

Article

Spectra of Phases of Clusterized Nuclei

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Abstract

In the theoretical understanding of nuclear structure, different models play a crucial role due to the complications of the quantum mechanical many-body problem. One of the fundamental models is the cluster model, based on the molecular picture. We investigate here the possible phases of nuclear clusterization in a quantitative way. In particular, the semimicroscopic algebraic cluster model is applied, in which the phases are defined by the dynamical symmetry limits. We construct detailed spectra corresponding to the shell-like, rigid-molecule-like and weak-coupled phases, and compare them. Special attention is focused on the role of the Pauli principle. The spectra of the limiting cases show dramatic differences, and only one of them harmonizes with this basic natural law. The experimentally observed $^{12}\text{C} + \alpha$ spectrum is considered for the discussion.

Keywords: clusterization; phases; dynamical symmetries; semimicroscopic algebraic cluster model

1. Introduction

The basic characteristics of atomic nuclei cannot be explained within a single model; rather, we have competing approaches to account for different aspects. The appearance of the magic proton and neutron numbers, e.g., is well understood within the shell model, which, however, has difficulties in describing the decay properties and nuclear reactions. For this purpose, the cluster model is very useful. It supposes that a nucleus is composed of smaller nuclei, having a structure like that of a molecule.

The cluster model is the oldest structure model of atomic nuclei, which was invented as an alpha-particle model in the early works of Gamow [1,2], before the discovery of the neutrons. Today, a very large variety of cluster models exists [3], and most of them take into account that the clusters are built up from nucleons. The nature of clusterization (i.e., the question of whether it is more like a rigid molecule, or more shell-like) seems to be an evergreen topic for discussion. Different aspects of the problem have been addressed within the framework of various models. Most of these disputes have taken place either (i) on a qualitative level or (ii) by using laborious approaches that are not applicable in a systematic way; see, e.g., [4–6], and references therein. With the present work, we wish to make the first step towards the systematic and quantitative study of the nature of clustering in light nuclei.

As of recent, this kind of problem can be investigated in terms of cold phases, both in nuclear physics and in other areas of many-body systems. They are not to be confused with thermal phases (and phase transitions), which are also present in nuclei. The best known



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examples for the latter kind are the liquid–gas phase transition of finite nuclear matter [7] and the transition from quark-gluon plasma to the hadronic phase [8].

In cold phases and their transitions, the temperature is not a relevant parameter; nevertheless, many of the concepts of phases and phase transitions can be described in a similar manner to those of the original thermodynamical problems. (The cold phases are also called shape phases or zero temperature phases.) Instead of the temperature and pressure, the control parameter is provided by relative weight(s) of the parts of the Hamiltonian with specific dynamical symmetries, which define the phase(s). Thus, when there are two limiting cases (phases), there is a single control parameter, and the phase diagram is one-dimensional, with the two dynamical symmetries at the end-points. For a model with three dynamical symmetries, the phase diagram is two-dimensional, which can be represented by a triangle, and there are two control parameters to specify the location of a system. In studying the cold phases, the application of algebraic models is especially useful [9–11].

A model is called fully algebraic when not only the basis states but also the physical operators carry group symmetries. In such models, dynamical symmetries appear as limiting cases, when the energy eigenvalue problem has an analytical solution. These limits define the phases, and the relative weights of the interactions with different dynamical symmetries serve as control parameters.

Algebraic models have been used successfully in the studies of phases of the quadrupole collective models [12] and the shell model [13]. Also, for the cluster model, the possible phases have been identified [14]. It turns out that there are three phases, with characteristic symmetries as follows. The shell-like (or soft vibrator) limit has a $U(3)$ symmetry, the rigid-molecule-like (or rigid rotor) phase with $O(4)$, and the weak-coupled limit of $O(3)$. Even the relation of the phase diagrams of the shell and cluster models has been revealed: the $U(3)$ limit is their common intersection. The detailed spectra of the three phases, however, have not been studied and compared yet. Here, we do so.

We apply the semimicroscopic algebraic cluster model (SACM) [15,16] in which the internal structure of the clusters is treated in terms of the Elliott shell model [17,18] while their relative motion is described by the modified vibron model [19,20]. Modification means the exclusion of the Pauli-forbidden states and the spurious excitation of the centre of mass. The model spaces are identical to those of the fully microscopic cluster models (e.g., RGM), when they apply the $U(3)$ basis. The physical operators are expressed in terms of the group generators, with some parameters to fit the experimental data.

SACM has already been applied for the investigation of the phase transitions of clusterization [21–24]. These studies concentrate mainly on the limit of the large number of particles (excitation quanta), where real phase transitions can be seen. For finite cases, some reminiscence can be observed. The question of where the experimentally observed cluster systems are located on the phase diagram is another aspect of the phases of clustering. We address this question with the work presented here, and some following studies.

The structure of this paper is organized as follows: In Section 2, we provide an overview of the dynamical symmetry limits within the SACM. We discuss the labeling of states, the role of the Pauli principle, and the Hamiltonians associated with these limits. In Section 3, we present schematic spectra of respective limits. In Section 4, we consider the experimental data of the $^{12}\text{C} + \alpha$ system, and a model calculation is performed. A summary is presented, and some conclusions are drawn in Section 5.

2. Semimicroscopic Algebraic Cluster Model

2.1. Labeling of States

In the simplest case of a binary cluster configuration with closed-shell clusters, the relative motion defines the phase space. It is described by the vibron model, which is an algebraic model of the dipole collectivity with a $U(4)$ group structure. $U(4)$ is a dynamical group, which contains a spectrum generating part in addition to the $U(3)$ symmetry group of the three-dimensional motion. It has two dynamical symmetries, defined by the group chains:

$$U_R(4) \supset U_R(3) \supset SU_R(3) \supset SO_R(3), \quad (1)$$

$$U_R(4) \supset O_R(4) \supset SO_R(4) \supset SO_R(3). \quad (2)$$

Here, the subscript R refers to the relative motion, and the three-dimensional orthogonal group $SO(3)$ is generated by the angular momentum. The four-dimensional orthogonal group and the unitary groups act in abstract spaces. These limits define the possible phases, and thus the phase diagram is one-dimensional. By considering the internal structure of the clusters, the phase diagram becomes two-dimensional (a triangle), as discussed in the next section.

For binary clusterization (C_1, C_2), the model has the algebraic structure $U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_R(4) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3)$. Here, $U_{C_i}^{ST}(4)$ denotes the spin-isospin group. When dealing with spin-isospin zero clusters, which is the case here, the $U^{ST}(4)$ groups play a role only in the construction of the model space, but they do not influence physical operators. The simplified group structure is $U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4)$ [15,16]. It reduces to three subgroup chains, which contain the $SO(3)$ group:

$$\begin{aligned} & U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4) \\ & \left| [n_1^{C_1}, n_2^{C_1}, n_3^{C_1}], [n_1^{C_2}, n_2^{C_2}, n_3^{C_2}], [N, 0, 0, 0], \right. \\ & \supset U_C(3) \otimes U_R(3) \supset U(3) \supset SU(3) \supset SO(3) \\ & \left. [n_1^C, n_2^C, n_3^C], [n_\pi, 0, 0], [n_1, n_2, n_3], (\lambda, \mu), K, L \right\rangle \end{aligned} \quad (3)$$

$$\begin{aligned} & U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4) \\ & \left| [n_1^{C_1}, n_2^{C_1}, n_3^{C_1}], [n_1^{C_2}, n_2^{C_2}, n_3^{C_2}], [N, 0, 0, 0], \right. \\ & \supset U_C(3) \otimes O_R(4) \supset SU_C(3) \otimes SO_R(4) \supset SO_C(3) \otimes SO_R(3) \supset SO(3) \\ & \left. [n_1^C, n_2^C, n_3^C], (\omega, 0), (\lambda_C, \mu_C), \omega, K_C, L_C, L_R, L \right\rangle \end{aligned} \quad (4)$$

$$\begin{aligned} & U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4) \\ & \left| [n_1^{C_1}, n_2^{C_1}, n_3^{C_1}], [n_1^{C_2}, n_2^{C_2}, n_3^{C_2}], [N, 0, 0, 0], \right. \\ & \supset U_C(3) \otimes U_R(3) \supset SU_C(3) \otimes SU_R(3) \supset SO_C(3) \otimes SO_R(3) \supset SO(3) \\ & \left. [n_1^C, n_2^C, n_3^C], [n_\pi, 0, 0], (\lambda_C, \mu_C), (n_\pi, 0), K_C, L_C, L_R, L \right\rangle \end{aligned} \quad (5)$$

We indicate here also the representation labels of the groups, which are quantum numbers in the physical application. These three group chains define the different dynamical symmetry limits. Group chain (3) and (4) are just extensions of the (1) and (2) symmetries of the vibron model. The $U(3)$ limit corresponds to a shell-like cluster configuration, while the $O(4)$ limit corresponds to a rigid rotor, or rigid-molecule-like cluster configuration. Group chain (5) is the weak coupling limit ($O(3)$ limit).

In Equations (3)–(5) C_i denotes the i^{th} cluster, with $[n_1^{C_i}, n_2^{C_i}, n_3^{C_i}]$ representing its $U(3)$ quantum numbers. The $U_C(3)$ representations $[n_1^C, n_2^C, n_3^C]$ are obtained from the outer

product of the individual cluster representations $[n_1^{C_1}, n_2^{C_1}, n_3^{C_1}] \otimes [n_1^{C_2}, n_2^{C_2}, n_3^{C_2}]$. $[N, 0, 0, 0]$ gives the $U_R(4)$ symmetry, where N denotes the total number of bosons. Since N is a conserved quantity, considering a single representation of $U_R(4)$ is sufficient for computing energy eigenstates. By adjusting N , the size of the basis can be modified accordingly.

We note here that $U_R(4)$ is a dynamical group of the relative motion (containing also a spectrum generating part), while $U_{C_i}(3)$ s are (dynamically broken) symmetry groups of the internal cluster structure.

