Alternative equations for generalized polynomials

Thesis for the degree
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written by Rayene Menzer
and supervised by Dr. Boros Zoltán

University of Debrecen
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Doctoral School of Mathematical and Computational Sciences
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Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences of the University of Debrecen in order to obtain a PhD Degree in Natural Sciences from the University of Debrecen.
I declare that the results published in this thesis are not reported in any other PhD thesis.

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Boros Zoltán
signature of the supervisor

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Dissertation advisor: Dr. Boros Zoltán

The official opponents of the dissertation:

Dr. ....
Dr. ....

The evaluation committee:

Chairperson: Dr. ....
Members: Dr. ....
Dr. ....
Dr. ....
Dr. ....

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Contents

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Introduction

Let $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ denote the set of real numbers, rationals, and positive integers, respectively. We call a function $f:\mathbb{R}\to\mathbb{R}$ *additive* on $\mathbb{R}$ if $f(x+y)=f(x)+f(y)$ holds for all $x,y\in\mathbb{R}$. The function f is called $\mathbb{Q}$-*homogeneous* if the equation $f(qx)=qf(x)$ is fulfilled by every $q\in\mathbb{Q}$ and $x\in\mathbb{R}$. As it is also well-known [**Kuc**, Theorem 5.2.1], if $f:\mathbb{R}\to\mathbb{R}$ is additive, then f is $\mathbb{Q}$-homogeneous as well.

We define the following sets:

$$S_0=\{(x,y)\in\mathbb{R}^2\,|\,xy=1\},$$
$$S_1=\{(x,y)\in\mathbb{R}^2\,|\,x^2-y^2=1\},$$
$$S_2=\{(x,y)\in\mathbb{R}^2\,|\,x^2+y^2=1\}.$$

Moreover, if p and q are regular (i.e., continuous), non-constant, real polynomials, while m is a real number (different from zero), we shall also consider the sets

$$\begin{aligned} R_{p,q} &= \{\,(p(t),q(t))\,|\,t\in\mathbb{R}\,\},\\ S_{1,m} &= \{(x,y)\in\mathbb{R}^2\,|\,x^2-my^2=1\}. \end{aligned}$$

Z. Kominek, L. Reich and J. Schwaiger [**KRS**] investigated additive real functions that satisfy the additional equation

$$\text{(E.1)}\qquad f(x)f(y)=0\quad\text{for every}\quad(x,y)\in D\,,$$

considering various subsets D of $\mathbb{R}^2$. In several cases, involving $D=R_{p,q}$ and $D=S_2$, as well as when D is measurable with a positive planar Lebesgue measure, they obtained $f(x)=0$ for every $x\in\mathbb{R}$.

The purpose of this dissertation is to generalize results involved in [**KRS**] to generalized monomials or generalized polynomials defined on the real line or more general domains. Therefore, it is reasonable to provide a short overview of the fundamental properties (involving the notions) of functions belonging to these classes. This short survey is contained in Chapter 1. We add further notable properties of these functions, that are not usually mentioned in monographs. These properties will be used in the subsequent chapters.

[**KRS**, Corollary 2] states the following theorem: Let v, $w : \mathbb{R} \to \mathbb{R}$ be arbitrary (ordinary) polynomials of degree at least one. If $f : \mathbb{R} \to \mathbb{R}$ is an additive function fulfilling

$$f(v(x))f(w(x)) = 0$$

for every $x \in \mathbb{R}$, then f equals zero identically. In Chapter 2 we extend this theorem to the case when f is a generalized polynomial. In addition, we formulate an analogous theorem involving two generalized polynomials.

As it was mentioned above, [**KRS**, Theorem 1] states that any additive function $f : \mathbb{R} \to \mathbb{R}$ fulfilling

$$\text{(E.2)} \qquad f(x)f(y) = 0 \quad \text{for every} \quad (x, y) \in S_2$$

has to be equal to zero identically. Boros and Fechner [**BF**, Theorem 1] proved the generalization of this result to the case when f is a generalized polynomial. On the other hand, P. Kutas [**Kut**] has recently established the existence of a non-zero additive function $f : \mathbb{R} \to \mathbb{R}$ fulfilling $f(x)f(y) = 0$ for all $(x, y) \in S_0$. So the case $D = S_0$ (a specific hyperbola) in (E.1) seems to be essentially different from the case $D = S_2$ (the unit circle). It is therefore a reasonable question whether one can extend any of the aforementioned results on $D = S_2$ to other particular conic sections, for instance, to the case $D = S_1$ (another hyperbola) or to the more general case $D = S_{1,m}$ for every non zero real number m. In Chapter 3 we investigate the conditional equation

$$\text{(E.3)} \qquad f(x)g(y) = 0 \quad \text{for every} \quad (x, y) \in S_{1,m}$$

under the assumption that f and g are generalized polynomials (of possibly different degrees).

In Chapters 4, 5 and 6, we consider non-algebraic constraints. These considerations are also motivated by a theorem due to Kominek, Reich and Schwaiger [**KRS**, Theorem 4], which states, for every positive integer N, that any additive function $f : \mathbb{R}^N \to \mathbb{R}$ fulfilling (E.1) has to be identically zero, if $D \subseteq \mathbb{R}^{2N}$ is a Lebesgue measurable subset with positive $2N$-dimensional Lebesgue measure.

In Chapter 4, we consider generalized polynomials $f : \mathbb{R}^N \to \mathbb{R}$ that satisfy the additional equation $f(x)f(y) = 0$ for the pairs $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N}$ has a positive $2N$ dimensional Lebesgue measure or it is a second category Baire set. We prove that $f(x) = 0$ for all $x \in \mathbb{R}^N$. In fact, some statements are established in a considerably more general setting. Namely, we investigate functions $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ satisfying the condition

$$f(x)g(y) = 0 \tag{E.4}$$

for all $(x, y) \in D$, where $D \subseteq X \times Y$ is large in some sense. We prove that f or g vanishes on a large subset of X or Y , respectively, in the same sense. Then we make use of a result by László Székelyhidi on the zeros of polynomials (and some of its counterparts) to conclude, in various settings, that whenever the product of generalized polynomials f and g vanishes, in the sense of equation (E.4), on a large set, then f or g equals zero identically. Another approach to this final conclusion was suggested by Eszter Gselmann. Some details of her suggestions are cited and presented as well.

In Chapter 5, we develop an analogy of these investigations for the sign of generalized monomials (of even degree) in real variables.

The analogy of Székelyhidi's theorem for the sign of real monomial functions can be formulated as follows: if a generalized monomial of an even degree in a real variable is non-negative on set of positive Lebesgue measure or on a second category Baire set, then it is non-negative everywhere. These results are discussed in Subsections 5.1.2 and and 5.2.2. The required tools from measure theory are enumerated in Subsection 5.1.1, while the topological versions of those ideas are discussed in Subsection 5.2.1. Applications for conditions involving the product space and the products of values are elaborated in

Subsections 5.1.3 and 5.2.3. Obvious counterexamples show that our results in Chapter 5 cannot be extended to more general domains.

Motivated by a reviewer's suggestion, we have extended several results of Chapters 4 and 5 to almost polynomial functions and almost monomial functions, respectively. This investigation is explored in Chapter 6.

Concerning Chapters 4, we concentrate on Corollary 4.12. (applied for $\mathcal{G} = \mathbb{R}^N$) and Corollary 4.22., which involve one polynomial function. The main tool in Section 6.1 is Ger's general result [**Ger**, Theorem 1] (see Theorem 6.6.), which states (in a more general setting) that every almost polynomial functions is almost everywhere equal to a polynomial function.

In Section 6.2, we extend our results, presented in Chapter 5, to almost monomial functions in a real variable. For this purpose, we introduce the corresponding concepts and establish a counterpart of Ger's theorem to this setting. Then we have to modify the remaining statements as well so that we can formulate our results in a convenient way. For instance, the original conclusion that $f(x)f(y) \geq 0$ everywhere has to be replaced with the property that $f(x) \geq 0$ almost everywhere or $f(x) \leq 0$ almost everywhere.

In the Summary, we conclude our study by synthesizing the results from previous chapters and discussing their broader implications. Finally, we close with a conclusion where we brush upon the core components of this work and state some potential future complementary research pursuits.

CHAPTER 1

Preliminaries

1.1. Concepts of monomial and polynomial functions

Let $(G, +)$ and $(H, +)$ denote Abelian groups. We call a function $f : G \to H$ *additive* if $f(x+y) = f(x) + f(y)$ holds for all $x, y \in G$. As it is well-known, the additivity of f implies the identity

$$f(rx) = rf(x) \tag{1.1}$$

for every $x \in G$ and for every integer r. Let X and Y denote linear spaces over the field of rationals. If $f : X \to Y$ is additive, then (1.1) is valid for every $x \in X$ and for every rational number r as well (cf. the proof of [**Kuc**, Theorem 5.2.1]).

Let m denote a positive integer. A function $F : G^m \to H$ is called *m-additive* if F is additive in each of its variables. Clearly, if $F : X^m \to Y$ is m-additive, then

$$\begin{aligned} & F(x_1, \dots, x_{j-1}, rx_j, x_{j+1}, \dots x_m) \\ = \; & rF(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots x_m) \end{aligned}$$

holds for every $j \in \{1, 2, \dots, m\}$, for every rational number r and for all $x_i \in X \quad (i = 1, 2, \dots, m)$. If $f : G \to H$ is defined as a diagonalization (or trace) of an m-additive mapping $F : G^m \to H$ as

$$f(x) = F(x, \dots, x) \tag{1.2}$$

for every $x \in G$, we say that f is a *generalized monomial of degree m* (or it is often called a *monomial function of degree m*). As a corollary, every generalized monomial $f : X \to Y$ of degree m fulfils the identity

$$f(rx) = r^m f(x) \tag{1.3}$$

for every $x \in X$ and for every rational number r. Replacing F with its symmetric part, we obtain that for every generalized monomial $f : G \to Y$ of degree m there exists a symmetric m-additive function $F : G^m \to Y$ such that (2) holds for every $x \in G$.

We may also call constant functions as (generalized) monomials of degree zero.

If f is a finite sum of generalized monomials, then f is called a *generalized polynomial* (or a *polynomial* function).

In case of real functions, basic examples of additive functions are the linear ones: $f(x) = cx$ for every $x \in \mathbb{R}$, where c denotes a fixed real number. The corresponding example for an m-additive function $F : \mathbb{R}^m \to \mathbb{R}$ is :

$$F(x_1, \ldots, x_m) = cx_1x_2\cdots x_m \qquad ((x_1, \ldots, x_m) \in \mathbb{R}^m). \tag{1.4}$$

Clearly, the diagolanization of F in (1.4) is of the form $f(x) = cx^m$ ($x \in \mathbb{R}$). We shall refer to these functions as *regular monomial*s. Finite sums of regular monomials will be called *regular polynomial*s. Hence a regular polynomial $P : \mathbb{R} \to \mathbb{R}$ can be expanded as

$$P(x) = \sum_{k=0}^{n} a_k x^k \qquad (x \in \mathbb{R}), \tag{1.5}$$

where n denotes a non-negative integer and $a_k \in \mathbb{R}$ ($k = 0, 1, \ldots, n$). However, it is well-known (established by Hamel [**Ham**]) that there exist non-linear additive functions $f : \mathbb{R} \to \mathbb{R}$. This implies the existence of monomial and polynomial functions that are different from regular monomials and regular polynomials, respectively.

1.2. Functional equations characterizing generalized monomials and polynomials

Let us suppose that f is a diagonalization of a symmetric m-additive function $F : G^m \to Y$. It is well known [**Kuc**, Lemma 15.9.2] that we have

$$\Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_m}f(x) = m!F(h_1, h_2, \ldots, h_m) \tag{1.6}$$

for all x, h_1, h_2, $\ldots$, $h_m \in G$, where $\Delta_y f(x) = f(x+y) - f(x)$ $(x, y \in G)$. This shows the uniqueness of F. Moreover (as it is also stated in [**Kuc**, Lemma 15.9.2]), equation (1.6) implies

$$\Delta_{h_1}\Delta_{h_2}\ldots\Delta_{h_n} f(x) = 0 \tag{1.7}$$

for all x, h_1, h_2, $\ldots$, $h_n \in G$ if n is an integer such that $n > m$.

It is a consequence of the identity (1.6) that any generalized monomial $f : G \to Y$ of degree m satisfies the m–monomial functional equation

$$\Delta_y^m f(x) = m! f(y) \qquad (x, y \in G), \tag{1.8}$$

where Δ_y^m denotes the m-th iterate of the operation Δ_y. In fact ([**Kuc**, Chapter 15], [**Sze91**, Chapter 1]), generalized monomials of degree m are characterized as the solutions of the m–monomial functional equation (1.8) if G is uniquely divisible by $(m+1)!$.

Let p denote a positive integer, $F_0 \in Y$, $f_0(x) = F_0$ for all $x \in G$, $F_k : G^k \to Y$ be a symmetric, k-additive function and

$$f_k(x) = F_k(\overbrace{x, x, \ldots, x}^{k}) \qquad (x \in G),$$

for every $k \in \{1, 2, \ldots, p\}$. Moreover, let

$$f(x) = \sum_{k=0}^{p} f_k(x) \qquad (x \in G). \tag{1.9}$$

Functions with representation (1.9) (defined for all $x \in G$) are called *generalized polynomials* (or *polynomial functions*) *of degree at most p*. As it follows from the above equation (1.7), if $f : G \to Y$ is a generalized polynomial of degree at most p, we have

$$\Delta_y^{p+1} f(x) = 0 \qquad (x, y \in G). \tag{1.10}$$

In fact ([**Kuc**, Chapter 15], [**Sze91**, Chapter 1]), generalized polynomials of degree at most p are characterized as solutions of the functional equation (1.10) if G is uniquely divisible by $(p+1)!$.

1.3. Further properties of generalized monomials and polynomials

In several subsequent chapters we shall investigate monomial or polynomial functions with real variables and values. So we enumerate further properties of real functions belonging to these families.

We begin this section by formulating the well known analogue of the celebrated binomial theorem for generalized monomials (established, for instance, in the monograph by L. Székelyhidi [**Sze91**, Chapter 1]).

1.1. Lemma. *If $m \in \mathbb{N}$, $F : \mathbb{R}^m \to \mathbb{R}$ is a symmetric m–additive function and f is defined by* (2)*, then*

$$f(x+y) = \sum_{j=0}^{m} \binom{m}{j} F_j(x,y) \tag{1.11}$$

for all $x, y \in \mathbb{R}$, where

$$F_j(x,y) = F(\underbrace{x,\ldots,x}_{m-j},\underbrace{y,\ldots,y}_{j})$$

As it was mentioned at the end of the previous section, the existence of discontinuous generalized monomials among real functions is a well known consequence of the axiom of choice (cf. [**Ham**]). However, we can establish the following limit property [**BM24a**, Lemma 2.2]:

1.2. Lemma. *If $f : \mathbb{R} \to \mathbb{R}$ is a generalized monomial of degree m and x, $y \in \mathbb{R}$, then*

$$\lim_{n\to\infty} f\left(x + \frac{y}{n}\right) = f(x).$$

Proof. Using the binomial theorem (Lemma 1.1.) we get

$$\begin{aligned} f\left(x+\frac{y}{n}\right) &= \sum_{j=0}^{m} \binom{m}{j} F_j\left(x\,,\,\frac{y}{n}\right) \\ &= f(x) + \sum_{j=1}^{m} \binom{m}{j} \frac{1}{n^j} F_j(x,y)\,. \end{aligned}$$

Clearly, each term $\frac{1}{n^j}F_j(x,y)$ tends to zero as n tends to infinity, hence the limit equals $f(x)$. □

In order to make use of the already mentioned $\mathbb{Q}$-homogeneity property of m-additive functions, in our arguments we shall repeatedly apply the following observation:

1.3. Lemma. *Let n denote a non-negative integer, $X \neq \emptyset$ (an arbitrary non-void set), $g_k : X \to \mathbb{R}$ $(k = 0, 1, \dots, n)$ and let T denote an infinite subset of the field of rational numbers. If*

$$\sum_{k=0}^{n} g_k(x) s^k = 0 \tag{1.12}$$

holds for every $s \in T$ and $x \in X$, then we have $g_k(x) = 0$ for every $x \in X$ and $k \in \{0, 1, \dots, n\}$.

Proof. If a regular real polynomial P equals zero for infinitely many distinct values of its variable s, then it is identically zero, i.e., the coefficient of s^k equals zero for every non-negative integer k (up to the degree of P). Clearly, it follows from the fact that for a not identically zero polynomial P the equation $P(s) = 0$ is satisfied only by a finite number of distinct values of s. We may apply this observation for each $x \in X$. □

A more general version of this lemma is explicitly stated in [**BG**, Lemma 1]. The application of this idea in the theory of functional equations goes back to the paper by Nishiyama and Horinouchi [**NH**].

Now we can formulate a useful observation concerning the set of zeros of a generalized polynomial [**BM23**, Lemma 2.1].

1.4. Lemma. *If $f : \mathbb{R} \to \mathbb{R}$ is a generalized polynomial, $I \subseteq \mathbb{R}$ is a non-degenerated interval and $f(x) = 0$ for every $x \in I$, then $f(x) = 0$ for all $x \in \mathbb{R}$.*

Proof. Due to our assumptions, there exist a positive integer n and k-additive mappings $A_k : \mathbb{R}^k \to \mathbb{R}$ $(k = 1, \dots, n)$ such that

$$f(x) = \sum_{k=0}^{n} A_k^*(x) \tag{1.13}$$

for every $x \in \mathbb{R}$, where $A_0^*(x) = A_0 \in \mathbb{R}$ and

$$A_k^*(x) = A_k(\underbrace{x, x, \dots, x}_{k}) \quad (x \in \mathbb{R}\,,\; k = 1, 2, \dots, n).$$

According to the hypothesis, $I \subseteq \mathbb{R}$ is an interval with positive length. From the density of $\mathbb{Q}$ in $\mathbb{R}$ we can see that for any real number $x \neq 0$ there exist infinitely many $r \in \mathbb{Q}$ such that $rx \in I$ and thus

$$0 = f(rx) = \sum_{k=0}^{n} A_k^*(rx) = \sum_{k=0}^{n} r^k A_k^*(x)\,.$$

We have just obtained a polynomial of degree (at most) n with infinitely many rational zeros. This implies that the polynomial is identically zero (cf. Lemma 1.3.), hence $A_k^*(x) = 0$ for every $k \in \{0, ,1, , \dots, n\}$, which yields $f(x) = 0$. In particular, we have $0 = A_0^*(x) = A_0 = f(0)$. Therefore, f vanishes on $\mathbb{R}$. $\square$

A generalization of this observation by L. Székelyhidi to generalized polynomials vanishing on a set of positive measure [**Sze85**, Theorem 2] as well as its category counterpart are discussed in Chapter 4 of this dissertation.

It is also clear that the set of all generalized monomials of degree m is a real linear space with respect to the point-wise operations for any non-negative integer m.

Concerning the structure of all generalized polynomials we can quote a result by Halter-Koch, Reich and Schwaiger [**HKRS**, Theorem 2] claiming that the set of generalized polynomials is an integral domain. Since the cited paper contains only a sketch of the proof of this fundamental property, we elaborate the details below in the particular case of real functions.

1.5. THEOREM. *The set of the generalized polynomials (among real functions) is an integral domain.*

PROOF. Let m and n denote positive integers, $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree m and $g : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree n. Define $h(x) = f(x)g(x)$ $(x \in \mathbb{R})$. Then there exist an m-additive mapping $F : \mathbb{R}^m \to \mathbb{R}$ and a n-additive mapping

$G : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = F(\underbrace{x, x, \ldots, x}_{m}) \qquad \text{and} \qquad g(x) = G(\underbrace{x, x, \ldots, x}_{n})$$

for every $x \in \mathbb{R}$. Now, for every $x_i \in \mathbb{R}$ $(i = 1, 2, \ldots, m, m+1, m+2, \ldots, m+n)$, let

$$\begin{aligned} & H(x_1\,,\, x_2\,,\, \ldots\,,\, x_m\,,\, x_{m+1}\,,\, x_{m+2}\,,\, \ldots\,,\, x_{m+n}) \\ = \;& F(x_1\,,\, x_2\,,\, \ldots\,,\, x_m)G(x_{m+1}\,,\, x_{m+2}\,,\, \ldots\,,\, x_{m+n}). \end{aligned}$$

Then $H : \mathbb{R}^{m+n} \to \mathbb{R}$ is $(m+n)$-additive and $h(x) = H(\underbrace{x, x, \ldots, x}_{m+n})$ for every $x \in \mathbb{R}$. Hence h is a generalized monomial of degree $m+n$.

