Simplest quartic and simplest sextic Thue equations over imaginary quadratic fields

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Abstract

The families of simplest cubic, simplest quartic and simplest sextic fields and the related Thue equations are well known, see [18], [17]. The family of simplest cubic Thue equations was already studied in the relative case, over imaginary quadratic fields. In the present paper we give a similar extension of simplest quartic and simplest sextic Thue equations over imaginary quadratic fields. We explicitly give the solutions of these infinite parametric families of Thue equations over arbitrary imaginary quadratic fields.

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1 Introduction

Let t be an integer parameter. The infinite parametric families of number fields generated by the roots of the polynomials

$$\begin{array}{ll} f_t^{(3)}(x) = x^3 - (t-1)x^2 - (t+2)x - 1, & (t \in \mathbb{Z}), \\ f_t^{(4)}(x) = x^4 - tx^3 - 6x^2 + tx + 1, & (t \in \mathbb{Z} \setminus \{-3, 0, 3\}), \\ f_t^{(6)}(x) = x^6 - 2tx^5 - (5t+15)x^4 - 20x^3 + 5tx^2 + (2t+6)x + 1, & (t \in \mathbb{Z} \setminus \{-8, -3, 0, 5\}), \end{array}$$

are called **simplest cubic, simplest quartic and simplest sextic fields**, respectively. They are extensively studied in algebraic number theory, starting with D.Shanks [20], in the cubic case. It was shown by G.Lettl, A.Pethő and P.Voutier [18] that these are all parametric families of number fields which are totally real cyclic with Galois group generated by a mapping of type $x \mapsto \frac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{Z}$.

Let $F(x, y) \in \mathbb{Z}[x, y]$ be an irreducible binary form of degree ≥ 3 and let $0 \neq k \in \mathbb{Z}$. There is an extensive literature of **Thue equations** of type

$$F(x, y) = k \text{ in } x, y \in \mathbb{Z}.$$

In 1909 A.Thue [22] proved that these equations admit only finitely many solutions. In 1967 A.Baker [1] gave effective upper bounds for the solutions. Later on authors constructed numerical methods to reduce the bounds and to explicitly calculate the solutions, see [5] for a summary.

In 1990 E.Thomas [21] considered an **infinite parametric family of Thue equations**, corresponding to the simplest cubic fields. For $t \in \mathbb{Z}$, let

$$F_t^{(3)}(x, y) = x^3 - (t-1)x^2y - (t+2)xy^2 - y^3$$

and consider

$$F_t^{(3)}(x, y) = \pm 1 \text{ in } x, y \in \mathbb{Z}.$$

E.Thomas described the solutions for large enough parameters t, later on the solutions were found for all parameters by M.Mignotte [19]. This was the first infinite parametric family of Thue equations that was completely solved. Instead of single equations the solutions were given for infinitely many equations, for all values of the parameter t. These equations are also called the infinite parametric family of the **simplest cubic Thue equations**.

A couple of other infinite parametric families of Thue equations were completely solved, see [9], [5], among others the parametric family of simplest quartic Thue equations [16], [4] and the parametric family of simplest sextic Thue equations [17], [12].

Let M be an algebraic number field with ring of integers $\mathbb{Z}_M$. Let $F(x, y) \in \mathbb{Z}_M[x, y]$ be an irreducible binary form of degree $n \geq 3$ and let $0 \neq \mu \in \mathbb{Z}_M$. As a generalization of Thue equations consider **relative Thue equations** of type

$$F(x, y) = \mu \text{ in } x, y \in \mathbb{Z}_M.$$

Using Baker's method S.V.Kotov and V.G.Sprindzuk [15] gave the first effective upper bounds for the solutions of relative Thue equations. Their theorem was extended by several authors. Applying Baker's method, reduction and enumeration algorithms I.Gaál and M.Pohst [7] (see also [5]) gave an efficient algorithm for solving relative Thue equations.

Authors considered infinite parametric families of Thue equations in the relative case, as well. Up to now all these families were considered over imaginary quadratic fields. The first of them was the family of simplest cubic Thue equations, [10], [11], [8], [14]. Later on other families of relative Thue equations were also studied, see e.g. [23], [24], [13].

