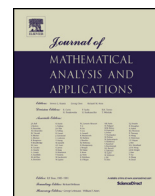




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Note

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ABSTRACT

The Kuratowski measure of noncompactness or the measure of nondensifiability provide direct approach to topological fixed point theorems or to existence issues of generalized fractals. We point out that these measures are not so distinguished as they appear at first glance: Requiring quite simple properties on a set-function, we can prove analogous results. The method behind (the main result of this note) is a reducing principle which allows to shrink bounded and closed domains to compact ones. In the approach, the Knaster–Tarski and the Kantorovich Fixed Point Theorems play a key role.

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1. Introduction

Although the classical result of Brouwer [4] might seem to be the simplest topological fixed point result, it plays a highly distinguished role in this theory. Indeed, many extensions apply Brouwer's result by reducing the situation to an essentially finite dimensional setting. For example, Schauder [18] completes the reduction with his famous approximation technique; Darbo [5] uses a Cantor-type intersection theorem; finally, Sadovskii [17] applies the Kuratowski measure of noncompactness [13]. In each case, the common idea behind is the construction of an invariant compact domain which allows the use of the Brouwer Theorem.

The most general result among the mentioned ones is due to Sadovskii [17]. The reason for the effectiveness of this approach is the Kuratowski measure of noncompactness. However, the role of this measure is not as distinguished as it seems at first glance. The aim of this note is to illustrate that some simple properties of a set-function provide the existence of a suitable “small” invariant domain, moreover: a “small” self-invariant subset in the invariant domain. The first statement is a consequence of the Knaster–Tarski fixed point theorem [12] while the second one follows from a result of Kantorovich [11]. Consult also the paper [1] for historical comments.

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As applications, we present topological fixed point theorems and revisit the existence part of Hutchinson's fundamental work in Fractal Theory [10]. Our investigations are partly based on the papers [2] and [3], and were inspired by the recent results of García and Mora [6], and Mena, Martín, and Sadarangani [16].

2. Basic concepts and the main result

For short, we use the convention $F: X \rightrightarrows X$ provided F is a self-map of the power set 2^X . For singletons, we simply write $F(x)$ instead of $F(\{x\})$.

Let us recall here two important fixed point results about set-valued functions since both play a crucial role in our investigations. The first one was obtained by Knaster and Tarski [12] while the other one is due to Kantorovich [11].

Lemma 1. (*Knaster–Tarski*) *If $F: X \rightrightarrows X$ is monotone (that is, preserves inclusion), then there exists $H \subseteq X$ such that $H = F(H)$.*

Lemma 2. (*Kantorovich*) *Assume that $F: X \rightrightarrows X$ commutes with countable chain union and there exists $x \in X$ such that $x \in F(x)$. Define the sequence (H_n) by*

$$H_1 = \{x\} \quad \text{and} \quad H_{n+1} = F(H_n).$$

Then $H = \bigcup_{n \in \mathbb{N}} H_n$ is the smallest set containing x and fulfilling $F(H) = H$.

In fact, one can change the set H_1 above to any set fulfilling $H \subseteq F(H)$ in order to arrive at the same conclusion. Now we collect the notions we need and list them in the next definitions.

Definition. A function $\chi: 2^X \rightarrow [0, +\infty]$ is called a *set-function* on X . A set function χ is

- *non-atomic* if $\chi(p) = 0$ for all $p \in X$;
- *finite* if it takes finite values, that is, $\chi(A) < +\infty$ for all $A \subseteq X$.

Based on the monograph of van de Vel [20], we recall the notion of convex structures. Observe that the convex hull in vector spaces or the closure operator in metric spaces fulfill all requirements.

Definition. A map $\Phi: X \rightrightarrows X$ is a *hull operator* on the nonempty set X , if it is

- *monotone*: $\Phi(A) \subseteq \Phi(B)$ whenever $A \subseteq B \subseteq X$;
- *idempotent*: $\Phi(\Phi(A)) = \Phi(A)$ for all $A \subseteq X$;
- *extensive*: $A \subseteq \Phi(A)$ for all $A \subseteq X$.

In this case, the pair (X, Φ) is a *convex structure*.