We shall also prove that h is not identically zero if neither f nor g equals zero identically. Without loss of generality we may suppose $0 \le m \le n$ and choose $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}$ so that $f(x_0) \ne 0$ and $g(y_0) \ne 0$. As

$$0 \ne f(x_0) = F(\underbrace{x_0, x_0, \ldots, x_0}_{m}),$$

there exist a unique k, $0 \le k \le m$ such that

$$F_{m-k}(x_0\,,\, y_0) = F(\underbrace{x_0, x_0, \ldots, x_0}_{k}, \underbrace{y_0, y_0, \ldots, y_0}_{m-k}) \ne 0\,,$$

while

$$F_{m-j}(x_0\,,\, y_0) = F(\underbrace{x_0, x_0, \ldots, x_0}_{j}, \underbrace{y_0, y_0, \ldots, y_0}_{m-j}) = 0$$

for $0 \le j < k$. For every $r \in \mathbb{Q}$ we have

$$\begin{aligned} & h(rx_0 + y_0) = f(rx_0 + y_0)g(rx_0 + y_0) \\ = \;& \sum_{j=0}^{m} \sum_{i=0}^{n} \binom{m}{j} \binom{n}{i} r^{j+i} F_{m-j}(x_0\,,\, y_0) G_{n-i}(x_0\,,\, y_0). \end{aligned}$$

Here the coefficient of r^k equals

$$\begin{aligned}
&\sum_{j=0}^{k}\binom{m}{j}\binom{n}{k-j}F_{m-j}(x_0\,,\,y_0)G_{n-k+j}(x_0\,,\,y_0)\\
=\;&\binom{m}{k}\binom{n}{0}F(\underbrace{x_0,x_0,\dots,x_0}_{k},\underbrace{y_0,y_0,\dots,y_0}_{m-k})G(\underbrace{y_0,y_0,\dots,y_0}_{n})\\
=\;&\binom{m}{k}F(\underbrace{x_0,x_0,\dots,x_0}_{k},\underbrace{y_0,y_0,\dots,y_0}_{m-k})g(y_0)\neq 0\,,
\end{aligned}$$

therefore the regular polynomial $P(r)=h(rx_0+y_0)$ is not identically equal to zero, hence the function h is not identically equal to zero.

This argument shows that the products of generalized monomials are generalized monomials and thus the products of generalized polynomials are generalized polynomials. We obtain that the set of generalized polynomials is a unitary ring (where the multiplicative unit is the constant 1).

We also proved that the product of two not identically zero generalized monomials is a not identically zero function. Making use of this result, we shall prove that the ring of generalized polynomials does not contain any divisor of zero.

Let f and g denote two non-zero generalized polynomials of degrees n and m, respectively. Then there exist symmetric k-additive mappings $A_k:\mathbb{R}^k\to\mathbb{R}$ $(k=1,\dots,n)$ such that (1.13) holds, A_n^* is not identically equal to zero, and there exist symmetric j-additive mappings $B_j:\mathbb{R}^j\to\mathbb{R}$ $(j=1,\dots,m)$ such that

$$g(x)=\sum_{j=0}^{m}B_j^*(x) \tag{1.14}$$

for every $x\in\mathbb{R}$, where $B_0^*(x)=B_0\in\mathbb{R}$,

$$B_j^*(x)=B_j(x,x,\dots,x)\quad(x\in\mathbb{R}\,,\ j=1,2,\dots,m)$$

and B_m^* is not identically equal to zero. Let r be a rational number and $h(x) = f(x)g(x)$. Then

$$\begin{aligned} h(rx) &= \sum_{k=0}^{n} \sum_{j=0}^{m} A_k^*(rx) B_j^*(rx) \\ &= \sum_{k=0}^{n} \sum_{j=0}^{m} r^{k+j} A_k^*(x) B_j^*(x) \\ &= \sum_{l=0}^{n+m} r^l \sum_{i=\max\{0,l-m\}}^{\min\{n,l\}} A_i^*(x) B_{l-i}^*(x). \end{aligned}$$

Since the coefficient of r^{m+n} is $A_n^*(x)B_m^*(x) \neq 0$ (due to our previous assertion), Lemma 1.3. yields that h is not identically equal to zero.

$\square$

CHAPTER 2

The alternative equation along pairs of polynomials

The motivation of this chapter is the following result by Kominek, Reich and Schwaiger [**KRS**, Corollary 2]: Let v, w : $\mathbb{R} \to \mathbb{R}$ be arbitrary (ordinary) polynomials of degree at least one. If $f : \mathbb{R} \to \mathbb{R}$ is an additive function fulfilling

$$f(v(x))f(w(x)) = 0$$

for every $x \in \mathbb{R}$, then f equals zero identically. In this chapter we extend this theorem to the case when f is a generalized polynomial. Moreover, we establish a generalization of this statement involving two generalized polynomials.

2.1. Tools

We begin with the verification of the following statement.

2.1. Lemma. *If* $f : \mathbb{R} \to \mathbb{R}$ *is a generalized monomial and* $p : \mathbb{R} \to \mathbb{R}$ *is a regular real polynomial, then* $f \circ p$ *is a generalized polynomial.*

Proof. Let j be a positive integer such that f is a monomial of degree j, i.e., f is the diagonalization of the j-additive mapping $A : \mathbb{R}^j \to \mathbb{R}$. Moreover, let n be a non-negative integer and $a_k \in \mathbb{R}$ $(k = 0, 1, \ldots .n)$ such that

$$p(x) = \sum_{k=0}^{n} a_k x^k$$

for every $x \in \mathbb{R}$. Then

$$\begin{aligned} f(p(x)) &= f\left(\sum_{k=0}^{n} a_k x^k\right) = A\left(\sum_{k_1=0}^{n} a_{k_1} x^{k_1}, \dots, \sum_{k_j=0}^{n} a_{k_j} x^{k_j}\right) \\ &= \sum_{k_1=0}^{n} \cdots \sum_{k_j=0}^{n} A\left(a_{k_1} x^{k_1}, \dots, a_{k_j} x^{k_j}\right). \end{aligned}$$

For any fixed non-negative integers $k_l \in \{\, 0, \, , 1, \, , \dots, \, n \,\}$ $(l = 1, 2, \dots, j)$, let $s = \sum\limits_{l=1}^{j} k_l$ and

$$G(t_1, \dots, t_s) = A\left(a_{k_1} t_1 \dots t_{k_1} \, , \, a_{k_2} t_{k_1+1} \dots t_{k_1+k_2} \, , \, \dots \, , \, a_{k_j} t_{s-k_j+1} \dots t_s\right),$$

where any empty product equals 1 (i.e., for $k_l = 0$ we have only a_{k_l} in the l-th entry of A). Due to the distributivity of multiplication of real numbers and the j-additivity of A, G is s-additive and

$$A\left(a_{k_1} x^{k_1} \, , \, a_{k_2} x^{k_2} \, , \, \dots \, , \, a_{k_j} x^{k_j}\right) = G(x, x, \dots, x).$$

Thus $f \circ p$ is a finite sum of generalized monomials, hence it is a generalized polynomial. □

2.2. Main result

Now we can establish our main theorem, which involves two generalized polynomials as well as two non-constant regular polynomials with possibly different degrees.

2.2. Theorem. *Let p and q be polynomials of degrees at least one. If the generalized polynomials $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$f(p(x))g(q(x)) = 0 \tag{2.15}$$

for every $x \in \mathbb{R}$, then $f = 0$ or $g = 0$ identically.

Proof. Since generalized polynomials are obtained as finite sums of generalized monomials, Lemma 2.1. implies that both $f \circ p$ and $g \circ q$ are generalized polynomials. Now we can use Theorem 1.5. (which is an elaboration of [**HKRS**, Theorem 2] by Halter-Koch, Reich and

Schwaiger) claiming that the set of generalized polynomials is an integral domain. In particular, if the product of two generalized polynomials is identically equal to zero, then one of those generalized polynomials has to be identically zero as well. Therefore the functional equation (2.15) implies that $f(p(x)) = 0$ identically or $g(q(x)) = 0$ identically. Due to our assumptions the ranges $p(\mathbb{R})$ and $q(\mathbb{R})$ are unbounded intervals, hence f or g vanishes on an unbounded interval. Applying Lemma 1.4. we obtain that $f(x) = 0$ holds for every $x \in \mathbb{R}$ or $g(x) = 0$ holds for every $x \in \mathbb{R}$. □

2.3. Corollary. *Let p and q be polynomials of degrees at least one. If the generalized polynomial $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation*

$$f(p(x))f(q(x)) = 0 \tag{2.16}$$

for every $x \in \mathbb{R}$, then $f(x) = 0$ identically.

However the statement is not true if we replace regular polynomials p and q with generalized polynomials. A counterexample can be given as the following:

2.4. Example. Let H be a Hamel base in $\mathbb{R}$ (over $\mathbb{Q}$), p_0, $q_0 : H \to \mathbb{Q}$ be non-zero functions, $p, q : \mathbb{R} \to \mathbb{R}$ be additive extensions of p_0 and q_0, respectively. Then $p(\mathbb{R}) \subseteq \mathbb{Q}$ and $q(\mathbb{R}) \subseteq \mathbb{Q}$. Moreover, let $f : \mathbb{R} \to \mathbb{R}$ be a non-zero additive mapping such that $f(s) = 0$ for every $s \in \mathbb{Q}$. Then $f \circ p = 0$ and $f \circ q = 0$, hence $f(p(t))f(q(t)) = 0$ for every $t \in \mathbb{R}$.

CHAPTER 3

The alternative equation with constraints along conic sections

As it was mentioned in the introduction, [**KRS**, Theorem 1] states that any additive function $f : \mathbb{R} \to \mathbb{R}$ fulfilling

$$f(x)f(y) = 0 \quad \text{for every} \quad (x, y) \in S_2 \tag{3.17}$$

has to be equal to zero identically. Boros and Fechner [**BF**, Theorem 1] proved the generalization of this result to the case when f is a generalized polynomial. On the other hand, P. Kutas [**Kut**] has recently established the existence of a non-zero additive function $f : \mathbb{R} \to \mathbb{R}$ fulfilling $f(x)f(y) = 0$ for all $(x, y) \in S_0$. The details of this example are elaborated in Section 3.1.

So the case $D = S_0$ (a specific hyperbola) in (E.1) seems to be essentially different from the case $D = S_2$ (the unit circle). It is therefore a reasonable question whether one can extend any of the aforementioned results in case $D = S_2$ to other particular conic sections, for instance, to the case $D = S_1$ (another hyperbola) or to the more general case $D = S_{1,m}$ for every non-zero real number m (involving ellipses as well). We note that, in some sense, $S_{1,1} = S_1$ is on a half way from S_0 to $S_{1,-1} = S_2$, as it is geometrically analogous to S_0 and algebraically analogous to S_2 .

In Section 3.3 we investigate the conditional equation

$$f(x)g(y) = 0 \quad \text{for every} \quad (x, y) \in S_{1,m} \tag{3.18}$$

under the assumption that m is a fixed non-zero real number, while f and g are generalized polynomials (of possibly different degrees). We prove that $f = 0$ or $g = 0$ identically. The major tool in our arguments is obtained by a family of linear transformations that leave such a hyperbola invariant. This observation is elaborated in Section 3.2.

3.1. The counterexample due to Péter Kutas

A real number α is said to be transcendental over $\mathbb{Q}$ if it is not a root of any nonzero polynomial with coefficients in $\mathbb{Q}$.

In other words, $\alpha \in \mathbb{R}$ is transcendental over $\mathbb{Q}$ if, for every polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (a_i \in \mathbb{Q},\ a_n \neq 0),$$

we have $p(\alpha) \neq 0$.

3.1. Definition. A valuation on a field K is a function $v : K \to \mathbb{R} \cup \{\infty\}$ satisfying the following properties for all $x, y \in K$:

- $v(x) = \infty$ if and only if $x = 0$;
- $v(xy) = v(x) + v(y)$;
- $v(x + y) \geq \min\{v(x), v(y)\}$.

Clearly, if v is a valuation on a field K with the multiplicative unit 1, we have

$$v(1) = v(1) + v(1) = 2v(1) \quad \text{and} \quad v(1) = v(-1) + v(-1) = 2v(-1),$$

and thus $v(1) = v(-1) = 0$. This implies $v(-x) = v((-1)x) = v(-1) + v(x) = v(x)$ for every $x \in K$.

3.2. Proposition. *Let K be a field and let v be a valuation on K. Define*

$$O = \{x \in K \mid v(x) \geq 0\}.$$

Then O has the following properties:

- *O is a subring of K;*
- *for every nonzero element $x \in K$, $x \in O$ or $\frac{1}{x} \in O$.*

Proof. To show that O is a subring of K, we need to verify that O is closed under subtraction and multiplication.

Let $x, y \in O$. By the properties of valuations:

$$v(x - y) = v(x + (-y)) \geq \min(v(x), v(-y)) = \min(v(x), v(y)) \geq 0.$$

Thus, $x-y \in O$, proving that O is closed under subtraction. Similarly, since $v(xy) = v(x) + v(y)$, $v(x) \geq 0$ and $v(y) \geq 0$ imply $v(xy) \geq 0$. Thus, $xy \in O$, proving that O is closed under multiplication.

For the second part, consider any nonzero element $x \in K$. By the properties of valuations:

$$v(x) + v\left(\frac{1}{x}\right) = v(1) = 0.$$

This implies that either $v(x) \geq 0$ (hence $x \in O$), or $v(x) < 0$, in which case $v\left(\frac{1}{x}\right) = -v(x) > 0$ (hence $\frac{1}{x} \in O$).

Thus, O satisfies both required properties. $\square$

3.3. Proposition. *There exists a not identically zero additive function* $f : \mathbb{R} \to \mathbb{R}$ *with the property that* $f(x)f\left(\frac{1}{x}\right) = 0$ *(for every* $x \neq 0$*) if and only if there exists a* $\mathbb{Q}$*-linear subspace* U *of the real numbers such that* $U \neq \mathbb{R}$ *(so it is a proper subspace) and* $x \in U$ *or* $\frac{1}{x} \in U$ *for every real number* $x \neq 0$.

Proof. If f is an additive function with this property, then the set

$$U = \{x \in \mathbb{R} \mid f(x) = 0\}$$

will suffice. Now consider the reverse direction. Assume such a U is given. Then there exists a Hamel-basis H_0 of U which can be extended to a Hamel-basis H of $\mathbb{R}$. Let us define $g : H \to \mathbb{R}$ by

$$g(h) = \begin{cases} 0 & \text{for} \quad h \in H_0\,, \\ 1 & \text{for} \quad h \in H \setminus H_0\,. \end{cases}$$

Let $f : \mathbb{R} \to \mathbb{R}$ denote the unique additive extension of g. Then f is nonzero and has the desired property. $\square$

3.4. Theorem. *There exists a non-identically-zero additive function* $f : \mathbb{R} \to \mathbb{R}$ *for which* $f(x)f\left(\dfrac{1}{x}\right) = 0$ *(whenever* $x \neq 0$*).*

Proof. Let α be a transcendental number over $\mathbb{Q}$. Consider the field $K = \mathbb{Q}(\alpha)$, the field extension of the rational numbers by α. The field K is isomorphic to the field $\mathbb{Q}(t)$, the field of rational functions in one variable. Hence every element of K is the quotient of two polynomials in α, and we can define a valuation on K such that for non-zero polynomials p and q with rational coefficients let

$$v\left(\frac{p(\alpha)}{q(\alpha)}\right) = \deg(q) - \deg(p)\,.$$

This is well-defined as α is transcendental over $\mathbb{Q}$. This valuation is zero on $\mathbb{Q}$ but nontrivial on K. The valuation v on K can be extended to a valuation w on $\mathbb{R}$. Let O be the valuation ring of w. Then O is a $\mathbb{Q}$-linear subspace by Proposition 3.2. and by the fact that the valuation w is zero on $\mathbb{Q}$. Proposition 3.2. implies that, for every nonzero $x \in \mathbb{R}$, $x \in O$ or $\frac{1}{x} \in O$. Now Proposition 3.3. implies the existence of a suitable f. □

3.2. A basic calculation

The following basic observation is a powerful tool in the proof of our main theorem.

3.5. Lemma. *Let m denote an arbitrary real number. Suppose that x, y, α and β are real numbers such that*

$$x^2 - my^2 = 1 \quad \textit{and} \quad \alpha^2 - m\beta^2 = 1\,.$$

Then we have

$$(\alpha x + \beta m y)^2 - m(\beta x + \alpha y)^2 = 1$$

as well.

Proof. It is obtained by a straightforward calculation:

$$\begin{aligned}
&(\alpha x + \beta m y)^2 - m(\beta x + \alpha y)^2 \\
&= (\alpha^2 x^2 + 2\alpha x \beta m y + \beta^2 m^2 y^2) - m(\beta^2 x^2 + 2\beta x \alpha y + \alpha^2 y^2) \\
&= \alpha^2(x^2 - my^2) - m\beta^2(x^2 - my^2) \\
&= \left(\alpha^2 - m\beta^2\right)\left(x^2 - my^2\right) = 1.
\end{aligned}$$

□

3.6. Remark. The geometric interpretation of this observation is that, for any $(\alpha, \beta) \in S_{1,m}$, the linear transformation on $\mathbb{R}^2$ given by the matrix

$$\begin{pmatrix} \alpha & m\beta \\ \beta & \alpha \end{pmatrix}$$

leaves the conic section $S_{1,m}$ invariant.

3.3. Main results

3.7. Theorem. *Let m denote a non-zero real number. Suppose that f, $g\colon \mathbb{R} \to \mathbb{R}$ are generalized polynomials and $f(x)g(y) = 0$ for all solutions of the equation*

$$x^2 - my^2 = 1\,. \tag{3.19}$$

Then f or g is identically equal to zero.

Proof. Let $\sigma = m/|m|$ and let us define a mapping $\tilde{g}\colon \mathbb{R} \to \mathbb{R}$ by

$$\tilde{g}(u) = g\left(\frac{u}{\sqrt{|m|}}\right) \qquad (u \in \mathbb{R}).$$

Clearly, we have $\sigma \in \{-1\,,\,1\}$. Suppose that x, $y \in \mathbb{R}$ such that

$$x^2 - \sigma y^2 = 1\,. \tag{3.20}$$

Using the notation

$$\bar{y} = \frac{y}{\sqrt{|m|}}\,,$$

we observe that the pair $(x, \bar{y})$ fulfills the condition (3.19), so our assumption yields

$$0 = f(x)g(\bar{y}) = f(x)\tilde{g}(y).$$

Hence the generalized polynomials f and $\tilde{g}$ fulfill the assumptions of the theorem in the particular case when m is replaced by σ. Moreover, if $\tilde{g}$ is identically equal to zero, then g is identically equal to zero as well. Therefore it is sufficient to verify the validity of the theorem for $m = -1$ and $m = 1$. The particular case $m = -1$ and $f = g$ is covered by [**BF**, Theorem 1]. In fact, our argument follows and extends the ideas presented in [**BF**].

So let us assume that $m \in \{\,-1\,,\,1\,\}$, f, $g\colon \mathbb{R} \to \mathbb{R}$ are generalized polynomials and $f(x)g(y) = 0$ for all solutions of the equation (3.19).

Since f is a generalized polynomial, there exists a positive integer K and k-additive and symmetric functionals $A_k\colon \mathbb{R}^k \to \mathbb{R}$ for $k = 0, 1, \ldots, K$ such that

$$f(x) = \sum_{k=0}^{K} A_k(x, \ldots, x) \tag{3.21}$$

for all $x \in \mathbb{R}$. Analogously, there exists a positive integer N and n-additive and symmetric functionals $B_n\colon \mathbb{R}^n \to \mathbb{R}$ for $n = 0, 1, \ldots, N$ such that

$$g(x) = \sum_{n=0}^{N} B_n(x, \ldots, x) \tag{3.22}$$

for all $x \in \mathbb{R}$. Moreover, let

$$h(x) = \sum_{n=0}^{N} (-1)^{\lfloor \frac{n}{2} \rfloor} B_n(x, x, \ldots, x)$$

for every $x \in \mathbb{R}$. Clearly, $h\colon \mathbb{R} \to \mathbb{R}$ is a generalized polynomial as well.