The $U_R(3)$ representation in (3) and (5), $[n_\pi, 0, 0]$, is related to N as follows:

$$n_\pi = N, N-1, \dots, 0. \quad (6)$$

The combined $U(3)$ representations $[n_1, n_2, n_3]$ are constructed from the outer product $[n_1^C, n_2^C, n_3^C] \otimes [n_\pi, 0, 0]$. The $SU(3)$ quantum numbers (λ, μ) are obtained from $U(3)$: $\lambda = n_1 - n_2$, $\mu = n_2 - n_3$. Similarly, the quantum numbers for other $SU_\gamma(3)$ groups are obtained using the same method, where γ represents C_1, C_2, C , or R . The angular momentum quantum number L is the label of $SO(3)$ group. For a (λ, μ) representation, it is defined as follows [17,18]:

$$\begin{aligned} L &= K, K+1, \dots, K + \max(\lambda, \mu), \\ K &= \min(\lambda, \mu), \min(\lambda, \mu) - 2, \dots, 1, \text{ or } 0 \end{aligned} \quad (7)$$

with the exception of $K = 0$, for which

$$L = \max(\lambda, \mu), \max(\lambda, \mu) - 2, \dots, 1, \text{ or } 0, \quad (8)$$

where K distinguishes multiple occurrences of a given L in (λ, μ) . Similarly, L_C and K_C can be obtained from (λ_C, μ_C) . L_R (of $SO_R(3)$) is unique within the $(n_\pi, 0)$; therefore, there is no need for an additional quantum number K_R . The label ω in the group chain (4) is used for $SO_R(4)$. It relates with N and L_R as [25]

$$\begin{aligned} \omega &= N, N-2, \dots, 1 \text{ or } 0, \\ L_R &= 0, 1, \dots, \omega. \end{aligned} \quad (9)$$

For the cases of $O(4)$ and $O(3)$ group chains ((4) and (5)), the label L of $SO(3)$ can be obtained from coupling, $L_C \oplus L_R = L$.

Keeping only the essential quantum numbers, the relevant basis states for these symmetries can be labeled as follows:

$$|[n_1^C, n_2^C, n_3^C], n_\pi, [n_1, n_2, n_3], (\lambda, \mu), K, L\rangle \quad (10)$$

$$|[n_1^C, n_2^C, n_3^C], (\lambda_C, \mu_C), \omega, K_C, L_C, L_R, L\rangle \quad (11)$$

$$|[n_1^C, n_2^C, n_3^C], (\lambda_C, \mu_C), n_\pi, K_C, L_C, L_R, L\rangle \quad (12)$$

The group chains (3)–(5) are expressed in a single sector of the spin–isospin space. If we consider the spin–isospin sector, the group chains are modified.

$$\begin{aligned} U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_R(4) &\supset U_C^{ST}(4) \otimes U_C(3) \otimes U_R(4) \supset \\ &U_C^S(2) \otimes U_C^T(2) \otimes U_C(3) \otimes U_R(3) \supset U_C^S(2) \otimes U_C^T(2) \otimes U(3) \supset \\ &U_C^S(2) \otimes U_C^T(2) \otimes O(3) \supset SU(2) \otimes SU_C^T(2) \end{aligned} \quad (13)$$

$$\begin{aligned}
U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_R(4) \supset U_C^{ST}(4) \otimes U_C(3) \otimes U_R(4) \supset \\
U_C^S(2) \otimes U_C^T(2) \otimes O_C(3) \otimes O_R(4) \supset U_C^S(2) \otimes U_C^T(2) \otimes O_C(3) \otimes O_R(3) \supset \\
U_C^S(2) \otimes U_C^T(2) \otimes O(3) \supset SU(2) \otimes SU_C^T(2)
\end{aligned} \quad (14)$$

$$\begin{aligned}
U_{C_1}^{ST}(4) \otimes U_{C_1}(3) \otimes U_{C_2}^{ST}(4) \otimes U_{C_2}(3) \otimes U_R(4) \supset U_C^{ST}(4) \otimes U_C(3) \otimes U_R(4) \supset \\
U_C^S(2) \otimes U_C^T(2) \otimes U_C(3) \otimes U_R(3) \supset U_C^S(2) \otimes U_C^T(2) \otimes O_C(3) \otimes O_R(3) \supset \\
U_C^S(2) \otimes U_C^T(2) \otimes O(3) \supset SU(2) \otimes SU_C^T(2)
\end{aligned} \quad (15)$$

Similarly to Equations (3)–(5), which were restricted to the space part only, these dynamical symmetries also describe to the shell-like, rigid molecule-like, and weak-coupled clusterization. Here, however, the spin and isospin degrees of freedom are also included, with a spin-orbit coupling for the angular momentum.

2.2. Pauli Principle

The internal wave functions of clusters are those of their completely antisymmetric shell model states. In particular, the antisymmetrization is performed by determining the $U^{ST}(4)$ irreducible representations [26] of different major shells. Then $U(\mathcal{N})$ ($\mathcal{N}$ is the number of orbitals in a single harmonic oscillator major shell) irreps in each major shell can be obtained through associate representation with $U^{ST}(4)$ (the Young patterns for each representation are connected by reflection: interchanging the rows and columns) [27]. The irreps of the permutation group and the $U(\mathcal{N})$ group are labeled by the same Young patterns. Those $U(3)$ irreps are considered that belong to the relevant $U(\mathcal{N})$ (and $U^{ST}(4)$) representations.

The product wave function of the two clusters and the relative motion still contains Pauli-forbidden components, because the antisymmetrization with respect to exchanging nucleons between the two clusters has not yet been taken into account.

The role of the Pauli principle and its treatment in the three dynamical symmetry limits are explained below.

- *$U(3)$ dynamical symmetry:*

In the $U(3)$ limit, the Pauli exclusion principle can be treated in a simple way. The detail procedure is discussed in Ref. [16]. The Wildermuth condition [28] gives a lower limit for the quantum number n_π of the relative motion of two clusters ($n_\pi \geq n_0$). In the shell model description n_π belongs to a major shell. Therefore, the application of this prescription makes a complete major shell either forbidden, or allowed, depending on whether the major quantum number is below or above the limit n_0 . If the clusters do not have internal structure, i.e., they are closed shell nuclei, this condition takes into account the Pauli principle. An example is $^{16}\text{O} + ^4\text{He}$, where $n_0 = 8$.

If, however, internal degrees of freedom couple to those of the relative motion, the situation is more complex. Within a major shell some $U(3)$ irreps can be forbidden while others are allowed. They can be distinguished by matching the cluster $U(3)$ basis with that of the shell model [16].

In this way, all the Pauli forbidden states can be excluded from the $U(3)$ basis of the cluster model.

An example is $^{12}\text{C} + ^4\text{He}$, as discussed in detail below. There, $n_0 = 4$, and for $n_\pi = 4, 5, 6, 7$ only some $U(3)$ irreps are allowed, others are forbidden. For $n_\pi \geq 8$, all $U(3)$ irreps are allowed.

- *$O(4)$ dynamical symmetry:*

In this case, we have no $U(3)$ quantum numbers (which keep the bookkeeping of the permutations) either in the relative motion or for the nuclear states, so we cannot exclude the forbidden states. The $O(4)$ basis states, which can be expressed as linear combination of $U(3)$ states [25], have contributions both from the forbidden and from the allowed $U(3)$ basis states.

- *$O(3)$ dynamical symmetry:*

In this case, n_π is a good quantum number, i.e., we have the $U(3)$ basis for the relative motion. Thus, the Wildermuth condition excludes some low-lying major shells which contain exclusively Pauli forbidden states. Above the n_0 limit, however, the selection between the forbidden and allowed states cannot be performed as before, because the states are not characterized by the $U(3)$ quantum numbers, but rather by the angular momenta of the internal cluster structure (L_C) and relative motion (L_R). Therefore, in this basis, some of the forbidden states are excluded, but not all of them.

The proper treatment of the exclusion principle can only be carried out by applying the $U(3)$ basis. The Hamiltonians can be chosen to have $O(3)$ or $O(4)$ dynamical symmetries, but in the truncated, i.e., Pauli-allowed basis, they are not diagonal anymore. Therefore, numerical diagonalization is needed.

2.3. Hamiltonian

The Hamiltonians associated with the dynamical-symmetry limits (Equations (3)–(5)) can be written in terms of the invariant operators of their respective group chains. If we consider up to two-body terms

$$H_{U(3)} = a\hat{C}_{U_C(3)}^1 + b\hat{C}_{U_C(3)}^2 + c\hat{C}_{U_R(3)}^1 + d\hat{C}_{U_R(3)}^2 + e\hat{C}_{U(3)}^1 + f\hat{C}_{SU(3)}^2 + g\hat{C}_{SO(3)}^2, \quad (16)$$

$$H_{O(4)} = a\hat{C}_{U_C(3)}^1 + b\hat{C}_{U_C(3)}^2 + h\hat{C}_{SO_R(4)}^2 + j\hat{C}_{SO_C(3)}^2 + k\hat{C}_{SO_R(3)}^2 + g\hat{C}_{SO(3)}^2, \quad (17)$$

$$H_{O(3)} = a\hat{C}_{U_C(3)}^1 + b\hat{C}_{U_C(3)}^2 + c\hat{C}_{U_R(3)}^1 + d\hat{C}_{U_R(3)}^2 + j\hat{C}_{SO_C(3)}^2 + k\hat{C}_{SO_R(3)}^2 + g\hat{C}_{SO(3)}^2. \quad (18)$$

In principle, in Equations (16)–(18) we need the invariant operators of $U_{C_1}(3) \otimes U_{C_2}(3) \otimes U_R(4)$, too. But their contribution is the same to the energy of each of the states within the spectrum (if we characterize the clusters with single internal $U(3)$ symmetries). Until we are interested in the excitation spectrum, i.e., the energies relative to the ground state, they can be neglected. They are important when, e.g., the binding energy of the ground state is considered. In Equation (17), the Casimir operator of $O_R(4)$ is not explicitly shown because the orthogonal and the special orthogonal groups have the same algebras [29].