Let t be an integer parameter, let $m \geq 1$ be a square-free positive integer, and set $M = \mathbb{Q}(i\sqrt{m})$ with ring of integers $\mathbb{Z}_M$. In the present paper we consider simplest quartic and simplest sextic Thue equations in the relative case, over M. Let

$$F_t^{(4)}(x,y) = x^4 - tx^3y - 6x^2y^2 + txy^3 + y^4$$

and let

$$F_t^{(6)}(x,y) = x^6 - 2tx^5y - (5t+15)x^4y^2 - 20x^3y^3 + 5tx^2y^4 + (2t+6)xy^5 + y^6.$$

We give all solutions of the **infinite parametric families of simplest quartic and simplest sextic relative Thue equations**. More precisely we give all solutions of the simplest quartic relative Thue inequalities

$$|F_t^{(4)}(x,y)| \leq 1 \text{ in } x,y \in \mathbb{Z}_M$$

and of the simplest sextic relative Thue inequalities

$$|F_t^{(6)}(x,y)| \leq 1 \text{ in } x,y \in \mathbb{Z}_M.$$

2 Results

We formulate now our main results. In both Theorems we exclude the parameters $t \in \mathbb{Z}$ for which the binary form involved is reducible over $\mathbb{Z}$. It is easily seen that over $\mathbb{Z}_M$ these form are reducible exactly for the same parameters $t \in \mathbb{Z}$.

Theorem 1 *Let $t \in \mathbb{Z}$ with $t \neq -3, 0, 3$. All solutions of*

$$|F_t^{(4)}(x,y)| \leq 1 \text{ in } x,y \in \mathbb{Z}_M \tag{1}$$

are up to sign given by the following:
for any m and any t: $(x,y) = (0,0), (0,1), (1,0)$,
for any m and any $t = 1$: $(x,y) = (1,2), (2,-1)$,
for any m and any $t = -1$: $(x,y) = (2,1), (-1,2)$,
for any m and any $t = 4$: $(x,y) = (2,3), (3,-2)$,
for any m and any $t = -4$: $(x,y) = (3,2), (-2,3)$,
for $m = 1$ and any t: $(x,y) = (0,i), (i,0)$,

for $m=3$ *and any* t*:* $(x,y)=(\omega,0),(0,\omega),(1-\omega,0),(0,1-\omega)$,
for $m=1$ *and* $t=1$*:* $(x,y)=(i,2i),(2i,-i)$,
for $m=1$ *and* $t=-1$*:* $(x,y)=(2i,i),(-i,2i)$,
for $m=1$ *and* $t=4$*:* $(x,y)=(2i,3i),(3i,-2i)$,
for $m=1$ *and* $t=-4$*:* $(x,y)=(3i,2i),(-2i,3i)$,
for $m=3$ *and* $t=1$*:* $(x,y)=(2\omega-2,-\omega+1),(\omega-1,2\omega-2),(-2\omega,\omega),(\omega,2\omega)$,
for $m=3$ *and* $t=-1$*:* $(x,y)=(-\omega+1,2\omega-2),(2\omega-2,\omega-1),(\omega,-2\omega),(2\omega,\omega)$,
for $m=3$ *and* $t=4$*:* $(x,y)=(3\omega-3,-2\omega+2),(2\omega-2,3\omega-3),(2\omega,3\omega),(3\omega,-2\omega)$,
for $m=3$ *and* $t=-4$*:* $(x,y)=(-2\omega+2,3\omega-3),(3\omega-3,2\omega-2),(3\omega,2\omega),(-2\omega,3\omega)$,
where $\omega=(1+i\sqrt{3})/2$.

Theorem 2 *Let* $t\in\mathbb{Z}, t\neq -8,-3,0,5$. *All solutions of*

$$|F_t^{(6)}(x,y)|\leq 1 \ \text{ in } \ x,y\in\mathbb{Z}_M \tag{2}$$

are up to sign given by the following:
for any m *and any* t*:* $(x,y)=(0,0),(0,1),(1,0),(1,-1)$,
for $m=1$ *and any* t*:* $(x,y)=(0,i),(i,0),(i,-i)$,
for $m=3$ *and any* t*:* $(x,y)=(\omega,0),(0,\omega),(\omega,-\omega),(1-\omega,0),(0,\omega-1),(\omega-1,-\omega+1)$.