Let us define the interval I such that I consists of all real numbers x fulfilling $x \geq 1$ if $m = 1$ and $0 \leq x \leq 1$ if $m = -1$, respectively. Now, let $x \in I$. Then there exists $0 \leq y \in \mathbb{R}$ such that $x^2 - my^2 = 1$. If α, β are real numbers such that $\alpha^2 - m\beta^2 = 1$, then Lemma 3.5. yields

$$(\alpha x + \beta m y)^2 - m(\beta x + \alpha y)^2 = 1\,.$$

According to our assumptions this implies the identity

$$f(\alpha x + \beta m y) g(\beta x + \alpha y) = 0\,. \tag{3.23}$$

Next, let us assume, in addition, that α and β are rationals. Moreover, let us denote

$$a_{k,l} = A_k(\underbrace{x, \ldots, x}_{l}, \underbrace{y, \ldots, y}_{k-l})$$

for $k = 0, 1, \ldots K$ and $l = 0, 1, \ldots k$, as well as

$$b_{n,r} = B_n(\underbrace{x, \ldots, x}_{r}, \underbrace{y, \ldots, y}_{n-r})$$

for $n = 0, 1, \ldots N$ and $r = 0, 1, \ldots n$. With this notation we can calculate that

$$f(\alpha x + \beta m y) = \sum_{k=0}^{K} A_k(\alpha x + \beta m y\,,\ \ldots\,,\ \alpha x + \beta m y)$$

$$= \sum_{k=0}^{K} \sum_{l=0}^{k} \binom{k}{l} (\alpha)^l (\beta m)^{k-l} a_{k,l} \tag{3.24}$$

and

$$g(\beta x + \alpha y) = \sum_{n=0}^{N} B_n(\beta x + \alpha y\,,\ \ldots\,,\ \beta x + \alpha y)$$

$$= \sum_{n=0}^{N} \sum_{r=0}^{n} \binom{n}{r} \alpha^{n-r} \beta^r b_{n,r}\,. \tag{3.25}$$

Due to equation (3.23), for every pair of rationals (α, β) fulfilling $\alpha^2 - m\beta^2 = 1$, at least one of the foregoing expressions is equal to zero.

What is more, we can find infinitely many distinct pairs (α_j, β_j) such that $\alpha_j^2 - m\beta_j^2 = 1$ and both α_j and β_j are rationals, so let us take

$$\alpha_j = \frac{mj^2 + 1}{mj^2 - 1} \quad \text{and} \quad \beta_j = \frac{2j}{mj^2 - 1} \tag{3.26}$$

for $j \in \mathbb{N}_1 \doteq \mathbb{N} \setminus \{1\}$.

Thus, for every $j \in \mathbb{N}_1$, we have

$$0 = \sum_{k=0}^{K} \sum_{l=0}^{k} \binom{k}{l} \left(\frac{mj^2 + 1}{mj^2 - 1}\right)^l \left(\frac{2mj}{mj^2 - 1}\right)^{k-l} a_{k,l} \tag{3.27}$$

or

$$0 = \sum_{n=0}^{N} \sum_{r=0}^{n} \binom{n}{r} \left(\frac{mj^2 + 1}{mj^2 - 1}\right)^{n-r} \left(\frac{2j}{mj^2 - 1}\right)^r b_{n,r}\,. \tag{3.28}$$

Multiplying (3.27) by $(mj^2-1)^K$, also multiplying (3.28) by $(mj^2-1)^N$, and introducing the functions

$$P(j)=\sum_{k=0}^{K}\sum_{l=0}^{k}\binom{k}{l}\left(mj^2+1\right)^l(2mj)^{k-l}(mj^2-1)^{K-k}a_{k,l}\,,$$

$$\tilde{P}(j)=\sum_{n=0}^{N}\sum_{r=0}^{n}\binom{n}{r}\left(mj^2+1\right)^{n-r}(2j)^r(mj^2-1)^{N-n}b_{n,r}\,,$$

we have $P(j)=0$ or $\tilde{P}(j)=0$ for each integer $j\in N_1$. Hence P or $\tilde{P}$ has infinitely many zeros. On the other hand, both P and $\tilde{P}$ are polynomials of degree not greater than $2K$ and $2N$, respectively. Therefore, one of them has to be identically equal to 0.

If P is identically equal to zero, we obtain

$$\begin{aligned}0 &= P(0)=\sum_{k=0}^{K}(-1)^{K-k}a_{k,k}=(-1)^K\sum_{k=0}^{K}(-1)^k a_{k,k}\\ &= (-1)^K\sum_{k=0}^{K}A(-x\,,\,-x\,,\,\ldots\,,\,-x)=(-1)^K f(-x)\,.\end{aligned}$$

Now let us investigate the case when $\tilde{P}$ is identically equal to zero. If $m=-1$, we have

$$\begin{aligned}0 &= \tilde{P}(-1)=\sum_{n=0}^{N}(-2)^n(-2)^{N-n}b_{n,n}=(-2)^N\sum_{n=0}^{N}b_{n,n}\\ &= (-2)^N g(x)\,,\end{aligned}$$

i.e., $g(x)=0$. So, we have $f(-x)=0$ or $g(x)=0$. Thus $f(-x)g(x)=0$ for every $x\in[0,1]$. It is a corollary of Lemma 2.1. that the mapping $x\mapsto f(-x)$ is a generalized polynomial as well. Since the family of all generalized polynomials constitutes an integral domain, the mapping $x\mapsto f(-x)g(x)$, being the product of two generalized polynomials, is a generalized polynomial as well. Since it vanishes on the interval $[0,1]$, Lemma 1.4. implies $f(-x)g(x)=0$ for every $x\in\mathbb{R}$. Recalling again that in an integral domain a product is zero if and only if one of its terms is zero, we get that $f(-x)=0$ for every $x\in\mathbb{R}$ or $g=0$ identically. This verifies the statement of the theorem.

If $m = 1$, we first note that the extention of $\tilde{P}$ to the field of complex numbers remains identically equal to zero. Denoting the imaginary unit by i, we have

$$\begin{aligned} 0 = \tilde{P}(i) &= \sum_{n=0}^{N} (2i)^n (-2)^{N-n} b_{n,n} \\ &= 2^N \sum_{n=0}^{N} i^n (-1)^{N-n} b_{n,n} = (-2)^N \sum_{n=0}^{N} (-i)^n b_{n,n} \,. \end{aligned}$$

which implies

$$0 = \sum_{n=0}^{N} (-i)^n b_{n,n} \,, \tag{3.29}$$

and

$$0 = \tilde{P}(-i) = \sum_{n=0}^{N} (-2i)^n (-2)^{N-n} b_{n,n} = (-2)^N \sum_{n=0}^{N} i^n b_{n,n} \,.$$

which implies

$$0 = \sum_{n=0}^{N} i^n b_{n,n} \,, \tag{3.30}$$

Taking the sum of the equations (3.29) and (3.30) (and dividing it by 2) we obtain

$$0 = \sum_{s=0}^{\lfloor \frac{N}{2} \rfloor} (-1)^s b_{2s,2s} \,. \tag{3.31}$$

Subtracting equation (3.29) from equation (3.30) (and multiplying it by $-i/2$) we obtain

$$0 = \sum_{s=0}^{\lfloor \frac{N-1}{2} \rfloor} (-1)^s b_{2s+1,2s+1} \,. \tag{3.32}$$

Now, considering the sum of the equations (3.31) and (3.32), we have

$$0 = \sum_{n=0}^{N} (-1)^{\lfloor \frac{n}{2} \rfloor} b_{n,n} = \sum_{n=0}^{N} (-1)^{\lfloor \frac{n}{2} \rfloor} B_n(x, x, \dots, x) = h(x) \,. \tag{3.33}$$

We have thus proved $f(-x)h(x) = 0$ for every real number $x \in I$. Since the product $x \mapsto f(-x) \cdot h(x)$ is a generalized polynomial of degree at most $K + N$,, we may apply Lemma 1.4. to obtain that $f(-x)h(x) = 0$ for all $x \in \mathbb{R}$. According to Theorem 1.5. (which is a particular case of [**HKRS**, Theorem 2]), the family of generalized polynomials has no divisor of zero, so we conclude that $f(-x) = 0$ for every $x \in \mathbb{R}$ or $h(x) = 0$ for every $x \in \mathbb{R}$. The fist option yields $f = 0$ identically. If $h = 0$ identically, we have

$$0 = h(rx) = \sum_{n=0}^{N} (-1)^{\lfloor \frac{n}{2} \rfloor} B_n(rx, rx, \ldots, rx) \tag{3.34}$$

$$= \sum_{n=0}^{N} (-1)^{\lfloor \frac{n}{2} \rfloor} B_n(x, x, \ldots, x) r^n . \tag{3.35}$$

for every rational r. So we have a polynomial that vanishes at every rational. Then this polynomial is identically equal to zero, hence each coefficient vanishes, i.e.,

$$0 = (-1)^{\lfloor \frac{n}{2} \rfloor} B_n(x, x, \ldots, x) ,$$

and thus $0 = B_n(x, x, \ldots, x)$ for every $n \in \{ 0, 1, \ldots, N \}$, In view of (3.22), this implies $g(x) = 0$ for every $x \in \mathbb{R}$. □

3.8. Corollary. *Let m denote a non-zero real number. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized polynomial and $f(x)f(y) = 0$ for all solutions of the equation $x^2 - my^2 = 1$. Then f is identically equal to zero.*

3.9. Corollary. *Let a and b denote positive real numbers and let $\sigma \in \{ -1 , 1 \}$. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized polynomial and $f(x)f(y) = 0$ for all solutions of the equation $\frac{x^2}{a^2} - \sigma \frac{y^2}{b^2} = 1$. Then f is identically equal to zero.*

Proof. Let u, w be real numbers fulfilling $u^2 - \sigma \frac{a^2}{b^2} w^2 = 1$. Moreover, let $g(t) = f(at)$ for all $t \in \mathbb{R}$. Clearly, then g is a generalized polynomial as well. For $x = au$ and $y = aw$ we have

$$\frac{x^2}{a^2} - \sigma \frac{y^2}{b^2} = u^2 - \sigma \frac{a^2}{b^2} w^2 = 1 ,$$

hence our assumption yields $g(u)g(w) = f(au)f(aw) = f(x)f(y) = 0$. Therefore g satisfies the assumptions in Corollary 3.8. with $m = \sigma\frac{a^2}{b^2}$, hence g is identically equal to zero, which yields $f(x) = g(x/a) = 0$ for every $x \in \mathbb{R}$ as well. $\square$

The above corollary involves hyperbolas and ellipses when $\sigma = 1$ or $\sigma = -1$, respectively.

CHAPTER 4

The alternative equation on big sets for generalized polynomials

In this chapter we consider generalized polynomials $f : \mathbb{R}^N \to \mathbb{R}$ that satisfy the additional equation $f(x)f(y) = 0$ for the pairs $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N}$ has a positive Lebesgue measure or it is a second category Baire set. We prove that $f(x) = 0$ for all $x \in \mathbb{R}^N$. In fact, some statements are established in a considerably more general setting. Namely, we consider Euclidean spaces (or even more general domains, for instance, σ-finite measure spaces) X, Y and we investigate functions $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ satisfying the condition

$$f(x)g(y) = 0 \tag{4.36}$$

for all $(x, y) \in D$, where $D \subseteq X \times Y$ is large in the sense of measure or category. We prove that f or g vanishes on a large subset of X or Y, respectively, in the same sense. In our arguments in this chapter and in the subsequent ones, we shall use the following notations several times.

4.1. Notation. For an arbitrary subset $D \subseteq X \times Y$, for any $x_0 \in X$ and for any $y_0 \in Y$, let us write

$$D_{x_0} = \{\, y \in Y \mid (x_0\,,\, y) \in D \,\}$$

and

$$D^*_{y_0} = \{\, x \in X \mid (x\,,\, y_0) \in D \,\}.$$

4.2. Remark. We shall use these notations for arbitrary subsets of Cartesian products of two sets (when we may use various notations for the involved sets). If we consider D as a relation, we could write

$$D_{x_0} = D(x_0) \quad \text{and} \quad D^*_{y_0} = D^{-1}(y_0)$$

as well.

Finally, we make use of a result by László Székelyhidi on the zeros of polynomials [**Sze85**, Theorem 2] (and some of its counterparts) to conclude, in various settings, that whenever the product of generalized polynomials f and g vanishes, in the sense of equation (4.36), on a large set, then f or g equals zero identically.

4.1. Equation with measure constraint

In this section we investigate measure constraints.

Our first theorem can be applied for the product of arbitrary functions over σ-finite measure spaces.

4.3. Theorem. *For each $j \in \{1, 2\}$, let $(X_j, \mathcal{A}_j, \mu_j)$ be a σ-finite measure space. Suppose that $f_j : X_j \to \mathbb{C}$ $(j = 1, 2)$ fulfill*

$$f_1(x)f_2(y) = 0 \tag{4.37}$$

for all $(x, y) \in D$, where $D \subseteq X_1 \times X_2$ is a $\mu_1 \times \mu_2$ measurable subset with positive product measure. Then there exist an index $j \in \{1, 2\}$ and $A_j \in \mathcal{A}_j$ such that $\mu_j(A_j) > 0$ and $f_j(x) = 0$ for every $x \in A_j$.

Proof. Let us consider the set

$$P_j = \{ x \in X_j \mid f_j(x) = 0 \} \quad (j = 1, 2).$$

We wish to prove the existence of $j \in \{1, 2\}$ and $A_j \subseteq P_j$ such that A_j is measurable and $\mu_j(A_j) > 0$.

Clearly, we have

$$D \subseteq (P_1 \times X_2) \cup (X_1 \times P_2).$$

Using our Notation 4.1., according to the properties of the product measure [**Hal**, § 34. Theorem A (p. 141), § 35. Theorems A and B (p. 143–144)], D_x is measurable for every $x \in X_1$, the mapping

$x \mapsto \mu_2(D_x)$ is μ_1-measurable and

$$(\mu_1 \times \mu_2)(D) = \int_{X_1} \mu_2(D_x) d\mu_1(x) .$$

Due to our assumption on the set D and the definition of the integral there exists a real number $r > 0$ and a measurable set $B \subseteq X_1$ such that $\mu_1(B) > 0$ and, for every $x \in B$, $\mu_2(D_x) \geq r$. If $D_x \subseteq P_2$ for some $x \in B$, we can verify the above statement with $A_2 = D_x$. Otherwise we have $B \subseteq P_1$ as $y \in D_x \backslash P_2$ implies $(x, y) \in D \backslash (X_1 \times P_2)$ and thus $(x, y) \in P_1 \times X_2$. Hence we can verify our statement with $A_1 = B$. $\square$

The statement is identical and the proof remains rather similar if, instead of the product measure $\mu_1 \times \mu_2$ (as defined by Halmos [**Hal**, § 34. Theorem A (p. 141), § 35. Theorems A and B (p. 143–144)]), we refer to its Lebesgue completion $\mu_1 \otimes \mu_2$ as defined, for instance, in a monograph by Bogachev [**Bog**, Theorem 3.3.1].

4.4. THEOREM. *For each $j \in \{1, 2\}$, let $(X_j, \mathcal{A}_j, \mu_j)$ be a σ-finite measure space. Suppose that $f_j : X_j \to \mathbb{C}$ $(j = 1, 2)$ fulfill*

$$f_1(x) f_2(y) = 0 \tag{4.38}$$

for all $(x, y) \in D$, where $D \subseteq X_1 \times X_2$ is a $\mu_1 \otimes \mu_2$ measurable subset with positive measure. Then there exist an index $j \in \{1, 2\}$ and $A_j \in \mathcal{A}_j$ such that $\mu_j(A_j) > 0$ and $f_j(x) = 0$ for every $x \in A_j$.

PROOF. Using the notation introduced in the proof of Theorem 4.3., we apply Fubini's theorem [**Bog**, Theorem 3.4.1 and Corollary 3.4.2] to conclude that D_x is measurable for μ_1-a.e. $x \in X_1$, the mapping $x \mapsto \mu_2(D_x)$ is μ_1-measurable and

$$(\mu_1 \otimes \mu_2)(D) = \int_{X_1} \mu_2(D_x) d\mu_1(x) .$$

The rest of the argument is identical to that in the proof of Theorem 4.3.. $\square$

In the sequel we apply our previous results to products of functions taken from particular families of functions.

4.5. DEFINITION. Let $(X\,,\,\mathcal{A}\,,\,\mu)$ denote a measure space (i.e., let $(X\,,\,\mathcal{A})$ denote a measurable space with a non-negative — possibly, but not identically, infinite — and σ-additive set function μ on $\mathcal{A}$). We call a family $\mathcal{F}$ of (possibly non-measurable) functions $f : X \to \mathbb{C}$ *algebraically measure regular* provided that the following implication is valid for every $f \in \mathcal{F}$: if $f(x) = 0$ for every $x \in B$, where $B \in \mathcal{A}$ and $\mu(B) > 0$, then $f(x) = 0$ for every $x \in X$.

So we call a family of functions *algebraically measure regular* if every member of this family that vanishes on a set of positive measure must be identically equal to zero.

Now we can formulate the main theorem of this section.

4.6. THEOREM. *For each $j \in \{\,1\,,\,2\,\}$, let $(X_j\,,\,\mathcal{A}_j\,,\,\mu_j)$ be a σ-finite measure space and let $\mathcal{F}_j$ denote an algebraically measure regular family of functions $f : X_j \to \mathbb{C}$. Let $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ such that*

$$f_1(x)f_2(y) = 0 \tag{4.39}$$

holds for all $(x, y) \in D$, where $D \subseteq X_1 \times X_2$ is a $\mu_1 \times \mu_2$ measurable subset with positive measure. Then f_1 or f_2 is identically equal to zero.

PROOF. Let us consider the set

$$P_j = \{\, x \in X_j \mid f_j(x) = 0 \,\} \quad (j = 1, 2).$$

Theorem 4.3. implies the existence of $j \in \{\,1\,,\,2\,\}$ and $A_j \subseteq P_j$ such that A_j is measurable and $\mu_j(A_j) > 0$. Now we may apply the assumptions on the families $\mathcal{F}_j$ $(j = 1, 2)$ to complete the proof. □

Analogously, as an applicaton of Theorem 4.4., we obtain the following result.

4.7. THEOREM. *For each $j \in \{\,1\,,\,2\,\}$, let $(X_j\,,\,\mathcal{A}_j\,,\,\mu_j)$ be a σ-finite measure space and let $\mathcal{F}_j$ denote an algebraically measure regular family of functions $f : X_j \to \mathbb{C}$. Let $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ such that*

$$f_1(x)f_2(y) = 0 \tag{4.40}$$

holds for all $(x, y) \in D$, *where* $D \subseteq X_1 \times X_2$ *is a* $\mu_1 \otimes \mu_2$ *measurable subset with positive measure. Then* f_1 *or* f_2 *is identically equal to zero.*

Since, for arbitrary positive integers k and m , the $k+m$ dimensional Lebesgue measure can be considered as the Lebesgue completion of the product of the k and m dimensional Lebesgue measures, we can establish the following particular case of Theorem 4.7..

4.8. COROLLARY. *For some* k , $m \in \mathbb{N}$, *let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *denote algebraically measure regular families of functions* $f_1 : \mathbb{R}^k \to \mathbb{C}$ *and* $f_2 : \mathbb{R}^m \to \mathbb{C}$ *related to the* k *and* m *dimensional Lebesgue measures, respectively. Suppose that* $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ *such that*

$$f_1(x) f_2(y) = 0 \tag{4.41}$$

holds for all $(x, y) \in D$, *where* $D \subseteq \mathbb{R}^{k+m}$ *is a measurable subset with a positive* $k+m$ *dimensional Lebesgue measure. Then* f_1 *or* f_2 *is identically equal to zero.*

Using induction, we can easily extend this corollary to arbitrary finite products.