It should be noted that the operator $\hat{C}_{U(3)}^2$ in Equation (16) is excluded, since it can be expressed in terms of the first-order Casimir operator of $U(3)$ and the second-order Casimir operator of $SU(3)$. Similarly, $\hat{C}_{SU_C(3)}^2$ in Equation (17), as well as $\hat{C}_{SU_C(3)}^2$ and $\hat{C}_{SU_R(3)}^2$ in Equation (18), are excluded because these operators can be written in terms of the first-order and second-order Casimir operators of $U_C(3)$ (or $U_R(3)$), which are discussed later. $\hat{C}_{U(3)}^1$, $\hat{C}_{U(3)}^2$, and $\hat{C}_{SU(3)}^2$ are uniquely related to each other (i.e., they are dependent).

From a practical point of view, keeping dependent operators leads to an overparameterized model Hamiltonian, and it will not be possible to find a unique best parameter set. In this case, many parameter sets can reproduce the same spectrum. We keep $\hat{C}_{U(3)}^1$ and $\hat{C}_{SU(3)}^2$ in the expressions because they have clear physical meaning. For example, $\hat{C}_{U(3)}^1$ corresponds to the number of oscillator quanta, and $\hat{C}_{SU(3)}^2$ refers to the quadrupole-quadrupole interaction.

The linear ($\hat{C}^1$) and quadratic ($\hat{C}^2$) Casimir operators can be expressed in terms of physical operators as follows (Equations (5.32), (5.38), (8.39), (5.43) and (5.44) of [25]):

$$\hat{C}_{U(3)}^1 = \hat{n} \quad (19)$$

$$\hat{C}_{U(3)}^2 = \frac{1}{3}\hat{n}^2 + \frac{1}{2}\hat{L}^2 + \hat{Q}^2 \quad (20)$$

$$\hat{C}_{SU(3)}^2 = \hat{Q}^2 + \frac{1}{2}\hat{L}^2 \quad (21)$$

$$\hat{C}_{SO(3)}^2 = \hat{L}^2 \quad (22)$$

$$\hat{C}_{SO(4)}^2 = \hat{L}^2 + \hat{D}^2 \quad (23)$$

where $\hat{n}$, $\hat{L}$, $\hat{D}$ and $\hat{Q}$ are the number operators, the angular, dipole and quadrupole momentum operators, respectively. The operators defined above can also be expressed in terms of other operators, and some of the expressions are given below:

$$\begin{aligned} \hat{C}_{U(3)}^2 &= \frac{1}{3}(\hat{C}_{U(3)}^1)^2 + \hat{C}_{SU(3)}^2, \\ \text{or, } \frac{1}{3}(\hat{C}_{U(3)}^1)^2 &+ \frac{1}{2}\hat{C}_{SO(3)}^2 + \hat{Q}^2, \\ \text{or, } \frac{1}{3}(\hat{C}_{U(3)}^1)^2 &+ \frac{1}{2}\hat{C}_{SO(4)}^2 + \hat{Q}^2 - \frac{1}{2}\hat{D}^2. \end{aligned} \quad (24)$$

$$\begin{aligned} \hat{C}_{SU(3)}^2 &= \hat{C}_{U(3)}^2 - \frac{1}{3}(\hat{C}_{U(3)}^1)^2, \\ \text{or, } \frac{1}{2}\hat{C}_{SO(3)}^2 &+ \hat{Q}^2, \\ \text{or, } \frac{1}{2}\hat{C}_{SO(4)}^2 &+ \hat{Q}^2 - \frac{1}{2}\hat{D}^2. \end{aligned} \quad (25)$$

$$\begin{aligned} \hat{C}_{SO(3)}^2 &= 2\hat{C}_{U(3)}^2 - \frac{2}{3}(\hat{C}_{U(3)}^1)^2 - 2\hat{Q}^2, \\ \text{or, } 2\hat{C}_{SU(3)}^2 &- 2\hat{Q}^2, \\ \text{or, } \hat{C}_{SO(4)}^2 &- \hat{D}^2. \end{aligned} \quad (26)$$

The expectation values of all the Casimir operators are given below (the expressions can be derived from Chapter 5 of [29], and some of them are also readily available in [16]).

$$\langle \hat{C}_{U(3)}^1 \rangle = n_1 + n_2 + n_3 \quad (27)$$

$$\langle \hat{C}_{U(3)}^2 \rangle = n_1^2 + n_2^2 + n_3^2 + 2(n_1 - n_3) \quad (28)$$

$$\langle \hat{C}_{SU(3)}^2 \rangle = \lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu) \quad (29)$$

$$\langle \hat{C}_{SO(3)}^2 \rangle = L(L + 1) \quad (30)$$

$$\langle \hat{C}_{SO(4)}^2 \rangle = \omega(\omega + 2) \quad (31)$$

The expression of $\langle \hat{C}_{SU(3)}^2 \rangle$ differs from that given in Ref. [29] by a factor of $\frac{6}{9}$. Here, we use the expression given in [16].

The eigenvalues of the three dynamically symmetric Hamiltonians (Equations (16)–(18)) are evaluated in their respective bases. By neglecting terms that contribute uniformly to all states, they read as

$$E_{U(3)} = c n_\pi + d n_\pi(n_\pi + 2) + e (n_1 + n_2 + n_3) + f (\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)) + g L(L + 1), \quad (32)$$

$$E_{O(4)} = h \omega(\omega + 2) + j L_C(L_C + 1) + k L_R(L_R + 1) + g L(L + 1), \quad (33)$$

$$E_{O(3)} = c n_\pi + d n_\pi(n_\pi + 2) + j L_C(L_C + 1) + k L_R(L_R + 1) + g L(L + 1). \quad (34)$$

2.4. Phase Diagram

Due to the presence of three dynamical symmetries, the phase diagram of the binary cluster model is two-dimensional and can be represented as a triangle [14].

One can express a general Hamiltonian: $H = xH_{U(3)} + (1 - x - y)H_{O(4)} + yH_{O(3)}$ for the study of phase transitions. When both x and y are zero, it corresponds to the $O(4)$ limit. If $x = 1$ and $y = 0$, it gives the $U(3)$ limit, while for $x = 0$ and $y = 1$, it corresponds to the $O(3)$ limit.

The phase diagram of the shell model also has a triangular structure [30]. The phase diagrams of the shell model and the cluster model are connected in the LS-coupling limit, where Elliott's $SU(3)$ symmetry naturally emerges. This link is shown in Figure 1. The $SU(2)$ limit of the shell model corresponds to the jj -coupling scheme.

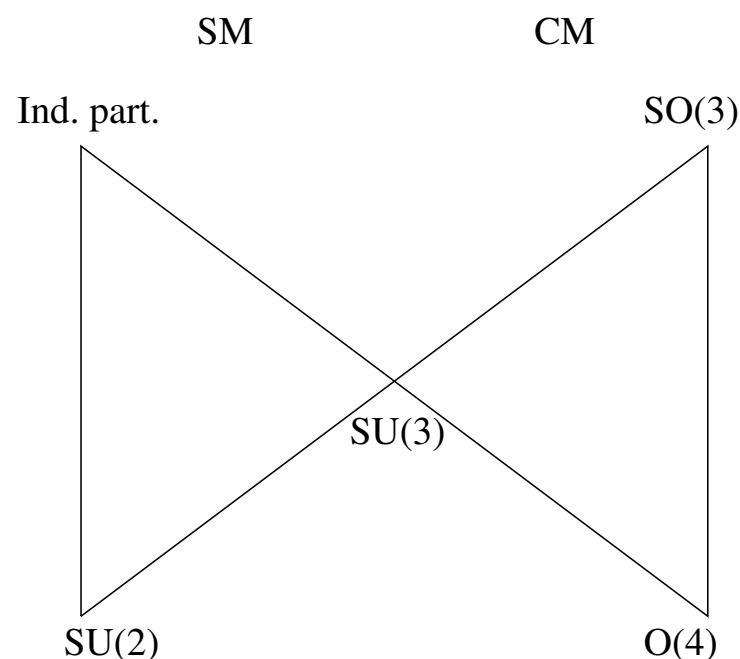


Figure 1. Connection between the phase diagrams of the shell model and the cluster model.

3. Schematic Spectra

We consider a core-plus- α cluster system in order to find the characteristic features of the phases of clusterization. As an example we take the $^{12}\text{C} + \alpha$ case, with the ^{12}C core in its intrinsic ground state, i.e., with the $[4,4,0]$ $U(3)$ symmetry. This means that not only the physical ground state is involved in the cluster space, but also its rotations, i.e., the ground-state band is taken into account. First, we construct the model space defined by the sets of basis of the three dynamical symmetries; then, we investigate the energy spectra.

3.1. Model Space

Here, we present the model space for each dynamical symmetry. We use different font styles to distinguish Pauli-forbidden and Pauli-allowed states. Pauli-forbidden states are

shown in italics. States that are partially contaminated by Pauli-forbidden components are shown in bold, while completely Pauli-allowed states are indicated in roman type.

We consider the $U_{C_1}(3)$ representation for ^{12}C as $[n_1^{C_1}, n_2^{C_1}, n_3^{C_1}] = [4, 4, 0]$ and the α particle is a $U_{C_2}(3)$ scalar: $[n_1^{C_2}, n_2^{C_2}, n_3^{C_2}] = [0, 0, 0]$. So the coupled $U_C(3)$ for the clusters ($^{12}\text{C} + \alpha$) is $[n_1^C, n_2^C, n_3^C] = [4, 4, 0]$ ($(\lambda_C, \mu_C) = (0, 4)$).

Within the shell model, ^{12}C contains eight oscillator quanta, whereas the α particle has zero. For ^{16}O , the total oscillator number is 12 quanta. To satisfy the Pauli exclusion principle, an additional 4 ($n_\pi = 4$) oscillator quanta must be added, as required by the Wildermuth condition. By applying the Wildermuth condition and matching the $U(3)$ basis with that of the shell model, the model space corresponding to the $U(3)$ limit is fully Pauli-allowed. The full model space of $U(3)$ limit is listed in Table 1 [15,16]. The $U(3)$ basis allows the proper treatment of both the Pauli principle and the spurious center-of-mass excitations.