3 An auxiliary result

Let $F(x,y)$ be a binary form of degree $n\geq 3$ with rational integer coefficients. Assume that $f(x)=F(x,1)$ has leading coefficient 1 and distinct real roots $\alpha_1,\ldots,\alpha_n$. Let $0<\varepsilon<1,\ 0<\eta<1$, and $K\geq 1$. Set

$$A=\min_{i\neq j}|\alpha_i-\alpha_j|,\quad B=\min_i\prod_{j\neq i}|\alpha_j-\alpha_i|,$$

$$C=\max\left(\frac{K}{(1-\varepsilon)^{n-1}B},1\right),\ C_1=\max\left(\frac{K^{1/n}}{\varepsilon A},\ (2C)^{1/(n-2)}\right),\ \ C_2=\max\left(\frac{K^{1/n}}{\varepsilon A},\ C^{1/(n-2)}\right),$$

$$D=\left(\frac{K}{\eta(1-\varepsilon)^{n-1}AB}\right)^{1/n},\ E=\frac{(1+\eta)^{n-1}K}{(1-\varepsilon)^{n-1}}.$$

Let $m\geq 1$ be a square-free positive integer, and set $M=\mathbb{Q}(i\sqrt{m})$.
If $m\equiv 3 \pmod 4$, then $x,y\in\mathbb{Z}_M$ can be written as

$$x=x_1+x_2\frac{1+i\sqrt{m}}{2}=\frac{(2x_1+x_2)+x_2i\sqrt{m}}{2},\ y=y_1+y_2\frac{1+i\sqrt{m}}{2}=\frac{(2y_1+y_2)+y_2i\sqrt{m}}{2}$$

with $x_1, x_2, y_1, y_2 \in \mathbb{Z}$.
If $m \equiv 1, 2 \pmod{4}$, then $x, y \in \mathbb{Z}_M$ can be written as

$$x = x_1 + x_2 i\sqrt{m}, \quad y = y_1 + y_2 i\sqrt{m}$$

with $x_1, x_2, y_1, y_2 \in \mathbb{Z}$.

Consider the relative Thue inequality

$$|F(x,y)| \leq K \quad \text{in} \quad x, y \in \mathbb{Z}_M. \tag{3}$$

We shall use the result of [6]:

Lemma 3 *Let $(x,y) \in \mathbb{Z}_M^2$ be solutions of (3). Assume that*

$$\begin{array}{lll} |y| > C_1 & \text{if} & m \equiv 3 \pmod{4}, \\ |y| > C_2 & \text{if} & m \equiv 1, 2 \pmod{4}. \end{array}$$

Then

$$x_2 y_1 = x_1 y_2.$$

I. Let $m \equiv 3 \pmod{4}$.

$$\begin{array}{ll} \text{IA1.} & \text{If } 2y_1 + y_2 = 0, \text{ then } 2x_1 + x_2 = 0 \text{ and } |F(x_2, y_2)| \leq \dfrac{2^n K}{(\sqrt{m})^n}. \\ \text{IA2.} & \text{If } |2y_1 + y_2| \geq 2D, \text{ then } |F(2x_1 + x_2, 2y_1 + y_2)| \leq 2^n E. \\ \text{IB1.} & \text{If } y_2 = 0, \text{ then } x_2 = 0 \text{ and } |F(x_1, y_1)| \leq K. \\ \text{IB2.} & \text{If } |y_2| \geq \dfrac{2}{\sqrt{m}} D, \text{ then } |F(x_2, y_2)| \leq \dfrac{2^n}{(\sqrt{m})^n} E. \end{array}$$

II. Let $m \equiv 1, 2 \pmod{4}$.

$$\begin{array}{ll} \text{IIA1.} & \text{If } y_1 = 0, \text{ then } x_1 = 0 \text{ and } |F(x_2, y_2)| \leq \dfrac{K}{(\sqrt{m})^n}. \\ \text{IIA2.} & \text{If } |y_1| \geq D, \text{ then } |F(x_1, y_1)| \leq E. \\ \text{IIB1.} & \text{If } y_2 = 0, \text{ then } x_2 = 0 \text{ and } |F(x_1, y_1)| \leq K. \\ \text{IIB2.} & \text{If } |y_2| \geq \dfrac{D}{\sqrt{m}}, \text{ then } |F(x_2, y_2)| \leq \dfrac{E}{(\sqrt{m})^n}. \end{array}$$

4 Simplest quartic Thue equations over imaginary quadratic fields

In this section we turn to the proof of Theorem 1. In our proof we shall use Lemma 3 and the corresponding results in the absolute case.

For right hand sides ± 1 J.Chen and P.Voutier [4] gave all solutions of simplest quartic Thue equations.

Lemma 4 *Let $t \in \mathbb{Z}$ with $t \geq 1, t \neq 3$. All solutions of*

$$F_t^{(4)}(x,y) = \pm 1 \ \text{ in } \ x,y \in \mathbb{Z}$$

are given by $(x,y) = (\pm 1, 0), (0, \pm 1)$.
Further, for $t = 1$ we have $(x,y) = (1,2), (-1,-2), (2,-1), (-2,1)$
and for $t = 4$ we have $(x,y) = (2,3), (-2,-3), (3,-2), (-3,2)$.