4.9. COROLLARY. *Let* $n \in \mathbb{N}$ *and* $k_j \in \mathbb{N}$ $(j = 1\,, 2\,\ldots, n)$ *and* $m = \sum_{j=1}^{n} k_j$. *For each* $j \in \{1\,, 2\,\ldots, n\}$, *let* $\mathcal{F}_j$ *denote an algebraically measure regular family of functions* $f : \mathbb{R}^{k_j} \to \mathbb{C}$ *with respect to the* k_j *dimensional Lebesgue measure. Let* $f_j \in \mathcal{F}_j$ $(j = 1, 2, \ldots, n)$ *such that*

$$f_1\,(x_1)\, f_2\,(x_2) \cdots f_n\,(x_n) = 0$$

holds for all $(x_1\,, x_2\,, \ldots\,, x_n) \in D$ *where* $D \subseteq \mathbb{R}^m$ *is measurable with positive* m *dimensional Lebesgue measure. Then there exists an index* $j^* \in \{1\,, 2\,\ldots, n\}$ *such that* f_{j^*} *is identically equal to zero.*

PROOF. Clearly, the statement can be interpreted in the following way: if the family $\mathcal{F}$ consists of functions $f : \mathbb{R}^m \to \mathbb{C}$ admitting a representation

$$f\,(x_1\,, x_2\,, \ldots\,, x_n) = f_1\,(x_1)\, f_2\,(x_2) \cdots f_n\,(x_n)\,,$$

for every $x_j \in \mathbb{R}^{k_j}$, with some functions $f_j \in \mathcal{F}_j$ $(j = 1, 2, \ldots, n)$, then $\mathcal{F}$ is an algebraically measure regular family of functions. This is obvious for $n = 1$, while we can apply Corollary 4.8. (which establishes the present statement for $n = 2$) to verify that the validity of the statement for $n = r$ implies its validity for $n = r + 1$ for any positive integer r. $\square$

Now we consider functions defined on particular locally compact Abelian groups. It is well known [**Hal**, § 58. Theorem B (p. 254)] that there exists a Haar measure on such a group (which is unique on Borel sets up to a constant factor [**Hal**, § 60. Theorem C (p. 263)]). In our statements and arguments we refer to such a measure.

We shall make use of Székelyhidi's result on the zeros of generalized polynomials ([**Sze85**, Theorem 2], [**Sze91**, Theorem 3.3]):

4.10. THEOREM. *Let $\mathcal{G}$ be a locally compact Abelian group which is generated by any neighborhood of zero and let Z be a complex linear space. If a generalized polynomial $p : \mathcal{G} \to Z$ vanishes on a Haar measurable set of positive Haar measure, then it vanishes everywhere.*

Clearly, Theorem 4.10. states that the family of all generalized polynomials $p : \mathcal{G} \to \mathbb{C}$ constitutes an algebraically measure regular family of functions. Therefore, we obtain the following theorem as a corollary of Theorem 4.6. and Theorem 4.10..

4.11. THEOREM. *For each $j \in \{1, 2\}$, let G_j be a locally compact Abelian group which is generated by any neighborhood of zero, let μ_j denote the Haar measure on G_j, and let us assume that μ_j is σ-finite. Let $f_j : G_j \to \mathbb{C}$ be generalized polynomials $(j = 1, 2)$ fulfilling*

$$f_1(x) f_2(y) = 0 \tag{4.42}$$

for all $(x, y) \in D$, where $D \subseteq G_1 \times G_2$ is a $\mu_1 \times \mu_2$ measurable subset with positive measure. Then f_1 or f_2 is identically equal to zero.

4.12. COROLLARY. *Let $\mathcal{G}$ be a locally compact Abelian group which is generated by any neighborhood of zero. Let μ denote the Haar measure on $\mathcal{G}$, and let us assume that μ is σ-finite. Let $f : \mathcal{G} \to \mathbb{C}$ be* *a generalized polynomial fulfilling*

$$f(x)f(y) = 0 \tag{4.43}$$

for all $(x, y) \in D$, *where* $D \subseteq \mathcal{G}^2$ *is a* $\mu \times \mu$ *measurable subset with positive measure. Then* $f(x) = 0$ *for every* $x \in \mathcal{G}$ *.*

Analogously, we can establish the following theorem as a consequence of Corollary 4.9. and Theorem 4.10..

4.13. Theorem. *Let* $n \in \mathbb{N}$ *and* $k_j \in \mathbb{N}$ $(j = 1\,,\, 2\, \ldots, n)$ *and* $m = \sum_{j=1}^{n} k_j$ *. Let* $f_j : \mathbb{R}^{k_j} \to \mathbb{C}$ *be generalized polynomials* $(j = 1, 2, \ldots, n)$ *fulfilling*

$$f_1\left(x_1\right) f_2\left(x_2\right) \cdots f_n\left(x_n\right) = 0$$

for all $(x_1\,,\, x_2\,,\, \ldots\,,\, x_n) \in D$ *where* $D \subseteq \mathbb{R}^m$ *is measurable with positive* m *dimensional Lebesgue measure. Then there exists an index* $j^* \in \{\,1\,,\, 2\, \ldots, n\,\}$ *such that* f_{j^*} *is identically equal to zero.*

4.2. Equation with category constraint

4.2.1. Second category Baire sets as level sets. In this section we elaborate an analogy of the previous results when sets of positive measure are replaced with second category Baire sets in Euclidean spaces. We recall that $B \subseteq \mathbb{R}^N$ has the Baire property (or shortly, B is a Baire set) is there exist an open set $G \subseteq \mathbb{R}^N$ and a first category set $T \subseteq \mathbb{R}^N$ such that $B = G \triangle T$ (where the set operation $\triangle$ denotes the symmetric difference, as usual [**Oxt**], [**Kuc**, Chapter 2]). Clearly, on open subset G in $\mathbb{R}^N$ is of the second category if, and only if $G \neq \emptyset$. Hence $B \subseteq \mathbb{R}^N$ is a second category Baire set if, and only if, there exist a non-void open set $G \subseteq \mathbb{R}^N$ and a first category set $T \subseteq \mathbb{R}^N$ such that $B = G \triangle T$.

We shall also make use of the following simple statement.

4.14. Proposition. *If* $B \subseteq \mathbb{R}^N$ *is a second category Baire set and* $A \subseteq \mathbb{R}^N$ *is of the first category, then* $B \setminus A$ *is a second category Baire set as well.*

Proof. Due to our assumptions, there exist a non-void open set $G \subseteq \mathbb{R}^N$ and a first category set $T \subseteq \mathbb{R}^N$ such that $B = G \triangle T$. Then the set

$$S = (T \setminus A) \cup (G \cap A)$$

is of the first category (as a subset of $T \cup A$) and

$$B \setminus A = G \triangle S\,.$$

□

In what follows, we wish to establish a category version of Székelyhidi's Theorem 4.10.. For this purpose we need an analogy of Steinhaus' Theorem involving finitely many second category Baire sets.

4.15. Lemma. *Let $m \in \mathbb{N}$ and $A \subseteq \mathbb{R}^N$ such that A is a second category Baire set. Then there exists a neighborhood U of zero such that, for every $y \in U$, there exists $x \in \mathbb{R}^N$ fulfilling*

$$x + ky \in A \quad (k = 0, 1, \ldots, m).$$

Proof. Let $A = G \triangle T$, where $G \subseteq \mathbb{R}^N$ is a non-void open set and $T \subseteq \mathbb{R}^N$ is of first category. Hence $G \subseteq A \cup T$.

Let $r > 0$ and $x_0 \in G$ such that $B_{(m+1)r}(x_0) \subseteq G$, and let $U = B_r(0)$. Note that, for all $x \in B_r(x_0)$, $y \in U$ and $k \in \{0, 1, \ldots, m\}$ we have

$$d(x + ky, x_0) \leq d(x, x_0) + \sum_{j=1}^{k} d(x + jy, x + (j-1)y) < (k+1)r\,,$$

which implies $x + ky \in G$. Then we have $x \in G - ky$, and thus $x \in A \cup T - ky$. Hence, for all $x \in B_r(x_0)$ and $y \in U$, we have

$$x \in \bigcap_{k=0}^{m} [(A \cup T) - ky] = \bigcap_{k=0}^{m} [(A - ky) \cup (T - ky)]\,,$$

which yields

$$x \in \bigcap_{k=0}^{m} (A - ky) \cup \bigcup_{k=0}^{m} (T - ky) = \bigcap_{k=0}^{m} (A - ky) \cup \tau(T, y, m)\,,$$

where $\tau(T, y, m) = \bigcup_{k=0}^{m} (T - ky)$ is of the first category for every $y \in U$. Then $B_r(x_0) \not\subseteq \tau(T, y, m)$, hence, for every $y \in U$, there exists $x \in B_r(x_0)$ such that

$$x \in \bigcap_{k=0}^{m}(A - ky)\,,$$

hence $x + ky \in A$ for every $k \in \{0, 1, \dots, m\}$. ☐

In the particular case $m = N = 1$ this result was proved by S. Piccard [**Pic**]. Piccard's theorem has been generalized by several authors. Perhaps, a general result of A. Járai [**Jar**] could be applied to our case, but the verification of the validity of the assumptions to our specific settings does not seem essentially shorter then the self contained proof above.

Now we can establish a category version of Theorem 4.10..

4.16. THEOREM. *If a generalized polynomial $f : \mathbb{R}^N \to \mathbb{C}$ vanishes on a second category Baire set, then f vanishes everywhere.*

PROOF. Suppose that f is not identically zero. Then f has representation (1.9) with a not identically zero f_p. Let A denote a second category Baire set such that $f(x) = 0$ for every $x \in A$. Let $m = p$ and choose the neighborhood U of zero as in Lemma 4.15.. Then, for every $y \in U$, we have

$$f_p(y) = \frac{\Delta_y^p f_p(x)}{p!} = \frac{\Delta_y^p f(x)}{p!} = \frac{1}{p!}\sum_{k=0}^{p}\binom{p}{k}(-1)^{p-k} f(x+ky) = 0\,.$$

Then f_p vanishes on U and then, due to the identity $f_p(ru) = r^p f_p(u)$ $(r \in \mathbb{Q},\ u \in U)$ f_p vanishes everywhere on $\mathbb{R}^N$, contrary to our assumption. ☐

4.2.2. An alternative equation with category constraints.

We shall also need the Kuratowski–Ulam theorem [**KU**] (see also, e.g., [**Kuc**, Theorem 2.1.7] and [**Oxt**, Theorem 15.1]). We apply our Notation 4.1. as well.

4.17. THEOREM. *Let k and m denote positive integers and let $T \subseteq \mathbb{R}^{k+m} = \mathbb{R}^k \times \mathbb{R}^m$ be a set of the first category. Then there exists a set $K \subseteq \mathbb{R}^k$ of the first category such that, for every $x \in \mathbb{R}^k \setminus K$, the set T_x is of the first category.*

Now we can establish the analogy of Theorem 4.3. for second category Baire sets.

4.18. THEOREM. *Let* $k\,,\,m \in \mathbb{N}$ *. Suppose that* $f_1 : \mathbb{R}^k \to \mathbb{C}$ *and* $f_2 : \mathbb{R}^m \to \mathbb{C}$ *such that*

$$f_1(x)f_2(y) = 0 \tag{4.44}$$

holds for all $(x, y) \in D$ *, where* $D \subseteq \mathbb{R}^{k+m}$ *is a second category Baire set. Then there exists a second category Baire set* $A_1 \subseteq \mathbb{R}^k$ *such that* $f_1(x) = 0$ *for every* $x \in A_1$ *, or there exists a second category Baire set* $A_2 \subseteq \mathbb{R}^m$ *such that* $f_2(y) = 0$ *for every* $y \in A_2$ *.*

PROOF. Let us consider the sets

$$\begin{aligned} P_1 &= \{\, x \in \mathbb{R}^k \mid f_1(x) = 0 \,\} \quad \text{and} \\ P_2 &= \{\, y \in \mathbb{R}^m \mid f_2(y) = 0 \,\}. \end{aligned}$$

We wish to prove the existence of $j \in \{\,1\,,\,2\,\}$ and $A_j \subseteq P_j$ such that A_j is a second category Baire set.

Clearly, we have

$$D \subseteq (P_1 \times \mathbb{R}^m) \cup (\mathbb{R}^k \times P_2)\,.$$

Due to our assumption on D, there exist a non-void open set $G \subseteq \mathbb{R}^{k+m}$ and a first category set $T \subseteq \mathbb{R}^{k+m}$ such that $D = G \triangle T$, hence we have $G \subseteq D \cup T$. On the other hand, there exist $(x_0\,,\,y_0) \in \mathbb{R}^k \times \mathbb{R}^m$ and $r > 0$ such that

$$B_r(x_0) \times B_r(y_0) \subseteq G\,.$$

Combining these inclusions we obtain

$$B_r(x_0) \times B_r(y_0) \subseteq \big((P_1 \times \mathbb{R}^m) \cup (\mathbb{R}^k \times P_2)\big) \cup T\,. \tag{4.45}$$

We shall also consider the set $K \subseteq \mathbb{R}^k$ with the properties described in Theorem 4.17..

Let

$$A = B_r(x_0) \cap P_1 \quad \text{and} \quad S = B_r(x_0) \setminus P_1\,.$$

Now, we consider two cases. First, let us assume that S is of the second category. Then $S \setminus K \neq \emptyset$, hence there exists $x_1 \in S \setminus K$.

Then, for every $y \in B_r(y_0)$, $y \in P_2$ or $y \in T_{x_1}$, hence we have

$$B_r(y_0) \subseteq P_2 \cup T_{x_1} .$$

Due to the choice of x_1 the set T_{x_1} is of the first category. Then the set

$$\tilde{T} = T_{x_1} \cap B_r(y_0)$$

is of the first category as well, hence $B = B_r(y_0) \cap P_2$ can be represented in the form

$$B = B_r(y_0) \setminus \tilde{T} = B_r(y_0) \triangle \tilde{T} .$$

Therefore, $B \subseteq \mathbb{R}^m$ is a second category Baire set and f_2 vanishes on B .

In the second case, when S is of the first category,

$$A = B_r(x_0) \setminus S = B_r(x_0) \triangle S$$

yields that $A \subseteq \mathbb{R}^k$ is a second category Baire set and f_1 vanishes on A . □

Now we introduce an analogy of Definition 4.5..

4.19. Definition. Let $k \in \mathbb{N}$. We call a family $\mathcal{F}$ of functions $f : \mathbb{R}^k \to \mathbb{C}$ *algebraically Baire regular* provided that the following implication is valid for every $f \in \mathcal{F}$: if $f(x) = 0$ for every $x \in B$, where $B \subseteq \mathbb{R}^k$ is a second category Baire set, then $f(x) = 0$ for every $x \in \mathbb{R}^k$.

So we call a family of functions *algebraically Baire regular* if every member of this family that vanishes on a second category Baire set must be identically equal to zero.

Now we are ready to formulate the category analogue of Theorem 4.6. and Corollary 4.8..

4.20. Theorem. *For some* $k\, ,\, m \in \mathbb{N}$ *, let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *denote algebraically Baire regular families of functions* $f_1 : \mathbb{R}^k \to \mathbb{C}$ *and* $f_2 : \mathbb{R}^m \to \mathbb{C}$ *, respectively. Suppose that* $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ *such that*

$$f_1(x)f_2(y) = 0 \tag{4.46}$$

holds for all $(x, y) \in D$, *where* $D \subseteq \mathbb{R}^{k+m}$ *is a second category Baire set. Then* f_1 *or* f_2 *is identically equal to zero.*

Proof. Let us consider the sets

$$\begin{aligned} P_1 &= \{\, x \in \mathbb{R}^k \mid f_1(x) = 0 \,\} \quad \text{and} \\ P_2 &= \{\, y \in \mathbb{R}^m \mid f_2(y) = 0 \,\}. \end{aligned}$$

Theorem 4.18. implies the existence of $j \in \{\,1\,,\,2\,\}$ and $A_j \subseteq P_j$ such that A_j is a second category Baire set. Thus $f_j \in \mathcal{F}_j$ yields that f_j is identically equal to zero. □

Clearly, Theorem 4.16. states that the set of all generalized polynomials $f : \mathbb{R}^N \to \mathbb{C}$ is an algebraically Baire regular family of functions for every positive integer N . Therefore, we have the following straightforward corollaries of Theorem 4.20. and Theorem 4.16..

4.21. Corollary. *For some* $k\,,\, m \in \mathbb{N}$ *, let* $f : \mathbb{R}^k \to \mathbb{C}$ *and* $g : \mathbb{R}^m \to \mathbb{C}$ *be generalized polynomials fulfilling*

$$f(x)g(y) = 0 \tag{4.47}$$

for all $(x, y) \in D$ *, where* $D \subseteq \mathbb{R}^{k+m}$ *is second category Baire set. Then* $f(x) = 0$ *for every* $x \in \mathbb{R}^k$ *or* $g(y) = 0$ *for every* $y \in \mathbb{R}^m$*.*

4.22. Corollary. *Let* $N \in \mathbb{N}$ *and let* $f : \mathbb{R}^N \to \mathbb{C}$ *be a generalized polynomial fulfilling*

$$f(x)f(y) = 0 \tag{4.48}$$

for all $(x, y) \in D$ *, where* $D \subseteq \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ *is second category Baire set. Then* $f(x) = 0$ *for every* $x \in \mathbb{R}^N$ *.*

4.3. Products of generalized polynomials

In this section we mention an alternative approach to our results for the products of generalized polynomials (Theorem 4.11., Theorem 4.13. and Corollary 4.21.). This argument was suggested by Eszter Gselmann [**Gse**].

We shall make use of the following particular case of [**Sze91**, Theorem 2.2 (p. 25)] (cf. [**Gse**, Theorem 2]).

4.23. Theorem. *Let G be a commutative semigroup and R be a commutative ring. The set of all generalized polynomials from G into the additive group of R forms a(n) (commutative) algebra over the field $\mathbb{K}$ if R is a(n) (commutative) algebra over $\mathbb{K}$.*

We note that the argument in this general setting is analogous to the arguments presented in the proof of Theorem 1.5. for the particular case when both G and R denote the set of real numbers.

We also recall the characterization of generalized polynomials via functional equations ([**Djo**], [**McK**], [**Sze79**], also stated as [**Gse**, Theorem 5]).

4.24. Theorem. *Let G be a commutative semigroup with identity, S a commutative group and n a positive integer. Let the multiplication by $n!$ be bijective in S. The function $f : G \to S$ is a solution of the Fréchet equation*

$$\Delta_{y_1,\dots,y_{n+1}} f(x) = 0 \quad (x, y_1, \dots, y_{n+1} \in G)$$

if and only if f is a generalized polynomial of degree at most n.

Now we can establish Eszter Gselmann's first remark concerning the structure of generalized polynomials (as presented and proved in [**Gse**, Proposition 1]).

4.25. Proposition. *Let G, H be commutative semigroups, n be a nonnegative integer such that the multiplication by $n!$ is bijective in the commutative group K. If $\varphi : G \to H$ is a homomorphism and $p : H \to K$ is a generalized polynomial of degree at most n, then $p \circ \varphi : G \to K$ is a generalized polynomial of degree at most n.*

Proof. Due to Theorem 4.24. and under the assumptions of the statement, we have

$$\Delta^{n+1}_{h_1,\dots,h_{n+1}} p(u) = 0$$

for all u, $h_1, \ldots, h_{n+1}$. Let now $x, g_1, \ldots, g_{n+1} \in G$ be arbitrary. Then

$$\begin{aligned}\Delta^{n+1}_{g_1,\ldots,g_{n+1}} p \circ \varphi(x) &= \Delta^{n+1}_{g_1,\ldots,g_{n+1}} p(\varphi(x)) \\ &= \sum_{\varepsilon_i \in \{0,1\}}^{\varepsilon_1 + \cdots \varepsilon_{n+1}} p\left(\varphi(x) + \varepsilon_1 \varphi\left(g_1\right) + \cdots + \varepsilon_{n+1} \varphi\left(g_{n+1}\right)\right) \\ &= \Delta^{n+1}_{\varphi(g_1),\ldots,\varphi(g_{n+1})} p(\varphi(x)) = 0\,.\end{aligned}$$

Thus, by Theorem 4.24., $p \circ \varphi : G \to K$ is a generalized polynomial of degree at most n. ☐

Applying the previous statement, Eszter Gselmann obtained the following notable result [**Gse**, Proposition 2].

4.26. PROPOSITION. *Let k be a positive integer, $G_1, \ldots, G_k$ be commutative semigroups and X be an algebra over the field $\mathbb{K}$. Let further $p_i : G_i \to X$ be a generalized polynomial for all $i = 1, \ldots, k$. Then the mapping P defined on $\times_{i=1}^k G_i$ by*

$$P\left(x_1, \ldots, x_k\right) = p_1\left(x_1\right) \cdots p_k\left(x_k\right) \qquad \left(\left(x_1, \ldots, x_k\right) \in \times_{i=1}^k G_i\right)$$

is a generalized polynomial on $\times_{i=1}^k G_i$.

