. 6.

In Appendix B, it was demonstrated that beyond the mass scale in the deep IR limit the RG flow of the polynomial scalar field theory (in the phase with unbroken  $Z_2$  symmetry) determined by the P-RG differs from the usual one obtained by other RG methods. It was also shown in Sec. III that the P-RG approach is not able to signal the appearance of SI for the weak-coupling phase of the SG model. Consequently, the P-RG is inappropriate to investigate the IR scaling of the MSG model quantitatively; hence, we do not discuss the RG flow of the MSG model in the framework of the P-RG method.

Let us turn to the discussion of the MSG model (14) in the framework of the EAA-RG. Considerations similar to those made in the case of the WH-RG lead to the linearized evolution equation

$$(2 + k\partial_k)\tilde{u}_1 = \frac{\beta^2}{4\pi}\tilde{u}_1 \frac{k^4}{(k^2 + M^2)^2} \quad (23)$$

for the fundamental Fourier amplitude when the optimized regulator is made of use. Here,  $M^2$  and  $\beta^2$  are again scale-independent parameters, and the analytic solution

$$\begin{aligned} \tilde{u}_1(k) &= \tilde{u}_1(\Lambda) \left(\frac{k}{\Lambda}\right)^{-2} \left(\frac{k^2 + M^2}{\Lambda^2 + M^2}\right)^{\beta^2/8\pi} \\ &\times \exp\left(\frac{\beta^2}{8\pi} \left[\frac{M^2}{k^2 + M^2} - \frac{M^2}{\Lambda^2 + M^2}\right]\right) \end{aligned} \quad (24)$$

provides qualitatively the same UV scaling law as that of Eq. (16) and indicates the absence of the KTB-type phase transition. Let us determine the critical RG trajectory on the  $\tilde{u}_1 - \tilde{M}^2$  plane at linearized level,

$$\begin{aligned} \tilde{u}_{1c}(\tilde{M}^2) &= \tilde{u}_{1c}(\Lambda) \left(\frac{\tilde{M}^2}{\tilde{M}_\Lambda^2}\right)^{1-(\beta^2/8\pi)} \left(\frac{1 + \tilde{M}^2}{1 + \tilde{M}_\Lambda^2}\right)^{\beta^2/8\pi} \\ &\times \exp\left[\frac{\beta^2}{8\pi} \left(\frac{\tilde{M}^2}{1 + \tilde{M}^2} - \frac{\tilde{M}_\Lambda^2}{1 + \tilde{M}_\Lambda^2}\right)\right]. \end{aligned} \quad (25)$$

The critical initial bare value  $\tilde{u}_{1c}(\Lambda)$  is fixed by the condition  $\tilde{u}_{1c}(\tilde{M}^2 = 1) = 2/\beta^2$  similarly to the WH-RG case, then one finds

$$\begin{aligned} \tilde{u}_{1c}(\tilde{M}^2) &= \frac{2}{\beta^2} \tilde{M}^2 \left(\frac{1 + \tilde{M}^2}{2\tilde{M}^2}\right)^{\beta^2/8\pi} \\ &\times \exp\left[\frac{\beta^2}{8\pi} \left(\frac{\tilde{M}^2}{1 + \tilde{M}^2} - \frac{1}{2}\right)\right]. \end{aligned} \quad (26)$$

From (26) one can read off the IR value of the critical ratio

$$\left[\frac{u}{M^2}\right]_c = \lim_{\tilde{M}^2 \rightarrow \infty} \frac{\tilde{u}_{1c}}{\tilde{M}^2} = \frac{2}{\beta^2} \left(\frac{1}{2}\right)^{\beta^2/8\pi} e^{\beta^2/8\pi}, \quad (27)$$

which is smaller than the superuniversal ratio  $[u/M^2]_c = 2/\beta^2$  obtained by the exact WH-RG method but gives a better result than Eq. (22) obtained by the linearized WH-RG equation, see the dashed-dotted line in Fig. 6. Indeed, the optimized regulator has been constructed to achieve the best convergence of the truncated RG equation; consequently, between various RG equations considered at the same order of the truncation, the optimized one gives the closest result to the exact one [6].

However, neither the linearized WH-RG nor the linearized optimized EAA-RG enables one to map the phase structure of the MSG model in a reliable manner, i.e. to determine the exact critical ratio  $[u/M^2]_c$ , which can only be obtained by solving the full-RG equation in the IR limit  $k \rightarrow 0$ . As shown in the framework of the WH-RG method, if SI arises, its appropriate treatment is necessary to determine the effective potential beyond the scale of SI and the critical ratio  $[u/M^2]_c$ . The flow Eq. (A18) has a pole at  $U_k'' + M^2 = -k^2$ ; consequently, the IR effective potential beyond the scale  $k_{SI}$  is found to be identical with that of Eq. (A4) obtained by the WH-RG method. Therefore, the

critical ratio  $[u/M^2]_c$  coincides with that determined by the WH-RG approach (see the solid line in Fig. 6).