For larger right hand sides we can use the statement of G.Lettl, A.Pethő and P.Voutier [18].

Lemma 5 *Let $t \in \mathbb{Z}, t \geq 58$ and consider the primitive solutions (i.e. solutions with $(x,y) = 1$) of*

$$|F_t^{(4)}(x,y)| \leq 6t + 7 \ \text{ in } x,y \in \mathbb{Z}. \tag{4}$$

If (x,y) is a solution of (4), then every pair in the orbit

$$\{(x,y), (y,-x), (-x,-y), (-y,x)\}$$

is also a solution. Every orbit has a solution with $y > 0$, $-y \leq x \leq y$. If an orbit contains a primitive solution, then all solutions in this orbit are primitive. All solutions of the above inequality with $y > 0, -y \leq x \leq y$ are $(0,1), (\pm 1, 1), (\pm 1, 2)$.

Remark 1. Since

$$F_t^{(4)}(x,y) = F_{-t}^{(4)}(y,x),$$

it is enough to solve the inequality (1) only for $t > 0$. Also, we have

$$F_t^{(4)}(x,y) = F_t^{(4)}(-x,-y) = F_t^{(4)}(y,-x) = F_t^{(4)}(-y,x).$$

Therefore, if $(x,y) \in \mathbb{Z}_M^2$ is solution, then $(y,-x), (-y,x), (-x,-y)$ are solutions, too.

4.1 Proof of Theorem 1 for $\mathbf{m \neq 1, 3}$

We use the notation of Lemma 3. Using the estimates of [18] for the roots of the polynomial $F_t^{(4)}(x,1)$ we obtain $A > 0.9833$ and $B > 58.1$ for $t \geq 58$. Calculating the roots for $0 < t < 58$ we obtain $A > 0.8284, B > 4.6114$ for any $t > 0, t \neq 3$. Set

$$\varepsilon = 0.1924, \qquad \eta = 0.169.$$

For $t > 0, t \neq 3$ and a square-free m with $m \neq 1, 3$ our Lemma 3 implies:

Corollary 6 *Let $(x,y) \in \mathbb{Z}_M^2$ be solutions of (1). Assume that*

$$|y| > 6.2741.$$

Then

$$x_2 y_1 = x_1 y_2.$$

I. Let $m \equiv 3 \pmod 4$.

$$\begin{array}{ll} \text{IA1.} & \text{If } 2y_1+y_2=0, \text{ then } 2x_1+x_2=0 \text{ and } |F_t^{(4)}(x_2,y_2)| \le 0.326. \\ \text{IA2.} & \text{If } |2y_1+y_2| \ge 2.618, \text{ then } |F_t^{(4)}(2x_1+x_2, 2y_1+y_2)| \le 48.526. \\ \text{IB1.} & \text{If } y_2=0, \text{ then } x_2=0 \text{ and } |F_t^{(4)}(x_1,y_1)| \le 1. \\ \text{IB2.} & \text{If } |y_2| \ge 0.989, \text{ then } |F_t^{(4)}(x_2,y_2)| \le 0.990. \end{array}$$

II. Let $m \equiv 1, 2 \pmod 4$.

$$\begin{array}{ll} \text{IIA1.} & \text{If } y_1=0, \text{ then } x_1=0 \text{ and } |F_t^{(4)}(x_2,y_2)| \le 0.25. \\ \text{IIA2.} & \text{If } |y_1| \ge 1.309, \text{ then } |F_t^{(4)}(x_1,y_1)| \le 3.032. \\ \text{IIB1.} & \text{If } y_2=0, \text{ then } x_2=0 \text{ and } |F_t^{(4)}(x_1,y_1)| \le 1. \\ \text{IIB2.} & \text{If } |y_2| \ge 0.925, \text{ then } |F_t^{(4)}(x_2,y_2)| \le 0.7582. \end{array}$$

Case I. $m \equiv 3 \pmod 4$

a) Assume that $|y| > 6.2741$. By Corollary 6 we have
If $y_2 = 0$, then by IB1 we have $x_2 = 0$ and using Lemma 4 $|F_t^{(4)}(x_1, y_1)| \le 1$ implies $|x_1|, |y_1| \le 3$. This contradicts to $|y| > 6.2741$.