Table 1. Complete model space of $^{12}\text{C} + \alpha$ system for $N = 5$ bosons (up to $n_\pi = 5$) in the $U(3)$ dynamical symmetry limit. The ground state $U_C(3)$ representation for clusters $[n_1^C, n_2^C, n_3^C] = [4, 4, 0]$ is considered. Representations shown in italics are spurious (either Pauli-forbidden, or excitation of the center of mass), whereas those shown in roman type are allowed states. Quantum numbers in bold indicate partially forbidden situation.

U(3) Limit			
n_π	$[n_1, n_2, n_3]$	$(\lambda, \mu)K^\pi$	L^π
0	<i>[4, 4, 0]</i>	<i>(0, 4)0⁺</i>	<i>0⁺, 2⁺, 4⁺</i>
1	<i>[5, 4, 0]</i>	<i>(1, 4)1⁻</i>	<i>1⁻, 2⁻, 3⁻, 4⁻, 5⁻</i>
	<i>[4, 4, 1]</i>	<i>(0, 3)0⁻</i>	<i>1⁻, 3⁻</i>
2	<i>[6, 4, 0]</i>	<i>(2, 4)0⁺</i> <i>(2, 4)2⁺</i>	<i>0⁺, 2⁺, 4⁺</i> <i>2⁺, 3⁺, 4⁺, 5⁺, 6⁺</i>
	<i>[5, 4, 1]</i>	<i>(1, 3)1⁺</i>	<i>1⁺, 2⁺, 3⁺, 4⁺</i>
	<i>[4, 4, 2]</i>	<i>(0, 2)0⁺</i>	<i>0⁺, 2⁺</i>
	<i>[7, 4, 0]</i>	<i>(3, 4)1⁻</i> <i>(3, 4)3⁻</i>	<i>1⁻, 2⁻, 3⁻, 4⁻, 5⁻</i> <i>3⁻, 4⁻, 5⁻, 6⁻, 7⁻</i>
3	<i>[6, 4, 1]</i>	<i>(2, 3)0⁻</i> <i>(2, 3)2⁻</i>	<i>1⁻, 3⁻</i> <i>2⁻, 3⁻, 4⁻, 5⁻</i>
	<i>[5, 4, 2]</i>	<i>(1, 2)1⁻</i>	<i>1⁻, 2⁻, 3⁻</i>
	<i>[4, 4, 3]</i>	<i>(0, 1)0⁻</i>	<i>1⁻</i>
	<i>[8, 4, 0]</i>	<i>(4, 4)0⁺</i> <i>(4, 4)2⁺</i> <i>(4, 4)4⁺</i>	<i>0⁺, 2⁺, 4⁺</i> <i>2⁺, 3⁺, 4⁺, 5⁺, 6⁺</i> <i>4⁺, 5⁺, 6⁺, 7⁺, 8⁺</i>
4	<i>[7, 4, 1]</i>	<i>(3, 3)1⁺</i> <i>(3, 3)3⁺</i>	<i>1⁺, 2⁺, 3⁺, 4⁺</i> <i>3⁺, 4⁺, 5⁺, 6⁺</i>
	<i>[6, 4, 2]</i>	<i>(2, 2)0⁺</i> <i>(2, 2)2⁺</i>	<i>0⁺, 2⁺</i> <i>2⁺, 3⁺, 4⁺</i>

Table 1. Cont.

U(3) Limit			
n_π	$[n_1, n_2, n_3]$	$(\lambda, \mu)K^\pi$	L^π
5	$[5, 4, 3]$	$(1, 1)1^+$	$1^+, 2^+$
	$[4, 4, 4]$	$(0, 0)0^+$	0^+
	$[9, 4, 0]$	$(5, 4)0^-$	$1^-, 3^-, 5^-$
		$(5, 4)2^-$	$2^-, 3^-, 4^-, 5^-, 6^-, 7^-$
		$(5, 4)4^-$	$4^-, 5^-, 6^-, 7^-, 8^-, 9^-$
	$[8, 4, 1]$	$(4, 3)1^-$	$1^-, 2^-, 3^-, 4^-, 5^-$
		$(4, 3)3^-$	$3^-, 4^-, 5^-, 6^-, 7^-$
	$[7, 4, 2]$	$(3, 2)0^-$	$1^-, 3^-$
		$(3, 2)2^-$	$2^-, 3^-, 4^-, 5^-$
	$[6, 4, 3]$	$(2, 1)1^-$	$1^-, 2^-, 3^-$
	$[5, 4, 4]$	$(1, 0)0^-$	1^-

As discussed earlier, the O(4) limit does not account for the Pauli principle. Once the number of bosons N is chosen, its model space can be obtained straightforwardly. Table 2 shows the model space for the O(4) limit with $N = 5$ bosons, which also includes Pauli-forbidden configurations. The total angular momentum L is obtained by coupling L_C and L_R while the total parity is determined by the product of their respective parities, $\pi = \pi_C \pi_R$.

Table 2. Model spaces of $^{12}\text{C} + \alpha$ system for $N = 5$ bosons for O(4) dynamical symmetry limit. The ground state $\text{SU}_C(3)$ representation for clusters $(\lambda_C, \mu_C) = (0, 4)$ and $K_C = 0$ is considered. The parity of L is determined by the product of respective parities of L_C and L_R , $\pi = \pi_C \pi_R$. ν is given by $(N - \omega)/2$. As we can not exclude the forbidden states, these basis states have contributions both from the forbidden and from the allowed U(3) basis states.

O(4) Limit				
ω	ν	L_C^π	L_R^π	L^π
1	2	0^+	0^+	0^+
		0^+	1^-	1^-
		2^+	0^+	2^+
		2^+	1^-	$1^-, 2^-, 3^-$
		4^+	0^+	4^+
		4^+	1^-	$3^-, 4^-, 5^-$
		0^+	0^+	0^+
		0^+	1^-	1^-
3	1	0^+	2^+	2^+
		0^+	3^-	3^-
		0^+	0^+	0^+
		0^+	1^-	1^-

Table 2. Cont.

O(4) Limit				
ω	ν	L_C^π	L_R^π	L^π
5	0	2^+	0^+	2^+
		2^+	1^-	$1^-, 2^-, 3^-$
		2^+	2^+	$0^+, 1^+, 2^+, 3^+, 4^+$
		2^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-$
		4^+	0^+	4^+
		4^+	1^-	$3^-, 4^-, 5^-$
		4^+	2^+	$2^+, 3^+, 4^+, 5^+, 6^+$
		4^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$
		0^+	0^+	0^+
		0^+	1^-	1^-
		0^+	2^+	2^+
		0^+	3^-	3^-
		0^+	4^+	4^+
		0^+	5^-	5^-
		2^+	0^+	2^+
		2^+	1^-	$1^-, 2^-, 3^-$
		2^+	2^+	$0^+, 1^+, 2^+, 3^+, 4^+$
		2^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-$
		2^+	4^+	$2^+, 3^+, 4^+, 5^+, 6^+$
		2^+	5^-	$3^-, 4^-, 5^-, 6^-, 7^-$
		4^+	0^+	4^+
		4^+	1^-	$3^-, 4^-, 5^-$
		4^+	2^+	$2^+, 3^+, 4^+, 5^+, 6^+$
		4^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$
		4^+	4^+	$0^+, 1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+$
		4^+	5^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-, 8^-, 9^-$

For the case of the O(3) limit, the Pauli forbidden states below $n_\pi = 4$ can be excluded. However, for $n_\pi \geq 4$, the states are contaminated by Pauli-forbidden states. In this case, too, L arises from the coupling of L_C and L_R , and the associated parity is determined as $\pi = \pi_C \pi_R$. The corresponding model space is listed in Table 3.

Table 3. Model spaces of $^{12}\text{C} + \alpha$ system for $N = 5$ bosons (up to $n_\pi = 5$), for O(3) dynamical symmetry limit. The ground state $\text{SU}_C(3)$ representation for clusters $(\lambda_C, \mu_C) = (0, 4)$ and $K_C = 0$ is considered. The parity is determined as $\pi = \pi_C \pi_R$. For $n_\pi < 4$, all states are Pauli-forbidden (shown in italics). For $n_\pi \geq 4$, the states are mixed, containing both Pauli-forbidden and Pauli-allowed components (shown in bold).

O(3) Limit			
n_π	L_C^π	L_R^π	L^π
<i>0</i>	<i>0^+</i>	<i>0^+</i>	<i>0^+</i>
	<i>2^+</i>	<i>0^+</i>	<i>2^+</i>
	<i>4^+</i>	<i>0^+</i>	<i>4^+</i>

Table 3. Cont.

n_π	O(3) Limit		
	L_C^π	L_R^π	L^π
1	0^+	1^-	1^-
	2^+	1^-	$1^-, 2^-, 3^-$
	4^+	1^-	$3^-, 4^-, 5^-$
2	0^+	0^+	0^+
	0^+	2^+	2^+
	2^+	0^+	2^+
	2^+	2^+	$0^+, 1^+, 2^+, 3^+, 4^+$
	4^+	0^+	4^+
	4^+	2^+	$2^+, 3^+, 4^+, 5^+, 6^+$
3	0^+	1^-	1^-
	0^+	3^-	3^-
	2^+	1^-	$1^-, 2^-, 3^-$
	2^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-$
	4^+	1^-	$3^-, 4^-, 5^-$
	4^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$
4	0^+	0^+	0^+
	0^+	2^+	2^+
	0^+	4^+	4^+
	2^+	0^+	2^+
	2^+	2^+	$0^+, 1^+, 2^+, 3^+, 4^+$
	2^+	4^+	$2^+, 3^+, 4^+, 5^+, 6^+$
	4^+	0^+	4^+
	4^+	2^+	$2^+, 3^+, 4^+, 5^+, 6^+$
	4^+	4^+	$0^+, 1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+$
5	0^+	1^-	1^-
	0^+	3^-	3^-
	0^+	5^-	5^-
	2^+	1^-	$1^-, 2^-, 3^-$
	2^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-$
	2^+	5^-	$3^-, 4^-, 5^-, 6^-, 7^-$
	4^+	1^-	$3^-, 4^-, 5^-$
	4^+	3^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-$
	4^+	5^-	$1^-, 2^-, 3^-, 4^-, 5^-, 6^-, 7^-, 8^-, 9^-$

3.2. Spectra

Figure 2 shows the spectra corresponding to the three dynamical symmetries. We use three different line styles: dotted, dashed, and solid lines indicate completely Pauli-

forbidden (or spurious excitation of the centre of mass), partially forbidden, and completely Pauli-allowed states, respectively.