PROOF. Let k be a positive integer, $G_1, \ldots, G_k$ be commutative semigroups and X be an algebra over the field $\mathbb{K}$. For all fixed $l \in \{1, \ldots, k\}$ let us consider the mapping π_l defined on $\times_{i=1}^k G_i$ by

$$\pi_l\left(x_1, \ldots, x_k\right) = x_l \qquad \left(\left(x_1, \ldots, x_k\right) \in \times_{i=1}^k G_i\right).$$

Then $\pi_l : \times_{i=1}^k G_i \to G_l$ is a homomorphism. Therefore, due to Proposition 4.25., for every $l \in \{1, \ldots, k\}$, the functions P_l defined on $\times_{i=1}^k G_i$ by

$$P_l\left(x_1, \ldots, x_k\right) = p_l\left(x_l\right)$$

are generalized polynomials on $\times_{i=1}^k G_i$. In view of Theorem 4.23., the family $\mathscr{P}\left(\times_{i=1}^k G_i, X\right)$ of all generalized polynomials mapping $\times_{i=1}^k G_i$ into X is an algebra over the field $\mathbb{K}$. So

$$\prod_{l=1}^{k} P_l\left(x_1, \ldots, x_k\right) = p_1\left(x_1\right) \cdots p_k\left(x_k\right) \tag{4.49}$$

is a generalized polynomial on $\times_{i=1}^{k} G_i$. $\square$

Now, if we assume that X denotes the field of (real or) complex numbers and the mapping (4.49) vanishes on a large set (in the sense of the Haar measure or category) with respect to the product topology on the Cartesian product of locally compact Abelian groups (or, in particular, Euclidean spaces), we may apply Theorem 4.10. or Theorem 4.16., respectively, to conclude that the mapping (4.49) vanishes identically on the product space. Then we may apply the formerly cited theorem by Halter-Koch, Reich and Schwaiger [**HKRS**, Theorem 2] claiming that the set of generalized polynomials is an integral domain. Hence, if the product is zero, one of the factors p_l must be identically equal to zero.

CHAPTER 5

Inequalities for generalized monomials on $\mathbb{R}$

In Chapter 4, we made use of Székelyhidi's theorem on the zeros of generalized polynomials (on quite general domain), as well as its category counterpart, in order to obtain analogous results for their products in product spaces. In this chapter, we elaborate an analogy of these investigations for the sign of generalized monomials (of even degree) in real variables.

The analogy of Székelyhidi's theorem for the sign of real monomial functions can be formulated as follows: if a generalized monomial of en even degree in a real variable is non-negative on set of positive Lebesgue measure or on a second category Baire set, then it is non-negative everywhere.

The following basic example shows that our results in this chapter cannot be extended to more general domains.

5.1. Example. Let us define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x_1\,,\,x_2) = x_1 x_2 \qquad ((x_1\,,\,x_2) \in \mathbb{R}^2).$$

Clearly, f is non-negative on

$$A = \{\,(x_1\,,\,x_2) \in \mathbb{R}^2 \mid x_j \geq 0 \ \ (j = 1, 2)\}.$$

Obviously, A is measurable with a positive (in fact, infinite) planar Lebesgue measure. At the same time, $A \subseteq \mathbb{R}^2$ is a second category Baire set. On the other hand, f takes negative values as well.

5.1. Inequalities for monomials with measure constraints

Prior to the investigation of real monomial functions that are non-negative on a measurable set with positive Lebesgue measure, we enumerate some powerful tools of Measure Theory.

5.1.1. Tools and lemmas involving the Lebesgue measure.

Here we recall the concept of an inner measure together with some of its basic properties [**Bog**, pp. 57, 70].

5.2. Definition. Let $(X, \mathcal{B}, \lambda)$ be a measure space with a finite non-negative measure λ. We define the inner measure of a set A by the formula

$$\lambda_i(A) = \sup\{\, \lambda(B) \,:\, B \subseteq A\,,\, B \in \mathcal{B}\,\}$$

5.3. Remark. Clearly, λ_i is a monotone function defined on all subsets of X and it has non-negative real values. Moreover, if $E_j \subseteq X$ $(j \in J)$ is a countable family of pairwise disjoint sets, then we have

$$\lambda_i\left(\bigcup_{j\in J} E_j\right) \geq \sum_{j\in J} \lambda_i(E_j)\,. \tag{5.50}$$

We shall apply the above properties in the particular case when X is a compact interval and λ is the restriction of the one dimensional Lebesgue measure to the σ-algebra of the Lebesgue measurable subsets of X.

We shall also apply Lebesgue's density theorem (for the one dimensional Lebesgue measure) ([**Bog**, Section 5.8 (ii) on page 366],[**Kuc**, Theorem 3.5.1]) in our arguments.

5.4. Definition. Let A be a measurable set in $\mathbb{R}$ equipped with Lebesgue measure λ. A point $x \in \mathbb{R}$ is called a density point (or a point of density) of A if

$$\lim_{r\to 0} \frac{\lambda(A \cap [x-r, x+r])}{\lambda([x-r, x+r])} = 1\,.$$

The following celebrated theorem guarantees the existence of density points for measurable sets with positive Lebesgue measure.

5.5. Theorem. [Lebesgue's Density Theorem]: *If $A \subseteq \mathbb{R}$ such that A is Lebesgue measurable, then almost every $x \in A$ is a density point of A.*

Investigations on the signs of monomial mappings (involving measure constraints) are based on the following preliminary results [**BM24a**, Lemmas 3.5 and 4.1]:

5.6. Lemma. *Suppose that* $P \subseteq \mathbb{R}$ *fulfills the following assumptions:*

- (i) *for all* $r \in \mathbb{Q}$ *and* $x \in P$ *we have* $rx \in P$ *;*
- (ii) *there exists a Lebesgue measurable set* A *such that* $A \subseteq P$ *and* $\lambda(A) > 0$ *.*

Then P *has full Lebesgue measure, that is,* $\mathbb{R} \setminus P$ *is Lebesgue measurable and* $\lambda(\mathbb{R} \setminus P) = 0$ *.*

Proof. Fix $0 < \varepsilon < 1$ and $0 < K \in \mathbb{R}$ such that $\lambda(A \cap [-K, K]) > 0$. According to Lebesgue's Density Theorem, there exists a closed interval I such that

$$\lambda(A \cap I) > (1 - \varepsilon)\lambda(I) .$$

In fact, several such intervals exist (the center of I can be taken from a set of positive Lebesgue measure; moreover, we may consider all sub-intervals of I with the same center). In addition, one may replace P with $-P$ (and, accordingly, A with $-A$) as well, whence we may assume that $\inf(I) > 0$ and that $q = \sup(I)/\inf(I)$ is rational (since, for any $x \in \mathbb{R}$ and $\delta > 0$, the set

$$\left\{ \frac{x+r}{x-r} \,|\, 0 < r < \delta \right\}$$

is proper interval, hence it contains rational numbers). Moreover, for any non-zero rational number s , we have $s \cdot P \subseteq P$, whence

$$\begin{aligned} \lambda_i(P \cap s \cdot I) &\geq \lambda(s \cdot A \cap s \cdot I) = |s|\lambda(A \cap I) \\ &> |s|(1 - \varepsilon)\lambda(I) = (1 - \varepsilon)\lambda(s \cdot I) . \end{aligned}$$

Clearly, the same results remain valid if we drop one of the endpoints of I . Then we can almost cover the interval $[-K, K]$ by countably many pairwise disjoint intervals of the form $s_n I$ ($n \in \mathbb{Z}$), with the uncovered part having measure less than $K\varepsilon$. Namely, we may find a rational number $t > 0$ such that

$$1 - \varepsilon < \frac{t \inf(I)}{K} \leq 1 .$$

Then we may take $s_n := tq^{-n}$ and $s_{-n} := -s_n$ for every $n \in \mathbb{N}$, adding $s_0 := 0$ as well.

Applying also the σ-superadditivity of λ_i, we obtain

$$\begin{aligned}
\lambda_i(P \cap [-K, K]) &\geq \sum_{k \in \mathbb{Z}} \lambda_i(P \cap s_k \cdot I) > \sum_{k \in \mathbb{Z}} (1 - \varepsilon)\, \lambda(s_k \cdot I) \\
&= 2\,(1 - \varepsilon) \sum_{n=1}^{\infty} tq^{-n} \inf(I)(q - 1) = 2\,(1 - \varepsilon)\, t \inf(I) \\
&> 2\,(1 - \varepsilon)^2\, K > (1 - \varepsilon)^2 \lambda([-K, K])
\end{aligned}$$

As $\varepsilon > 0$ and $K > 0$ were chosen arbitrarily, we have proved that P is of full measure, i.e., $\mathbb{R} \setminus P$ is measurable and $\lambda(\mathbb{R} \setminus P) = 0$. □

5.7. Lemma. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree $m \geq 1$ and let H denote a closed subset of $\mathbb{R}$ such that*

$$P = \{\, x \in \mathbb{R} \mid f(x) \in H \,\}$$

has full Lebesgue measure. Then $f(x) \in H$ for every $x \in \mathbb{R}$.

Proof. Let $S = \mathbb{R} \setminus P$. Suppose, on the contrary, that $S \neq \emptyset$, i.e., $f(x) \in \mathbb{R} \setminus H$ for some $x \in \mathbb{R}$. For each $n \in \mathbb{N}$ let

$$U_n = \{\, y \in \mathbb{R} \mid f\left(x + \frac{y}{n}\right) \in \mathbb{R} \setminus H \,\}.$$

Then we have $U_n = n \cdot S - nx$ (i.e., U_n is an affine transform of S), hence U_n has Lebesgue measure zero for every positive integer n. On the other hand, the set $\mathbb{R} \setminus H$ is open, hence Lemma 1.2. yields

$$\mathbb{R} = \bigcup_{k \in \mathbb{N}} \bigcap_{n \geq k} U_n ,$$

which is a contradiction. □

5.1.2. Conditional inequalities with measure constraints.

The following results are motivated by a representation theorem due to Gy. Maksa [**Mak**]: If a quadratic function f is non-negative on $\mathbb{R}$, then there exists a Hilbert space H and an additive mapping

$\varphi : \mathbb{R} \to H$ such that $f(x) = \|\varphi(x)\|^2$ for every $x \in \mathbb{R}$. We establish, in a more general context, that the assumption in Maksa's theorem involving the non-negativity of f everywhere could be replaced by the weaker condition that f is non-negative on a set with positive Lebesgue measure. [**BM24a**, Theorem 4.3].

5.8. Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of even degree $m = 2k$ with some positive integer k fulfilling*

$$f(x) \geq 0 \tag{5.51}$$

for all $x \in A$, where $A \subseteq \mathbb{R}$ is a Lebesgue measurable subset with positive Lebesgue measure. Then (5.51) *holds for every $x \in \mathbb{R}$.*

Proof. Let us consider the set

$$P := \{\, x \in \mathbb{R} \,|\, f(x) \geq 0 \,\}.$$

Due to the rational homogeneity property and our assumptions in the theorem, the set P satisfies assumptions (i) and (ii) in Lemma 5.6.. Hence P is a subset with full measure in $\mathbb{R}$.

Clearly, f and P satisfy the assumptions of Lemma 5.7. if H denotes the set of non-negative real numbers. Thus we have $f(x) \geq 0$ for every $x \in \mathbb{R}$. □

We may obtain an analogous statement for monomial functions with odd degree [**BM24a**, Corollary 4.4].

5.9. Corollary. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of even degree $m = 2k - 1$ with some positive integer k fulfilling*

$$xf(x) \geq 0 \tag{5.52}$$

for all $x \in A$, where $A \subseteq \mathbb{R}$ is a Lebesgue measurable subset with positive Lebesgue measure. Then (5.52) *holds for every $x \in \mathbb{R}$.*

Proof. Let $F : \mathbb{R}^m \to \mathbb{R}$ be an *m-additive* mapping such that (2) holds. For every

$$(x_1\,, x_2\,, \ldots\,, x_m\,, x_{m+1}) \in \mathbb{R}^{m+1},$$

let

$$G(x_1\,, x_2\,, \ldots\,, x_m\,, x_{m+1}) = x_{m+1}F(x_1\,, x_2\,, \ldots\,, x_m).$$

Moreover, let g denote the diagonalization of G. Then G is an $(m+1)$-additive mapping, hence $g : \mathbb{R} \to \mathbb{R}$ is a generalized monomial of degree $m + 1 = 2k$. On the other hand, we have

$$g(x) = G(x, x, \ldots , x, x) = xF(x, x, \ldots , x) = xf(x)$$

for every $x \in \mathbb{R}$, hence g satisfies the assumptions of Theorem 5.8.. $\square$

5.1.3. Inequalities for products with measure constraints. Now we can establish a sufficient condition for a monomial function of even degree to assure that it preserves its sign.

5.10. THEOREM. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of even degree $m = 2k$ fulfilling*

$$f(x)f(y) \geq 0 \tag{5.53}$$

for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a Lebesgue measurable subset with positive planer Lebesgue measure. Then (5.53) *holds for all $(x, y) \in \mathbb{R}^2$ (i.e., f does not change sign).*

PROOF. Let us consider the sets

$$\begin{aligned} P_f &= \{\, x \in \mathbb{R} \mid f(x) \geq 0 \,\}, \\ N_f &= \{\, x \in \mathbb{R} \mid f(x) \leq 0 \,\}. \end{aligned}$$

We note that both P_f and N_f obviously satisfy assumption (i) in Lemma 5.6.. We are going to verify assumption (ii) of the same lemma for one of these sets.

Clearly, we have

$$D \subseteq (P_f \times P_f) \cup (N_f \times N_f)\,.$$

Let E denote a measurable subset of D such that

$$0 < \lambda^2(E) < +\infty$$

(such a subset E exists since the planar Lebesgue measure is σ-finite).

For every real number x we define

$$E_x = \{\, y \in \mathbb{R} \mid (x, y) \in E \,\}.$$

According to Fubini's theorem [**Bog**, Theorem 3.4.1 and Corollary 3.4.2] E_x is Lebesgue measurable for λ-a.e. x , the mapping $x \mapsto \lambda(E_x)$

is measurable and

$$\lambda^2(E) = \int_{\mathbb{R}} \lambda(E_x) d\lambda(x)\,.$$

Due to our assumption on the set E and the definition of the integral there exists a real number $r > 0$ and a Lebesgue measurable set $B \subseteq \mathbb{R}$ such that $\lambda(B) > 0$ and, for every $x \in B$, $\lambda(E_x) \geq r$. Now we have to distinguish three cases.

Case I: If $E_x \subseteq P_f$ for some $x \in B$, we can verify assumption (ii) of Lemma 5.6. for P_f with $A = E_x$. Clearly, f and P_f satisfy the assumptions of Lemma 5.7. with $H = \{\, t \in \mathbb{R} \,|\, t \geq 0 \,\}$. Thus we have $f(x) \geq 0$ for every $x \in \mathbb{R}$.

Case II: If $E_x \subseteq N_f$ for some $x \in B$, we can verify assumption (ii) of Lemma 5.6. for N_f with $A = E_x$. Clearly, f and N_f satisfy the assumptions of Lemma 5.7. with $H = \{\, t \in \mathbb{R} \,|\, t \leq 0 \,\}$. Thus we have $f(x) \leq 0$ for every $x \in \mathbb{R}$.

Case III: For every $x \in B$, we have

$$E_x \setminus P_f \neq \emptyset \quad \text{and} \quad E_x \setminus N_f \neq \emptyset\,.$$

That is, for each $x \in B$, there exist $y_j \in E_x$ $(j = 1, 2)$ such that $f(y_1) < 0$ and $f(y_2) > 0$. However, we have $(x, y_j) \in E$ and thus $f(x)f(y_j) \geq 0$ for both $j = 1$ and $j = 2$. This yields $f(x) = 0$. Hence $B \subseteq P_f \cap N_f$. Hence we can verify assumption (ii) of Lemma 5.6. for $P_f \cap N_f$ with $A = B$.

Due to the statement of Lemma 5.6., $Z_f = P_f \cap N_f$ is a subset with full measure in $\mathbb{R}$.

Clearly, f and Z_f satisfy the assumptions of Lemma 5.7. with $H = \{0\}$. Thus we have $f(x) = 0$ for every $x \in \mathbb{R}$. □

5.2. Inequalities for monomials with category constraints

It is natural to investigate the topological analogues of our previous results, when the required properties are satisfied for arguments taken from a Baire set of second category, which is the subject of Section 5.2.

5.2.1. Category versions of some preliminary results. In order to investigate inequalities for monomial functions on a second category Baire set, we have to elaborate analogies of our Lemma 5.6. and Lemma 5.7. with category constraints.

5.11. Lemma. *Suppose that $P \subseteq \mathbb{R}$ fulfills the following assumptions:*

(i) *for all $r \in \mathbb{Q}$ and $x \in P$ we have $rx \in P$;*

(ii) *there exists a second category Baire set A such that $A \subseteq P$.*

Then $\mathbb{R} \setminus P$ is of the first category.

Proof. Let $A = G \triangle T$, where $G \subseteq \mathbb{R}$ is a non-void open set and $T \subseteq \mathbb{R}$ is of first category. Hence $G \subseteq A \cup T$. Let I denote a closed subinterval of G with positive length. In fact, several such intervals exist. We may take the center x of I arbitrary from any open subinterval of G. So we may assume $x \neq 0$. We may consider all sub-intervals of I with the same center. In addition, one may replace P with $-P$ (and, accordingly, A with $-A$, G with $-G$ and T with $-T$) as well, whence we may assume that $\inf(I) > 0$ and that $q = \sup(I)/\inf(I)$ is rational (since, for any $x \in \mathbb{R}$ and $0 < \delta \leq x$, the set

$$\left\{ \frac{x+r}{x-r} \,|\, 0 < r < \delta \right\}$$

is proper interval, hence it contains rational numbers). Let

$$A_0 = A \cap I \quad \text{and} \quad T_0 = T \cap I \,.$$

Then we have

$$I = A_0 \cup T_0 \,,$$

where $A_0 \subseteq P$ and T_0 is of the first category. Now, for every integer k let

$$I_k = q^k I \,, \quad A_k = q^k A_0 \,, \quad T_k = q^k T_0 \,, \quad \tilde{I}_k = -q^k I \,, \quad \tilde{A}_k = -q^k A_0$$
$$\text{and} \quad \tilde{T}_k = -q^k T_0 \,.$$

Then we have

$$\mathbb{R} \setminus \{0\} = \bigcup_{k \in \mathbb{Z}} I_k \cup \bigcup_{k \in \mathbb{Z}} \tilde{I}_k$$

as well as

$$I_k \subseteq A_k \cup T_k \quad \text{and} \quad \tilde{I}_k \subseteq \tilde{A}_k \cup \tilde{T}_k$$

for every integer k. Moreover, each T_k and $\tilde{T}_k$ is of the first category, hence the set

$$T^\diamond = \{0\} \cap \bigcup_{k\in\mathbb{Z}} T_k \cup \bigcup_{k\in\mathbb{Z}} \tilde{T}_k$$

is of the first category in $\mathbb{R}$ as well. On the other hand, we have $A_k \subseteq P$ and $\tilde{A}_k \subseteq P$ for every integer k , hence we have

$$\mathbb{R} = P \cup T^\diamond$$

and thus $\mathbb{R} \setminus P \subseteq T^\diamond$, which implies that $\mathbb{R} \setminus P$ is of the first category.

□

The proof of the following Lemma is quite similar to the proof of Lemma 5.7..

5.12. Lemma. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree $m \geq 1$ and let H denote a closed subset of $\mathbb{R}$ such that the set*

$$S = \{\, x \in \mathbb{R} \mid f(x) \notin H \,\}$$

is of the first category. Then $f(x) \in H$ for every $x \in \mathbb{R}$.

Proof. Suppose, on the contrary, that $f(x) \in \mathbb{R} \setminus H$ for some $x \in \mathbb{R}$. For each $n \in \mathbb{N}$ let

$$U_n = \{\, y \in \mathbb{R} \mid f\left(x + \frac{y}{n}\right) \in \mathbb{R} \setminus H \,\}.$$

Then we have $U_n = n \cdot S - nx$ (i.e., U_n is an affine transform of S), hence U_n is of the first category for every positive integer n . On the other hand, the set $\mathbb{R} \setminus H$ is open, hence Lemma 1.2. yields

$$\mathbb{R} = \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} U_n\,,$$

which is a contradiction. □

5.2.2. Conditional inequalities with category constraints.

Now we can establish the category analogue of Theorem 5.8..

5.13. Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of even degree $m = 2k$ with some positive integer k fulfilling*

$$f(x) \geq 0 \tag{5.54}$$

for all $x \in H$, where $H \subseteq \mathbb{R}$ is a second category Baire set. Then (5.54) holds for every $x \in \mathbb{R}$.

Proof. Let us consider the set

$$P = \{\, x \in \mathbb{R} \,|\, f(x) \geq 0 \,\}.$$

Due to the rational homogeneity property (1.3) and our assumptions in the theorem, the set P satisfies assumptions (i) and (ii) in Lemma 5.11.. Hence $\mathbb{R} \setminus P$ is of the first category.