Finally, in the framework of the EAA-RG with the quartic regulator the linearization of Eq. (A17) results in the flow equation

$$\begin{aligned} (2 + k\partial_k)\tilde{u}_1 &= \frac{\beta^2}{4\pi}\tilde{u}_1 \frac{k^4}{\frac{1}{4}M^4} \mu_k^4 \\ &\times \begin{cases} 1 + \mu_k^2(\arctg \mu_k^2 - \frac{\pi}{2}), & \text{for } \frac{1}{2}M^2 < k^2 \\ -1 - \frac{\mu_k^2}{2} \ln(\frac{\mu_k^2 - 1}{\mu_k^2 + 1}), & \text{for } \frac{1}{2}M^2 > k^2 \end{cases} \end{aligned} \quad (28)$$

with  $\mu_k^2 = 1/2M^2/\sqrt{k^4 - 1/4M^4}$  and the scale-independent parameters  $M^2$  and  $\beta^2$ . Equation (28) provides again an UV scaling law that is very much like those of Eqs. (16) and (24). In order to find the validity range of the mass-corrected UV scaling law and to map the phase structure, one has to solve Eq. (A17) numerically. For sufficiently large initial values of  $\tilde{u}_1(\Lambda)$  an IR singularity appears in the RG flow at some scale  $k_{SI}$ , similarly to what happened in the framework of the WH-RG and the EAA-RG with the optimized regulator. One has to use Eq. (A16), i.e.  $M^2 + U_k'' = -C(b)k^2$  to treat the SI explicitly and arrives at  $U_k = -1/2[C(b)k^2 + M^2]\phi^2$  and the IR effective potential (A15) with  $C(2) = 2$ , i.e.  $V_{k \rightarrow 0} = -k^2\phi^2$ . Since  $U_{k=0} = -\frac{1}{2}M^2\phi^2$  is independent of the scheme-dependent constant  $C(b)$ , the critical ratio turns out to be independent of the scheme as well. Further on the potential  $U_{k=0}$  and, consequently the critical ratio  $[u_1/M^2]_c = 2/\beta^2$  are just the same as those found by the WH-RG method.

One can conclude, on the one hand, that the WH-RG, the CS-RG (being equivalent now with the WH-RG), and the EAA-RG methods enable one to determine the same phase structure and critical ratio  $[u/M^2]_c$  for the MSG model. On the other hand, the P-RG method is found to be inappropriate to follow the RG trajectories beyond the mass scale and to perform a quantitative analysis of the IR behavior of the MSG model. It was also shown that the truncated RG equations produce a scheme-dependent critical ratio but any exact RG (LPA is the only approximation used) gives the same (scheme-independent) result, which coincides with the results of density matrix and lattice methods.

## V. LAYERED SINE-GORDON MODEL

In this section we show that the critical frequency that separates the phases of an SG-type model, which undergoes a Kosterlitz-Thouless-Berezinskii type phase transition similar to the one for the two-dimensional SG model, can be obtained exactly (and scheme independently) by the linearized-RG flow. As an example, we consider the renormalization scheme dependence of the LSG model, i.e. the multicomponent scalar field theory where the two-dimensional periodic interaction terms are coupled by an

explicit mass matrix

$$\tilde{V}_k(\underline{\phi}) = \frac{1}{2} \underline{\phi}^T \tilde{M}_k^2 \underline{\phi} + \tilde{u}(k) \sum_{n=1}^{N_L} \cos(\beta \phi_n), \quad (29)$$

where  $N_L$  is the number of the coupled fields (i.e. number of “layers”) with the  $O(N_L)$  multiplet  $\underline{\phi} = (\phi_1, \dots, \phi_{N_L})$ . The mass matrix describes the interaction between the fields and is chosen here to be of the form

$$\frac{1}{2} \underline{\phi}^T \tilde{M}_k^2 \underline{\phi} = \frac{1}{2} \tilde{G}_k \left( \sum_{n=1}^{N_L} \phi_n \right)^2, \quad (30)$$

where  $\tilde{G}_k$  is the strength of the interfield interactions.

The WH-RG equation in LPA for the multicomponent LSG model presented in our previous publications [22–25] reads as

$$(2 + k\partial_k) \tilde{V}_k(\underline{\phi}) = -\frac{1}{4\pi} \ln[\det(\delta_{ij} + \tilde{V}_k^{ij}(\underline{\phi}))], \quad (31)$$

where  $\tilde{V}_k^{ij}(\underline{\phi})$  denotes the second derivatives of the potential with respect to  $\phi_i, \phi_j$ . Inserting the ansatz (29) into the RG Eq. (31) the right-hand side becomes periodic, while the left-hand side contains both periodic and nonperiodic parts. The nonperiodic part contains only mass terms, so that we obtain a trivial tree-level RG flow equation for the dimensionless mass matrix

$$(2 + k\partial_k) \tilde{M}_k^2 = 0, \quad (32)$$

which provides the trivial scaling  $\tilde{G}_k = k^{-2}G$ , where the dimensionful interfield coupling  $G$  remains constant during the blocking. We recall that in LPA there is no wavefunction renormalization; thus, the parameter  $\beta$  also remains constant during the blocking. Although the solution of Eq. (31) can only be obtained numerically, however, analytical results are also available using an approximation of Eq. (31) similarly to the MSG model. This is achieved by linearizing the WH-RG equation in the periodic piece of the blocked potential (not in the full potential),

$$(2 + k\partial_k) \tilde{U}_k(\phi_1, \dots, \phi_{N_L}) \approx -\frac{1}{4\pi} \frac{F_1(\tilde{U}_k)}{C}, \quad (33)$$

where  $\tilde{U}_k(\phi_1, \dots, \phi_{N_L}) = \tilde{u}(k) \sum_{n=1}^{N_L} \cos(\beta \phi_n)$  and  $C$  and  $F_1(\tilde{U}_k)$  stand for the constant and linear pieces of the determinant

$$\det[\delta_{ij} + \tilde{V}_k^{ij}] \approx C + F_1(\tilde{U}_k) + \mathcal{O}(\tilde{U}_k^2). \quad (34)$$