We have to link set-functions and hull operators properly and (motivated by the notion of condensing maps) clarify the mapping class we shall deal with:

Definition. A set-function χ on a convex structure (X, Φ) is Φ -*compatible* if, for all $A, B \subseteq X$,

$$\chi(\Phi(A \cup B)) \leq \max\{\chi(A), \chi(B)\}.$$

Definition. A map $F: X \rightrightarrows X$ is *decreasing* with respect to the set function χ (or simply: χ -decreasing) if $\chi(F(A)) < \chi(A)$ whenever $A \subseteq X$ and $0 < \chi(A) < +\infty$.

In the rest of this note we shall concentrate only on special types of set-valued functions called invariance operators. Their definition appear in the fundamental work of Hutchinson [10] and they help to capture self-similarity in an abstract form.

Definition. Let $\mathcal{F}$ be a family of self-maps of X . The invariance operator $F: X \rightrightarrows X$ induced by the family $\mathcal{F}$ is defined by

$$F(H) := \bigcup_{f \in \mathcal{F}} f(H).$$

We say that a set $H \subseteq X$ is an $\mathcal{F}$ -invariant if it fulfills the *invariance equation* $H = F(H)$.

Let us emphasize that an invariance operator is monotone, moreover it commutes with increasing chain union. However, we shall use frequently these evident properties.

Now we can formulate our main result. It states that, under suitable requirements, we can change the domain of an invariance operator to a “small” one such that it inherits the properties of the original domain and on which the invariance operator is still a self-map.

Theorem. Assume that (X, Φ) is a convex structure, $D \subseteq X$ is nonempty, Φ -invariant, and χ is Φ -compatible, finite, non-atomic on D . If $\mathcal{F}$ is a finite set of χ -decreasing self-maps of D and $F: D \rightrightarrows D$ is the induced invariance operator, then there exists a nonempty Φ -invariant set $H \subseteq D$ such that $F: H \rightrightarrows H$ and $\chi(H) = 0$.

Proof. Fix an arbitrary point $p \in D$. For a set $A \subseteq D$, define $T(A) \subseteq X$ as

$$T(A) = \Phi(F(A) \cup \{p\}).$$

Then $T: D \rightrightarrows D$ since $F(A) \cup \{p\} \subseteq D$ and D is Φ -invariant. Furthermore, T inherits the monotonicity of Φ and F . By the Knaster–Tarski fixed point theorem, there exists a subset H of D such that $T(H) = H$. We prove that H is the set we are looking for. Since Φ is extensive, $p \in H$ yielding $H \neq \emptyset$. Using that Φ is idempotent, we arrive at

$$\Phi(H) = \Phi(T(H)) = T(H) = H.$$

Thus H is Φ -invariant. On the other hand, Φ is monotone and extensive; hence

$$H = T(H) = \Phi(F(H) \cup \{p\}) \supseteq \Phi(F(H)) \supseteq F(H).$$

Therefore $F: H \rightrightarrows H$. Finally, assume to the contrary that $\chi(H) > 0$. Then $\chi(f_k(H)) < \chi(H)$ for all $k \in \{1, \dots, n\}$ since χ is finite valued. Using that χ is Φ -compatible and non-atomic,

$$\begin{aligned} \chi(H) &= \chi(T(H)) = \chi(\Phi(F(H) \cup \{p\})) \\ &= \chi(\Phi(f_1(H) \cup \dots \cup f_n(H) \cup \{p\})) \\ &\leq \max\{\chi(f_1(H)), \dots, \chi(f_n(H)), \chi(p)\} < \chi(H). \end{aligned}$$

This contradiction shows that $\chi(H) = 0$ and the proof is completed. $\square$

3. Applications

Our main result provides to shrink closed and bounded domains to compact ones. Using this method, we present fixed point and fractal results in various situations. First we study the next construction invented by Kuratowski [13].

Definition. Let (X, d) be a metric space. The *Kuratowski measure of noncompactness* of a set $H \subseteq X$ is the (extended) nonnegative real number

$$\chi_k(H) := \inf \left\{ \varepsilon > 0 \mid \exists (A_k)_{k=1}^n \subseteq X : H \subseteq \bigcup_{k=1}^n A_k, \text{diam}(A_k) \leq \varepsilon, k = 1, \dots, n \right\}.$$

Clearly, $\chi_k(H)$ is finite if and only if H is bounded; $\chi_k(H) = 0$ if and only if H is totally bounded. Further properties of this measure can be found in the monograph of Granas and Dugundji [9]. Not claiming completeness, we just subsume them. If $A, B \subseteq X$, then

- $\chi_k(A) \leq \chi_k(B)$ whenever $A \subseteq B$;
- $\chi_k(A \cup B) = \max\{\chi_k(A), \chi_k(B)\}$;
- $\chi_k(A) = \chi_k(\overline{A})$;
- $\chi_k(A) = \chi_k(\text{conv}(A))$ provided X is a normed space.

