Well-chained, connected and simple relators

doktori (PhD) értekezés

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jelölt

Tanúsítom, hogy Pataki Gergely doktorjelölt 2001-2004 között a fent megnevezett Doktori Iskola analízis programjának keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

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Dr. Száz Árpád
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Köszönetnyilvánítás

Ezúton mondok köszönetet Dr. Száz Árpádnak lelkiismeretes témavezetői munkájáért. Közreműködése és tanácsai nélkül ez a disszertáció nem készülhetett volna el.
Továbbá a Debreceni Egyetem Matematikai Intézetének, ezen belül is az Analízis tanszéknek.

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Contents

Introduction

If X and Y are arbitrary sets and $R \subset X \times Y$, then we say that R is a binary relation on X to Y.

If R is a relation on X to Y, and moreover $x \in X$ and $A \subset X$, then the sets $R(x) = \{y \in Y : (x, y) \in R\}$ and $R[A] = \bigcup_{x \in A} R(x)$ are called the images of x and A under R, respectively. Whenever $A \in X$ is unlikely, we may write $R(A)$ in place of $R[A]$.

If $R(x) \neq \emptyset$ for all $x \in X$ (and $Y = R(X)$), then we say that R is a relation of X into (onto) Y.

A relation R on X to Y is said to be a function if for each $x \in X$ there exists $y \in Y$ such that $R(x) \subset \{y\}$. If $R(x) = \{y\}$, then by identifying singletons with their elements we usually write $R(x) = y$.

In this dissertation, we only need the case $X = Y$. In this particular case an $R \subset X^2$ is called a relation on X for short. Note that if R is a relation on X to Y, then R is also a relation on $X \cup Y$. Therefore, it is frequently not a severe restriction to assume that $X = Y$.

In particular, the relations $\Delta_X = \{(x, x) : x \in X\}$ and $X^2 = X \times X$ are called the identity and the universal relations on X, respectively.

If R is a relation on X, then the values $R(x)$, where $x \in X$, uniquely determine R since we have $R = \bigcup_{x \in X} \{x\} \times R(x)$. This allows one to use the following

Definition 1. If R and S are relations on X, then the inverse R^{-1} of R, the composition $R \circ S$ and the box product $R \boxtimes S$ of R and S can be defined such that

$$\begin{aligned} R^{-1}(x) &= \{y \in X : x \in R(y)\} && \text{for all } x \in X, \\ (R \circ S)(x) &= R\big(S(x)\big) && \text{for all } x \in X, \\ (R \boxtimes S)(x, y) &= R(x) \times S(y) && \text{for all } x, y \in X. \end{aligned}$$

Thus, we have $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$ and $(R \boxtimes S)^{-1} = R^{-1} \boxtimes S^{-1}$. Moreover, $(R \circ S)(A) = R\big(S(A)\big)$ for all $A \subset X$, and $(R \boxtimes S)(B) = S \circ B \circ R^{-1}$ for all $B \subset X^2$. And hence, in particular, $R \circ S = (S^{-1} \boxtimes R)(\Delta_X)$.

Now, we can briefly define some important type of relations.

Definition 2. If R is a relation on X, then we say that:

(1) R is reflexive if $\Delta_X \subset R$;
(2) R is partial if $R^{-1}(X) \neq X$;
(3) R is symmetric if $R = R^{-1}$;
(4) R is transitive if $R \circ R \subset R$;
(5) R is a tolerance if it is reflexive and symmetric;
(6) R is a preorder if it is reflexive and transitive;
(7) R is an equivalence if it is a symmetric preorder;

Moreover, we need the following unary operations for relations.

Definition 3. If R is a relation on X, then the relation

$$R^{\infty} = \bigcup_{n=0}^{\infty} R^n,$$

where $R^n = R \circ R^{n-1}$ for all $n \in \mathbb{N}$ and $R^0 = \Delta_X$, is called the preorder hull of R.

Moreover, the complement R^c of the relation R on X can be defined such that

$$R^c = X^2 \setminus R.$$

Namely, we have the following well-known

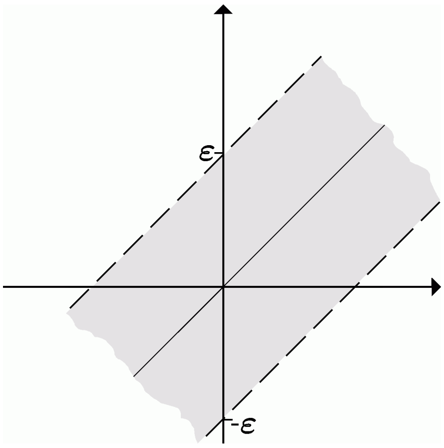
Theorem 4. *If R is a relation on X, then R^{∞} is the smallest preorder on X such that $R \subset R^{\infty}$.*

Therefore, $R = R^{\infty}$ if and only if R is a preorder. Moreover, $R^{\infty} = R^{\infty\infty}$ and $\left(R^{\infty}\right)^{-1} = \left(R^{-1}\right)^{\infty}$.

Now, we shall list some important examples for relations.

Example 5 (The **ε**-sized d-surroundings)**.** Let d be a certain distance function on X, and let $\varepsilon > 0$. Define the relation B^d_{ε} on X such that $(x, y) \in B^d_{\varepsilon}$, that is $y \in B^d_{\varepsilon}(x)$ if $d(x, y) < \varepsilon$, for all $x, y \in X$.

If, in particular, d is a metric on X, then we can see that B^d_{ε} is a tolerance on X for all $\varepsilon > 0$. Moreover $\bigcup_{\varepsilon>0} B^d_{\varepsilon} = X^2$ and $\bigcap_{\varepsilon>0} B^d_{\varepsilon} = \Delta_X$.



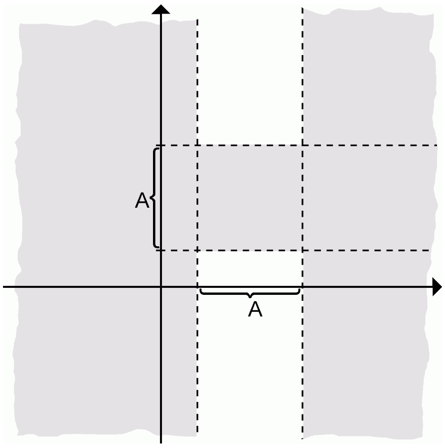
The figure shows the B^d_{ε} relation on $\mathbb{R}$, where, in particular, $d(x, y) = |x - y|$.

Example 6 (The Davis–Pervin relations)**.** Let A be a subset of X. Then, the relation

$$R_A = A^2 \cup (X \setminus A) \times X$$

is called the Davis–Pervin relation on X generated by A.

Remark 7. Namely, the relations R_A were first used by Davis [9] and Pervin [55] in their uniformization procedures of topological spaces.

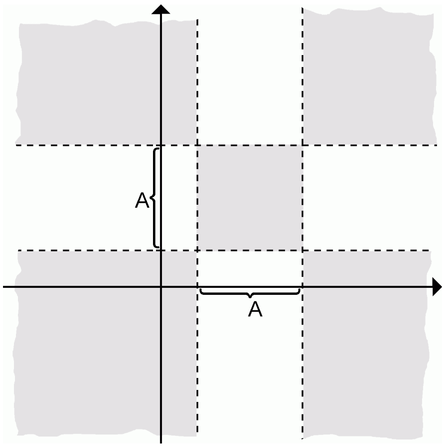


The figure shows the Davis–Pervin relation R_A on $\mathbb{R}$, generated by the closed interval A.

Example 8 (The symmetrization of the Davis–Pervin relations)**.** Let A be a subset of X. Then, the relation

$$S_A = R_A \cap R_A^{-1}$$

is called the symmetrization of the Davis–Pervin relation on X generated by A.



The figure shows the symmetrization of the Davis–Pervin relation S_A on $\mathbb{R}$, generated by the closed interval A.

Now, we can define the main object of the dissertation.

Definition 9. A nonvoid family $\mathcal{R}$ of binary relations on a nonvoid set X is called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a relator space.

Note, that relator spaces are straightforward generalizations of ordered sets and uniform spaces.

Definition 10. If $\mathcal{R}$ is a relator on X, then we say that:

(1) $\mathcal{R}$ is reflexive if R is a reflexive relation on X for all $R \in \mathcal{R}$;

(2) $\mathcal{R}$ is total if R is a non-partial relation on X for all $R \in \mathcal{R}$;

(3) $\mathcal{R}$ is tolerance if R is a tolerance on X for all $R \in \mathcal{R}$;

(4) $\mathcal{R}$ is preorder if R is a preorder on X for all $R \in \mathcal{R}$;

(5) $\mathcal{R}$ is equivalence if R is an equivalence on X for all $R \in \mathcal{R}$;

Remark 11. Symmetric and transitive relators will be defined later in some special senses.

Definition 12. If $\mathcal{R}$ is a relator on X, then the relator

$$\mathcal{R}^{-1} = \{R^{-1} : R \in \mathcal{R}\}$$

is called the inverse of $\mathcal{R}$.

Example 13 (Relators induced by distance functions)**.** If d is a certain distance function on X, then the relator $\mathcal{R}^d$ induced by d can be defined such that

$$\mathcal{R}^d = \{B_\varepsilon^d : \varepsilon > 0\}.$$

Theorem 14. *If $X(d)$ is a metric space, then $\mathcal{R}^d$ is a tolerance relator on X.*

Example 15 (Relators induced by families of sets)**.** If $\mathcal{T}$ is a nonvoid family of subsets of X, then the relator $\mathcal{R}_\mathcal{T}$ induced by $\mathcal{T}$ can be defined such that

$$\mathcal{R}_\mathcal{T} = \{R_A : A \in \mathcal{T}\}.$$

Before the following two theorems, we need some definition.

Definition 16. If $\mathcal{R}$ is a relator on X and $A \subset X$, then the set

$$\operatorname{int}_\mathcal{R}(A) = \{x \in X : \exists R \in \mathcal{R} : R(x) \subset A\}$$

is called the topological interior of A induced by $\mathcal{R}$ on X.

Moreover, the members of the families

$$\mathcal{T}_\mathcal{R} = \{A \subset X : A \subset \operatorname{int}_\mathcal{R}(A)\} \qquad \text{and} \qquad \mathcal{E}_\mathcal{R} = \{A \subset X : \operatorname{int}_\mathcal{R}(A) \neq \emptyset\}$$

are called the topologically open and the fat subsets of the relator space $X(\mathcal{R})$, respectively.

And, the relators

$$\mathcal{R}^\wedge = \big\{S \subset X^2 : \forall x \in X : x \in \operatorname{int}_\mathcal{R}\big(S(x)\big)\big\}$$

and

$$R^\vartriangle = \big\{S \subset X^2 : \forall x \in X : S(x) \in \mathcal{E}_\mathcal{R}\big\}$$

are called the topological and the paratopological refinements of $\mathcal{R}$, respectively.

Theorem 17. *If $\mathcal{T}$ is a nonvoid family of subsets of X, then $\mathcal{R}_\mathcal{T}$ is a preorder relator on X, such that the following assertions are equivalent:*

(1) $X \in \mathcal{T}$ and $\mathcal{T}$ is closed under arbitrary union;

(2) $\mathcal{T} = \mathcal{T}_{\mathcal{R}_\mathcal{T}}$;

(3) $\mathcal{T} = \mathcal{T}_\mathcal{S}$ for some relator $\mathcal{S}$ on X.

Hint. If $A \in \mathcal{T}_{\mathcal{R}_{\mathcal{T}}}$, then for all $x \in A$ there exists $R_x \in \mathcal{R}_{\mathcal{T}}$, such that $R_x(x) \subset A$. Since $R_x \in \mathcal{R}_{\mathcal{T}}$ means that $R_x = R_{B_x}$ for some $B_x \in \mathcal{T}$, we have that $x \in B_x \subset A$ for some $B_x \in \mathcal{T}$ or $A = X$. Therefore, if the assertion (1) holds, then since $\bigcup_{x \in A} B_x \in \mathcal{T}$ and $X \in \mathcal{T}$, we have that $A \in \mathcal{T}$ for all $A \in \mathcal{T}_{\mathcal{R}_{\mathcal{T}}}$, that is $\mathcal{T}_{\mathcal{R}_{\mathcal{T}}} \subset \mathcal{T}$. Since the converse inclusion is quite obvious, the assertion (2) also holds.

Theorem 18. *If $\mathcal{E}$ is a nonvoid family of subsets of X, then $\mathcal{R}_{\mathcal{E}}$ is a preorder relator on X, such that the following assertions are equivalent:*

(1) $\mathcal{E}$ is an ascending family in $\mathcal{P}(X)$;

(2) $\mathcal{E} = \mathcal{E}_{\mathcal{R}_{\mathcal{E}}}$;

(3) $\mathcal{E} = \mathcal{E}_{\mathcal{S}}$ for some relator $\mathcal{S}$ on X.

Hint. If $A \in \mathcal{E}_{\mathcal{R}_{\mathcal{E}}}$, then there exist $R \in \mathcal{R}_{\mathcal{E}}$ and $x \in X$, such that $R(x) \subset A$. Since $R \in \mathcal{R}_{\mathcal{E}}$ means that $R = R_B$ for some $B \in \mathcal{E}$ and $B \subset R_B(x)$, we have that $B \subset A$. Therefore, if the assertion (1) holds, then we have that $A \in \mathcal{E}$ for all $A \in \mathcal{E}_{\mathcal{R}_{\mathcal{E}}}$, that is $\mathcal{E}_{\mathcal{R}_{\mathcal{E}}} \subset \mathcal{E}$. Since the converse inclusion is quite obvious, the assertion (2) also holds.

I

By establishing some intimate connections between unary operations and set-valued functions for relators, we greatly extend and supplement some of the former results of Á. Száz and J. Mala on the various refinements and modifications of relators.

A function $\square$ of the family of all relators on X into itself is called a unary operation for relators on X. And we write $\mathcal{R}^{\square} = \square(\mathcal{R})$ for every relator $\mathcal{R}$ on X. Moreover, a function $\mathfrak{F}$ of the family of all relators on X into a family of sets is called a set-valued function for relators on X. And we write $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

If $\square$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then we say that:

(1) $\square$ is expansive if $\mathcal{R} \subset \mathcal{R}^{\square}$ for every relator $\mathcal{R}$ on X;

(2) $\square$ is idempotent if $\mathcal{R}^{\square} = \mathcal{R}^{\square\square}$ for every relator $\mathcal{R}$ on X;

(3) $\mathfrak{F}$ is increasing (decreasing) if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$) for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X with $\mathcal{S} \subset \mathcal{R}$.

(4) $\square$ is a modification if it is idempotent and increasing;

(5) $\square$ is a refinement if it is expansive, idempotent and increasing;

(6) $\mathfrak{F}$ is $\square$-increasing ($\square$-decreasing) if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$).

If $\mathfrak{F}$ is an increasing (decreasing) set-valued function for relators on X, then the induced unary operation $\square_{\mathfrak{F}}$ is defined by

$$\mathcal{R}^{\square_{\mathfrak{F}}} = \left\{ S \subset X^2 : \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}} \right\} \qquad \left(\mathcal{R}^{\square_{\mathfrak{F}}} = \left\{ S \subset X^2 : \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\{S\}} \right\} \right)$$

for every relator $\mathcal{R}$ on X.

Moreover, a monotone set-valued function for relators on X is called regular if

$$\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}$$

for every relator $\mathcal{R}$ on X. And, an increasing (decreasing) set-valued function for relators is called normal if

$$\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \qquad \Big(\mathfrak{F}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \Big)$$

for every relator $\mathcal{R}$ on X.

The most important theorems of this chapter will show, that a normal set-valued function for relators on X is, in particular, regular. On the other hand, a unary operation induced by a regular set-valued function is a refinement. Moreover, if $\mathfrak{F}$ is a regular set-valued function for relators on X and $\mathcal{R}$ is a relator on X, then $\mathcal{R}^{\Box_{\mathfrak{F}}}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}$.

Now, we can see that, since int and $\mathcal{E}$ are normal increasing set-valued functions for relators on X, the induced unary operations $\wedge = \Box_{\text{int}}$ and $\vartriangle = \Box_{\mathcal{E}}$ are refinements.

Finally, the preorder modification of the relator $\mathcal{R}$ on X is defined by

$$\mathcal{R}^{\infty} = \{R^{\infty} : R \in \mathcal{R}\}.$$

Note, that the unary operation ∞ is not expansive, therefore it is not a refinement.

II

In this chapter, a unified treatment of some old and new well-chainedness and connectedness properties of the most basic topological structures (such as closures, proximities and uniformities, for instance) is offered in the framework of relators and their fundamental refinements.

The results obtained show that the various connectedness properties are actually particular cases of Cantor's well-chainedness property neglected by several authors. Moreover, they show that the hyperconnectedness introduced by L. A. Steen and J. A. Seebach is a particular case of our paratopological connectedness.

A relator $\mathcal{R}$ on X will be called properly well-chained or chain-connected if

$$\mathcal{R}^{\infty} = \{X^2\}.$$

Moreover, if $\Box$ is a unary operation for relators on X, then the relator $\mathcal{R}$ on X will be called $\Box$-well-chained if the relator $\mathcal{R}^{\Box}$ is properly well-chained.

The condition $\mathcal{R}^{\infty} = \{X^2\}$, in a detailed form, means only that for every $R \in \mathcal{R}$ and $x, y \in X$, with $x \neq y$, there exists an $n \in \mathbb{N}$ such that $(x, y) \in R^n$. That is, there exists a family $(x_i)_{i=0}^{n}$ in X such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R$ for all $i = 1, \dots, n$.

In the dissertation, we prove that a relator $\mathcal{R}$ on X is properly well-chained if and only if $\mathcal{P}(R_A) \cap \mathcal{R} = \emptyset$ for every proper nonvoid subset A of X, or equivalently $\mathcal{P}(R) \cap \mathcal{R} = \emptyset$ for every proper preorder R on X.

Moreover, a relator $\mathcal{R}$ on X is topologically (paratopologically) well-chained if and only if $R_A \notin \mathcal{R}^\wedge$ ($R_A \notin \mathcal{R}^\vartriangle$) for every proper nonvoid subset A of X, or equivalently $R \notin \mathcal{R}^\wedge$ ($R \notin \mathcal{R}^\vartriangle$) for every proper preorder R on X.

Furthermore, a relator $\mathcal{R}$ on X is topologically well-chained if and only if $\mathcal{T}_\mathcal{R} = \{\emptyset, X\}$. And, if $\operatorname{card}(X) > 1$, then the relator $\mathcal{R}$ on X is paratopologically well-chained if and only if $\mathcal{E}_\mathcal{R} = \{X\}$, or equivalently $\mathcal{R} = \{X^2\}$.

A relator $\mathcal{R}$ on X will be called properly connected if the relator $\{R \cup R^{-1} : R \in \mathcal{R}\}$ is properly well-chained. Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ will be called $\square$-connected if the relator $\mathcal{R}^\square$ is properly connected.

In the dissertation, we prove that a relator $\mathcal{R}$ on X is properly connected if and only if $\mathcal{P}(S_A) \cap \mathcal{R} = \emptyset$ for every proper nonvoid subset A of X, or equivalently $\mathcal{P}(S) \cap \mathcal{R} = \emptyset$ for every proper equivalence S on X.

Moreover, a relator $\mathcal{R}$ on X is topologically (paratopologically) connected if and only if $S_A \notin \mathcal{R}^\wedge$ ($S_A \notin \mathcal{R}^\vartriangle$) for every proper nonvoid subset A of X, or equivalently $S \notin \mathcal{R}^\wedge$ ($S \notin \mathcal{R}^\vartriangle$) for every proper equivalence S on X.

Furthermore, a relator $\mathcal{R}$ on X is topologically connected if and only if the complement of any proper nonvoid topologically open set is not topologically open. And, if $\operatorname{card}(X) > 1$, then the relator $\mathcal{R}$ on X is paratopologically connected if and only if the intersection of any two fat sets is non empty.

At the end of this chapter, a diagram can be found which shows the main implications among the various well-chainedness and connectedness properties of relators.

III

In chapter III, some published and unpublished results of Árpád Száz, József Mala and Jenő Deák on simple and quasi-simple relators are illustrated and supplemented.

A relator $\mathcal{R}$ on X is called properly simple if it is a singleton. Moreover, if $\square$ is a unary operation for relators on X, then the relator is called $\square$-simple if there exists a relation R on X such that $\mathcal{R}^\square = \{R\}^\square$. On the other hand, a relator is called quasi-$\square$-simple, if it is $\square\infty$-simple. We remark, that for instance, the topological well-chainedness is a particular case of quasi-topological simplicity.

In the dissertation, we prove that a relator $\mathcal{R}$ on X is quasi-properly simple if and only if $R^\infty = S^\infty$ for all $R, S \in \mathcal{R}$, or equivalently $\mathcal{R}^\infty$ is properly simple. And, if a relator is $\square$-simple, then it is also quasi-$\square$-simple.

Moreover, for instance, we prove that a relator $\mathcal{R}$ is (quasi-)topologically simple if and only if $\bigcap \mathcal{R} \in \mathcal{R}^\wedge$ ($\bigcap \mathcal{R}^{\wedge\infty} \in \mathcal{R}^\wedge$). And, we state that the paratopological simplicity is equivalent to the quasi-paratopological simplicity. After this, we characterize paratopological simple relators, to construct an equivalence relator which is not paratopologically simple.

At the end of this chapter, a diagram can be found which shows the main implications among the various simplicity and quasi-simplicity properties of relators.

1 General set-valued functions and unary operations for relators

1.1 Set-valued functions and unary operations for relators

Definition 1.1.1. A function $\Box$ of the family of all relators on X into itself is called a unary operation for relators on X. And we write $\mathcal{R}^{\Box} = \Box(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

Moreover, a function $\mathfrak{F}$ of the family of all relators on X into a family of sets is called a set-valued function for relators on X. And we write $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

Remark 1.1.2. Note that thus a unary operation for relators is, in particular, a set-valued function for relators.

Important examples for set-valued functions and unary operations for relators have been given by Száz and Mala in [71], [73], [34], and [39].

Definition 1.1.3. If $\Box$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then we say that:

(1) $\Box$ is stable if $\{X^2\}^{\Box} = \{X^2\}$;

(2) $\Box$ is expansive if $\mathcal{R} \subset \mathcal{R}^{\Box}$ for every relator $\mathcal{R}$ on X;

(3) $\Box$ is idempotent if $\mathcal{R}^{\Box} = \mathcal{R}^{\Box\Box}$ for every relator $\mathcal{R}$ on X;

(4) $\mathfrak{F}$ is increasing if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X with $\mathcal{S} \subset \mathcal{R}$.

Remark 1.1.4. Analogously, the set-valued function $\mathfrak{F}$ is called decreasing if $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X with $\mathcal{S} \subset \mathcal{R}$.

Note that the expansivity property (2) implies that $\mathcal{R}^{\Box} \subset \mathcal{R}^{\Box\Box}$ for every relator $\mathcal{R}$ on X. Therefore, an expansive operation $\Box$ for relators is idempotent if the converse inclusion holds.

Definition 1.1.5. If $\mathfrak{F}$ is a set-valued function for relators on X, then we say that the relators $\mathcal{R}$ and $\mathcal{S}$ on X are $\mathfrak{F}$-equivalent if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.

Moreover, if $\Box$ is a unary operation for relators on X, then we say that the relator $\mathcal{R}$ on X is $\Box$-fine if $\mathcal{R} = \mathcal{R}^{\Box}$.

Remark 1.1.6. Note that $\Box$ is stable if and only if $\{X^2\}$ is $\Box$-fine.

Moreover, $\Box$ is idempotent if and only if $\mathcal{R}^{\Box}$ is the only one $\Box$-fine relator on X, which $\Box$-equivalent to $\mathcal{R}$ for every relator $\mathcal{R}$ on X.

The usefulness the expansivity and idempotency properties is already apparent from the following

Theorem 1.1.7. *If $\Box$ is a unary operation for relators on X, then the following assertions are equivalent:*

(1) $\Box$ is expansive and idempotent;

(2) for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square}$ is the largest relator on X which $\square$-equivalent to $\mathcal{R}$;

(3) there exists a set-valued function $\mathfrak{F}$ for relators on X such that, for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$.

Proof. Assume that the assertion (1) holds, and let $\mathcal{R}$ be a relator on X. If $\mathcal{S}$ is a relator on X such that $\mathcal{R}^{\square} = \mathcal{S}^{\square}$, then by the expansivity property of the operation $\square$, we also have $\mathcal{S} \subset \mathcal{R}^{\square}$. Therefore, since Remark 1.1.6 the assertion (2) also holds.

While if the assertion (2) holds, then we can observe that the set-valued function $\mathfrak{F} = \square$ has the properties required in the assertion (3). Therefore, the implication (2) $\Longrightarrow$ (3) is obviously true.

Assume now that the assertion (3) holds, and let $\mathcal{R}$ be a relator on X. Then, from the obvious equality $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}}$, by using the corresponding maximality property of the relator $\mathcal{R}^{\square}$, we can infer that $\mathcal{R} \subset \mathcal{R}^{\square}$. Therefore, the operation $\square$ is expansive. Moreover, by writing $\mathcal{R}^{\square}$ in place of $\mathcal{R}$ in the assertion (3), we can at once see that $\mathfrak{F}_{\mathcal{R}^{\square}} = \mathfrak{F}_{\mathcal{R}^{\square\square}}$. Hence, since $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$, we also have $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square\square}}$. Hence, by using the corresponding maximality property of the relator $\mathcal{R}^{\square}$, we can infer that $\mathcal{R}^{\square\square} \subset \mathcal{R}^{\square}$. Therefore, by Remark 1.1.4, the operation $\square$ is idempotent. And thus the assertion (1) also holds.

Remark 1.1.8. Despite the above theorem, the most important property of a unary operation for relators is certainly the monotonicity property.

Namely, all the important set-valued functions for relators are monotone. But, some useful operations for relators are neither expansive nor idempotent.

Definition 1.1.9. If $\square$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then we say that:

(1) $\square$ is an extension if it is expansive and increasing;

(2) $\square$ is a modification if it is idempotent and increasing;

(3) $\square$ is a refinement if it is expansive, idempotent and increasing;

(4) $\mathfrak{F}$ is $\square$-increasing if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$.

Remark 1.1.10. Analogously, the set-valued function $\mathfrak{F}$ is called $\square$-decreasing if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$.

Moreover, in particular, the operation $\square$ will be called self-increasing if it is $\square$-increasing. That is, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathcal{S}^{\square} \subset \mathcal{R}^{\square}$.

The appropriateness of the above definitions is already apparent from the following

Theorem 1.1.11. *If $\square$ is a unary operation for relators on X, then the following assertions are equivalent:*

(1) $\square$ is a refinement; *(2) $\square$ is self-increasing;*

(3) there exists a $\square$-increasing set-valued function $\mathfrak{F}$ for relators on X.

Proof. Assume that the assertion (1) holds, and let $\mathcal{R}$ and $\mathcal{S}$ be relators on X. If $\mathcal{S} \subset \mathcal{R}^{\Box}$, then by the increasingness and the idempotency properties of the operation $\Box$, it is clear that $\mathcal{S}^{\Box} \subset \mathcal{R}^{\Box\Box} = \mathcal{R}^{\Box}$. While, if $\mathcal{S}^{\Box} \subset \mathcal{R}^{\Box}$, then by the expansivity property of the operation $\Box$ alone it is clear that $\mathcal{S} \subset \mathcal{R}^{\Box}$. Therefore, the assertion (2) also holds.

While if the assertion (2) holds, then we can note that the set-valued function $\mathfrak{F} = \Box$ is $\Box$-increasing. Therefore, the implication (2) $\Longrightarrow$ (3) is obviously true.

Assume now that the assertion (3) holds, and let $\mathcal{R}$ and $\mathcal{S}$ be relators on X. Then, from the obvious inclusion $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}}$, by using the assumption (3) we can infer that $\mathcal{R} \subset \mathcal{R}^{\Box}$. Therefore, the operation $\Box$ is expansive. On the other hand, from the obvious inclusion $\mathcal{R}^{\Box} \subset \mathcal{R}^{\Box}$, by using the assumption (3), we can infer that $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, by writing $\mathcal{R}^{\Box}$ in place of $\mathcal{R}$, we can see that $\mathfrak{F}_{\mathcal{R}^{\Box\Box}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, by using the assumption (3), we can infer that $\mathcal{R}^{\Box\Box} \subset \mathcal{R}^{\Box}$. Thus, by Remark 1.1.4, the operation $\Box$ is idempotent.

Finally, if $\mathcal{S} \subset \mathcal{R}$, then by the expansivity property of the operation $\Box$ it is clear that $\mathcal{S} \subset \mathcal{R}^{\Box}$. Hence, by using the assumption (3), we can infer that $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, since $\mathfrak{F}_{\mathcal{S}^{\Box}} \subset \mathfrak{F}_{\mathcal{S}}$, it is clear that we also have $\mathfrak{F}_{\mathcal{S}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, by using the assumption (3), we can infer that $\mathcal{S}^{\Box} \subset \mathcal{R}^{\Box}$. Therefore, the operation $\Box$ is increasing. And thus the assertion (1) also holds.

Concerning refinements, we can also easily prove the following

Theorem 1.1.12. *If $\Box$ is a refinement for relators on X and $\big(\mathcal{R}_i\big)_{i \in I}$ is a nonvoid family of relators on X, then*

$$\Big(\bigcup_{i \in I} \mathcal{R}_i\Big)^{\Box} = \Big(\bigcup_{i \in I} \mathcal{R}_i^{\Box}\Big)^{\Box}.$$

Hence, it is clear that, in particular, we have

Corollary 1.1.13. *If $\Box$ is a refinement for relators on X and $\mathcal{R}$ is a relator on X, then*

$$\mathcal{R}^{\Box} = \Big(\bigcup_{R \in \mathcal{R}} \{R\}^{\Box}\Big)^{\Box}.$$

However, it is now more important to note that, in addition to Theorem 1.1.11, we can also prove the following

Theorem 1.1.14. *If $\Box$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then the following assertions are equivalent:*

(1) $\mathfrak{F}$ is $\Box$-increasing;

(2) $\mathfrak{F}$ is increasing and, for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\Box}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$.

Proof. Assume that the assertion (1) holds, and let $\mathcal{R}$ and $\mathcal{S}$ be relators on X. Then, if $\mathcal{S} \subset \mathcal{R}$, then by the assertion (1) and Theorem 1.1.11, it is clear that $\mathcal{S} \subset \mathcal{R}^{\square}$, and hence $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$. Therefore, the function $\mathfrak{F}$ is increasing. On the other hand, from the obvious inclusion $\mathcal{R}^{\square} \subset \mathcal{R}^{\square}$, by using the assertion (1), we can infer that $\mathfrak{F}_{\mathcal{R}^{\square}} \subset \mathfrak{F}_{\mathcal{R}}$. Moreover, if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$, then by using the assertion (1) we can see that $\mathcal{S} \subset \mathcal{R}^{\square}$. Therefore, the assertion (2) also holds.

Conversely, assume now that the assertion (2) holds, and let $\mathcal{R}$ and $\mathcal{S}$ be relators on X. Then, if $\mathcal{S} \subset \mathcal{R}^{\square}$, then by the assertion (2) it is clear that $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}^{\square}} \subset \mathfrak{F}_{\mathcal{R}}$. While, if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$, then by the corresponding maximality of the relator $\mathcal{R}^{\square}$ it is clear that $\mathcal{S} \subset \mathcal{R}^{\square}$. Therefore, the assertion (1) also holds.

Now, as an immediate consequence of Theorems 1.1.11 and 1.1.14, we can also state

Corollary 1.1.15. *If $\square$ is a unary operation and $\mathfrak{F}$ is a $\square$-increasing set-valued function for relators on X, then for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$.*

Proof. If $\mathcal{R}$ is a relator on X, then by Theorem 1.1.11 we have $\mathcal{R} \subset \mathcal{R}^{\square}$. Hence, by Theorem 1.1.14, it follows that $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}^{\square}}$. Moreover, by Theorem 1.1.14, we also have $\mathfrak{F}_{\mathcal{R}^{\square}} \subset \mathfrak{F}_{\mathcal{R}}$. Therefore, $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$.

On the other hand, if $\mathcal{S}$ is a relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$, then in particular $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$. Hence, since $\mathfrak{F}$ is $\square$-increasing, it follows that $\mathcal{S} \subset \mathcal{R}^{\square}$. Therefore, $\mathcal{R}^{\square}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$.

Remark 1.1.16. Note that if $\square$ is a unary operation and $\mathfrak{F}$ is a $\square$-decreasing set-valued function for relators on X, then the pointwise complement $\mathfrak{F}^c$ of $\mathfrak{F}$, defined by

$$\mathfrak{F}^c_{\mathcal{R}} = \Big(\bigcup \operatorname{rng}(\mathfrak{F})\Big) \setminus \mathfrak{F}_{\mathcal{R}}$$

for every relator $\mathcal{R}$ on X, is $\square$-increasing. Therefore, the corresponding duals of Theorems 1.1.11 and 1.1.14 and Corollary 1.1.15 are also true.

1.2 The induced unary operations and regular set-valued functions

Definition 1.2.1. If $\mathfrak{F}$ is an increasing set-valued function for relators on X, then the operation $\square_{\mathfrak{F}}$, defined by

$$\mathcal{R}^{\square_{\mathfrak{F}}} = \big\{ S \subset X^2 : \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}} \big\}$$

for every relator $\mathcal{R}$ on X, is called the operation induced by $\mathfrak{F}$.

Remark 1.2.2. Analogously, if $\mathfrak{F}$ is a decreasing set-valued function for relators on X, then the function $\square_{\mathfrak{F}}$, defined by $\mathcal{R}^{\square_{\mathfrak{F}}} = \big\{ S \subset X^2 : \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\{S\}} \big\}$ for every relator $\mathcal{R}$ on X, is called the operation induced by $\mathfrak{F}$.

Note that if $\mathfrak{F}$ is a set-valued function for relators on X such that $\mathfrak{F}$ is both increasing and decreasing, then $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{P}(X^2)}$ for every relator $\mathcal{R}$ on X. Therefore, the two possible definitions cannot lead to confusions.

Definition 1.2.3. A monotone set-valued function $\mathfrak{F}$ for relators on X is called regular if

$$\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}$$

for every relator $\mathcal{R}$ on X.

The appropriateness of Definition 1.2.1 is already apparent from the following

Theorem 1.2.4. *If $\Box$ is a unary operation and $\mathfrak{F}$ is a $\Box$-increasing set-valued function for relators on X, then $\Box = \Box_{\mathfrak{F}}$. And thus, in particular, $\mathfrak{F}$ is regular.*

Proof. Let $\mathcal{R}$ be an arbitrary relator on X. $S \in \mathcal{R}^{\Box}$, i.e., $\{S\} \subset \mathcal{R}^{\Box}$ if and only if $\mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}}$ since $\mathfrak{F}$ is $\Box$-increasing. By the definition of $\mathcal{R}^{\Box_{\mathfrak{F}}}$ it is equivalent to $S \in \mathcal{R}^{\Box_{\mathfrak{F}}}$. It follows that $\mathcal{R}^{\Box} = \mathcal{R}^{\Box_{\mathfrak{F}}}$ for every relator $\mathcal{R}$ on X, that is $\Box = \Box_{\mathfrak{F}}$.

Moreover, since Corollary 1.1.15 $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}$, i.e., $\mathfrak{F}$ is regular.

Now, as an immediate consequence of Theorem 1.2.4, we can also state

Corollary 1.2.5. *If $\mathfrak{F}$ is a set-valued function for relators on X, then there exists at most one unary operation $\Box$ for relators on X such that $\mathfrak{F}$ is $\Box$-increasing.*

Remark 1.2.6. Later we shall see that even for a refinement $\Box$ for relators there may exist more than one $\Box$-increasing set-valued function.

Moreover, there are important increasing set-valued functions for relators which are not regular.

However, despite this, we can still prove the following

Theorem 1.2.7. *If $\mathfrak{F}$ is an increasing set-valued function for relators on X, then*

(1) $\Box_{\mathfrak{F}}$ is an extension for relators on X;

(2) $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ implies $\mathcal{S} \subset \mathcal{R}^{\Box_{\mathfrak{F}}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X.

Proof. By using Definition 1.2.1, we can easily see that

$$\mathcal{R}^{\Box_{\mathfrak{F}}} = \bigcup \{\mathcal{S} : \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}\}$$

for every relator $\mathcal{R}$ on X. And hence, the required assertions are quite obvious.

Remark 1.2.8. Later, we shall see that the operation $\Box_{\mathfrak{F}}$, induced by an increasing set-valued function $\mathfrak{F}$ for relators, need not be idempotent.

However, despite this, we can still prove the following

Theorem 1.2.9. *If $\mathfrak{F}$ is an increasing set-valued function for relators on X, then the following assertions are equivalent:*

(1) $\mathfrak{F}$ is regular;

(2) $\mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}} \subset \mathfrak{F}_{\mathcal{R}}$ for every relator $\mathcal{R}$ on X;

(3) $\mathcal{S} \subset \mathcal{R}^{\Box_{\mathfrak{F}}}$ implies $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X;

(4) $\mathfrak{F}$ is $\Box$-increasing for some unary operation $\Box$ for relators on X;

(5) $\mathfrak{F}$ is $\Box_{\mathfrak{F}}$-increasing.

Proof. If the assertion (2) holds, then since $\mathfrak{F}$ is increasing it is clear that $\mathcal{S} \subset \mathcal{R}^{\square_{\mathfrak{F}}}$ implies $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} \subset \mathfrak{F}_{\mathcal{R}}$ for any two relator $\mathcal{R}$ and $\mathcal{S}$ on X. That is, the assertion (3) also holds.

While, if the assertion (3) holds, then from Theorem 1.2.7, it is clear that $\mathfrak{F}$ is $\square_{\mathfrak{F}}$-increasing. Therefore, the assertion (4) also holds.

Moreover, if the assertion (4) holds, then by Theorem 1.2.4 the assertion (5) also holds.

Finally, if the assertion (5) holds, i.e., $\mathfrak{F}$ is $\square_{\mathfrak{F}}$-increasing, then from Corollary 1.1.15 we can see that the assertion (1) also holds.

Now, to guarantee the regularity of increasing set-valued functions for relators, we may naturally have

Definition 1.2.10. An increasing set-valued function $\mathfrak{F}$ for relators on X is called normal if, for every relator $\mathcal{R}$ on X, we have

$$\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}}.$$

Remark 1.2.11. Analogously, a decreasing set-valued function $\mathfrak{F}$ for relators on X is called normal if $\mathfrak{F}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}}$ for every relator $\mathcal{R}$ on X.

Note that if $\mathfrak{F}$ is an increasing set-valued function for relators on X, then $\bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \subset \mathfrak{F}_{\mathcal{R}}$ for every relator $\mathcal{R}$ on X, and if $\mathfrak{F}$ is a decreasing set-valued function for relators on X, then $\mathfrak{F}_{\mathcal{R}} \subset \bigcap_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}}$ for every relator $\mathcal{R}$ on X. Therefore a set-valued function $\mathfrak{F}$ for relators on X is normal if and only if it is monotone and

$$\bigcap_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \subset \mathfrak{F}_{\mathcal{R}} \subset \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}}.$$

Now, as a useful consequence of Theorem 1.2.9, we can also state

Theorem 1.2.12. *A normal set-valued function for relators on X is, in particular, regular.*

Hint. If $\mathfrak{F}$ is a normal increasing set-valued function for relators on X, then by Definitions 1.2.10 and 1.2.1, we evidently have

$$\mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} = \bigcup_{S \in \mathcal{R}^{\square_{\mathfrak{F}}}} \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}}$$

for every relator $\mathcal{R}$ on X. Therefore, by Theorem 1.2.9, $\mathfrak{F}$ is regular.

Moreover, concerning normal set-valued functions, we can also easily prove the following

Theorem 1.2.13. *If $\mathfrak{F}$ is a normal increasing set-valued function for relators on X and $\left(\mathcal{R}_i\right)_{i \in I}$ is a nonvoid family of relators on X, then*

$$\mathfrak{F}_{\bigcup_{i \in I} \mathcal{R}_i} = \bigcup_{i \in I} \mathfrak{F}_{\mathcal{R}_i}.$$

The importance of regular set-valued functions lies mainly in the following consequence of Theorems 1.1.11 and 1.1.14 and Corollary 1.1.15.

Theorem 1.2.14. *If $\mathfrak{F}$ is a regular increasing set-valued function for relators on X, then*

(1) $\square_{\mathfrak{F}}$ is a refinement (self-increasing operation) for relators on X;

(2) for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square_{\mathfrak{F}}}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$ $\big(\mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} \subset \mathfrak{F}_{\mathcal{R}}\big)$.

Remark 1.2.15. Note that because of Remark 1.1.16 the corresponding duals of the above results are also true.

Moreover, as an immediate consequence of Theorems 1.1.11 and 1.2.4, we can also state

Theorem 1.2.16. *If $\Diamond$ is a refinement for relators on X, then $\Diamond = \square_{\Diamond}$. And thus, in particular, $\Diamond$ is regular.*

Somewhat more generally, we can also easily prove the following

Theorem 1.2.17. *If $\Diamond$ is an increasing operation for relators on X, then*

(1) $\mathcal{R}^{\square_{\Diamond}} \subset \mathcal{R}^{\Diamond}$ for every relator $\mathcal{R}$ on X whenever $\Diamond$ is expansive;

(2) $\mathcal{R}^{\Diamond} \subset \mathcal{R}^{\square_{\Diamond}}$ for every relator $\mathcal{R}$ on X whenever $\Diamond$ is idempotent.

Proof. If $S \in \mathcal{R}^{\square_{\Diamond}}$ and $\Diamond$ is expansive, then by the corresponding definitions we have $S \in \{S\}^{\Diamond} \subset \mathcal{R}^{\Diamond}$. Therefore, the assertion (1) holds.

While, $\Diamond$ is idempotent, then in particular we have $\big(\mathcal{R}^{\Diamond}\big)^{\Diamond} \subset \mathcal{R}^{\Diamond}$ for every relator $\mathcal{R}$ on X. And hence, by Theorem 1.2.7, the assertion (2) follows immediately.

Remark 1.2.18. Note that, by Theorem 1.2.7, the converse of the assertion (1) is also true.

Moreover, by Theorem 1.2.9, $\Diamond$ is regular if and only if $\mathcal{R}^{\Diamond} = \big(\mathcal{R}^{\square_{\Diamond}}\big)^{\Diamond}$ or equivalently $\big(\mathcal{R}^{\square_{\Diamond}}\big)^{\Diamond} \subset \mathcal{R}^{\Diamond}$.

1.3 Comparison of unary operations

Definition 1.3.1. If $\Diamond$ and $\square$ are unary operations for relators on X, then we say that:

(1) $\square$ is $\Diamond$-dominating if $\mathcal{R}^{\Diamond} \subset \mathcal{R}^{\square}$ for every relator $\mathcal{R}$ on X;

(2) $\square$ is $\Diamond$-invariant if $\mathcal{R}^{\square} = \mathcal{R}^{\square\Diamond}$ for every relator $\mathcal{R}$ on X;

(3) $\square$ is $\Diamond$-absorbing if $\mathcal{R}^{\square} = \mathcal{R}^{\Diamond\square}$ for every relator $\mathcal{R}$ on X;

(4) $\square$ is $\Diamond$-compatible if $\mathcal{R}^{\square\Diamond} = \mathcal{R}^{\Diamond\square}$ for every relator $\mathcal{R}$ on X.

Remark 1.3.2. In particular, the operation $\Box$ will be called inversion compatible if $\big(\mathcal{R}^{\Box}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{\Box}$ for every relator $\mathcal{R}$ on X.

Moreover, by Definition 1.1.5 we can at once see that $\Box$ is $\Diamond$-invariant if and only if $\mathcal{R}^{\Box}$ is $\Diamond$-fine for every relator $\mathcal{R}$ on X, and $\Box$ is $\Diamond$-absorbing if and only if $\mathcal{R}$ and $\mathcal{R}^{\Diamond}$ are $\Box$-equivalent for every relator $\mathcal{R}$ on X.

Now, as some useful consequences of the corresponding definitions, we can also prove the following theorems.

Theorem 1.3.3. *If $\Diamond$ is an expansive and $\Box$ is a $\Diamond$-dominating idempotent operation for relators on X, then $\Box$ is $\Diamond$-invariant.*

Proof. For every relator $\mathcal{R}$ on X, we have $\mathcal{R}^{\Box} \subset \mathcal{R}^{\Box\Diamond} \subset \mathcal{R}^{\Box\Box} = \mathcal{R}^{\Box}$, and hence $\mathcal{R}^{\Box} = \mathcal{R}^{\Box\Diamond}$.

Theorem 1.3.4. *If $\Diamond$ is an expansive and $\Box$ is a $\Diamond$-dominating modification for relators on X, then $\Box$ is $\Diamond$-absorbing.*

Proof. For any relator $\mathcal{R}$ on X, we have $\mathcal{R}^{\Box} \subset \mathcal{R}^{\Diamond\Box} \subset \mathcal{R}^{\Box\Box} = \mathcal{R}^{\Box}$, and hence $\mathcal{R}^{\Box} = \mathcal{R}^{\Diamond\Box}$.

Remark 1.3.5. Note that if $\Diamond$ is an expansive and $\Box$ is a $\Diamond$-dominating operation for relators on X, then $\Box$ is also expansive.

Moreover, if $\Diamond$ is an arbitrary (increasing) and $\Box$ is an expansive operation for relators on X such that $\mathcal{R}^{\Diamond\Box} \subset \mathcal{R}^{\Box}$ $\big(\mathcal{R}^{\Box\Diamond} \subset \mathcal{R}^{\Box}\big)$ for every relator $\mathcal{R}$ on X, then $\Box$ is $\Diamond$-dominating.

Note the following important consequences of Theorem 1.2.14

Theorem 1.3.6. *If $\mathfrak{F}$ and $\mathfrak{G}$ are regular set-valued functions for relators on X such that $\mathfrak{F} \circ \mathfrak{G}^{-1}$ is a function, then $\Box_{\mathfrak{F}}$ is $\Box_{\mathfrak{G}}$-dominating.*

Proof. Since $\mathfrak{G}$ is a regular set-valued function, by using the assertion (2) of Theorem 1.2.14, we have that $\mathfrak{G}_{\mathcal{R}^{\Box_{\mathfrak{G}}}} = \mathfrak{G}_{\mathcal{R}}$ for every relator $\mathcal{R}$ on X. Therefore, we also have that $\mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{G}}}} = \mathfrak{F}_{\mathcal{R}}$ for every relator $\mathcal{R}$ on X, because $\mathfrak{F} \circ \mathfrak{G}^{-1}$ is a function. Now, by using the assertion (2) of Theorem 1.2.14 again, we have that $\mathcal{R}^{\Box_{\mathfrak{G}}} \subset \mathcal{R}^{\Box_{\mathfrak{F}}}$ for every relator $\mathcal{R}$ on X which proves the theorem.

Remark 1.3.7. Note, that $\mathfrak{F} \circ \mathfrak{G}^{-1}$ is a function if and only if $\mathfrak{G}$ determines $\mathfrak{F}$.