The mass-corrected linearized WH-RG for the coupled periodic model (29) reads as

$$(2 + k\partial_k) \tilde{u}(k) = \frac{\beta^2}{4\pi} \tilde{u}(k) \frac{k^2 + (N_L - 1)G}{k^2 + N_L G}, \quad (35)$$

where  $G$  and  $\beta^2$  are scale-independent parameters, and the solution can be obtained analytically

$$\tilde{u}(k) = \tilde{u}(\Lambda) \left( \frac{k}{\Lambda} \right)^{((N_L-1)\beta^2)/(N_L 4\pi) - 2} \left( \frac{k^2 + N_L G}{\Lambda^2 + N_L G} \right)^{\beta^2/N_L 8\pi}, \quad (36)$$

where  $\tilde{u}(\Lambda)$  is the initial value for the Fourier amplitude at the UV cutoff  $\Lambda$ . The critical frequency that separates the two phases of the model can be read out directly as

$$\beta_c^2(N_L) = \frac{8\pi N_L}{N_L - 1}. \quad (37)$$

For  $N_L = 1$  the coupled model (29) reduces to the massive 2D SG model with  $\beta_c^2 = \infty$ , which indicates the absence of the KTB phase transition. For  $N_L \rightarrow \infty$  the coupled SG model behaves like a massless 2D-SG model with the critical frequency  $\beta_c^2 = 8\pi$ . Our goal is to consider whether the determination of the critical frequency (37) does depend on the choice of renormalization scheme.

Since the P-RG method fails to determine the correct IR behavior of the one-component MSG model, here we do not apply the P-RG method to study the properties of the multicomponent LSG model. Therefore, let us turn directly to the RG analysis of the LSG model (29) by the EAA-RG method with the optimized regulator. Then the RG flow equation for the local potential of the  $N_L$ -component scalar field reads as

$$(2 + k\partial_k) \tilde{V}_k = \frac{1}{4\pi} \text{Tr} \left( \frac{\delta_{ij}}{\delta_{ij} + \tilde{V}_k^{ij}} \right), \quad (38)$$

where  $\tilde{V}_k^{ij} = \partial_{\phi_i} \partial_{\phi_j} \tilde{V}_k$ . Using the same machinery as in case of the WH-RG, the mass-corrected linearized form of the optimized regulator RG for the LSG model (29) reads as

$$(2 + k\partial_k) \tilde{u} = \frac{\beta^2}{4\pi} \tilde{u} \frac{k^4 + 2(N_L - 1)Gk^2 + N_L(N_L - 1)G^2}{(k^2 + N_L G)^2}, \quad (39)$$

where  $G$  and  $\beta^2$  are scale-independent parameters, and the solution can be obtained analytically,

$$\tilde{u}(k) = \tilde{u}(\Lambda) \left( \frac{k}{\Lambda} \right)^{((N_L-1)\beta^2)/(N_L 4\pi) - 2} \left( \frac{k^2 + N_L G}{\Lambda^2 + N_L G} \right)^{\beta^2/N_L 8\pi} \exp \left( \frac{\beta^2}{8\pi} \left[ \frac{G}{k^2 + N_L G} - \frac{G}{\Lambda^2 + N_L G} \right] \right), \quad (40)$$

where  $\tilde{u}(\Lambda)$  is the initial value for the Fourier amplitude at the UV cutoff  $\Lambda$ . This gives the same  $N_L$ -dependent critical frequency (37) as that obtained by the WH-RG method.

In general, any RG method, which is applicable to predict the IR behavior for the one-component massless and massive SG models, like the WH-RG, the EAA-RG with either the optimized or the quartic regulators, and the functional CS-RG methods, is assumed to produce the same  $N_L$ -dependent critical frequency (37) for the LSG model (29). This can be understood by using a suitable

$O(N_L)$  rotated form of the coupled LSG model where the mass matrix is diagonal. Since the rotation leaves the phase structure unchanged, therefore, instead of the original model one can investigate the rotated one where the single massive 2D-SG field can be considered perturbatively, which results in an effective SG-type model. This strategy can be applied for the multicomponent coupled LSG model with arbitrary number of components ( $N_L > 1$ ); hence, in the lowest order of the perturbative treatment the corresponding effective theory is always an SG-type model and, consequently, the various linearized-RG equations produce the same critical frequency.

## VI. SUMMARY

The renormalization of SG-type periodic scalar field models with explicit mass terms, i.e. the MSG and the multicomponent LSG models have been investigated by various functional RG methods using the LPA. Our aim was to compare the Wegner-Houghton, the Polchinski, the functional Callan-Symanzik methods, and the effective average action RG method with various regulator functions and to investigate the scheme dependence of the low-energy behavior of SG-type models. In particular, our goal was to consider under which conditions is it possible to determine scheme independently three physical parameters, the critical frequency  $\beta_c^2$  of the SG theory, the critical ratio  $u/M^2$  of the MSG model, and the layer-dependent critical value  $\beta_c^2(N_L)$  of the LSG model.

Even if the effects of the truncation of the functional subspace in which the effective potential is sought for are under control, the RG flow depends on the particular choice of the renormalization scheme due to the different ways the quantum fluctuations are eliminated. Nevertheless, no scheme dependence is expected if the UV scaling laws or the (exact) IR physics are considered. UV scaling laws are invoked from the linearized flow equations being insensitive to the choice of the renormalization scheme. The IR physics is obtained by integrating out all the quantum fluctuations, and the result does not depend on the manner this happened if no approximations (e.g. a truncation of the functional subspace) were used.

We demonstrated that the RG flows of the Wegner-Houghton, the functional Callan-Symanzik, and the effective average action RG methods are similar for polynomial models (in the phase with unbroken  $Z_2$  symmetry), for the generalized SG model (in both of its phases) as well as for the MSG model. Those methods enable one to determine the IR physics, the phase structure of the MSG model in a reliable manner, in good agreement with density matrix and lattice results. It was also shown that the Polchinski RG method is inappropriate to determine the RG flow beyond the mass scale in the deep IR limit. While this is not a drawback in the case of polynomial models and the generalized SG model (with vanishing mass) it disables

one to determine the phase structure of the MSG model quantitatively.