In the basis associated with the $U(3)$ dynamical symmetry (Equation (3)), the bands are labeled $n_\pi(\lambda, \mu)K^\pi$, whereas bands associated with $O(4)$ dynamical symmetry (Equation (4)) are defined by $\nu(L_C, L_R)$, where ν is $(N - \omega)/2$. Bands of $O(3)$ dynamical symmetry limit (Equation (5)) are characterized by $n_\pi(L_C, L_R)$ quantum number.

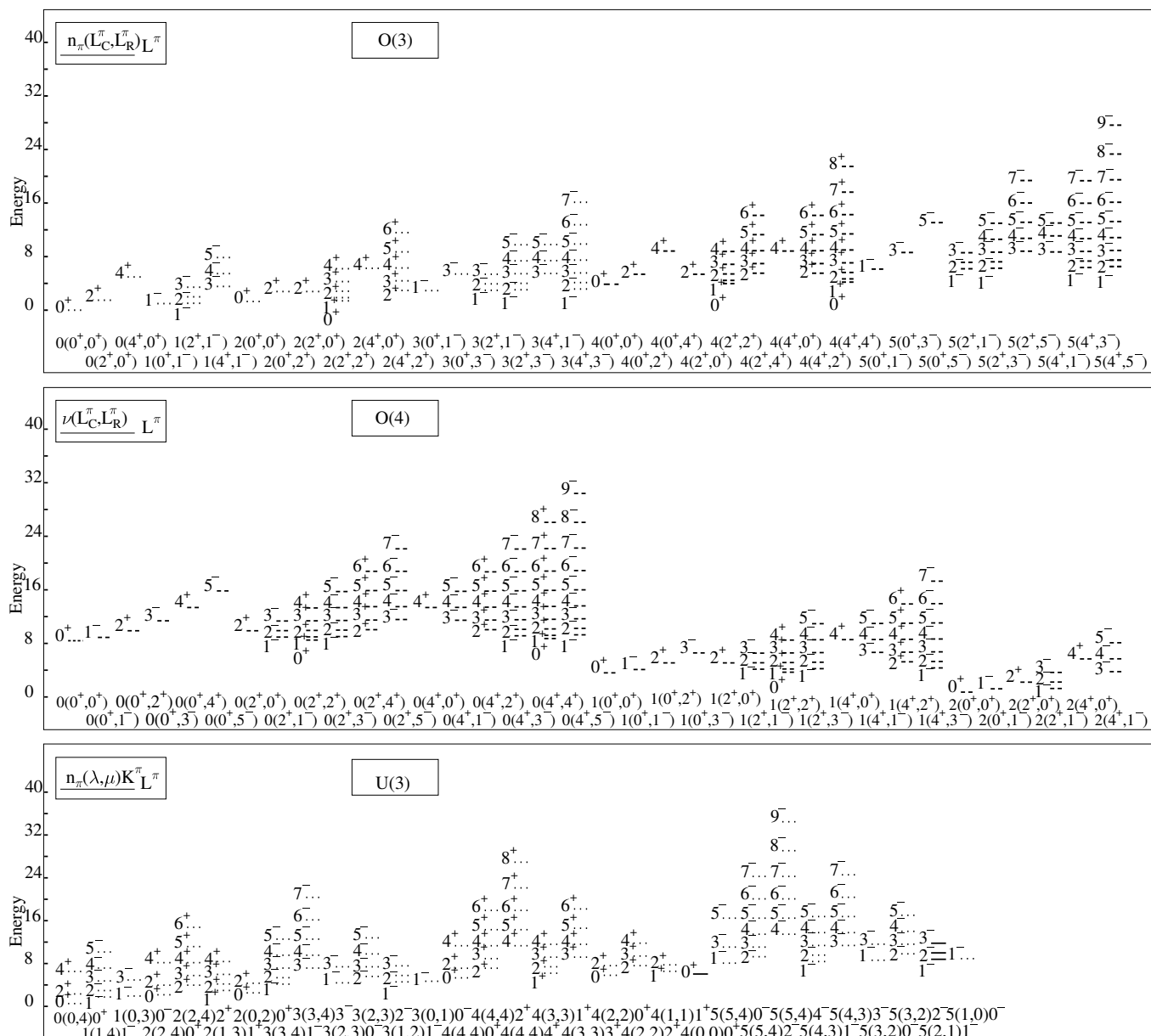


Figure 2. Schematic spectra of the $U(3)$ (bottom), $O(4)$ (middle), and $O(3)$ (top) symmetry limits (see Equations (3)–(5)), obtained using the eigenvalues $E_{U(3)}$, $E_{O(4)}$, and $E_{O(3)}$ (see Equations (32)–(34)), respectively. The parameters are set to $d = 0.2$, $f = -0.01$, $g = 0.3$, $h = 0.3$, $e = 0.1$ and $c = j = k = 0.01$. The eigenstates of the $U(3)$, $O(4)$, and $O(3)$ dynamical symmetry limits are labeled by the quantum numbers $n_\pi(\lambda, \mu)K^\pi$, $\nu(L_C, L_R)$, and $n_\pi(L_C, L_R)$, respectively, where $\nu = (N - \omega)/2$ and N is the number of bosons. The spectra are shown for $N = 5$ bosons. Energies are given in arbitrary units. Dotted lines correspond to completely forbidden states, dashed lines correspond to states contaminated with Pauli-forbidden components, and solid lines correspond to completely Pauli-allowed states.

Figure 2 illustrates the crucial role of the Pauli principle. The three spectra are qualitatively different. A nucleus is a multi-fermion system; thus, it obeys the Pauli exclusion

principle. The U(3) dynamical symmetry fulfills this requirement; therefore, we use this limit for the calculations of the alpha-cluster bands of the ^{16}O nucleus.

4. Realistic $^{12}\text{C} + \alpha$ Spectrum

4.1. Experimental Data of ^{16}O

The experimentally established rotational bands are presented in [31,32]. One of them is based on the 0^+ state at 6.05 MeV. Further states are 2^+ at 6.92, 4^+ at 10.35, 6^+ at 16.28 MeV. An 8^+ state was suggested to be a member of this band with higher excitation energy. From the extrapolation of the rotational band, the 8^+ was predicted to be at $E_x = 20$ MeV [32]. In [31] it was found at $E_x = 29$ MeV.

A negative parity band was presented in both [31] and [32]. It involves 1^- at 9.59, 3^- at 11.6, 5^- at 14.67, 7^- at 20.85 MeV. A 9^- state was added in this band at 30 MeV by Katsuma [31]. These are shown as 0_1^+ and 0^- bands in Figure 3.

Several rotational bands have been arranged in the study of alpha cluster model [33]. The experimentally established bands are also present in this arrangement except last members (8^+ , 9^-). The (2_1^+) , (2_2^+) , (0_2^+) , (0_4^+) , and the (2_1^-) bands (in Figure 3) correspond to these band arrangements. We consider the excitation energies of the states of these bands by comparing Figure 7 of [33] and the experimental data compilation [34]. There was a (0^+) state as the lowest member of (0_4^+) . However, no corresponding state is listed in the compilation.

We organize three rotational bands using the experimental compilations [34]. When at least three experimental states with well-established spin-parity lie approximately on a straight line in the E vs. $J(J+1)$ plot, it is assumed to be a band. The (1_1^+) , (0_3^+) , and (1_1^-) bands are constructed by us. The experimental energies of states as a function of $J(J+1)$ in the ^{16}O nucleus are shown in Figure 3. The experimental spectrum is presented in Figure 4.

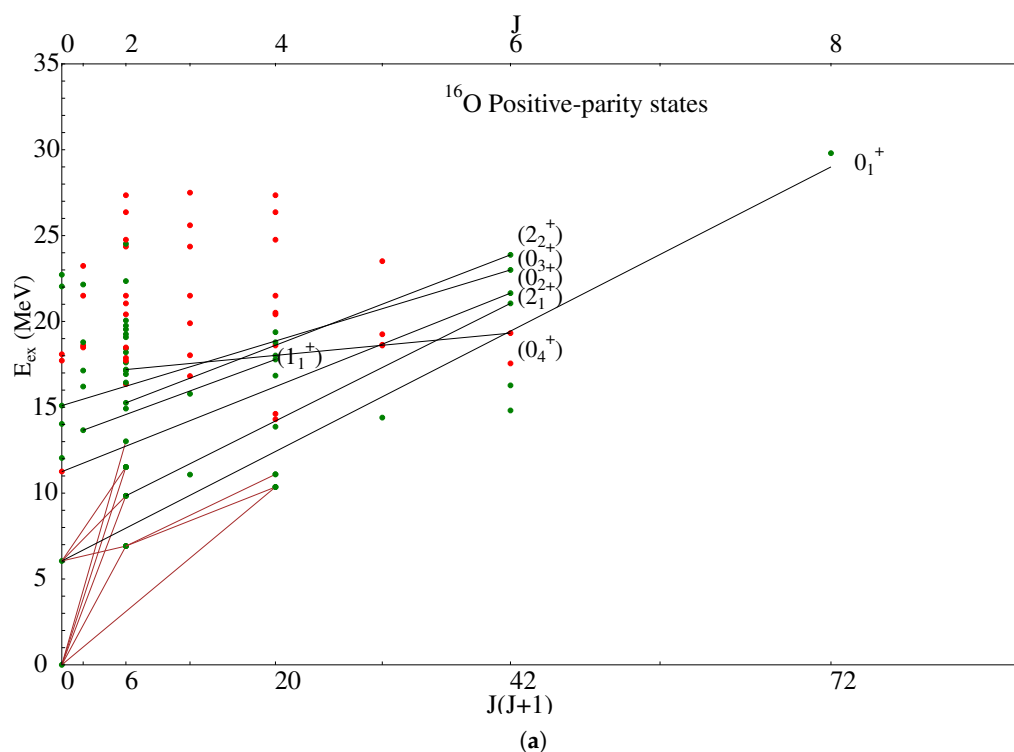


Figure 3. Cont.