Corollary 1.3.8. *If $\mathfrak{F}$ and $\mathfrak{G}$ are regular set-valued functions for relators on X such that $\mathfrak{F} \circ \mathfrak{G}^{-1}$ and $\mathfrak{G} \circ \mathfrak{F}^{-1}$ are both functions, then $\Box_{\mathfrak{F}} = \Box_{\mathfrak{G}}$.*

Corollary 1.3.9. *If $\mathfrak{F}$ and $\mathfrak{G}$ are regular set-valued functions for relators on X such that $\mathfrak{F} \circ \mathfrak{G}^{-1}$ is a function, then $\Box_{\mathfrak{F}}$ is $\Box_{\mathfrak{G}}$-invariant, $\Box_{\mathfrak{G}}$-absorbing and $\Box_{\mathfrak{G}}$-compatible.*

Hint. By Theorem 1.2.14 we have that $\Box_{\mathfrak{F}}$ and $\Box_{\mathfrak{G}}$ are refinements, therefore Theorems 1.3.3, 1.3.4 and 1.3.6 follow the required assertions.

2 Important set-valued functions and unary operations for relators

2.1 The most important set-valued functions for relators

Definition 2.1.1. If $\mathcal{R}$ is a relator on X, then for any $A, B \subset X$ and $x, y \in X$ we write:

(1) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ if $R(B) \subset A$ for some $R \in \mathcal{R}$;

(2) $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ if $R(B) \cap A \neq \emptyset$ for all $R \in \mathcal{R}$;

(3) $x \in \operatorname{int}_{\mathcal{R}}(A)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(A)$;

(4) $x \in \operatorname{cl}_{\mathcal{R}}(A)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(A)$;

(5) $y \in \sigma_{\mathcal{R}}(x)$ if $y \in \operatorname{int}_{\mathcal{R}}\big(\{x\}\big)$;

(6) $y \in \rho_{\mathcal{R}}(x)$ if $y \in \operatorname{cl}_{\mathcal{R}}\big(\{x\}\big)$.

The relations $\operatorname{Int}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$, and $\sigma_{\mathcal{R}}$ are called the proximal, the topological, and the infinitesimal interiors induced by $\mathcal{R}$ on X, respectively.

While the relations $\operatorname{Cl}_{\mathcal{R}}$, $\operatorname{cl}_{\mathcal{R}}$, and $\rho_{\mathcal{R}}$ are called the proximal, the topological, and the infinitesimal closures induced by $\mathcal{R}$ on X, respectively.

Remark 2.1.2. If $\mathcal{R}$ is, in particular, a uniformity, then the relations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ are just the inverses of the induced proximity $\delta_{\mathcal{R}}$ and strong inclusion $\Subset_{\mathcal{R}}$, respectively. (See [14, p. 12]).

Concerning the above relations, we shall only quote here the following theorems from [71].

Theorem 2.1.3. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

$$\operatorname{Cl}_{\mathcal{R}}(A) = \mathcal{P}(X) \setminus \operatorname{Int}_{\mathcal{R}}(X \setminus A) \qquad \textit{and} \qquad \operatorname{cl}_{\mathcal{R}}(A) = X \setminus \operatorname{int}_{\mathcal{R}}(X \setminus A).$$

Theorem 2.1.4. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

$$\operatorname{cl}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} R^{-1}(A) \qquad \textit{and} \qquad \rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = \Big(\bigcap \mathcal{R}\Big)^{-1}.$$

Remark 2.1.5. The proximal and infinitesimal closures are usually more convenient tools than the topological closures since we have $\operatorname{Cl}_{\mathcal{R}}^{-1} = \operatorname{Cl}_{\mathcal{R}^{-1}}$ and $\rho_{\mathcal{R}}^{-1} = \rho_{\mathcal{R}^{-1}}$.

Definition 2.1.6. If $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we write:

(1) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$;

(2) $A \in \reflectbox{$\tau$}_{\mathcal{R}}$ if $X \setminus A \notin \operatorname{Cl}_{\mathcal{R}}(A)$;

(3) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subset \operatorname{int}_{\mathcal{R}}(A)$;

(4) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subset A$;

(5) $A \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(A) \neq \emptyset$;

(6) $A \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) = X$.

The members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$, and $\mathcal{E}_{\mathcal{R}}$ are called the proximally open, the topologically open, and the fat subsets of $X(\mathcal{R})$, respectively.

While the members of the families $\reflectbox{$\tau$}_{\mathcal{R}}$, $\mathcal{F}_{\mathcal{R}}$, and $(\mathcal{D}_{\mathcal{R}})$ are called the proximally closed, the topologically closed, and the dense subsets of $X(\mathcal{R})$, respectively.

Remark 2.1.7. The fat sets are usually more important tools than the open sets. For instance, if $\prec$ is a preorder on X, then $\mathcal{T}_{\{\prec\}}$ and $\mathcal{E}_{\{\prec\}}$ are just the families of all ascending and residual subsets of the preordered set $X(\prec)$, respectively.

Moreover, if for instance $X = \mathbb{R}$ and R is a relation on X such that $R(x) =]-\infty, x] \cup \{x+1\}$ for all $x \in X$, then $\mathcal{T}_{\{R\}} = \{\emptyset, X\}$, but $\mathcal{E}_{\{R\}} \neq \{X\}$. Thus, the fat sets may be useful tools even if there is not non trivial topologically open sets [26]. In subsection 4.4 we shall investigate the topologically indiscrete relators.

Concerning the above families, we shall only quote here the following theorems from [71].

Theorem 2.1.8. *If $\mathcal{R}$ is a relator on X, then*

$$\tau_{\mathcal{R}} = \big\{ A \subset X : X \setminus A \in \tau_{\mathcal{R}} \big\} \qquad \textit{and} \qquad \mathcal{F}_{\mathcal{R}} = \big\{ A \subset X : X \setminus A \in \mathcal{T}_{\mathcal{R}} \big\}.$$

Theorem 2.1.9. *If $\mathcal{R}$ is a relator on X, then*

$$\mathcal{D}_{\mathcal{R}} = \big\{ A \subset X : X \setminus A \notin \mathcal{E}_{\mathcal{R}} \big\} = \big\{ A \subset X : \forall B \in \mathcal{E}_{\mathcal{R}} : A \cap B \neq \emptyset \big\}.$$

Remark 2.1.10. The proximally open sets are usually more convenient tools than the topologically open sets and the fat sets since we also have $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$.

Definition 2.1.11. If $\mathcal{R}$ is a relator on X, then we write

$$E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} \qquad \text{and} \qquad D_{\mathcal{R}} = \bigcup \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \big).$$

To reveal the relationship between the relation $\rho_{\mathcal{R}}$ and the set $E_{\mathcal{R}}$, we can prove the following theorem and remark.

Theorem 2.1.12. *If $\mathcal{R}$ is a relator on X, then*

$$E_{\mathcal{R}} = \bigcap_{x \in X} \rho_{\mathcal{R}}^{-1}(x) \qquad \textit{and} \qquad D_{\mathcal{R}} = X \setminus E_{\mathcal{R}}.$$

Proof. By the corresponding definitions and Theorem 2.1.4, we have

$$E_{\mathcal{R}} = \bigcap_{A \in \mathcal{E}_{\mathcal{R}}} A = \bigcap_{x \in X} \bigcap_{R \in \mathcal{R}} R(x) = \bigcap_{x \in X} \Big(\bigcap_{R \in \mathcal{R}} R \Big)(x) = \bigcap_{x \in X} \rho_{\mathcal{R}}^{-1}(x).$$

Moreover, by the corresponding definitions and Theorem 2.1.9, we also have

$$D_{\mathcal{R}} = \bigcup_{A \notin \mathcal{D}_{\mathcal{R}}} A = \bigcup_{X \setminus A \in \mathcal{E}_{\mathcal{R}}} A = \bigcup_{B \in \mathcal{E}_{\mathcal{R}}} X \setminus B = X \setminus \bigcap_{B \in \mathcal{E}_{\mathcal{R}}} B = X \setminus E_{\mathcal{R}},$$

where it is tacitly assumed that $A \subset X$.

Remark 2.1.13. Note that if R is a relation and $\mathcal{R}$ is a relator on X, then by Theorems 2.1.12 and 2.1.4 we have

$$E_{\{R\}} = \bigcap_{x \in X} \rho_{\{R\}}^{-1}(x) = \bigcap_{x \in X} R(x) \qquad \text{and} \qquad E_{\mathcal{R}} = \bigcap_{x \in X} \rho_{\mathcal{R}}^{-1}(x) = E_{\{\rho_{\mathcal{R}}^{-1}\}}.$$

Definition 2.1.14. A function x of a preordered set Γ into a set X is called a Γ-net in X. The Γ-net x is said to be fatly (densely) in a subset A of X if $x^{-1}(A)$ is a fat (dense) subset of Γ.

If $\mathcal{R}$ is a relator on X, then for any Γ-nets x and y in X and $a \in X$ we write:

(1) $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$ if the net (y, x) is fatly (densely) in each $R \in \mathcal{R}$;

(2) $a \in \lim_{\mathcal{R}}(x)$ $\big(a \in \operatorname{adh}_{\mathcal{R}}(x)\big)$ if $a_{\Gamma} \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(a_{\Gamma} \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$, where $a_{\Gamma} = \Gamma \times \{a\}$.

Remark 2.1.15. Note that the above definitions would actually allow Γ to be an arbitrary relator space.

However, the present generality is sufficient for several purposes since as a slight extension of [64, Theorem 3.1] we have the following

Theorem 2.1.16. *If $\mathcal{R}$ is a relator on X, then for any $A, B \subset X$, we have $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ if and only if there exist nets x and y in A and B, respectively, such that $y \in \operatorname{Lim}_{\mathcal{R}}(x) \big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$.*

Hence, it is clear that in particular we also have

Corollary 2.1.17. *If $\mathcal{R}$ is a relator on X, then for any $a \in X$ and $A \subset X$, we have $a \in \operatorname{cl}_{\mathcal{R}}(A)$ if and only if there exists a net x in A such that $a \in \lim_{\mathcal{R}}(x)$ $\big(a \in \operatorname{adh}_{\mathcal{R}}(x)\big)$.*

In this respect, it is also worth mentioning that as a slight extension of [64, Theorem 3.10] we also have

Theorem 2.1.18. *If $\mathcal{R}$ is a relator on X, then for any Γ-net x in X we have*

$$\lim_{\mathcal{R}}(x) = \bigcap_{A \in \mathcal{D}_{\Gamma}} \operatorname{cl}_{\mathcal{R}}\big(x[A]\big) \qquad \textit{and} \qquad \operatorname{adh}_{\mathcal{R}}(x) = \bigcap_{A \in \mathcal{E}_{\Gamma}} \operatorname{cl}_{\mathcal{R}}\big(x[A]\big).$$

Remark 2.1.19. In connection with Definition 2.1.14, it should also be noted that the family of all nets in the set X is not, even in the case of a simpler definition of nets, a well-defined collection.

Therefore, by defining the limit and adherence relations induced by a relator $\mathcal{R}$ on X, we should rather make some cardinality restrictions on the domains of nets considered in X than to allow them to be arbitrary relator spaces.

Even so, these operations will be considered like real set-valued functions. To avoid any confusions, we shall not use these for any conclusions.

Concerning the basic tools defined up till now, we shall mainly need here the following

Theorem 2.1.20. Int, int, *σ, τ, $\tau\!\!\!\!-$, $\mathcal{E}$, and D are normal increasing set-valued functions for relators on X.*

Moreover, Lim, Adh, lim, adh, Cl, cl, *ρ, $\mathcal{D}$, and E are normal decreasing set-valued functions for relators on X.*

Hint. To prove the monotonicity and the normality of the set-valued functions E and D, note that for every relator $\mathcal{R}$ on X we have

$$E_{\mathcal{R}} = \bigcap_{A \in \mathcal{E}_{\mathcal{R}}} A = \bigcap_{A \in \bigcup\limits_{R \in \mathcal{R}} \mathcal{E}_{\{R\}}} A = \bigcap_{R \in \mathcal{R}} \bigcap_{A \in \mathcal{E}_{\{R\}}} A = \bigcap_{R \in \mathcal{R}} E_{\{R\}}$$

and

$$D_{\mathcal{R}} = X \setminus E_{\mathcal{R}} = X \setminus \bigcap_{R \in \mathcal{R}} E_{\{R\}} = \bigcup_{R \in \mathcal{R}} \big(X \setminus E_{\{R\}}\big) = \bigcup_{R \in \mathcal{R}} D_{\{R\}}.$$

Remark 2.1.21. Later we shall see that the increasing set-valued functions $\mathcal{T}$ and $\mathcal{F}$ are not even regular in general.

Therefore, if $\mathcal{R}$ is a relator on X, then in general there does not exist a largest relator $\mathcal{R}^{\Box}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\Box}}$ $\big(\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\Box}}\big)$.

This reveals a further serious disadvantage of the topologically open sets in comparison to the proximally open sets and the fat sets.

By the required definitions, we have the following

Theorem 2.1.22. *If $\mathfrak{F}$ and $\mathfrak{G}$ are set-valued functions for relators on X such that*

$$(\mathfrak{G}, \mathfrak{F}) \in \big\{(\mathrm{Int}, \mathrm{int}), (\mathrm{Cl}, \mathrm{cl}), (\mathrm{int}, \mathcal{E}), (\mathrm{cl}, \mathcal{D}), (\mathrm{int}, \sigma), (\mathrm{cl}, \rho), (\mathcal{E}, E), (\mathcal{D}, D)\big\},$$

then $\mathfrak{G}$ determines $\mathfrak{F}$.

Remark 2.1.23. Note, that for instance quite similarly Lim determines Cl, Cl determines lim.

By the required theorems, we can also easily prove the following

Theorem 2.1.24. *If $\mathfrak{F}$ and $\mathfrak{G}$ are set-valued functions for relators on X such that*

$$(\mathfrak{G}, \mathfrak{F}) \in \big\{(\mathrm{Int}, \mathrm{Cl}), (\mathrm{int}, \mathrm{cl}), (\mathcal{E}, \mathcal{D}), (E, D)\big\},$$

then $\mathfrak{G}$ and $\mathfrak{F}$ determines each other.

Remark 2.1.25. Note, that by the above two theorems we have that for instance Int determines cl and $\mathcal{D}$ determines E.

2.2 The most important unary operations for relators

Definition 2.2.1. If $\mathcal{R}$ is a relator on X, then the relators

$$\begin{aligned}
\mathcal{R}^{*} &= \big\{S \subset X^2 : \exists R \in \mathcal{R} : R \subset S\big\},\\
\mathcal{R}^{\#} &= \big\{S \subset X^2 : \forall A \subset X : A \in \mathrm{Int}_{\mathcal{R}}\big(S(A)\big)\big\},\\
\mathcal{R}^{\wedge} &= \big\{S \subset X^2 : \forall x \in X : x \in \mathrm{int}_{\mathcal{R}}\big(S(x)\big)\big\},\\
R^{\triangle} &= \big\{S \subset X^2 : \forall x \in X : S(x) \in \mathcal{E}_{\mathcal{R}}\big\}
\end{aligned}$$

are called the uniform, the proximal, the topological, and the paratopological refinements of $\mathcal{R}$, respectively. Moreover, the relators

$$\mathcal{R}^{\bullet} = \big\{\rho_{\mathcal{R}}^{-1}\big\}^{*} \qquad \text{and} \qquad \mathcal{R}^{\blacktriangle} = \big\{X \times E_{\mathcal{R}}\big\}^{*}$$

are called the infinitesimal and the parainfinitesimal refinements of $\mathcal{R}$, respectively.

Remark 2.2.2. Our notations of the paratopological and the infinitesimal refinements differ here from those of Á. Száz [73] in order that, analogously to $\mathcal{R}^{\vee} = \big(\mathcal{R}^{\wedge}\big)^{-1}$, we could also write $\mathcal{R}^{\triangledown} = \big(\mathcal{R}^{\vartriangle}\big)^{-1}$ and $\mathcal{R}^{\blacktriangledown} = \big(\mathcal{R}^{\blacktriangle}\big)^{-1}$.

Definition 2.2.3. If $\mathcal{R}$ is a relator on X, then the relator $\mathcal{R}^{\bigstar} = \{\rho_{\mathcal{R}}^{-1}\}^{\vartriangle}$ is called the ultrainfinitesimal extension of $\mathcal{R}$.

Remark 2.2.4. The parainfinitesimal refinement and the ultrainfinitesimal extension of relators on X were first investigated in [52].

Simple applications of the corresponding definitions and Remark 2.1.13 give the following

Theorem 2.2.5. *If R is a relation on X, then*

(1) $\big\{R\big\}^{*} = \big\{R\big\}^{\#} = \big\{R\big\}^{\wedge} = \big\{R\big\}^{\bullet}$; *(2)* $\big\{R\big\}^{\vartriangle} = \big\{R\big\}^{\bigstar} = \Big(R \circ X^{X}\Big)^{*}$;

(3) $\big\{R\big\}^{\blacktriangle} = \Big\{X \times \bigcap_{x \in X} R(x)\Big\}^{*}$.

Hint. Note that if $S \in \{R\}^{\vartriangle}$, then for each $x \in X$ we have $S(x) \in \mathcal{E}_{\{R\}}$. Therefore, there exists $f_x \in X$ such that $R\left(f_x\right) \subset S(x)$. Hence, we can see that $R \circ f \subset S$ for some $f \in X^{X}$, and thus $S \in \big(R \circ X^{X}\big)^{*}$. Therefore, $\big\{R\big\}^{\vartriangle} \subset \big(R \circ X^{X}\big)^{*}$. The converse inclusion can be proved quite similarly.

From the above theorem, it is clear that in particular we have

Corollary 2.2.6. *The operations given in Definitions 2.2.1 and 2.2.3 are stable.*

Hint. If $\mathcal{R} = \{X^2\}$ and $S \in \mathcal{R}^{*}$, then $X^2 \subset S$. It follows that $\mathcal{R}^{*} = \{X^2\}$, that is $*$ is a stable unary operation for relators on X.

Therefore, we may also naturally introduce the following

Definition 2.2.7. If $\mathcal{R}$ is a relator on X, then the relator $\mathcal{R}^{\blacklozenge}$, given by

$$\mathcal{R}^{\blacklozenge} = \big\{X^2\big\} \quad \text{if} \quad \mathcal{R} = \big\{X^2\big\} \qquad \text{and} \qquad \mathcal{R}^{\blacklozenge} = \mathcal{P}\big(X^2\big) \quad \text{if} \quad \mathcal{R} \neq \big\{X^2\big\},$$

is called the ultimate stable refinement of $\mathcal{R}$.

Theorem 2.2.8. *The operations $*$ and $\blacklozenge$ are normal.*

Proof. The normality of $*$ is followed by [73, Remark 1.11], and normality of $\blacklozenge$ is also quite obvious.

The appropriateness of the corresponding definitions is already apparent from the following theorems.

Theorem 2.2.9. *We have the following equalities.*

(1) $* = \Box_{\mathrm{Lim}} = \Box_{\mathrm{Adh}}$*;*
(2) $\# = \Box_{\mathrm{Int}} = \Box_{\mathrm{Cl}}$*;*
(3) $\wedge = \Box_{\mathrm{lim}} = \Box_{\mathrm{adh}}$*;*
(4) $\wedge = \Box_{\mathrm{int}} = \Box_{\mathrm{cl}}$*;*
(5) $\triangle = \Box_{\mathcal{E}} = \Box_{\mathcal{D}}$*;*
(6) $\bullet = \Box_{\rho}$*.*
(7) $\blacktriangle = \Box_{E} = \Box_{D}$*;*
(8) $\blacklozenge = \Box_{\blacklozenge}$*.*

Hint. By using Theorem 2.1.24, Corollary 1.3.8 follows the following equalities: $\Box_{\mathrm{Int}} = \Box_{\mathrm{Cl}}$, $\Box_{\mathrm{int}} = \Box_{\mathrm{cl}}$, $\Box_{\mathcal{E}} = \Box_{\mathcal{D}}$, $\Box_{E} = \Box_{D}$.

Since we can state similar assertions for Lim, Adh, lim and adh, and since the corresponding definitions, the other equalities of the theorem are also quite obvious.

Theorem 2.2.10. *The operations given in Definitions 2.2.1 and 2.2.7 are stable refinements for relators on X such that, for any relator $\mathcal{R}$ on X,*

(1) $\mathcal{R}^{*}$ *is the largest relator on X such that* $\mathrm{Lim}_{\mathcal{R}} = \mathrm{Lim}_{\mathcal{R}^{*}}$*, or equivalently* $\mathrm{Adh}_{\mathcal{R}} = \mathrm{Adh}_{\mathcal{R}^{*}}$*;*

(2) $\mathcal{R}^{\#}$ *is the largest relator on X such that* $\mathrm{Int}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R}^{\#}}$*, or equivalently* $\mathrm{Cl}_{\mathcal{R}} = \mathrm{Cl}_{\mathcal{R}^{\#}}$*;*

(3) $\mathcal{R}^{\wedge}$ *is the largest relator on X such that* $\mathrm{lim}_{\mathcal{R}} = \mathrm{lim}_{\mathcal{R}^{\wedge}}$*, or equivalently* $\mathrm{adh}_{\mathcal{R}} = \mathrm{adh}_{\mathcal{R}^{\wedge}}$*;*

(4) $\mathcal{R}^{\wedge}$ *is the largest relator on X such that* $\mathrm{int}_{\mathcal{R}} = \mathrm{int}_{\mathcal{R}^{\wedge}}$*, or equivalently* $\mathrm{cl}_{\mathcal{R}} = \mathrm{cl}_{\mathcal{R}^{\wedge}}$*;*

(5) $\mathcal{R}^{\triangle}$ *is the largest relator on X such that* $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{\triangle}}$*, or equivalently* $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{\triangle}}$*;*

(6) $\mathcal{R}^{\bullet}$ *is the largest relator on X such that* $\rho_{\mathcal{R}} = \rho_{\mathcal{R}^{\bullet}}$*.*

(7) $\mathcal{R}^{\blacktriangle}$ *is the largest relator on X such that* $E_{\mathcal{R}} = E_{\mathcal{R}^{\blacktriangle}}$*, or equivalently* $D_{\mathcal{R}} = D_{\mathcal{R}^{\blacktriangle}}$*.*

(8) $\mathcal{R}^{\blacklozenge}$ *is the largest relator on X such that* $\mathcal{R} = \mathcal{R}^{\blacklozenge}$ *whenever* $\mathcal{R} = \{X^2\}$*.*

Proof. The stability of the required operations was proved in Corollary 2.2.6. Since Theorems 2.1.20 and 2.2.8, moreover 1.2.12, 1.2.14 and 2.2.9 the other assertions are quite obvious.

To fulfil our engagement in Remark 2.1.19, note that $* = \Box_{*}$, and since Theorem 2.2.8 we have that $*$ is also a refinement.

From Theorem 1.2.9, and its dual, we can also at once get the following

Theorem 2.2.11. *(1)* Lim *and* Adh *are $*$-decreasing set-valued functions for relators on X.

(2) Int *is a* $\#$*-increasing and* Cl *is a* $\#$*-decreasing set-valued function for relators on* X*.*

(3) lim *and* adh *are* $\wedge$*-decreasing set-valued functions for relators on* X*.*

(4) int *is a* $\wedge$*-increasing and* cl *is a* $\wedge$*-decreasing set-valued function for relators on* X*.*

(5) $\mathcal{E}$ *is a* $\vartriangle$*-increasing and* $\mathcal{D}$ *is a* $\vartriangle$*-decreasing set-valued function for relators on* X*.*

(6) ρ *is a* $\bullet$*-decreasing set-valued function for relators on* X*.*

(7) E *is a* $\blacktriangle$*-decreasing and* D *is a* $\blacktriangle$*-increasing set-valued function for relators on* X*.*

(8) $\blacklozenge$ *is a self-increasing unary operation for relators on* X*.*

By Definition 1.1.9 and its dual we have the following

Theorem 2.2.12. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are relators on* X*, then we have*

(1) $\mathcal{S} \subset \mathcal{R}^* \iff \operatorname{Lim}_{\mathcal{R}} \subset \operatorname{Lim}_{\mathcal{S}} \iff \operatorname{Adh}_{\mathcal{R}} \subset \operatorname{Adh}_{\mathcal{S}}$

(2) $\mathcal{S} \subset \mathcal{R}^\# \iff \operatorname{Int}_{\mathcal{S}} \subset \operatorname{Int}_{\mathcal{R}} \iff \operatorname{Cl}_{\mathcal{R}} \subset \operatorname{Cl}_{\mathcal{S}}$

(3) $\mathcal{S} \subset \mathcal{R}^\wedge \iff \operatorname{lim}_{\mathcal{R}} \subset \operatorname{lim}_{\mathcal{S}} \iff \operatorname{adh}_{\mathcal{R}} \subset \operatorname{adh}_{\mathcal{S}}$

(4) $\mathcal{S} \subset \mathcal{R}^\wedge \iff \operatorname{int}_{\mathcal{S}} \subset \operatorname{int}_{\mathcal{R}} \iff \operatorname{cl}_{\mathcal{R}} \subset \operatorname{cl}_{\mathcal{S}}$

(5) $\mathcal{S} \subset \mathcal{R}^\vartriangle \iff \mathcal{E}_{\mathcal{S}} \subset \mathcal{E}_{\mathcal{R}} \iff \mathcal{D}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{S}}$

(6) $\mathcal{S} \subset \mathcal{R}^\bullet \iff \rho_{\mathcal{R}} \subset \rho_{\mathcal{S}}$

(7) $\mathcal{S} \subset \mathcal{R}^\blacktriangle \iff E_{\mathcal{R}} \subset E_{\mathcal{S}} \iff D_{\mathcal{S}} \subset D_{\mathcal{R}}$

(8) $\mathcal{S} \subset \mathcal{R}^\blacklozenge \iff \mathcal{S}^\blacklozenge \subset \mathcal{R}^\blacklozenge$

The following theorem was proved in [73], but now we can give a different proof.

Theorem 2.2.13. *If* $\mathcal{R}$ *is a relator on* X*, then*

$$\mathcal{R} \subset \mathcal{R}^* \subset \mathcal{R}^\# \subset \mathcal{R}^\wedge \subset \mathcal{R}^\vartriangle \cap \mathcal{R}^\bullet .$$

Proof. The expansivity of the operation $*$ gives the first inclusion. On the other hand, since $B \in \operatorname{Int}_{\mathcal{R}}(A)$ if and only if $R(B) \subset A$ for some $R \in \mathcal{R}^*$, we can see that $*$ determines Int, therefore Theorems 2.2.9 and 1.3.6 give the second inclusion. Finally, by Theorem 2.1.22, Theorem 1.3.6 follows the other inclusions.

By using the new unary operations in [52] we can enlarge the above chain of inclusions. For this we need the following theorem of [52].

Theorem 2.2.14. *If* $\mathcal{R}$ *is a relator on* X*, then*

$$\mathcal{R}^\blacktriangle = \mathcal{R}^{\vartriangle\bullet} \qquad \textit{and} \qquad \mathcal{R}^\bigstar = \mathcal{R}^{\bullet\vartriangle} .$$

Proof. $S \in (X \times \mathcal{E}_{\mathcal{R}})^{\wedge}$ if and only if for all $x \in X$ there exists an $A \in \mathcal{E}_{\mathcal{R}}$ such that $A \subset S(X)$, that is $S(x) \in \mathcal{E}_{\mathcal{R}}$ for all $x \in X$. Therefore $\mathcal{R}^{\vartriangle} = (X \times \mathcal{E}_{\mathcal{R}})^{\wedge}$, hence since the corresponding definition and Theorems 2.2.13, 1.3.4 and 2.2.5

$$\mathcal{R}^{\vartriangle \bullet} = (X \times \mathcal{E}_{\mathcal{R}})^{\wedge \bullet} = (X \times \mathcal{E}_{\mathcal{R}})^{\bullet} = \{X \times E_{\mathcal{R}}\}^{*} = \mathcal{R}^{\blacktriangle}.$$

On the other hand, by the corresponding definitions and Theorems 2.2.13 and 1.3.4, we evidently have that

$$\mathcal{R}^{\bullet \vartriangle} = \{\rho_{\mathcal{R}}^{-1}\}^{* \vartriangle} = \{\rho_{\mathcal{R}}^{-1}\}^{\vartriangle} = \mathcal{R}^{\bigstar}.$$

Corollary 2.2.15. $\bigstar$ *is a stable extension for relators on* X.

Theorem 2.2.16. *If* $\mathcal{R}$ *is a relator on* X, *then*

$$\mathcal{R}^{\vartriangle} \cup \mathcal{R}^{\bullet} \subset \mathcal{R}^{\bigstar} \subset \mathcal{R}^{\blacktriangle} \subset \mathcal{R}^{\blacklozenge}.$$

Proof. Since the expansivity of $\vartriangle$, we have $\mathcal{R}^{\bullet} \subset \mathcal{R}^{\bullet \vartriangle} = \mathcal{R}^{\bigstar}$. On the other hand by using the expansivity of $\bullet$ and the increasingness of $\vartriangle$ we have $\mathcal{R}^{\vartriangle} \subset \mathcal{R}^{\bullet \vartriangle} = \mathcal{R}^{\bigstar}$, that is the first inclusion is proved.

Quite similarly, we can see that $\mathcal{R}^{\bullet} \subset \mathcal{R}^{\blacktriangle}$. Since $\bullet$ and $\blacktriangle$ are refinements, by using 1.3.4 we have that $\mathcal{R}^{\bigstar} \subset \mathcal{R}^{\bigstar \bullet} = \mathcal{R}^{\bullet \blacktriangle} = \mathcal{R}^{\blacktriangle}$, that is the second inclusion is also proved.

Finally, by Theorems 2.1.22 and 2.2.9 and Corollary 1.3.6, we can see that the third inclusion also holds.

The following example shows that unfortunately $\mathcal{R}^{\vartriangle}$ and $\mathcal{R}^{\bullet}$ are, in general incomparable, and $\bigstar$ is not, in general, idempotent.

Example 2.2.17. If $X = \{1, 2, 3\}$ and $R_i \subset X^2$ for all $i = 1, 2$ such that:

$$R_1(1) = \{1\}, \qquad R_1(2) = \{2, 3\}, \qquad R_1(3) = \{2, 3\};$$
$$R_2(1) = \{1, 3\}, \qquad R_2(2) = \{2\}, \qquad R_2(3) = \{1, 3\};$$

then $\mathcal{R} = \{R_1, R_2\}$ is an equivalence relator on X such that:

(1) $\mathcal{R}^{\bullet} \not\subset \mathcal{R}^{\vartriangle}$; (2) $\mathcal{R}^{\vartriangle} \not\subset \mathcal{R}^{\bullet}$; (3) $\left(\mathcal{R}^{\bigstar}\right)^{\bigstar} \not\subset \mathcal{R}^{\bigstar}$.

Note that $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = \Delta_X$, and hence $\Delta_X \in \left\{\rho_{\mathcal{R}}^{-1}\right\}^{*} = \mathcal{R}^{\bullet}$. But, $\{3\} \notin \mathcal{E}_{\mathcal{R}}$, and hence $\Delta_X \notin \mathcal{R}^{\vartriangle}$. Moreover, if $S = X \times \{2\}$, then $S \in \mathcal{R}^{\vartriangle}$, since $\{2\} \in \mathcal{E}_{\mathcal{R}}$. But, $S \notin \mathcal{R}^{\bullet}$, since $\mathcal{R}^{\bullet} = \left\{\Delta_X\right\}^{*}$ and $\Delta_X \not\subset S$. Therefore, the assertions (1) and (2) are true.

On the other hand, $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} = \emptyset$. Therefore,

$$\left(\mathcal{R}^{\bullet \vartriangle}\right)^{\bullet \vartriangle} = \mathcal{R}^{\bullet \blacktriangle \vartriangle} = \mathcal{R}^{\blacktriangle} = \left\{X \times E_{\mathcal{R}}\right\}^{*} = \left\{\emptyset\right\}^{*} = \mathcal{P}\left(X^2\right).$$

But,

$$\mathcal{R}^{\bullet \vartriangle} = \left\{\Delta_X\right\}^{* \vartriangle} = \left\{\Delta_X\right\}^{\vartriangle} = \left\{\Delta_X \circ X^X\right\}^{*} = \left\{X^X\right\}^{*}.$$

Therefore, the assertion (3) is also true.

Therefore, by Theorems 1.3.3 and 1.3.4, we can only state the following

Theorem 2.2.18. *If* $\Diamond, \Box \in \{*, \#, \wedge, \circ, \blacktriangle, \blacklozenge\}$*, where* $\circ = \triangle$ *or* $\bullet$ *or* $\bigstar$*, such that* $\Diamond$ *precedes* $\Box$ *in the above list, then* $\Box$ *is both* $\Diamond$*-invariant and* $\Diamond$*-absorbing.*

Moreover $\bigstar$ *is* $\bullet$*-absorbing, and* $\bigstar$ *is* $\triangle$*-invariant.*

The assertions of the above theorem , except for which refer to the $\bigstar$ operation can be easily followed by Theorem 2.1.22 and 2.2.9 and Corollary 1.3.9

Remark 2.2.19. From the equality $\mathcal{R}^{\bullet} = \left\{\rho_{\mathcal{R}}^{-1}\right\}^{*}$, by using the above theorem, we can infer that $\mathcal{R}^{\blacktriangle} = \left\{\rho_{\mathcal{R}}^{-1}\right\}^{\blacktriangle}$ and $\mathcal{R}^{\blacklozenge} = \left\{\rho_{\mathcal{R}}^{-1}\right\}^{\blacklozenge}$.

In addition to Theorem 2.2.18, it is also worth proving the following

Theorem 2.2.20. *The operations* $*, \#$ *and* $\bullet$ *are inversion compatible. While, the operation* $\blacklozenge$ *is both inversion invariant and inversion absorbing.*

Hint. Everything stated here is quite obvious, except that the operation $\#$ is inversion compatible. The latter fact was first established in [64] and [73].

Remark 2.2.21. The above theorem shows in particular that the operation $\blacklozenge$ is also inversion compatible, and we have $\mathcal{R}^{-1} \subset \mathcal{R}^{\blacklozenge}$ for every relator $\mathcal{R}$ on X.

Remark 2.2.22. It can be shown that the operations $\wedge, \triangle, \bigstar$, and $\blacktriangle$ are not, in general, inversion compatible. Moreover, the operations $\#, \wedge, \triangle, \bullet, \bigstar$, and $\blacktriangle$ are not, in general, normal.

Theorem 2.2.23. *If* $\mathcal{R}$ *is a relator on* X*, then*

(1) $\mathcal{R}^{\bullet} = \mathcal{R}^{\vee\vee}$*;* *(2)* $\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\triangledown\triangledown} = \mathcal{R}^{\blacktriangledown\blacktriangledown}$*.*

Proof. For each $x \in X$, let V_x be a relation on X such that

$$V_x(y) = X \text{ if } y \in \rho_{\mathcal{R}}(x) \qquad \text{and} \qquad V_x(y) = X \setminus \{x\} \text{ if } y \in X \setminus \rho_{\mathcal{R}}(x).$$

Then, by the corresponding definitions, it is clear that $V_x \in \mathcal{R}^{\wedge}$. Namely, if $y \in \rho_{\mathcal{R}}(x)$, then $R(y) \subset X = V_x(y)$ for any $R \in \mathcal{R}$. While, if $y \in X$ such that $y \notin \rho_{\mathcal{R}}(x)$, then since $\rho_{\mathcal{R}}(x) = \operatorname{cl}_{\mathcal{R}}\big(\{x\}\big)$ there exists an $R \in \mathcal{R}$ such that $R(y) \cap \{x\} = \emptyset$, and hence $R(y) \subset X \setminus \{x\} = V_x(y)$.

Now, since $V_x \in \mathcal{R}^{\wedge}$, it is clear that $V_x^{-1} \in \mathcal{R}^{\wedge -1} = \mathcal{R}^{\vee}$. Moreover, we can easily see that

$$V_x^{-1}(x) = \rho_{\mathcal{R}}(x).$$

Namely, for any $y \in X$, we have $y \in V_x^{-1}(x)$, i.e., $x \in V_x(y)$ if and only if $y \in \rho_{\mathcal{R}}(x)$. Now, since $V_x^{-1} \in \mathcal{R}^{\vee}$ and $V_x^{-1}(x) \subset \rho_{\mathcal{R}}(x)$ for all $x \in X$, it is clear that

$$\rho_{\mathcal{R}} \in \mathcal{R}^{\vee\wedge},$$

and hence $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^{\vee\vee}$. Hence, by using the corresponding properties of the operations $*, -1$ and $\wedge$, we can infer that

$$\mathcal{R}^{\bullet} = \left\{\rho_{\mathcal{R}}^{-1}\right\}^{*} \subset \mathcal{R}^{\vee\vee *} = \mathcal{R}^{\vee\vee}.$$

Moreover, by Theorem 2.1.4, it is clear that $\mathcal{R} \subset \big\{\rho_{\mathcal{R}}^{-1}\big\}^{*}$. Hence, by using the corresponding properties of the operations $\wedge, -1$ and $*$, we can infer that

$$\mathcal{R}^{\vee\vee} \subset \big\{\rho_{\mathcal{R}}^{-1}\big\}^{*\vee\vee} = \big\{\rho_{\mathcal{R}}^{-1}\big\}^{*} = \mathcal{R}^{\bullet}.$$

Therefore, the first equality is true.

If in particular $\mathcal{R} = \big\{X^2\big\}$, then by Corollary 2.2.6 we have $\mathcal{R}^{\vartriangle} = \mathcal{R}$. Hence, it is clear that $\mathcal{R}^{\triangledown} = \mathcal{R}^{\vartriangle -1} = \mathcal{R}^{-1} = \mathcal{R}$ is also true. Therefore, we also have $\mathcal{R}^{\triangledown\vartriangle} = \mathcal{R}^{\vartriangle} = \mathcal{R} = \mathcal{R}^{\blacklozenge}$.

While, if $\mathcal{R} \neq \big\{X^2\big\}$, then there exist $R \in \mathcal{R}$ and $x, y \in X$ such that $y \notin R(x)$. Hence, it follows that $R(x) \subset X \setminus \{y\}$, and thus $X \setminus \{y\} \in \mathcal{E}_{\mathcal{R}}$. Therefore, under the notation $S = \big(X \setminus \{y\}\big) \times X$, we have

$$S^{-1} = X \times \big(X \setminus \{y\}\big) \in \mathcal{R}^{\vartriangle},$$

and hence $S \in \mathcal{R}^{\vartriangle -1} = \mathcal{R}^{\triangledown}$. Hence, since $S(y) = \emptyset$, it is clear that $\mathcal{E}_{\mathcal{R}^{\triangledown}} = \mathcal{P}(X)$. Therefore, $\mathcal{R}^{\triangledown\vartriangle} = \mathcal{P}\big(X^2\big) = \mathcal{R}^{\blacklozenge}$.

Thus, we have proved that $\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\triangledown\vartriangle}$. Hence, by Theorem 2.2.20, it is clear that we also have $\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\blacklozenge -1} = \mathcal{R}^{\triangledown\vartriangle -1} = \mathcal{R}^{\triangledown\triangledown}$. Moreover, by using Theorem 2.2.23 and some of the basic properties of corresponding refinements, we can easily see that

$$\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\blacktriangle\blacklozenge\bullet} = \mathcal{R}^{\blacktriangle\triangledown\vartriangle\bullet} = \mathcal{R}^{\blacktriangle\vartriangle -1\vartriangle\bullet} = \mathcal{R}^{\blacktriangle -1\blacktriangle} = \mathcal{R}^{\blacktriangledown\blacktriangle},$$

and thus $\mathcal{R}^{\blacklozenge} = \mathcal{R}^{\blacklozenge -1} = \mathcal{R}^{\blacktriangledown\blacktriangle -1} = \mathcal{R}^{\blacktriangledown\blacktriangledown}$ is also true.

Remark 2.2.24. The first statement of the above theorem was already proved by J. Mala [33, Theorem 1] and Á. Száz [73, Theorem 3.14]. However, our proof is more direct than the ones given by the above mentioned authors.

2.3 Some further useful unary operations for relators

Definition 2.3.1. If $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^{\infty} = \{R^{\infty} : R \in \mathcal{R}\} \qquad \text{and} \qquad \mathcal{R}^{\partial} = \{S \subset X^2 : S^{\infty} \in \mathcal{R}^{\infty}\}$$

are called the (direct) preorder modification and the inverse preorder refinement of $\mathcal{R}$, respectively.

Simple applications of the corresponding definitions give the following

Theorem 2.3.2. *∞ is an inversion compatible, normal and stable modification for relators on X such that, for every relator $\mathcal{R}$ on X, we have*

$$\mathcal{R}^{\infty} \subset \mathcal{R}^{*\infty} \subset \mathcal{R}^{\infty *} \subset \mathcal{R}^{*}.$$

Moreover, $\partial = \Box_{\infty}$ is an inversion compatible, normal refinement for relators on X such that, ∞ is ∂-absorbing and ∂ is ∞-absorbing, and for every relator $\mathcal{R}$ on X, we have

$$\mathcal{R}^{\partial} \cup \mathcal{R}^{*} \subset \mathcal{R}^{\partial *} \subset \mathcal{R}^{*\partial}.$$

In [54] we can find a false terminology for ∂, but this does not occur any difference since the following

Corollary 2.3.3. *If $\mathcal{R}$ is a relator on X and $\Box$ is $*$-dominating idempotent operation for relators on X, then $\mathcal{R}^{\Box\infty} \subset \mathcal{R}^{\Box}$.*

Moreover, if R is a preorder on X, then

$$R \in \mathcal{R}^{\Box} \iff R \in \mathcal{R}^{\Box\infty}.$$

Proof. Since the above theorem, by using Theorem 1.3.3 the first statement is quite obvious, and this inclusion follows one part of the equivalence.

For the proof of the other implication suppose that R is a preorder in $\mathcal{R}^{\Box}$. Since we have that $R = R^{\infty} \in \mathcal{R}^{\Box\infty}$ the theorem is proved.

Definition 2.3.4. If $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^{\sharp} = \mathcal{R}^{\#\partial} \qquad \text{and} \qquad \mathcal{R}^{\Yup} = \mathcal{R}^{\wedge\partial}$$

are called the superproximal refinement and the supertopological extension of $\mathcal{R}$, respectively.

Definition 2.3.5. If $\mathcal{R}$ is a relator on X, then the relator

$$\mathcal{R}^{\star} = \{S \subset X^2 : \sigma_{\{S\}} \subset \sigma_{\mathcal{R}}\}$$

is called the σ-infinitesimal refinement of $\mathcal{R}$.

By the corresponding definitions and Theorem 2.2.5, it is clear that in particular we also have

Theorem 2.3.6. *If R is a relation on X, then*

$$\{R\}^{\sharp} = \{R\}^{\Yup} = \{S \subset X^2 : R \subset S^{\infty}\}$$

and

$$\mathcal{R}^{\star} = \big\{S \subset X^2 : \forall x \in X : \operatorname{card}(S(x)) \leq 1 \implies \exists R \in \mathcal{R} : R(x) \subset S(x)\big\}.$$

Corollary 2.3.7. *Hence, in particular, it is clear that*

$$\{X^2\}^{\sharp} = \{X^2\}^{\Yup} = \{S \subset X^2 : X^2 = S^{\infty}\}$$

and if $\operatorname{card}(X) \geq 2$, *then*

$$\{X^2\}^{\star} = \{S \subset X^2 : \forall x \in X : \operatorname{card}(S(x)) \geq 2\}$$

Therefore, the operations $\sharp$, $\Yup$ and $\star$ are not, in general, stable. That is, these operations is not, in general, dominated by $\blacklozenge$.

The appropriateness of the above operations is apparent from the following

Theorem 2.3.8. *We have the following equalities.*

(1) $\sharp = \Box_{\scriptscriptstyle\mathcal{T}} = \Box_{\scriptscriptstyle\mathcal{F}}$; *(2)* $\star = \Box_{\sigma}$;

(3) $\Yup = \Box_{\mathcal{T}} = \Box_{\mathcal{F}}$.

Theorem 2.3.9. $\sharp$ *and* $\star$ *are refinements for relators on* X *such that, for any relator* $\mathcal{R}$ *os* X,

(1) $\mathcal{R}^{\sharp}$ *is the largest relator on* X *such that* ${\scriptstyle\mathcal{T}}_{\mathcal{R}} = {\scriptstyle\mathcal{T}}_{\mathcal{R}^{\sharp}}$*, or equivalently* ${\scriptstyle\mathcal{F}}_{\mathcal{R}} = {\scriptstyle\mathcal{F}}_{\mathcal{R}^{\sharp}}$;

(2) $\mathcal{R}^{\star}$ *is the largest relator on* X *such that* $\sigma_{\mathcal{R}} = \sigma_{\mathcal{R}^{\star}}$.

Proof. Since Theorems 2.1.20, 1.2.12, 1.2.14 and 2.3.8 the theorem is proved.

From Theorem 1.2.9, and its dual, we can also at once get the following

Theorem 2.3.10. *(1)* ${\scriptstyle\mathcal{T}}$ *and* ${\scriptstyle\mathcal{F}}$ *are* $\sharp$*-increasing set-valued functions for relators on* X.

(2) σ *is a* $\star$*-increasing set-valued function for relators on* X.

By Definition 1.1.9 and its dual we have the following

Theorem 2.3.11. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are relators on* X *then we have that*

(1) $\mathcal{S} \subset \mathcal{R}^{\sharp} \iff {\scriptstyle\mathcal{T}}_{\mathcal{S}} \subset {\scriptstyle\mathcal{T}}_{\mathcal{R}} \iff {\scriptstyle\mathcal{F}}_{\mathcal{S}} \subset {\scriptstyle\mathcal{F}}_{\mathcal{R}}$

(2) $\mathcal{S} \subset \mathcal{R}^{\star} \iff \sigma_{\mathcal{S}} \subset \sigma_{\mathcal{R}}$

By Theorem 2.3.8 with Theorem 1.2.7 we can get the following

Theorem 2.3.12. $\Yup$ *is an extension for relators on* X *such that, for any relators* $\mathcal{R}$ *and* $\mathcal{S}$ *on* X,

$$\mathcal{S} \subset \mathcal{R}^{\Yup} \iff \mathcal{T}_{\mathcal{S}} \subset \mathcal{T}_{\mathcal{R}} \iff \mathcal{F}_{\mathcal{S}} \subset \mathcal{F}_{\mathcal{R}}$$

Theorem 2.3.13. *If* $\mathcal{R}$ *is a relator on* X*, then* $\mathcal{R}^{\#} \subset \mathcal{R}^{\sharp}$, $\mathcal{R}^{\wedge} \subset \mathcal{R}^{\Yup}$ *and* $\mathcal{R}^{\wedge} \subset \mathcal{R}^{\star}$.

By Theorem 2.3.2 we have the following

Theorem 2.3.14. *If* $\mathcal{R}$ *is a relator on* X*, then* $\mathcal{R}^{\#\infty} = \mathcal{R}^{\sharp\infty}$ *and* $\mathcal{R}^{\wedge\infty} = \mathcal{R}^{\Yup\infty}$.