The phase structure of the MSG model (which is the Bose form of  $\text{QED}_2$ ) obtained by classical and quantum analysis were also compared. It is known that  $\text{QED}_2$  has two phases, the strong  $g \gg m$  and the weak  $g \ll m$  coupling phase, where  $m$  is the fermion mass and  $g$  is the coupling between the gauge and the fermion field. The critical ratio  $[m/g]_c$ , which separates the phases of  $\text{QED}_2$  obtained at the classical level, was found to be 2 times smaller than that determined by RG methods. Above (below) the critical ratio, the periodicity of the bosonized model (i.e. for the MSG model) is broken spontaneously (explicitly). Therefore, the region of spontaneous symmetry breaking in the parameter space is reduced at quantum level as compared to the classical one, which is in agreement with the general assumption that the effect of quantum fluctuations always shrinks the region of spontaneous symmetry breaking.

On the one hand, we showed that the critical frequency  $\beta_c^2$ , which separates the phases of an SG-type model that undergoes a KTB-type phase transition similar to the two-dimensional SG model, can be obtained exactly (and scheme independently) by the RG flow linearized around the UV Gaussian fixed point. This is the consequence of the extension of the UV scaling region down to the vicinity of the crossover region separating the UV and IR scaling regions. As examples we considered the renormalization of the one-component two-dimensional SG theory and the multicomponent LSG model where the two-dimensional SG fields of the various layers are coupled by an explicit mass matrix (29). On the other hand, it was also shown that the linearized-RG equations produce a scheme-dependent critical ratio  $[u/M^2]_c$  for the MSG model (i.e.  $[m/g]_c$  for  $\text{QED}_2$ ) and only the exact RG (where LPA was the only approximation used) gives the same (scheme-independent) result, which coincides with the critical ratio determined by density matrix and lattice methods. Therefore, this demonstrates that a KTB-type phase transition of an SG-type model is a crossover between the UV and IR scaling regimes where the linearization around the Gaussian UV fixed point gives reliable results but the Ising-type phase transition of the MSG model is found to be an IR one where one has to solve the full-RG equation in order to determine the exact transition point.

Finally, let us conclude with two general results that are not restricted to the particular models investigated. We have shown that in some cases it is possible to extend the validity of the UV linearized-RG flow down to the vicinity of the crossover region separating the UV and IR scales, which can produces physical parameters [such as the critical frequency  $\beta_c^2$  of the SG and the layer number dependent critical value  $\beta_c^2(N_L)$  of the LSG models] independently of the particular choice of the renormalization scheme. This is opposed to the general assumption, namely, that once



approximations are used nontrivial fixed points and their critical behavior should be scheme dependent. It was also shown that in case of spontaneous symmetry breaking the Maxwell construction represents a strong constraint on the RG flow, which results in a superuniversal IR behavior and as a consequence, scheme-independent IR results (such as the critical ratio  $[u/M^2]_c$  of the MSG model) can be obtained even if the local potential approximation is used. This receives important application in any case where the low-energy effective theory can only be determined by nonperturbative methods like functional RG approaches and a spontaneous symmetry breaking influences the low-energy behavior. Our results indicate that scheme-independent results can be obtained even in the LPA.

### ACKNOWLEDGMENTS

Fruitful discussions with G. Delfino are warmly acknowledged. We also thank G. Mussardo and P. Sodano for several discussions. A.T. has been supported by the ESF grant INSTANS and by the MIUR projects “Quantum Field Theory and Statistical Mechanics in Low Dimensions” and “Quantum Noise in Mesoscopic Systems.”

### APPENDIX A: RENORMALIZATION GROUP METHODS FOR $d = 2$ IN THE LOCAL POTENTIAL APPROXIMATION

In this appendix we settle our notations and remind the reader on some well-known features of the frequently used renormalization schemes in the LPA for two-dimensional Euclidean one-component scalar field theories. In principle, the various renormalization schemes are constructed in such a manner that the RG flow starts at the bare action and provides the effective action in the IR limit, so that the physical predictions (e.g. the critical exponents) are independent of the renormalization scheme particularly used. Nevertheless, scheme dependence may appear in the RG flow at intermediate scales due to the different manners the quantum fluctuations are eliminated in the various RG approaches. But even the physical predictions at the IR scales may depend on the used renormalization scheme if additional approximations are involved like the improperly strong reduction of the functional subspace in which the local potential is sought for or the linearization of the RG flow equations at the Gaussian fixed point. Therefore, it is of relevance to clarify how far the results obtained are independent of the particular choice of the renormalization scheme used.

The use of the sharp cutoff, i.e. the WH-RG approach [28] makes the blocking transformation transparent and simple since the modes to be eliminated are well defined. The price of this clarity is the incompatibility with the gradient expansion. One possible solution for this problem could be the usage of the smooth momentum cutoff, where

the higher frequency modes of the field are suppressed partially, but not eliminated. This can be realized by the P-RG [29] method for the bare action and by the EAA-RG approach [6,31–33] with various types of regulator functions. As a rule, the solution of the truncated RG flow depends on the particular choice of the regulator. Another treatment of the RG, which handles the effective action obtained by a suitable Legendre-transformation, is represented by the functional CS-RG or also called the internal space RG approach [30], where the quantum fluctuations are separated according to their amplitudes instead of their frequencies or length scales. In this case the blocking procedure is performed in the space of the field variable and not in the space-time, which is understood as the external space.