If $y_2 \ne 0$, then IB2 implies $|F_t^{(4)}(x_2, y_2)| \le 0.990$, whence $y_2 = 0$, a contradiction.

Therefore $|y| > 6.2741$ is not possible.

b) Consider now $|y| \le 6.2741$. By Remark 1 if (x, y) is a solution then so also is $(y, -x)$. As above we obtain that $|x| > 6.2741$ is not possible, hence $|x| \le 6.2741$.

We enumerate all x, y with $|x| \le 6.2741$ and $|y| \le 6.2741$ and we obtain the solutions

$$(x, y) = (0, 0), (0, \pm 1), (\pm 1, 0).$$

Additionally we have up to sign
for $t = 1$: $(x, y) = (1, 2), (2, -1)$ and
for $t = 4$: $(x, y) = (2, 3), (3, -2)$.

Case II. $m \equiv 1, 2 \pmod 4$
Similar to Case I, we obtain the same solutions.

According to Remark 1 we proved Theorem 1 for all t with $t \ne -3, 0 - 3$ and for $m \ne 1, 3$.

4.2 Proof of Theorem 1 for m = 1

Set

$$\varepsilon = 0.1792, \qquad \eta = 0.0308.$$

For $t > 0, t \ne 3$ and $m = 1$ Lemma 3 implies:

Corollary 7 *Let $(x, y) \in \mathbb{Z}_M^2$ be a solution of (1). Assume*

$$|y| > 6.736.$$

Then

$$x_2 y_1 = x_1 y_2.$$

Further,

$$\begin{array}{ll} \text{IIA1.} & \text{if } y_1 = 0, \text{ then } x_1 = 0 \text{ and } |F_t^{(4)}(x_2, y_2)| \leq 1, \\ \text{IIA2.} & \text{if } |y_1| \geq 1.98, \text{ then } |F_t^{(4)}(x_1, y_1)| \leq 1.981, \\ \text{IIB1.} & \text{if } y_2 = 0, \text{ then } x_2 = 0 \text{ and } |F_t^{(4)}(x_1, y_1)| \leq 1, \\ \text{IIB2.} & \text{if } |y_2| \geq 1.98, \text{ then } |F_t^{(4)}(x_2, y_2)| \leq 1.981. \end{array}$$

a) Assume $|y| > 6.736$. By the above Corollary we deduce:

If $y_1 = 0$, then by IIA1 we have $|F_t^{(4)}(x_2, y_2)| \leq 1$, whence by Lemma 4 $|y_2| \leq 3$, contradicting $|y| > 6.736$.

If $|y_1| > 3$, then by IIA2 we have $|F_t^{(4)}(x_1, y_1)| \leq 1$, whence by Lemma 4 $|y_1| \leq 3$, a contradiction.

Therefore only $|y_1| = 1, 2, 3$ is possible.

Using IIB1 and IIB2 we similarly obtain that only $|y_2| = 1, 2, 3$ is possible. But $|y_1| = 1, 2, 3$, $|y_2| = 1, 2, 3$ contradicts $|y| > 6.736$.

b) Hence only $|y| \leq 6.736$ is possible. If (x, y) is a solution, then so also is $(y, -x)$ therefore we must also have $|x| \leq 6.736$. Enumerating the set $(x, y) \in \mathbb{Z}_M^2$ with $|x|, |y| \leq 6.736$ we obtain

$$(x, y) = (0, 0), (0, \pm 1), (\pm 1, 0), (0, \pm i), (\pm i, 0).$$

Additionally we have up to sign
for $t = 1$ $(x, y) = (1, 2), (i, 2i), (2, -1), (2i, -i)$ and
for $t = 4$ $(x, y) = (2, 3), (2i, 3i), (3, -2), (3i, -2i)$.

According to Remark 1 we have proved Theorem 1 for all t with $t \neq -3, 0, 3$ and for $m = 1$.