The following example shows that the operation $\Yup$ is not, in general idempotent. Therefore, the assertions (3) of Theorems 1.1.7 and 1.1.11, cannot, in general, hold for the operation $\Box = \Yup$.

Example 2.3.15. If $X = \{1,2,3\}$ and $\mathcal{R} = \big\{X^2\big\}$, then

(1) $\mathcal{T}_{\mathcal{R}^{\Yup}} \not\subset \mathcal{T}_{\mathcal{R}}$; (2) $\mathcal{R}^{\Yup\Yup} \not\subset \mathcal{R}^{\Yup}$; (3) $\mathcal{R}^{\Yup} \not\subset \mathcal{R}^{\blacklozenge}$.

Moreover, there is no largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$. And thus, there is no unary operation $\Box$ for relators on X such that $\mathcal{T}$ is $\Box$-increasing. That is, $\mathcal{T}$ is not a regular set-valued function for relators on X.

To prove the above assertions, for each $i = 1, 2, 3$, define $R_i \subset X^2$ such that:

$$\begin{array}{lll} R_1(1) = \{1,2\}, & R_1(2) = X, & R_1(3) = X; \\ R_2(1) = X, & R_2(2) = \{1,2\}, & R_2(3) = X; \\ R_3(1) = \{1,2\}, & R_3(2) = \{1,2\}, & R_3(3) = X. \end{array}$$

Then, by the definitions of open sets, we evidently have

$$\mathcal{T}_{\{R_i\}} = \{\emptyset, X\} = \mathcal{T}_{\mathcal{R}},$$

for each $i = 1, 2$. Hence, by Definition 1.2.1 and Theorem 2.3.8, it is clear that

$$R_1, R_2 \in \mathcal{R}^{\Box\mathcal{T}} = \mathcal{R}^{\curlywedge}.$$

Thus, in particular, the assertion (3) is true.

Moreover, now we can also easily see that

$$\mathcal{T}_{\{R_3\}} = \big\{\emptyset, \{1,2\}, X\big\} = \mathcal{T}_{\{R_1,R_2\}} \subset \mathcal{T}_{\mathcal{R}^{\curlywedge}}.$$

Therefore, the assertion (1) is true. Moreover, $R_3 \in \mathcal{R}^{\curlywedge\Box\mathcal{T}} = \mathcal{R}^{\curlywedge\curlywedge}$. But, since $\mathcal{T}_{\{R_3\}} \not\subset \mathcal{T}_{\mathcal{R}}$, it is clear that $R_3 \notin \mathcal{R}^{\Box\mathcal{T}} = \mathcal{R}^{\curlywedge}$. Therefore, the assertion (2) is also true.

Finally, to prove the remaining assertions, assume on the contrary that $\mathcal{S}$ is a largest relator on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$. Then, since $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\{R_i\}}$ for each $i = 1, 2$, we necessarily have $R_1, R_2 \in \mathcal{S}$. Hence, by the increasingness of $\mathcal{T}$, it follows that $\mathcal{T}_{\{R_1,R_2\}} \subset \mathcal{T}_{\mathcal{S}} = \mathcal{T}_{\mathcal{R}}$, and this is a contradiction. Hence, by Theorem 1.1.14, it is clear that there is no unary operation $\Box$ for relators on X such that $\mathcal{T}$ is $\Box$-increasing. That is, by Theorem 1.2.9, $\mathcal{T}$ is not a regular set-valued function for relators on X.

By Theorems 1.3.3 and 1.3.4 we can state the following

Theorem 2.3.16. *If $\Diamond \in \{*, \#\}$, then $\sharp$, $\curlywedge$ and $\star$ are both $\Diamond$-invariant and $\Diamond$-absorbing. Moreover, $\curlywedge$ and $\star$ are both $\wedge$-invariant and $\wedge$-absorbing.*

Proof. By Theorem 2.3.13 it will be sufficient to show that $\curlywedge$ is $\Diamond$-invariant if $\Diamond \in \{*, \#, \wedge\}$.

$R \in \mathcal{R}^{\curlywedge\wedge}$ means that for all $x \in X$ there exists an $S_x \in \mathcal{R}^{\curlywedge}$, such that $S_x(x) \subset R(x)$, hence $S_x^{\infty}(x) \subset R^{\infty}(x)$. Since $S_x \in \mathcal{R}^{\curlywedge}$, that is $S_x^{\infty} \in \mathcal{R}^{\wedge\infty}$ by using Corollary 2.3.3 we have that $S_x^{\infty} \in \mathcal{R}^{\wedge}$, and hence $R^{\infty} \in \mathcal{R}^{\wedge\wedge} = \mathcal{R}^{\wedge}$. Therefore, by using Corollary 2.3.3 again $R^{\infty} \in \mathcal{R}^{\wedge\infty}$, that is $R \in \mathcal{R}^{\curlywedge}$.

Since the converse inclusion is quite obvious, we have that $\mathcal{R}^{\curlywedge\wedge} = \mathcal{R}^{\curlywedge}$, and since Theorem 2.2.13 we have that

$$\mathcal{R}^{\curlywedge} \subset \mathcal{R}^{\curlywedge *} \subset \mathcal{R}^{\curlywedge \#} \subset \mathcal{R}^{\curlywedge\wedge} = \mathcal{R}^{\curlywedge}.$$

Therefore, the theorem is proved.

In addition to the above theorem, it is also worth proving the following

Theorem 2.3.17. *The operation $\sharp$ is inversion compatible.*

Hint. By Theorems 2.3.2 and 2.2.20 the assertion is quite obvious.

Remark 2.3.18. It can be shown that the operations $\curlywedge$ and $\star$ are not, in general, inversion compatible. Moreover, the operations $\sharp$, $\curlywedge$ and $\star$ are not, in general, normal.

3 Some other tools for relators

3.1 Some important binary operations for relators

Definition 3.1.1. If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then we define

$$\begin{aligned} \mathcal{R} \circ \mathcal{S} &= \big\{ R \circ S : R \in \mathcal{R},\, S \in \mathcal{S} \big\}, & \mathcal{R} \boxdot \mathcal{S} &= \big\{ R \boxdot S : R \in \mathcal{R},\, S \in \mathcal{S} \big\}, \\ \mathcal{R} \wedge \mathcal{S} &= \big\{ R \cap S : R \in \mathcal{R},\, S \in \mathcal{S} \big\}, & \mathcal{R} \vee \mathcal{S} &= \big\{ R \cup S : R \in \mathcal{R},\, S \in \mathcal{S} \big\}. \end{aligned}$$

Remark 3.1.2. By the corresponding definitions, it is clear that $\mathcal{R} \cap \mathcal{S} \subset \mathcal{R} \wedge \mathcal{S}$ and $\mathcal{R} \cap \mathcal{S} \subset \mathcal{R} \vee \mathcal{S}$.

Moreover, concerning the binary operations $\circ$ and $\vee$, we can easily prove the following

Theorem 3.1.3. *If $\mathcal{R}$ and $\mathcal{S}$ are reflexive relators on X, then*

$$\big(\mathcal{R} \circ \mathcal{S} \big)^{*} \subset \big(\mathcal{R} \vee \mathcal{S} \big)^{*}.$$

Proof. In this case, for any $R \in \mathcal{R}$ and $S \in \mathcal{S}$, we have $\Delta_X \subset R$ and $\Delta_X \subset S$. Hence, it follows that $R = R \circ \Delta_X \subset R \circ S$ and $S = \Delta_X \circ S \subset R \circ S$, and thus $R \cup S \subset R \circ S$. Therefore, $\mathcal{R} \circ \mathcal{S} \subset \big(\mathcal{R} \vee \mathcal{S} \big)^{*}$, and hence by the monotonicity and the idempotency of $*$ it is clear that the required inclusion is also true.

Remark 3.1.4. Note that in the above theorem we may write any increasing $*$-absorbing operation $\square$ in place of $*$.

Therefore, it is also of some interest to prove the following

Theorem 3.1.5. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X and $\square \in \{*, \#\}$, then*

$$\big(\mathcal{R} \circ \mathcal{S} \big)^{\square} = \big(\mathcal{R}^{\square} \circ \mathcal{S}^{\square} \big)^{\square}.$$

Hint. Since $\mathcal{R} \subset \mathcal{R}^{\#}$ and $\mathcal{S} \subset \mathcal{S}^{\#}$, we evidently have $\mathcal{R} \circ \mathcal{S} \subset \mathcal{R}^{\#} \circ \mathcal{S}^{\#}$. And hence, by the monotonicity of $\#$, it is clear that $\big(\mathcal{R} \circ \mathcal{S} \big)^{\#} \subset \big(\mathcal{R}^{\#} \circ \mathcal{S}^{\#} \big)^{\#}$.

On the other hand, if $W \in \big(\mathcal{R}^{\#} \circ \mathcal{S}^{\#} \big)^{\#}$, then for each $A \subset X$ there exist $U \in \mathcal{R}^{\#}$ and $V \in \mathcal{S}^{\#}$ such that $(U \circ V)(A) \subset W(A)$. Moreover, there exists $S \in \mathcal{S}$ such that $S(A) \subset V(A)$, and there exists $R \in \mathcal{R}$ such that $R\big(S(A)\big) \subset U\big(S(A)\big)$. Hence, it is clear that

$$(R \circ S)(A) = R\big(S(A)\big) \subset U\big(S(A)\big) \subset U\big(V(A)\big) = (U \circ V)(A) \subset W(A).$$

Therefore, $W \in \big(\mathcal{R} \circ \mathcal{S} \big)^{\#}$, and thus $\big(\mathcal{R}^{\#} \circ \mathcal{S}^{\#} \big)^{\#} \subset \big(\mathcal{R} \circ \mathcal{S} \big)^{\#}$ also holds.

By using a similar argument, concerning the unary operation $\wedge$, we can only prove the following

Theorem 3.1.6. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then*

$$\big(\mathcal{R}\circ\mathcal{S}\big)^{\wedge}=\big(\mathcal{R}^{\#}\circ\mathcal{S}^{\wedge}\big)^{\wedge}.$$

Hence, by writing $\mathcal{R}^{\wedge}$ in place of $\mathcal{R}$, we can immediately get

Corollary 3.1.7. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then*

$$\big(\mathcal{R}^{\wedge}\circ\mathcal{S}\big)^{\wedge}=\big(\mathcal{R}^{\wedge}\circ\mathcal{S}^{\wedge}\big)^{\wedge}.$$

Moreover, analogously to Theorem 3.1.5, we can also easily prove the following

Theorem 3.1.8. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X and $\square\in\{*,\#\}$, then*

$$\big(\mathcal{R}\wedge\mathcal{S}\big)^{\square}=\big(\mathcal{R}^{\square}\wedge\mathcal{S}^{\square}\big)^{\square}.$$

The binary operation $\vee$ has some more satisfactory properties than $\circ$ and $\wedge$ since we have the following

Theorem 3.1.9. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X and $\square\in\{*,\#,\wedge\}$, then*

$$\big(\mathcal{R}\vee\mathcal{S}\big)^{\square}=\mathcal{R}^{\square}\cap\mathcal{S}^{\square}.$$

Hint. $V\in\big(\mathcal{R}\vee\mathcal{S}\big)^{\#}$ if and only if for every $A\subset X$ there exist $R\in\mathcal{R}$ and $S\in\mathcal{S}$ such that $(R\cup S)(A)\subset V(A)$.

On the other hand, $V\in\mathcal{R}^{\#}\cap\mathcal{S}^{\#}$, that is $V\in\mathcal{R}^{\#}$ and $V\in\mathcal{S}^{\#}$ if and only if for every $A\subset X$ there exist $R\in\mathcal{R}$ and $S\in\mathcal{S}$ such that $R(A)\subset V(A)$ and $S(A)\subset V(A)$.

Finally, $(R\cup S)(A)=R(A)\cup S(A)$ follows the equality.

Now, as a close analogue of Theorem 3.1.5, we can also easily establish

Corollary 3.1.10. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X and $\square\in\{*,\#,\wedge\}$, then*

$$\big(\mathcal{R}\vee\mathcal{S}\big)^{\square}=\big(\mathcal{R}^{\square}\vee\mathcal{S}^{\square}\big)^{\square}\qquad\textit{and}\qquad\mathcal{R}^{\square}\cap\mathcal{S}^{\square}=\big(\mathcal{R}^{\square}\cap\mathcal{S}^{\square}\big)^{\square}.$$

Proof. By using Theorem 3.1.9, we can see that

$$\big(\mathcal{R}\vee\mathcal{S}\big)^{\square}=\mathcal{R}^{\square}\cap\mathcal{S}^{\square}=\mathcal{R}^{\square\square}\cap\mathcal{S}^{\square\square}=\big(\mathcal{R}^{\square}\vee\mathcal{S}^{\square}\big)^{\square}$$

and

$$\mathcal{R}^{\square}\cap\mathcal{S}^{\square}=\big(\mathcal{R}\vee\mathcal{S}\big)^{\square}=\big(\mathcal{R}\vee\mathcal{S}\big)^{\square\square}=\big(\mathcal{R}^{\square}\cap\mathcal{S}^{\square}\big)^{\square}.$$

In this respect, it is also worth proving the following

Theorem 3.1.11. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X and $\square\in\{*,\#,\wedge\}$, then the following assertions are equivalent:*

(1) $\mathcal{R}\vee\mathcal{S}\subset\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$; *(2)* $\big(\mathcal{R}\vee\mathcal{S}\big)^{\square}=\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$;

(3) $\mathcal{R}^{\square}\cap\mathcal{S}^{\square}\subset\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$; *(4)* $\mathcal{R}^{\square}\cap\mathcal{S}^{\square}=\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$.

Proof. By the self-increasingness of $\square$, it is clear the assertion (1) is equivalent to the inclusion $\big(\mathcal{R}\vee\mathcal{S}\big)^{\square}\subset\big(\mathcal{R}\cap\mathcal{S}\big)^{\square}$. Moreover, from Remark 3.1.2 we can at once see that the converse inclusion is always true. Therefore, the assertions (1) and (2) are equivalent. On the other hand, by Theorem 3.1.9, it is clear that the equivalences (1) $\iff$ (3) and (2) $\iff$ (4) are also true.

The importance of the binary operation $\vee$ lies mainly in the following

Theorem 3.1.12. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then*

$$\operatorname{Int}_{\mathcal{R}\vee\mathcal{S}}=\operatorname{Int}_{\mathcal{R}}\cap\operatorname{Int}_{\mathcal{S}}\qquad\textit{and}\qquad\operatorname{Cl}_{\mathcal{R}\vee\mathcal{S}}=\operatorname{Cl}_{\mathcal{R}}\cup\operatorname{Cl}_{\mathcal{S}}\,.$$

Proof. $A\subset X$ and $B\in\operatorname{Int}_{\mathcal{R}\vee\mathcal{S}}(A)$ if and only if there exist $R\in\mathcal{R}$ and $S\in\mathcal{S}$ such that $(R\cup S)(B)\subset A$, that is $R(B)\cup S(B)\subset A$.

On the other hand, $A\subset X$ and $B\in(\operatorname{Int}_{\mathcal{R}}\cap\operatorname{Int}_{\mathcal{S}})(A)=\operatorname{Int}_{\mathcal{R}}(A)\cap\operatorname{Int}_{\mathcal{S}}(A)$ if and only if there exist $R\in\mathcal{R}$ and $S\in\mathcal{S}$ such that $R(B)\subset A$ and $S(B)\subset A$, that is $R(B)\cup S(B)\subset A$.

It follows the first assertion. Moreover, the second assertion of the theorem can be derived from the first one by using Theorem 2.1.3.

Now, as an immediate consequence of Theorem 3.1.12, we can also state

Corollary 3.1.13. *If $\mathcal{R}$ is a relator on X, then $\mathscr{T}_{\mathcal{R}\vee\mathcal{S}}=\mathscr{T}_{\mathcal{R}}\cap\mathscr{T}_{\mathcal{S}}$ and $\mathscr{F}_{\mathcal{R}\vee\mathcal{S}}=\mathscr{F}_{\mathcal{R}}\cap\mathscr{F}_{\mathcal{S}}$.*

Moreover, combining Theorems 3.1.11 and 3.1.12, we can also easily establish the following

Theorem 3.1.14. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the following assertions are equivalent:*

(1) $\mathcal{R}\vee\mathcal{S}\subset\big(\mathcal{R}\cap\mathcal{S}\big)^{\#}$; *(2)* $\mathcal{R}^{\#}\cap\mathcal{S}^{\#}\subset\big(\mathcal{R}\cap\mathcal{S}\big)^{\#}$

(3) $\operatorname{Int}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Int}_{\mathcal{R}}\cap\operatorname{Int}_{\mathcal{S}}$; *(4)* $\operatorname{Cl}_{\mathcal{R}\cap\mathcal{S}}=\operatorname{Cl}_{\mathcal{R}}\cup\operatorname{Cl}_{\mathcal{S}}$.

Proof. By Theorems 3.1.11, 2.2.10 (2) and 3.1.12 the assertions (1) and (3) are equivalent.

Finally, to complete the proof, we note that the equivalences (1) $\iff$ (2) and (3) $\iff$ (4) are immediate from Theorems 3.1.11 and 2.1.3, respectively.

Definition 3.1.15. If $\mathcal{R}=\{R_i\}_{i\in I}$ and $\mathcal{S}=\{S_i\}_{i\in I}$ are relators on X, then by trusting to the reader's good sense to avoid confusions we also define

$$\mathcal{R}\odot\mathcal{S}=\big\{R_i\circ S_i:i\in I\big\},\qquad\mathcal{R}\boxdot\mathcal{S}=\big\{R_i\boxdot S_i:i\in I\big\},$$
$$\mathcal{R}\vartriangle\mathcal{S}=\big\{R_i\cap S_i:i\in I\big\},\qquad\mathcal{R}\nabla\mathcal{S}=\big\{R_i\cup S_i:i\in I\big\}.$$

Remark 3.1.16. Note, that, in particular, we have $\mathcal{R}\odot\mathcal{S}\subset\mathcal{R}\circ\mathcal{S}$ and $\mathcal{R}\nabla\mathcal{S}\subset\mathcal{R}\vee\mathcal{S}$.

Moreover, if $\mathcal{R}$ is a relator on X, then by considering $\mathcal{R}=\{R\}_{R\in\mathcal{R}}$ and $\mathcal{R}^{-1}=\{R^{-1}\}_{R\in\mathcal{R}}$ we have

$$\mathcal{R}\odot\mathcal{R}^{-1}=\big\{R\circ R^{-1}:R\in\mathcal{R}\big\}\quad\text{and}\quad\mathcal{R}\nabla\mathcal{R}^{-1}=\big\{R\cup R^{-1}:R\in\mathcal{R}\big\}.$$

In addition to Theorem 3.1.12, it is also worth proving the following

Theorem 3.1.17. *If $\mathfrak{F}$ is a normal increasing (decreasing) set-valued function for relators on X, and moreover $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{U}$ are relators on X, then the following assertions are equivalent:*

(1) $\mathfrak{F}_{\mathcal{U}} = \mathfrak{F}_{\mathcal{R}} \cup \mathfrak{F}_{\mathcal{S}}$ $\big(\mathfrak{F}_{\mathcal{U}} = \mathfrak{F}_{\mathcal{R}} \cap \mathfrak{F}_{\mathcal{S}}\big)$;

(2) $\mathcal{U}^{\square_{\mathfrak{F}}} = \big(\mathcal{R} \cup \mathcal{S}\big)^{\square_{\mathfrak{F}}}$; *(3)* $\mathcal{U}^{\square_{\mathfrak{F}}} = \big(\mathcal{R}^{\square_{\mathfrak{F}}} \cup \mathcal{S}^{\square_{\mathfrak{F}}}\big)^{\square_{\mathfrak{F}}}$;

(4) $\mathcal{U} \subset \mathcal{V}^{\square_{\mathfrak{F}}} \iff \mathcal{R} \cup \mathcal{S} \subset \mathcal{V}^{\square_{\mathfrak{F}}}$ *for every relator $\mathcal{V}$ on X.*

Proof. Define $\mathcal{W} = \mathcal{R} \cup \mathcal{S}$. Then, by the increasingness and the normality of $\mathfrak{F}$ it is clear that $\mathfrak{F}_{\mathcal{W}} = \mathfrak{F}_{\mathcal{R}} \cup \mathfrak{F}_{\mathcal{S}}$. Moreover, by Theorem 1.2.14, it is clear that $\mathfrak{F}_{\mathcal{U}} = \mathfrak{F}_{\mathcal{W}}$ if and only if $\mathcal{U}^{\square_{\mathfrak{F}}} = \mathcal{W}^{\square_{\mathfrak{F}}}$. Therefore, the first part of the assertion (1) is equivalent to the assertion (2). Hence, since $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$ and $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{S}^{\square_{\mathfrak{F}}}}$, is is clear that the assertions (2) and (3) are also equivalent.

On the other hand, if the assertion (2) holds and $\mathcal{V}$ is a relator on X, then by using Theorem 1.1.11 we can easily see that

$$\mathcal{U} \subset \mathcal{V}^{\square_{\mathfrak{F}}} \iff \mathcal{U}^{\square_{\mathfrak{F}}} \subset \mathcal{V}^{\square_{\mathfrak{F}}} \iff \mathcal{W}^{\square_{\mathfrak{F}}} \subset \mathcal{V}^{\square_{\mathfrak{F}}} \iff \mathcal{W} \subset \mathcal{V}^{\square_{\mathfrak{F}}}.$$

Therefore, the assertion (4) also holds. Finally, if the assertion (4) holds, then by putting $\mathcal{U}$ and $\mathcal{W}$ in place of $\mathcal{V}$ in the inclusions of (4) we can immediately see that $\mathcal{W} \subset \mathcal{U}^{\square_{\mathfrak{F}}}$ and $\mathcal{U} \subset \mathcal{W}^{\square_{\mathfrak{F}}}$, and hence $\mathcal{U}^{\square_{\mathfrak{F}}} = \mathcal{W}^{\square_{\mathfrak{F}}}$. Therefore, the assertion (2) also holds.

Remark 3.1.18. From Theorem 3.1.17, by using Theorem 1.2.16, we can at once see that if $\Diamond$ is a refinement for relators on X, and moreover $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then

$$\big(\mathcal{R} \cup \mathcal{S}\big)^{\Diamond} = \big(\mathcal{R}^{\Diamond} \cup \mathcal{S}^{\Diamond}\big)^{\Diamond}.$$

Unfortunately, most of the basic unary operations for relators are not normal. However, for instance, by Theorems 2.2.8 and 2.3.2we have that the refinements $*$, ∂ and $\blacklozenge$ and the modification ∞ are normal.

As an immediate consequence of Theorem 3.1.17 we can also state the following

Corollary 3.1.19. *If $\mathfrak{F}$ is a normal increasing (decreasing) set-valued function for relators on X, and moreover $\mathcal{R}$ and $\mathcal{U}$ are relators on X, then the following assertions are equivalent:*

(1) $\mathfrak{F}_{\mathcal{U}} = \mathfrak{F}_{\mathcal{R}} \cup \mathfrak{F}_{\mathcal{R}^{-1}}$ $\big(\mathfrak{F}_{\mathcal{U}} = \mathfrak{F}_{\mathcal{R}} \cap \mathfrak{F}_{\mathcal{R}^{-1}}\big)$;

(2) $\mathcal{U}^{\square_{\mathfrak{F}}} = (\mathcal{R} \cup \mathcal{R}^{-1})^{\square_{\mathfrak{F}}}$; *(3)* $\mathcal{U}^{\square_{\mathfrak{F}}} = \Big(\mathcal{R}^{\square_{\mathfrak{F}}} \cup (\mathcal{R}^{-1})^{\square_{\mathfrak{F}}}\Big)^{\square_{\mathfrak{F}}}$;

(4) $\mathcal{U} \subset \mathcal{V}^{\square_{\mathfrak{F}}} \iff \mathcal{R} \subset \mathcal{V}^{\square_{\mathfrak{F}}} \cap \big(\mathcal{V}^{\square_{\mathfrak{F}}}\big)^{-1}$ *for every relator $\mathcal{V}$ on X.*

Proof. By Theorem 3.1.17, with writing $\mathcal{R}^{-1}$ in place of $\mathcal{S}$, it is enough to show that

$$\mathcal{R} \cup \mathcal{R}^{-1} \subset \mathcal{V}^{\square_{\mathfrak{F}}} \iff \mathcal{R} \subset \mathcal{V}^{\square_{\mathfrak{F}}} \cap \big(\mathcal{V}^{\square_{\mathfrak{F}}}\big)^{-1}.$$

From Theorem 3.1.17, by Theorem 2.1.20, it is clear that in particular we also have the following

Theorem 3.1.20. *If $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{U}$ are relators on X, then the following assertions are equivalent:*

(1) $\operatorname{Int}_{\mathcal{U}} = \operatorname{Int}_{\mathcal{R}} \cup \operatorname{Int}_{\mathcal{S}}$ $\big(\operatorname{Cl}_{\mathcal{U}} = \operatorname{Cl}_{\mathcal{R}} \cap \operatorname{Cl}_{\mathcal{S}}\big)$*;*

(2) $\mathcal{U}^{\#} = \big(\mathcal{R} \cup \mathcal{S}\big)^{\#}$*;* *(3)* $\mathcal{U}^{\#} = \big(\mathcal{R}^{\#} \cup \mathcal{S}^{\#}\big)^{\#}$*;*

(4) $\mathcal{U} \subset \mathcal{V}^{\#} \iff \mathcal{R} \cup \mathcal{S} \subset \mathcal{V}^{\#}$ *for every relator* $\mathcal{V}$ *on* X.

Hence, it is clear that in particular we also have

Corollary 3.1.21. *If $\mathcal{R}$ and $\mathcal{U}$ are relators on X, then the following assertions are equivalent:*

(1) $\operatorname{Int}_{\mathcal{U}} = \operatorname{Int}_{\mathcal{R}} \cup \operatorname{Int}_{\mathcal{R}^{-1}}$ $\big(\operatorname{Cl}_{\mathcal{U}} = \operatorname{Cl}_{\mathcal{R}} \cap \operatorname{Cl}_{\mathcal{R}^{-1}}\big)$*;*

(2) $\mathcal{U}^{\#} = (\mathcal{R} \cup \mathcal{R}^{-1})^{\#}$*;* *(3)* $\mathcal{U}^{\#} = \Big(\mathcal{R}^{\#} \cup (\mathcal{R}^{-1})^{\#}\Big)^{\#}$*;*

(4) $\mathcal{U} \subset \mathcal{V}^{\#} \iff \mathcal{R} \subset \mathcal{V}^{\#} \cap \big(\mathcal{V}^{\#}\big)^{-1}$ *for every relator* $\mathcal{V}$ *on* X.

Remark 3.1.22. Note that, because of Theorem 3.1.12, we may naturally write $\mathcal{R} \wedge \mathcal{R}^{-1}$ or $\mathcal{R} \triangle \mathcal{R}^{-1}$ in place of $\mathcal{U}$ in Corollary 3.1.21.

Moreover, it is also worth noticing that some of the results of this subsection can be naturally extended to arbitrary families of relators.

3.2 Topological, symmetric, transitive and filtered relators

Definition 3.2.1. If $\mathcal{R}$ is a relator on X, then we say that:

(1) $\mathcal{R}$ is quasi-topological if $x \in \operatorname{int}_{\mathcal{R}}(\operatorname{int}_{\mathcal{R}}(R(x)))$ for all $x \in X$ and $R \in \mathcal{R}$;

(2) $\mathcal{R}$ is topological if for all $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subset R(x)$.

Remark 3.2.2. Quite similarly, a relator $\mathcal{R}$ on X may be called proximal if for all $A \subset X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $A \subset V \subset R(A)$.

The appropriateness of Definition 3.2.1 is already quite obvious from the following four basic theorems which have been mostly proved in [68].

Theorem 3.2.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is total; *(2)* $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ $\big(X \in \mathcal{D}_{\mathcal{R}}\big)$*;*

(3) $\mathcal{E}_{\mathcal{R}} \neq \mathcal{P}(X)$ $\big(\mathcal{D}_{\mathcal{R}} \neq \emptyset\big)$.

Theorem 3.2.4. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is reflexive; *(2) $\rho_{\mathcal{R}}$ is reflexive;*

(3) $\operatorname{int}_{\mathcal{R}}(A) \subset A$ $\big(A \subset \operatorname{cl}_{\mathcal{R}}(A)\big)$ *for all* $A \subset X$*;*

(4) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ $\big(B \cap A \neq \emptyset\big)$ *implies* $B \subset A$ $\big(B \in \operatorname{Cl}_{\mathcal{R}}(A)\big)$ *for all* $A, B \subset X$.

Theorem 3.2.5. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topological;*

(2) $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{T}_{\mathcal{R}}$ *for all* $x \in X$ *and* $R \in \mathcal{R}$*;*

(3) $\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$ $\big(\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}\big)$ *for all* $A \subset X$.

Theorem 3.2.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topological;*

(2) *$\mathcal{R}$ is reflexive and quasi-topological;*

(3) $\operatorname{int}_{\mathcal{R}}(A) = \bigcup\{V \in \mathcal{T}_{\mathcal{R}} : V \subset A\}$ $\big(\operatorname{cl}_{\mathcal{R}}(A) = \bigcap\{W \in \mathcal{F}_{\mathcal{R}} : A \subset W\}\big)$ *for all* $A \subset X$.

Remark 3.2.7. By Theorem 3.2.5, a relator $\mathcal{R}$ on X may be called weakly (strongly) quasi-topological if $\rho_{\mathcal{R}}(x) \in \mathcal{F}_{\mathcal{R}}$ for all $x \in X$ $\big(R(x) \in \mathcal{T}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}\big)$.

Moreover, by Theorem 3.2.6, the relator $\mathcal{R}$ may be called weakly (strongly) topological if it is reflexive and weakly (strongly) quasi-topological.

Also by Theorems 3.2.5 and 3.2.6, it is clear that in particular we have the following

Theorem 3.2.8. *If $\mathcal{R}$ is a topological relator on X, then*

$$\mathcal{E}_{\mathcal{R}} = \big\{A \subset X : \exists V \in \mathcal{T}_{\mathcal{R}} : \emptyset \neq V \subset A\big\}.$$

Remark 3.2.9. Unfortunately, if the above equality holds, then we can only state that $\operatorname{int}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}\big(R(x)\big)\big) \neq \emptyset$, and hence $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{E}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$.

Now, in addition to Theorem 3.2.6, we can also state the following theorem of [68].

Theorem 3.2.10. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topological;* *(2)* *$\mathcal{R}$ is topologically equivalent to* $\mathcal{R}_{\mathcal{T}_{\mathcal{R}}}$ $\big(\mathcal{R}^{\wedge\infty}\big)$*;*

(3) *$\mathcal{R}$ is topologically equivalent to a preorder (topological) relator on X.*

Remark 3.2.11. Moreover, it is also worth mentioning that a unary operation $\square$ for relators on X is a refinement (modification) if and only if there exists a topological (quasi-topological) relator $\mathfrak{R}$ on $\mathcal{P}(X^2)$ such that $\mathcal{R}^{\square} = \operatorname{cl}_{\mathfrak{R}}(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

Definition 3.2.12. A relator $\mathcal{R}$ on X is called weakly (strongly) symmetric if $\rho_{\mathcal{R}}$ (each member of $\mathcal{R}$) is a symmetric relation.

Moreover, the relator $\mathcal{R}$ is called properly symmetric if $\mathcal{R} = \mathcal{R}^{-1}$. And if $\square$ is a unary operation for relators on X, then $\mathcal{R}$ is called $\square$-symmetric if the relator $\mathcal{R}^{\square}$ is properly symmetric.

Remark 3.2.13. We note that the relator $\mathcal{R}$ is properly symmetric if and only if $\mathcal{R} \subset \mathcal{R}^{-1}$, or equivalently $\mathcal{R}^{-1} \subset \mathcal{R}$.

Moreover, the relator $\mathcal{R}$ is, for instance, to be called proximally (quasi-proximally) symmetric if it is $\#$-symmetric ($\#\infty$-symmetric).

Concerning the latter notions, we shall only quote here the following two theorems which have been mostly established in [68].

Theorem 3.2.14. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-proximally symmetric; *(2) $\big(\mathcal{R}^{\infty}\big)^{-1} \subset \mathcal{R}^{\#}$;*

(3) $\big(\mathcal{R}^{\#\infty}\big)^{-1} \subset \mathcal{R}^{\infty\#}$; *(4) $\mathcal{R}$ is $\sharp$-symmetric;*

(5) $\mathcal{R}$ is $\infty\#$-symmetric.

Remark 3.2.15. In addition to this theorem, we can also state that the relator $\mathcal{R}$ is quasi-proximally symmetric if and only if the relator $\mathcal{R}^{\infty}$ is proximally symmetric.

Moreover, by calling a relator quasi-properly (pseudo-properly) symmetric if it is ∞-symmetric (∂-symmetric), we can also state that $\mathcal{R}$ is quasi-proximally symmetric if and only if $\mathcal{R}^{\#}$ is quasi-properly (pseudo-properly) symmetric.

Theorem 3.2.16. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-proximally symmetric;

(2) $\mathcal{R}$ and $\mathcal{R}^{-1}$ are quasi-proximally equivalent;

(3) $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{-1}}$ $\big(\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{-1}}\big)$; *(4) $\mathcal{T}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}}$.*

Remark 3.2.17. Note that, in contrast to the proximal symmetry, the topological symmetry is already a rather restrictive property.

Namely, by Theorem 2.2.23, a relator $\mathcal{R}$ is topologically symmetric if and only if $\mathcal{R}^{\wedge} = \big\{\rho_{\mathcal{R}}\big\}^{\wedge}$, that is, $\mathcal{R}$ is topologically simple (see Theorem 7.1.6) and weakly symmetric.

Therefore, a relator $\mathcal{R}$ has, in addition, to be called topologically semi-symmetric if $\mathcal{R}^{-1} \subset \mathcal{R}^{\wedge}$, or equivalently $\big(\mathcal{R}^{-1}\big)^{\wedge} \subset \mathcal{R}^{\wedge}$.

The importance of the binary operation $\circ$ lies mainly in the following

Definition 3.2.18. A relator $\mathcal{R}$ on X is called weakly (strongly) transitive if $\rho_{\mathcal{R}}$ (each member of $\mathcal{R}$) is transitive.

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ is called $\square$-transitive (strictly $\square$-transitive) if $\mathcal{R}^{\square} \subset \big(\mathcal{R}^{\square} \circ \mathcal{R}^{\square}\big)^{\square}$ $\Big(\mathcal{R}^{\square} \subset \big(\mathcal{R}^{\square} \odot \mathcal{R}^{\square}\big)^{\square}\Big)$.

Remark 3.2.19. Thus, the relator $\mathcal{R}$ is, for instance, to be called topologically transitive if $\mathcal{R}^{\wedge} \subset \big(\mathcal{R}^{\wedge} \circ \mathcal{R}^{\wedge}\big)^{\wedge}$.

Moreover, the relator $\mathcal{R}$ may, for instance, be called strictly topologically transitive if $\mathcal{R}^{\wedge} \subset \big(\mathcal{R}^{\#} \circ \mathcal{R}^{\wedge}\big)^{\wedge}$.

By using Theorems 1.1.11 and 3.1.6 and Corollary 3.1.7, we can easily establish the following

Theorem 3.2.20. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{R}$ is topologically transitive if and only if $\mathcal{R} \subset \big(\mathcal{R}^{\wedge} \circ \mathcal{R}\big)^{\wedge}$;

(2) $\mathcal{R}$ is strictly topologically transitive if and only if $\mathcal{R} \subset \big(\mathcal{R} \circ \mathcal{R}\big)^{\wedge}$.

Remark 3.2.21. If $\mathcal{R}$ is a reflexive relator on X and $\square \in \{*, \#, \wedge\}$, then by Theorems 3.1.3 and 3.1.9, we have $\big(\mathcal{R} \circ \mathcal{R}\big)^{\square} \subset \big(\mathcal{R} \vee \mathcal{R}\big)^{\square} = \mathcal{R}^{\square} \cap \mathcal{R}^{\square} = \mathcal{R}^{\square}$.

Therefore, in addition to Theorem 3.2.20, we can also state the following

Theorem 3.2.22. *If $\mathcal{R}$ is a reflexive relator on X, then*

(1) $\mathcal{R}$ is topologically transitive if and only if $\mathcal{R}^{\wedge} = \big(\mathcal{R}^{\wedge} \circ \mathcal{R}\big)^{\wedge}$;

(2) $\mathcal{R}$ is strictly topologically transitive if and only if $\mathcal{R}^{\wedge} = \big(\mathcal{R} \circ \mathcal{R}\big)^{\wedge}$.

Moreover, to let the reader feel the appropriateness of the above concepts, we can also state the following theorem of [68].

Theorem 3.2.23. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topological;

(2) $\mathcal{R}$ is topologically transitive;

(3) $\mathcal{R}^{\wedge}$ is quasi-topological;

(4) $\mathcal{R}^{\wedge}$ is strictly proximally transitive.

Remark 3.2.24. In [68], it was also proved that a relator $\mathcal{R}$ is topological if and only if the relator $\mathcal{R}^{\wedge}$ is proximal.

The importance of the binary operation $\wedge$ lies mainly in the following

Definition 3.2.25. A relator $\mathcal{R}$ on X is called properly filtered if $\mathcal{R} = \mathcal{R} \wedge \mathcal{R}$.

Moreover, if $\square$ is a unary operation on relators on X, then the relator $\mathcal{R}$ called $\square$-filtered if the relator $\mathcal{R}^{\square}$ is properly filtered.

Remark 3.2.26. Note that, by Remark 3.1.2, we always have $\mathcal{R} \subset \mathcal{R} \wedge \mathcal{R}$. Therefore, the relator $\mathcal{R}$ is properly filtered if and only if $\mathcal{R} \wedge \mathcal{R} \subset \mathcal{R}$.

Moreover, the relator $\mathcal{R}$ is, for instance, to be called uniformly, proximally and topologically filtered if it is $\square$-filtered with $\square = *, \#$ and $\wedge$, respectively.

Concerning the latter notions, we shall only quote here the following two theorems of [68].

Theorem 3.2.27. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is uniformly filtered;

(2) $\mathcal{R} \wedge \mathcal{R} \subset \mathcal{R}^{*}$;

(3) $\mathcal{R}^{*} = \big(\mathcal{R} \wedge \mathcal{R}\big)^{*}$.

Theorem 3.2.28. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically filtered;

(2) $\mathcal{R} \wedge \mathcal{R} \subset \mathcal{R}^{\wedge}$;

(3) $\mathcal{R}^{\wedge} = \big(\mathcal{R} \wedge \mathcal{R}\big)^{\wedge}$.

Remark 3.2.29. Unfortunately, an analogue of the above theorems fails to hold for the proximal filteredness taken in the sense of Definition 3.2.25.

Therefore, a relator $\mathcal{R}$ has, in addition, to be called weakly proximally filtered if $\mathcal{R} \wedge \mathcal{R} \subset \mathcal{R}^{\#}$, or equivalently $\mathcal{R}^{\#} = \big(\mathcal{R} \wedge \mathcal{R}\big)^{\#}$ $\big(\mathrm{Int}_{\mathcal{R}} = \mathrm{Int}_{\mathcal{R} \wedge \mathcal{R}}\big)$.

Moreover, a relator $\mathcal{R}$ has to be called properly proximally filtered if for any $A \subset X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T(A) \subset R(A) \cap S(A)$, or equivalently $\mathrm{Int}_{\mathcal{R}}(A \cap B) = \mathrm{Int}_{\mathcal{R}}(A) \cap \mathrm{Int}_{\mathcal{R}}(B)$ for all $A, B \subset X$.

Now, we can easily prove the following concerning the relators in Definitions 3.1.1 and 3.1.15

Theorem 3.2.30. *If* $\mathcal{R}$ *is a uniformly filtered relator on* X*, then*

$$\big(\mathcal{R} \odot \mathcal{R}^{-1}\big)^* = \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^* \qquad \textit{and} \qquad \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^* = \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*.$$

Hint. In this case, for any $R, S \in \mathcal{R}$ there exists a $T \in \mathcal{R}$ such that $T \subset R \cap S$. Hence, it follows that $T \subset R$ and $T^{-1} \subset S^{-1}$, and thus $T \circ T^{-1} \subset R \circ S^{-1}$. Therefore, $\mathcal{R} \circ \mathcal{R}^{-1} \subset \big(\mathcal{R} \odot \mathcal{R}^{-1}\big)^*$, and hence $\big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^* \subset \big(\mathcal{R} \odot \mathcal{R}^{-1}\big)^*$. Moreover, since $\mathcal{R} \odot \mathcal{R}^{-1} \subset \mathcal{R} \circ \mathcal{R}^{-1}$, it is clear that the converse inclusion is always true.

Now, by writing $\mathcal{R}^{-1}$ in place of $\mathcal{R}$, we can immediately get

Corollary 3.2.31. *If* $\mathcal{R}$ *is a uniformly filtered relator on* X*, then*

$$\big(\mathcal{R}^{-1} \odot \mathcal{R}\big)^* = \big(\mathcal{R}^{-1} \circ \mathcal{R}\big)^* \qquad \textit{and} \qquad \big(\mathcal{R}^{-1} \vartriangle \mathcal{R}\big)^* = \big(\mathcal{R}^{-1} \vee \mathcal{R}\big)^*.$$

Moreover, in addition to Theorem 3.2.30 and Corollary 3.2.31, we can also easily prove the following theorem and corollary

Theorem 3.2.32. *If* $\mathcal{R}$ *is a reflexive relator on* X*, then*

$$\big(\mathcal{R} \odot \mathcal{R}^{-1}\big)^* \subset \big(\mathcal{R} \vartriangle \mathcal{R}^{-1}\big)^* \qquad \textit{and} \qquad \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^* \subset \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*.$$

Corollary 3.2.33. *If* $\mathcal{R}$ *is a reflexive relator on* X*, then*

$$\big(\mathcal{R}^{-1} \odot \mathcal{R}\big)^* \subset \big(\mathcal{R} \vartriangle \mathcal{R}^{-1}\big)^* \qquad \textit{and} \qquad \big(\mathcal{R}^{-1} \circ \mathcal{R}\big)^* \subset \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*.$$

Remark 3.2.34. Note that in Theorems 3.2.30 and 3.2.32 and their corollaries, we may again write any increasing $*$-absorbing operation $\square$ in place of $*$.

3.3 Mild continuities in relator spaces

Definition 3.3.1. If F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square$ is a unary operation for relators, then the relation F is said to be $\square$-continuous, or more precisely mildly $\square$-continuous [80] if

$$\big(F^{-1} \circ \mathcal{S}^{\square} \circ F\big)^{\square} \subset \mathcal{R}^{\square}.$$

Remark 3.3.2. We should use two different unary operations, $\square_X$ for relators on X and $\square_Y$ for relators on Y. However, since later we use, for instance, $\wedge$ instead of both $\wedge_X$ and $\wedge_Y$, here we also use $\square$ instead of both $\square_X$ and $\square_Y$.

Remark 3.3.3. Now, the relation F may be naturally called properly continuous if it is $\square$-continuous with $\square$ being the identity operation for relators.

By Theorem 1.1.11, we have the following specialization of Definition 3.3.1.

Theorem 3.3.4. *If F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square$ is a refinement for relators, then the following assertions are equivalent:*

(1) F is $\square$-continuous; *(2) $F^{-1} \circ \mathcal{S}^{\square} \circ F \subset \mathcal{R}^{\square}$.*

Remark 3.3.5. Therefore, in this case, F is $\square$-continuous if and only if F is a properly continuous as a relation on $X\big(\mathcal{R}^{\square}\big)$ to $Y\big(\mathcal{S}^{\square}\big)$.

Moreover, as some further specializations of Definition 3.3.1, we can also prove the following theorems.

Theorem 3.3.6. *If F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square \in \{*, \#\}$, then the following assertions are equivalent:*

(1) F is $\square$-continuous; *(2) $F^{-1} \circ \mathcal{S} \circ F \subset \mathcal{R}^{\square}$.*

Hint. If $V \in \mathcal{S}^{\#}$, then for each $A \subset X$ there exists an $S \in \mathcal{S}$ such that $S\big(F(A)\big) \subset V\big(F(A)\big)$. Hence, $F^{-1}\big(S\big(F(A)\big)\big) \subset F^{-1}\big(V\big(F(A)\big)\big)$, and thus $\big(F^{-1} \circ S \circ F\big)(A) \subset \big(F^{-1} \circ V \circ F\big)(A)$. Moreover, if the assertion (2) holds with $\square = \#$, then $F^{-1} \circ S \circ F \in \mathcal{R}^{\#}$. Therefore, there exists an $R \in \mathcal{R}$ such that $R(A) \subset \big(F^{-1} \circ S \circ F\big)(A)$. Consequently, we also have $R(A) \subset \big(F^{-1} \circ V \circ F\big)(A)$. Hence, it is clear that $F^{-1} \circ V \circ F \in \mathcal{R}^{\#}$, and thus $F^{-1} \circ \mathcal{S}^{\#} \circ F \subset \mathcal{R}^{\#}$. Thus, by Theorem 3.3.4, the assertion (1) also holds with $\square = \#$.

Theorem 3.3.7. *If f is a function on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square \in \{\wedge, \bullet\}$, then the following assertions are equivalent:*

(1) f is $\square$-continuous; *(2) $f^{-1} \circ \mathcal{S} \circ f \subset \mathcal{R}^{\square}$.*

Hint. If the assertion (2) holds with $\square = \bullet$, then for each $S \in \mathcal{S}$ we have $f^{-1} \circ S \circ f \in \mathcal{R}^{\bullet}$. Hence, since $\mathcal{R}^{\bullet} = \big\{\rho_{\mathcal{R}}^{-1}\big\}^{*} = \big\{\bigcap \mathcal{R}\big\}^{*}$, it follows that $\bigcap \mathcal{R} \subset f^{-1} \circ S \circ f$. Therefore, we also have $\bigcap \mathcal{R} \subset \bigcap_{S \in \mathcal{S}} f^{-1} \circ S \circ f$. Hence, by using that $\bigcap_{S \in \mathcal{S}} f^{-1} \circ S \circ f = f^{-1} \circ \big(\bigcap \mathcal{S}\big) \circ f$, we can infer that $\bigcap \mathcal{R} \subset f^{-1} \circ \big(\bigcap \mathcal{S}\big) \circ f$. Therefore, we also have $f^{-1} \circ \big(\bigcap \mathcal{S}\big) \circ f \in \big(\bigcap \mathcal{R}\big)^{*}$. Hence, by Theorem 3.3.6, it is clear that $f^{-1} \circ \big\{\bigcap \mathcal{S}\big\}^{*} \circ f \subset \big(\bigcap \mathcal{R}\big)^{*}$. Therefore, we also have $f^{-1} \circ \mathcal{S}^{\bullet} \circ f \subset \mathcal{R}^{\bullet}$. Thus, by Theorem 3.3.4, the assertion (1) also holds with $\square = \bullet$.