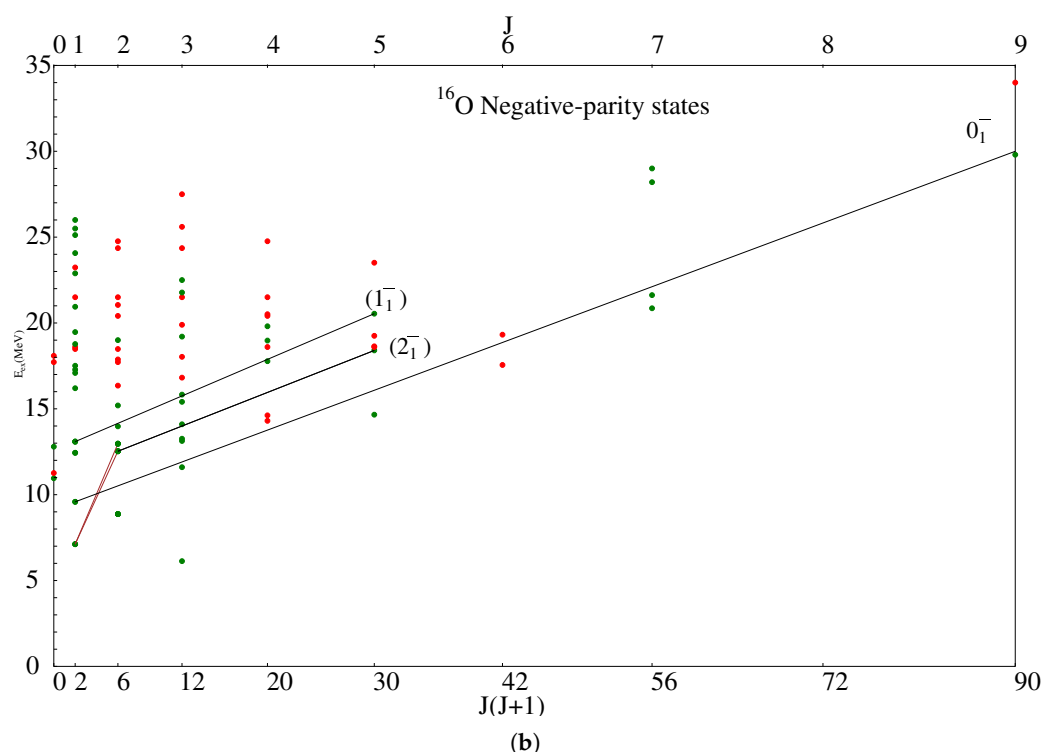


Figure 3. Experimental spectra of positive- and negative-parity states of the ^{16}O nucleus as a function of $J(J+1)$, based on data from Ref. [34]. (a) Positive-parity states. The states identified with small green circles correspond to well-known spin-parities, while the small red circles refer to states with uncertain spin-parities. The brown lines connecting the circles represent electromagnetic transitions. The black lines indicate the rotational bands, and we labeled them with K^π quantum numbers. The 0_1^+ band was presented in [31,32], while the (2_1^+) , (0_2^+) , (2_2^+) , and (0_4^+) bands were arranged by Bauhoff et al. [33]. (1_1^+) and (0_3^+) bands are organized by us. (b) Negative-parity states. The notation follows (a). The 0_1^- band was presented in [31,32], while the (2_1^-) band was arranged by Bauhoff et al. [33]. (1_1^-) band is organized by us. (a) Positive-parity states of ^{16}O . (b) Negative-parity states of ^{16}O .

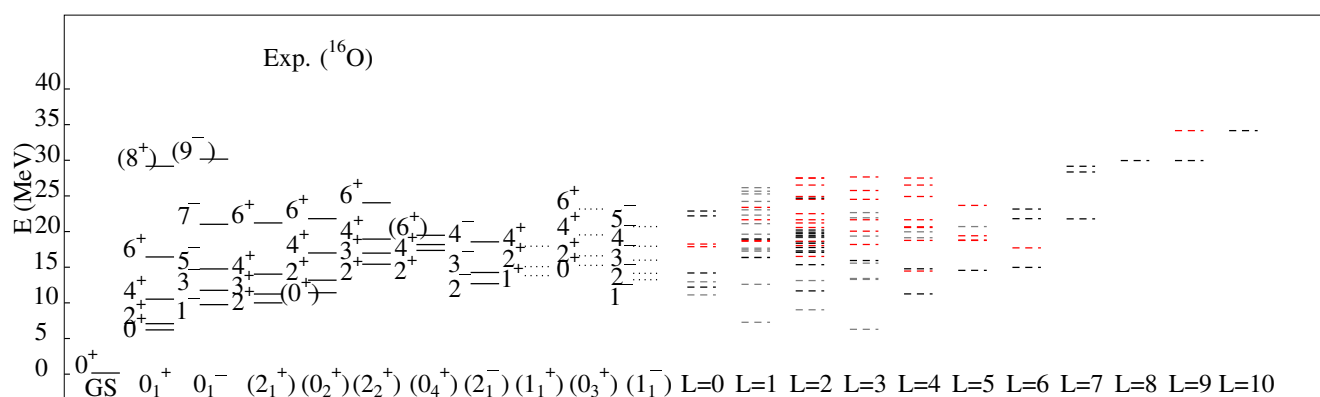


Figure 4. Complete experimental spectrum of ^{16}O nucleus. We labeled the rotational bands with K^π quantum numbers. The 0_1^- , 0_1^+ bands were presented in [31,32], while the (2_1^+) , (0_2^+) , (2_2^+) , (0_4^+) , and (2_1^-) bands (solid lines) were arranged by Bauhoff et al. [33]. The (1_1^+) , (0_3^+) , and (1_1^-) bands are organized by us (dotted lines). The dashed lines denote the states ($L = 0$ to $L = 10$) that do not belong to any rotational band. Black dashed lines correspond to positive parity, grey dashed lines to negative parity, and red dashed lines indicate states with uncertain spin-parity.

4.2. Spectrum Calculation

A completely Pauli allowed model space of U(3) dynamical symmetry limit, which is also free from the spurious excitation of the centre of mass, is presented in Table 4.

Table 4. Pauli allowed model space of $^{12}\text{C} + \alpha$ system (up to $n_\pi = 10$) in the U(3) dynamical symmetry limit. The ground state $U_C(3)$ representation for clusters $[n_1^C, n_2^C, n_3^C] = [4, 4, 0]$ is considered.

n_π	$[n_1, n_2, n_3]$	$(\lambda, \mu) \mathbf{K}^\pi$	$\mathbf{L}^\pi$
4	[4,4,4]	(0,0)0 ⁺	0 ⁺
5	[6,4,3]	(2,1)1 ⁻	1 ⁻ , 2 ⁻ , 3 ⁻
6	[8,4,2]	(4,2)0 ⁺	0 ⁺ , 2 ⁺ , 4 ⁺
		(4,2)2 ⁺	2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺
	[7,4,3]	(3,1)1 ⁺	1 ⁺ , 2 ⁺ , 3 ⁺ , 4 ⁺
	[6,4,4]	(2,0)0 ⁺	0 ⁺ , 2 ⁺
7	[10,4,1]	(6,3)1 ⁻	1 ⁻ , 2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻
		(6,3)3 ⁻	3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻ , 8 ⁻ , 9 ⁻
	[9,4,2]	(5,2)0 ⁻	1 ⁻ , 3 ⁻ , 5 ⁻
		(5,2)2 ⁻	2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻
	[8,4,3]	(4,1)1 ⁻	1 ⁻ , 2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻
	[7,4,4]	(3,0)0 ⁻	1 ⁻ , 3 ⁻
8	[12,4,0]	(8,4)0 ⁺	0 ⁺ , 2 ⁺ , 4 ⁺ , 6 ⁺ , 8 ⁺
		(8,4)2 ⁺	2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺ , 9 ⁺ , 10 ⁺
		(8,4)4 ⁺	4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺ , 9 ⁺ , 10 ⁺ , 11 ⁺ , 12 ⁺
	[11,4,1]	(7,3)1 ⁺	1 ⁺ , 2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺
		(7,3)3 ⁺	3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺ , 9 ⁺ , 10 ⁺
	[10,4,2]	(6,2)0 ⁺	0 ⁺ , 2 ⁺ , 4 ⁺ , 6 ⁺
	(6,2)2 ⁺	2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺ , 7 ⁺ , 8 ⁺	
	[9,4,3]	(5,1)1 ⁺	1 ⁺ , 2 ⁺ , 3 ⁺ , 4 ⁺ , 5 ⁺ , 6 ⁺
	[8,4,4]	(4,0)0 ⁺	0 ⁺ , 2 ⁺ , 4 ⁺
9	[13,4,0]	(9,4)0 ⁻	1 ⁻ , 3 ⁻ , 5 ⁻ , 7 ⁻ , 9 ⁻
		(9,4)2 ⁻	2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻ , 8 ⁻ , 9 ⁻ , 10 ⁻ , 11 ⁻
		(9,4)4 ⁻	4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻ , 8 ⁻ , 9 ⁻ , 10 ⁻ , 11 ⁻ , 12 ⁻ , 13 ⁻
	[12,4,1]	(8,3)1 ⁻	1 ⁻ , 2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻ , 8 ⁻ , 9 ⁻
		(8,3)3 ⁻	3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻ , 8 ⁻ , 9 ⁻ , 10 ⁻ , 11 ⁻
	[11,4,2]	(7,2)0 ⁻	1 ⁻ , 3 ⁻ , 5 ⁻ , 7 ⁻
	(7,2)2 ⁻	2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻ , 8 ⁻ , 9 ⁻	
	[10,4,3]	(6,1)1 ⁻	1 ⁻ , 2 ⁻ , 3 ⁻ , 4 ⁻ , 5 ⁻ , 6 ⁻ , 7 ⁻
	[9,4,4]	(5,0)0 ⁻	1 ⁻ , 3 ⁻ , 5 ⁻