Clearly, f satisfies the assumptions of Lemma 5.12. if H denotes the set of non-negative real numbers. Thus we have $f(x) \geq 0$ for every $x \in \mathbb{R}$. □

The proof of the following corollary of Theorem 5.13. is identical to the proof of Corollary 5.9..

5.14. Corollary. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of odd degree $m = 2k - 1$ with some positive integer k fulfilling*

$$xf(x) \geq 0 \tag{5.55}$$

for all $x \in A$, where $A \subseteq \mathbb{R}$ is a second category Baire set. Then (5.55) holds for every $x \in \mathbb{R}$.

5.2.3. Inequalities for products with category constraints. Applying Lemma 5.11. and Lemma 5.12., we can investigate the category analogue of Theorem 5.10..

5.15. Theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of even degree $m = 2k$ fulfilling*

$$f(x)f(y) \geq 0 \tag{5.56}$$

for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a second category Baire set. Then (5.56) holds for all $(x, y) \in \mathbb{R}^2$ (i.e., f does not change sign).

PROOF. Let us consider the sets

$$\begin{aligned} P_f &= \{\, x \in \mathbb{R} \,|\, f(x) \geq 0 \,\}, \\ N_f &= \{\, x \in \mathbb{R} \,|\, f(x) \leq 0 \,\}. \end{aligned}$$

We note that both P_f and N_f obviously satisfy assumption (i) in Lemma 5.11.. We are going to verify assumption (ii) of the same lemma for one of these sets.

Clearly, we have

$$D \subseteq (P_f \times P_f) \cup (N_f \times N_f)\,.$$

Due to our assumption on D, there exist a non-void open set $G \subseteq \mathbb{R}\times\mathbb{R}$ and a first category set $T \subseteq \mathbb{R}\times\mathbb{R}$ such that $D = G \triangle T$, hence we have $G \subseteq D \cup T$. On the other hand, there exist $(x_0\,,\, y_0) \in \mathbb{R}\times\mathbb{R}$ and $r > 0$ such that

$$B_r(x_0) \times B_r(y_0) \subseteq G\,.$$

Combining these inclusions we obtain

$$B_r(x_0) \times B_r(y_0) \subseteq ((P_f \times P_f) \cup (N_f \times N_f)) \cup T\,. \tag{5.57}$$

We shall also consider the set $K \subseteq \mathbb{R}$ with the properties described in Theorem 4.17. (aaplied to the case $N = 1$).

Let

$$A = B_r(x_0) \cap P_f \quad \text{and} \quad S = B_r(x_0) \setminus P_f\,.$$

Let us assume that S is of the second category. Then $S \setminus K \neq \emptyset$, hence there exists $x_1 \in S \setminus K$. Then $f(x_1) < 0$ and, for every $y \in B_r(y_0)$, inclusion (5.57) yields $y \in N_f$ or $y \in T_{x_1}$, hence we have

$$B_r(y_0) \subseteq N_f \cup T_{x_1}\,.$$

Due to the choice of x_1, the set T_{x_1} is of the first category. Then the set

$$T^{\triangledown} = (T_{x_1} \cap B_r(y_0)) \setminus N_f$$

is of the first category as well, hence $B = B_r(y_0) \cap N_f$ can be represented in the form

$$B = B_r(y_0) \setminus T^{\triangledown} = B_r(y_0) \,\triangle\, T^{\triangledown}\,.$$

Therefore, B is a second category Baire set and $B \subseteq N_f$, hence N_f satisfies the assumptions in Lemma 5.11.. Thus $\mathbb{R} \setminus N_f$ is of the first category. Then f fulfills the conditions in Lemma 5.12. with $H = \{\, t \in \mathbb{R} \,|\, t \leq 0 \,\}$, hence $f(x) \leq 0$ for every $x \in \mathbb{R}$.

If S is of the first category,

$$A = B_r(x_0) \setminus S = B_r(x_0) \triangle S$$

yields that A is a second category Baire set and $A \subseteq P_f$, hence P_f satisfies the assumptions in Lemma 5.11.. Thus $\mathbb{R} \setminus P_f$ is of the first category. Then f fulfills the conditions in Lemma 5.12. with $H = \{\, t \in \mathbb{R} \,|\, t \geq 0 \,\}$, hence $f(x) \geq 0$ for every $x \in \mathbb{R}$. $\square$

CHAPTER 6

Results for almost polynomials and almost monomials

6.1. Equations for almost polynomials

In this final chapter we extend our results for almost polynomial functions and almost monomial functions, respectively.

The subsequent concepts and results are collected in our accepted manuscript [**BM25a**] and partially (covering some particular cases) in our recently published paper [**BM24a**].

In order to cover the concept and the description of almost polynomial functions both in the sense of measure and in the sense of category, we recall the concepts of conjugate proper linearly independent ideals ([**Ger**, Definitions 2 and 3], [**Kuc**, Definitions in Section 17.5]).

6.1. Definition. Let N denote a positive integer. A family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is called a *proper linearly independent ideal* if

(i) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;
(ii) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$;
(iii) $\mathbb{R}^N \notin \mathcal{I}$;
(iv) $\alpha \in \mathbb{R}$, $A \in \mathcal{I}$ and $u \in \mathbb{R}^N$ implies $\alpha A + u \in \mathcal{I}$.

For an arbitrary set $M \subseteq \mathbb{R}^N \times \mathbb{R}^N$ and for every $x \in \mathbb{R}^N$, let

$$M_x = \{\, y \in \mathbb{R}^N \mid (x, y) \in M \,\}.$$

6.2. Definition. Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}^N$ and $\mathcal{I}_2$ be a proper linearly independent ideal in $\mathbb{R}^N \times \mathbb{R}^N$. We say that the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are *conjugate* if, for every set $M \in \mathcal{I}_2$ there exists a set $U \in \mathcal{I}_1$ such that $M_x \in \mathcal{I}_1$ for every $x \in \mathbb{R}^N \setminus U$.

Given a proper linearly independent ideal $\mathcal{I}$ in $\mathbb{R}^N$, we say that a condition is satisfied $\mathcal{I}$*-almost everywhere* in $\mathbb{R}^N$ (written $\mathcal{I}$-(a.e.)) if there exists a set $S \in \mathcal{I}$ such that the condition holds for every $x \in \mathbb{R}^N \setminus S$. Using this terminology, we may say that the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate if, for every set $M \in \mathcal{I}_2$, the condition $M_x \in \mathcal{I}_1$ holds $\mathcal{I}_1$-almost everywhere in $\mathbb{R}^N$.

The following examples are corollaries of Fubini's theorem [**Bog**, Theorem 3.4.1 and Corollary 3.4.2] and the Kuratowski–Ulam theorem (see Theorem 4.17.), respectively.

6.3. Example. Let $\mathcal{L}_0^N$ denote the family of Lebesgue measurable subsets A of $\mathbb{R}^N$ fulfilling $\lambda^N(A) = 0$, where λ^N denotes the N-dimensional Lebesgue measure. Then $\mathcal{L}_0^N$ and $\mathcal{L}_0^{2N}$ are conjugate proper linearly independent ideals.

6.4. Example. Let $\mathcal{F}^N$ denote the family of first category subsets of $\mathbb{R}^N$. Then $\mathcal{F}^N$ and $\mathcal{F}^{2N}$ are conjugate proper linearly independent ideals.

These examples motivate the following concept ([**Ger**, Definition 4], [**Kuc**, Section 17.7]).

6.5. Definition. Let $\mathcal{I}$ be a proper linearly independent ideal in $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ and let $p \in \mathbb{N}$. We call a function $f : \mathbb{R}^N \to \mathbb{R}$ an $\mathcal{I}$*-almost polynomial function of degree at most* p if there exists $S \in \mathcal{I}$ such that

$$\Delta_y^{p+1} f(x) = 0 \tag{6.58}$$

holds for every $(x, y) \in \mathbb{R}^{2N} \setminus S$.

This concept was introduced by Roman Ger in order to establish an abstract description of almost polynomial functions as follows ([**Ger**, Theorem 1], [**Kuc**, Theorem 17.7.2]).

6.6. Theorem. [R. Ger] *Let* $\mathcal{I}_1$ *be a proper linearly independent ideal in* $\mathbb{R}^N$ *and* $\mathcal{I}_2$ *be a proper linearly independent ideal in* $\mathbb{R}^{2N}$ *such that* $\mathcal{I}_1$ *and* $\mathcal{I}_2$ *are conjugate. If* $f : \mathbb{R}^N \to \mathbb{R}$ *is an* $\mathcal{I}_2$*-almost polynomial function of degree at most* p*, then there exists a unique* *polynomial function* $g : \mathbb{R}^N \to \mathbb{R}$ *such that* $f = g$ $\mathcal{I}_1$*-almost everywhere in* $\mathbb{R}^N$*.*

As immediate consequences of Theorem 6.6., Examples 6.3 and 6.4, Corollary 4.12. (applied for $\mathcal{G} = \mathbb{R}^N$) and Corollary 4.22., respectively, we can establish the following corollaries.

6.7. Corollary. *Let* $f : \mathbb{R}^N \to \mathbb{R}$ *be an* $\mathcal{L}_0^{2N}$*-almost polynomial function of degree at most* $p \geq 1$ *fulfilling*

$$f(x)f(y) = 0 \tag{6.59}$$

for all $(x, y) \in D$*, where* $D \subseteq \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ *is Lebesgue-measurable and* $\lambda^{2N}(D) > 0$*. Then* $f = 0$ $\mathcal{L}_0^N$*(-a.e.) in* $\mathbb{R}^N$ *(i.e.,* $f(x) = 0$ *for* λ^N*-almost every* $x \in \mathbb{R}^N$ *).*

Proof. From Theorem 6.6. we conclude that there exists a unique generalized polynomial g of degree at most p such that

$$f = g \quad \mathcal{L}_0^N\,(-a.e.) \text{ in } \mathbb{R}^N.$$

Namely, there exists a Lebesgue measurable set $A \subset \mathbb{R}^N$ fulfilling $\lambda^N(A) = 0$ such that $f(x) = g(x)$ for every $x \in \mathbb{R}^N \setminus A$. Clearly, the set

$$S_0 = (A \times \mathbb{R}^N) \cup (\mathbb{R}^N \times A)$$

is Lebesgue measurable and fulfills $\lambda^{2N}(S_0) = 0$, hence the set $D_0 = D \setminus S_0$ is also Lebesgue measurable with

$$\lambda^{2N}(D_0) = \lambda^{2N}(D) > 0\,.$$

Moreover, for every $(x, y) \in D_0$ we have

$$g(x)g(y) = f(x)f(y) = 0\,.$$

Applying Corollary 4.12. (or Theorem 4.13.) we get that $g = 0$ identically, therefore $f = 0$ for λ^N-almost every $x \in \mathbb{R}^N$. ☐

6.8. Corollary. *Let* $f : \mathbb{R}^N \to \mathbb{R}$ *be an* $\mathcal{F}^{2N}$*-almost polynomial function of degree at most* $p \geq 1$ *fulfilling* (6.59) *for all* $(x, y) \in D$*, where* $D \subseteq \mathbb{R}^{2N}$ *is a second category Baire set. Then* $f = 0$ $\mathcal{F}^N$*(-a.e.) in* $\mathbb{R}^N$*.*

Proof. From Theorem 6.6. we conclude that there exists a unique generalized polynomial g of degree at most p such that

$$f = g \quad \mathcal{F}^N\,(-a.e.) \ \text{ in } \mathbb{R}^N.$$

Namely, there exists a set $A \subset \mathbb{R}^N$ of the first category such that $f(x) = g(x)$ for every $x \in \mathbb{R}^N \setminus A$. Clearly, the set

$$S_0 = (A \times \mathbb{R}^N) \cup (\mathbb{R}^N \times A)$$

is of the first category in $\mathbb{R}^{2N}$, hence (according to Proposition 4.14.) the set $D_0 = D \setminus S_0$ is a second category Baire set. Moreover, for every $(x, y) \in D_0$ we have

$$g(x)g(y) = f(x)f(y) = 0\,.$$

Applying Corollary 4.22. we get that $g = 0$ identically, therefore $f = 0$ $\mathcal{F}^N$(-a.e.) in $\mathbb{R}^N$. □

6.2. Inequalities for almost monomials

Since we have considered conditional inequalities for monomial functions in a real variable, we may wish to extend our related results to almost monomial functions defined on $\mathbb{R}$. Combining our definition for almost monomial functions [**BM24a**] with Definition 6.5., we may introduce the required concept as follows.

6.9. Definition. Let $\mathcal{I}$ be a proper linearly independent ideal in $\mathbb{R}^2$ and let $m \in \mathbb{N}$. We call a function $f : \mathbb{R} \to \mathbb{R}$ an *$\mathcal{I}$-almost monomial function of degree m* if there exists $S \in \mathcal{I}$ such that

$$\Delta_y^m f(x) = m! f(y) \tag{6.60}$$

holds for every $(x, y) \in \mathbb{R}^2 \setminus S$.

In order to extend [**BM24a**, Theorem 5.1] to the ideal setting (possibly preserving the ideas in its proof), we need a stronger invariance property with respect to the linear transformations of the members of the ideal. So we consider the following concept.

6.10. Definition. Let N denote a positive integer. A family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is called a *proper linearly invariant ideal* if $\mathcal{I}$ is a proper linearly independent ideal such that $S \in \mathcal{I}$ implies $\phi(S) \in \mathcal{I}$ for every linear bijection $\phi : \mathbb{R}^N \to \mathbb{R}^N$.

In other words, we could say that a non-void family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is a proper linearly invariant ideal if $\mathcal{I}$ fulfills properties (i)–(iii) in Definition 6.1, and (instead of (iv)) it satisfies

(iv*) $S \in \mathcal{I}$ implies $\phi(S) + u \in \mathcal{I}$ for every $u \in \mathbb{R}^N$ and for every linear bijection $\phi : \mathbb{R}^N \to \mathbb{R}^N$.

Using the notation introduced in Examples 6.3 and 6.4, we can claim that sets having Lebesgue measure zero as well as sets of the first category form proper linearly invariant ideals. The first example follows immediately from the identity for the Lebesgue measure of linear transforms of measurable sets [**Bog**, Corollary 3.6.4].

6.11. Example. $\mathcal{L}_0^N$ is a proper linearly invariant ideal.

The second example can be established as follows.

6.12. Proposition. *$\mathcal{F}^N$ is a proper linearly invariant ideal.*

Proof. Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ denote a linear bijection. Then ϕ and ϕ^{-1} are continuous linear mappings. If

$$V \subseteq A \subseteq E \subseteq \mathbb{R}^N$$

such that V is open and E is closed, we conclude that $\phi(V)$ and $\phi^{-1}(V)$ are open, $\phi(E)$ and $\phi^{-1}(E)$ are closed; moreover, we have

$$\phi(V) \subseteq \phi(A) \subseteq \phi(E) \quad \text{and} \quad \phi^{-1}(V) \subseteq \phi^{-1}(A) \subseteq \phi^{-1}(E)\,.$$

In view of the definitions of the closure $\mathrm{cl}(A)$ and the interior $\mathrm{int}(A)$ of a set A, we obtain

$$\mathrm{int}(\phi(A)) = \phi(\mathrm{int}(A)) \quad \text{and} \quad \mathrm{cl}(\phi(A)) = \phi(\mathrm{cl}(A)) \tag{6.61}$$

for every $A \subseteq \mathbb{R}^N$.

Now $S \in \mathcal{F}^N$ yields that there exist a countable set J and nowhere dense sets $C_j \subseteq \mathbb{R}^N$ $(j \in J)$ such that

$$S = \bigcup_{j\in J} C_j\,.$$

Then, for every $j \in J$, (6.61) implies

$$\operatorname{int}(\operatorname{cl}(\phi(C_j))) = \phi(\operatorname{int}(\operatorname{cl}(C_j))) = \phi(\emptyset) = \emptyset\,,$$

hence $\phi(C_j)$ is nowhere dense and

$$\phi(S) = \bigcup_{j\in J} \phi(C_j)\,.$$

This yields $\phi(S) \in \mathcal{F}^N$. □

In view of these examples, it is reasonable to formulate the following generalization of [**BM24a**, Theorem 5.1].

6.13. Theorem. *Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}$ and $\mathcal{I}_2$ be a proper linearly invariant ideal in $\mathbb{R}^2$ such that $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate. Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{I}_2$-almost monomial function of degree $m \geq 1$. Then there exists a unique monomial function $g : \mathbb{R} \to \mathbb{R}$ of degree m such that $f = g$ $\mathcal{I}_1$-almost everywhere*

Proof. Due to our assumptions, there exist $M \in \mathcal{I}_2$ such that f satisfies (6.60) for every $(x, y) \in \mathbb{R}^2 \setminus M$. Let

$$E = \{\, (x, y) \in \mathbb{R}^2 \,|\, (x, y) \in \mathbb{R}^2 \setminus M \quad \text{and} \quad (x + y, y) \in \mathbb{R}^2 \setminus M \,\}.$$

Clearly, $(x, y) \in \mathbb{R}^2 \setminus E$ means $(x, y) \in M$ or $(x + y, y) \in M$. In the latter case there exists $(u, y) \in M$ such that $x = u - y$. Let

$$\phi(u, y) = (u - y, y) \qquad ((u, y) \in \mathbb{R}^2).$$

According to the previous argument and our assumptions on $\mathcal{I}_2$, we have

$$\mathbb{R}^2 \setminus E = M \cup \phi(M) \in \mathcal{I}_2\,.$$

For every $(x, y) \in E$ we have

$$\begin{aligned} \Delta_y^{m+1} f(x) &= \Delta_y \Delta_y^m f(x) = \Delta_y^m f(x + y) - \Delta_y^m f(x) \\ &= m! f(y) - m! f(y) = 0\,, \end{aligned}$$

hence f is an $\mathcal{I}_2$-almost polynomial function of degree at most m. According to Theorem 6.6., there exists a unique polynomial function $g : \mathbb{R} \to \mathbb{R}$ of degree at most m such that $f = g$ $\mathcal{I}_1$-almost everywhere

By definition, g can be represented in the form

$$g(x) = \sum_{k=0}^{m} g_k(x)$$

for every $x \in \mathbb{R}$, where $g_k : \mathbb{R} \to \mathbb{R}$ is a monomial function of degree k for every $k \in \{0, 1, \ldots, m\}$ (in particular, g_0 is a constant function). Equations (1.8) and (1.7) imply, for every $(x, y) \in \mathbb{R}^2$,

$$\Delta_y^m g_m(x) = m! g_m(y) \quad \text{and} \quad \Delta_y^m g_k(x) = 0$$

for every $k \in \{0, 1, \ldots, m-1\}$, respectively. Hence we have

$$\Delta_y^m g(x) = \Delta_y^m \left(\sum_{k=0}^{m} g_k(x) \right) = \sum_{k=0}^{m} \Delta_y^m g_k(x) = m! g_m(y)$$

for every $(x, y) \in \mathbb{R}^2$.

On the other hand, there exists $B \in \mathcal{I}_1$ such that $f(x) = g(x)$ for every $x \in \mathbb{R} \setminus B$.

For every $y \in \mathbb{R}$, let us consider the sections

$$M_y^* = \{ x \in \mathbb{R} \,|\, (x, y) \in M \}$$

(cf. Notation 4.1.). Applying the linear bijection $\psi(x, y) = (y, x)$ on $\mathbb{R}^2$, we have $\psi(M) \in \mathcal{I}_2$ and thus there exists $A \in \mathcal{I}_1$ such that

$$M_y^* = \psi(M)_y \in \mathcal{I}_1$$

holds for every $y \in \mathbb{R} \setminus A$.

Now let us consider an arbitrary $y \in \mathbb{R} \setminus A$. Let

$$C[y] = M_y^* \cup \bigcup_{k=0}^{m} (B - ky).$$

Then $C[y] \in \mathcal{I}_1$. Hence we may choose $x \in \mathbb{R} \setminus C[y]$. Clearly, $(x, y) \in \mathbb{R}^2 \setminus M$ and $x + ky \in \mathbb{R} \setminus B$ for every $k \in \{0, 1, \ldots, m\}$. Thus, applying also [**Kuc**, Corollary 15.1.2] we have

$$\begin{aligned} m!f(y) &= \Delta_y^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+ky) \\ &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g(x+ky) = \Delta_y^m g(x) = m! g_m(y). \end{aligned}$$

We have obtained $f(y) = g_m(y)$, where y is an arbitrary element of $\mathbb{R} \setminus A$ and $g_m : \mathbb{R} \to \mathbb{R}$ is a monomial function of order m. ☐

Now we can establish the analogies of Theorem 5.10. and Theorem 5.15. for $\mathcal{L}_0^2$-almost monomial functions and $\mathcal{F}^2$-almost monomial functions, respectively.