Theorem 3.3.8. *If f is a function on one relator space $X(\mathcal{R})$ onto another $Y(\mathcal{S})$ and $\square \in \{\triangle, \blacktriangle, \blacklozenge\}$, then the following assertions are equivalent:*

(1) f is $\square$-continuous; *(2) $f^{-1} \circ \mathcal{S} \circ f \subset \mathcal{R}^{\square}$.*

Hint. If $\mathcal{S} = \big\{Y^2\big\}$, then $\mathcal{S}^{\blacklozenge} = \big\{Y^2\big\}$. Hence, by Theorem 3.3.4, it is clear that the implication (2) $\Longrightarrow$ (1) holds true with $\square = \blacklozenge$.

While, if $\mathcal{S} \neq \big\{Y^2\big\}$, then there exists an $S \in \mathcal{S}$ such that $S \neq Y^2$. Therefore, there exist $y, z \in Y$ such that $(y, z) \notin S$, and hence $z \notin S(y)$. Moreover, since $Y = f(X)$, there exist $u, v \in X$ such that $y = f(u)$ and $z = f(v)$. Therefore, we also have $f(v) \notin S\big(f(u)\big)$. Hence, it follows that $v \notin f^{-1}\big(S\big(f(u)\big)\big)$, and thus $v \notin \big(f^{-1} \circ S \circ f\big)(u)$. Therefore, $(u, v) \notin f^{-1} \circ S \circ f$, and thus $f^{-1} \circ S \circ f \neq X^2$.

On the other hand, if the assertion (2) holds with $\square = \blacklozenge$, then $f^{-1} \circ S \circ f \in \mathcal{R}^{\blacklozenge}$. Therefore, we necessarily have $\mathcal{R}^{\blacklozenge} = \mathcal{P}\big(X^2\big)$. And thus, the assertion (1) also holds.

Remark 3.3.9. Note that if, for instance, $X = \{0,1\}$ and $f = X \times \{0\}$, then $f^{-1} \circ \Delta_X \circ f = X^2 \in \big\{\Delta_X\big\}^{\vartriangle}$, but f is not a $\vartriangle$-continuous function of $X\big(\Delta_X\big)$ into itself. Namely, if $V = X^2 \setminus \Delta_X$, then $V \in \big\{\Delta_X\big\}^{\vartriangle}$, but $f^{-1} \circ V \circ f = \emptyset \notin \big\{\Delta_X\big\}^{\vartriangle}$.

Moreover, it is also worth noticing that an analogue of Theorem 3.3.8 does not hold for the operation $\bigstar$ since it is not idempotent by Example 2.2.17.

The appropriateness of Definition 3.3.1 is apparent from the following particular cases of the results of [64], [68] and [80].

Theorem 3.3.10. *If f is a function on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) f is uniformly continuous;

(2) $x \in \operatorname{Lim}_{\mathcal{R}}(y)$ implies $f \circ x \in \operatorname{Lim}_{\mathcal{S}}(f \circ y)$;

(3) $x \in \operatorname{Adh}_{\mathcal{R}}(y)$ implies $f \circ x \in \operatorname{Adh}_{\mathcal{S}}(f \circ y)$.

Theorem 3.3.11. *If F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) F is proximally continuous;

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ implies $F(A) \in \operatorname{Cl}_{\mathcal{S}}\big(F(B)\big)$;

(3) $F(A) \in \operatorname{Int}_{\mathcal{S}}(B)$ implies $A \in \operatorname{Int}_{\mathcal{R}}\big(F^{-1}(B)\big)$.

Theorem 3.3.12. *If f is a function on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) f is topologically continuous;

(2) $x \in \operatorname{lim}_{\mathcal{R}}(y)$ implies $f(x) \in \operatorname{lim}_{\mathcal{S}}(f \circ y)$;

(3) $x \in \operatorname{adh}_{\mathcal{R}}(y)$ implies $f(x) \in \operatorname{adh}_{\mathcal{S}}(f \circ y)$.

Theorem 3.3.13. *If f is a function of one relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) f is topologically continuous;

(2) $a \in \operatorname{cl}_{\mathcal{R}}(B)$ implies $f(a) \in \operatorname{cl}_{\mathcal{S}}\big(f(B)\big)$;

(3) $f(a) \in \operatorname{int}_{\mathcal{S}}(B)$ implies $a \in \operatorname{int}_{\mathcal{R}}\big(f^{-1}(B)\big)$.

Corollary 3.3.14. *If f is a function of an arbitrary relator space $X(\mathcal{R})$ into a topological one $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) f is topologically continuous;

(2) $U \in \mathcal{T}_{\mathcal{S}}$ implies $f^{-1}(U) \in \mathcal{T}_{\mathcal{R}}$; *(3) $V \in \mathcal{F}_{\mathcal{S}}$ implies $f^{-1}(V) \in \mathcal{F}_{\mathcal{R}}$.*

Remark 3.3.15. If f is a function of an arbitrary relator space $X(\mathcal{R})$ into a proximal one $Y(\mathcal{S})$, then we can also state that f is proximally continuous if and only if $U \in \tau_{\mathcal{S}}$ $\big(V \in \bar{\tau}_{\mathcal{S}}\big)$ implies $f^{-1}(U) \in \tau_{\mathcal{R}}$ $\big(f^{-1}(V) \in \bar{\tau}_{\mathcal{R}}\big)$.

Theorem 3.3.16. *If f is a function of one relator space $X(\mathcal{R})$ onto another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) f is paratopologically continuous;

(2) $A \in \mathcal{D}_{\mathcal{R}}$ implies $f(A) \in \mathcal{D}_{\mathcal{S}}$; *(3) $B \in \mathcal{E}_{\mathcal{S}}$ implies $f^{-1}(B) \in \mathcal{E}_{\mathcal{R}}$.*

Remark 3.3.17. By using Theorem 2.1.9, it can be easily seen that the assertions (2) and (3) are equivalent for any relation f on $X(\mathcal{R})$ to $Y(\mathcal{S})$.

However, the implications (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are not, in general, true. Therefore, it is of some importance to point out that the following theorem is true.

Theorem 3.3.18. *If F is a paratopologically continuous relation on a total relator space $X(\mathcal{R})$ to an arbitrary one $Y(\mathcal{S})$, then $F(A) \in \mathcal{D}_{\mathcal{S}}$ for all $A \in \mathcal{D}_{\mathcal{R}}$.*

Proof. If this not the case, then there exists $A \in \mathcal{D}_{\mathcal{R}}$ such that $F(A) \notin \mathcal{D}_{\mathcal{S}}$. Then, by Theorem 2.1.9, we necessarily have $Y \setminus F(A) \in \mathcal{E}_{\mathcal{S}}$. Hence, by defining $V = Y \times \big(Y \setminus F(A)\big)$ and $U = F^{-1} \circ V \circ F$, we can at once see that $V \in \mathcal{S}^{\triangle}$, and thus $U \in \mathcal{R}^{\triangle}$. Moreover, we can also at once see that

$$U(x) = F^{-1}\big(V\big(F(x)\big) = F^{-1}\big(Y \setminus F(A)\big)$$

for all $x \in X$ with $F(x) \neq \emptyset$. Therefore, if $u \in U(x)$ for some $x \in X$, then we necessarily have $F(u) \cap \big(Y \setminus F(A)\big) \neq \emptyset$. Thus, there exists $w \in F(u)$ such that $w \notin F(A)$, i.e., $w \notin F(a)$ for all $a \in A$. This shows that $U(x) \cap A = \emptyset$, and hence $A \notin \mathcal{D}_{\mathcal{R}}$, which is a contradiction. Therefore, we actually have $U = \emptyset$, and hence $\emptyset \in \mathcal{R}^{\triangle}$. Thus, by Theorem 3.2.3, the relator $\mathcal{R}$ cannot be total, which is again a contradiction.

Now, as an immediate consequence of Theorems 3.2.3 and 3.3.18, we can also state

Corollary 3.3.19. *If F is a paratopologically continuous relation on a total relator space $X(\mathcal{R})$ to an arbitrary one $Y(\mathcal{S})$, then $F(X) \in \mathcal{D}_{\mathcal{S}}$, and thus in particular $Y(\mathcal{S})$ is also total.*

Hence, it is clear that in particular we also have

Corollary 3.3.20. *If F is a paratopologically continuous relation on a total relator space $X(\mathcal{R})$ to an arbitrary one $Y(\mathcal{S})$ such that $F(X) \in \mathcal{F}_{\mathcal{S}}$, then $Y = F(X)$.*

Moreover, by noticing that $\mathcal{F}_{\mathcal{S}} = \mathcal{P}(Y)$ whenever $\Delta_Y \in \mathcal{S}^{\wedge}$, and moreover $\operatorname{card}(X) < \operatorname{card}(Y)$ whenever there is a function of X onto Y, we can also state

Corollary 3.3.21. *If $\operatorname{card}(X) < \operatorname{card}(Y)$, and $\mathcal{R}$ and $\mathcal{S}$ are relators on X and Y, respectively, such that $\mathcal{R}$ is total and $\Delta_Y \in \mathcal{S}^{\wedge}$, then there is no paratopologically continuous function of $X(\mathcal{R})$ into $Y\big(\mathcal{S}\big)$.*

4 Well-chained relators

4.1 Comparison of the important set-valued functions

In addition to the results of Section 2, we shall also need the following theorems.

Theorem 4.1.1. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

(1) $\operatorname{Int}_{\mathcal{R}^{\wedge}}(A) = \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(A)\big)$; *(2)* $\operatorname{Cl}_{\mathcal{R}^{\wedge}}(A) = \mathcal{P}(X) \setminus \mathcal{P}\big(X \setminus \operatorname{cl}_{\mathcal{R}}(A)\big)$.

Corollary 4.1.2. *If $\mathcal{R}$ is a relator on X, then $\tau_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$ and $\bar{\tau}_{\mathcal{R}^{\wedge}} = \mathcal{F}_{\mathcal{R}}$.*

Theorem 4.1.3. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

(1) $\operatorname{Int}_{\mathcal{R}^{\vartriangle}}(A) = \{\emptyset\}$ *if* $A \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{Int}_{\mathcal{R}^{\vartriangle}}(A) = \mathcal{P}(X)$ *if* $A \in \mathcal{E}_{\mathcal{R}}$;

(2) $\operatorname{Cl}_{\mathcal{R}^{\vartriangle}}(A) = \emptyset$ *if* $A \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^{\vartriangle}}(A) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $A \in \mathcal{D}_{\mathcal{R}}$.

Corollary 4.1.4. *If $\mathcal{R}$ is a relator on X, then $\tau_{\mathcal{R}^{\vartriangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$ and $\bar{\tau}_{\mathcal{R}^{\vartriangle}} = \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}$.*

Theorem 4.1.5. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

(1) $\operatorname{Int}_{\mathcal{R}^{\bullet}}(A) = \mathcal{P}\big(X \setminus \rho_{\mathcal{R}}(X \setminus A)\big)$; *(2)* $\operatorname{Cl}_{\mathcal{R}^{\bullet}}(A) = \mathcal{P}(X) \setminus \mathcal{P}\big(X \setminus \rho_{\mathcal{R}}(A)\big)$.

Corollary 4.1.6. *If $\mathcal{R}$ is a relator on X, then $\tau_{\mathcal{R}^{\bullet}} = \bar{\tau}_{\{\rho_{\mathcal{R}}\}}$ and $\bar{\tau}_{\mathcal{R}^{\bullet}} = \tau_{\{\rho_{\mathcal{R}}\}}$.*

Proof. By Theorems 4.1.5 and 2.1.8, it is clear that

$$A \in \tau_{\mathcal{R}^{\bullet}} \iff A \subset X \setminus \rho_{\mathcal{R}}(X \setminus A) \iff \rho_{\mathcal{R}}(X \setminus A) \subset X \setminus A \iff$$
$$X \setminus A \in \tau\{\rho_{\mathcal{R}}\} \iff A \in \bar{\tau}\{\rho_{\mathcal{R}}\}$$

Theorem 4.1.7. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

(1) $\operatorname{Int}_{\mathcal{R}^{\star}}(A) = \mathcal{P}(X)$ *if* $\operatorname{card}(A) \geq 2$ *and* $\operatorname{Int}_{\mathcal{R}^{\star}}(A) = \operatorname{Int}_{\mathcal{R}}(A)$ *if* $\operatorname{card}(A) \leq 1$;

(2) $\operatorname{Cl}_{\mathcal{R}^{\star}}(A) = \emptyset$ *if* $\operatorname{card}(X \setminus A) \geq 2$ *and* $\operatorname{Cl}_{\mathcal{R}^{\star}}(A) = \operatorname{Cl}_{\mathcal{R}}(A)$ *if* $\operatorname{card}(X \setminus A) \leq 1$.

Corollary 4.1.8. *If $\mathcal{R}$ is a relator on X, then $\tau_{\mathcal{R}^{\star}} = \tau_{\mathcal{R}} \cup \{A \subset X : \operatorname{card}(A) \neq 1\}$ and $\bar{\tau}_{\mathcal{R}^{\star}} = \bar{\tau}_{\mathcal{R}} \cup \{A \subset X : \operatorname{card}(X \setminus A) \neq 1\}$.*

Theorem 4.1.9. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

(1) $\operatorname{Int}_{\mathcal{R}^{\bigstar}}(A) = \emptyset$ *if* $X = \rho_{\mathcal{R}}(X \setminus A)$ *and* $\operatorname{Int}_{\mathcal{R}^{\bigstar}}(A) = \mathcal{P}(X)$ *if* $X \neq \rho_{\mathcal{R}}(X \setminus A)$;

(2) $\operatorname{Cl}_{\mathcal{R}^{\bigstar}}(A) = \emptyset$ *if* $X \neq \rho_{\mathcal{R}}(A)$ *and* $\operatorname{Cl}_{\mathcal{R}^{\bigstar}}(A) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $X = \rho_{\mathcal{R}}(A)$.

Corollary 4.1.10. *If $\mathcal{R}$ is a relator on X, then*
$\tau_{\mathcal{R}^{\bigstar}} = \{A \subset X : X \neq \rho_{\mathcal{R}}(X \setminus A)\} \cup \{\emptyset\}$ and $\bar{\tau}_{\mathcal{R}^{\bigstar}} = \{A \subset X : X \neq \rho_{\mathcal{R}}(A)\} \cup \{X\}$.

Theorem 4.1.11. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

(1) $\operatorname{Int}_{\mathcal{R}^{\blacktriangle}}(A) = \{\emptyset\}$ *if* $E_{\mathcal{R}} \not\subset A$ *and* $\operatorname{Int}_{\mathcal{R}^{\blacktriangle}}(A) = \mathcal{P}(X)$ *if* $E_{\mathcal{R}} \subset A$;

(2) $\operatorname{Cl}_{\mathcal{R}^{\blacktriangle}}(A) = \emptyset$ *if* $A \subset D_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^{\blacktriangle}}(A) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $A \not\subset D_{\mathcal{R}}$.

Corollary 4.1.12. *If $\mathcal{R}$ is a relator on X, then $\tau_{\mathcal{R}^{\blacktriangle}} = \{A \subset X : E_{\mathcal{R}} \subset A\} \cup \{\emptyset\}$ and $\digamma_{\mathcal{R}^{\blacktriangle}} = \mathcal{P}(D_{\mathcal{R}}) \cup \{X\}$.*

Theorem 4.1.13. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then*

(1) $\operatorname{Int}_{\mathcal{R}^{\blacklozenge}}(A) = \{\emptyset\}$ *if $A \neq X$ and $\mathcal{R} = \{X^2\}$ and* $\operatorname{Int}_{\mathcal{R}^{\blacklozenge}}(A) = \mathcal{P}(X)$ *if $A = X$ or $\mathcal{R} \neq \{X^2\}$;*

(2) $\operatorname{Cl}_{\mathcal{R}^{\blacklozenge}}(A) = \emptyset$ *if $A = \emptyset$ or $\mathcal{R} \neq \{X^2\}$ and* $\operatorname{Cl}_{\mathcal{R}^{\blacklozenge}}(A) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if $A \neq \emptyset$ and $\mathcal{R} = \{X^2\}$.*

Corollary 4.1.14. *If $\mathcal{R}$ is a relator on X, then $\tau_{\mathcal{R}^{\blacklozenge}} = \{\emptyset, X\}$ if $\mathcal{R} = \{X^2\}$ and $\tau_{\mathcal{R}^{\blacklozenge}} = \mathcal{P}(X)$ if $\mathcal{R} \neq \{X^2\}$, and moreover $\digamma_{\mathcal{R}^{\blacklozenge}} = \tau_{\mathcal{R}^{\blacklozenge}}$.*

4.2 Some basic properties of the Davis–Pervin relations

Definition 4.2.1. For each $A \subset X$, the relation

$$R_A = A^2 \cup (X \setminus A) \times X$$

is called the Davis–Pervin relation on X generated by A.

Remark 4.2.2. Namely, the relations R_A were first used by Davis [9] and Pervin [55] in their uniformization procedures of topological spaces.

In the sequel, we shall often need the following simple propositions about the inverses, complements and images of the relations R_A.

Proposition 4.2.3. *If $A \subset X$, then R_A is a preorder on X such that*

$$R_A^{-1} = R_{X \setminus A} \qquad \text{and} \qquad R_A^c = A \times (X \setminus A).$$

Proposition 4.2.4. *If $A, B \subset X$, then*

$$\begin{array}{ll} R_A(B) = \emptyset \text{ if } B = \emptyset, & R_A(B) = A \text{ if } \emptyset \neq B \subset A, \\ R_A(B) = X \text{ if } B \not\subset A. & \end{array}$$

Remark 4.2.5. The relations R_A are important particular cases of the relations $R_{(A,B)} = A \times B \cup (X \setminus A) \times X$ considered first by Császár [7, pp. 42] and Hunsaker and Lindgren [18] for some $A \subset B \subset X$.

Moreover, the following theorem is an important particular case of [68, Theorems 2.6.1 and 2.9.1]. However, since we are now not interested in the relations $R_{(A,B)}$, it seems appropriate to provide here a direct proof.

Theorem 4.2.6. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^*$; *(2)* $A \in \tau_{\mathcal{R}}$.

Proof. If the assertion (1) holds, then there exists $R \in \mathcal{R}$ such that $R \subset R_A$. Hence, it follows that $R(A) \subset R_A(A) = A$. Therefore, the assertion (2) also holds.

While, if the assertion (2) holds, then there exists $R \in \mathcal{R}$ such that $R(A) \subset A$. Hence, it follows that $R(x) \subset R(A) \subset A = R_A(x)$ for all $x \in A$. Moreover, it is clear that $R(x) \subset X = R_A(x)$ for all $x \in X \setminus A$. Therefore, $R \subset R_A$, and thus the assertion (1) also holds.

Corollary 4.2.7. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^*$; *(2) $R_A \in \mathcal{R}^{\#}$.*

Proof. By Theorems 4.2.6, 2.2.10 (2) and 2.2.18, it is clear that

$$R_A \in \mathcal{R}^* \iff A \in \tau_{\mathcal{R}} \iff A \in \tau_{\mathcal{R}^{\#}} \iff R_A \in \mathcal{R}^{\#*} \iff R_A \in \mathcal{R}^{\#}.$$

From Theorem 4.2.6, we can also easily get the following more particular theorems.

Theorem 4.2.8. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\wedge}$; *(2) $A \in \mathcal{T}_{\mathcal{R}}$.*

Proof. By Theorems 2.2.18 and 4.2.6 and Corollary 4.1.2, it is clear that

$$R_A \in \mathcal{R}^{\wedge} \iff R_A \in \mathcal{R}^{\wedge *} \iff A \in \tau_{\mathcal{R}^{\wedge}} \iff A \in \mathcal{T}_{\mathcal{R}}.$$

Theorem 4.2.9. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $A \neq \emptyset$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\vartriangle}$; *(2) $A \in \mathcal{E}_{\mathcal{R}}$.*

Proof. By Theorems 2.2.18 and 4.2.6 and Corollary 4.1.4, it is clear that

$$R_A \in \mathcal{R}^{\vartriangle} \iff R_A \in \mathcal{R}^{\vartriangle *} \iff A \in \tau_{\mathcal{R}^{\vartriangle}} \iff A \in \mathcal{E}_{\mathcal{R}}.$$

Theorem 4.2.10. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\bullet}$; *(2) $A \in \tau_{\{\rho_{\mathcal{R}}\}}$.*

Proof. By Theorems 2.2.18 and 4.2.6 and Corollary 4.1.6, it is clear that

$$R_A \in \mathcal{R}^{\bullet} \iff R_A \in \mathcal{R}^{\bullet *} \iff A \in \tau_{\mathcal{R}^{\bullet}} \iff A \in \tau_{\{\rho_{\mathcal{R}}\}}.$$

Theorem 4.2.11. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\star}$; *(2)* $\operatorname{card}(A) \neq 1$ *or* $A \in \tau_{\mathcal{R}}$ $\big(A \in \mathcal{T}_{\mathcal{R}}\big)$.

Proof. By Theorems 2.2.18 and 4.2.6 and Corollary 4.1.8, it is clear that

$$R_A \in \mathcal{R}^{\star} \iff R_A \in \mathcal{R}^{\star *} \iff A \in \tau_{\mathcal{R}^{\star}} \iff \operatorname{card}(A) \neq 1 \text{ or } A \in \tau_{\mathcal{R}}.$$

Theorem 4.2.12. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $A \neq \emptyset$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\bigstar}$; *(2)* $X \neq \rho_{\mathcal{R}}(X \setminus A)$.

Proof. By Theorems 2.2.18 and 4.2.6 and Corollary 4.1.10, it is clear that

$$R_A \in \mathcal{R}^{\bigstar} \iff R_A \in \mathcal{R}^{\bigstar *} \iff A \in \tau_{\mathcal{R}^{\bigstar}} \iff X \neq \rho_{\mathcal{R}}(X \setminus A).$$

Theorem 4.2.13. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $A \neq \emptyset$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\blacktriangle}$; *(2)* $E_{\mathcal{R}} \subset A$.

Proof. By Theorems 2.2.18 and 4.2.6 and Corollary 4.1.12, it is clear that

$$R_A \in \mathcal{R}^{\blacktriangle} \iff R_A \in \mathcal{R}^{\blacktriangle *} \iff A \in \tau_{\mathcal{R}^{\blacktriangle}} \iff E_{\mathcal{R}} \subset A.$$

Remark 4.2.14. Since Theorems 2.2.14, 4.2.12 and 4.2.13 can also be proved with the help of Theorems 4.2.9 and 4.2.10. For this, it is enough to note only that

$$\mathcal{E}_{\mathcal{R}^{\bullet}} = \mathcal{E}_{\{\rho_{\mathcal{R}}^{-1}\}^{*}} = \mathcal{E}_{\{\rho_{\mathcal{R}}^{-1}\}} = \big\{A \subset X : X \neq \rho_{\mathcal{R}}(X \setminus A)\big\}$$

and

$$\rho_{\mathcal{R}^{\vartriangle}} = \rho_{\mathcal{R}^{\vartriangle \bullet}} = \rho_{\mathcal{R}^{\blacktriangle}} = \rho_{\{X \times E_{\mathcal{R}}\}^{*}} = \rho_{\{X \times E_{\mathcal{R}}\}} = E_{\mathcal{R}} \times X.$$

Theorem 4.2.15. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $\emptyset \neq A \neq X$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\blacklozenge}$; *(2)* $\mathcal{R} \neq \{X^2\}$.

Proof. If the assertion (2) does not hold, then we have $\mathcal{R}^{\blacklozenge} = \{X^2\}$. Hence, since $R_A \neq X^2$, it is clear that the assertion (1) does not also holds.

On the other hand, if the assertion (2) holds, then we have $\mathcal{R}^{\blacklozenge} = \mathcal{P}(X^2)$. Therefore, the assertion (1) also holds.

Remark 4.2.16. In this respect, it is also worth noticing that $R_A \in \mathcal{R}^{\partial}$ if and only if $R_A \in \mathcal{R}$. Thus, in particular, $R_A \in \mathcal{R}^{\sharp}$ $\big(R_A \in \mathcal{R}^{\curlywedge}\big)$ if and only if $R_A \in \mathcal{R}^{\#}$ $\big(R_A \in \mathcal{R}^{\wedge}\big)$.

Moreover, in addition to Corollary 4.2.7, we can also prove the following

Theorem 4.2.17. *If $\mathcal{R}$ is a relator on X, $A \subset X$ and $\square \in \{*, \#\}$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\square}$; *(2)* $R_A \in \mathcal{R}^{\square\infty}$; *(3)* $R_A \in \mathcal{R}^{\infty\square}$.

Proof. By Proposition 4.2.3, is clear that (1) $\Longrightarrow$ (2). Moreover, by the inclusions $\mathcal{R}^{\square\infty} \subset \mathcal{R}^{\infty\square} \subset \mathcal{R}^{\square}$, is clear that implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1) are also true.

Remark 4.2.18. Note that in Remark 4.2.16 and Theorem 4.2.17 we may write any preorder in place of R_A.

4.3 Well-chainedness of arbitrary relators

Definition 4.3.1. A relator $\mathcal{R}$ on X will be called properly well-chained or chain-connected if $\mathcal{R}^{\infty} = \{X^2\}$.

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ will be called $\square$-well-chained if the relator $\mathcal{R}^{\square}$ is properly well-chained.

Remark 4.3.2. The condition $\mathcal{R}^{\infty} = \{X^2\}$, in a detailed form, means only that for every $R \in \mathcal{R}$ we have $X^2 = R^{\infty} = \Delta_X \cup \bigcup_{n=1}^{\infty} R^n$. That is, for every $x, y \in X$, with $x \neq y$, there exists an $n \in \mathbb{N}$ such that $(x, y) \in R^n$. That is, there exists a family $(x_i)_{i=0}^n$ in X such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R$ for all $i = 1, \ldots, n$.

Therefore, our present definition of proper well-chainedness is a straightforward generalization of Cantor's chain-connectedness. (See, for instance, Thron [82, p. 29] and Wilder [86, p. 721].)

By a reformulation of the above definition we have the following

Remark 4.3.3. A relator $\mathcal{R}$ on X is well-chained if and only if $\mathcal{R} \subset \mathcal{R}^{\partial}$.

Preliminary forms of some of the following theorems, for Weil uniformities and reflexive relators, have already proved by Levine [31] and Kurdics and Száz [26], respectively. However, for the readers convenience, we shall now give some improved proofs.

Theorem 4.3.4. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) $R_A \notin \mathcal{R}^*$ for every proper nonvoid subset A of X;

(3) $R \notin \mathcal{R}^*$ for every proper preorder R on X;

(4) $R \notin \mathcal{R}^*$ for every proper nonvoid transitive relation R on X.

Proof. If $A \subset X$ such that $R_A \in \mathcal{R}^*$, then there exists an $R \in \mathcal{R}$ such that $R \subset R_A$. Hence, it follows that $R^{\infty} \subset R_A^{\infty} = R_A$. Moreover, if the assertion (1) holds, then we have $R^{\infty} = X^2$. Therefore, we also have $R_A = X^2$. This implies that $A = \emptyset$ or $A = X$. Therefore, the assertion (2) also holds.

While, if R is a preorder on X and $A = R(x)$ for some $x \in X$, then $x \in A$ and $R(A) = R\big(R(x)\big) \subset R(x) = A$. Therefore, if $R \in \mathcal{R}^*$ holds, then $A \in \tau_{\mathcal{R}^*} = \tau_{\mathcal{R}}$ also holds. Hence, by Theorem 4.2.6, it follows that $R_A \in \mathcal{R}^*$. Therefore, if the assertion (2) holds, then since $A \neq \emptyset$ we necessarily have $A = X$, and thus $R(x) = X$. Hence, it is clear that $R = X^2$, and thus the assertion (3) also holds.

On the other hand, if R is a transitive relation on X, then $S = \Delta_X \cup R$ is a preorder on X such that $R \subset S$. Therefore, if $R \in \mathcal{R}^*$, then we also have $S \in \mathcal{R}^*$. Hence, if the assertion (3) holds, we can infer that $S = X^2$, and thus $X^2 = \Delta_X \cup R$. Therefore, if $u \in X$ and $v \in X \setminus \{u\}$, then we necessarily have $(u, v) \in R$ and $(v, u) \in R$. Hence, by the transitivity of R, it follows that $(u, u) \in R$. Therefore, if $\operatorname{card}(X) > 1$, then we necessarily have $R = X^2$ even if R was not supposed to be nonvoid. Therefore, the assertion (4) also holds. Namely, if $\operatorname{card}(X) = 1$, then $\emptyset$ and X^2 are the only relations on X.

Finally, to complete the proof, we note that if $R \in \mathcal{R}$, then R^∞ is, in particular, a nonvoid transitive relation on X. Therefore, if the assertion (4) holds, then we necessarily have $R^\infty = X^2$. And thus, the assertion (1) also holds.

Remark 4.3.5. The assertion (3) of Theorem 4.3.4 can be briefly verbalized by saying that X^2 is the only preorder being contained in $\mathcal{R}^*$.

A simple application of the assertion (4) of Theorem 4.3.4 gives the following

Corollary 4.3.6. *If $\mathcal{R}$ is a properly well-chained relator on X and $\operatorname{card}(X) > 1$, then $\mathcal{R}$ is a total relator on X.*

Proof. If this not the case, then there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) = \emptyset$. Hence, it is clear that $S = (X \setminus \{x\}) \times X$ is a proper nonvoid transitive relation on X such that $R \subset S$, and thus $S \in \mathcal{R}^*$. And this is a contradiction by Theorem 4.3.4.

The following simple proposition shows that the extra cardinality condition on X cannot be omitted from the above corollary.

Proposition 4.3.7. *If X is a nonvoid set, then the following assertions are equivalent:*

(1) $\operatorname{card}(X) = 1$*;*

(2) $\{\emptyset\}$ is a properly well-chained relator on X;

(3) every relator $\mathcal{R}$ on X is properly well-chained.

Now, by using Theorem 4.3.4 and Corollary 4.3.6, we can also easily establish a useful reformulation of Definition 4.3.1.

Proposition 4.3.8. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained; *(2) $X^2 = \bigcup_{n=1}^{\infty} R^n$ for all $R \in \mathcal{R}$.*

Proof. If $R \in \mathcal{R}$, then it is clear that $S = \bigcup_{n=1}^{\infty} R^n$ is a transitive relation on X such that $R \subset S$, and hence $S \in \mathcal{R}^*$. Moreover, if the assertion (1) holds, then by Corollary 4.3.6 in particular we have $R \neq \emptyset$, and hence $S \neq \emptyset$. Therefore, by Theorem 4.3.4, we necessarily have $S = X^2$, and thus the assertion (2) also holds. Now, since the converse implication (2) $\Longrightarrow$ (1) is quite obvious, the proof is complete.

Moreover, as an immediate consequence of Theorems 4.3.4 and 4.2.6, we can also state the following

Theorem 4.3.9. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) $\tau_{\mathcal{R}} = \{\emptyset, X\}$; *(3) $\tau\!\!\!-_{\mathcal{R}} = \{\emptyset, X\}$.*

Proof. Namely, by Theorem 4.3.4, we have (1) if and only if $R_A \notin \mathcal{R}^*$ for all proper nonvoid subset A of X. Moreover, by Theorem 4.2.6, for any $A \subset X$ we have $R_A \notin \mathcal{R}^*$ if and only if $A \notin \tau_{\mathcal{R}}$. Therefore, the assertions (1) and (2) are equivalent. Moreover, by Theorem 2.1.8, it is clear that the assertions (2) and(3) are also equivalent.

Remark 4.3.10. The assertion (2) of Theorem 4.3.9 can be briefly verbalized by saying that no proper nonvoid subset of $X(\mathcal{R})$ is proximally open.

From Theorem 4.3.9, by the definitions of the families $\tau_{\mathcal{R}}$ and $\tau\!\!\!-_{\mathcal{R}}$, it is clear that the proper well-chainedness of a relator $\mathcal{R}$ can also be expressed in terms of the relations $\mathrm{Int}_{\mathcal{R}}$ and $\mathrm{Cl}_{\mathcal{R}}$.

Therefore, it is rather surprising that the following theorem has formerly been overlooked by the authors of the papers [26] and [27].

Theorem 4.3.11. *If $\mathcal{R}$ is a relator on X and $\mathrm{card}(X) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) $B \in \mathrm{Int}_{\mathcal{R}}(A)$ implies $A \not\subset B$ for all $A, B \subset X$ with $A \neq X$ and $B \neq \emptyset$;

(3) $X = A \cup B$ implies $B \in \mathrm{Cl}_{\mathcal{R}}(A)$ for all $A, B \subset X$ with $A \neq \emptyset$ and $B \neq \emptyset$.

Proof. If the assertion (1) holds, then by Theorem 4.3.9 we have $\tau_{\mathcal{R}} = \{\emptyset, X\}$. Moreover, if the assertion (2) does not hold, then there exist $A, B \subset X$, with $A \neq X$ and $B \neq \emptyset$, such that $B \in \mathrm{Int}_{\mathcal{R}}(A)$ and $A \subset B$. Hence, by the corresponding definitions, it is clear that $A \in \mathrm{Int}_{\mathcal{R}}(A)$ and $B \in \mathrm{Int}_{\mathcal{R}}(B)$, and thus $A, B \in \tau_{\mathcal{R}}$. Hence, since $\tau_{\mathcal{R}} = \{\emptyset, X\}$ and $A \neq X$ and $B \neq \emptyset$, we can infer that $A = \emptyset$ and $B = X$. Therefore, we actually have $X \in \mathrm{Int}_{\mathcal{R}}(\emptyset)$, and hence $\emptyset \in \mathcal{R}$. Hence, by the assertion (1) and Proposition 4.3.7, it follows that $\mathrm{card}(X) = 1$, which is a contradiction. Therefore, the implication (1) $\Longrightarrow$ (2) is true.

Now, to prove the converse implication (2) $\Longrightarrow$ (1), we note that if $A \in \tau_{\mathcal{R}}$, then $A \in \mathrm{Int}_{\mathcal{R}}(A)$. Therefore, if the assertion (2) holds then we necessarily have $A = X$ or $A = \emptyset$. Consequently, $\tau_{\mathcal{R}} = \{\emptyset, X\}$, and thus by Theorem 4.3.9, the assertion (1) also holds.

Finally, to complete the proof, we note that the equivalence of the assertions (2) and (3) is immediate from Theorem 2.1.3. Namely, for $A, B \subset X$, the conditions $X \setminus A \subset B$ and $X = A \cup B$ are equivalent.

Remark 4.3.12. By Proposition 4.3.7, it is clear that not only the equivalence of the assertions (2) and (3), but also the implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) are true without the extra cardinality condition on X.

However, if $\operatorname{card}(X) = 1$ and $\mathcal{R} = \{\emptyset\}$, and moreover $A = B = X$, then $X = A \cup B$, with $A \neq \emptyset$ and $B \neq \emptyset$, such that $B \notin \operatorname{Cl}_{\mathcal{R}}(A)$. Therefore, in this case, the converses of the above implications fail to hold.

Now, as an immediate consequence of Theorems 4.3.9, 4.3.11 and 2.1.16, we can also state the following

Theorem 4.3.13. *If $\mathcal{R}$ is a relator on X and $\operatorname{card}(X) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) for each proper nonvoid subset A of X there exist nets x and y in A and $X \setminus A$, respectively, such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$;

(3) for any two nonvoid subsets A and B of X, with $X = A \cup B$, there exist nets x and y in A and B, respectively, such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$.

Remark 4.3.14. Later, we shall see that the proper well-chainedness of a relator $\mathcal{R}$ cannot, in general, be expressed in terms of the relations $\operatorname{cl}_{\mathcal{R}}$ or $\lim_{\mathcal{R}}$.

Therefore, it is of some interest to point out that we still have the following

Theorem 4.3.15. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained; *(2) $\rho_{\mathcal{R}^\infty} = X^2$.*

Proof. Note that, by Theorem 2.1.4, we have $\rho_{\mathcal{R}^\infty}^{-1} = \bigcap \mathcal{R}^\infty$. Therefore, the equality $\mathcal{R}^\infty = \{X^2\}$ can hold if and only if $\rho_{\mathcal{R}^\infty}^{-1} = X^2$, that is, $\rho_{\mathcal{R}^\infty} = X^2$.

Moreover, as an immediate consequence of Definition 4.3.1 and the inversion compatibility of the operation ∞, we can also at once state

Theorem 4.3.16. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained; *(2) $\mathcal{R}^{-1}$ is properly well-chained.*

Finally, as some immediate consequences of the corresponding definitions and Theorems 4.3.4 and 4.3.16, we can also state the following two theorems.

Theorem 4.3.17. *If $\mathcal{R}$ is a relator on X and $\square$ is an $*$-invariant operation on relators, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$-well-chained;

(2) $R_A \notin \mathcal{R}^\square$ for every proper nonvoid subset A of X;

(3) $R \notin \mathcal{R}^\square$ for every proper preorder R on X;

(4) $R \notin \mathcal{R}^\square$ for every proper nonvoid transitive relation R on X.

Theorem 4.3.18. *If $\mathcal{R}$ is a relator on X and $\square$ is an inversion compatible operation on relators, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$-well-chained; *(2) $\mathcal{R}^{-1}$ is $\square$-well-chained.*

4.4 Well-chainedness of refinement relators

Definition 4.4.1. A relator $\mathcal{R}$ on X will be called uniformly, proximally, topologically, paratopologically, infinitesimally, σ-infinitesimally, ultrainfinitesimally, parainfinitesimally, and ultimately well-chained if it is $\square$-well-chained with $\square = *, \#, \wedge, \triangle, \bullet, \star, \bigstar, \blacktriangle$, and $\blacklozenge$, respectively.

Remark 4.4.2. From the inclusion relations of the above operations, it is clear that 'paratopologically or infinitesimally well-chained' $\Longrightarrow$ 'topologically well-chained' $\Longrightarrow$ 'proximally well-chained' $\Longrightarrow$ 'uniformly well-chained' $\Longrightarrow$ 'properly well-chained'. And 'ultimately well-chained' $\Longrightarrow$ 'parainfinitesimally well-chained' $\Longrightarrow$ 'ultrainfinitesimally well-chained' $\Longrightarrow$ 'paratopologically and infinitesimally well-chained'.

Moreover, in addition to the corresponding particular cases of Theorems 4.3.17 and 4.3.18, we can also easily establish the following theorems.

Theorem 4.4.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) $\mathcal{R}$ is uniformly well-chained; *(3) $\mathcal{R}$ is proximally well-chained.*

Proof. Note that, by Theorem 2.2.10(2), we have $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^*} = \tau_{\mathcal{R}^\#}$. Therefore, by Theorem 4.3.9, the required assertions are also equivalent.

Remark 4.4.4. Later we shall see that 'proximally well-chained' $\not\Longleftrightarrow$ 'topologically well-chained' $\not\Longleftrightarrow$ 'infinitesimally well-chained' $\not\Longleftrightarrow$ 'paratopologically well-chained'.

But, 'paratopologically well-chained' $\Longrightarrow$ 'ultimately well-chained'. Therefore, 'paratopologically well-chained' is equivalent to 'ultrainfinitesimally, parainfinitesimally and ultimately well-chained'. Moreover, 'paratopologically well-chained' $\Longrightarrow$ 'infinitesimally well-chained'.

Theorem 4.4.5. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically well-chained;

(2) $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$; *(3) $\mathcal{F}_{\mathcal{R}} = \{\emptyset, X\}$.*

Proof. By Theorem 2.2.18 and 4.3.9, we have (1) if and only if $\tau_{\mathcal{R}^\wedge} = \big\{\emptyset, X\big\}$. Moreover, by Corollary 4.1.2, we also have $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$. Therefore, the assertions (1) and (2) are equivalent. Moreover, by Theorem 2.1.8, the assertions (2) and (3) are also equivalent.

Theorem 4.4.6. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$ *, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topologically well-chained;*

(2) $B \subset \operatorname{int}_{\mathcal{R}}(A)$ *implies* $A \not\subset B$ *for all* $A, B \subset X$ *with* $A \neq X$ *and* $B \neq \emptyset$*;*

(3) $X = A \cup B$ *implies* $B \cap \operatorname{cl}_{\mathcal{R}}(A) \neq \emptyset$ *for all* $A, B \subset X$ *with* $A \neq \emptyset$ *and* $B \neq \emptyset$*.*

Proof. By Theorem 2.2.18, it is clear that $\mathcal{R}$ is topologically well-chained if and only if $\mathcal{R}^{\wedge}$ is proximally well-chained. Moreover, by Theorem 4.1.1, for any $A, B \subset X$, we have $B \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(A)$ if and only if $B \subset \operatorname{int}_{\mathcal{R}}(A)$. Therefore, by Theorem 4.3.11, the assertions (1) and (2) are equivalent. Moreover, by Theorem 2.1.3, it is clear that the assertions (2) and (3) are also equivalent.

Theorem 4.4.7. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topologically well-chained;*

(2) for each proper nonvoid subset A *there exists a net* x *in* A *and a point* y *in* $X \setminus A$ *such that* $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$*;*

(3) for any two nonvoid subsets A *and* B *of* X*, with* $X = A \cup B$*, there exists a net* x *in* A *and a point* y *in* B *such that* $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$*.*

Remark 4.4.8. Later we shall see that the inverse of a topologically well-chained relator need not be topologically well-chained.

Concerning paratopological well-chainedness, we first prove the following analogue of Theorem 4.4.5.

Theorem 4.4.9. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically well-chained;*

(2) $\mathcal{E}_{\mathcal{R}} = \{X\}$*;* *(3)* $\mathcal{D}_{\mathcal{R}} = \mathcal{P}(X) \setminus \{\emptyset\}$*.*

Proof. By Theorem 2.2.18 and 4.3.9, we have (1) if and only if $\tau_{\mathcal{R}^{\triangle}} = \big\{\emptyset, X\big\}$. Moreover, by Corollary 4.1.4, we also have $\tau_{\mathcal{R}^{\triangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$. Therefore, by Theorem 3.2.3 and Corollary 4.3.6 the assertions (1) and (2) are equivalent. Moreover, by Theorem 2.1.9, the assertions (2) and (3) are also equivalent.

The latter theorem allows us to easily prove that paratopologically well-chained relators need not actually be studied since we have the following

Theorem 4.4.10. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically well-chained;* *(2)* $\mathcal{R} = \{X^2\}$*.*

Proof. If $R \in \mathcal{R}$ and $x \in X$, then $R(x) \in \mathcal{E}_{\mathcal{R}}$. Therefore, if the assertion (1) holds, then by Theorem 4.4.9 we necessarily have $R(x) = X$. Hence, it is clear that $R = X^2$, and thus the assertion (2) also holds.

On the other hand, if the assertion (2) holds, then by the corresponding definitions it is clear that $\mathcal{E}_{\mathcal{R}} = \{X\}$. Therefore, again by Theorem 4.4.9, the assertion (1) also holds.

Remark 4.4.11. Note that if $\operatorname{card}(X)=1$, then by Proposition 4.3.7 any relator on X is paratopologically well chained.

Moreover, if $\operatorname{card}(X)=1$ and $\mathcal{R}$ is a relator on X, then we actually have $\mathcal{E}_{\mathcal{R}}=\{X\}$ if and only if $\emptyset \notin \mathcal{R}$.

From Theorem 4.4.10, it is clear that, in contrast to Remark 4.4.8, we have

Corollary 4.4.12. *A relator $\mathcal{R}$ on X is paratopologically well-chained if and only if its inverse $\mathcal{R}^{-1}$ is paratopologically well-chained.*

Moreover, by using Theorem 4.4.10, we can also easily establish the following

Theorem 4.4.13. *If $\mathcal{R}$ is a relator on X and $\operatorname{card}(X)>1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically well-chained;

(2) $\operatorname{int}_{\mathcal{R}}(A)=\emptyset$ $\big(\operatorname{Int}_{\mathcal{R}}(A)=\{\emptyset\}\big)$ for all $A\subset X$ with $A\neq X$;

(3) $\operatorname{cl}_{\mathcal{R}}(A)=X$ $\big(\operatorname{Cl}_{\mathcal{R}}(A)=\mathcal{P}(X)\setminus\{\emptyset\}\big)$ for all $A\subset X$ with $A\neq\emptyset$.

Hint. Note that if the assertion (2) holds, then $x\notin\operatorname{int}_{\mathcal{R}}(A)$ for all $x\in X$ and $A\subset X$ with $A\neq X$. This implies that $R(x)\not\subset A$ for all $R\in\mathcal{R}$, $x\in X$ and $A\subset X$ with $A\neq X$. Therefore, we necessarily have $R(x)=X$ for all $R\in\mathcal{R}$ and $x\in X$. Consequently, $\mathcal{R}=\{X^2\}$, and thus the assertion (1) also holds.

Theorem 4.4.14. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically well-chained;

(2) $\mathcal{R}$ is ultrainfinitesimally well-chained;

(3) $\mathcal{R}$ is parainfinitesimally well-chained;

(4) $\mathcal{R}$ is ultimately well-chained.

Proof. If the assertion (1) holds and $\operatorname{card}(X)>1$, then by Theorem 4.4.10 we have $\mathcal{R}=\{X^2\}$, and hence $\mathcal{R}^{\blacklozenge}=\{X^2\}$. Therefore, by the corresponding definitions, the assertion (4) also holds.

Now, by Remarks 4.4.11 and 4.4.2, it is clear that required assertions are equivalent even if $\operatorname{card}(X)=1$. Moreover, we have the following

Corollary 4.4.15. *If $\mathcal{R}$ is a paratopologically well-chained relator on X, then $\mathcal{R}$ is, in particular, infinitesimally well-chained.*

In this respect, it is also worth mentioning that analogously to Theorem 4.3.15 we also have

Theorem 4.4.16. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is infinitesimally well-chained;

(2) $\mathcal{T}_{\{\rho_{\mathcal{R}}\}}=\{\emptyset,X\}$; *(3) $\mathcal{F}_{\{\rho_{\mathcal{R}}\}}=\{\emptyset,X\}$;*

(4) $\{\rho_{\mathcal{R}}\}$ is properly well-chained; *(5) $\rho_{\mathcal{R}}^{\infty}=X^2$.*

Proof. By Theorem 2.2.18 and 4.3.9, we have (1) if and only if $\tau_{\mathcal{R}^\bullet} = \big\{\emptyset, X\big\}$. Moreover, by Corollary 4.1.6, we also have $\tau_{\mathcal{R}^\bullet} = \tau_{\{\rho_{\mathcal{R}}\}}$. Therefore, the assertions (1) and (3) are equivalent. Moreover, by Theorem 2.1.8, the assertions (2) and (3) are also equivalent.

Therefore, we have that the relator $\mathcal{R}$ is infinitesimally well-chained if and only if the relator $\{\rho_{\mathcal{R}}\}$ is properly well-chained, that is the assertions (1) and (4) are also equivalent.

Finally, by calling a relator σ-infinitesimally well-chained if it is $\star$-well-chained, Corollary 4.1.8 follows the following

Theorem 4.4.17. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is σ-infinitesimally well-chained;

(2) $\operatorname{card}(X) < 3$ *and* $\tau_{\mathcal{R}} = \{\emptyset, X\}$ $\big(\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}\big)$.