4.3 Proof of Theorem 1 for m = 3

First we assume $t \geq 58$. Then $A > 0.9833$, and $B > 58.1$. Set

$$\varepsilon = 0.6273, \qquad \eta = 0.0361.$$

Corollary 8 *Let $(x, y) e Z_M^2$ be solutions of (1) and let $m = 3$. Assume $t \geq 58$ and*

$$|y| > 1.621.$$

Then

$$x_2y_1 = x_1y_2.$$

Further

$$\begin{array}{ll} \text{IA1.} & \text{if } 2y_1+y_2=0, \text{ then } 2x_1+x_2=0 \text{ and } |F_t^{(4)}(x_2,y_2)| \le 1.778, \\ \text{IA2.} & \text{if } |2y_1+y_2| \ge 3.497, \text{ then } |F_t^{(4)}(2x_1+x_2, 2y_1+y_2)| \le 343.753, \\ \text{IB1.} & \text{if } y_2=0, \text{ then } x_2=0 \text{ and } |F_t^{(4)}(x_1,y_1)| \le 1, \\ \text{IB2.} & \text{if } |y_2| \ge 2.019, \text{ then } |F_t^{(4)}(x_2,y_2)| \le 38.195. \end{array}$$

a)Assume $|y| > 1.621$. Then by the above Corollary:

If $2y_1 + y_2 = 0$, then by IA1 $2x_1 + x_2 = 0$ and $|F_t^{(4)}(x_2, y_2)| \le 1.778$. By Lemma 4 this later inequality implies $(x_2, y_2) = (0, 0), (0, \pm 1), (\pm 1, 0)$. However for $(x_2, y_2) = (0, \pm 1), (\pm 1, 0)$ one of the the equations $2y_1 + y_2 = 0$ and $2x_1 + x_2 = 0$ have no integer solutions in y_1, resp. x_1. If $(x_2, y_2) = (0, 0)$ then $2y_1 + y_2 = 0$ implies $y_1 = 0$, but $(y_1, y_2) = (0, 0)$ contradicts $|y| > 1.621$.

If $|2y_1 + y_2| > 3.497$, then IA2 implies $|F_t^{(4)}(2x_1 + x_2, 2y_1 + y_2)| \le 343.753$. Using Lemma 5 we can easily list all primitive and non-primitive solutions of this inequality and we always have $|2y_1 + y_2| \le 4$.

Therefore only $|2y_1 + y_2| = 1, 2, 3, 4$ is possible.

Using IB1 and IB2 we similarly obtain that only $|y_2| = 1, 2$ is possible. The equations $|2y_1 + y_2| = 1, 2, 3, 4$, $|y_2| = 1, 2$ leave only a few possible values for (y_1, y_2).

b) If $|x| > 1.621$, then we similarly obtain $|2x_1 + x_2| = 1, 2, 3, 4$, $|x_2| = 1, 2$, since if (x, y) is a solution, then so also is $(y, -x)$.

c1) If $|x| > 1.621$ and $|y| > 1.621$ then we test the finite set $|2x_1+x_2| = 1, 2, 3, 4$, $|x_2| = 1, 2$, $|2y_1 + y_2| = 1, 2, 3$, $|y_2| = 1, 2$.

c2) If $|x| > 1.621$ and $|y| \le 1.621$ then we test the finite set $|2x_1+x_2| = 1, 2, 3, 4$, $|x_2| = 1, 2$, $|y| \le 1.621$.

c3) If $|x| \le 1.621$ and $|y| > 1.621$ then we test the finite set $|x| \le 1.621$, $|2y_1+y_2| = 1, 2, 3, 4$, $|y_2| = 1, 2$.

c4) Finally, if $|x| \le 1.621$ and $|y| \le 1.621$ then we test this finite set.

All together up to sign we get the following solutions for arbitrary $t \ge 58$:
$(x, y) = (0, 0), (1, 0), (0, 1), (\omega, 0), (0, \omega), (1 - \omega, 0), (0, 1 - \omega)$.

Let now $0 < t < 58$**.** Considering the roots of the polynomial $F_t^{(4)}(x, 1) = 0$ for these parameters we obtain $A > 0.8284$, $B > 4.6114$. Set

$$\varepsilon = 0.0348, \qquad \eta = 0.0005.$$

Corollary 9 *Let $m = 3$ and $0 < t < 58$. Let $(x, y) \in \mathbb{Z}_M^2$ be a solution of (1) and assume*

$$|y| > 34.688.$$

Then

$$x_2y_1 = x_1y_2.$$

Further

$$\begin{array}{ll} \text{IA1.} & \text{if } 2y_1+y_2=0, \text{ then } 2x_1+x_2=0 \text{ and } |F_t^{(4)}(x_2,y_2)| \le 1.778, \\ \text{IA2.} & \text{if } |2y_1+y_2| \ge 9.824, \text{ then } |F_t^{(4)}(2x_1+x_2, 2y_1+y_2)| \le 17.825, \\ \text{IB1.} & \text{if } y_2=0, \text{ then } x_2=0 \text{ and} |F_t^{(4)}(x_1,y_1)| \le 1, \\ \text{IB2.} & \text{if } |y_2| \ge 5.672, \text{ then } |F_t^{(4)}(x_2,y_2)| \le 1.981. \end{array}$$

a) Assume $|y| > 34.688$. Then by the above Corollary we have:

If $y_2 = 0$, then by IB1 $|F_t^{(4)}(x_1, y_1)| \le 1$. By Lemma 4 we know the possible solutions y_1. These, together with $y_2 = 0$ contradict $|y| > 34.688$.