One of the advantages of the Wilsonian RG method is that the condensates, which are generated in the RG flow, can be treated in a simple manner. In fact, the condensates appear as nontrivial saddle points in the bare functional integral, which can easily be detected and handled by expanding around the maximum of the integrand [34]. On the contrary, one always finds a convex effective action if the effects of the possible nontrivial saddle points are correctly incorporated during the Legendre transformation; consequently, the Maxwell construction hides a large part of the dynamics generated by them. However, the RG methods based on the effective average action, which interpolates between the bare action and the full quantum effective action, provide generally an RG flow exhibiting singularity in a truncated functional subspace. The IR singularity of the functional RG equation is supposed to be related to the convexity of the effective action for theories within a phase of spontaneous symmetry breaking [32].

#### 1. Wegner-Houghton RG

The blocking in momentum space, i.e. the integration over the field fluctuations with momenta of the magnitude between the UV scale  $\Lambda$  and zero is performed in successive blocking steps over infinitesimal momentum intervals  $k \rightarrow k - \Delta k$ , each of which consists of the splitting the field variable  $\phi = \varphi + \phi'$  in such a manner that  $\varphi$  and  $\phi'$  contain Fourier modes with  $|p| < k - \Delta k$  and  $k - \Delta k < |p| < k$ , respectively, and the integration over  $\phi'$  leads to the WH-RG equation [28]

$$(2 + k\partial_k)\tilde{V}_k(\phi) = -\frac{1}{4\pi} \ln(1 + \tilde{V}_k''(\phi)), \quad (\text{A1})$$

with  $\tilde{V}_k''(\phi) = \partial_\phi^2 \tilde{V}_k(\phi)$  for the dimensionless local potential  $\tilde{V}_k = k^{-2}V_k$  for  $d = 2$  dimensions in the leading order of the derivative expansion, in the LPA when  $\phi$  reduces to a constant. (Below we suppress the notation of the field dependence of the local potential.) The differentiation with respect to the field variable and the multiplication with  $1 + \tilde{V}_k''$  leads to the derivative form of the WH-RG

equation

$$(2 + k\partial_k)\tilde{V}'_k = -\tilde{V}''_k(2 + k\partial_k)\tilde{V}'_k - \frac{1}{4\pi}\tilde{V}'''_k. \quad (\text{A2})$$

This equation is obtained by assuming the absence of instabilities for the modes around the gliding cutoff  $k$ . The WH-RG scheme, which uses the sharp gliding cutoff  $k$ , can also account for the spinodal instability, which appears when the restoring force acting on the field fluctuations to be eliminated vanishes,  $1 + \tilde{V}''_k(\phi) = 0$  at some finite scale  $k_{\text{SI}}$ , and the resulting condensate generates tree-level contributions to the evolution equation. The saddle point  $\phi'_0$  for the single blocking step  $k \rightarrow k - \Delta k$  is obtained by minimizing the action  $S_{k-\Delta k}[\phi] = \min_{\phi'_0}(S_k[\phi + \phi'_0])$ . The restriction of the space of saddle-point configurations to that of the plane waves  $\phi'_0 = \rho \cos(k_1 x)$  gives [34]

$$\tilde{V}_{k-\Delta k}(\phi) = \min_{\rho} \left[ \rho^2 + \frac{1}{2} \int_{-1}^1 du \tilde{V}_k(\phi + 2\rho \cos(\pi u)) \right] \quad (\text{A3})$$

in LPA, where the minimum is sought for the amplitude  $\rho$  only. It was shown that the tree-level RG Eq. (A3) leads to the local potential

$$\tilde{V}_k = -\frac{1}{2}\phi^2, \quad (\text{A4})$$

which can also be obtained as the solution of

$$1 + \tilde{V}''_k(\phi) = 0. \quad (\text{A5})$$

If SI occurs during the RG flow at some scale  $k_{\text{SI}} > 0$ , then Eqs. (A1) and (A2) can be applied only for scales  $k > k_{\text{SI}}$ , and the tree-level renormalization should be performed at scales  $k < k_{\text{SI}}$ . The right-hand side of Eq. (A1) develops a singularity when the SI occurs, but Eq. (A2) does not develop such a singularity, and its solution mathematically extends to  $k \rightarrow 0$ . It is interesting to notice, that Eq. (A2) yields the fixed-point equation

$$2\tilde{V}_* + [\tilde{V}'_*]^2 + \frac{1}{4\pi}\tilde{V}''_* = c_1, \quad (\text{A6})$$

with the arbitrary constant  $c_1$ , exhibiting the trivial solution  $\tilde{V}_* = c_1/2$  (Gaussian fixed point) and

$$\tilde{V}_* = -\frac{1}{2}\phi^2 + \frac{1}{8\pi} + \frac{1}{2}c_1, \quad (\text{A7})$$

which is equivalent to the fixed-point potential (A4).

## 2. Polchinski's RG

In the P-RG method [29] the realization of the differential RG transformations is based on a nonlinear generalization of the blocking procedure using a smooth momentum cutoff. In the infinitesimal blocking step the field variable  $\phi$  is split again into the sum of a slowly oscillating IR and fast oscillating UV components, but both

fields contain now low- and high-frequency modes, as well, due to the smoothness of the cutoff. Above the moving momentum scale  $k$  the propagator for the IR component is suppressed by a properly chosen smooth regulator function  $K(y)$  with  $y = p^2/k^2$ ,  $K(y) \rightarrow 0$  if  $y \gg 1$ , and  $K(y) \rightarrow 1$  if  $y \ll 1$ . The P-RG equation in LPA for  $d = 2$  dimensions reads as

$$(2 + k\partial_k)\tilde{V}_k = -[\tilde{V}'_k]^2 K'_0 + \tilde{V}''_k I_2, \quad (\text{A8})$$