Proof. By Theorem 2.3.16 and 4.3.9, we have (1) if and only if $\tau_{\mathcal{R}^\star} = \big\{\emptyset, X\big\}$. Moreover, by Corollary 4.1.8, we also have $\tau_{\mathcal{R}^\star} = \tau_{\mathcal{R}} \cup \{A \subset X : \operatorname{card}(A) \neq 1\}$. Therefore, the assertions (1) and (2) are equivalent.

Remark 4.4.18. From the equality $\mathcal{R}^\infty = \mathcal{R}^{\infty\infty}$ we can at once see that the relator $\mathcal{R}$ is ∞-well-chained if and only if it is properly well-chained. Therefore, the 'quasi well-chainedness properties' of relators need not be studied.

Moreover, from Theorem 4.3.7, by Theorem 4.2.17 and Remark 4.2.16, we can at once see that the 'almost uniform (almost proximal) and the superproximal (supertopological) well-chainedness properties' of relators need not also be studied.

However, note that a relator $\mathcal{R}$ on X may be naturally called properly well-chained at a point x of X if $R^\infty(x) = X$ for all $R \in \mathcal{R}$. Therefore, localized forms of the corresponding well-chainedness properties may also be investigated.

5 Connected relators

5.1 Symmetrization of the Davis–Pervin relations

Definition 5.1.1. For each $A \subset X$, the relation

$$S_A = R_A \cap R_A^{-1}$$

is called the symmetrization of the Davis–Pervin relation R_A.

Concerning the relations S_A, we can easily establish the following propositions.

Proposition 5.1.2. *If $A \subset X$, then S_A is an equivalence on X such that*

$$S_A = A^2 \cup (X \setminus A)^2 \qquad \textit{and} \qquad S_A^c = A \times (X \setminus A) \cup (X \setminus A) \times A .$$

Proposition 5.1.3. *If $A, B \subset X$, then*

$$\begin{array}{ll} S_A(B) = \emptyset \ \textit{if}\ B = \emptyset, & S_A(B) = X \setminus A \ \textit{if}\ \emptyset \neq B \subset X \setminus A, \\ S_A(B) = A \ \textit{if}\ \emptyset \neq B \subset A, & S_A(B) = X \ \textit{if}\ B \not\subset A \ \textit{and}\ B \not\subset X \setminus A. \end{array}$$

Proposition 5.1.4. *If $A \subset X$, then*

$$\operatorname{Int}_{S_A} = \operatorname{Int}_{R_A} \cup \operatorname{Int}_{R_A^{-1}} \qquad \textit{and} \qquad \operatorname{Cl}_{S_A} = \operatorname{Cl}_{R_A} \cap \operatorname{Cl}_{R_A^{-1}} .$$

Proof. By the corresponding definitions and Propositions 5.1.3 and 4.2.3, it is clear that for any $B, C \subset X$ we have

$$\begin{aligned} B \in \operatorname{Int}_{S_A}(C) \iff S_A(B) \subset C \iff R_A(B) \subset C \text{ or } R_{X \setminus A}(B) \subset C \iff \\ \iff B \in \operatorname{Int}_{R_A}(C) \text{ or } B \in \operatorname{Int}_{R_A^{-1}}(C) \iff B \in \big(\operatorname{Int}_{R_A} \cup \operatorname{Int}_{R_A^{-1}}\big)(C). \end{aligned}$$

Therefore, the first assertion of the theorem is true. The second assertion of the theorem can be easily derived from the first one by using Theorem 2.1.3.

Moreover, concerning the relations S_A, we can also easily prove the following

Theorem 5.1.5. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$; *(2)* $R_A \in \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^*$.

Proof. If the assertion (1) holds, then there exists $R \in \mathcal{R}$ such that $R \subset S_A$. Hence, it follows that $R \cup R^{-1} \subset S_A \cup S_A^{-1} = S_A \subset R_A$. Therefore, the assertion (2) also holds.

While, if the assertion (2) holds, then there exists $R \in \mathcal{R}$ such that $R \cup R^{-1} \subset R_A$. Hence, it follows that $R \subset R_A \cap R_A^{-1} = S_A$. Therefore, the assertion (1) also holds.

From Theorem 5.1.5, by Theorem 3.2.30, it is clear that we also have

Corollary 5.1.6. *If $\mathcal{R}$ is a uniformly filtered relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$; *(2)* $R_A \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*$.

Moreover, analogously to Theorem 5.1.5, we can also easily prove the following

Theorem 5.1.7. *If $\mathcal{R}$ is a reflexive relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$;

(2) $R_A \in \big(\mathcal{R} \odot \mathcal{R}^{-1}\big)^*$; *(3)* $R_A \in \big(\mathcal{R}^{-1} \odot \mathcal{R}\big)^*$.

Proof. If the assertion (1) holds, then there exists $R \in \mathcal{R}$ such that $R \subset S_A$. Hence, by Proposition 5.1.2, it is clear that $R \circ R^{-1} \subset S_A \circ S_A^{-1} = S_A \subset R_A$ and $R^{-1} \circ R \subset S_A^{-1} \circ S_A = S_A \subset R_A$. Therefore, the assertions (2) and (3) also hold.

While if the assertion (2) or (3) hold, then by Theorem 3.2.32 and Corollary 3.2.33, we have $R_A \in \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^*$. Therefore, by Theorem 5.1.5, the assertion (1) also holds.

Remark 5.1.8. Note that the implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) do not require the relator $\mathcal{R}$ to be reflexive.

Therefore, the inclusion $R_A \in \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^*$ implies $R_A \in \big(\mathcal{R} \odot \mathcal{R}^{-1}\big)^*$ and $R_A \in \big(\mathcal{R}^{-1} \odot \mathcal{R}\big)^*$ even if the relator $\mathcal{R}$ is not reflexive.

From Theorem 5.1.7, by Theorem 3.2.30 and Corollary 3.2.31, it is clear that we also have

Corollary 5.1.9. *If $\mathcal{R}$ is a uniformly filtered reflexive relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$;

(2) $R_A \in \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^*$; *(3)* $R_A \in \big(\mathcal{R}^{-1} \circ \mathcal{R}\big)^*$.

On the other hand from Proposition 5.1.4, by Corollaries 3.1.21 and 4.2.7, it is clear that we also have the following

Theorem 5.1.10. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\#}$;

(2) $R_A \in \mathcal{R}^* \cap \big(\mathcal{R}^*\big)^{-1}$; *(3)* $R_A \in \mathcal{R}^{\#} \cap \big(\mathcal{R}^{\#}\big)^{-1}$.

Remark 5.1.11. Note that if $X = \{0, 1\}$, $A = \{0\}$ and $\mathcal{R} = \{R_A, R_A^{-1}\}$, then $R_A \in \mathcal{R}^* \cap \big(\mathcal{R}^*\big)^{-1}$, but $S_A \notin \mathcal{R}^*$. Therefore, an analogue of Theorem 5.1.10 does not, in general, hold for the operation $*$.

However, as an immediate consequence of Theorem 5.1.10, we can also state that if $\Box$ is a $\#$-invariant operation for relators on X, then for any set $A \subset X$ and any relator $\mathcal{R}$ on X we have $S_A \in \mathcal{R}^{\Box}$ if and only if $R_A \in \mathcal{R}^{\Box} \cap \big(R^{\Box}\big)^{-1}$.

From Theorem 5.1.10, by using Theorem 4.2.6, we can also quite easily get the following

Theorem 5.1.12. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\#}$; *(2)* $A \in \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$.

Proof. If $S_A \in \mathcal{R}^{\#}$, then by Theorem 5.1.10 we also have $R_A \in \mathcal{R}^* \cap (\mathcal{R}^*)^{-1}$. This implies that $R_A \in \mathcal{R}^*$ and $R_A \in (\mathcal{R}^*)^{-1}$, i.e., $R_{X \setminus A} \in \mathcal{R}^*$. Hence, by Theorem 4.2.6, it follows that $A \in \tau_{\mathcal{R}}$ and $X \setminus A \in \tau_{\mathcal{R}}$, i.e., $A \in \digamma_{\mathcal{R}}$. Therefore, the implication (1) $\Longrightarrow$ (2) is true. The converse implication can be proved quite similarly, by reversing the above argument.

In addition to Theorem 5.1.12, it is also worth proving the following

Theorem 5.1.13. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\#}$; *(2)* $R_A \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^{-}$.

Proof. If the assertion (1) holds, then by Theorem 5.1.12 we have $A \in \tau_{\mathcal{R}}$ and $A \in \digamma_{\mathcal{R}}$, and hence $A \in \tau_{\mathcal{R}^{-1}}$. Hence, by Theorem 4.2.6, it follows that $R_A \in \mathcal{R}^*$ and $R_A \in \big(\mathcal{R}^{-1}\big)^*$. Therefore, by Theorem 3.1.9, we have $R_A \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*$. That is, the assertion (2) also holds.

The converse implication (2) $\Longrightarrow$ (1) can be proved quite similarly, by reversing the above argument.

Now, as an immediate consequence of Theorem 5.1.13 and Corollary 5.1.6, we can also state

Corollary 5.1.14. *If $\mathcal{R}$ is a uniformly filtered relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$; *(2)* $S_A \in \mathcal{R}^{\#}$.

Moreover, analogously to the Theorem 5.1.13, we can also prove the following

Theorem 5.1.15. *If $\mathcal{R}$ is a reflexive relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\#}$;

(2) $R_A \in \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^*$; *(3)* $R_A \in \big(\mathcal{R}^{-1} \circ \mathcal{R}\big)^*$.

Proof. If the assertion (1) holds, then as in the proof of Theorem 5.1.9 we have $R_A \in \mathcal{R}^*$ and $R_A \in \big(\mathcal{R}^{-1}\big)^*$. Hence, by Proposition 4.2.3 and Theorem 3.1.5, it is clear that

$$R_A = R_A \circ R_A \in \mathcal{R}^* \circ \big(\mathcal{R}^{-1}\big)^* \subset \Big(\mathcal{R}^* \circ \big(\mathcal{R}^{-1}\big)^*\Big)^* = \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^*$$

and

$$R_A = R_A \circ R_A \in \big(\mathcal{R}^{-1}\big)^* \circ \mathcal{R}^* \subset \Big(\big(\mathcal{R}^{-1}\big)^* \circ \mathcal{R}^*\Big)^* = \big(\mathcal{R}^{-1} \circ \mathcal{R}\big)^*.$$

That is, the assertions (2) and (3) also hold.

While if the assertion (2) or (3) holds, then by Theorem 3.2.32 and Corollary 3.2.33 we have $R_A \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*$. Therefore, by Theorem 5.1.13, the assertion (1) also holds.

Remark 5.1.16. Note that the implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) do not require the relator $\mathcal{R}$ to be reflexive.

Therefore, the inclusion $R_A \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*$ implies $R_A \in \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^*$ and $R_A \in \big(\mathcal{R}^{-1} \circ \mathcal{R}\big)^*$ even if the relator $\mathcal{R}$ is not reflexive.

From Theorem 5.1.12, we can also easily get the following more particular theorems.

Theorem 5.1.17. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\wedge$; *(2) $A \in \mathcal{T}_\mathcal{R} \cap \mathcal{F}_\mathcal{R}$.*

Proof. By Theorems 2.2.18 and 5.1.12 and Corollary 4.1.2, it is clear that

$$S_A \in \mathcal{R}^\wedge \iff S_A \in \mathcal{R}^{\wedge\#} \iff A \in \tau_{\mathcal{R}^\wedge} \cap \digamma_{\mathcal{R}^\wedge} \iff A \in \mathcal{T}_\mathcal{R} \cap \mathcal{F}_\mathcal{R}.$$

Theorem 5.1.18. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $\emptyset \neq A \neq X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\vartriangle$; *(2) $A \in \mathcal{E}_\mathcal{R} \setminus \mathcal{D}_\mathcal{R}$.*

Proof. By Theorems 2.2.18 and 5.1.12, and Corollary 4.1.4, it is clear that

$$\begin{aligned} S_A \in \mathcal{R}^\vartriangle \iff S_A \in \mathcal{R}^{\vartriangle\#} \iff A \in \tau_{\mathcal{R}^\vartriangle} \cap \digamma_{\mathcal{R}^\vartriangle} \iff \\ \iff A \in \mathcal{E}_\mathcal{R} \cap \big(\mathcal{P}(X) \setminus \mathcal{D}_\mathcal{R}\big) \iff A \in \mathcal{E}_\mathcal{R} \setminus \mathcal{D}_\mathcal{R}. \end{aligned}$$

Theorem 5.1.19. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\bullet$; *(2) $A \cap \rho_\mathcal{R}(X \setminus A) = \emptyset$ and $(X \setminus A) \cap \rho_\mathcal{R}(A) = \emptyset$.*

Proof. By Theorems 2.2.18 and 5.1.12 and Corollary 4.1.6, it is clear that

$$\begin{aligned} S_A \in \mathcal{R}^\bullet \iff S_A \in \mathcal{R}^{\bullet\#} \iff A \in \tau_{\mathcal{R}^\bullet} \text{ and } A \in \digamma_{\mathcal{R}^\bullet} \iff \\ \iff A \cap \rho_\mathcal{R}(X \setminus A) = \emptyset \text{ and } (X \setminus A) \cap \rho_\mathcal{R}(A) = \emptyset. \end{aligned}$$

Theorem 5.1.20. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\star$;

(2) $\big(\operatorname{card}(A) \neq 1$ or $A \in \tau_\mathcal{R} (A \in \mathcal{T}_\mathcal{R})\big)$ and $\big(\operatorname{card}(X \setminus A) \neq 1$ or $A \in \digamma_\mathcal{R} (A \in \mathcal{F}_\mathcal{R})\big)$.

Proof. By Proposition 5.1.4, it is clear that $\sigma_{S_A} = \sigma_{R_A} \cup \sigma_{R_A^{-1}}$. Hence, by Theorem 2.1.20, Corollary 3.1.19 and Proposition 4.2.3, it is clear that

$$S_A \in \mathcal{R}^\star \iff R_A \in \mathcal{R}^\star \cap \left(\mathcal{R}^\star\right)^{-1} \iff R_A \in \mathcal{R}^\star \text{ and } R_{X\setminus A} \in \mathcal{R}^\star.$$

Therefore, by Theorems 4.2.11 and 2.1.8, the required assertions are also equivalent.

Theorem 5.1.21. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $\emptyset \neq A \neq X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\bigstar$; *(2)* $X \neq \rho_{\mathcal{R}}(A)$ *and* $X \neq \rho_{\mathcal{R}}(X \setminus A)$.

Proof. By Theorems 2.2.18 and 5.1.12 and Corollary 4.1.10, it is clear that

$$S_A \in \mathcal{R}^\bigstar \iff S_A \in \mathcal{R}^{\bigstar\#} \iff A \in \tau_{\mathcal{R}^\bigstar} \text{ and } A \in \tau\!\!\!-_{\mathcal{R}^\bigstar} \iff$$
$$\iff X \neq \rho_{\mathcal{R}}(X \setminus A) \text{ and } X \neq \rho_{\mathcal{R}}(A).$$

Theorem 5.1.22. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $\emptyset \neq A \neq X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\blacktriangle$; *(2)* $E_{\mathcal{R}} = \emptyset$.

Proof. By Theorems 2.2.18 and 5.1.12, Corollary 4.1.12 and Theorem 2.1.12, it is clear that

$$S_A \in \mathcal{R}^\blacktriangle \iff S_A \in \mathcal{R}^{\blacktriangle\#} \iff A \in \tau_{\mathcal{R}^\blacktriangle} \text{ and } A \in \tau\!\!\!-_{\mathcal{R}^\blacktriangle} \iff$$
$$\iff E_{\mathcal{R}} \subset A \text{ and } A \subset D_{\mathcal{R}} \iff E_{\mathcal{R}} \subset A \text{ and } E_{\mathcal{R}} \subset X \setminus A \iff E_{\mathcal{R}} = \emptyset.$$

Finally, we note that analogously to Theorems 4.2.15 and 4.2.11, the following two theorems are also true.

Theorem 5.1.23. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $\emptyset \neq A \neq X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\blacklozenge$; *(2)* $\mathcal{R} \neq \{X^2\}$.

Remark 5.1.24. Moreover, note that, by Remark 4.2.18, we can write S_A in place of R_A in Remark 4.2.16 and Theorem 4.2.17.

5.2 Connectedness of arbitrary relators

Definition 5.2.1. A relator $\mathcal{R}$ on X will be called properly connected if the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is properly well-chained.

Moreover, if $\Box$ is a unary operation for relators on X, then the relator $\mathcal{R}$ will be called $\Box$-connected if the relator $\mathcal{R}^\Box$ is properly connected.

Remark 5.2.2. The appropriateness of the above apparently very strange definition should have already been quite obvious from the results of Kurdics [23]. However, despite this, it has later been still overlooked even by Kurdics and Száz [27].

The proper connectedness of $\mathcal{R}$, i.e., the condition $\big(\mathcal{R}\triangledown\mathcal{R}^{-1}\big)^{\infty} = \{X^2\}$, by Remark 4.3.2, means only that for every $x, y \in X$, with $x \neq y$, and every $R \in \mathcal{R}$ there exist a finite family $(x_i)_{i=0}^{n}$ in X such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R \cup R^{-1}$, i.e., $(x_{i-1}, x_i) \in R$ or $(x_i, x_{i-1}) \in R$ for all $i = 1, \ldots, n$.

Moreover, as a close analogue of Theorem 4.3.4, we can also easily prove the following

Theorem 5.2.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected;

(2) $S_A \notin \mathcal{R}^*$ for every proper nonvoid subset A of X;

(3) $S \notin \mathcal{R}^*$ for every proper equivalence S on X;

(4) $S \notin \mathcal{R}^*$ for every proper nonvoid symmetric and transitive relation S on X.

Proof. From Definition 5.2.1 and Theorem 4.3.4, we can at once see that the assertion (1) holds if and only if $R_A \notin \big(\mathcal{R}\triangledown\mathcal{R}^{-1}\big)^*$ for all proper nonvoid subset A of X. Moreover, from Theorem 5.1.5, we know that for any $A \subset X$ we have $R_A \notin \big(\mathcal{R}\triangledown\mathcal{R}^{-1}\big)^*$ if and only if $S_A \notin \mathcal{R}^*$. Therefore, the assertions (1) and (2) are equivalent.

On the other hand, it is clear that the implications (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) are true. Namely, S_A is a proper equivalence on X whenever A is a proper nonvoid subset of X. Therefore, to complete proof, we need only show that the implication (1) $\Longrightarrow$ (4) is also true.

For this, note that if the assertion (4) does not hold, then there exists a proper nonvoid symmetric and transitive relation S on X such that $S \in \mathcal{R}^*$. Therefore, there exists an $R \in \mathcal{R}$ such that $R \subset S$. Hence, it follows that $R \cup R^{-1} \subset S \cup S^{-1} = S$. Therefore, $S \in \big(\mathcal{R}\triangledown\mathcal{R}^{-1}\big)^*$. And thus, by Theorem 4.3.4 and Definition 5.2.1, the assertion (1) does not also holds.

Remark 5.2.4. The assertion (3) of Theorem 5.2.3 can be briefly verbalized by saying that X^2 is the only equivalence being contained in $\mathcal{R}^*$.

Now, as a useful consequence of Theorem 5.2.3, we can also state the following

Theorem 5.2.5. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(Y) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected;

(2) $f^{-1} \circ f \notin \mathcal{R}^*$ for every non-constant function f of X into Y.

Proof. If the assertion (2) does not hold, then there exists a non-constant function f of X into Y such that $f^{-1} \circ f \in \mathcal{R}^*$. Hence, since

$$f^{-1} \circ f = \big\{(u, v) \in X^2 : f(u) = f(v)\big\}$$

is a proper equivalence on X, Theorem 5.2.3 shows that the assertion (1) does not also holds. Therefore, the implication (1) $\Longrightarrow$ (2) is true.

While, if the assertion (1) does not hold, then by Theorem 5.2.3 there exists a proper nonvoid subset A of X such that $S_A \in \mathcal{R}^*$. Hence, by choosing $y, z \in Y$ such that $y \neq z$, and defining a function f on X such that $f(x) = y$ for all $x \in A$ and $f(x) = z$ for all $x \in X \setminus A$, we can at once see that $f^{-1} \circ f = S_A \in \mathcal{R}^*$. That is, the assertion (2) does not also hold. Therefore, the implication (2) $\Longrightarrow$ (1) is also true.

Hence, by Theorem 3.3.6, it is clear that we also have the following

Theorem 5.2.6. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(Y) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected;

(2) every uniformly continuous function f of $X(\mathcal{R})$ into $Y(\Delta_Y)$ is constant.

Proof. Note that if f is a function of X into Y, then $f^{-1} \circ f = f^{-1} \circ \Delta_Y \circ f$. Moreover, by Theorem 3.3.6, f is a uniformly continuous function of $X(\mathcal{R})$ into $Y(\Delta_Y)$ if and only if $f^{-1} \circ \Delta_Y \circ f \in \mathcal{R}^*$. Therefore, the assertion (2) of Theorem 5.2.6 is equivalent to that of Theorem 5.2.5.

From Theorem 5.2.3, we can also easily get the following

Theorem 5.2.7. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected;

(2) for each proper nonvoid subset A of X there exists a net (x, y) in $A \times (X \setminus A) \cup (X \setminus A) \times A$ such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$;

(3) for any two nonvoid subsets A and B of X, with $X = A \cup B$, there exists a net (x, y) in $A \times B \cup B \times A$ such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$.

Hint. If the assertion (1) holds, then by Theorem 5.2.3, for each proper nonvoid subset A of X, we have $S_A \notin \mathcal{R}^*$. Therefore, for each $R \in \mathcal{R}$, there exists a pair $(y_R, x_R) \in R$ such that $(y_R, x_R) \notin S_A$, and hence by Proposition 5.1.2 $(y_R, x_R) \in A \times (X \setminus A) \cup (X \setminus A) \times A$. Now, by defining $x = (x_R)_{R \in \mathcal{R}}$ and $y = (y_R)_{R \in \mathcal{R}}$, and preordering $\mathcal{R}$ with the reverse set inclusion (the discrete preorder), we can easily see that (x, y) is a partially ordered (directed) net in $A \times (X \setminus A) \cup (X \setminus A) \times A$ such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$. Therefore, the assertion (2) also holds.

On the other hand, by using Corollary 4.3.6, we can also easily prove the following

Theorem 5.2.8. *If $\mathcal{R}$ is a properly connected relator on X and* $\operatorname{card}(X) > 1$, *then $X = R(X) \cup R^{-1}(X)$ for all $R \in \mathcal{R}$.*

Proof. In this case, by Corollary 4.3.6, the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is total. Therefore, we have $X = \big(R \cup R^{-1}\big)^{-1}(X) = \big(R \cup R^{-1}\big)(X) = R(X) \cup R^{-1}(X)$ for all $R \in \mathcal{R}$.

From the equality $\mathcal{R}\triangledown\mathcal{R}^{-1}=\mathcal{R}^{-1}\triangledown\big(\mathcal{R}^{-1}\big)^{-1}$, by Definition 5.2.1, it is clear that now we also have

Theorem 5.2.9. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected; *(2) $\mathcal{R}^{-1}$ is properly connected.*

Moreover, by using Theorem 5.2.5, we can also easily prove the following

Theorem 5.2.10. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X such that $\mathcal{R}$ is reflexive and properly connected and $\mathcal{S}$ is uniformly refined by $\mathcal{R}$, then $\mathcal{S}$ is also properly connected.*

Proof. If this is not the case, then by Theorem 5.2.5 there exists a function f of X onto $\{0,1\}$ such that $f^{-1}\circ f\in\mathcal{S}^*$. Hence, since $\mathcal{S}^*$ is also uniformly refined by $\mathcal{R}$, we can infer that there exists a function g on X to X such that $f^{-1}\circ f\circ g\in\mathcal{R}^*$. Hence, since $\mathcal{R}^*$ is also reflexive, we can infer that $x\in f^{-1}\big(f\big(g(x)\big)\big)$, and thus $f(x)=f\big(g(x)\big)$ for all $x\in X$. Therefore, we have $f=f\circ g$, and thus $f^{-1}\circ f=f^{-1}\circ f\circ g\in\mathcal{R}^*$. Hence, by Theorem 5.2.5, it follows that $\mathcal{R}$ is also not properly connected, and this contradiction proves the theorem.

Remark 5.2.11. Later we shall see that the counterpart of Theorem 5.2.10 with 'connected' replaced by 'well-chained' is not true.

Now, as some immediate consequence of the corresponding definitions and Theorems 5.2.3 and 5.2.5, we can also state the following theorems.

Theorem 5.2.12. *If $\mathcal{R}$ is a relator on X and $\Box$ is an $*$-invariant operation for relators on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\Box$-connected;

(2) $S_A\notin\mathcal{R}^{\Box}$ for every proper nonvoid subset A of X;

(3) $S\notin\mathcal{R}^{\Box}$ for every proper equivalence S on X;

(4) $S\notin\mathcal{R}^{\Box}$ for every proper nonvoid symmetric and transitive relation S on X.

Theorem 5.2.13. *If $\mathcal{R}$ is a relator on X, $\Box$ is an $*$-invariant operation for relators on X and* $\operatorname{card}(Y)>1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\Box$-connected;

(2) $f^{-1}\circ f\notin\mathcal{R}^{\Box}$ for every non-constant function f of X into Y.

Hence, by using Theorems 3.3.6, 3.3.7 and 3.3.8, we can also easily get the following two theorems.

Theorem 5.2.14. *If $\mathcal{R}$ is a relator on X, $\Box\in\{*,\#,\wedge,\bullet\}$ and* $\operatorname{card}(Y)>1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\Box$-connected;

(2) every $\Box$-continuous function f of $X(\mathcal{R})$ into $Y(\Delta_Y)$ is constant.

Theorem 5.2.15. *If $\mathcal{R}$ is a relator on X, $\square \in \{\vartriangle, \blacktriangle, \blacklozenge\}$ and* $\operatorname{card}(Y) = 2$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$-connected;

(2) every $\square$-continuous function f of $X(\mathcal{R})$ into $Y(\Delta_Y)$ is constant.

Hint. If the assertion (1) does not hold, then by Theorem 5.2.13 there exists a function f of X onto Y such that $f^{-1} \circ \Delta_Y \circ f = f^{-1} \circ f \in \mathcal{R}^{\square}$. Hence, by Theorem 3.3.8, it follows that f is a $\square$-continuous function of $X(\mathcal{R})$ onto $Y(\Delta_Y)$. Therefore, the assertion (2) does not also hold. Consequently, the implication (2) $\Longrightarrow$ (1) is true.

Remark 5.2.16. Note that if, for instance, $1 < \operatorname{card}(X) < \operatorname{card}(Y)$, then by Corollary 3.3.21 every $\vartriangle$-continuous function f of $X(\Delta_X)$ into $Y(\Delta_Y)$ is constant, but by the equivalence of the assertions (1) and (3) of Theorem 5.2.12 the relator $\{\Delta_X\}$ is not $\vartriangle$-connected.

In this respect, it is also worth mentioning that, by using the corresponding results of Subsection 5.4, it can be easily shown that if $\square \in \{\vartriangle, \blacktriangle, \blacklozenge\}$ and f is a $\square$-continuous function on a $\square$-connected relator space $X(\mathcal{R})$ to a $\square$-separated relator space $Y(\mathcal{S})$, then f is necessarily constant.

Finally, as some immediate consequence of the corresponding definitions and Theorem 5.2.9, we can also state

Theorem 5.2.17. *If $\mathcal{R}$ is a relator on X and $\square$ is an inversion compatible operation for relators on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$-connected; *(2) $\mathcal{R}^{-1}$ is $\square$-connected.*

5.3 Connectedness of refinement relators

Definition 5.3.1. A relator $\mathcal{R}$ on X will be called uniformly, proximally, topologically, paratopologically, infinitesimally, ultrainfinitesimally, parainfinitesimally, and ultimately connected if it is $\square$-connected with $\square = *, \#, \wedge, \vartriangle, \bullet, \bigstar, \blacktriangle$, and $\blacklozenge$, respectively.

Remark 5.3.2. From the inclusions relations of the above operations, it is clear that 'paratopologically or infinitesimally connected' $\Longrightarrow$ 'topologically connected' $\Longrightarrow$ 'proximally connected' $\Longrightarrow$ 'uniformly connected' $\Longrightarrow$ 'properly connected'.

And 'ultimately connected' $\Longrightarrow$ 'parainfinitesimally connected' $\Longrightarrow$ 'ultrainfinitesimally connected' $\Longrightarrow$ 'paratopologically and infinitesimally connected'.

Moreover, as an immediate consequence of Theorems 5.2.3 and 5.2.12, we can at once state the following

Theorem 5.3.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected; *(2) $\mathcal{R}$ is uniformly connected.*

Remark 5.3.4. Later we shall see that 'uniformly connected' $\not\Leftarrow$ 'proximally connected' $\not\Leftarrow$ 'topologically connected' $\not\Leftarrow$ ' paratopologically or infinitesimally connected'. And 'paratopologically and infinitesimally connected' are independent notions.

Moreover, 'paratopologically and infinitesimally connected' $\not\Leftarrow$ 'ultrainfinitesimally connected' $\not\Leftarrow$ 'parainfinitesimally connected' $\not\Leftarrow$ 'ultimately connected'.

However, as an immediate consequence of Theorems 5.2.3 and 5.2.12 and Corollary 5.1.14, we still have the following counterpart of Theorem 4.4.3.

Theorem 5.3.5. *If $\mathcal{R}$ is a uniformly filtered relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected; *(2) $\mathcal{R}$ is proximally connected.*

Moreover, as an immediate consequence of Theorems 5.2.12 and 5.1.12, we can also at once state

Theorem 5.3.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally connected; *(2) $\tau_{\mathcal{R}} \cap \tau_{\mathcal{R}} = \{\emptyset, X\}$.*

In addition to this theorem, it is also worth proving the following

Theorem 5.3.7. *If $\mathcal{R}$ is a uniformly filtered relator on X and $\operatorname{card}(X) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally connected;

(2) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ and $X \setminus A \in \operatorname{Int}_{\mathcal{R}}(X \setminus B)$ imply $A \not\subset B$ for all $A, B \subset X$ with $A \neq X$ and $B \neq \emptyset$;

(3) $X = A \cup B$ implies $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ or $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ for all $A, B \subset X$ with $A \neq \emptyset$ and $B \neq \emptyset$.

Proof. From Theorem 5.3.5 we know that $\mathcal{R}$ is proximally connected if and only if $\mathcal{R}$ is properly connected. Moreover, from Corollary 5.1.6, by using Theorems 5.2.3 and 4.3.4, we can see $\mathcal{R}$ is properly connected if and only if $\mathcal{R} \vee \mathcal{R}^{-1}$ is properly well-chained.

On the other hand, from Theorem 4.3.11 we know that $\mathcal{R} \vee \mathcal{R}^{-1}$ is properly well-chained if and only if $X = A \cup B$ implies $B \in \operatorname{Cl}_{\mathcal{R} \vee \mathcal{R}^{-1}}(A)$ for all $A, B \subset X$ with $A \neq \emptyset$ and $B \neq \emptyset$. Moreover, from Theorem 3.1.12 we know that $B \in \operatorname{Cl}_{\mathcal{R} \vee \mathcal{R}^{-1}}(A)$ if and only if $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ or $B \in \operatorname{Cl}_{\mathcal{R}^{-1}}(A)$. Furthermore, from Remark 2.1.5, we know that $B \in \operatorname{Cl}_{\mathcal{R}^{-1}}(A)$ if and only if $A \in \operatorname{Cl}_{\mathcal{R}}(B)$. Therefore the assertion (1) and (3) are equivalent.

The equivalence of the assertions (2) and (3) is again immediate from Theorem 2.1.3.

From Theorems 5.3.6 and 5.3.7, by Theorem 2.1.16, it is clear that we also have the following

Theorem 5.3.8. *If $\mathcal{R}$ is a uniformly filtered relator on X and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally connected;

(2) for each proper nonvoid subset A of X, there exists a net (x, y) in $A \times (X \setminus A)$ or $(X \setminus A) \times A$ such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$;

(3) for any two nonvoid subsets A and B of X, with $X = A \cup B$, there exists a net (x, y) in $A \times B$ or $B \times A$ such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$.

From Theorem 5.3.6, by Theorem 2.2.18 and Corollary 4.1.2, it is clear that we also have the following

Theorem 5.3.9. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically connected; *(2) $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} = \{\emptyset, X\}$.*

In addition to Theorem 5.3.9, we can also easily prove

Theorem 5.3.10. *If $\mathcal{R}$ is a topologically filtered relator on X and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically connected;

(2) $B \subset \operatorname{int}_{\mathcal{R}}(A)$ and $X \setminus A \subset \operatorname{int}_{\mathcal{R}}(X \setminus B)$ imply $A \not\subset B$ for all $A, B \subset X$ with $A \neq X$ and $B \neq \emptyset$;

(3) $X = A \cup B$ implies $B \cap \operatorname{cl}_{\mathcal{R}}(A) \neq \emptyset$ or $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ for all $A, B \subset X$ with $A \neq \emptyset$ and $B \neq \emptyset$.

Proof. By Theorem 2.2.18, it is clear that $\mathcal{R}$ is topologically connected if and only if $\mathcal{R}^{\wedge}$ is proximally connected. Moreover, since $\mathcal{R}^{\wedge}$ is now uniformly filtered, from Theorem 5.3.7 we know that $\mathcal{R}^{\wedge}$ is proximally connected if and only if $B \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(A)$ and $X \setminus A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(X \setminus B)$ imply $A \not\subset B$ for all $A, B \subset X$ with $A \neq X$ and $B \neq \emptyset$. Moreover, from Theorem 4.1.1, we know that $B \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(A)$ if and only if $B \subset \operatorname{int}_{\mathcal{R}}(A)$. Therefore, the assertions (1) and (2) are equivalent.

Moreover, by Theorem 2.1.3, it is clear that the assertions (2) and (3) are also equivalent.

Remark 5.3.11. The assertion (2) of Theorem 5.3.9 can be briefly verbalized by saying that no proper nonvoid subset of X is topologically clopen.

While, the assertion (3) of Theorem 5.3.10 can be briefly verbalized by saying that X cannot be decomposed into the union of two nonvoid separated sets.

From Theorems 5.3.9 and 5.3.10, by using Corollary 2.1.17, we can also get the following

Theorem 5.3.12. *If $\mathcal{R}$ is a topologically filtered relator on X and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically connected;

(2) for each proper nonvoid subset A of X there exist a net x in A and a point y in $X \setminus A$, or a net x in $X \setminus A$ and a point y in A such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$;

(3) for any two nonvoid subsets A and B of X, with $X = A \cup B$, there exist a net x in A and a point y in B, or a net x in B and a point y in A such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$.

Hence, analogously to Ward [83, Theorem 82, p. 66], we can also state

Corollary 5.3.13. *If $\mathcal{R}$ is a topologically filtered relator on X and $\operatorname{card}(X) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically connected;

(2) for each proper nonvoid topologically open subset A of $X(\mathcal{R})$ there exist a net x in A and a point y in $X \setminus A$ such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$;

(3) for each proper nonvoid topologically closed subset A of $X(\mathcal{R})$ there exist a net x in $X \setminus A$ and a point y in A such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$.

Hint. To prove the implication (2) $\Longrightarrow$ (1), suppose on the contrary that the assertion (2) holds, but the assertion (1) does not hold. Then, by Theorem 5.3.9 there exists a proper nonvoid subset A of X such that A is both topologically open and closed in $X(\mathcal{R})$. Therefore, by the assertion (2), there exists a net x in A and point y in $X \setminus A$ such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$. Hence, by Theorem 2.1.17, it follows that $y \in \operatorname{cl}_{\mathcal{R}}(A) \subset A$, and this contradicts to the fact that $y \in X \setminus A$.

5.4 Some further results on the connectedness of refinement relators

Theorem 5.4.1. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected; *(2) $\mathcal{E}_{\mathcal{R}} \setminus \{\emptyset\} \subset \mathcal{D}_{\mathcal{R}} \cup \{X\}$.*

Proof. From Theorem 5.3.6, by Theorem 2.2.18 and Corollary 4.1.4, it is clear that the assertion (1) holds if and only if

$$\big(\mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}\big) \cap \big((\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}) \cup \{X\}\big) = \{\emptyset, X\}.$$

Since $\emptyset \notin \mathcal{D}_{\mathcal{R}}$ and $X \in \mathcal{E}_{\mathcal{R}}$ the assertions (1) and (2) are also equivalent.

Corollary 5.4.2. *If $\mathcal{R}$ is a paratopologically connected relator on X such that $\operatorname{card}(X) > 1$, then $\mathcal{R}$ is a total relator on X.*

Proof. If this is not the case, then by Theorem 3.2.3 we have $\mathcal{E}_{\mathcal{R}} = \mathcal{P}(X)$ and $\mathcal{D}_{\mathcal{R}} = \emptyset$. Hence, using that $\operatorname{card}(X) > 1$, we can see that the assertion (2) of Theorem 5.4.1 fails to hold. And this contradicts the paratopological connectedness of $\mathcal{R}$.

Remark 5.4.3. Later, we shall see that even an infinitesimally connected relator $\mathcal{R}$ on X need not be total.

Theorem 5.4.4. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected;

(2) $\mathcal{E}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}}$; *(3) $\mathcal{R}(x) \subset \mathcal{D}_{\mathcal{R}}$ for all $x \in X$.*

Proof. If the assertion (1) holds, then by Corollary 5.4.2 and Theorem 3.2.3 we have $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ and $X \in \mathcal{D}_{\mathcal{R}}$. And hence, by Theorems 5.4.1, it is clear that the assertion (2) also holds.

Moreover, by Theorem 5.4.1, it is clear that the converse implication is true even if X is a singleton.

Finally, to complete the proof, we note that the equivalence of the assertions (2) and (3) is immediate from the facts that $\mathcal{E}_{\mathcal{R}}$ is the smallest ascending family in X containing $\mathcal{R}(x) = \{R(x) : R \in \mathcal{R}\}$ for all $x \in X$, and moreover $\mathcal{D}_{\mathcal{R}}$ is also an ascending family in X.

Theorem 5.4.5. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected;

(2) $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{E}_{\mathcal{R}}$;

(3) $R(x) \cap S(y) \neq \emptyset$ for all $x, y \in X$ and $R, S \in \mathcal{R}$.

Proof. From Theorem 5.4.4 we know that the assertion (1) is equivalent to the inclusion $\mathcal{E}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}}$. Moreover, from Theorem 2.1.9 we can see that the inclusion $\mathcal{E}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}}$ equivalent to the assertion (2).

On the other hand, by the corresponding definitions, it is clear that the assertions (2) and (3) are also equivalent.

Corollary 5.4.6. *If $\mathcal{R}$ is a topological relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected;

(2) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subset \mathcal{D}_{\mathcal{R}}$; *(3) $U \cap V \neq \emptyset$ for all $U, V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.*

Proof. Now, by Theorem 3.2.8, for any $A \subset X$, we have $A \in \mathcal{E}_{\mathcal{R}}$ if and only if there exists a $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subset A$. Therefore, by Theorems 5.4.4 and 5.4.5, the required assertions are equivalent whenever $\operatorname{card}(X) > 1$.

However, since a topological relator is in particular total, the above cardinality condition can be omitted.

Remark 5.4.7. Hence, it is clear that the hyperconnectedness of Steen and Seebach [62, p. 29], studied also by Levine [28] and several other people (see [60], [47] and [1]), is a particular case of our paratopological connectedness.

Moreover, from Theorem 5.4.5 we can also at once see that the semi-directedness of Száz [68] coincides with our paratopological connectedness. Therefore, according to the corresponding results of [68], we also have the following theorems.

Theorem 5.4.8. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected;

(2) $X \setminus A \notin \mathcal{E}_{\mathcal{R}}$ for all $A \in \mathcal{E}_{\mathcal{R}}$;

(3) $A \in \mathcal{D}_{\mathcal{R}}$ whenever $A \subset X$ such that $A \cap B \neq \emptyset$ for all $B \in \mathcal{D}_{\mathcal{R}}$.

Theorem 5.4.9. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected;

(2) $A \in \mathcal{D}_{\mathcal{R}}$ or $X \setminus A \in \mathcal{D}_{\mathcal{R}}$ for all $A \subset X$;

(3) $A \in \mathcal{D}_{\mathcal{R}}$ or $B \in \mathcal{D}_{\mathcal{R}}$ whenever $X = A \cup B$.

Theorem 5.4.10. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected; *(2) $\mathcal{R}^{-1} \circ \mathcal{R} = \{X^2\}$;*

(3) $\rho_{\mathcal{R}^{-1} \circ \mathcal{R}} = X^2$; *(4) $cl_{\mathcal{R} \square \mathcal{R}}(\Delta_X) = X^2$.*

Proof. If the assertion (1) holds, then by Theorem 5.4.5 for any $x, y \in X$ and $R, S \in \mathcal{R}$ we have $R(x) \cap S(y) \neq \emptyset$. Hence, it follows that

$$y \in S^{-1}\big(R(x)\big) = \big(S^{-1} \circ R\big)(x).$$

Therefore, we have $\big(S^{-1} \circ R\big)(x) = X = X^2(x)$ for all $x \in X$. And hence, it follows that $S^{-1} \circ R = X^2$. Therefore, $\mathcal{R}^{-1} \circ \mathcal{R} = \{X^2\}$. That is, the assertion (2) also holds.

The converse implication (2) $\Longrightarrow$ (1) can be proved quite similarly by reversing the above argument.

While, to prove the equivalences (2) $\Longleftrightarrow$ (3) and (3) $\Longleftrightarrow$ (4), it is enough to note that

$$\rho_{\mathcal{R}^{-1} \circ \mathcal{R}} = \bigcap \mathcal{R}^{-1} \circ \mathcal{R} \qquad \text{and} \qquad \rho_{\mathcal{R}^{-1} \circ \mathcal{R}} = \mathrm{cl}_{\mathcal{R} \square \mathcal{R}}(\Delta_X).$$

Remark 5.4.11. The above theorem is a counterpart of the more familiar statement that a relator $\mathcal{R}$ on X is reflexive and properly separating if and only if $\rho_{\mathcal{R}^{-1} \circ \mathcal{R}} = \Delta_X$, or equivalently $cl_{\mathcal{R} \square \mathcal{R}}(\Delta_X) = \Delta_X$.

From Theorem 5.3.6, by Theorem 2.2.18 and Corollary 4.1.6, it is clear that we also have the following

Theorem 5.4.12. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is infinitesimally connected;

(2) $\tau_{\{\rho_{\mathcal{R}}\}} \cap \tau\!\!\!-_{\{\rho_{\mathcal{R}}\}} = \{\emptyset, X\}$; *(3) $\{\rho_{\mathcal{R}}\}$ is properly connected.*

Hint. By Theorem 5.3.6, assertion (2) holds if and only if the relator $\{\rho_{\mathcal{R}}\}$ is proximally connected. On the other hand, by Theorems 2.2.5 and 5.3.3 it is equivalent to assertion (3).

From Theorem 5.3.6, by Theorem 2.2.18 and Corollary 4.1.8, it is clear that we also have the following

Theorem 5.4.13. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is σ-infinitesimally connected;

(2) $\big(\text{card}(X) \leq 2$ and $\tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}} = \{\emptyset, X\}\big)$ or $\big(\text{card}(X) = 3$ and $\{x\} \notin \tau_{\mathcal{R}}$ for all $x \in X\big)$.

Theorem 5.4.14. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is ultrainfinitesimally connected;

(2) $X = \rho_{\mathcal{R}}(A)$ or $X = \rho_{\mathcal{R}}(X \setminus A)$ for every proper nonvoid subset A of X;

Proof. From Theorem 5.3.6, by Theorem 2.2.18 and Corollary 4.1.10, it is clear that the assertion (1) holds if and only if

$$\big(\{A \subset X : X \neq \rho_{\mathcal{R}}(X \setminus A)\} \cup \{\emptyset\}\big) \cap \big(\{A \subset X : X \neq \rho_{\mathcal{R}}(A)\} \cup \{X\}\big) = \{\emptyset, X\}.$$

Since $\rho_{\mathcal{R}}(\emptyset) = \emptyset \neq X$ the assertions (1) and (2) are also equivalent.

Theorem 5.4.15. *If $\mathcal{R}$ is a relator on X and* $\text{card}(x) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is parainfinitesimally connected;

(2) $E_{\mathcal{R}} \neq \emptyset$;

(3) $D_{\mathcal{R}} \neq X$.

Proof. From Theorem 5.3.6, by Theorem 2.2.18 and Corollary 4.1.12, it is clear that the assertion (1) holds if and only if

$$\big(\{A \subset X : E_{\mathcal{R}} \subset A\} \cup \{\emptyset\}\big) \cap \big(\{A \subset X : A \subset D_{\mathcal{R}}\} \cup \{X\}\big) = \{\emptyset, X\}.$$

Since $E_{\mathcal{R}} \subset X$ and $\emptyset \subset D_{\mathcal{R}}$ it is equivalent to

$$\{A \subset X : E_{\mathcal{R}} \subset A \subset D_{\mathcal{R}}\} \subset \{\emptyset, X\}.$$

By Theorem 2.1.12the assertions (1), (2) and (2) are also equivalent.

Remark 5.4.16. Later, we shall see that the inverse of a topologically, paratopologically, or parainfinitesimally connected relator need not have the same connectedness property.

However, the following counterpart of Theorem 4.4.10 shows that the inverse of an ultimately connected relator is still ultimately connected.

Theorem 5.4.17. *If $\mathcal{R}$ is a relator on X and* $\text{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is ultimately connected;

(2) $\mathcal{R}$ is ultimately well-chained;

(3) $\mathcal{R} = \{X^2\}$.

Proof. From Theorem 5.3.6, by Theorem 2.2.18 and Corollary 4.1.12, it is clear that the assertions (1) and (2) are equivalent. Moreover, by Theorem 4.4.14 and 4.4.10 assertions (2) and (3) are also equivalent.

Remark 5.4.18. From Theorem 5.2.3, by Remark 4.2.18, we can at once see that a relator $\mathcal{R}$ on X is ∞-connected if and only if it is properly connected. Therefore, the 'quasi-connectedness properties' of relators need not also be studied.

Moreover, from Theorem 5.2.12, by Remark 4.2.18, we can at once see that the 'almost uniform (almost proximal) and the superproximal (supertopological) connectedness properties' of relators need not also be studied.

However, by Remark 4.4.18, a relator $\mathcal{R}$ on X may naturally be called properly connected at a point $x \in X$ if the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is properly well-chained at x. Therefore, localized forms of the corresponding connectedness properties of relators may also be investigated.

6 Comparison of well-chainednesses and connectednesses

6.1 Relationships between well-chainedness and connectedness properties

Theorem 6.1.1. *If $\mathcal{R}$ is a properly well-chained relator on X, then $\mathcal{R}$ is, in particular, properly connected.*

Proof. If $\mathcal{R}$ is properly well-chained, then $R^{\infty} = X^2$ for all $R \in \mathcal{R}$. Hence, since $R \subset R \cup R^{-1}$, it is clear that $\big(R \cup R^{-1}\big)^{\infty} = X^2$ for all $R \in \mathcal{R}$. Therefore, the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is also properly well-chained, and thus $\mathcal{R}$ is properly connected.