If $|y_2| \ge 5.672$, then by IB2 $|F_t^{(4)}(x_2, y_2)| \le 1$, whence by Lemma 4 $|y_2| \le 3$, a contradiction.

Therefore only $|y_2| = 1, 2, 3, 4, 5$ is possible.

If $2y_1 + y_2 = 0$, then by IA1 we have $2x_1 + x_2 = 0$ and $|F_t^{(4)}(x_2, y_2)| \le 1$. From Lemma 4 we get the possible values of y_2 and we calculate y_1 from $2y_1 + y_2 = 0$. These are in contradiction with $|y| > 34.688$.

If $|2y_1 + y_2| \ge 9.824$ then by IA2 we have $|F_t^{(4)}(2x_1 + x_2, 2y_1 + y_2)| \le 17$. Using Magma [2] we solve the equation $F_t^{(4)}(2x_1 + x_2, 2y_1 + y_2) = d$ for all $t \le 58$ and $|d| \le 17$ and list the solutions. All these solutions contradict $|2y_1 + y_2| > 9.824$.

Therefore only $|2y_1 + y_2| = 1, \ldots, 9$ is possible.

In the set $|y_2| = 1, 2, 3, 4, 5$, $|2y_1 + y_2| = 1, \ldots, 9$ all $y = y_1 + \omega y_2$ have absolute values less than 34.688 which is in contradiction with $|y| > 34.688$.

b) Therefore only $|y| \le 34.688$ is possible. If (x, y) is a solution, then so also is $(y, -x)$ therefore we similarly obtain $|x| \le 34.688$. Enumerating all x, y with these properties we obtain up to sign the following solutions:

for arbitrary t: $(1, 0), (0, 1), (\omega, 0), (0, \omega), (1 - \omega, 0), (0, 1 - \omega)$,

for $t = 1$: $(1, 2), (2, -1), (2\omega - 2, -\omega + 1), (\omega - 1, 2\omega - 2), (-2\omega, \omega), (\omega, 2\omega)$,

for $t = 4$: $(2, 3), (3, -2), (3\omega - 3, -2\omega + 2), (2\omega - 2, 3\omega - 3), (2\omega, 3\omega), (3\omega, -2\omega)$.

According to Remark 1 we have proved Theorem 1 for all t with $t \ne -3, 0, 3$ and $m = 3$.

5 Simplest sextic Thue equations over imaginary quadratic fields

In this section we turn to the proof of Theorem 2. In our proof we shall use Lemma 3 and the corresponding results in the absolute case.

G.Lettl, A.Pethő, and P.Voutier [17] and A.Hoshi [12] gave all solutions in rational integers of the equation $F_t^{(6)}(x, y) = \pm 1$ for all parameters.

Lemma 10 *Let $t \in \mathbb{Z}, t \neq -8, -3, 0, 5$. All solutions of*

$$F_t^{(6)}(x, y) = \pm 1 \text{ in } x, y \in \mathbb{Z}$$

are given by

$$(x, y) = (\pm 1, 0), (0, \pm 1), (1, -1), (-1, 1).$$

For larger right hand sides we shall use the statement of G.Lettl, A.Pethő and P.Voutier [18].

Lemma 11 *Let $t \in \mathbb{Z}, t \geq 89$ and consider the primitive solutions of*

$$|F_t^{(6)}(x, y)| \leq 120t + 323 \text{ in } x, y \in \mathbb{Z}. \tag{5}$$

If (x, y) is a solution of the above inequality, then every pair in the orbit

$$\{(x, y), (-y, x + y), (-x - y, x), (-x, -y), (y, -x - y), (x + y, -x)\}$$

is also a solution. Every orbit has a solution with $y > 0$, $-y/2 < x \leq y$. If an orbit contains one primitive solution, then all solutions in this orbit are primitive. All solutions of the above inequality with $y > 0, -y/2 < x \leq y$ are $(0, 1), (1, 1), (1, 2), (-1, 3)$.