where  $K' = \partial_y K(y)$ ,  $K'_0 = \partial_y K(y)|_{y=0}$ , and  $I_2 = (1/4\pi) \times \int_0^\infty K'(y)$ . The parameters  $K'_0$  and  $I_2$  can be eliminated by the rescaling of the potential and the field variable; consequently, they do not influence the physics. In order to make the comparison of the RG flows obtained by various RG methods straightforward, we choose  $I_2 \equiv -\frac{1}{4\pi}$  and  $K'_0 = -1$  for which the linearized forms of Eqs. (A1) and (A8) and the UV scaling laws obtained by WH-RG and P-RG are identical. Then the differentiation of both sides of Eq. (A8) with respect to the field variable  $\phi$  yields

$$(2 + k\partial_k)\tilde{V}'_k = 2\tilde{V}''_k\tilde{V}'_k - \frac{1}{4\pi}\tilde{V}'''_k \quad (\text{A9})$$

being independent of the regulator function  $K(y)$  and differing from the WH-RG Eq. (A2) by the term  $-\tilde{V}''_k k \partial_k \tilde{V}'_k$  with opposite sign for the nonlinear term.

Let us note on the one hand, that the P-RG method treats all quantum fluctuations below and above the scale  $k$  on the same footing. Therefore, even if there occurs a scale  $k_{\text{SI}}$  at which  $1 + \tilde{V}''_k$  exhibits zeros, one cannot decide unambiguously at what scale should one turn to tree-level renormalization. On the other hand, Eq. (A8) with the choice of the parameters  $K'_0 = -1$ ,  $I_2 = -\frac{1}{4\pi}$  leads to the fixed-point equation similar to that of (A6), with  $c_1 = 0$

$$2\tilde{V}_* - [\tilde{V}'_*]^2 + \frac{1}{4\pi}\tilde{V}''_* = 0 \quad (\text{A10})$$

and exhibits trivial fixed-point solutions: the Gaussian one ( $\tilde{V}_* = \text{const}$ ) and the high-temperature (or infinitely massive) fixed point

$$\tilde{V}_* = +\frac{1}{2}\phi^2 - \frac{1}{8\pi}, \quad (\text{A11})$$

which is only accounted for by the P-RG method. Let us note, that (A11) is similar to (A7), which is obtained in the framework of the WH-RG method but with opposite sign. For dimensions  $d = 2$  it was shown in [38] that nontrivial fixed-point solutions in LPA are either singular at finite  $\phi$  or periodic.

## 3. Effective average action RG

The EAA-RG method [31–33] has grown out of the idea of coarse graining the quantum fields, and it interpolates between the bare action and the full quantum effective action. The scale-dependent effective average action  $\Gamma_k$  satisfies the functional differential equation

$$k\partial_k\Gamma_k = \frac{1}{2}\text{Tr}(\Gamma_k^{(2)} + R_k)^{-1}k\partial_k R_k, \quad (\text{A12})$$

where  $\Gamma_k^{(2)}$  denotes the second functional derivative of the effective action. Here,  $R_k$  is a properly chosen IR regulator function that fulfills a few basic constraints to ensure that  $\Gamma_k$  approaches the bare action in the UV limit ( $k \rightarrow \Lambda$ ) and the full quantum effective action in the IR limit ( $k \rightarrow 0$ ) and to guarantee that no IR divergences are encountered in the presence of massless modes. In the present paper we shall use  $R_k(p^2) \equiv p^2 r(y)$  with the power-law regulator  $r(y) = y^{-b}$  ( $b > 1$ ) [33] and the optimized regulator [6]  $r(y) = (\frac{1}{y} - 1)\Theta(1 - y)$  [ $\Theta$  denotes the Heaviside step-function], where  $y = p^2/k^2$ . For  $d = 2$  dimensions, Eq. (A12) can be rewritten in the LPA as

$$(2 + k\partial_k)\tilde{V}_k = -\frac{1}{4\pi} \int_0^{\Lambda^2/k^2} dy \frac{r'y^2}{y(1+r) + \tilde{V}_k''}, \quad (\text{A13})$$

with  $r' = dr/dy$  for the dimensionless local potential. The truncation of the basis set of functions in which the local potential is expanded may introduce a dependence of the physical results (e.g. the critical exponents) on the particular choice of the regulator function. In the literature several optimization procedures have been proposed in order to achieve better convergence of the critical exponents with the removal of that truncation, e.g. Refs. [6,8]. For example, if one compares various RG schemes, it has been argued that the fastest convergence can be achieved by using the optimized regulator [6]. Concerning the power-law regulator it was argued that the optimal choice of the parameter  $b$  is met when the minimum value

$$C(b) = \frac{b}{(b-1)^{(b-1)/b}} \quad (\text{A14})$$

is maximal, which happens for  $b = 2$ , i.e. for the quartic regulator. For arbitrary parameter value  $b$ , the propagator in the right-hand side of Eq. (A13) may develop a pole at some scale  $k_{\text{SI}}$  and at some value of the field  $\phi$  for which  $\tilde{V}_k''(\phi) = -C(b)$  holds, which signals the occurrence of SI. It was shown that in such a case one has to seek the local potential for  $k < k_{\text{SI}}$  by minimizing  $\Gamma_k$  in the subspace of inhomogeneous (soliton like) field configurations and ends up with the result [32]

$$\tilde{V}_k = -\frac{1}{2}C(b)\phi^2 \quad (\text{A15})$$

of parabolic shape, the solution of the equation

$$C(b) + \tilde{V}_k''(\phi) = 0. \quad (\text{A16})$$

For the quartic regulator the integral over  $y$  can be performed and Eq. (A13) rewritten in the explicit form