Remark 6.1.2. Later we shall see that even a parainfinitesimally connected relator need not be properly well-chained. However, as an immediate consequence of the corresponding definitions, we still have the following

Theorem 6.1.3. *If $\mathcal{R}$ is a strongly symmetric relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected; *(2) $\mathcal{R}$ is properly well-chained.*

Proof. If $\mathcal{R}$ is strongly symmetric, then $R^{-1} = R$, and hence $R \cup R^{-1} = R$ for all $R \in \mathcal{R}$. Therefore, $\mathcal{R} \triangledown \mathcal{R}^{-1} = \mathcal{R}$, and hence in particular $\big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^{\infty} = \mathcal{R}^{\infty}$. Thus, the assertions (1) and (2) are equivalent.

Concerning properly connected relators, we can also easily prove the following theorems.

Theorem 6.1.4. *If $\mathcal{R}$ is a uniformly filtered relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected; *(2) $\mathcal{R} \vee \mathcal{R}^{-1}$ is properly well-chained.*

Proof. In this case, by Corollary 5.1.6, for any $A \subset X$ we have $S_A \in \mathcal{R}^*$ if and only if $R_A \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*$. Therefore, by Theorems 5.2.3 and 4.3.4, the required assertions are equivalent.

Theorem 6.1.5. *If $\mathcal{R}$ is a reflexive relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected;

(2) $\mathcal{R} \odot \mathcal{R}^{-1}$ is properly well-chained; *(3) $\mathcal{R}^{-1} \odot \mathcal{R}$ is properly well-chained.*

Proof. In this case, by Theorem 5.1.7, for any $A \subset X$ we have $S_A \in \mathcal{R}^*$ if and only if $R_A \in \big(\mathcal{R} \odot \mathcal{R}^{-1}\big)^*$, or equivalently $R_A \in \big(\mathcal{R}^{-1} \odot \mathcal{R}\big)^*$. Therefore, by Theorems 5.2.3 and 4.3.4, the required assertions are equivalent.

Remark 6.1.6. The proper well-chainedness of the relator $\mathcal{R}^{-1} \odot \mathcal{R}$, that is, the condition $\big(\mathcal{R}^{-1} \odot \mathcal{R}\big)^{\infty} = \{X^2\}$, by Remark 4.3.2, means only that for any $x, y \in X$, with $x \neq y$, and any $R \in \mathcal{R}$ there exists a finite family $(x_i)_{i=0}^{n}$ in X such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R^{-1} \circ R$, that is, $R(x_{i-1}) \cap R(x_i) \neq \emptyset$ for all $i = 1, \ldots, n$.

Theorem 6.1.7. *If $\mathcal{R}$ is a uniformly filtered reflexive relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected;

(2) $\mathcal{R} \circ \mathcal{R}^{-1}$ is properly well-chained; *(3) $\mathcal{R}^{-1} \circ \mathcal{R}$ is properly well-chained.*

Proof. In this case, by Corollary 5.1.9, for any $A \subset X$ we have $S_A \in \mathcal{R}^{*}$ if and only if $R_A \in \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^{*}$, or equivalently $R_A \in \big(\mathcal{R}^{-1} \circ \mathcal{R}\big)^{*}$. Therefore, by Theorems 5.2.3 and 4.3.4, the required assertions are equivalent.

As an immediate consequence of Theorem 6.1.1, we can at once state

Theorem 6.1.8. *If $\square$ is a unary operation for relators on X and $\mathcal{R}$ is a $\square$-well-chained relator on X, then $\mathcal{R}$ is, in particular, $\square$-connected.*

Moreover, in addition to Theorem 6.1.3, we can also easily prove the following

Theorem 6.1.9. *If $\mathcal{R}$ is a quasi-proximally symmetric relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally connected; *(2) $\mathcal{R}$ is proximally well-chained.*

Proof. In this case, by Theorem 3.2.16, we have $\digamma_{\mathcal{R}} = \tau_{\mathcal{R}}$, and hence $\tau_{\mathcal{R}} \cap \digamma_{\mathcal{R}} = \tau_{\mathcal{R}}$. Therefore, by Theorems 5.3.6, 4.3.9 and 4.4.3, the required assertions are equivalent.

From Theorem 6.1.9, by Theorems 5.3.3, 5.3.5 and 4.4.3, it is clear that in particular we also have the following

Corollary 6.1.10. *If $\mathcal{R}$ is a uniformly filtered and quasi-proximally symmetric relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly (uniformly) connected;

(2) $\mathcal{R}$ is properly (uniformly) well-chained.

Moreover, from Theorem 6.1.9, we can also easily get the following

Theorem 6.1.11. *If $\square$ is a $\#$-invariant operation for relators on X and $\mathcal{R}$ is a quasi-$\square$-symmetric relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\square$-connected; *(2) $\mathcal{R}$ is $\square$-well-chained.*

Proof. Since $\mathcal{R}$ is quasi-$\square$-symmetric, we have $\big(\mathcal{R}^{\square\infty}\big)^{-1} = \mathcal{R}^{\square\infty}$. Hence, since $\mathcal{R}^{\square\#} = \mathcal{R}^{\square}$, we can infer that $\Big(\big(\mathcal{R}^{\square}\big)^{\#\infty}\Big)^{-1} = \big(\mathcal{R}^{\square}\big)^{\#\infty}$. Therefore, the relator $\mathcal{R}^{\square}$ is quasi-proximally symmetric.

Now, by Theorem 6.1.9, it is clear that the relator $\mathcal{R}^{\square}$ is proximally connected if and only if it is proximally well-chained. That is, the relator $\mathcal{R}^{\square\#}$ is properly connected if and only if it is properly well-chained. And hence, since $\mathcal{R}^{\square\#} = \mathcal{R}^{\square}$, it is clear that the required assertions are equivalent.

Remark 6.1.12. From Theorem 6.1.11, in particular, it is clear that a quasi-topologically symmetric relator is topologically connected if and only if it is topologically well-chained.

6.2 A few illustrating examples

The following example shows that even a parainfinitesimally connected preorder relator need not be properly well-chained.

Example 6.2.1. If $X = \{1, 2\}$ and $R \subset X^2$ such that $R(1) = \{1\}$ and $R(2) = X$, then $\mathcal{R} = \{R\}$ is a parainfinitesimally connected preorder relator on X such that $\mathcal{R}$ is neither properly well-chained nor ultimately connected.

Note that $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} = R(1) \cap R(2) = \{1\} \neq \emptyset$. Therefore, by Theorem 5.4.15, $\mathcal{R}$ is parainfinitesimally connected. But, $R^{\infty} = R \neq X^2$. Therefore, by Definition 4.3.1, $\mathcal{R}$ is not properly well-chained. Moreover, by Theorem 5.4.17, $\mathcal{R}$ is not ultimately connected.

The following example shows that even a uniformly connected preorder relator need not be proximally connected. Moreover, the hypotheses of Theorems 5.3.5 and 6.1.3 cannot be significantly weakened.

Example 6.2.2. If $X = \{1, 2\}$ and $R \subset X^2$ such that $R(1) = \{1\}$ and $R(2) = X$, then $\mathcal{R} = \{R, R^{-1}\}$ is a uniformly connected, proximally filtered and properly symmetric preorder relator on X such that $\mathcal{R}$ is neither properly well-chained nor proximally connected.

Note that $S_{\{1\}} = S_{\{2\}} = \Delta_X$ is not contained in $\mathcal{R}^*$. Therefore, by Theorem 5.2.12, $\mathcal{R}$ is uniformly connected. But, $\{1\} \in \tau_{\mathcal{R}}$, and moreover $\{2\} \in \tau_{\mathcal{R}}$, and hence $\{1\} \in \mathscr{F}_{\mathcal{R}}$. Therefore, by Theorem 5.3.6, $\mathcal{R}$ isnot proximally connected. The required filteredness property of $\mathcal{R}$ is immediate from the fact that $\Delta_X \in \mathcal{R}^{\#}$.

The following two examples show that even a paratopologically connected tolerance or preorder relator need not be infinitesimally connected. Hence, the counterpart of Corollary 4.4.15 with 'well-chained' replaced by 'connected' is not true.

Example 6.2.3. If $X = \{1, 2, 3\}$ and $R_i \subset X^2$ for $i = 1, 2$ such that

$$
\begin{array}{lll}
R_1(1) = X, & R_1(2) = \{1, 2\}, & R_1(3) = \{1, 3\}, \\
R_2(1) = \{1, 2\}, & R_2(2) = X, & R_2(3) = \{2, 3\},
\end{array}
$$

then $\mathcal{R} = \{R_1, R_2\}$ is a proximally well-chained and paratopologically connected tolerance relator on X such that $\mathcal{R}$ is neither topologically well-chained nor infinitesimally connected.

Note that $\tau_{\mathcal{R}} = \{\emptyset, X\}$. Therefore, by Theorems 4.3.9 and 4.4.3, $\mathcal{R}$ is proximally well-chained. But, $\{1, 2\} \in \mathcal{T}_{\mathcal{R}}$. Therefore, by Theorem 4.4.5, $\mathcal{R}$ is not topologically well-chained.

Moreover, we evidently have $R_i(x) \cap R_j(y) \neq \emptyset$ for all $i, j \in \{1, 2\}$ and $x, y \in X$. Therefore, by Theorem 5.4.5, $\mathcal{R}$ is paratopologically connected.

But, $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = S_{\{3\}}$, and hence $S_{\{3\}} \in \left\{\rho_{\mathcal{R}}^{-1}\right\}^* = \mathcal{R}^\bullet$. Therefore, by Theorem 5.2.12, $\mathcal{R}$ is not infinitesimally connected.

Example 6.2.4. If $X = \{1, 2, 3\}$ and $R_i \subset X^2$ for all $i \in X$ such that

$$\begin{array}{lll} R_1(1) = X, & R_1(2) = \{2,3\}, & R_1(3) = \{2,3\}, \\ R_2(1) = \{1,3\}, & R_2(2) = X, & R_2(3) = \{1,3\}, \\ R_3(1) = \{1,2\}, & R_3(2) = \{1,2\}, & R_3(3) = X, \end{array}$$

then $\mathcal{R} = \{R_1, R_2, R_3\}$ is a paratopologically connected preorder relator on X such that $\mathcal{R}$ is neither properly well-chained nor infinitesimally connected.

Note that $R_i(x) \cap R_j(y) \neq \emptyset$ for all $i, j \in X$ and $x, y \in X$. Therefore, by Theorem 5.4.5, $\mathcal{R}$ is paratopologically connected. But, $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = \Delta_X$, and hence $\Delta_X \in \left\{\rho_{\mathcal{R}}^{-1}\right\}^* = \mathcal{R}^\bullet$. Therefore, by Theorem 5.2.12, $\mathcal{R}$ is not infinitesimally connected.

The following example shows that even an infinitesimally well-chained and ultrainfinitesimally connected reflexive relator need not be parainfinitesimally connected.

Example 6.2.5. If $X = \{1, 2, 3\}$ and $R \subset X^2$ such that

$$R(1) = \{1,2\}, \qquad R(2) = \{2,3\}, \qquad R(3) = \{1,3\},$$

then $\mathcal{R} = \{R\}$ is an infinitesimally well-chained and ultrainfinitesimally connected reflexive relator on X such that $\mathcal{R}$ is neither paratopologically well-chained nor parainfinitesimally connected.

Note that $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = R^{-1}$. Moreover, $R^2 = X^2$, and hence $R^\infty = X^2$. Therefore, $\rho_{\mathcal{R}}^\infty = \left(R^{-1}\right)^\infty = \left(R^\infty\right)^{-1} = X^2$, and thus by Theorem 4.4.16 $\mathcal{R}$ is infinitesimally well-chained. But, by Theorem 4.4.10, $\mathcal{R}$ is not paratopologically well-chained.

Moreover, $R^{-1}(1) = \{1,3\}$, $R^{-1}(2) = \{1,2\}$ and $R^{-1}(3) = \{2,3\}$. Therefore, $\rho_{\mathcal{R}}(A) = R^{-1}(A) = X$ for all $A \subset X$ with $\operatorname{card}(A) = 2$. Thus, by Theorem 5.4.14, $\mathcal{R}$ is ultrainfinitesimally connected. But, now we also have $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} = \bigcap_{i=1}^3 R(i) = \emptyset$. Therefore, by Theorem 5.4.15, $\mathcal{R}$ is not parainfinitesimally connected.

The following example shows that even an infinitesimally well-chained tolerance relator need not be paratopologically connected.

Example 6.2.6. If $X = \{1, 2, 3, 4\}$ and $R \subset X^2$ such that

$$R(1) = \{1,2\}, \qquad R(2) = \{1,2,3\}, \qquad R(3) = \{2,3,4\}, \qquad R(4) = \{3,4\},$$

then $\mathcal{R} = \{R\}$ is an infinitesimally well-chained and infinitesimally connected tolerance relator on X such that $\mathcal{R}$ is neither paratopologically well-chained nor paratopologically connected.

Note that $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = R^{-1} = R$. Moreover, $R^3 = X^2$, and hence $R^\infty = X^2$. Therefore, $\rho_{\mathcal{R}}^\infty = X^2$, and thus by Theorem 4.4.16 $\mathcal{R}$ is infinitesimally well-chained. And hence, by Theorem 6.1.8, $\mathcal{R}$ is infinitesimally connected. But, $R(1) \cap R(4) = \emptyset$, and thus by Theorem 5.4.5 $\mathcal{R}$ is not paratopologically connected. Moreover, by Theorem 4.4.10, $\mathcal{R}$ is not paratopologically well-chained.

The following example shows that even a topologically well-chained tolerance relator need not be infinitesimally connected.

Example 6.2.7. If $X = \{1,2,3,4\}$ and $R_i \subset X^2$ for $i = 1,2$ such that

$$\begin{array}{llll} R_1(1) = \{1,2,3\}, & R_1(2) = \{1,2,4\}, & R_1(3) = \{1,3,4\}, & R_1(4) = \{2,3,4\}, \\ R_2(1) = \{1,2,4\}, & R_2(2) = \{1,2,3\}, & R_2(3) = \{2,3,4\}, & R_2(4) = \{1,3,4\}, \end{array}$$

then $\mathcal{R} = \{R_1, R_2\}$ is a topologically well-chained and topologically connected tolerance relator on X such that $\mathcal{R}$ is neither infinitesimally well-chained nor infinitesimally connected.

Note that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$. Therefore, by Theorem 4.4.5, $\mathcal{R}$ is topologically well-chained. And hence, by Theorem 6.1.8, $\mathcal{R}$ is topologically connected. But, if $A = \{1,2\}$, then $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = S_A$, and hence $S_A \in \big\{\rho_{\mathcal{R}}^{-1}\big\}^* = \mathcal{R}^\bullet$. Therefore, by Theorem 5.2.12, $\mathcal{R}$ is not infinitesimally connected. And thus, by Theorem 6.1.8, $\mathcal{R}$ is not infinitesimally well-chained.

The following example shows that even a proximally well-chained tolerance relator need not be topologically connected.

Example 6.2.8. If $X = \{1,2,3,4\}$ and $R_i \subset X^2$ for all $i \in X$ such that

$$\begin{array}{llll} R_1(1) = \{1,2\}, & R_1(2) = X, & R_1(3) = R_1(4) = \{2,3,4\}, & \\ R_2(1) = X, & R_2(2) = \{1,2\}, & R_2(3) = R_2(4) = \{1,3,4\}, & \\ R_3(1) = R_3(2) = \{1,2,4\}, & & R_3(3) = \{3,4\}, & R_3(4) = X, \\ R_4(1) = R_4(2) = \{1,2,3\}, & & R_4(3) = X, & R_4(4) = \{3,4\}, \end{array}$$

then $\mathcal{R} = \big\{R_i\big\}_{i=1}^{4}$ is a proximally well-chained and proximally connected tolerance relator on X such that $\mathcal{R}$ is neither topologically well-chained nor topologically connected.

Note that $\tau_{\mathcal{R}} = \{\emptyset, X\}$. Therefore, by Theorems 4.3.9 and 4.4.3, $\mathcal{R}$ is proximally well-chained. And hence, by Theorem 6.1.8, $\mathcal{R}$ is proximally connected. But, $\{1,2\} \in \mathcal{T}_{\mathcal{R}}$, and moreover $\{3,4\} \in \mathcal{T}_{\mathcal{R}}$, and hence $\{1,2\} \in \mathcal{F}_{\mathcal{R}}$. Therefore, by Theorem 5.3.9, $\mathcal{R}$ is not topologically connected. And hence, by Theorem 6.1.8, $\mathcal{R}$ is not topologically well-chained.

The following example shows that the inverse of even a topologically well-chained, paratopologically and infinitesimally connected reflexive relator need not be topologically well-chained.

Example 6.2.9. If $X = \{1,2,3\}$ and $R_i \subset X^2$ for $i = 1,2$ such that

$$\begin{array}{lll} R_1(1) = \{1,2\}, & R_1(2) = \{2,3\}, & R_1(3) = X, \\ R_2(1) = \{1,3\}, & R_2(2) = \{2,3\}, & R_2(3) = X, \end{array}$$

then $\mathcal{R} = \{R_1, R_2\}$ is a topologically well-chained, paratopologically and infinitesimally connected reflexive relator on X such that $\mathcal{R}$ is neither infinitesimally well-chained nor ultrainfinitesimally connected. Moreover, $\mathcal{R}^{-1}$ is a proximally well-chained and parainfinitesimally connected reflexive relator on X such that $\mathcal{R}^{-1}$ is neither topologically well-chained nor ultimately connected.

Note that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$. Therefore, by Theorem 4.4.5, $\mathcal{R}$ is topologically well-chained. Hence, by Remark 4.4.2 and Theorem 4.3.18, $\mathcal{R}^{-1}$ is proximally well-chained. Moreover, $R_i(x) \cap R_j(y) \neq \emptyset$ for all $i, j \in \{1,2\}$ and $x, y \in X$. Therefore, by Theorem 5.4.5, $\mathcal{R}$ is paratopologically connected.

Moreover, it can be easily seen that

$$\begin{array}{lll} R_1^{-1}(1) = \{1,3\}, & R_1^{-1}(2) = X, & R_1^{-1}(3) = \{2,3\}, \\ R_2^{-1}(1) = \{1,3\}, & R_2^{-1}(2) = \{2,3\}, & R_2^{-1}(3) = X. \end{array}$$

Hence, since $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = R_1^{-1} \cap R_2^{-1}$, it is clear that

$$\rho_{\mathcal{R}}(1) = \{1,3\} \qquad \text{and} \qquad \rho_{\mathcal{R}}(2) = \rho_{\mathcal{R}}(3) = \{2,3\}.$$

Therefore, $\{2,3\} \cap \rho_{\mathcal{R}}(1) \neq \emptyset$, $\{1,3\} \cap \rho_{\mathcal{R}}(2) \neq \emptyset$ and $\{1,2\} \cap \rho_{\mathcal{R}}(3) \neq \emptyset$. Thus, by Theorem 5.4.12, $\mathcal{R}$ is infinitesimally connected.

Moreover, we can also easily see that

$$E_{\mathcal{R}^{-1}} = \bigcap \mathcal{E}_{\mathcal{R}^{-1}} = \bigcap \{ R_i^{-1}(x) : x \in X, i \in \{1,2\} \} = \{3\} \neq \emptyset.$$

Therefore, by Theorem 5.4.15, $\mathcal{R}^{-1}$ is parainfinitesimally connected.

On the other hand, if $A = \{1\}$, then we can at once see that $\rho_{\mathcal{R}}(A) = \{1,3\}$ and $\rho_{\mathcal{R}}(X \setminus A) = \{2,3\}$. Thus, by Theorem 5.4.14, $\mathcal{R}$ is not ultrainfinitesimally connected. Moreover, we can also at once see that $\{2,3\} \in \mathcal{T}_{\mathcal{R}^{-1}}$. Therefore, by Theorem 4.4.5, $\mathcal{R}^{-1}$ is not topologically well-chained. Thus, by Remark 4.4.2 and Theorem 4.3.18, $\mathcal{R}$ is not infinitesimally well-chained. Finally, by Theorem 5.4.17, it is clear that $\mathcal{R}^{-1}$ is not ultimately connected.

Remark 6.2.10. From Example 6.2.9 we can at once see that the inverse of even a proximally well-chained and parainfinitesimally connected reflexive relator need not be either infinitesimally well-chained or ultrainfinitesimally connected.

Moreover, by using Example 6.2.4, we can easily show that the inverse of even a paratopologically connected preorder relator need not be topologically connected. Therefore, neither the topological nor the paratopological connectedness is inverse invariant.

Example 6.2.11. If X and $\mathcal{R}$ are as in Example 6.2.4, then $\mathcal{R}$ is a paratopologically connected preorder relator on X such that $\mathcal{R}^{-1}$ is a preorder relator on X such that $\mathcal{R}^{-1}$ is not even topologically connected.

Note that, under the notation of Example 6.2.4, we have

$$\begin{array}{lll} R_1^{-1}(1) = \{1\}, & R_1^{-1}(2) = X, & R_1^{-1}(3) = X, \\ R_2^{-1}(1) = X, & R_2^{-1}(2) = \{2\}, & R_2^{-1}(3) = X, \\ R_3^{-1}(1) = X, & R_3^{-1}(2) = X, & R_3^{-1}(3) = \{3\}, \end{array}$$

Hence, since $\mathcal{R}^{-1} = \{R_1^{-1}, R_2^{-1}, R_3^{-1}\}$, we can see that $\{1\} \in \mathcal{T}_{\mathcal{R}^{-1}}$, and moreover $\{2,3\} \in \mathcal{T}_{\mathcal{R}^{-1}}$, and hence $\{1\} \in \mathcal{F}_{\mathcal{R}^{-1}}$. Therefore, by Theorem 5.3.9, $\mathcal{R}^{-1}$ is not topologically connected.

The following example shows that the hypothesis of the reflexivity of the relator $\mathcal{R}$ in Theorem 5.2.10 is essential.

Example 6.2.12. If $X = \{1,2\}$ and $R = X \times \{1\}$, then $\mathcal{R} = \{R\}$ is a parainfinitesimally connected, strongly transitive relator and $\mathcal{S} = \{\Delta_X\}$ is an equivalence relator on X such that $\mathcal{S}$ is properly refined by $\mathcal{R}$, but $\mathcal{S}$ is not properly connected.

Note that $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} = R(1) \cap R(2) = \{1\} \neq \emptyset$. Therefore, by Theorem 5.4.15, $\mathcal{R}$ is parainfinitesimally connected. Moreover, $f = R$ is a function of X into itself such that $\Delta_X \circ f = R \in \mathcal{R}$. Therefore, $\mathcal{S}$ is properly refined by $\mathcal{R}$. But, for instance, $S_{\{1\}} = \Delta_X \in \mathcal{S} \subset \mathcal{S}^*$, and thus by Theorem 5.2.3 $\mathcal{S}$ is not properly connected.

The following example shows that the reflexivity of the relator $\mathcal{S}$ in Theorem 5.2.10 cannot be stated.

Remark 6.2.13. If $X = \{1,2\}$ and $S \subset X^2$ such that $S(1) = \{2\}$ and $S(2) = X$, then $\mathcal{R} = \{X^2\}$ is a paratopologically well-chained and ultimately connected equivalence relator and $\mathcal{S} = \{S\}$ is an infinitesimally well-chained and parainfinitesimally connected strongly symmetric relator on X such that $\mathcal{S}$ is properly refined by $\mathcal{R}$, but $\mathcal{S}$ is not reflexive.

By Theorems 4.4.10 and 5.4.17, it is clear that the relator $\mathcal{R}$ is paratopologically well-chained and ultimately connected. On the other hand, we can easily see that $\rho_{\mathcal{S}} = \bigcap \mathcal{S}^{-1} = S^{-1} = S$. Moreover, $S^2 = X^2$, and hence $S^\infty = X^2$. Therefore, $\rho_{\mathcal{S}}^\infty = X^2$, and thus by Theorem 4.4.16 $\mathcal{S}$ is infinitesimally well-chained. Moreover, it is clear that $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{S}} = S(1) \cap S(2) = \{2\} \neq \emptyset$. Therefore, by Theorem 5.4.15, $\mathcal{S}$ is parainfinitesimally connected. Finally, we can observe that $f = X \times \{2\}$ is a function of X into itself such that $S \circ f = X^2 \in \mathcal{R}$. Therefore, $\mathcal{S}$ is properly refined by $\mathcal{R}$. But, despite this, $\mathcal{S}$ is not reflexive.

The following similar example shows that the counterpart of Theorem 5.2.10 with 'connected' replaced by 'well-chained' is not true.

Example 6.2.14. If $X = \{1,2\}$ and $S \subset X^2$ such that $S(1) = \{1\}$ and $S(2) = X$, then $\mathcal{R} = \{X^2\}$ is a paratopologically well-chained and ultimately connected equivalence relator and $\mathcal{S} = \{S\}$ is a parainfinitesimally connected preorder relator on X such that $\mathcal{S}$ is properly refined by $\mathcal{R}$, but $\mathcal{S}$ is not properly well-chained.

As a more delicate example of the above types, we can also at once state

Example 6.2.15. If $X = \{1,2\}$ and $S_i \subset X^2$ for all $i \in X$ such that

$$S_1(1) = \{1\}, \qquad S_1(2) = X, \qquad \text{and} \qquad S_2(1) = X, \qquad S_2(2) = \{2\},$$

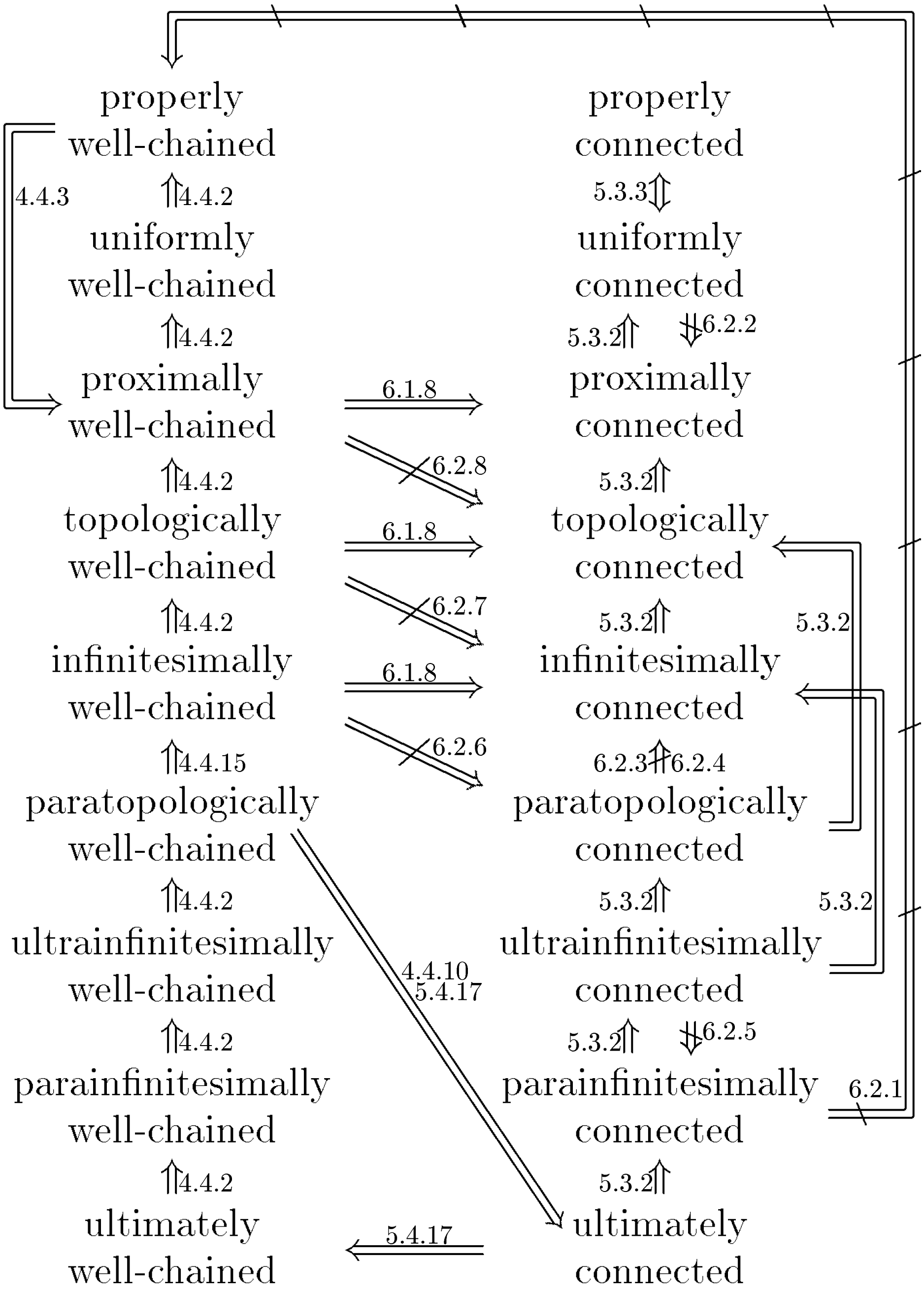
then $\mathcal{R} = \{X^2\}$ is a paratopologically well-chained and ultimately connected equivalence relator and $\mathcal{S} = \{S_1, S_2\}$ is a uniformly connected, properly filtered and properly symmetric preorder relator on X such that $\mathcal{S}$ is properly refined by $\mathcal{R}$, but $\mathcal{S}$ is neither properly well-chained nor proximally connected. Thus, in particular, the relator $\mathcal{S}^{\#}$ cannot be refined by $\mathcal{R}^{\blacklozenge}$.

The following example shows that, in contrast to Corollaries 4.3.6 and 5.4.2, an infinitesimally connected relator need not be total.

Example 6.2.16. If $X = \{1, 2\}$ and $R = \{2\} \times X$, then $\mathcal{R} = \{R\}$ is an infinitesimally connected relator on X such that $\mathcal{R}$ is not total. Thus, in particular, $\mathcal{R}$ cannot be properly well-chained and paratopologically connected.

Note that $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = R^{-1} = X \times \{2\}$, and hence $\{2\} \cap \rho_{\mathcal{R}}(1) \neq \emptyset$. Therefore, by Theorem 5.4.12, $\mathcal{R}$ is infinitesimally connected. But, despite this $\mathcal{R}$ is not total.

The following diagram shows the main implications among the various wellchainedness and connectedness properties of relators.



7 Simple relators

7.1 Simple relators and their basic characterizations

Definition 7.1.1. A relator $\mathcal{R}$ on X is called properly simple if it is a singleton. That is, there exists a relation R on X such that $\mathcal{R} = \{R\}$.

Moreover, if $\square$ is a unary operation for relators on X, then a relator $\mathcal{R}$ on X is called $\square$-simple if it is $\square$-equivalent to a properly simple relator.

Finally, if $\square$ is a unary operation for relators on X, then a relator $\mathcal{R}$ on X is called quasi-$\square$-simple if it is $\square\infty$-simple.

In particular the relator $\mathcal{R}$ on X is quasi-properly simple if it is ∞-simple.

Remark 7.1.2. We could define the $\mathfrak{F}$-simplicity of relators on X for an arbitrary set-valued function $\mathfrak{F}$ for relators on X, but we would not get any new notion. Namely, for instance, the Int-, cl- and $\mathcal{E}$-simplicity would be equivalent to the $\#$-, $\wedge$- and $\triangle$-simplicity, respectively. Moreover, the τ- and $\mathcal{F}$-simplicity would be equivalent to the quasi-$\#$- and quasi-$\wedge$-simplicity, respectively.

Furthermore, we could not define the quasi-$\mathfrak{F}$-simplicity for any set-valued functions $\mathfrak{F}$ for relators on X.

The above basic simplicity properties of relators, except some pathological ones, were most systematically investigated in [68]. However, some particular results were also proved in [37], [33], [38], [35], and also in [26], [53], and in the previous part of this dissertation. Namely, the properly (topologically) well-chained relators are, in particular quasi-properly (quasi-topologically) simple, for instance.

First of all, in [68], it has been established the following four basic theorems.

Theorem 7.1.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly simple;

(2) $\mathcal{R} = \{\rho_{\mathcal{R}}^{-1}\}$.

Theorem 7.1.4. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is uniformly simple;

(2) $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^*$;

(3) $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}$;

(4) $\mathcal{R}^* = \{\rho_{\mathcal{R}}^{-1}\}^*$.

Hint. To prove the equivalence of the assertions (2) and (3), which has not been observed in [24], we note that if the assertion (2) holds, then $\bigcap \mathcal{R} = \rho_{\mathcal{R}}^{-1} \in \mathcal{R}^*$. Therefore, there exists an $R \in \mathcal{R}$ such that $R \subset \bigcap \mathcal{R}$. Hence, since $\bigcap \mathcal{R} \subset R$, it follows that $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = R \in \mathcal{R}$. That is, the assertion (3) also holds. The converse implication (3) $\Longrightarrow$ (2) is immediate from the fact that $\mathcal{R} \subset \mathcal{R}^*$.

Theorem 7.1.5. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally simple;

(2) $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^{\#}$;

(3) $\mathcal{R}^{\#} = \{\rho_{\mathcal{R}}^{-1}\}^{\#}$.

Theorem 7.1.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically simple;

(2) $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^{\wedge}$; *(3) $\mathcal{R}^{\wedge} = \{\rho_{\mathcal{R}}^{-1}\}^{\wedge}$.*

Concerning the paratopological simplicity of a relator $\mathcal{R}$, we can only prove the following

Theorem 7.1.7. *If $\mathcal{R}$ is a relator on X such that $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^{\vartriangle}$, then $\mathcal{R}^{\vartriangle} = \{\rho_{\mathcal{R}}^{-1}\}^{\vartriangle}$, and thus $\mathcal{R}$ is paratopologically simple.*

Proof. In this case, since $\vartriangle$ is a refinement and thus increasing and idempotent, we have $\{\rho_{\mathcal{R}}^{-1}\}^{\vartriangle} \subset \mathcal{R}^{\vartriangle\vartriangle} = \mathcal{R}^{\vartriangle}$. Moreover, since $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R}$, we also have $\mathcal{R} \subset \{\rho_{\mathcal{R}}^{-1}\}^{*}$, and hence by the increasingness of $\vartriangle$ and by Theorem 2.2.18 $\mathcal{R}^{\vartriangle} \subset \{\rho_{\mathcal{R}}^{-1}\}^{*\vartriangle} = \{\rho_{\mathcal{R}}^{-1}\}^{\vartriangle}$. Therefore, the required equality is also true.

Corollary 7.1.8. *If $\mathcal{R}$ is a relator on X such that $\mathcal{R}$ is not paratopologically simple, then $\mathcal{R}^{\bullet} \not\subset \mathcal{R}^{\vartriangle}$.*

Proof. Namely, we evidently have $\{\rho_{\mathcal{R}}^{-1}\} \subset \{\rho_{\mathcal{R}}^{-1}\}^{*} = \mathcal{R}^{\bullet}$, that is $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^{\bullet}$.

Moreover, as an immediate consequence of the Definition of $\mathcal{R}^{\bullet}$, we can also state

Theorem 7.1.9. *If $\mathcal{R}$ is a relator on X, then $\mathcal{R}$ is infinitesimally simple and $\mathcal{R}^{\vee}$ is topologically simple.*

Proof. By Theorem 2.2.5, we have $\mathcal{R}^{\bullet} = \{\rho_{\mathcal{R}}^{-1}\}^{\bullet}$ that is $\mathcal{R}$ is infinitesimally simple.

Moreover, by Theorem 2.2.23, 2.2.20 and 2.2.5 we have that

$$\mathcal{R}^{\vee\wedge} = \mathcal{R}^{\bullet-1} = \{\rho_{\mathcal{R}}^{-1}\}^{*-1} = \{\rho_{\mathcal{R}}^{-1}\}^{-1*} = \{\rho_{\mathcal{R}}\}^{*} = \{\rho_{\mathcal{R}}\}^{\wedge}$$

that is $\mathcal{R}^{\vee}$ is topologically simple.

Remark 7.1.10. Hence, it is clear that the infinitesimally and quasi-infinitesimally simple relators need not be studied any more.

Moreover, from Theorem 2.2.18, it is clear that 'properly simple' $\Longrightarrow$ 'uniformly simple' $\Longrightarrow$ 'proximally simple' $\Longrightarrow$ 'topologically simple' $\Longrightarrow$ 'paratopologically simple'.

On the other hand, it is also clear that 'infinitesimally simple' $\Longrightarrow$ 'ultrainfinitesimally simple' $\Longrightarrow$ 'parainfinitesimally simple' $\Longrightarrow$ 'ultimately simple', hence the (quasi-)ultrainfinitesimally simple, (quasi-)parainfinitesimally simple and (quasi-)ultimately simple relators also need not be studied any more.

In addition to Theorem 7.1.3, we can also easily establish the following

Theorem 7.1.11. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-properly simple;

(2) $\mathcal{R}^{\infty} = \{\rho_{\mathcal{R}^{\infty}}^{-1}\}$; *(3) $\mathcal{R}^{\infty} = \{\rho_{\mathcal{R}^{\infty}}^{-1}\}^{\infty}$.*

Proof. By the corresponding definition, it is clear that the assertion (1) holds if and only if the relator $\mathcal{R}^{\infty}$ is properly simple. Therefore, by Theorem 7.1.3, the assertion (1) is equivalent to the assertion (2). Moreover, since $\rho_{\mathcal{R}^{\infty}}^{-1} = \bigcap \mathcal{R}^{\infty}$ is a preorder, it is clear that the assertion (2) and (3) are also equivalent.

The following three basic theorems have also been mostly established in [68].

Theorem 7.1.12. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-uniformly simple;

(2) $\rho_{\mathcal{R}^{\infty}}^{-1} \in \mathcal{R}^{*}$;

(3) $\rho_{\mathcal{R}^{\infty}}^{-1} \in \mathcal{R}^{\infty}$;

(4) $\mathcal{R}^{*\infty} = \left\{ \rho_{\mathcal{R}^{\infty}}^{-1} \right\}^{*\infty}$.

Hint. To prove the equivalence of the assertion (2) and (3), which has not been observed in [24], note that if the assertion (2) holds, then $\bigcap \mathcal{R}^{\infty} = \rho_{\mathcal{R}^{\infty}}^{-1} \in \mathcal{R}^{*}$. Therefore, there exists an $R \in \mathcal{R}$ such that $R \subset \bigcap \mathcal{R}^{\infty}$. Hence, since $\bigcap \mathcal{R}^{\infty}$ is a preorder on X, it follows that $R^{\infty} \subset \bigcap \mathcal{R}^{\infty}$. Now, since $\bigcap \mathcal{R}^{\infty} \subset R^{\infty}$, it is clear that $\rho_{\mathcal{R}^{\infty}}^{-1} = \bigcap \mathcal{R}^{\infty} = R^{\infty} \in \mathcal{R}^{\infty}$. That is, the assertion (3) also holds. The converse implication (3) $\Longrightarrow$ (2) is immediate from the fact that $\mathcal{R}^{\infty} \subset \mathcal{R}^{*}$.

Theorem 7.1.13. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-proximally simple;

(2) $\rho_{\mathcal{R}^{\infty}}^{-1} \in \mathcal{R}^{\#}$;

(3) $\mathcal{R}^{\#\infty} = \left\{ \rho_{\mathcal{R}^{\infty}}^{-1} \right\}^{\#\infty}$.

Theorem 7.1.14. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically simple;

(2) $\rho_{\mathcal{R}^{\wedge\infty}}^{-1} \in \mathcal{R}^{\wedge}$;

(3) $\mathcal{R}^{\wedge\infty} = \left\{ \rho_{\mathcal{R}^{\wedge\infty}}^{-1} \right\}^{\wedge\infty}$.

Remark 7.1.15. Note that if $\mathcal{R}$ is a $\square$-simple relator on X, then $\mathcal{R}$ is also $\square\infty$-simple, that is quasi-$\square$-simple.

In addition to the above theorems, we can now also prove the following Theorem and Corollary.

Theorem 7.1.16. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X such that $\mathcal{S}$ is total, then the following assertions are equivalent:*

(1) $\mathcal{S}^{\triangle} \subset \mathcal{R}^{\triangle}$;

(2) $\mathcal{S}^{\triangle\infty} \subset \mathcal{R}^{\triangle\infty}$.

Proof. If the assertion (2) holds and $A \in \mathcal{E}_{\mathcal{S}}$, then $R_A \in \mathcal{S}^{\triangle}$ and by using Proposition 4.2.3 we have $R_A \in \mathcal{S}^{\triangle\infty} \subset \mathcal{R}^{\triangle\infty} \subset \mathcal{R}^{\triangle}$. And since $\mathcal{S}$ is total, hence $A \neq \emptyset$, therefore $A \in \mathcal{E}_{\mathcal{R}}$. By using Theorem 2.2.10 and 2.2.12 $\mathcal{E}_{\mathcal{S}} \subset \mathcal{E}_{\mathcal{R}}$ follows that $\mathcal{S}^{\triangle} \subset \mathcal{R}^{\triangle}$, that is the assertion (1) also holds.

The converse implication is quite obvious.

Corollary 7.1.17. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically simple;

(2) $\mathcal{R}$ is quasi-paratopologically simple.

Proof. If the assertion (2) holds, then there exists a relation R on X such that $\mathcal{R}^{\triangle\infty} = \{R\}^{\triangle\infty}$. Hence, if both $\mathcal{R}$ and $\{R\}$ are total relators on X, then by using Theorem 7.1.16, we can infer that $\mathcal{R}^{\triangle} = \{R\}^{\triangle}$. That is, the assertion (1) also holds.

On the other hand, if $\mathcal{R}$ is a non-total relator on X, then from Remark 3.2.3 we know that $\emptyset \in \mathcal{E}_{\mathcal{R}}$. Hence, by the corresponding definitions, it is clear that $\mathcal{E}_{\mathcal{R}} = \mathcal{P}(X) = \mathcal{E}_{\{\emptyset\}}$. Therefore, by Theorem 2.2.10, we now have $\mathcal{R}^{\triangle} = \{\emptyset\}^{\triangle}$. That is, the assertion (1) again holds, even if the assertion (2) is not supposed to be true.

Now, it remains to consider only the case when $\mathcal{R}$ is a total, but $\{R\}$ is a non-total relator on X. In this case, by Remark 3.2.3, have $\emptyset \in \mathcal{E}_{\{R\}}$. Hence, by the corresponding definitions, it is clear $\mathcal{E}_{\{R\}} = \mathcal{P}(X) = \mathcal{E}_{\mathcal{P}(X^2)}$. Therefore, by Theorem 2.2.10, we have $\{R\}^{\triangle} = \mathcal{P}(X^2)^{\triangle} = \mathcal{P}(X^2)$. Hence, it follows that $\mathcal{R}^{\triangle\infty} = \{R\}^{\triangle\infty} = \mathcal{P}(X^2)^{\infty}$. Therefore, we have $\Delta_X \in \mathcal{R}^{\triangle\infty} \subset \mathcal{R}^{\triangle *} = \mathcal{R}^{\triangle}$. Hence, by using the corresponding definitions and Theorem 2.2.10 and Remark 3.2.3, we can infer that $\mathcal{E}_{\Delta_X} \subset \mathcal{E}_{\mathcal{R}^{\triangle}} = \mathcal{E}_{\mathcal{R}} \subset \mathcal{P}(X) \setminus \{\emptyset\} = \mathcal{E}_{\Delta_X}$, and thus $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\Delta_X}$. Therefore, by Theorem 2.2.10, we have $\mathcal{R}^{\bullet} = \{\Delta_X\}^{\bullet}$. That is, the assertion (1) again holds.

Remark 7.1.18. Hence, it is clear that the quasi-paratopologically simple relators need not be studied any more.

Moreover, from Theorems 7.1.11 through 7.1.13, by the inclusion $\mathcal{R}^{\infty} \subset \mathcal{R}^{*} \subset \mathcal{R}^{\#}$, it is clear that 'quasi-properly simple' $\Longrightarrow$ 'quasi-uniformly simple' $\Longrightarrow$ 'quasi-proximally simple'.

7.2 A few illustrating examples

The following example shows that even a uniformly simple equivalence relator need not be quasi-properly simple.

Example 7.2.1. If $X = \{1, 2\}$, then $\mathcal{R} = \{\Delta_X, X^2\}$ is a uniformly simple equivalence relator on X such that $\mathcal{R}$ is not quasi-properly simple.

Namely, $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = \Delta_X \in \mathcal{R}$, but $\mathcal{R}^{\infty} = \mathcal{R} \neq \{\Delta_X\} = \{\rho_{\mathcal{R}^{\infty}}^{-1}\}$. And thus Theorems 7.1.4 and 7.1.11 can be applied to get the required assertions.

The following example shows that even a proximally simple equivalence relator need not be quasi-uniformly simple.

Example 7.2.2. If $X = \{1, 2, 3\}$, and $R_i \subset X^2$ for all $i \in X$ such that

$$\begin{array}{lll} R_1(1) = \{1\}, & R_1(2) = \{2, 3\}, & R_1(3) = \{2, 3\}; \\ R_2(1) = \{1, 3\}, & R_2(2) = \{2\}, & R_2(3) = \{1, 3\}; \\ R_3(1) = \{1, 2\}, & R_3(2) = \{1, 2\}, & R_3(3) = \{3\}; \end{array}$$

then $\mathcal{R} = \{R_1, R_2, R_3\}$ is a proximally simple equivalence relator on X such that R is not quasi-uniformly simple.

Namely, now it can be easily seen that $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = \Delta_X \in \mathcal{R}^{\#}$, but $\rho_{\mathcal{R}^{\infty}}^{-1} = \rho_{\mathcal{R}}^{-1} = \Delta_X \notin \mathcal{R}^{*}$. And thus, Theorems 7.1.5 and 7.1.12 can be applied to get the required assertions. (To see that $\Delta_X \in \mathcal{R}^{\#}$, note that we have $R_i[X \setminus \{i\}] = \Delta_X[X \setminus \{i\}]$ for all $i \in X$.)

The following example shows that even a topologically simple equivalence relator need not be quasi-proximally simple.

Example 7.2.3. If $X = \{1,2,3,4\}$ and $R_i \subset X^2$ for all $i = 1,2$ such that

$$\begin{array}{lll} R_1(2) = \{2\}, & R_1(4) = \{4\}, & R_1(1) = R_1(3) = \{1,3\}; \\ R_2(1) = \{1\}, & R_2(3) = \{3\}, & R_2(2) = R_2(4) = \{2,4\}; \end{array}$$

then $\mathcal{R} = \{R_1, R_2\}$ is a topologically simple equivalence relator on X such that R is not quasi-proximally simple.

Namely, now it can be easily seen that $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = \Delta_X \in \mathcal{R}^{\wedge}$, but $\rho_{\mathcal{R}^{\infty}}^{-1} = \rho_{\mathcal{R}}^{-1} = \Delta_X \notin \mathcal{R}^{\#}$. And thus, Theorems 7.1.6 and 7.1.13 can be applied to get the required assertions. (To see that $\Delta_X \notin \mathcal{R}^{\#}$, note that we have $R_i[\{1,2\}] \not\subset \Delta_X[\{1,2\}]$ for all $i \in \{1,2\}$.)