6.14. COROLLARY. *Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{L}_0^2$-almost monomial function of even degree m fulfilling (5.53) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a Lebesgue measurable subset with positive planar Lebesgue measure. Then $f(x) \geq 0$ for $\mathcal{L}_0^1$-almost every $x \in \mathbb{R}$ or $f(x) \leq 0$ for $\mathcal{L}_0^1$-almost every $x \in \mathbb{R}$.*

PROOF. From Theorem 6.13. we conclude that there exists a unique generalized monomial $g : \mathbb{R} \to \mathbb{R}$ of degree m such that $f = g$ almost everywhere (with respect to the Lebesgue measure on the real line). Namely, there exists a Lebesgue measurable set $A \subseteq \mathbb{R}$ fulfilling $\lambda(A) = 0$ such that $f(x) = g(x)$ for every $x \in \mathbb{R} \setminus A$. Clearly, the set

$$S = (A \times \mathbb{R}) \cup (\mathbb{R} \times A)$$

is Lebesgue measurable and fulfills $\lambda^2(S) = 0$, hence the set $D^\bullet = D \setminus S$ is also Lebesgue measurable with

$$\lambda^2(D^\bullet) = \lambda^2(D) > 0 .$$

Moreover, for every $(x, y) \in D^\bullet$ we have

$$g(x)g(y) = f(x)f(y) \geq 0 .$$

Applying Theorem 5.10. we get that g does not change its sign, i.e., $g \geq 0$ identically or $g \leq 0$ identically. Therefore, $f \geq 0$ holds $\mathcal{L}_0^1$-almost everywhere or $f \leq 0$ holds $\mathcal{L}_0^1$-almost everywhere. ☐

6.15. Corollary. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an* $\mathcal{F}^2$*-almost monomial function of even degree* m *fulfilling* (5.56) *for all* $(x, y) \in D$, *where* $D \subseteq \mathbb{R}^2$ *is a second category Baire set. Then* $f(x) \geq 0$ *for* $\mathcal{F}^1$*-almost every* $x \in \mathbb{R}$ *or* $f(x) \leq 0$ *for* $\mathcal{F}^1$*-almost every* $x \in \mathbb{R}$.

Proof. From Theorem 6.13. we conclude that there exists a unique generalized monomial $g : \mathbb{R} \to \mathbb{R}$ of degree m such that $f = g$ holds $\mathcal{F}^1$-almost everywhere. Namely, there exists a first category set $A \subseteq \mathbb{R}$ such that $f(x) = g(x)$ for every $x \in \mathbb{R} \setminus A$. We observe (cf. [**Kuc**, Theorem 2.1.8]) that the set

$$T = (A \times \mathbb{R}) \cup (\mathbb{R} \times A)$$

fulfills $T \in \mathcal{F}^2$, hence (according to Proposition 4.14.) the set $D^\bullet = D \setminus T$ is a second category Baire set in $\mathbb{R}^2$ as well. Moreover, for every $(x, y) \in D^\bullet$ we have

$$g(x)g(y) = f(x)f(y) \geq 0\,.$$

Applying Theorem 5.15. we get that g does not change its sign, i.e., $g \geq 0$ identically or $g \leq 0$ identically. Therefore, $f \geq 0$ holds $\mathcal{F}^1$-almost everywhere or $f \leq 0$ holds $\mathcal{F}^1$-almost everywhere. $\square$

Summary

A synopsis of the main results of the doctoral dissertation is presented here. Our investigation has produced various significant lemmas, propositions, theorems and corollaries that have been comprehensively explained in the previous chapters as well as in our papers [**BM23**], [**BM24a**],[**BM24b**] and [**BM25a**].

Motivation and research plan

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ denote the sets of complex numbers, real numbers, rational numbers, and positive integers, respectively.

We introduce the following subsets of $\mathbb{R}^2$ to describe certain geometric structures:

$$\begin{aligned} S_0 &= \{(x,y) \in \mathbb{R}^2 \,|\, xy = 1\}, \\ S_1 &= \{(x,y) \in \mathbb{R}^2 \,|\, x^2 - y^2 = 1\}, \\ S_2 &= \{(x,y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 1\}. \end{aligned}$$

Additionally, given two regular, non-constant real polynomials p and q, and a nonzero real number m, we define:

$$\begin{aligned} R_{p,q} &= \{(p(t), q(t)) \,|\, t \in \mathbb{R}\}, \\ S_{1,m} &= \{(x,y) \in \mathbb{R}^2 \,|\, x^2 - my^2 = 1\}. \end{aligned}$$

The existence of non-linear additive real functions $f : \mathbb{R} \to \mathbb{R}$ has been established by G. Hamel [**Ham**]. Clearly, these functions are irregular in several senses (for instance, the graph of such a function is dense in the plane). Therefore, the study of additive real functions satisfying additional (possibly conditional) functional equations has been an active area of research. Such investigations were extended to quadratic functions by Z. Boros and Edit Garda-Mátyás (see [**BG**] and the references therein). Z. Kominek, L. Reich, and J. Schwaiger [**KRS**] investigated additive functions $f : \mathbb{R} \to \mathbb{R}$ fulfilling the additional condition:

$$f(x)f(y) = 0 \quad \text{for all } (x, y) \in D\,, \tag{1}$$

and proved that the only solution is $f(x) = 0$ for all $x \in \mathbb{R}$, where D represents various subsets of $\mathbb{R}^2$:

- $D = R_{p,q}$, with p and q being regular, non-constant polynomials;
- $D = S_2$, the unit circle;
- D is any measurable set in $\mathbb{R}^2$ with a positive planar Lebesgue measure.

Further, Z. Boros and W. Fechner [**BF**] extended these findings for $D = S_2$ to include generalized polynomials. However, P. Kutas [**Kut**] recently established the existence of a nonzero additive function $f : \mathbb{R} \to \mathbb{R}$ that satisfies the same condition (1) for $D = S_0$. The purpose of this dissertation was to extend various results involved in [**KRS**] to generalized polynomials defined on the real line or more general domains.

Let $(G, +)$ denote an Abelian group and m be a positive integer. A function $F : G^m \to \mathbb{R}$ is called *m-additive* if F is additive in each of its variables. If $f : G \to \mathbb{R}$ is defined as a diagonalization (or trace) of an m-additive mapping $F : G^m \to \mathbb{R}$ as

$$f(x) = F(x, \ldots, x) \tag{2}$$

for every $x \in G$, we say that f is a *generalized monomial of degree m*.

We note that generalized monomials of degree m are characterized as the solutions of the functional equation

$$\Delta_y^m f(x) = m! f(y) \qquad (x, y \in G), \tag{3}$$

if G is uniquely divisible by $(m + 1)!$ ([**Kuc**, Chapter 15], [**Sze91**, Chapter 1]).

Moreover, let

$$f(x) = \sum_{k=0}^{p} f_k(x) \qquad (x \in G), \tag{4}$$

where each $f_k : G \to \mathbb{R}$ is a generalized monomial of degree k (in particular f_0 is a constant function). Functions with representation (4) are called *generalized polynomials* (or *polynomial functions*) *of degree at most p*.

As a consequence, generalized polynomials of degree at most p are characterized as solutions of the functional equation

$$\Delta_y^{p+1} f(x) = 0 \qquad (x, y \in G) \tag{5}$$

if G is uniquely divisible by $(p+1)!$.

First we collect some notable properties of these functions that are established in order to provide powerful tools for our subsequent investigations. We establish the following continuity type property of generalized monomials under limits involving linear perturbations.

Lemma. *Let $m \in \mathbb{N}$, and let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree m. For any $x, y \in \mathbb{R}$, the following holds:*

$$\lim_{n\to\infty} f\left(x + \frac{y}{n}\right) = f(x).$$

We recall a key result by Halter-Koch, Reich, and Schwaiger [**HKRS**, Theorem 2], claiming that the set of generalized polynomials is an integral domain. Since the cited paper contains only a sketch of the proof of this fundamental property, the details of the argument were discussed in the particular case of real functions.

Equations along pairs of polynomials and conic sections

In the first part, we generalize these results by considering the condition:

$$f(x)g(y) = 0 \quad \text{for all } (x, y) \in D,$$

where f and g are generalized polynomials. Specifically, we examine cases where:

- $D = R_{p,q}$, with p and q as regular, non-constant polynomials;
- $D = S_{1,m}$, where m is an arbitrary nonzero real number.

Concerning the first setting we obtained that when the product of the compositions $f(p(x))$ and $g(q(x))$ vanishes identically, one of the generalized polynomials must be identically zero.

THEOREM. *Let p and q be polynomials of degrees at least one. If the generalized polynomials $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$f(p(x))g(q(x)) = 0 \tag{6}$$

for every $x \in \mathbb{R}$, then $f(x) = 0$ for all $x \in \mathbb{R}$ or $g(x) = 0$ for all $x \in \mathbb{R}$.

We may formulate the following corollary, demonstrating that a single generalized polynomial cannot satisfy such a product equation unless it is identically zero.

COROLLARY. *Let p and q be polynomials of degrees at least one. If the generalized polynomial $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation*

$$f(p(x))f(q(x)) = 0 \tag{7}$$

for every $x \in \mathbb{R}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

Given the conditional equation

$$f(x)g(y) = 0 \quad \text{for every} \quad (x, y) \in S_{1,m} \tag{8}$$

with a non-zero real number m, and two generalized polynomials (of possibly different degrees) f and g, we proved the following statement:

THEOREM. *Let m denote a non-zero real number. Suppose that f , g: $\mathbb{R} \to \mathbb{R}$ are generalized polynomials and $f(x)g(y) = 0$ for all $(x, y) \in S_{1,m}$. Then f or g is identically equal to zero.*

Corollary. *Let a and b denote positive real numbers and let $\sigma \in \{ -1 , 1 \}$. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a generalized polynomial and $f(x)f(y) = 0$ for all solutions of the equation $\frac{x^2}{a^2} - \sigma\frac{y^2}{b^2} = 1$. Then f is identically equal to zero.*

The previous corollary involves hyperbolas and ellipses when $\sigma = 1$ or $\sigma = -1$, respectively.

Investigations involving non-algebraic constraints

In the second part of the dissertation we consider non-algebraic constraints where $f : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \to \mathbb{R}$ are generalized polynomials fulfilling the conditional equation

$$f(x)g(y) = 0 \tag{9}$$

for every $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N}$ has a positive $2N$ dimensional Lebesgue measure or it is a second category Baire set. We prove that $f(x) = 0$ for every $x \in \mathbb{R}^N$ or $g(x) = 0$ for every $x \in \mathbb{R}^N$.

In fact, some statements are established in a considerably more general setting. Namely, we consider Euclidean spaces (or even more general domains, for instance, σ-finite measure spaces) X , Y and we investigate arbitrary functions $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ satisfying the condition (9) for all $(x, y) \in D$, where $D \subseteq X \times Y$ is large in the sense of measure or category. We prove that f or g vanishes on a large subset of X or Y, respectively, in the same sense.

Finally, we make use of a result by László Székelyhidi on the zeros of polynomials [**Sze85**, Theorem 2] (and some of its counterparts) to conclude, in various settings, that whenever the product of generalized polynomials f and g vanishes, in the sense of equation (9), on a large set, then f or g equals zero identically.

Then we investigate generalized monomials $f : \mathbb{R} \to \mathbb{R}$ of even degree fulfilling $f(x)f(y) \geq 0$ for the pairs $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ has a positive planar Lebesgue measure or it is a second category Baire set. We prove that f cannot change its sign.

All these investigations are extended to almost polynomial (respectively, almost monomial) functions as well.

Equation with measure constraint

The first theorem can be applied for the product of arbitrary functions over σ-finite measure spaces.

Theorem. *For each* $j \in \{1, 2\}$*, let* $(X_j, \mathcal{A}_j, \mu_j)$ *be a* σ*-finite measure space. Suppose that* $f_j : X_j \to \mathbb{C}$ $(j = 1, 2)$ *fulfill*

$$f_1(x) f_2(y) = 0 \tag{10}$$

for all $(x, y) \in D$*, where* $D \subseteq X_1 \times X_2$ *is a* $\mu_1 \times \mu_2$ *measurable subset with positive product measure. Then there exist an index* $j \in \{1, 2\}$ *and* $A_j \in \mathcal{A}_j$ *such that* $\mu_j(A_j) > 0$ *and* $f_j(x) = 0$ *for every* $x \in A_j$.

The statement is identical if, instead of the product measure $\mu_1 \times \mu_2$ (as defined by Halmos [**Hal**, § 34. Theorem A (p. 141), § 35. Theorems A and B (p. 143–144)]), we refer to its Lebesgue completion $\mu_1 \otimes \mu_2$ as defined, for instance, in a monograph by Bogachev [**Bog**, Theorem 3.3.1].

In the sequel we apply our previous results to products of functions taken from particular families of functions.

Definition. Let $(X, \mathcal{A}, \mu)$ denote a measure space (i.e., let $(X, \mathcal{A})$ denote a measurable space with a non-negative — possibly, but not identically, infinite — and σ-additive set function μ on $\mathcal{A}$). We call a family $\mathcal{F}$ of (possibly non-measurable) functions $f : X \to \mathbb{C}$ *algebraically measure regular* provided that the following implication is valid for every $f \in \mathcal{F}$: if $f(x) = 0$ for every $x \in B$, where $B \in \mathcal{A}$ and $\mu(B) > 0$, then $f(x) = 0$ for every $x \in X$.

So we call a family of functions *algebraically measure regular* if every member of this family that vanishes on a set of positive measure must be identically equal to zero.

THEOREM. *For each* $j \in \{1, 2\}$, *let* $(X_j, \mathcal{A}_j, \mu_j)$ *be a* σ*-finite measure space and let* $\mathcal{F}_j$ *denote an algebraically measure regular family of functions* $f : X_j \to \mathbb{C}$. *Let* $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ *such that (10) holds for all* $(x, y) \in D$, *where* $D \subseteq X_1 \times X_2$ *is a* $\mu_1 \otimes \mu_2$ *measurable subset with positive measure. Then* f_1 *or* f_2 *is identically equal to zero.*

Since, for arbitrary positive integers k and m, the $k + m$ dimensional Lebesgue measure can be considered as the Lebesgue completion of the product of the k and m dimensional Lebesgue measures, we can establish the following particular case of the previous theorem.

COROLLARY. *For some* $k, m \in \mathbb{N}$, *let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *denote algebraically measure regular families of functions* $f_1 : \mathbb{R}^k \to \mathbb{C}$ *and* $f_2 : \mathbb{R}^m \to \mathbb{C}$ *related to the* k *and* m *dimensional Lebesgue measures, respectively. Suppose that* $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ *such that (10) holds for all* $(x, y) \in D$, *where* $D \subseteq \mathbb{R}^{k+m}$ *is a measurable subset with a positive* $k+m$ *dimensional Lebesgue measure. Then* f_1 *or* f_2 *is identically equal to zero.*

Using induction, we can easily extend this corollary to arbitrary finite products.

COROLLARY. *Let* $n \in \mathbb{N}$, $k_j \in \mathbb{N}$ $(j = 1, 2 \ldots, n)$ *and* $m = \sum_{j=1}^{n} k_j$. *For each* $j \in \{1, 2 \ldots, n\}$, *let* $\mathcal{F}_j$ *denote an algebraically measure regular family of functions* $f : \mathbb{R}^{k_j} \to \mathbb{C}$ *with respect to the* k_j *dimensional Lebesgue measure. Let* $f_j \in \mathcal{F}_j$ $(j = 1, 2, \ldots, n)$ *such that*

$$f_1(x_1)\, f_2(x_2) \cdots f_n(x_n) = 0 \tag{11}$$

holds for all $(x_1, x_2, \ldots, x_n) \in D$ *where* $D \subseteq \mathbb{R}^m$ *is measurable with positive* m *dimensional Lebesgue measure. Then there exists an index* $j^* \in \{1, 2 \ldots, n\}$ *such that* f_{j^*} *is identically equal to zero.*

Now we consider functions defined on particular locally compact Abelian groups. It is well known [**Hal**, § 58. Theorem B (p. 254)] that there exists a Haar measure on such a group (which is unique on Borel sets up to a constant factor [**Hal**, § 60. Theorem C (p. 263)]). In our statements and arguments we refer to such a measure.

We shall make use of Székelyhidi's result on the zeros of generalized polynomials ([**Sze85**, Theorem 2], [**Sze91**, Theorem 3.3]):

THEOREM. *Let $\mathcal{G}$ be a locally compact Abelian group which is generated by any neighborhood of zero and let Z be a complex linear space. If a generalized polynomial $p : \mathcal{G} \to Z$ vanishes on a Haar measurable set of positive Haar measure, then it vanishes everywhere.*

Clearly, Székelyhidi's theorem states that the family of all generalized polynomials $p : \mathcal{G} \to \mathbb{C}$ constitutes an algebraically measure regular family of functions. Therefore, we obtain the following theorem as a corollary.

THEOREM. *For each $j \in \{1, 2\}$, let G_j be a locally compact Abelian group which is generated by any neighborhood of zero, let μ_j denote the Haar measure on G_j, and let us assume that μ_j is σ-finite. Let $f_j : G_j \to \mathbb{C}$ be generalized polynomials ($j = 1, 2$) fulfilling (10) for all $(x, y) \in D$, where $D \subseteq G_1 \times G_2$ is a $\mu_1 \times \mu_2$ measurable subset with positive measure. Then f_1 or f_2 is identically equal to zero.*

COROLLARY. *Let $\mathcal{G}$ be a locally compact Abelian group which is generated by any neighborhood of zero. Let μ denote the Haar measure on $\mathcal{G}$, and let us assume that μ is σ-finite. Let $f : \mathcal{G} \to \mathbb{C}$ be a generalized polynomial fulfilling*

$$f(x)f(y) = 0 \tag{12}$$

for all $(x, y) \in D$, where $D \subseteq \mathcal{G}^2$ is a $\mu \times \mu$ measurable subset with positive measure. Then $f(x) = 0$ for every $x \in \mathcal{G}$.

The following theorem is a straightforward corollary of our results as well.

Theorem. *Let* $n \in \mathbb{N}$, $k_j \in \mathbb{N}$ $(j = 1\,, 2\,\ldots, n)$ *and* $m = \sum_{j=1}^{n} k_j$. *Let* $f_j : \mathbb{R}^{k_j} \to \mathbb{C}$ *be generalized polynomials* $(j = 1, 2, \ldots, n)$ *fulfilling (11) for all* $(x_1\,, x_2\,, \ldots\,, x_n) \in D$ *where* $D \subseteq \mathbb{R}^m$ *is measurable with positive* m *dimensional Lebesgue measure. Then there exists an index* $j^* \in \{\,1\,, 2\,\ldots, n\,\}$ *such that* f_{j^*} *is identically equal to zero.*

Equation with category constraint

In this section we elaborate an analogy of the previous results when sets of positive measure are replaced with second category Baire sets in Euclidean spaces. We recall that $B \subseteq \mathbb{R}^N$ has the Baire property (or shortly, B is a Baire set) if there exist an open set $G \subseteq \mathbb{R}^N$ and a first category set $T \subseteq \mathbb{R}^N$ such that $B = G \bigtriangleup T$ (where the set operation $\bigtriangleup$ denotes the symmetric difference, as usual [**Oxt**], [**Kuc**, Chapter 2]).

In what follows, we wish to establish a category version of Székelyhidi's theorem on the zeros of generalized polynomials. For this purpose we need an analogy of Steinhaus' Theorem.

Lemma. *Let* $m \in \mathbb{N}$ *and* $A \subseteq \mathbb{R}^N$ *such that* A *is a second category Baire set. Then there exists a neighborhood* U *of zero such that, for every* $y \in U$, *there exists* $x \in \mathbb{R}^N$ *fulfilling*

$$x + ky \in A \quad (k = 0, 1, \ldots, m).$$

In the particular case $m = N = 1$ this result was proved by S. Piccard [**Pic**]. Piccard's theorem has been generalized by several authors (e.g. [**Jar**]).

Now we can establish a category version of Székelyhidi's theorem.

Theorem. *If a generalized polynomial* $f : \mathbb{R}^N \to \mathbb{C}$ *vanishes on a second category Baire set, then* f *vanishes everywhere.*

In the rest of this section, results for alternative equations with category constraints are presented. In our first theorem we consider arbitrary functions.