In complete metric spaces, $\chi_k(A) = 0$ if and only if A is relatively compact. This measure allows to extend the notion of contractivity: We say that $f: X \rightarrow X$ is *condensing*, if it is continuous and $\chi_k(f(H)) < \chi_k(H)$ holds for all bounded, but not totally bounded subsets. These properties guarantee that we can apply the main result to prove the Sadovskii fixed point theorem [17].

In all arguments, the closure-convex operator plays a key role. Although the composition of two hull operators is monotone and extensive, but need not be idempotent. However, the closure of a convex set is convex, and this property ensures that the composition in our setting remains a hull operator, indeed.

Corollary 1. *If D is a bounded, convex and closed subset of a Banach space X and $f: D \rightarrow D$ is condensing, then f has a fixed point in D .*

Proof. Let $\Phi := \overline{\text{conv}}$ and let χ_k be the Kuratowski measure of noncompactness. Then χ_k is non-atomic and finite (being D bounded), while Φ is a hull operator on D (as we have just mentioned). Moreover, the properties of χ_k show that

$$\chi_k(\overline{\text{conv}}(A \cup B)) = \chi_k(A \cup B) = \max\{\chi_k(A), \chi_k(B)\}.$$

Thus χ_k is Φ -compatible. If $\mathcal{F} = \{f\}$, then it consists of a χ_k -decreasing member and the induced invariance operator $F: D \rightrightarrows D$ is evidently monotone. By the main result, there exists a nonempty subset H of D such that H is Φ -invariant, that is, convex and closed in D ; moreover, $F: H \rightrightarrows H$ and $\chi_k(H) = 0$. Equivalently, H is convex and closed in X since D is closed, $f: H \rightarrow H$, and H is totally bounded. In particular, H is compact by the completeness of X . Hence the statement follows from the Schauder Fixed Point Theorem. $\square$

The main result has impacts in Fractal Theory, as well. Let (X, d) be a metric space and let $\mathcal{F}$ be a family of self-maps on X . Under an $\mathcal{F}$ -fractal we mean a nonempty, compact, and $\mathcal{F}$ -invariant subset of X . The question arises, quite evidently: Under what conditions can we state the existence and uniqueness of an $\mathcal{F}$ -fractal?

Hutchinson [10] proved that there exists a unique fractal provided that the underlying space is complete and $\mathcal{F}$ consists of finitely many contractions. Contractivity can be weakened if we are interested in the existence part only as Leśniak illustrates [14]. We refer to the monograph [15] by Leśniak, Snigireva, and Stobin as an excellent summary of the topic.

The mentioned achievements use the Blaschke completeness theorem. A completely different approach is suggested in the papers [2] and [3] via the Kuratowski measure of noncompactness. Now we revisit one of the main results [2, Theorem 2.]:

Corollary 2. *If D is a bounded, convex and closed subset of a Banach space X and $\mathcal{F}$ is a finite family of condensing self-maps of D , then there exists an $\mathcal{F}$ -fractal.*

Proof. Let $\Phi := \overline{\text{conv}}$ and let χ_k be the Kuratowski measure of noncompactness. As we have seen in the previous proof, Φ and χ_k fulfill the requirements of the main result. Consider the invariance operator $F: D \rightrightarrows D$ induced by $\mathcal{F}$. The main result ensures that there exists a nonempty, Φ -invariant (that is: closed and convex) set $H \subseteq D$ such that $F: H \rightrightarrows H$ and $\chi_k(H) = 0$. Thus H is totally bounded. The completeness of the underlying space results in the compactness of H .

The invariance operator commutes with countable chain union and Corollary 1 provides a fixed point of a member in $\mathcal{F}$. By Lemma 2, there exists a smallest $\mathcal{F}$ -invariant subset H containing that fixed point. The closure of the $\mathcal{F}$ -invariant set is compact (being a closed subset of the compact set H) and it is also $\mathcal{F}$ -invariant by [2, Lemma 3.] or by [3, Lemma 5.]. In other words: it is an $\mathcal{F}$ -fractal. $\square$

Contractions are condensing, therefore Corollary 2 extends the existence part of Hutchinson's result in Banach spaces. However, uniqueness cannot be guaranteed now.