The following example shows that even a paratopologically simple equivalence relator need not be quasi-topologically simple.

Example 7.2.4. If X and $\mathcal{R}$ are as in Example 2.2.17, then $\mathcal{R}$ is a paratopologically simple equivalence relator on X such that $\mathcal{R}$ is not quasi-topologically simple.

Namely, if $R \subset X^2$ such that

$$R(1) = R(2) = \{2\} \qquad \text{and} \qquad R(3) = \{1\},$$

then we can at once see that $\mathcal{R}^{\triangle} = \{R\}^{\triangle}$, since $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\{R\}}$. Therefore, $\mathcal{R}$ is paratopologically simple. Moreover, by using [39, Theorem 4.13], we can easily see that

$$\rho_{\mathcal{R}^{\wedge\infty}}^{-1}(i) = \bigcap_{i \in A \in \mathcal{T}_{\mathcal{R}}} A = \{i\} = \Delta_X(i)$$

for all $i \in X$, since now we have $\mathcal{T}_{\mathcal{R}} = \mathcal{P}(X) \setminus \{\{3\}\}$. Therefore, $\rho_{\mathcal{R}^{\wedge\infty}}^{-1} = \Delta_X \notin \mathcal{R}^{\wedge}$. And thus, by Theorem 7.1.14, the relator $\mathcal{R}$ is not quasi-topologically simple.

Remark 7.2.5. In connection with the above relator $\mathcal{R}$, it is also worth noticing that $\mathcal{R}^{\bullet} = \not\subset \mathcal{R}^{\triangle}$ (see Example 2.2.17). Therefore, the converses of Theorem 7.1.7 and Corollary 7.1.8 need not be true.

The following example shows that even a quasi-properly simple, properly symmetric, reflexive relator need not be quasi-topologically simple.

Example 7.2.6. If $X = \{1,2,3\}$ and $R_i \subset X^2$ for all $i = 1,2$ such that

$$\begin{array}{lll} R_1(1) = \{1,2\}, & R_1(2) = \{2,3\}, & R_1(3) = \{1,3\}; \\ R_2(1) = \{1,3\}, & R_2(2) = \{1,2\}, & R_2(3) = \{2,3\}; \end{array}$$

then $\mathcal{R} = \{R_1, R_2\}$ is a quasi-properly simple (properly, but not topologically well-chained) properly symmetric reflexive relator on X such that $\mathcal{R}$ is not quasi-topologically simple.

Namely, it can be easily seen that $R_1^2 = X^2$ and $R_2^2 = X^2$, and thus $\mathcal{R}^\infty = \{X^2\}$. Therefore, $\mathcal{R}$ is properly well-chained, and thus in particular it is quasi-properly simple. Moreover, by using [39, Theorem 4.13], we can easily see that

$$\rho^{-1}_{\mathcal{R}^{\wedge\infty}}(i) = \bigcap_{i \in A \in \mathcal{T}_\mathcal{R}} A = \{i\} = \Delta_X(i)$$

for all $i \in X$, since now we have $\mathcal{T}_\mathcal{R} = \{\emptyset, \{1,2\}, \{1,3\}, \{2,3\}, X\}$. Therefore, $\rho^{-1}_{\mathcal{R}^{\wedge\infty}} = \Delta_X \notin \mathcal{R}^\wedge$, and thus by Theorem 7.1.6 the relator $\mathcal{R}$ is not quasi-topologically simple. (Moreover, from Theorem 4.4.5, we can see that $\mathcal{R}$ is not topologically well-chained. $R_1^{-1} = R_2$ follows the proper symmetry of $\mathcal{R}$.)

The following example shows that even a quasi-properly and quasi-topologically simple reflexive relator need not be paratopologically simple.

Example 7.2.7. If $X = \{1,2,3,4\}$ and $R_i \subset X^2$ for all $i = 1,2$ such that

$$\begin{array}{llll} R_1(1) = \{1,2\}, & R_1(2) = \{2,3\}, & R_1(3) = \{3,4\}, & R_1(4) = \{1,4\}; \\ R_2(1) = \{1,3\}, & R_2(i) = X \text{ for } i \in X \setminus \{1\} & & \end{array}$$

then $\mathcal{R} = \{R_1, R_2\}$ is a quasi-properly and quasi-topologically simple (properly, but not topologically well-chained) reflexive relator on X such that $\mathcal{R}$ is not paratopologically simple.

Namely, it can be easily seen that $R_1^3 = X^2$ and $R_2^2 = X^2$, and thus $\mathcal{R}^\infty = \{X^2\}$. Therefore, $\mathcal{R}$ is properly well-chained, and thus in particular it is quasi-properly simple. Moreover, by using [39, Theorem 4.13], we can easily see that

$$\rho^{-1}_{\mathcal{R}^{\wedge\infty}}(i) = \bigcap_{i \in A \in \mathcal{T}_\mathcal{R}} A = \begin{cases} X & \text{for } i = 2, \\ X \setminus \{2\} & \text{for } i \in X \setminus \{2\}, \end{cases}$$

since now we have $\mathcal{T}_\mathcal{R} = \{\emptyset, X \setminus \{2\}, X\}$. Therefore, $\rho^{-1}_{\mathcal{R}^{\wedge\infty}} \in \mathcal{R}^\wedge$, and thus by Theorem 7.1.6 the relator $\mathcal{R}$ is quasi-topologically simple. (Moreover, from [53, Theorem 3.7] we can see that $\mathcal{R}$ is not topologically well-chained.)

The crucial fact that the relator $\mathcal{R}$ is not paratopologically simple will only be cleared up by our forthcoming Theorem 7.4.10 and Example 7.4.11 which rely upon the observations of the following preparatory section.

7.3 Two natural operations on families of sets

Definition 7.3.1. If $\mathcal{A} \subset \mathcal{P}(X)$, then we write

$$\mathcal{A}^* = \big\{B \subset X : \exists A \in \mathcal{A} : A \subset B\big\}.$$

Remark 7.3.2. Note that the notion of the uniform refinement of relators is a special case of the above definition.

As a useful reformulation of the above definition, we can at once state the following

Proposition 7.3.3. *If $\mathcal{A} \subset \mathcal{P}(X)$, then*

$$\mathcal{A}^* = \big\{ B \subset X : \mathcal{A} \cap \mathcal{P}(B) \neq \emptyset \big\}.$$

Remark 7.3.4. Hence, by noticing that for any $C \subset X$ we have $C \in \mathcal{P}(B)$ if and only if $B \supset C$, we can at once see that $\mathcal{A}^*$ is just the family of all adherence points of the set $\mathcal{A}$ in the simple relator space $\mathcal{P}(X)(\supset)$.

Therefore, $\mathcal{A}^* = \mathrm{cl}_{\{\supset\}}(\mathcal{A})$, and thus it is not surprising that we have the following theorem which can however also be easily proved directly by using only Definition 7.3.1 or Proposition 7.3.3.

Theorem 7.3.5. *The operation $*$ has the following properties:*

- *(1) $\emptyset^* = \emptyset$ and $\mathcal{P}(X)^* = \mathcal{P}(X)$;*
- *(2) $\mathcal{A} \subset \mathcal{A}^*$ and $\mathcal{A}^* = \mathcal{A}^{**}$ for all $\mathcal{A} \subset \mathcal{P}(X)$;*
- *(3) $\big(\bigcup_{i \in I} \mathcal{A}_i\big)^* = \bigcup_{i \in I} \mathcal{A}_i^*$ whenever $\mathcal{A}_i \subset \mathcal{P}(X)$ for all $i \in I$.*

Hint. The assertion (3) is also immediate from the fact that

$$\left(\bigcup_{i \in I} \mathcal{A}_i \right) \cap \mathcal{P}(B) = \bigcup_{i \in I} \big(\mathcal{A}_i \cap \mathcal{P}(B) \big)$$

for every $B \subset X$.

Remark 7.3.6. Note that one half of the assertion (3) is equivalent to the monotonicity property of the operation $*$.

Definition 7.3.7. If $\mathcal{B} \subset \mathcal{A} \subset \mathcal{P}(X)$, then we say that:

- (1) $\mathcal{B}$ is ascending in $\mathcal{A}$ if $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $B \subset A$ imply $A \in \mathcal{B}$;
- (2) $\mathcal{B}$ is descending in $\mathcal{A}$ if $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $A \subset B$ imply $A \in \mathcal{B}$.

Remark 7.3.8. Note that thus the family $\mathcal{B}$ is descending in $\mathcal{A}$ if and only if the family $X \setminus \mathcal{B}$ is ascending in $X \setminus \mathcal{A}$.

Moreover, as a useful reformulation of the first part of the above definition, we can at once state the following

Theorem 7.3.9. *If $\mathcal{B} \subset \mathcal{A} \subset \mathcal{P}(X)$, then the following assertions are equivalent:*

(1) $\mathcal{B}$ is ascending in $\mathcal{A}$; *(2) $\mathcal{A} \cap \mathcal{B}^* \subset \mathcal{B}$;* *(3) $\mathcal{B} = \mathcal{A} \cap \mathcal{B}^*$.*

Remark 7.3.10. Note that if $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ such that $\mathcal{B} = \mathcal{A} \cap \mathcal{B}^*$, then $\mathcal{B} \subset \mathcal{A}$ and $\mathcal{B}$ is ascending in $\mathcal{A}$.

From Theorem 7.3.9, it is clear that the following characterization of the ascendingness of the family $\mathcal{A}$ in $\mathcal{P}(X)$ is also true.

Corollary 7.3.11. *If $\mathcal{A} \subset \mathcal{P}(X)$, then the following assertions are equivalent:*

(1) $\mathcal{A}$ is ascending; *(2) $\mathcal{A}^* \subset \mathcal{A}$;* *(3) $\mathcal{A} = \mathcal{A}^*$.*

Definition 7.3.12. If $\mathcal{A}$ is an ascending family in $\mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{A}$ such that for each $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $B \subset A$, then $\mathcal{B}$ will be called a base for $\mathcal{A}$.

Theorem 7.3.13. *If $\mathcal{A}$ is an ascending family in $\mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{A}$, then the following assertions are equivalent:*

(1) $\mathcal{B}$ is a base for $\mathcal{A}$; *(2) $\mathcal{A} \subset \mathcal{B}^*$;* *(3) $\mathcal{A} = \mathcal{B}^*$.*

Remark 7.3.14. Note that if $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$ such that $\mathcal{A} = \mathcal{B}^*$, then $\mathcal{A}$ is ascending and $\mathcal{B}$ is a base for $\mathcal{A}$.

In addition to Definition 7.3.1, it is also worth introducing the following

Definition 7.3.15. If $\mathcal{A} \subset \mathcal{P}(X)$, then we write

$$\mathcal{A}^\diamond = \big\{ B \in \mathcal{A} : A \in \mathcal{A}, A \subset B \implies A = B \big\}.$$

Remark 7.3.16. Note that thus $\mathcal{A}^\diamond$ is just the family of the minimal members of $\mathcal{A}$.

Moreover, as a useful reformulation of the above definition, we can at once state

Proposition 7.3.17. *If $\mathcal{A} \subset \mathcal{P}(X)$, then*

$$\mathcal{A}^\diamond = \big\{ B \subset X : \mathcal{A} \cap \mathcal{P}(B) = \{B\} \big\}.$$

Remark 7.3.18. Hence, by noticing that for any $C \subset X$ we have $C \in \mathcal{P}(B)$ if and only if $B \supset C$, we can at once see that $\mathcal{A}^\diamond$ is just the family of all isolated points of the set $\mathcal{A}$ in the simple relator space $\mathcal{P}(X)(\supset)$.

Therefore, $\mathcal{A}^\diamond$ can only be a very small portion of the genuine interior $\operatorname{int}_{\{\supset\}}(\mathcal{A}) = \mathcal{P}(X) \setminus (\mathcal{P}(X) \setminus \mathcal{A})^*$. However, despite this, by using Definition 7.3.15 or Proposition 7.3.17, we can still prove the following vague analogue of Theorem 7.3.5.

Theorem 7.3.19. *The operation $\diamond$ has the following properties:*

(1) $\emptyset^\diamond = \emptyset$ and $\mathcal{P}(X)^\diamond = \{\emptyset\}$;

(2) $\mathcal{A}^\diamond \subset \mathcal{A}$ and $\mathcal{A}^\diamond = \mathcal{A}^{\diamond\diamond}$ for all $\mathcal{A} \subset \mathcal{P}(X)$;

(3) $\mathcal{A} \cap \mathcal{B}^\diamond \subset (\mathcal{A} \cap \mathcal{B})^\diamond$ for all $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(X)$.

Hint. To prove the inclusion $\mathcal{A}^\diamond \subset \mathcal{A}^{\diamond\diamond}$, note that if $B \in \mathcal{A}^\diamond$, then $\mathcal{A} \cap \mathcal{P}(B) = \{B\}$. Hence, since $\mathcal{A}^\diamond \subset \mathcal{A}$, it follows that $\mathcal{A}^\diamond \cap \mathcal{P}(B) \subset \{B\}$. Now, since $B \in \mathcal{A}^\diamond \cap \mathcal{P}(B)$, it is clear that $\mathcal{A}^\diamond \cap \mathcal{P}(B) = \{B\}$, and thus $B \in \mathcal{A}^{\diamond\diamond}$ is also true.

Remark 7.3.20. Note that if $\mathcal{A}$ is a chain in $\mathcal{P}(X)$ (i.e., $\mathcal{A} \subset \mathcal{P}(X)$ such that $A \subset B$ or $B \subset A$ for all $A, B \in \mathcal{A}$), then $\mathcal{A}^\diamond = \{\bigcap \mathcal{A}\}$ whenever $\mathcal{A}^\diamond \neq \emptyset$.

Moreover, the operation $\diamond$ is not, in general, monotone. And even the converse of the inclusion established in (3) need not be true.

Namely, if for instance X is a nonvoid set, and moreover $\mathcal{A} = \{X\}$ and $\mathcal{B} = \{\emptyset, X\}$, then $\mathcal{A} \cap \mathcal{B}^\diamond = \emptyset$ and $(\mathcal{A} \cap \mathcal{B})^\diamond = \{\emptyset\}$.

However, despite the above inconveniences, we can still prove the following useful theorems.

Theorem 7.3.21. *If $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B}$ is either a descending family or a maximal chain in $\mathcal{A}$, then $\mathcal{B}^{\diamond} \subset \mathcal{A}^{\diamond}$.*

Hint. If $\mathcal{B}$ is a chain in $\mathcal{A}$ and $C \in \mathcal{B}^{\diamond}$, then by Remark 7.3.20, we necessarily have $C = \bigcap \mathcal{B}$. Therefore, if $D \in \mathcal{A} \cap \mathcal{P}(C)$, then $\mathcal{C} = \mathcal{B} \cup \{D\}$ is a chain in $\mathcal{A}$ such that $\mathcal{B} \subset \mathcal{C}$. Hence, if $\mathcal{B}$ is a maximal chain in $\mathcal{A}$, we can infer that $\mathcal{B} = \mathcal{C}$, and thus $D \in \mathcal{B}$. Now, since $D \subset C$ and $C = \bigcap \mathcal{B}$, it is clear that $D = C$. Therefore, $\mathcal{A} \cap \mathcal{P}(C) \subset \{C\}$. Hence, since $C \in \mathcal{B} \subset \mathcal{A}$, we can already see that $\mathcal{A} \cap \mathcal{P}(C) = \{C\}$, and thus $C \in \mathcal{A}^{\diamond}$ is also true.

Theorem 7.3.22. *If $\mathcal{B} \subset \mathcal{A} \subset \mathcal{P}(X)$, then the following assertions are equivalent:*

(1) $\mathcal{A}^{\diamond} \subset \mathcal{B}^{\diamond}$; *(2) $\mathcal{A}^{\diamond} \subset \mathcal{B}$;* *(3) $\mathcal{A}^{\diamond} \subset \mathcal{B}^{*}$.*

Hint. If $A \in \mathcal{A}^{\diamond}$ and the assertion (3) holds, then we evidently have

$$\emptyset \neq \mathcal{B} \cap \mathcal{P}(A) \subset \mathcal{A} \cap \mathcal{P}(A) = \{A\},$$

and hence $\mathcal{B} \cap \mathcal{P}(A) = \{A\}$. Therefore, $A \in \mathcal{B}^{\diamond}$, and thus the assertion (1) also holds.

Theorem 7.3.23. *If $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B}$ is a base for $\mathcal{A}$, then $\mathcal{A}^{\diamond} = \mathcal{B}^{\diamond}$.*

Proof. In this case, we have $\mathcal{B} \subset \mathcal{A}$, and moreover $\mathcal{A} \subset \mathcal{B}^{*}$, and hence $\mathcal{A}^{\diamond} \subset \mathcal{B}^{*}$. Now, by Theorem 7.3.22, it is clear that $\mathcal{A}^{\diamond} \subset \mathcal{B}^{\diamond}$. Therefore, we need only show that the converse inclusion is also true.

For this, note that if $C \in \mathcal{B}^{\diamond}$, then $\mathcal{B} \cap \mathcal{P}(C) = \{C\}$. Hence, since $\mathcal{B} \subset \mathcal{A}$, it follows that $\{C\} \subset \mathcal{A} \cap \mathcal{P}(C)$. On the other hand, if $D \in \mathcal{A} \cap \mathcal{P}(C)$, then $D \in \mathcal{A}$ and $D \subset C$. Hence, since $\mathcal{A} \subset \mathcal{B}^{*}$, it follows that there exists a $B \in \mathcal{B}$ such that $B \subset D \subset C$. Therefore, $B \in \mathcal{B} \cap \mathcal{P}(C) = \{C\}$, and thus $B = C$. Hence, it is clear that $D = C$, and thus $\mathcal{A} \cap \mathcal{P}(C) \subset \{C\}$. Therefore, we also have $\mathcal{A} \cap \mathcal{P}(C) = \{C\}$, and hence $C \in \mathcal{A}^{\diamond}$. And thus, the inclusion $\mathcal{B}^{\diamond} \subset \mathcal{A}^{\diamond}$ is also true.

Corollary 7.3.24. *If $\mathcal{A} \subset \mathcal{P}(X)$ and $\mathcal{B}$ is a base for $\mathcal{A}$, then $\mathcal{A}^{\diamond} \subset \mathcal{B}$.*

Theorem 7.3.25. *If $\mathcal{A} \subset \mathcal{P}(X)$ such that each nonvoid chain contained in $\mathcal{A}$ has a minimal element, then $\mathcal{A}^{*} = \mathcal{A}^{\diamond *}$.*

Proof. Assume that $A \in \mathcal{A}$, and define $\mathcal{B} = \mathcal{A} \cap \mathcal{P}(A)$. Then $A \in \mathcal{B}$, and thus $\mathcal{B} \neq \emptyset$. Therefore, by the Hausdorff maximality principle [17, p. 14], the family $\mathcal{B}$ contains a maximal chain $\mathcal{C}$. Moreover, by the hypothesis of the theorem, the family $\mathcal{C}$ contains a minimal element C.

Hence, by Remark 7.3.16, it is clear that $C \in \mathcal{C}^{\diamond}$. Moreover, by using Theorem 7.3.21, we can see that $\mathcal{C}^{\diamond} \subset \mathcal{B}^{\diamond} \subset \mathcal{A}^{\diamond}$. Namely, $\mathcal{C}$ is a maximal chain in $\mathcal{B}$ and $\mathcal{B}$ is a descending family in $\mathcal{A}$. Therefore, we also have $C \in \mathcal{A}^{\diamond}$. Hence, since $C \in \mathcal{B}$, and thus $C \subset A$, it is clear that $A \in \mathcal{A}^{\diamond *}$ also holds.

Therefore, we have $\mathcal{A} \subset \mathcal{A}^{\diamond *}$. Hence, by using Theorems 7.3.5 and 7.3.19, we can infer that

$$\mathcal{A}^* \subset \mathcal{A}^{\diamond **} = \mathcal{A}^{\diamond *} \subset \mathcal{A}^*.$$

And thus the required equality $A^* = \mathcal{A}^{\diamond *}$ is also true.

Corollary 7.3.26. *If $\mathcal{A}$ is an ascending subfamily of $\mathcal{P}(X)$ such that each nonvoid chain contained in $\mathcal{A}$ has a minimal element, then $\mathcal{A}^\diamond$ is the smallest base for $\mathcal{A}$.*

Proof. In this case, by Corollary 7.3.11 and Theorem 7.3.25, we have $\mathcal{A} = \mathcal{A}^* = \mathcal{A}^{\diamond *}$. Therefore, by Remark 7.3.14, $\mathcal{A}^\diamond$ is a base for $\mathcal{A}$. Moreover, if $\mathcal{B}$ is a base for $\mathcal{A}$, then from Corollary 7.3.24 we know that $\mathcal{A}^\diamond \subset \mathcal{B}$.

7.4 Some further characterizations of simple relators

The importance of the simple and quasi-simple relators lies mainly in the following theorems proved also in [68].

Theorem 7.4.1. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is proximally simple;*

(2) $\operatorname{Cl}_{\mathcal{R}}(A) = \big\{B \subset X : B \cap \rho_{\mathcal{R}}(A) \neq \emptyset\big\}$ *for all* $A \subset X$*;*

(3) $\operatorname{Int}_{\mathcal{R}}\big(\bigcap \mathcal{A}\big) = \bigcap_{A \in \mathcal{A}} \operatorname{Int}_{\mathcal{R}}(A)$ *and* $\operatorname{Int}_{\mathcal{R}}^{-1}\big(\bigcup \mathcal{A}\big) = \bigcap_{A \in \mathcal{A}} \operatorname{Int}_{\mathcal{R}}^{-1}(A)$ *for all* $\mathcal{A} \subset \mathcal{P}(X)$*.*

Theorem 7.4.2. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topologically simple;*

(2) $\mathcal{R}^\wedge = \mathcal{R}^\bullet$*;* *(3)* $\big(\mathcal{R}^{-1}\big)^\wedge \subset \big(\mathcal{R}^\wedge\big)^{-1}$*.*

Theorem 7.4.3. *If $\mathcal{R}$ is a relator on X and $\mathcal{S} = \mathcal{R}^{-1}$ or $\mathcal{R}^\vee$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topologically simple;*

(2) $A \subset \operatorname{int}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{S}}(A)\big)$ *for all* $A \subset X$*;*

(3) $B \cap \operatorname{cl}_{\mathcal{R}}(A) \neq \emptyset$ *implies* $A \cap \operatorname{cl}_{\mathcal{S}}(B) \neq \emptyset$ *for all* $A, B \subset X$*.*

Theorem 7.4.4. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topologically simple;*

(2) $\rho_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}(A)$ *for all* $A \subset X$*;*

(3) $\operatorname{int}_{\mathcal{R}}\big(\bigcap \mathcal{A}\big) = \bigcap_{A \in \mathcal{A}} \operatorname{int}_{\mathcal{R}}(A)$ *for all* $\mathcal{A} \subset \mathcal{P}(X)$*.*

Theorem 7.4.5. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-proximally simple;*

(2) $\bigcap \mathcal{A} \in \tau_{\mathcal{R}}$ *and* $\bigcup \mathcal{A} \in \tau_{\mathcal{R}}$ *for all* $\mathcal{A} \subset \tau_{\mathcal{R}}$*.*

Theorem 7.4.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is quasi-topologically simple;

(2) $\bigcap \mathcal{A} \in \mathcal{T}_{\mathcal{R}}$ for all $\mathcal{A} \subset \mathcal{T}_{\mathcal{R}}$.

Remark 7.4.7. In view of the above theorems, Árpád Száz asked for similar characterizations for the paratopologically simple relators at several conferences. Moreover, he also urged everybody to find a non-paratopologically simple relator, in order that his theory of generalized nets could be justified.

The above and some other similar problems of Árpád Száz were mostly solved by Jenő Deák [11], who essentially proved the following theorem which makes use only of Theorem 2.2.10 and the following almost self-evident, but important lemma. (See [17, p. 23].)

Lemma 7.4.8. *If $A, B \subset X$ such that $B \neq \emptyset$, then $\operatorname{card}(B) \leq \operatorname{card}(A)$ if and only if there exists a function f of A onto B.*

Now, we can prove the above mentioned theorem of Jenő Deák which was originally stated in terms of the minimum of the cardinalities of the bases of the family $\mathcal{E}_{\mathcal{R}}$.

Theorem 7.4.9. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically simple;

(2) $\mathcal{E}_{\mathcal{R}}$ has a base $\mathcal{B}$ with $\operatorname{card}(\mathcal{B}) \leq \operatorname{card}(X)$.

Proof. If the assertion (1) holds, then there exists a relation R on X such that $\mathcal{R}^{\triangle} = \{R\}^{\triangle}$. Hence, by Theorem 2.2.10, it follows that $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\{R\}}$. Now, by noticing that $\mathcal{B} = \big\{ R(x) : x \in X \big\}$ is a base for $\mathcal{E}_{\{R\}}$, and by making use of the mapping $x \mapsto R(x)$ of X onto $\mathcal{B}$ and Lemma 7.4.8, we can see that the assertion (2) also holds.

On the other hand, if the assertion (2) holds, then again by Lemma 7.4.8 there exists a function F of X onto $\mathcal{B}$. Hence, by defining a relation R on X such that $R(x) = F(x)$ for all $x \in X$, we can at once see that $\mathcal{E}_{\{R\}} = \mathcal{B}^{*} = \mathcal{E}_{\mathcal{R}}$. And thus, by Theorem 2.2.10, the assertion (1) also holds.

Hence, by using Theorems 7.4.9 and 7.3.23 and Corollary 7.3.26, we can also easily derive the following

Theorem 7.4.10. *If $\mathcal{R}$ is a relator on X such that each nonvoid chain contained in $\mathcal{E}_{\mathcal{R}}$ has a minimal element, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically simple;

(2) $\operatorname{card}(\mathcal{E}_{\mathcal{R}}^{\diamond}) \leq \operatorname{card}(X)$;

(3) $\operatorname{card}\big(\{R(x) : x \in X, R \in \mathcal{R}\}^{\diamond}\big) \leq \operatorname{card}(X)$.

Proof. If the assertion (1) holds, then by Theorem 7.4.9 there exists a base $\mathcal{B}$ for $\mathcal{E}_{\mathcal{R}}$ such that $\operatorname{card}(\mathcal{B}) \leq \operatorname{card}(X)$. Moreover, from Corollary 7.3.24, we know that $\mathcal{E}_{\mathcal{R}}^{\diamond} \subset \mathcal{B}$. Therefore, the assertion (2) also holds.

On the other hand, from Corollary 7.3.26, we know that $\mathcal{E}_{\mathcal{R}}^{\diamond}$ is a base for $\mathcal{E}_{\mathcal{R}}$. Therefore, if the assertion (2) holds, then by Theorem 7.4.9 the assertion (1) also holds.

Finally, to complete the proof, we note that the equivalence of the assertions (2) and (3) is an immediate consequence of Theorem 7.3.23.

The above theorem allows us to easily check the main assertion of the following

Example 7.4.11. If $X = \{1, 2, 3, 4\}$ and $R_i \subset X^2$ for all $i = 1, 2, 3$ such that

$$\begin{aligned} &R_1(1) = R_1(2) = \{1, 2\}, \qquad && R_1(3) = R_1(4) = \{3, 4\};\\ &R_2(1) = R_2(3) = \{1, 3\}, && R_2(2) = R_2(4) = \{2, 4\};\\ &R_3(1) = R_3(4) = \{1, 4\}, && R_3(2) = R_3(3) = \{2, 3\}; \end{aligned}$$

then $\mathcal{R} = \{R_1, R_2, R_3\}$ is an equivalence relator on X such that $\mathcal{R}$ is not paratopologically simple.

Since $\mathcal{E}_{\mathcal{R}} \subset \mathcal{P}(X)$, we can at once see that $\mathcal{E}_{\mathcal{R}}$ is finite. Hence, it is clear that each nonvoid chain contained in $\mathcal{E}_{\mathcal{R}}$ has a minimal element. Therefore, to prove that $\mathcal{R}$ is not paratopologically simple Theorem 7.4.10 can be applied.

Namely, by using Remark 7.3.16, we can easily see that

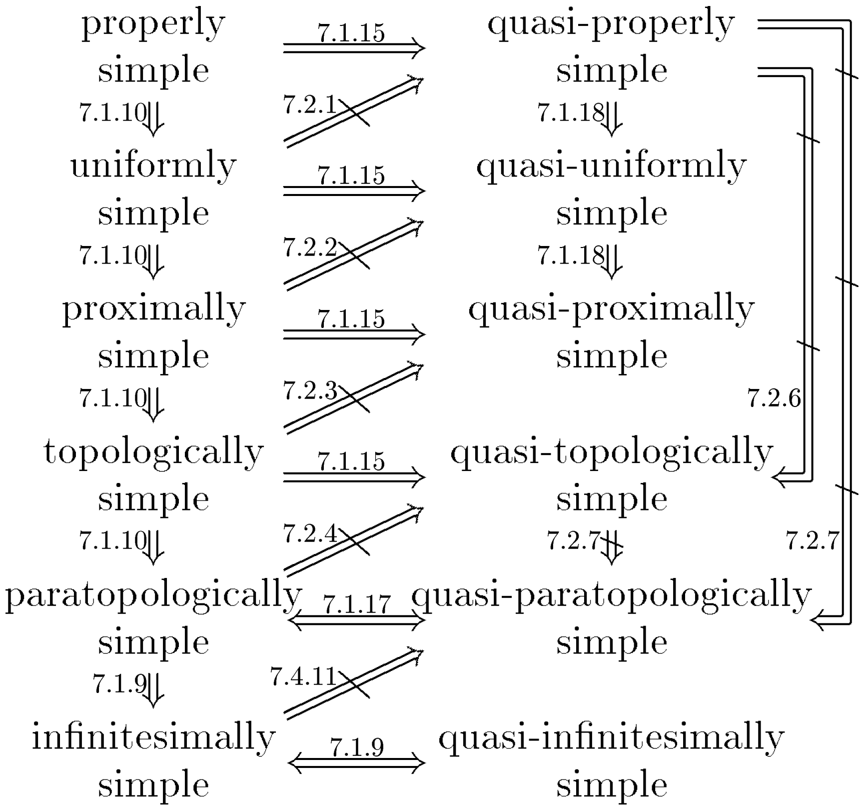
$$\mathcal{E}_{\mathcal{R}}^{\diamond} = \big\{\{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\big\},$$

and thus $\operatorname{card}(\mathcal{E}_{\mathcal{R}}^{\diamond}) \not\leq \operatorname{card}(X)$.

Remark 7.4.12. The above example substantially improves an example of Jenő Deák [11], which gives only a non-paratopologically simple, non-reflexive relator on a four element set.

In this respect, it is also worth noticing that Theorem 7.1.17 was stated by Jenő Deák as an immediate consequence of [39, Theorem 5.7] which later turned out to be true only for total relators.

The following diagram shows the main implications among the various simplicity and quasi-simplicity properties of relators.



Summary

Introduction

The PhD dissertation consists of three chapters.Most of the results of the dissertation have been published in our papers [52], [54] and [51]. In the introduction of the dissertation, we define relators and relator spaces.

A nonvoid family $\mathcal{R}$ of binary relations on a nonvoid set X is called a **relator** on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a **relator space**.

I. Relators and their induced basic tools

By establishing some intimate connections between unary operations and set-valued functions for relators, we greatly extend and supplement some of the former results of Á. Száz and J. Mala on the various refinements and set-valued functions of relators.

To provide a general framework for the investigation of the above mentioned basic tools, we have introduced in [52] several new definitions and established their most important consequences.

In *section 1*, we define unary operations (for instance refinements, modifications) and set-valued functions for relators.

A function $\Box$ of the family of all relators on X into itself is called a **unary operation** for relators on X. And we write $\mathcal{R}^{\Box} = \Box(\mathcal{R})$ for every relator $\mathcal{R}$ on X. Moreover, a function $\mathfrak{F}$ of the family of all relators on X into a family of sets is called a **set-valued function** for relators on X. And we write $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ for every relator $\mathcal{R}$ on X. If $\Box$ is a unary operation and $\mathfrak{F}$ is a set-valued function for relators on X, then we say that:

(1) $\Box$ is **expansive** if $\mathcal{R} \subset \mathcal{R}^{\Box}$ for every relator $\mathcal{R}$ on X;

(2) $\Box$ is **idempotent** if $\mathcal{R}^{\Box} = \mathcal{R}^{\Box\Box}$ for every relator $\mathcal{R}$ on X;

(3) $\mathfrak{F}$ is **increasing** (**decreasing**) if $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$) for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X with $\mathcal{S} \subset \mathcal{R}$.

(4) $\Box$ is a **modification** if it is idempotent and increasing;

(5) $\Box$ is a **refinement** if it is expansive, idempotent and increasing;

(6) $\mathfrak{F}$ is **$\Box$-increasing** (**$\Box$-decreasing**) if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\Box} \iff \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ ($\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}}$).

We prove that a unary operation $\Box$ for relators on X is a refinement if and only if there exists a set-valued function for relators on X, which is $\Box$-increasing ($\Box$-decreasing). Moreover, we also prove that if $\Box$ is a unary operation for relators on X, then the set-valued function $\mathfrak{F}$ for relators on X is $\Box$-increasing ($\Box$-decreasing) if and only if it is increasing (decreasing) and for every relator are on X $\mathcal{R}^{\Box}$ is the largest relator such that $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$.

For an increasing (decreasing) set-valued function $\mathfrak{F}$ for relators on X, we define the

induced unary operation $\square_{\mathfrak{F}}$ by

$$\mathcal{R}^{\square_{\mathfrak{F}}} = \big\{ S \subset X^2 : \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}} \big\} \qquad \Big(\mathcal{R}^{\square_{\mathfrak{F}}} = \big\{ S \subset X^2 : \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\{S\}} \big\} \Big)$$

for every relator $\mathcal{R}$ on X.

After this we can define the regular set-valued functions for relators. A monotone set-valued function for relators on X is called **regular** if

$$\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$$

for every relator $\mathcal{R}$ on X. We prove a characterization of regular set-valued functions for relators, which shows that a unary operation induced by a regular set-valued function is a refinement. Moreover the characterization shows also that if $\mathfrak{F}$ is a regular set-valued function for relators and $\mathcal{R}$ is a relator, then $\mathcal{R}^{\square_{\mathfrak{F}}}$ is the largest relator which is $\mathfrak{F}$-equivalent to $\mathcal{R}$. (The $\mathcal{R}$ and $\mathcal{S}$ relators are called **$\mathfrak{F}$-equivalent**, if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.)

We have to show the regularity of the above mentioned basic set-valued functions for relators. In order to verify this, we define the normality of set-valued functions for relators. An increasing (decreasing) set-valued function for relators is called **normal** if

$$\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \qquad \Big(\mathfrak{F}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}} \Big)$$

for every relator $\mathcal{R}$ on X. We prove that a normal set-valued function for relators on X is, in particular, regular.

We define some relation of unary operations for relators. For instance, if $\square$ and $\diamondsuit$ are unary operations for relators on X such that $\mathcal{R}^{\diamondsuit} \subset \mathcal{R}^{\square}$ for every relator $\mathcal{R}$ on X, then the unary operation $\square$ is called **$\diamondsuit$-dominant**. We state some theorem about the above relationships. Moreover, we prove that if $\mathfrak{F}$ and $\mathfrak{G}$ are regular set-valued functions for relators on X, such that $\mathfrak{F}$ determines $\mathfrak{G}$, then the unary operation $\square_{\mathfrak{F}}$ is $\square_{\mathfrak{G}}$-dominant.

After this, in *section 2*, we list the above mentioned basic set-valued functions and unary operations for relators. If $\mathcal{R}$ is a relator on X, then for any $A, B \subset X$ and $x, y \in X$ we write:

(1) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ if $R(B) \subset A$ for some $R \in \mathcal{R}$;

(2) $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ if $R(B) \cap A \neq \emptyset$ for all $R \in \mathcal{R}$;

(3) $x \in \operatorname{int}_{\mathcal{R}}(A)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(A)$; (4) $x \in \operatorname{cl}_{\mathcal{R}}(A)$ if $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(A)$;

(5) $y \in \sigma_{\mathcal{R}}(x)$ if $y \in \operatorname{int}_{\mathcal{R}}\big(\{x\}\big)$; (6) $y \in \rho_{\mathcal{R}}(x)$ if $y \in \operatorname{cl}_{\mathcal{R}}\big(\{x\}\big)$;

The relations $\operatorname{Int}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$, and $\sigma_{\mathcal{R}}$ are called the **proximal, the topological, and the infinitesimal interiors** induced by $\mathcal{R}$, respectively. While the relations $\operatorname{Cl}_{\mathcal{R}}$, $\operatorname{cl}_{\mathcal{R}}$, and $\rho_{\mathcal{R}}$ are called the **proximal, the topological, and the infinitesimal closures** induced by $\mathcal{R}$, respectively. Moreover,

(1) $A \in \tau_{\mathcal{R}}$ if $A \in \operatorname{Int}_{\mathcal{R}}(A)$; (2) $A \in \bar{\tau}_{\mathcal{R}}$ if $X \setminus A \notin \operatorname{Cl}_{\mathcal{R}}(A)$;

(3) $A \in \mathcal{T}_{\mathcal{R}}$ if $A \subset \operatorname{int}_{\mathcal{R}}(A)$;
(4) $A \in \mathcal{F}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) \subset A$;
(5) $A \in \mathcal{E}_{\mathcal{R}}$ if $\operatorname{int}_{\mathcal{R}}(A) \neq \emptyset$;
(6) $A \in \mathcal{D}_{\mathcal{R}}$ if $\operatorname{cl}_{\mathcal{R}}(A) = X$.
(7) $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}}$;
(8) $D_{\mathcal{R}} = \bigcup \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big)$.

The members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$, and $\mathcal{E}_{\mathcal{R}}$ are called the **proximally open**, the **topologically open**, and the **fat subsets** of $X(\mathcal{R})$, respectively. While the members of the families $\tau\!\!\!-_{\mathcal{R}}$, $\mathcal{F}_{\mathcal{R}}$, and $(\mathcal{D}_{\mathcal{R}})$ are called the **proximally closed**, the **topologically closed**, and the **dense** subsets of $X(\mathcal{R})$, respectively.

All of the above set-valued functions are normal except $\mathcal{T}$ and $\mathcal{F}$. Unfortunately, the increasing set-valued functions $\mathcal{T}$ and $\mathcal{F}$, on which topology was based on, are not even regular, in general. Therefore, if $\mathcal{R}$ is a relator on X, then in general there does not exist a largest relator on X, such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$ $\big(\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\square}}\big)$.

If $\mathcal{R}$ is a relator on X, then the relators

(1) $\mathcal{R}^{*} = \big\{ S \subset X^2 : \exists R \in \mathcal{R} : R \subset S \big\}$;
(2) $\mathcal{R}^{\#} = \big\{ S \subset X^2 : \forall A \subset X : A \in \operatorname{Int}_{\mathcal{R}}\big(S(A)\big) \big\}$;
(3) $\mathcal{R}^{\wedge} = \big\{ S \subset X^2 : \forall x \in X : x \in \operatorname{int}_{\mathcal{R}}\big(S(x)\big) \big\}$;
(4) $R^{\triangle} = \big\{ S \subset X^2 : \forall x \in X : S(x) \in \mathcal{E}_{\mathcal{R}} \big\}$;
(5) $\mathcal{R}^{\bullet} = \big\{ \rho_{\mathcal{R}}^{-1} \big\}^{*}$;
(6) $\mathcal{R}^{\blacktriangle} = \big\{ X \times E_{\mathcal{R}} \big\}^{*}$

are called **the uniform, the proximal, the topological, the paratopological, the infinitesimal and the parainfinitesimal refinements** of $\mathcal{R}$, respectively.

Since the results of section 1, we state the relationships between the required unary operations and set-valued functions. Namely, we have

$$\# = \square_{\mathrm{Int}} = \square_{\mathrm{Cl}}, \qquad \wedge = \square_{\mathrm{int}} = \square_{\mathrm{cl}}, \qquad \triangle = \square_{\mathcal{E}} = \square_{\mathcal{D}}, \qquad \bullet = \square_{\rho}, \qquad \blacktriangle = \square_{E} = \square_{D}.$$

Indeed, by virtue of the required theorems, the normality of the corresponding set-valued functions follows that the above defined unary operations are refinements. (We have to prove only, that $*$ is a normal set-valued function for relators on X, and $* = \square_{*}$.) Moreover, since the corresponding relations between the required set-valued functions for relators on X, we also have that if $\mathcal{R}$ is a relator on X, then

$$\begin{array}{ccccccccc} \mathcal{R} & \subset & \mathcal{R}^{*} & \subset & \mathcal{R}^{\#} & \subset & \mathcal{R}^{\wedge} & \subset & \mathcal{R}^{\triangle} \\ & & & & & & \cap & & \cap \\ & & & & & & \mathcal{R}^{\bullet} & \subset & \mathcal{R}^{\blacktriangle}. \end{array}$$

Only the last inclusion of the above figure needs some comment, and with a counterexample we prove that $\mathcal{R}^{\triangle}$ and $\mathcal{R}^{\bullet}$ are incomparable, in general. Moreover, we give a more direct proof for the equality $\mathcal{R}^{\bullet} = \mathcal{R}^{\vee\vee}$, which was already proved by J. Mala and Á. Száz.

Finally, we investigate some other set-valued functions and unary operations for relators on X. Now, we mention only one of these, which will be necessary in the following chapter. The **preorder modification** of the relator $\mathcal{R}$ on X defined by

$$\mathcal{R}^\infty = \{R^\infty : R \in \mathcal{R}\},$$

where R^∞ means the preorder hull of the relation R on X. Since this unary operation is not expansive, we can only prove that this is a modification for relators on X. Moreover, we compare it with the uniform refinement. Here we show, that the set-valued functions $\mathcal{T}$ and $\mathcal{F}$ are not even regular, in general.

In *section 3*, we investigate some special relators, which are interesting by the point of view of topology. In this summary we mention only one of these. The relator $\mathcal{R}$ on X is called **topological**, if for all $x \in X$ and $R \in \mathcal{R}$, there exists a topologically open set V such that $x \in V \subset R(x)$. We cite two important theorem without their proofs. Namely, a relator space is topological if and only if the topologically interior of an arbitrary set is equal to the union of its topologically open subsets. On the other hand, a subset of a topological relator space is fat if and only if there exists a nonvoid topologically open subset of it.

Farther, we write about continuous relations in relator spaces. The above and the following notion, we need some binary operation for relators. Now, we mention one of these. If $\mathcal{R} = \{R_i\}_{i \in I}$ and $\mathcal{S} = \{S_i\}_{i \in I}$ are relators on X, then

$$\mathcal{R} \triangledown \mathcal{S} = \{R_i \cup S_i : i \in I\}.$$

In particular, since $\mathcal{R} = \{R\}_{R \in \mathcal{R}}$ and $\mathcal{R}^{-1} = \{R^{-1}\}_{R \in \mathcal{R}}$, we have that

$$\mathcal{R} \triangledown \mathcal{R}^{-1} = \{R \cup R^{-1} : R \in \mathcal{R}\}.$$

II. Well-chainednesses and connectednesses of relators

In this chapter, a unified treatment of some old and new well-chainedness and connectedness properties of the most basic topological structures (such as closures, proximities and uniformities, for instance) is offered in the framework of relators and their fundamental refinements.

The results obtained show that the various connectedness properties are actually particular cases of Cantor's well-chainedness property neglected by several authors. Moreover, they show that the hyperconnectedness introduced by L. A. Steen and J. A. Seebach is a particular case of our paratopological connectedness.

In *section 4*, according to the results of [26] we define well-chainedness properties in the following way.

At first we need some relationship between set-valued functions, which are not listed here, later we note the needful ones.

Moreover, we need the notion of the Davis–Pervin relations. If $A \subset X$, then the relation

$$R_A = A^2 \cup (X \setminus A) \times X$$

is called the **Davis–Pervin relation** generated by A.

After this we define well-chained relators. The relator $\mathcal{R}$ on X is called **properly well-chained** if

$$\mathcal{R}^{\infty} = \{X^2\}.$$

The condition $\mathcal{R}^{\infty} = \{X^2\}$, in a detailed form, means only that for every $R \in \mathcal{R}$ we have $X^2 = R^{\infty} = \Delta_X \cup \bigcup_{n=1}^{\infty} R^n$. That is, for every $x, y \in X$, with $x \neq y$, there exists an $n \in \mathbb{N}$ such that $(x, y) \in R^n$. That is, there exists a family $(x_i)_{i=0}^{n}$ in X such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R$ for all $i = 1, \dots, n$. Note, that if $\operatorname{card}(X) \geq 2$, then the $x \neq y$ condition is omissible. By the previous theorems we give the following, detailed characterizations of well-chained relators. If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:

(1) $\mathcal{R}$ is properly well-chained;

(2) $R_A \notin \mathcal{R}^*$ for every proper nonvoid subset A of X;

(3) $R \notin \mathcal{R}^*$ for every proper preorder R on X;

(4) $R \notin \mathcal{R}^*$ for every proper nonvoid transitive relation R on X.

(5) $\mathscr{T}_{\mathcal{R}} = \{\emptyset, X\}$; (6) $\mathscr{F}_{\mathcal{R}} = \{\emptyset, X\}$.

Moreover, we investigate the various well-chainedness properties by the refinements from chapter I. If $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ on X is called **$\square$-well-chained**, if the relator space $X(\mathcal{R}^{\square})$ is properly well-chained. We can easily see, that if $\square$ and $\diamondsuit$ are unary operations for relators such that $\square$ is $\diamondsuit$-dominating, then the $\square$-well-chainedness follows the $\diamondsuit$-well-chainedness. We characterize the $\square$-well-chained relators like the properly well-chained relators. If $\square$ is a $*$-invariant refinement for relators (all refinements from the previous chapter are $*$-invariant), then the relator $\mathcal{R}$ on X is $\square$-well-chained if and only if X^2 is the only preorder in $\mathcal{R}^{\square}$.

Since $\mathscr{T}_{\mathcal{R}} = \mathscr{T}_{\mathcal{R}^*} = \mathscr{T}_{\mathcal{R}^\#}$, or equivalently $\mathscr{F}_{\mathcal{R}} = \mathscr{F}_{\mathcal{R}^*} = \mathscr{F}_{\mathcal{R}^\#}$, we can see that the proper, uniform and proximal well-chainedness properties are equivalent.

Since $\mathscr{T}_{\mathcal{R}^{\wedge}} = \mathcal{T}_{\mathcal{R}}$, or equivalently $\mathscr{F}_{\mathcal{R}^{\wedge}} = \mathcal{F}_{\mathcal{R}}$, we can see that the relator $\mathcal{R}$ on X is topologically well-chained if and only if $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$, or equivalently $\mathcal{F}_{\mathcal{R}} = \{\emptyset, X\}$.

Since $\mathscr{T}_{\mathcal{R}^{\vartriangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$, or equivalently $\mathscr{F}_{\mathcal{R}^{\vartriangle}} = (\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}) \cup \{X\}$, we can see that if $\operatorname{card}(X) > 1$ then the relator $\mathcal{R}$ on X is paratopologically well-chained if and only if $\mathcal{E}_{\mathcal{R}} = \{X\}$, or equivalently $\mathcal{R} = \{X^2\}$.