$$(2 + k\partial_k)\tilde{V}_k = -\frac{1}{2\pi\sqrt{|1 - (\frac{1}{2}\tilde{V}_k'')^2|}} \times \begin{cases} \arctg(\frac{\frac{1}{2}\tilde{V}_k''}{\sqrt{1 - (\frac{1}{2}\tilde{V}_k'')^2}}) - \frac{\pi}{2}, & \text{for } |(\frac{1}{2}\tilde{V}_k'')| < 1 \\ \frac{1}{2} \ln(\frac{\frac{1}{2}\tilde{V}_k'' - \sqrt{(\frac{1}{2}\tilde{V}_k'')^2 - 1}}{\frac{1}{2}\tilde{V}_k'' + \sqrt{(\frac{1}{2}\tilde{V}_k'')^2 - 1}}), & \text{for } (\frac{1}{2}\tilde{V}_k'') > 1 \end{cases}, \quad (\text{A17})$$

when the limit  $\Lambda \rightarrow \infty$  is taken. Making use of l'Hospital's rule it is straightforward to show that the right-hand side of (A17) tends to  $\frac{1}{4\pi}$  for  $1/2\tilde{V}_k'' \rightarrow 1$  from either below or above and, consequently, the couplings behave nonanalytically at such scale  $k_+$ , but do not diverge to infinity. However, if there exists a scale  $k_- > 0$  for which  $1/2\tilde{V}_k'' \rightarrow -1$ , then the right-hand side of (A17) becomes infinite, which indicates the appearance of SI due to vanishing of the inverse of the IR regulated propagator. Therefore, the scale  $k_-$  can be identified with the scale of the SI,  $k_- \equiv k_{\text{SI}}$ . The scale  $k_+$  may exist even if no SI arises during the flow, it is an artifact of the regulator used, but of no physical significance. We shall demonstrate this latter case for the one-component scalar field theory with polynomial interaction. The two-dimensional generalized sine-Gordon model shall demonstrate another interesting case when the scales  $k_{\pm}$  are equal.

It is worthwhile noticing that Eq. (A13) with the power-law regulator leads to the WH-RG Eqs. (A1) and (A16) leads to Eq. (A5) for  $b = 1$  as well as for  $b \rightarrow \infty$  in the limit  $\Lambda \rightarrow \infty$ . This feature of the case  $b = 1$  holds only for  $d = 2$ .

The RG equation with the optimized regulator in the LPA reads as

$$(2 + k\partial_k)\tilde{V}_k = -\frac{1}{4\pi} \frac{\tilde{V}_k''}{1 + \tilde{V}_k''}, \quad (\text{A18})$$

where a field-independent constant has been added to the right-hand side of the equation. Let us note that if SI occurs, the right-hand side of Eq. (A18) becomes infinite at the scale  $k_{\text{SI}}$ , so that all or some of the couplings should tend to infinity at this scale. One should again turn to the explicit treatment of SI. The equation obtained by deriving (A18) with respect to the field and multiplying its both sides by  $(1 + \tilde{V}_k'')^2$  leads to the fixed-point equation  $2(1 + \tilde{V}_k'')^2\tilde{V}_k' = -\frac{1}{4\pi}\tilde{V}_k'''$ , which exhibits the trivial solution  $\tilde{V}_* = \text{const}$  (Gaussian fixed point), and also the parabolic solution (A7). Let us note, it was demonstrated that Eqs. (A18) and (A8) give the same critical exponents for the O(N) symmetric scalar theory in three dimensions [10], and the two RG equations can be transformed onto each other via a suitable Legendre transformation [9]. However, the RG trajectories and the singularity structures of (A18) and (A8) could be different.

#### 4. Functional Callan-Symanzik RG

In the functional CS-RG-type internal space RG method [30], the successive elimination of the field fluctuations is performed in the space of the field variable (internal space) as opposed to the usual RG methods where the blocking transformations are realized in either the momentum or the real (external) space. This can be achieved by introducing an additional mass term into the bare action

$$S_\lambda[\phi] = S_B[\phi] + \frac{1}{2}\lambda^2\phi^2, \quad (\text{A19})$$

with the control parameter  $\lambda$ . For  $\lambda = \lambda_0$  being of the order of the UV cutoff the large-amplitude fluctuations are suppressed and decreasing the control parameter  $\lambda$  toward zero, they are gradually accounted for. The functional evolution equation for the effective action is

$$\lambda\partial_\lambda\Gamma_\lambda = \frac{1}{2}\text{Tr}[\lambda^2 + \Gamma_\lambda^{(2)}]^{-1}2\lambda^2, \quad (\text{A20})$$

where  $\Gamma_\lambda^{(2)} = \delta^2\Gamma_\lambda/\delta\phi\delta\phi$ . Equation (A20) is equivalent to the RG Eq. (A12) for the EAA with the power-law regulator function with  $b = 1$  for  $\lambda = k$ . By using Eq. (A13) the functional Callan-Symanzik RG equation for the one-component scalar field theory for dimensions  $d = 2$  in the LPA reads

$$(2 + \lambda\partial_\lambda)\tilde{V}_\lambda = -\frac{1}{4\pi}\ln\left(\frac{1 + \tilde{V}_\lambda''}{\Lambda^2/\lambda^2 + 1 + \tilde{V}_\lambda''}\right), \quad (\text{A21})$$

where  $\tilde{V}_\lambda \equiv \lambda^{-2}V_\lambda$  is the dimensionless scale-dependent effective potential. Adding the field-independent term  $-1/(4\pi)\ln(\Lambda^2/\lambda^2)$  to the right-hand side of Eq. (A21), one can take the limit  $\Lambda \rightarrow \infty$

$$(2 + \lambda\partial_\lambda)\tilde{V}_\lambda = -\frac{1}{4\pi}\ln(1 + \tilde{V}_\lambda''). \quad (\text{A22})$$