Based on the concepts given by García and Mora [6], now we present another branch of applications of our main results.

Definition. Let $H \neq \emptyset$ be a bounded subset in a metric space (X, d) and let $\alpha \geq 0$. A continuous curve $c: [0, 1] \rightarrow H$ is α -dense in H if, for all $x \in H$, there exists a parameter $t \in [0, 1]$ such that $d(x, c(t)) \leq \alpha$. Denote the set of α -dense curves in H by $\Gamma_\alpha(H)$. The *degree of nondensifiability* of H is given by the real number

$$\chi_d(H) := \inf\{\alpha \geq 0 \mid \Gamma_\alpha(H) \neq \emptyset\}.$$

If $H \neq \emptyset$ is bounded, then $\chi_d(H)$ is finite. If additionally H is arc-connected, then $\chi_d(H) = 0$ if and only if H is totally bounded. Contrarily to the Kuratowski measure of noncompactness, the degree of nondensifiability is not a monotone set-function. However, some characteristic properties coincide or are similar. Let $A, B \subseteq X$ be nonempty, bounded sets. Then

- $\chi_d(A \cup B) \leq \max\{\chi_d(A), \chi_d(B)\}$ if A, B are arc-connected, and intersecting;
- $\chi_d(A) = \chi_d(\overline{A})$;
- $\chi_d(\text{conv}(A)) \leq \chi_d(A)$ provided X is a normed space;
- $\chi_d(\text{conv}(A \cup B)) \leq \max\{\chi_d(\text{conv}(A)), \chi_d(\text{conv}(B))\}$ provided X is a normed space.

In complete metric spaces, $\chi_d(H) = 0$ if and only if H is the relatively compact, provided that $H \neq \emptyset$ is bounded and arc-connected. This measure induces a condensing property, as well: We say that the map $f: X \rightarrow X$ is χ_d -condensing, if it is continuous and $\chi_d(f(H)) < \chi_d(H)$ holds for all nonempty, arc-connected, bounded, but not totally bounded subsets H .

The first consequence of the main result in this setting is an analogue of the Sadovskii Fixed Point Theorem obtained by García and Mora [7]. Since there are a few subtle differences in the proof compared to the proof of Corollary 1, we present the arguments behind.

Corollary 3. *If D is a bounded, convex and closed subset of a Banach space X and χ_d is the degree of nondensifiability on X , and $f: D \rightarrow D$ is χ_d -condensing, then f has a fixed point in D .*

Proof. Let $\Phi := \overline{\text{conv}}$ and let χ_d be the degree of nondensifiability. Since the subsets of a bounded subset are bounded, χ_d is finite and clearly nonatomic on D . The properties of χ_d directly show that χ_d is compatible with respect to the hull operator Φ :

$$\begin{aligned}\chi_d(\overline{\text{conv}}(A \cup B)) &= \chi_d(\text{conv}(A \cup B)) \\ &\leq \max\{\chi_d(\text{conv}(A)), \chi_d(\text{conv}(B))\} \leq \max\{\chi_d(A), \chi_d(B)\}.\end{aligned}$$

The same arguments as in the proof of Corollary 1 result in the existence of a nonempty, closed, convex set $H \subseteq D$ such that $f: H \rightarrow H$ and $\chi_d(H) = 0$. The underlying space is complete and H is arc-connected by its convexity, thus the last property means the total boundedness of H . But H is also complete, therefore it is compact. Hence the statement follows from the Schauder Fixed Point Theorem. $\square$

Recently García and Mora [8] and Mena, Martín, and Sadarangani [16] have demonstrated that the degree of nondensifiability is applicable in Fractal Theory. One can prove their result (see below) with appropriate modifications on the proof of Corollary 2 thus we omit the details.

Corollary 4. *If D is a bounded, convex and closed subset of a Banach space X and χ_d is the degree of nondensifiability on X , and $\mathcal{F}$ is a finite family of χ_d -condensing self-maps of D , then there exists an $\mathcal{F}$ -fractal.*

Finally let us note that the monograph [19] by Toledano, Benavides, and Acedo presents an axiomatic approach to measures of noncompactness. The required properties guarantee that these measures are compatible with the closed-convex hull operator. Thus the main result may be applied in such situations including for example the Hausdorff measure of noncompactness.

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