By using our above results, we can see that parainfinitesimally well-chained relators need not be studied anymore.

Since $\mathscr{T}_{\mathcal{R}^{\bullet}} = \mathscr{F}_{\{\rho_{\mathcal{R}}\}}$, or equivalently $\mathscr{F}_{\mathcal{R}^{\bullet}} = \mathscr{T}_{\{\rho_{\mathcal{R}}\}}$, we can see that the relator $\mathcal{R}$ on X is infinitesimally well-chained if and only if $\mathscr{T}_{\{\rho_{\mathcal{R}}\}} = \{\emptyset, X\}$, or equivalently $\mathscr{F}_{\{\rho_{\mathcal{R}}\}} = \{\emptyset, X\}$. Moreover, we can see that the the infinitesimally well-chainedness of the relator $\mathcal{R}$ on X is equivalent to the properly well-chainedness of the relator $\{\rho_{\mathcal{R}}\}$ on X.

In *section 5*, according to the results of [23] and [27], we define connectedness properties in the following way.

We need the symmetrization of the Davis–Pervin relations. If $A \subset X$, then the relation

$$S_A = R_A \cap R_A^{-1}$$

is called the **symmetrization of the Davis–Pervin relation** R_A.

A relator $\mathcal{R}$ on X will be called **properly connected** if the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is properly well-chained.

The condition $\left(\mathcal{R} \triangledown \mathcal{R}^{-1}\right)^{\infty} = \{X^2\}$, means only that for every $x, y \in X$, with $x \neq y$, and every $R \in \mathcal{R}$ there exist a finite family $(x_i)_{i=0}^{n}$ in X such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R \cup R^{-1}$, i.e., $(x_{i-1}, x_i) \in R$ or $(x_i, x_{i-1}) \in R$ for all $i = 1, \ldots, n$. Note, that if $\operatorname{card}(X) \geq 2$, then the $x \neq y$ condition is omissible. By the previous theorems we give the following, detailed characterizations of connected relators. If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:

(1) $\mathcal{R}$ is properly connected;

(2) $S_A \notin \mathcal{R}^*$ for every proper nonvoid subset A of X;

(3) $S \notin \mathcal{R}^*$ for every proper equivalence S on X;

(4) $S \notin \mathcal{R}^*$ for every proper nonvoid symmetric and transitive relation S on X.

Moreover, we investigate the various connectedness properties by the refinements from chapter I. If $\Box$ is a unary operation for relators on X, then the relator $\mathcal{R}$ on X is called **$\Box$-connected** if the relator $\mathcal{R}^{\Box}$ is properly connected. We can easily see, that if $\Box$ and $\Diamond$ are unary operations for relators such that $\Box$ is $\Diamond$-dominating, then the $\Box$-connectedness follows the $\Diamond$-connectedness. We characterize the $\Box$-connected relators like the properly connected relators. If $\Box$ is a $*$-invariant refinement for relators (all refinements from the previous chapter are $*$-invariant), then the relator $\mathcal{R}$ on X is $\Box$-connected if and only if X^2 is the only equivalence in $\mathcal{R}^{\Box}$. It follows that proper and uniform connectedness properties are equivalent.

Since proximal connectedness is not equivalent to the proper one, we can state only that a relator $\mathcal{R}$ is on X is proximally connected if and only if $\tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}} = \{\emptyset, X\}$.

Since the above listed equalities, we have the following characterizations of $\Box$-connected properties. The relator $\mathcal{R}$ on X is topologically connected if and only if $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} = \{\emptyset, X\}$.

If $\operatorname{card}(X) > 1$ then the relator $\mathcal{R}$ on X is paratopologically connected if and only if $\mathcal{E}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}}$, or equivalently $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{E}_{\mathcal{R}}$.

The relator $\mathcal{R}$ on X is infinitesimally connected if and only if $\tau_{\{\rho_{\mathcal{R}}\}} \cap \bar{\tau}_{\{\rho_{\mathcal{R}}\}} = \{\emptyset, X\}$, or equivalently $\{\rho_{\mathcal{R}}\}$ is proximally connected. By our former results it holds if and only if the relator $\{\rho_{\mathcal{R}}\}$ is properly connected.

And, since $\tau_{\mathcal{R}^{\blacktriangle}} = \{A \subset X : E_{\mathcal{R}} \subset A\} \cup \{\emptyset\}$, or equivalently $\bar{\tau}_{\mathcal{R}^{\blacktriangle}} = \mathcal{P}(D_{\mathcal{R}}) \cup \{X\}$, we can see that the relator $\mathcal{R}$ on X is parainfinitesimally connected if and only if $E_{\mathcal{R}} \neq \emptyset$, or equivalently $D_{\mathcal{R}} \neq X$.

We further investigate the connected relators. If $\mathcal{R}$ is a topological relator on X, then it is paratopologically connected if and only if $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subset \mathcal{D}_{\mathcal{R}}$, or equivalently

$U \cap V \neq \emptyset$ for all $U, V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.

Moreover, we investigate the properties of continuous relations and functions of a relator space into another.

In *section 6*, we compare the various well-chainedness and connectedness properties. For instance, if $\Box$ is a unary operation for relators on X and $\mathcal{R}$ is a $\Box$-well-chained relator on X, then $\mathcal{R}$ is, in particular, $\Box$-connected.

We show counterexamples for implications between the well-chainedness and connectedness properties which do not hold, in general.

At the end of this chapter, a diagram can be found which shows the main implications among the various simplicity and quasi-simplicity properties of relators.

III. Simplicity of relators

In chapter III, some published and unpublished results of Árpád Száz, József Mala and Jenő Deák on simple and quasi-simple relators are illustrated and supplemented. Simple and quasi-simple relators were mainly investigated by Árpád Száz, but several interesting problems have been left open. The most exciting ones were solved by József Mala and Jenő Deák. However, the results of the latter author have not been published because of his early and tragic death.

Therefore, the main purpose of [51] was not only to solve some of the remaining open problems of Árpád Száz, but also to present the relevant results of Jenő Deák. The latter author provided a useful characterization of paratopologically simple relators, which lead us to the investigation of two natural operations on families of sets.

In *section 7* we define the simplicity properties of relators. A relator $\mathcal{R}$ on X is called **properly simple** if it is a singleton. Moreover, if $\Box$ is a unary operation for relators on X, then the relator $\mathcal{R}$ is called **$\Box$-simple**, if it is $\Box$-equivalent to a properly simple relator. Further, a relator is called **quasi-$\Box$-simple**, if it is $(\Box\infty)$-equivalent to a properly simple relator. We remark, that for instance, the topologically well-chainedness is a particular case of quasi-topologically simplicity.

In the dissertation, we prove that a relator $\mathcal{R}$ on X is quasi-properly simple if and only if $R^{\infty} = S^{\infty}$ for all $R, S \in \mathcal{R}$, or equivalently $\mathcal{R}^{\infty}$ is properly simple. And, if a relator is $\Box$-simple, then it is also quasi-$\Box$-simple.

After this, we characterize $\Box$-simple relators. If $\Box \in \{*, \#, \wedge, \bullet\}$, then the relator $\mathcal{R}$ on X is $\Box$-simple if and only if $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^{\Box}$.

By the above theorem, we can see that every relator is infinitesimally simple. Therefore, quasi-infinitesimally, parainfinitesimally and quasi-parainfinitesimally simple relators need not be study anymore. Moreover, 'paratopologically simple' $\Longrightarrow$ 'infinitesimally simple'.

Moreover, we characterize quasi-simple relators. The relator $\mathcal{R}$ on X is quasi-uniformly, quasi-proximally and quasi-topologically simple if and only if $\rho_{\mathcal{R}^{\infty}}^{-1} \in \mathcal{R}^{*}$, $\rho_{\mathcal{R}^{\infty}}^{-1} \in \mathcal{R}^{\#}$ and $\rho_{\mathcal{R}^{\wedge\infty}}^{-1} \in \mathcal{R}^{\wedge}$, respectively.

After this, we only have to study paratopological and quasi-paratopological simplicities. At first, we prove that this two properties are equivalent. And, we introduce two operations on families of sets.

If $\mathcal{A} \subset \mathcal{P}(X)$, then we write

$$\mathcal{A}^* = \big\{ B \subset X : \exists A \in \mathcal{A} : A \subset B \big\}.$$

Note that the notion of the uniform refinement of relators is a special case of the above definition.

If $\mathcal{A} \subset \mathcal{P}(X)$, then we write

$$\mathcal{A}^\diamond = \big\{ B \in \mathcal{A} : A \in \mathcal{A}, A \subset B \implies A = B \big\}.$$

Note that thus $\mathcal{A}^\diamond$ is just the family of the minimal members of $\mathcal{A}$.

Moreover, we define the base of families of sets. If $\mathcal{A}$ is an ascending family in $\mathcal{P}(X)$ and $\mathcal{B} \subset \mathcal{A}$ such that for each $A \in \mathcal{A}$ there exists a $B \in \mathcal{B}$ such that $B \subset A$, then $\mathcal{B}$ will be called a **base** for $\mathcal{A}$.

Now, we can characterize paratopological simple relators. The relator $\mathcal{R}$ on X is paratopologically simple if and only if $\mathcal{E}_{\mathcal{R}}$ has a base $\mathcal{B}$ with $\operatorname{card}(\mathcal{B}) \leq \operatorname{card}(X)$.

For farther characterization we need the following well-known assertion. If $\mathcal{A}$ is an ascending subfamily of $\mathcal{P}(X)$ such that each nonvoid chain contained in $\mathcal{A}$ has a minimal element, then $\mathcal{A}^\diamond$ is the smallest base for $\mathcal{A}$.

By the above two assertions we can see that, if $\mathcal{R}$ is a relator on X such that each nonvoid chain contained in $\mathcal{E}_{\mathcal{R}}$ has a minimal element, then the paratopological simplicity of the relator $\mathcal{R}$ is equivalent to $\operatorname{card}(\mathcal{E}_{\mathcal{R}}^\diamond) \leq \operatorname{card}(X)$.

Since the extra condition according to the minimal element holds if X is a finite set, we can easily construct a non-paratopologically simple, equivalence relator.

At the end of this chapter, a diagram can be found which shows the main implications among the various simplicity and quasi-simplicity properties of relators.

Összefoglaló

Bevezetés

A disszertáció bevezetésében definiáljuk a reláció, a relátor és a relátor tér fogalmát, illetve rögzítjük a velük kapcsolatos jelöléseket és elnevezéseket. Ezek után megmutatjuk, hogyan általánosítja a relátor terek elmélete fontos topológiai struktúrákét, úgymint metrikus terekét az ε-sugarú környékekkel, és topologikus terekét a Davis–Pervin relációkkal. Megemlítjük továbbá, hogy a rendezett halmazok és az uniform terek a relátor terek speciális esetei.

Ebben az összefoglalóban főleg az új eredményekre koncentrálunk, a disszertációban szereplő korábbi eredmények közül csak azokat emeljük ki, melyeket feltétlenül szükséges.

Egy nem üres X halmazon tekintett binér relációk egy $\mathcal{R}$ nem üres rendszerét az X-en tekintett **relátor**nak nevezzük, az $X(\mathcal{R}) = (X, \mathcal{R})$ rendezett párt pedig **relátor tér**nek.

I. rész: Relátorok és alapvető eszközeik

A disszertáció három részre tagolódik. Ezek közül az elsőben megismerkedünk a legfontosabb eszközökkel, melyek a disszertáció későbbi részeiben, illetve más, a relátor terek elméletével foglalkozó kutatásokban alapvető jelentőségűek. Bár az első részben a téma alapjait tárgyaljuk, mégis itt találjuk a disszertációban szereplő új eredmények nagyobb részét. A Mala J. és Száz Á. által részletesen vizsgált különböző kifinomítások, és az őket indukáló halmazértékű függvények tárgyalása sok hasonlóságot mutat, ez tette lehetővé egységes tárgyalásukat.

Az *1. szakasz* az [52] cikkem alapján íródott, de tartalmaz néhány publikálatlan állítást, melyek egy része elhangzott a 2004-ben rendezett, X. Síkfőkúti Analízis Szemináriumon, „A relátorok jólláncoltságáról” című előadásomban.

Ebben a szakaszban bevezetjük a kifinomítás és más unér operációk, például módosítás, általános fogalmát, illetve megvizsgáljuk a relátorokon értelmezett monoton halmazértékű függvények kapcsolatát unér operációkkal. (Nyilván az unér operációk speciális halmazértékű függvények.)

Az X-en tekintett relátorokon értelmezett **halmazértékű függvény**nek olyan függvényt nevezünk, melynek értelmezési tartománya az X-en tekintett relátorok halmaza, értékei pedig halmazok. Fontos pontosan meghatározni az X alaphalmazt, de a rövidség kedvéért ezt most elhagyjuk, és csak halmazértékű függvényekként említjük ezeket. Ebben az összefoglalóban ez nem okoz félreértést, mivel relátor alatt mindig X-en tekintett relátort értünk, a halmazértékű függvényeket pedig mindig relátorokon értelmezzük. Külön foglalkozunk növekvő és csökkenő halmazértékű függvényekkel, de mivel a monoton halmazértékű függvények két típusa teljesen párhuzamosan tárgyalható, most csak a növekvőkről beszélünk. **Unér operáció**nak olyan, halmazértékű függvényt nevezünk, melynek értékei szintén relátorok. (X-en tekintett relátorokon értelmezett unér operáció helyett, a fentihez hasonló megfontolásból, csak unér operációt írunk.) Egy $\mathfrak{F}$

halmazértékű függvényt a $\Box$ unér operáció szerint **$\Box$-növekvő**nek nevezünk, ha

$$\mathcal{S} \subset \mathcal{R}^{\Box} \iff \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$$

teljesül minden $\mathcal{S}$ és $\mathcal{R}$ relátor esetén. Végül egy $\Box$ unér operációt **kifinomítás**nak nevezünk, ha

(1) **expanzív**, azaz $\mathcal{R} \subset \mathcal{R}^{\Box}$ minden $\mathcal{R}$ relátor esetén,

(2) **növekvő**, azaz $\mathcal{S}^{\Box} \subset \mathcal{R}^{\Box}$ minden $\mathcal{S}$ és $\mathcal{R}$ relátorra, melyekre $\mathcal{S} \subset \mathcal{R}$,

(3) **idempotens**, azaz $(\mathcal{R}^{\Box})^{\Box} = \mathcal{R}^{\Box}$ szintén minden $\mathcal{R}$ relátor esetén.

Megmutatjuk, hogy egy $\Box$ unér operáció pontosan akkor kifinomítás, ha van $\Box$-növekvő halmazértékű függvény. Megmutatjuk továbbá, hogy ha $\Box$ egy unér operáció, akkor egy $\mathfrak{F}$ halmazértékű függvény pontosan akkor $\Box$-növekvő, ha növekvő és minden $\mathcal{R}$ relátor esetén $\mathcal{R}^{\Box}$ a legbővebb relátor, melyre $\mathfrak{F}_{\mathcal{R}^{\Box}} \subset \mathfrak{F}_{\mathcal{R}}$.

Értelmezzük az $\mathfrak{F}$ növekvő halmazértékű függvény által **indukált $\Box_{\mathfrak{F}}$ unér operáció**t a következőképpen:

$$\mathcal{R}^{\Box_{\mathfrak{F}}} = \{S \subset X^2 : \mathfrak{F}_{\{S\}} \subset \mathfrak{F}_{\mathcal{R}}\},$$

ahol $\mathcal{R}$ egy tetszőleges relátor. Ezek után definiálhatjuk a reguláris halmazértékű függvény fogalmát. Egy növekvő halmazértékű függvényt **reguláris**nak nevezünk, ha minden $\mathcal{R}$ relátor esetén

$$\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}.$$

Megadjuk reguláris halmazértékű függvények egy jellemzését, melyből a fent említett ekvivalenciák alapján kiderül, hogy reguláris halmazértékű függvény által indukált unér operáció mindig kifinomítás. Kiderül továbbá, hogy ha $\mathfrak{F}$ egy reguláris halmazértékű függvény, $\mathcal{R}$ pedig egy relátor, akkor $\mathcal{R}^{\Box_{\mathfrak{F}}}$ a legbővebb $\mathcal{R}$-rel $\mathfrak{F}$-ekvivalens relátor. (Az $\mathcal{R}$ és $\mathcal{S}$ relátorokat **$\mathfrak{F}$-ekvivalens**nek nevezzük, ha $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$.) Azonban ezzel még nem tudjuk az egységes tárgyalás speciális eseteiként kezelni a fent említett szerzők által vizsgált eseteket. Ehhez az általuk vizsgált halmazértékű függvényekről meg kell mutatnunk, hogy regulárisak. Ennek érdekében bevezetjük a normális halmazértékű függvény fogalmát. Egy $\mathfrak{F}$ növekvő halmazértékű függvényt **normális**nak nevezünk, ha

$$\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_{\{R\}}$$

minden $\mathcal{R}$ relátor esetén. Megmutatjuk, hogy egy normális halmazértékű függvény mindig reguláris, és a fenti konkrét halmazértékű függvények normalitása pedig már könnyen ellenőrizhető.

Végül kimondunk néhány definíciót unér operációk közötti kapcsolatokról. Például, ha a $\Box$ és $\Diamond$ unér operációk úgy, hogy $\mathcal{R}^{\Diamond} \subset \mathcal{R}^{\Box}$ minden $\mathcal{R}$ relátor esetén, akkor a $\Box$ unér operációt **$\Diamond$-domináns**nak nevezzük. Kimondunk még néhány tételt arról, hogy a fent definiált kapcsolatok mely esetekben implikálják egymást. Megmutatjuk továbbá, hogy ha az $\mathfrak{F}$ reguláris halmazértékű függvényből kifejezhető a $\mathfrak{G}$ reguláris halmazértékű függvény, akkor a $\Box_{\mathfrak{F}}$ unér operáció $\Box_{\mathfrak{G}}$-domináns.

A *2. szakasz* szintén az [52] alapján íródott, de az itt szereplő állítások közül sokat megtalálhatunk a témával kapcsolatos korábbi ([39], [64] és [71]) cikkekben is. Ezeken kívül ez a szakasz tartalmaz néhány olyan állítást, melyek a Száz Árpáddal közös [54] cikkben szerepeltek először, ezekről az összefoglalóban nem lesz szó.

Definiáljuk a jól ismert halmazértékű függvényeket, továbbá néhány újat, melyek közül most csak a legfontosabbakat soroljuk fel. Ha $\mathcal{R}$ egy relátor, $A, B \subset X$ és $x, y \in X$, akkor

(1) $B \in \mathrm{Int}_{\mathcal{R}}(A)$, ha $R(B) \subset A$ valamely $R \in \mathcal{R}$ relációra;

(2) $B \in \mathrm{Cl}_{\mathcal{R}}(A)$, ha $R(B) \cap A \neq \emptyset$ minden $R \in \mathcal{R}$ relációra;

(3) $x \in \mathrm{int}_{\mathcal{R}}(A)$, ha $\{x\} \in \mathrm{Int}_{\mathcal{R}}(A)$; (4) $x \in \mathrm{cl}_{\mathcal{R}}(A)$, ha $\{x\} \in \mathrm{Cl}_{\mathcal{R}}(A)$;

(5) $y \in \sigma_{\mathcal{R}}(x)$, ha $y \in \mathrm{int}_{\mathcal{R}}\big(\{x\}\big)$; (6) $y \in \rho_{\mathcal{R}}(x)$, ha $y \in \mathrm{cl}_{\mathcal{R}}\big(\{x\}\big)$.

Az $\mathrm{Int}_{\mathcal{R}}$, $\mathrm{int}_{\mathcal{R}}$ és $\sigma_{\mathcal{R}}$ relációkat rendre az $\mathcal{R}$ által indukált **proximális, topologikus és infinitezimális belsőképzés**nek nevezzük, a $\mathrm{Cl}_{\mathcal{R}}$, $\mathrm{cl}_{\mathcal{R}}$ és $\rho_{\mathcal{R}}$ relációkat pedig rendre az $\mathcal{R}$ által indukált **proximális, topologikus és infinitezimális lezárás**nak. Továbbá

(7) $A \in \tau_{\mathcal{R}}$, ha $A \in \mathrm{Int}_{\mathcal{R}}(A)$; (8) $A \in \tau\!\!\!{-}_{\mathcal{R}}$, ha $X \setminus A \notin \mathrm{Cl}_{\mathcal{R}}(A)$;

(9) $A \in \mathcal{T}_{\mathcal{R}}$, ha $A \subset \mathrm{int}_{\mathcal{R}}(A)$; (10) $A \in \mathcal{F}_{\mathcal{R}}$, ha $\mathrm{cl}_{\mathcal{R}}(A) \subset A$;

(11) $A \in \mathcal{E}_{\mathcal{R}}$, ha $\mathrm{int}_{\mathcal{R}}(A) \neq \emptyset$; (12) $A \in \mathcal{D}_{\mathcal{R}}$, ha $\mathrm{cl}_{\mathcal{R}}(A) = X$.

A $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ és $\mathcal{E}_{\mathcal{R}}$ halmazok elemeit rendre az $X(\mathcal{R})$ relátor tér **proximálisan nyílt**, **topologikusan nyílt** és **kövér** részhalmazainak nevezzük, a $\tau\!\!\!{-}_{\mathcal{R}}$, $\mathcal{F}_{\mathcal{R}}$ és $\mathcal{D}_{\mathcal{R}}$ halmazok elemeit pedig rendre az $X(\mathcal{R})$ relátor tér **proximálisan zárt**, **topologikusan zárt** és **sűrű** részhalmazainak. Definiálunk két további halmazértékű függvényt:

(13) $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}}$; (14) $D_{\mathcal{R}} = \bigcup \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big)$.

Hivatkozunk arra, hogy a $\mathcal{T}$ és $\mathcal{F}$ kivételével mind normálisak, illetve megmutatjuk a közöttük fennálló kapcsolatokat, ahol azok nem derülnek ki a definíciókból.

Ezután bevezetjük az előbbiekben definiált normális halmazértékű függvények által indukált kifinomításokat, melyek közül most szintén csak a legfontosabbakat soroljuk fel. Ha az $\mathcal{R}$ egy relátor az X-en, akkor a következő relátorokat rendre az $\mathcal{R}$ **egyenletes, proximális, topologikus, paratopologikus**, **infinitezimális** illetve **parainfinitezimális kifinomítás**ának nevezzük:

$$\begin{aligned}
\mathcal{R}^{*} &= \big\{ S \subset X^2 : \exists R \in \mathcal{R} : R \subset S \big\}, \\
\mathcal{R}^{\#} &= \big\{ S \subset X^2 : \forall A \subset X : A \in \mathrm{Int}_{\mathcal{R}}\big(S(A)\big) \big\}, \\
\mathcal{R}^{\wedge} &= \big\{ S \subset X^2 : \forall x \in X : x \in \mathrm{int}_{\mathcal{R}}\big(S(x)\big) \big\}, \\
R^{\vartriangle} &= \big\{ S \subset X^2 : \forall x \in X : S(x) \in \mathcal{E}_{\mathcal{R}} \big\}, \\
\mathcal{R}^{\bullet} &= \big\{ \rho_{\mathcal{R}}^{-1} \big\}^{*} \qquad \text{és} \qquad \mathcal{R}^{\blacktriangle} = \big\{ X \times E_{\mathcal{R}} \big\}^{*}.
\end{aligned}$$

Az 1. szakasz eredményei alapján, javarészt bizonyításuk nélkül közöljük a tárgyalt kifinomítások kapcsolatát a megfelelő halmazértékű függvényekkel. Nevezetesen, hogy

$$\# = \square_{\mathrm{Int}} = \square_{\mathrm{Cl}}, \qquad \wedge = \square_{\mathrm{int}} = \square_{\mathrm{cl}}, \qquad \vartriangle = \square_{\mathcal{E}} = \square_{\mathcal{D}}, \qquad \bullet = \square_{\rho}, \qquad \blacktriangle = \square_{E} = \square_{D}.$$

Mivel a megfelelő halmazértékű függvények normálisak, így az imént definiált unér operációk kifinomítások. (Azt kell csak belátnunk, hogy $*$, mint halmazértékű függvény normális, és hogy $* = \Box_*$.) Továbbá a halmazértékű függvények közötti megfelelő kapcsolatok szerint, melyekre az összefoglalóban csak hivatkoztunk, ha az $\mathcal{R}$ egy relátor, akkor

$$\begin{array}{ccccccccc} \mathcal{R} & \subset & \mathcal{R}^* & \subset & \mathcal{R}^\# & \subset & \mathcal{R}^\wedge & \subset & \mathcal{R}^\triangle \\ &&&&&& \cap && \cap \\ &&&&&& \mathcal{R}^\bullet & \subset & \mathcal{R}^\blacktriangle. \end{array}$$

Az ábra utolsó tartalmazásának, az $\mathcal{R}^\bullet \subset \mathcal{R}^\blacktriangle$ tartalmazásnak igazolásához van szükségünk csupán konkrét vizsgálatokra, illetve egy ellenpélda konstruálásával igazoljuk, hogy $\mathcal{R}^\triangle$ és $\mathcal{R}^\bullet$ általában nem összehasonlíthatóak. Továbbá a Mala J. és Száz Á. által már korábban bizonyított tételnek, mely szerint $\mathcal{R}^\bullet = \mathcal{R}^{\vee\vee}$, adjuk egy új, az előzőnél közvetlenebb bizonyítását.

Végül kitérünk néhány egyéb halmazértékű függvény és unér operáció vizsgálatára, melyek közül most csak egyet említünk, amely nélkülözhetetlen lesz a második részben. Egy $\mathcal{R}$ relátor **prerendezési módosítás**án az

$$\mathcal{R}^\infty = \{R^\infty : R \in \mathcal{R}\}$$

relátort értjük, ahol R^∞ az R reláció prerendezési burka. Minthogy ez az unér operáció nem expanzív, így csak annyit tudunk róla bizonyítani, hogy módosítás, illetve az egyenletes kifinomítással való kapcsolatát említjük még meg, amely szintén hasznos lesz a későbbiekben. Itt mutatjuk meg azt is egy ellenpéldán keresztül, hogy a fenti $\mathcal{T}$ és $\mathcal{F}$ halmazértékű függvények még csak nem is regulárisak általában.

A *3. szakasz* az [54] cikk topológiai szempontból érdekes relátorok vizsgálatával foglakozó részét idézi. Az ebben a szakaszban szereplő tételek, Mala J. és Száz Á. korábbi eredményeinek átfogalmazásai, a fenti általánosításoknak megfelelően.

Ebben az összefoglalóban ezekre nem térünk ki, csak példaként említünk közülük egyet. Egy X-en tekintett $\mathcal{R}$ relátort **topologikus**nak nevezünk, ha minden $x \in X$ és $R \in \mathcal{R}$ esetén létezik V topologikusan nyílt halmaz, melyre $x \in V \subset R(x)$. Bizonyítás nélkül idézünk két fontos tételt. Nevezetesen egy relátor tér pontosan akkor topologikus, ha tetszőleges halmaz topologikus belseje megegyezik topologikusan nyílt részhalmazainak uniójával. Illetve topologikus relátor térben egy halmaz pontosan akkor kövér, ha létezik nem üres, topologikusan nyílt részhalmaza.

Ezeken kívül beszélünk még relátor terekben értelmezett folytonos relációkról. A fenti és későbbi fogalmak tárgyalásához szükségünk van néhány relátorok közti binér operációra, melyek közül példaként megemlítünk egyet. Ha $\mathcal{R} = \{R_i\}_{i \in I}$ és $\mathcal{S} = \{S_i\}_{i \in I}$ relátorok az X-en, akkor

$$\mathcal{R} \nabla \mathcal{S} = \{R_i \cup S_i : i \in I\}.$$

Speciálisan, ha az $\mathcal{R}$ egy relátor, akkor $\mathcal{R} = \{R\}_{R \in \mathcal{R}}$ és $\mathcal{R}^{-1} = \{R^{-1}\}_{R \in \mathcal{R}}$, így

$$\mathcal{R} \nabla \mathcal{R}^{-1} = \{R \cup R^{-1} : R \in \mathcal{R}\}.$$

II. rész: Relátorok jólláncoltsága és összefüggősége

A II. részben a topológiai struktúrákban nagyon fontos jólláncoltságot és összefüggőséget vizsgáljuk relátor terekben. Ez a rész szinte teljes egészében az [54] cikk alapján íródott, melyben ismertetjük Kurdics J. és Száz Á. [25], [26], [27] és [76] cikkekben szereplő korábbi eredményeit, majd kiegészítjük, és példákkal illusztrálva megmutatjuk a különböző jólláncoltságok és összefüggőségek közötti kapcsolatokat.

A *4. szakasz*ban a jólláncolt relátorokat vizsgáljuk. Ehhez először is szükségünk van néhány halmazértékű függvény közötti kapcsolatra. Melyeket most nem sorolunk fel, később majd a szükségeseket megemlítjük.

Szükségünk van továbbá a Davis–Pervin relációkra. Ha $A \subset X$, akkor az

$$R_A = A^2 \cup (X \setminus A) \times X$$

relációt az A által generált **Davis–Pervin relációnak** nevezzük.

Ezek után bevezetjük a jólláncoltság fogalmát. Az $X(\mathcal{R})$ relátor teret **ténylegesen jólláncoltnak** nevezzük, ha

$$\mathcal{R}^{\infty} = \{X^2\}.$$

Ez részletezve azt jelenti, hogy minden $R \in \mathcal{R}$, és $x, y \in X$, $x \neq y$ esetén, létezik $n \in \mathbb{N}$ és $x_0, x_1, \ldots, x_n \in X$ úgy, hogy $x_0 = x$, $x_n = y$, $x_i \in R(x_{i-1})$ minden $i = 1, \ldots, n$-re. Jegyezzük még meg, hogy ha X legalább kételemű, akkor az $x \neq y$ feltétel elhagyható. Korábbi tételeink alapján többek között a jólláncolt relátorok következő, részletes jellemzését bizonyítjuk. Ha $\mathcal{R}$ egy relátor az X-en, akkor a következő állítások ekvivalensek:

(1) $\mathcal{R}$ jólláncolt;

(2) $R_A \notin \mathcal{R}^*$ az X minden valódi, nem üres A részhalmazára;

(3) $R \notin \mathcal{R}^*$ minden az X-en tekintett R valódi prerendezésre;

(4) $R \notin \mathcal{R}^*$ minden az X-en tekintett R valódi, nem üres tranzitív relációra;

(5) $\tau_{\mathcal{R}} = \{\emptyset, X\}$; (6) $\bar{\tau}_{\mathcal{R}} = \{\emptyset, X\}$.

Végül megvizsgáljuk az előző részben tárgyalt különböző kifinomítások szerinti jólláncoltságokat. Az $X(\mathcal{R})$ relátor teret **$\Box$-jólláncoltnak** nevezzük, ha az $X(\mathcal{R}^{\Box})$ relátor tér jólláncolt, ahol $\Box$ egy unér operáció. Természetesen, ha $\Box$ és $\Diamond$ is unér operációk úgy, hogy $\Box$ $\Diamond$-dominált, akkor a $\Box$-jólláncoltságból következik a $\Diamond$-jólláncoltság. Jellemezzük a $\Box$-jólláncoltságot a tényleges jólláncoltsághoz hasonlóan. Ha $\Box$ egy $*$-invariáns kifinomítás (az előző részben tárgyaltak mind ilyenek), akkor az $X(\mathcal{R})$ relátor tér pontosan akkor $\Box$-jólláncolt, ha X^2 az egyetlen prerendezés az $\mathcal{R}^{\Box}$-ban. Illetve felhasználva a 4. szakasz elején már említett tételeket, egyrészt mivel $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^*} = \tau_{\mathcal{R}^\#}$, így kapjuk, hogy a tényleges, az egyenletes és a proximális jólláncoltság ekvivalens fogalmak. Másrészt például $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$ miatt egy relátor pontosan, akkor topologikusan jólláncolt, ha nincs nem triviális topologikusan nyílt halmaz, illetve $\tau_{\mathcal{R}^\vartriangle} = \mathcal{E}_{\mathcal{R}} \cup \{0\}$ miatt egy legalább kételemű X halmazon egy $\mathcal{R}$ relátor pontosan akkor paratopologikusan jólláncolt, ha $\mathcal{R} = \{X^2\}$. Amiből könnyen következik, hogy a paratopologikus, és a parainfinitezimális jólláncoltság egymással szintén ekvivalens fogalmak, valamint, hogy az infinitezimális jólláncoltság következik a paratopologikus jólláncoltságból. Továbbá $\tau_{\mathcal{R}^\bullet} = \digamma_{\{\rho_\mathcal{R}\}}$ miatt egy $\mathcal{R}$ relátor pontosan akkor infinitezimálisan jólláncolt, ha a $\{\rho_\mathcal{R}\}$ relátor ténylegesen jólláncolt.

Az *5. szakasz*ban, az előzőhöz hasonlóan az összefüggő relátorokat vizsgáljuk. Ehhez szükségünk van a Davis–Pervin relációk szimmetrizációjára. Ha $A \subset X$, akkor az

$$S_A = R_A \cap R_A^{-1}$$

relációt az R_A **Davis–Pervin reláció szimmetrizációjá**nak nevezzük.

Ezek után, a jólláncoltságra támaszkodva bevezetjük az összefüggőség fogalmát. Az $X(\mathcal{R})$ relátor teret **ténylegesen összefüggő**nek nevezzük, ha az $\mathcal{R} \triangledown \mathcal{R}^{-1}$ relátor jólláncolt. Ez részletezve azt jelenti, hogy minden $R \in \mathcal{R}$, és $x, y \in X$, $x \neq y$ esetén, létezik $n \in \mathbb{N}$ és $x_0, x_1, \ldots, x_n \in X$ úgy, hogy $x_0 = x$, $x_n = y$, $x_i \in R(x_{i-1})$ vagy $x_{i-1} \in R(x)$ minden $i = 1, \ldots, n$-re. Jegyezzük még meg, hogy ha X legalább kételemű, akkor az $x \neq y$ feltétel elhagyható. Korábbi tételeink alapján többek között az összefüggő relátorok következő, részletes jellemzését bizonyítjuk. Ha $\mathcal{R}$ egy relátor az X-en, akkor a következő állítások ekvivalensek:

(1) $\mathcal{R}$ összefüggő;

(2) $S_A \notin \mathcal{R}^*$ az X minden valódi, nem üres A részhalmazára;

(3) $S \notin \mathcal{R}^*$ minden az X-en tekintett S valódi ekvivalenciára;

(4) $S \notin \mathcal{R}^*$ minden az X-en tekintett S valódi, nem üres szimmetrikus és tranzitív relációra;

Ezek után megvizsgáljuk az előző részben tárgyalt különböző kifinomítások szerinti összefüggőségeket. Az $X(\mathcal{R})$ relátor teret **$\Box$-összefüggő**nek nevezzük, ha az $X(\mathcal{R}^\Box)$ relátor tér összefüggő, ahol $\Box$ egy unér operáció. Természetesen, ha $\Box$ és $\Diamond$ is unér operációk úgy, hogy $\Box$ $\Diamond$-dominált, akkor a $\Box$-összefüggőségből következik a $\Diamond$-összefüggőség. Jellemezzük a $\Box$-összefüggőséget a tényleges összefüggőséghez hasonlóan. Ha $\Box$ egy $*$-invariáns kifinomítás (az előző részben tárgyaltak mind ilyenek), akkor az $X(\mathcal{R})$ relátor tér pontosan akkor $\Box$-összefüggő, ha X^2 az egyetlen ekvivalencia az $\mathcal{R}^\Box$-ban. Ezt felhasználva kapjuk, hogy a tényleges összefüggőség és az egyenletes összefüggőség ekvivalens fogalmak.

Azonban a proximális összefüggőség nem ekvivalens a tényleges összefüggőséggel, így csak a következőt tudjuk állítani. Az $X(\mathcal{R})$ relátor tér pontosan akkor proximálisan összefüggő, ha nincs olyan valódi, nem üres részhalmaza az X-nek, ami proximálisan nyílt és proximálisan zárt. Illetve, mivel $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_\mathcal{R}$ és $\digamma_{\mathcal{R}^\wedge} = \mathcal{F}_\mathcal{R}$, egy relátor pontosan akkor topologikusan összefüggő, ha nincs olyan nem triviális topologikusan nyílt halmaz, amely topologikusan zárt. Továbbá, mivel $\tau_{\mathcal{R}^\vartriangle} = \mathcal{E}_\mathcal{R} \cup \{\emptyset\}$ és $\digamma_{\mathcal{R}^\vartriangle} = (\mathcal{P}(X) \setminus \mathcal{D}_\mathcal{R}) \cup \{X\}$, egy relátor pontosan akkor paratopologikusan összefüggő, ha $\mathcal{E}_\mathcal{R} \setminus \{\emptyset\} \subset \mathcal{D}_\mathcal{R} \cup \{X\}$. Így egy legalább kételemű halmazon a paratopologikus összefüggőség pontosan azt jelenti, hogy minden kövér halmaz sűrű, illetve azzal is ekvivalens, hogy bármely két kövér halmaz metszete nem üres.

Ezen kívül $\tau_{\mathcal{R}^\bullet} = \overline{\tau}_{\{\rho_\mathcal{R}\}}$ és $\overline{\tau}_{\mathcal{R}^\bullet} = \tau_{\{\rho_\mathcal{R}\}}$ miatt egy $\mathcal{R}$ relátor pontosan akkor infinitezimálisan összefüggő, ha a $\{\rho_\mathcal{R}\}$ relátor ténylegesen összefüggő. Illetve a $\tau_{\mathcal{R}^\blacktriangle}$-ra és a $\overline{\tau}_{\mathcal{R}^\blacktriangle}$-ra vonatkozó tételek szerint egy legalább kételemű halmazon tekintett $\mathcal{R}$ relátor pontosan akkor parainfinitezimálisan összefüggő, ha $E_\mathcal{R} \neq \emptyset$.

Az összefüggő relátorokat, a jólláncoltakkal ellentétben folytonos relációkkal is jellemezzük, illetve megvizsgáljuk a 3. szakaszban tárgyalt, topologikus, szimmetrikus, tranzitív és filtrált relátorok összefüggőségét is. Az említett tételek közül most csak egyet említünk. Egy topologikus relátor tér pontosan akkor paratopologikusan összefüggő, ha minden nem üres topologikusan nyílt halmaz sűrű.

A *6. szakasz*ban összehasonlítjuk az előzőekben tárgyalt különféle jólláncoltságokat és összefüggőségeket. Megmutatjuk, hogy minden $\square$ unér operáció esetén a $\square$-jólláncoltságból következik a $\square$-összefüggőség. Illetve vizsgáljuk, hogy a fordított implikáció milyen esetekben teljesül.

Végül, ha valamely jólláncoltság vagy összefüggőség általában nem implikálja valamely másikat, akkor ezt szemléltetjük egy ellenpélda segítségével. Igyekszünk lehetőleg olyan példákat konstruálni, melyek egyes speciális esetekben, például prerendezési, tolerancia vagy reflexív relátorok esetében, cáfolják a kérdéses implikációt. Ettől is fontosabb szempont azonban a példák egyszerűsége. Igyekszünk minél kisebb elemszámú halmazon, minél kevesebb relációból álló relátorokon bemutatni, az egyes implikációk hiányát.

A II. rész legvégén, egy ábrán szemléltettük a különféle jólláncoltságok és összefüggőségek közötti kapcsolatokat. (Lásd 79. oldal.)

III. rész: Relátorok egyszerűsége

A disszertáció utolsó részében a relátorok egyszerűségével foglalkozunk. A relációk elmélete és az egyszerű relátorok elmélete közti párhuzam mutatja az egyszerű relátorok vizsgálatának szükségességét. Például a parciálisan rendezett halmazokat egyszerű relátor térként tudjuk tárgyalni. Ez a rész az [51] cikkre épül, melyben idézzük Száz Á. és Deák J. korábbi eredményeit (lásd: [68] és [11]), továbbá ezeket kiegészítjük, és az 1. szakasznak megfelelően, általános kereteket között tárgyaljuk. Továbbá a vizsgált eredményeket példákkal illusztráljuk.

A *7. szakasz*t az egyszerű relátorok definíciójával kezdjük. Az X-en tekintett $\mathcal{R}$ relátort **ténylegesen egyszerű**nek nevezzük, ha egyetlen relációból áll. Illetve, ha még $\square$ egy unér operáció, akkor az $\mathcal{R}$ relátort **$\boldsymbol{\square}$-egyszerű**nek nevezzük, ha $\square$-ekvivalens egy ténylegesen egyszerű relátorral. Továbbá az $\mathcal{R}$ relátort **$\boldsymbol{\square}$-kvázi egyszerű**nek nevezzük, ha $(\square\infty)$-ekvivalens egy ténylegesen egyszerű relátorral. Megjegyezzük, hogy például a topologikus jólláncoltság a kvázi-topologikus egyszerűség speciális esete.

Ha $\mathcal{R}$ egy relátor az X-en és $\square \in \{*, \#, \wedge, \bullet\}$, akkor az $\mathcal{R}$ $\square$-egyszerűsége ekvivalens a $\rho_\mathcal{R}^{-1} \in \mathcal{R}^\square$ tartalmazással. A $\square = \bullet$ eset jól mutatja, hogy minden relátor infinitezimálisan egyszerű, így a $\bullet$-domináns unér operációkkal nem is foglalkozunk.

Ezek után felhasználva a különböző kifinomítások közötti tartalmazásokat, könnyen beláthatjuk a következő implikációláncot: 'tényleges egyszerűség' $\Longrightarrow$ 'egyenletes egyszerűség' $\Longrightarrow$ 'proximális egyszerűség' $\Longrightarrow$ 'topologikus egyszerűség' $\Longrightarrow$ 'paratopologikus egyszerűség'. Továbbá, mivel láttuk, hogy minden relátor infinitezimálisan egyszerű kapjuk, hogy 'paratopologikus egyszerűség' $\Longrightarrow$ 'infinitezimális egyszerűség', illetve, hogy például a parainfinitezimális egyszerűséggel nem érdemes foglalkoznunk.

Foglalkozunk még a $\Box$-kvázi egyszerűséggel is. Kimondjuk például, hogy egy $\mathcal{R}$ relátor pontosan akkor kvázi-ténylegesen egyszerű, ha $\mathcal{R}^{\infty} = \{\rho^{-1}_{\mathcal{R}^{\infty}}\}$, illetve a kvázi-egyenletes és kvázi-proximális egyszerűség jellemzését. Nevezetesen, ha $\Box \in \{*, \#\}$, akkor az $\mathcal{R}$ relátor pontosan akkor $\Box$-kvázi egyszerű, ha $\rho^{-1}_{\mathcal{R}^{\infty}} \in \mathcal{R}^{\Box}$.

A fenti jellemzésekből látható, hogy a kvázi-tényleges illetve a kvázi-egyenletes egyszerűség implikálja rendre a kvázi-egyenletes és a kvázi-proximális egyszerűséget. Ezen kívül megmutatjuk, hogy ha $\Box$ egy unér operáció, akkor a $\Box$-egyszerűségből következik a $\Box$-kvázi egyszerűség. Továbbá, hogy a paratopologikus és a kvázi-paratopologikus egyszerűség ekvivalens fogalmak.

Az előző részhez hasonlóan, ellenpéldákon keresztül mutatjuk meg, hogy a különféle egyszerűségek és kvázi egyszerűségek közötti bizonyos implikációk általában nem igazak. Most is igyekszünk lehetőleg olyan példákat konstruálni, melyek egyes speciális esetekben, például prerendezési, tolerancia vagy reflexív relátorok esetében, cáfolják a kérdéses implikációt. A relátorok paratopologikus egyszerűségének vizsgálata érdekében bevezetünk két operációt $\mathcal{A} \subset \mathcal{P}(X)$ halmazrendszerekre:

$$\mathcal{A}^* = \{B \subset X : \exists A \in \mathcal{A} : A \subset B\}$$

$$\mathcal{A}^{\diamond} = \{B \in \mathcal{A} : A \in \mathcal{A}, A \subset B \implies A = B\}.$$

Ezekkel kapcsolatban kimondunk néhány tételt, melyek közül most csak az utolsót említjük, melyhez szükségünk van a bázis fogalmára. Az $\mathcal{A} \subset \mathcal{P}(X)$ felszálló halmazrendszer $\mathcal{B}$ részhalmazát **bázis**nak nevezzük, ha $\mathcal{A} = \mathcal{B}^*$. Ezek után a már említett tétel a következő. Ha az $\mathcal{A} \subset \mathcal{P}(X)$ egy felszálló halmazrendszerben minden nem üres láncnak van minimális eleme, akkor $\mathcal{A}^{\diamond}$ a legszűkebb bázisa $\mathcal{A}$-nak.

Végül tovább vizsgáljuk a relátorok egyszerűségét, kimondjuk például a következő tételt. Ha az $X(\mathcal{R})$ relátor tér kövér halmazainak minden nem üres láncában van minimális elem, akkor a relátor paratopologikus egyszerűsége ekvivalens a $\operatorname{card}(\mathcal{E}^{\diamond}_{\mathcal{R}}) \leq \operatorname{card}(X)$ egyenlőtlenséggel. (Ha például X véges, akkor ez nyilván teljesül.) Ezt kihasználva konstruálunk példát nem paratopologikusan egyszerű, ekvivalencia relátorra.

A III. rész legvégén, egy ábrán szemléltettük a különféle egyszerűségek és kvázi egyszerűségek közötti kapcsolatokat. (Lásd 93. oldal.)

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[2] On the extensions, refinements and modifications of relators, *Math. Balk.* **15** (2001), 155–186. [MR 2002j:08001]

[3] A unified treatment of well-chainedness and connectedness properties, *Acta Math. Acad. Paedagog. Nyházi. (N.S.)* **19** (2003), 101–166 (electronic). (With Á. Száz.)

Refereed papers on other subjects

[4] Characterizations of nonexpansive multipliers on partially ordered sets, *Math. Slovaca* **51/4** (2001), 371–382 (With Á. Száz.) [MR 2002i:06001] [Zbl. 0991.06001]

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[11] On the convergence of some particular series, *Abstracts of The 17th Summer Conference on Real Functions Theroy* (Stará Lesná, 2002), 56–56.

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WELL-CHAINED, CONNECTED AND SIMPLE RELATORS

Értekezés a doktori (Ph.D.) fokozat megszerzése érdekében
a matematika tudományágban

Írta: Pataki Gergely okleveles matematikus

Készült a Debreceni Egyetem TTK Matematika Doktori Iskolája
(Analízis programja) keretében

Témavezető: Dr. Száz Árpád

A doktori szigorlati bizottság:
elnök: Dr. …………………… …………………
tagok: Dr. …………………… …………………
Dr. …………………… …………………

A doktori szigorlat időpontja: 200… . ……………… … .

Az értekezés bírálói:
Dr. …………………… …………………
Dr. …………………… …………………
Dr. …………………… …………………

A bírálóbizottság:
elnök: Dr. …………………… …………………
tagok: Dr. …………………… …………………
Dr. …………………… …………………
Dr. …………………… …………………
Dr. …………………… …………………

Az értekezés védésének időpontja: 200… . ……………… … .