This equation is mathematically equivalent to the two-dimensional WH-RG equation in the LPA assuming the equivalence of the scales  $\lambda \equiv k$ . However, for dimensions  $d \neq 2$  the functional Callan-Symanzik RG and the WH-RG differ from each other. Assuming the above mentioned equivalence of the scales  $\lambda$  and  $k$ , there occurs the same singularity in the right-hand side of (A22) as the one in the WH-RG approach. Therefore, the functional Callan-Symanzik RG signals the SI with the vanishing of the argument of the logarithm in the right-hand side of (A22). The solution of (A22) provides the scaling laws down to the scale  $k_{\text{SI}}$ , and one has to turn to the tree-level renormalization with the help of the WH-RG approach in order to determine the IR scaling laws.

#### APPENDIX B: COMPARISON OF RG SCHEMES FOR POLYNOMIAL INTERACTION

Here, we compare the applicability of the frequently used functional RG methods to two-dimensional one-component Euclidean scalar fields with polynomial self-interaction. We show that the flow of the polynomial model

determined by the P-RG differs of that obtained by other RG methods in the deep IR regime. For the polynomial local potential

$$\tilde{V}_k(\phi) = \sum_{n=1}^N \frac{1}{(2n)!} \tilde{g}_{2n}(k) \phi^{2n} \quad (\text{B1})$$

the scale dependence is encoded in the dimensionless coupling constants  $\tilde{g}_{2n}(k)$ , related to their dimensionful counterparts via  $k^2\tilde{g}_{2n} = g_{2n}$ . The bare action exhibits the  $Z_2$  symmetry  $\phi \rightarrow -\phi$ , and it is well known that there exist two phases of such a model with either unbroken or broken  $Z_2$  symmetry.

Let us start with the discussion of the RG flow in the phase with unbroken  $Z_2$  symmetry. Inserting the potential (B1) into any of the RG Eqs. (A2), (A18), (A17), and (A9) and expanding both sides of them into Taylor-series in the dimensionless field variable  $\phi$ , one arrives at a coupled set of ordinary, nonlinear differential equations for the couplings  $\tilde{g}_{2n}(k)$ . These have been solved numerically for the same initial condition  $g_2(\Lambda) = -0.001$ ,  $g_4(\Lambda) = 0.01$ ,  $g_{2n>4} = 0$  at the UV cutoff  $\Lambda$ , which ensured that no SI occurred, i.e. the inequality  $k^2 + g_2(k) > 0$  has been kept during the flow. The RG flow of the couplings  $g_2(k)$  and  $g_4(k)$  for various RG methods and for  $N = 10$  is plotted in Fig. 8. There were no appreciable changes in the results when we increased  $N$  further. The WH-RG, the EAA-RG with optimized regulator, and the EAA-RG with quartic regulator give qualitatively the same results, the dimensionful couplings become constant in the IR limit. Let us note that  $g_4(k)$  is nonanalytic at the scale for which  $k_+^2 = g_2(k_+)$  if the flow is determined by the quartic regulator RG.

For the P-RG method the flow of the couplings in the IR region ( $k \rightarrow 0$ ) differs of that obtained by WH-RG and EAA-RG methods (see Fig. 8). The RG trajectories tend to the trivial high-temperature (or infinitely massive) fixed point (A11), which is accounted only for the P-RG method. (Let us note, that in three dimensions the critical behavior is dominated by the nontrivial Wilson-Fisher fixed point.) However, one may argue that the RG analysis is physically uninteresting in the IR limit since below the momentum scale  $k^2 = g_2(k)$  the mass term suppresses the quantum fluctuations in the propagator and after a rather small transient domain, the RG flow obtained by the other RG methods becomes trivial, i.e. all the dimensionless couplings ( $\tilde{g}_2, \tilde{g}_4$ ) scale as  $\sim k^{-2}$  (for  $d = 2$ ); consequently, the corresponding dimensionful parameters ( $g_2, g_4$ ) tend to constants, see Fig. 8. It is also demonstrated in Fig. 8, that even the P-RG shows up the change of the sign of  $g_2(k)$  because it occurs above the mass scale. However, in this paper we show that there may be special situations such as, for example, the case of the MSG model where one has to go beyond the mass scale in order to map out the phase structure of the model in a reliable manner. In such a



case, the P-RG method is inappropriate for quantitative analysis.

The discussion above has been restricted to the phase with unbroken  $Z_2$  symmetry, when no SI occurs during the RG flow. The RG flow has also been investigated in the literature for the phase with spontaneously broken  $Z_2$  symmetry in the frameworks of both the WH-RG involving tree-level renormalization and the EAA-RG. The truncated WH-RG flow becomes numerically unstable, i.e. the couplings  $\tilde{g}_{2n}(k)$  start to heavily oscillate approaching the scale  $k_{\text{SI}}$  from above where the SI occurs. But it has been shown that the tree-level evolution for RG trajectories

started below the scale  $k_{\text{SI}}$  run into the universal IR fixed-point potential (A4). Applying the explicit treatment of SI proposed in [32], one would obtain similar results by the EAA-RG. It has been argued in [32] that the dimensionless IR effective potential for  $k < k_{\text{SI}}$  reads as the fixed-point potential (A15) [cf. Appendix A 3]. It was also demonstrated in [39] that the appearance of SI can be avoided by a suitable rescaling of the RG equations at least in case of the polynomial scalar theory. However, an attractive IR fixed point appears in the rescaled RG flow, which can be identified as the effective potential (A15).

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