Theorem. *Let* $k\,,\, m \in \mathbb{N}\,$. *Suppose that* $f_1 : \mathbb{R}^k \to \mathbb{C}$ *and* $f_2 : \mathbb{R}^m \to \mathbb{C}$ *such that (10) holds for all* $(x, y) \in D\,$, *where* $D \subseteq \mathbb{R}^{k+m}$ *is a second category Baire set. Then there exists a second category Baire set* $A_1 \subseteq \mathbb{R}^k$ *such that* $f_1(x) = 0$ *for every* $x \in A_1\,$, *or there exists a second category Baire set* $A_2 \subseteq \mathbb{R}^m$ *such that* $f_2(y) = 0$ *for every* $y \in A_2\,$.

Definition. Let $k \in \mathbb{N}\,$. We call a family $\mathcal{F}$ of functions $f : \mathbb{R}^k \to \mathbb{C}$ *algebraically Baire regular* provided that the following implication is valid for every $f \in \mathcal{F}$: if $f(x) = 0$ for every $x \in B\,$, where $B \subseteq \mathbb{R}^k$ is a second category Baire set, then $f(x) = 0$ for every $x \in \mathbb{R}^k\,$.

So we call a family of functions *algebraically Baire regular* if every member of this family that vanishes on a second category Baire set must be identically equal to zero.

Theorem. *For some* $k\,,\, m \in \mathbb{N}\,$, *let* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *denote algebraically Baire regular families of functions* $f_1 : \mathbb{R}^k \to \mathbb{C}$ *and* $f_2 : \mathbb{R}^m \to \mathbb{C}\,$. *Suppose that* $f_j \in \mathcal{F}_j$ $(j = 1, 2)$ *such that (10) holds for all* $(x, y) \in D\,$, *where* $D \subseteq \mathbb{R}^{k+m}$ *is a second category Baire set. Then* f_1 *or* f_2 *is identically equal to zero.*

Corollary. *For some* $k\,,\, m \in \mathbb{N}\,$, *let* $f : \mathbb{R}^k \to \mathbb{C}$ *and* $g : \mathbb{R}^m \to \mathbb{C}$ *be generalized polynomials fulfilling (9) for all* $(x, y) \in D\,$, *where*

$D \subseteq \mathbb{R}^{k+m}$ is second category Baire set. Then $f(x) = 0$ for every $x \in \mathbb{R}^k$ or $g(y) = 0$ for every $y \in \mathbb{R}^m$.

Corollary. *Let $N \in \mathbb{N}$ and let $f : \mathbb{R}^N \to \mathbb{C}$ be a generalized polynomial fulfilling (12) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ is second category Baire set. Then $f(x) = 0$ for every $x \in \mathbb{R}^N$.*

Products of generalized polynomials

In this section we mention an alternative approach to some of our results for the products of generalized polynomials. This argument was suggested by Eszter Gselmann [**Gse**].

Proposition. *Let G, H be commutative semigroups, n be a nonnegative integer such that the multiplication by $n!$ is bijective in the commutative group K. If $\varphi : G \to H$ is a homomorphism and $p : H \to K$ is a generalized polynomial of degree at most n, then $p \circ \varphi : G \to K$ is a generalized polynomial of degree at most n.*

Applying the previous statement, Eszter Gselmann obtained the following notable result as well.

Proposition. *Let k be a positive integer, $G_1, \ldots, G_k$ be commutative semigroups and X be an algebra over the field $\mathbb{K}$. Let further $p_i : G_i \to X$ be a generalized polynomial for all $i = 1, \ldots, k$. Then the mapping P defined on $\times_{i=1}^k G_i$ by*

$$P(x_1, \ldots, x_k) = p_1(x_1) \cdots p_k(x_k) \quad \big((x_1, \ldots, x_k) \in \times_{i=1}^k G_i\big) \tag{13}$$

is a generalized polynomial on $\times_{i=1}^k G_i$.

Now, if we assume that X denotes the field of (real or) complex numbers and the mapping (13) vanishes on a large set (in the sense of the Haar measure or category) with respect to the product topology on the Cartesian product of locally compact Abelian groups (or, in particular, Euclidean spaces), we may apply Székelyhidi's theorem or its category counterpart to conclude that the mapping (13) vanishes identically on the product space. Then we may apply the formerly cited theorem by Halter-Koch, Reich and Schwaiger [**HKRS**, Theorem 2] claiming that the set of generalized polynomials is an integral domain. Hence, if the product is zero, one of the factors p_l must be identically equal to zero.

Inequalities for monomials with measure constraint

Investigations on the signs of monomial mappings (involving measure constraints) are based on the following preliminary results.

LEMMA. *Suppose that $P \subseteq \mathbb{R}$ fulfills the following assumptions:*

(i) *for all $r \in \mathbb{Q}$ and $x \in P$ we have $rx \in P$;*

(ii) *there exists a Lebesgue measurable set A such that $A \subseteq P$ and $\lambda(A) > 0$.*

Then P has full Lebesgue measure, that is, $\mathbb{R} \setminus P$ is Lebesgue measurable and $\lambda(\mathbb{R} \setminus P) = 0$.

LEMMA. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree $m \geq 1$ and let H denote a closed subset of $\mathbb{R}$ such that*

$$P = \{\, x \in \mathbb{R} \mid f(x) \in H \,\}$$

has full Lebesgue measure. Then $f(x) \in H$ for every $x \in \mathbb{R}$.

Now we can establish a sufficient condition for the non-negativity of monomial functions of even degree [**BM24a**, Theorem 4.3].

THEOREM. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of even degree $m = 2k$ (with some positive integer k) fulfilling*

$$f(x) \geq 0 \tag{14}$$

for all $x \in A$, *where* $A \subseteq \mathbb{R}$ *is a Lebesgue measurable subset with positive Lebesgue measure. Then (14) holds for every* $x \in \mathbb{R}$.

This theorem can be combined with a representation theorem due to Gy. Maksa [**Mak**]: If a quadratic function f is non-negative on $\mathbb{R}$, then there exists a Hilbert space H and an additive mapping $\varphi : \mathbb{R} \to H$ such that $f(x) = \|\varphi(x)\|^2$ for every $x \in \mathbb{R}$. In view of our above theorem, the assumption in Maksa's theorem involving the non-negativity of f everywhere could be replaced by the weaker condition that f is non-negative on a set with positive Lebesgue measure (since quadratic functions are generalized monomials of degree 2).

We may obtain an analogous statement for monomial functions with odd degree.

Corollary. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a generalized monomial of odd degree* $m = 2k - 1$ *(with some positive integer* k*) fulfilling*

$$xf(x) \geq 0 \tag{15}$$

for all $x \in A$, *where* $A \subseteq \mathbb{R}$ *is a Lebesgue measurable subset with positive Lebesgue measure. Then (15) holds for every* $x \in \mathbb{R}$.

Now we can establish a sufficient condition for a monomial function of even degree to assure that it preserves its sign.

Theorem. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a generalized monomial of even degree fulfilling*

$$f(x)f(y) \geq 0 \tag{16}$$

for all $(x, y) \in D$, *where* $D \subseteq \mathbb{R}^2$ *is a Lebesgue measurable subset with positive planar Lebesgue measure. Then* (16) *holds for all* $(x, y) \in \mathbb{R}^2$ *(i.e.,* f *does not change sign).*

Inequalities for monomials with category constraint

It is natural to investigate the topological analogues of our previous results, when the required properties are satisfied for arguments taken from a second category Baire set.

LEMMA. *Suppose that $P \subseteq \mathbb{R}$ fulfills the following assumptions:*

(i) *for all $r \in \mathbb{Q}$ and $x \in P$ we have $rx \in P$;*
(ii) *there exists a second category Baire set A such that $A \subseteq P$.*

Then $\mathbb{R} \setminus P$ is of the first category.

LEMMA. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of degree $m \geq 1$ and let H denote a closed subset of $\mathbb{R}$ such that the set*

$$S = \{\, x \in \mathbb{R} \,|\, f(x) \notin H \,\}$$

is of the first category. Then $f(x) \in H$ for every $x \in \mathbb{R}$.

THEOREM. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of even degree $m = 2k$ (with some positive integer k) fulfilling (14) for all $x \in A$, where $A \subseteq \mathbb{R}$ is a second category Baire set. Then (14) holds for every $x \in \mathbb{R}$.*

COROLLARY. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of odd degree $m = 2k - 1$ (with some positive integer k) fulfilling (15) for all $x \in A$, where $A \subseteq \mathbb{R}$ is a second category Baire set. Then (15) holds for every $x \in \mathbb{R}$.*

THEOREM. *Let $f : \mathbb{R} \to \mathbb{R}$ be a generalized monomial of even degree fulfilling (16) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^2$ is a second category Baire set. Then (16) holds for all $(x, y) \in \mathbb{R}^2$ (i.e., f does not change sign).*

Equations for almost polynomials

Finally it is possible to extend the previous results for almost polynomial functions and almost monomial functions, respectively.

In order to cover the concept and the description of almost polynomial functions both in the sense of measure and in the sense of category, we recall Ger's concepts of conjugate proper linearly independent ideals ([**Ger**, Definitions 2 and 3], [**Kuc**, Definitions in Section 17.5]).

Definition. Let N denote a positive integer. A family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is called a *proper linearly independent ideal* if

(i) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;
(ii) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$;
(iii) $\mathbb{R}^N \notin \mathcal{I}$;
(iv) $\alpha \in \mathbb{R}$, $A \in \mathcal{I}$ and $u \in \mathbb{R}^N$ implies $\alpha A + u \in \mathcal{I}$.

For an arbitrary set $M \subseteq \mathbb{R}^N \times \mathbb{R}^N$ and for every $x \in \mathbb{R}^N$, let

$$M_x = \{\, y \in \mathbb{R}^N \mid (x, y) \in M \,\}.$$

Definition. Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}^N$ and $\mathcal{I}_2$ be a proper linearly independent ideal in $\mathbb{R}^N \times \mathbb{R}^N$. We say that the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are *conjugate* if, for every set $M \in \mathcal{I}_2$ there exists a set $U \in \mathcal{I}_1$ such that $M_x \in \mathcal{I}_1$ for every $x \in \mathbb{R}^N \setminus U$.

Given a proper linearly independent ideal $\mathcal{I}$ in $\mathbb{R}^N$, we say that a condition is satisfied $\mathcal{I}$-*almost everywhere* in $\mathbb{R}^N$ (written $\mathcal{I}$-(a.e.)) if there exists a set $S \in \mathcal{I}$ such that the condition holds for every $x \in \mathbb{R}^N \setminus S$. Using this terminology, we may say that the ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate if, for every set $M \in \mathcal{I}_2$, the condition $M_x \in \mathcal{I}_1$ holds $\mathcal{I}_1$-almost everywhere in $\mathbb{R}^N$.

The following examples are corollaries of Fubini's theorem [**Bog**, Theorem 3.4.1 and Corollary 3.4.2] and the Kuratowski–Ulam theorem [**KU**], respectively.

Example. Let $\mathcal{L}_0^N$ denote the family of Lebesgue measurable subsets A of $\mathbb{R}^N$ fulfilling $\lambda^N(A) = 0$, where λ^N denotes the N-dimensional Lebesgue measure. Then $\mathcal{L}_0^N$ and $\mathcal{L}_0^{2N}$ are conjugate proper linearly independent ideals.

Example. Let $\mathcal{F}^N$ denote the family of first category subsets of $\mathbb{R}^N$. Then $\mathcal{F}^N$ and $\mathcal{F}^{2N}$ are conjugate proper linearly independent ideals.

These examples motivate the following concept ([**Ger**, Definition 4], [**Kuc**, Section 17.7]).

Definition. Let $\mathcal{I}$ be a proper linearly independent ideal in $\mathbb{R}^{2N} = \mathbb{R}^N \times \mathbb{R}^N$ and let $p \in \mathbb{N}$. We call a function $f : \mathbb{R}^N \to \mathbb{R}$ an *$\mathcal{I}$-almost polynomial function of degree at most p* if there exists $S \in \mathcal{I}$ such that

$$\Delta_y^{p+1} f(x) = 0 \tag{17}$$

holds for every $(x, y) \in \mathbb{R}^{2N} \setminus S$.

This concept was introduced by Roman Ger in order to establish an abstract description of almost polynomial functions as follows ([**Ger**, Theorem 1], [**Kuc**, Theorem 17.7.2]).

Theorem. [R. Ger] *Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}^N$ and $\mathcal{I}_2$ be a proper linearly independent ideal in $\mathbb{R}^{2N}$ such that $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate. If $f : \mathbb{R}^N \to \mathbb{R}$ is an $\mathcal{I}_2$-almost polynomial function of degree at most p, then there exists a unique polynomial function $g : \mathbb{R}^N \to \mathbb{R}$ such that $f = g$ $\mathcal{I}_1$-almost everywhere in $\mathbb{R}^N$.*

As immediate consequences of the previous examples and Ger's theorem, as well as the corresponding results for generalized polynomials, we can establish the following corollaries.

Corollary. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be an $\mathcal{L}_0^{2N}$-almost polynomial function of degree at most $p \geq 1$ fulfilling (12) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^N \times \mathbb{R}^N$ is Lebesgue-measurable and $\lambda^{2N}(D) > 0$. Then $f = 0$ $\mathcal{L}_0^N$(-a.e.) in $\mathbb{R}^N$ (i.e., $f(x) = 0$ for λ^N-almost every $x \in \mathbb{R}^N$).*

Corollary. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be an $\mathcal{F}^{2N}$-almost polynomial function of degree $m \geq 1$ fulfilling (12) for all $(x, y) \in D$, where $D \subseteq \mathbb{R}^N \times \mathbb{R}^N$ is a second category Baire set. Then $f = 0$ $\mathcal{F}^N$(-a.e.) in $\mathbb{R}^N$.*

Inequalities for almost monomials

Since we have considered conditional inequalities for monomial functions in a real variable, we may wish to extend our related results to almost monomial functions defined on $\mathbb{R}$. Motivated by the approach in the previous section, we may introduce the required concept as follows.

Definition. Let $\mathcal{I}$ be a proper linearly independent ideal in $\mathbb{R}^2$ and let $m \in \mathbb{N}$. We call a function $f : \mathbb{R} \to \mathbb{R}$ an *$\mathcal{I}$-almost monomial function of degree m* if there exists $S \in \mathcal{I}$ such that

$$\Delta_y^m f(x) = m! f(y) \tag{18}$$

holds for every $(x, y) \in \mathbb{R}^2 \setminus S$.

In order to establish an analogy of Ger's theorem to almost monomials, we need a stronger invariance property with respect to the linear transformations of the members of the ideal. So we consider the following concept.

Definition. Let N denote a positive integer. A family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is called a *proper linearly invariant ideal* if $\mathcal{I}$ is a proper linearly independent ideal such that $S \in \mathcal{I}$ implies $\phi(S) \in \mathcal{I}$ for every linear bijection $\phi : \mathbb{R}^N \to \mathbb{R}^N$.

In other words, we could say that a non-void family $\mathcal{I}$ of subsets of $\mathbb{R}^N$ is a proper linearly invariant ideal if $\mathcal{I}$ fulfills properties (i)–(iii) in the definition of proper linearly independent ideals and (instead of (iv)) it satisfies

(iv*) $S \in \mathcal{I}$ implies $\phi(S) + u \in \mathcal{I}$ for every $u \in \mathbb{R}^N$ and for every linear bijection $\phi : \mathbb{R}^N \to \mathbb{R}^N$.

Using the notation introduced in the former examples, we can claim that sets having Lebesgue measure zero as well as sets of the first category form proper linearly invariant ideals. The first example follows immediately from the identity for the Lebesgue measure of linear transforms of measurable sets [**Bog**, Corollary 3.6.4].

Example. $\mathcal{L}_0^N$ is a proper linearly invariant ideal.

The second example can be established as follows.

Proposition. *$\mathcal{F}^N$ is a proper linearly invariant ideal.*

In view of these examples, it is reasonable to formulate the following theorem.

Theorem. *Let $\mathcal{I}_1$ be a proper linearly independent ideal in $\mathbb{R}$ and $\mathcal{I}_2$ be a proper linearly invariant ideal in $\mathbb{R}^2$ such that $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate. Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{I}_2$-almost monomial function of degree $m \geq 1$. Then there exists a unique monomial function $g : \mathbb{R} \to \mathbb{R}$ of degree m such that $f = g$ $\mathcal{I}_1$-almost everywhere*

Now we can establish the analogies of our former results for generalized monomials for $\mathcal{L}_0^2$-almost monomial functions and $\mathcal{F}^2$-almost monomial functions.

Corollary. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be an* $\mathcal{L}_0^2$*-almost monomial function of degree* $m=2k$ *fulfilling (16) for all* $(x,y)\in D$ *, where* $D\subseteq\mathbb{R}^2$ *is a Lebesgue measurable subset with positive planar Lebesgue measure. Then* $f(x)\geq 0$ *for* $\mathcal{L}_0^1$*-almost every* $x\in\mathbb{R}$ *or* $f(x)\leq 0$ *for* $\mathcal{L}_0^1$*-almost every* $x\in\mathbb{R}$ *.*

Corollary. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be an* $\mathcal{F}^2$*-almost monomial function of degree* $m=2k$ *fulfilling (16) for all* $(x,y)\in D$ *, where* $D\subseteq\mathbb{R}^2$ *is a second category Baire set. Then* $f(x)\geq 0$ *for* $\mathcal{F}^1$*-almost every* $x\in\mathbb{R}$ *or* $f(x)\leq 0$ *for* $\mathcal{F}^1$*-almost every* $x\in\mathbb{R}$ *.*

Perspectives of this research project

Concerning the algebraic conditions (along conic sections) for the product of generalized polynomials, the problem is completely solved for parabolas, ellipses and hyperbolas in standard positions. The counterexample by Kutas [**Kut**] demonstrates that our results cannot be extended to arbitrary hyperbolas even in the case when a single additive function is involved. It is a natural (but possibly difficult) research project to give a description of conic sections $S\subseteq\mathbb{R}^2$ that admit the implication that whenever a generalized polynomial $f:\mathbb{R}\to\mathbb{R}$ fulfills the condition $f(x)f(y)=0$ for all $(x,y)\in S$, then $f=0$ identrically. The problem is open even in the particular case when f is additive.

Another natural idea is to consider the additional equation

$$f(x_1)f(x_2)\dots f(x_k)=0\,,$$

fulfilled under the condition $(x_1\,,\,x_2\,,\,\ldots\,,\,x_k) \in D$, where $D \subseteq \mathbb{R}^k$ is given by an appropriate algebraic equation and $f : \mathbb{R} \to \mathbb{R}$ is additive (or a generalized monomial, respectively, polynomial).

Our results for products vanishing on large subsets of the product space admit applications for algebraically measure/Baire regular families of functions. In the thesis we mention the family of generalized polynomials (on various domains) as an example for such families. It is a natural idea to look for other (different or more general) interesting families of functions fulfilling these properties. Clearly, new examples admit new applications of our main results in this direction. It is another interesting question (suggested by Peter Eliaš at the 38^{th} ISCRFT, September 15–20, 2024, Stará Lesná, Slovakia) whether we can replace the product $f(x)g(y)$ by some other appropriate operation $F(f(x), g(y))$ in our investigations.

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List of talks

1. *Phd Qualification at the end of the first year*, Institute of Mathematics, University of Debrecen, June 18, 2021.
2. *Alternative equations for quadratic functions*, Síkfőkút seminar of the department of analysis, August 27–29, 2021.
3. *Alternative equations for quadratic functions* 19th International Conference on Functional Equations and Inequalities, September 12–18, 2021 (on-line).
4. *An alternative equation for generalized monomials*, 21st Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Brenna, Poland, February 2–5, 2022.
5. *Complex Exam*, Institute of Mathematics, University of Debrecen, June 29, 2022.
6. *An alternative equation for generalized monomials involving measure*, 22nd Debrecen–Katowice Winter Seminar on Functional Equations and Inequalities, Hajdúszoboszló, Hungary, February 1–4, 2023.
7. *Phd Qualification at the End of the Third Year*, Institute of Mathematics, University of Debrecen, June 6, 2023.
8. *An alternative equation for generalized polynomials of degree two*, The 59th International Symposium on Functional Equations, Hajdúszoboszló, Hungary, June 18–25, 2023.
9. *An alternative equation for almost polynomials and related inequalities for generalized monomials*, 37th International Summer Conference on Real Functions Theory, Rowy, Poland, September 10–15, 2023.

(10) *An alternative equation for polynomial functions on locally compact abelian groups*, 23rd Katowice–Debrecen Winter Seminar on Functional Equations and Inequalities, Brenna, Poland, January 31 – February 3, 2024.

(11) *An alternative equation for two polynomial functions on locally compact abelian groups*, 60th International Symposium on Functional Equations, Kościelisko, Poland, June 9–15, 2024.