Table 4. Cont.

n_π	$[n_1, n_2, n_3]$	$(\lambda, \mu)K^\pi$	L^π
10	[14,4,0]	$(10,4)0^+$	$0^+, 2^+, 4^+, 6^+, 8^+, 10^+$
		$(10,4)2^+$	$2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+, 11^+, 12^+$
		$(10,4)4^+$	$4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+, 11^+, 12^+, 13^+, 14^+$
	[13,4,1]	$(9,3)1^+$	$1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+$
		$(9,3)3^+$	$3^+, 4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+, 11^+, 12^+$
	[12,4,2]	$(8,2)0^+$	$0^+, 2^+, 4^+, 6^+, 8^+$
		$(8,2)2^+$	$2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+, 9^+, 10^+$
	[11,4,3]	$(7,1)1^+$	$1^+, 2^+, 3^+, 4^+, 5^+, 6^+, 7^+, 8^+$
	[10,4,4]	$(6,0)0^+$	$0^+, 2^+, 4^+, 6^+$

For the calculation of realistic spectrum, we used a simpler Hamiltonian

$$\hat{H} = \hbar\omega\hat{n} + a\hat{C}_{SU(3)}^{(2)} + bK^2 + \frac{1}{2\theta}L(L+1), \quad (35)$$

than the earlier Equation (32). This Hamiltonian is expressed in terms of the invariant operators of the $U(3) \supset SU(3) \supset SO(3)$ group chain and it is known as a Hamiltonian of multi-configurational dynamical symmetry (MUSY) [35]. Several features are common to both Hamiltonians: (i) Both correspond to $U(3)$ dynamical symmetry limit. (ii) The harmonic oscillator (linear invariant of $U(3)$), the quadrupole-quadrupole interaction ($\hat{C}_{SU(3)}^{(2)}$) and the angular momentum terms are common to both Hamiltonians. Nevertheless, there are also differences: (i) The earlier Hamiltonian [Equation (32)] has five parameters, whereas the MUSY Hamiltonian has three. (ii) The advantage of using Hamiltonian (Equation (35)) is that it remains invariant under transformations between different configurations [35–37]. (iii) In addition, it has greater physical content; $\hat{K}^2$ lifts the degeneracy of states with the same angular momentum within an $SU(3)$ representation and θ represents the moment of inertia (in units of $\frac{\hbar^2}{MeV}$) calculated for ellipsoid with cylindrical symmetry defined by the $U(3)$ quantum numbers [38]. $\hbar\omega$, a , b are obtained by fitting the experimental data.

In qualitative description of energy states within the shell model [39], the lowest positive-parity excited state (0^+ at 6.05 MeV) came from the leading $4p-4h$ $SU(3)$ irrep. And this state is the lowest state of a rotational band (0_1^+ band in Figure 4) of positive parity. It belongs to the leading representation of $4\hbar\omega$ (model band $n_\pi(\lambda, \mu)K^\pi = 8(8,4)0^+$) here.

The function that is minimized to determine the Hamiltonian parameters is

$$F = \sum_i w(i) \frac{(E_i^{exp} - E_i^{Th.})^2}{(E_i^{exp.})^2}. \quad (36)$$

In the fitting process, states belonging to the two experimental established bands [31,32] with well-defined spin-parity were assigned a unit weight, whereas those with uncertain spin-parity were given a weight of 0.5. The band assignments in Bauhoff et al. [33] were mainly determined by the $J(J+1)$ energy spacing, which aligns with the approach we used in our own arrangements. For this reason, we assigned an equal weight of 0.2 to the states with a certain spin-parity for both our bands and theirs. And those with uncertain spin-parity were given a weight of 0.1.

The calculation was performed in three stages:

- (i) the two experimentally established bands (est.) [31,32] were first considered;

- (ii) subsequently, the bands arranged by Bauhoff et al. (est. + Bauhoff) [33] were included; and
- (iii) finally, the bands organized in the present work were added (est. + Bauhoff + our).

As discussed earlier, the lowest excited positive-parity state, which is part of the 0_1^+ band, originates qualitatively from the leading $4p\text{--}4h$ $SU(3)$ irrep [39]. According to the calculations of Katsuma [31], this band corresponds to a total harmonic oscillator quantum number ($N = 8$) for the relative motion between the α -particle and ^{12}C , while for the 0_1^- band it is ($N = 9$). Here, we assigned the 0_1^+ and 0_1^- bands to the leading $SU(3)$ representations of $4\hbar\omega$ and $5\hbar\omega$, respectively (or, $n_\pi = 8$ and $n_\pi = 9$).

For the arranged bands of Bauhoff et al. [33], we assigned the lowest lying, most deformed configurations that have proper K^π and spin-parity content. Finally, we assigned our arranged bands in the same way as for the bands of Bauhoff et al.

The 0_4^+ band is assigned to the $10(8,2)0^+$ quantum number. Although lower-excitation representations could also be assigned to this band, it contains only even-spin states. Therefore, it is reasonable to assign $10(8,2)0^+$ from the list of model space.

In addition, the 2_1^- and 1_1^- bands are assigned to the $7(5,2)2^-$ and $7(6,3)1^-$ configurations, respectively. Specifically, the less deformed $7(5,2)2^-$ representation is associated with the lower 2_1^- band since the smallest angular momentum for this representation is $L = 2$, consistent with the experimental bandhead. The more deformed $7(6,3)1^-$ representation is then assigned to the higher 1_1^- band.

The fitted parameters of the Hamiltonian (Equation (35)) are listed in Table 5.

Table 5. The parameters of the Hamiltonian (Equation (35)). $\hbar\omega$, a , b are in MeV. F is a minimization function used to determine the parameters of the Hamiltonian, and $N_{exp.}$ is the number of experimental states considered in the fitting. The fitting is performed by three successive steps: (i) First, experimentally well-established bands [31,32], were considered (est.), (ii) then the states of arranged bands of Bauhoff et al. [33] were included (est. + Bauhoff), (iii) finally, the bands we arranged were included in the fitting process (est. + Bauhoff + our).

	$\hbar\omega$	a	b	$\frac{F}{N_{exp.}}$
est.	5.27635	−0.10213	0	0.00378
est. + Bauhoff	3.75356	−0.05886	1.04696	0.00409
est. + Bauhoff + our	4.08735	−0.06776	1.18675	0.00783

Concerning the comparison of the model calculation with the experimental data the following can be said. The band structure of ^{16}O , including its ground state, can be described with good approximation by the $U(3)$ dynamically symmetric $^{12}\text{C} + \alpha$ model, as it is illustrated by Figure 5. The band structure was established by the previous studies and extended with further candidate bands here. As for the states of ^{16}O , without established band assignment, many of them may very well correspond to $^{12}\text{C} + \alpha$ cluster structure, as it is illustrated by Figure 6. But in order to make definite statement in this respect, further careful investigations are needed especially from the experimental side.

The agreement between the experimental observation and the model calculation is obviously a consequence of the incorporation of the Pauli principle. The spectra of the other cluster phases, with states contaminated by Pauli forbidden components, shown in Figure 2, have drastically different nature than the experimental one.

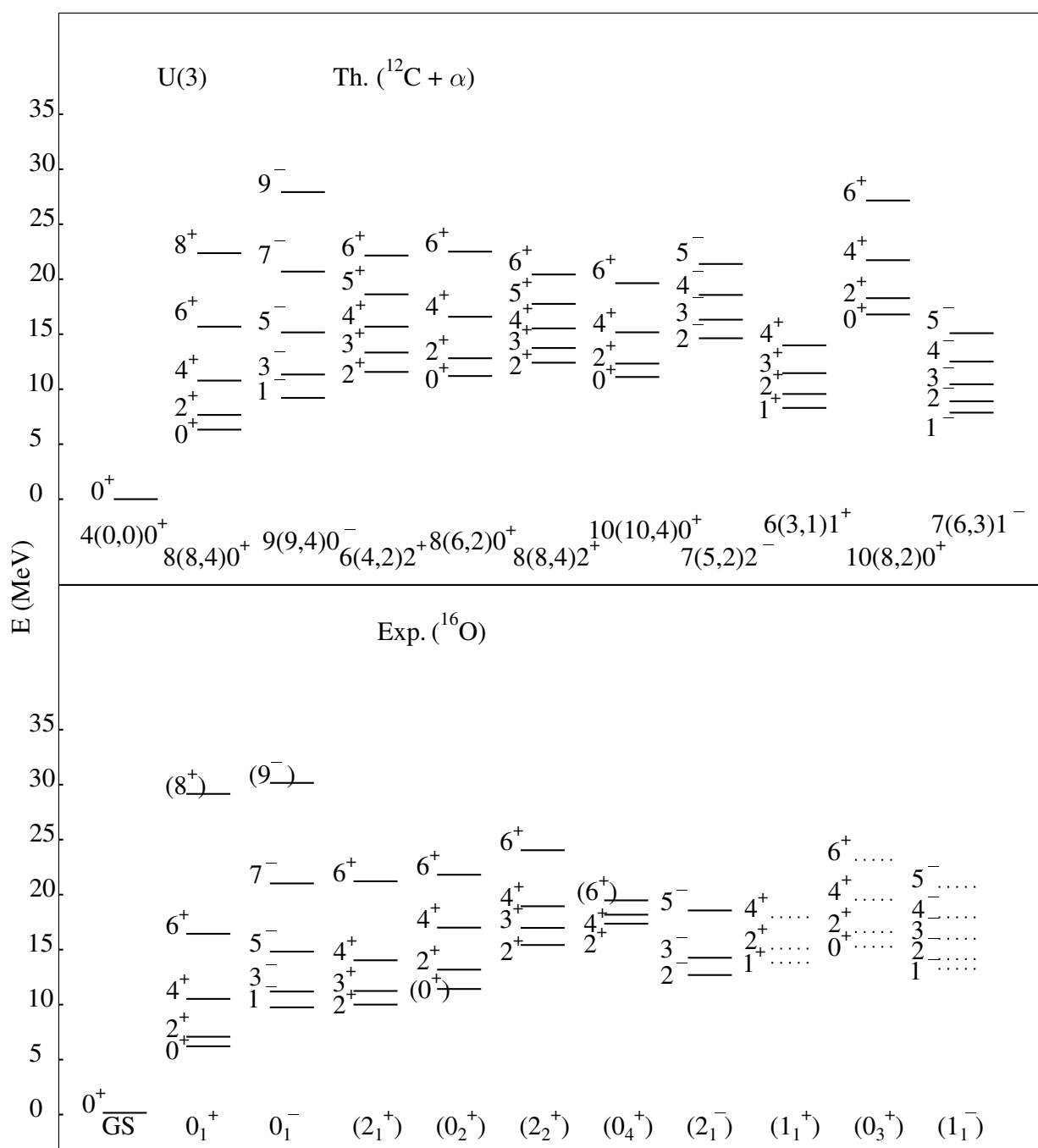


Figure 5. U(3) dynamically symmetric spectrum of $^{12}\text{C} + \alpha$ cluster system (top) in comparison with experimental spectrum (bottom) of ^{16}O . The bands from [31–33] are shown with solid lines, while the bands arranged by us are shown with dotted lines. We labeled the experimental bands with K^π quantum numbers, while the model bands are labeled by $n_\pi(\lambda, \mu)K^\pi$ labels. The figure shows the results obtained when the two experimentally established bands [31,32], the arranged bands of Bauhoff et al. [33], and the bands we arranged are considered in the fitting process.

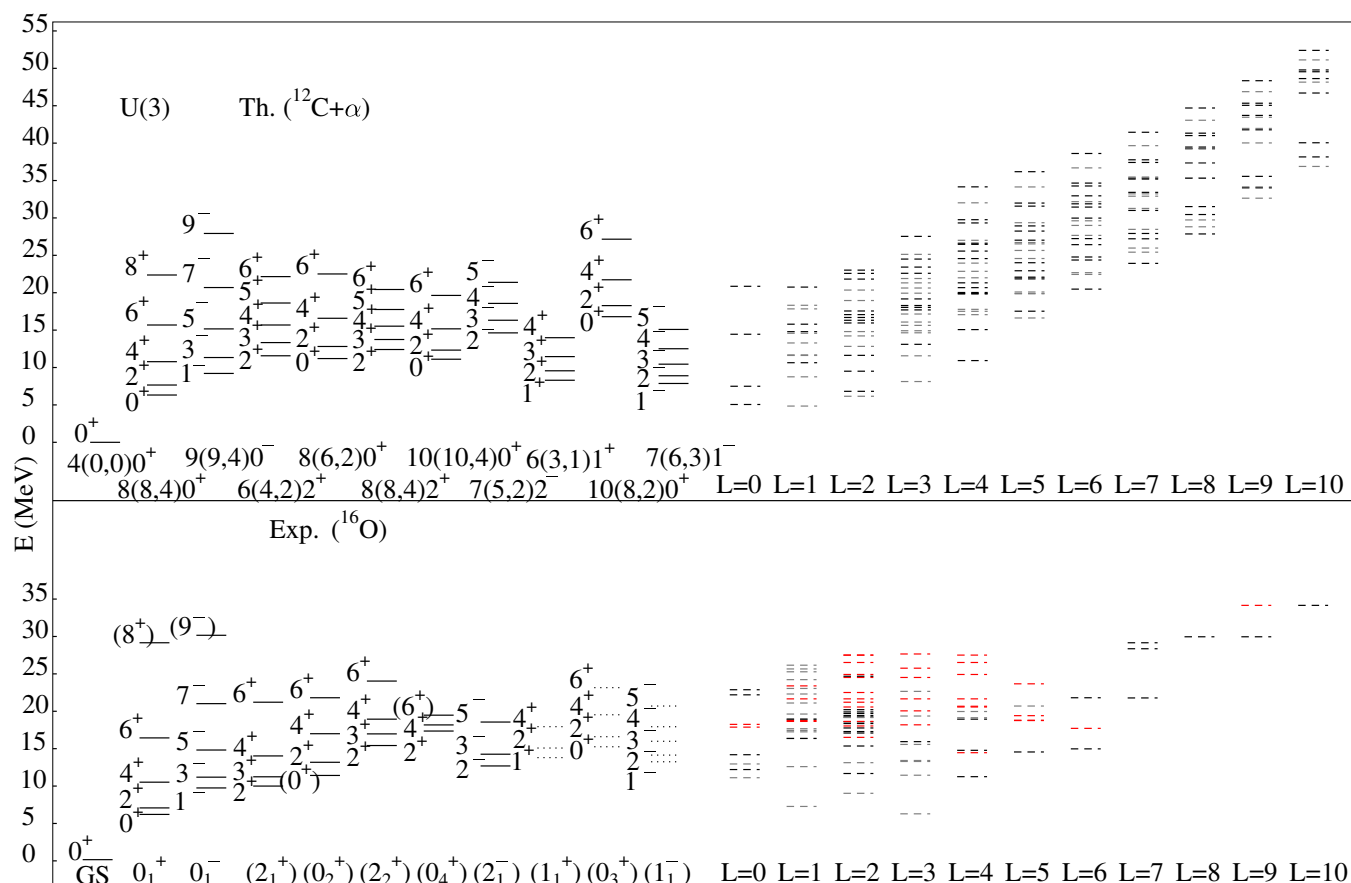


Figure 6. U(3) dynamical spectrum (using Hamiltonian (Equation (35))) of $^{12}\text{C} + \alpha$ cluster system (**top**) in comparison with experimental spectrum (**bottom**) of ^{16}O . The bands from [31–33] are shown with solid lines, while the bands arranged by us are shown with dotted lines. We have labeled the experimental bands with K^π quantum numbers, while the model bands are labeled by $n_\pi(\lambda, \mu)K^\pi$ labels. The figure shows the results obtained when the two experimentally established bands [31,32], the arranged bands of Bauhoff et al. [33], and the bands we arranged are considered in the fitting process. The dashed lines denote the states ($L = 0$ to $L = 10$) that do not belong to any rotational band. Black dashed lines correspond to positive parity, grey dashed lines to negative parity, and red dashed lines indicate states with uncertain spin-parity. The corresponding theoretical states (**top**) are obtained using the model space up to $n_\pi = 10$ with fixed model parameters. Black dashed lines denote positive-parity states, and grey dashed lines denote negative-parity states.

5. Summary and Conclusions

In this paper, we studied the spectra of the phases of clustering in light nuclei by applying the semimicroscopic algebraic cluster model. This approach combines microscopic model spaces with phenomenological operators. There are three phases of clusterization, which are defined by dynamical symmetries: the shell-like (or soft vibrator), the rigid molecule-like (or rotator), and the weak coupled limits. They have U(3), O(4), and O(3) symmetries, respectively. The phase diagram of clustering is two-dimensional, which can be illustrated by a triangle. Its U(3) corner coincides with that of the shell model.

For binary cluster systems, there are no more phases (dynamical symmetries). For multi-cluster systems, there are more groups, with further possibilities for their coupling; nevertheless, the basic phases remain these three ones. The large number of group-chains represent subcases of these major categories.

At this point, we wish to stress that the shell-like and rigid-molecule-like limits are different phases of clustering, in the same sense as the soft vibration and the rigid rotation

are two different phases of the quadrupole collectivity. Therefore, when the clustering is interpreted exclusively in terms of a special shell-like configuration, or in terms of rigid molecule-like picture, it is too restrictive by excluding the other natural phases of the phenomenon.

The spectra of the three phases are qualitatively different. For example, the U(3) dynamical symmetry produces spectra with similar structure to that of the shell model. On the other hand, the O(4), i.e., rigid-molecule-like limit, gives bands with alternating positive and negative parities.

The most drastic difference originates from the role of the Pauli principle. When applying the (strong coupled) U(3) basis, the consequences of the antisymmetrization can be taken into account exactly. All the Pauli-forbidden states can be excluded, and all the allowed states are included in the model space. Thus, the shell-like phase, when the U(3) basis is combined with a U(3) dynamically symmetric interaction, may result in a fully physical description of the system. This is illustrated by the example of the $^{12}\text{C} + \alpha$ system discussed here.

For the rigid-molecule-like phase, i.e., for the O(4) basis, all the states are partially Pauli forbidden and partially allowed. In the case of the weak-coupled phase, i.e., the O(3)-coupled basis, an element of the Pauli-exclusion effect can be taken into account (by the truncation of the relative motion U(3) quantum numbers); however, it is not possible to distinguish between allowed and forbidden states within a single major shell. The contribution of the spurious Pauli-forbidden states decreases with increasing model space, but it is definitely considerable in the small energy region.

As a conclusion, we can say that when a dynamically symmetric (pure phase) approach is applied for the description of clustering, then the shell-like configuration (U(3)) is the reliable choice, due to the importance of the Pauli principle. When a symmetry-breaking is taken into account, then the exact location on the phase diagram can be obtained in the following way. The calculation should be carried out based on the U(3) basis, and excluding the Pauli-forbidden states. The interactions, however, should contain contributions with U(3), as well as with O(4) and O(3) dynamical symmetries. Their relative weights (obtained from the best fit to the experimental data) give us the control parameters which define the position on the phase diagram. Work is in progress in order to carry out a systematic investigation of this kind.

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