Remark 2. Since

$$F_t^{(6)}(x, y) = F_{-t-3}^{(6)}(y, x),$$

it is enough to solve the inequality (2) only for $t \geq -1, t \neq 0, 5$. Also, we have

$$F_t^{(6)}(x, y) = F_t^{(6)}(-y, x+y) = F_t^{(6)}(-x-y, x) = F_t^{(6)}(-x, -y) = F_t^{(6)}(y, -x-y) = F_t^{(6)}(x+y, -x).$$

Therefore, if $(x, y) \in \mathbb{Z}_M^2$ is solution then, $(-y, x + y), (-x - y, x), (-x, -y), (y, -x - y), (x + y, -x)$ are solutions, as well.

5.1 Proof of Thereom 2 for $m \neq 1, 3$

Using the estimates of [18] for $t \geq 89$, we obtain $A > 0.4986$ and $B > 101.83$. Calculating the roots of $F_t^{(6)}(x, 1)$ for $-1 \leq t < 89, t \neq 0, 5$ finally we get $A > 0.4646$, $B > 3.3121$. Set

$$\varepsilon = 0.12, \qquad \eta = 0.23.$$

Our Lemma 3 implies for $m \neq 1, 3$:

Corollary 12 *Let $t \geq -1, t \neq 0, 5$ and let $(x, y) \in \mathbb{Z}_M^2$ be solutions of (2). Assume that*

$$|y| > 17.937.$$

Then

$$x_2 y_1 = x_1 y_2.$$

I. Let $m \equiv 3 \pmod 4$.

$$\begin{array}{ll} \text{IA1.} & \text{If } 2y_1+y_2=0, \text{ then } 2x_1+x_2=0 \text{ and } |F_t^{(6)}(x_2,y_2)| \le 0.1866. \\ \text{IA2.} & \text{If } |2y_1+y_2| \ge 2.6453, \text{ then } |F_t^{(6)}(2x_1+x_2, 2y_1+y_2)| \le 341.42. \\ \text{IB1.} & \text{If } y_2=0, \text{ then } x_2=0 \text{ and} |F_t^{(6)}(x_1,y_1)| \le 1. \\ \text{IB2.} & \text{If } |y_2| \ge 0.99983, \text{ then } |F_t^{(6)}(x_2,y_2)| \le 0.9954. \end{array}$$

II. Let $m \equiv 1,2 \pmod 4$.

$$\begin{array}{ll} \text{IIA1.} & \text{If } y_1=0, \text{ then } x_1=0 \text{ and } |F_t^{(6)}(x_2,y_2)| \le 0.125. \\ \text{IIA2.} & \text{If } |y_1| \ge 1.3227, \text{ then } |F_t^{(6)}(x_1,y_1)| \le 5.3347. \\ \text{IIB1.} & \text{If } y_2=0, \text{ then } x_2=0 \text{ and } |F_t^{(6)}(x_1,y_1)| \le 1. \\ \text{IIB2.} & \text{If } |y_2| \ge 0.93526, \text{ then } |F_t^{(6)}(x_2,y_2)| \le 0.66684. \end{array}$$

Case I. $m \equiv 3 \pmod 4$
a) Assume that $|y| > 17.937$. Then by Corollary 12 we have:
If $y_2 = 0$, then by IB1 we have $|F_t^{(6)}(x_1,y_1)| \le 1$. By Lemma 10 the possible values of y_1 and $y_2 = 0$ contradict $|y| > 17.937$.
If $|y_2| \ge 1$ then by IB2 we have $|F_t^{(6)}(x_2,y_2)| = 0$, whence $y_2 = 0$, a contradiction.
b) Therefore only $|y| \le 17.937$ is possible. If (x,y) is a solution, then by Lemma 11 $(-x-y,x)$ is also a solution. Hence we also must have $|x| \le 17.937$.
Case II. $m \equiv 1,2 \pmod 4$
We similarly obtain that only $|x| \le 17.9365$, $|y| \le 17.9365$ is possible.

In both cases we enumerate all possible solutions with $|x| \le 17.9365$ and $|y| \le 17.9365$ and finally up to sign obtain

$$(x,y) = (0,0),(0,1),(1,0),(1,-1)$$

According to Remark 2 these are solutions for $m \ne 1,3$ for all t with $t \ne -8,-3,0,5$.

5.2 Proof of Thereom 2 for m = 1

We set

$$\varepsilon = 0.11, \qquad \eta = 0.02.$$

For $t > -1, t \ne 0,5$ and $m = 1$ Lemma 3 implies:

Corollary 13 *Let* $(x,y) \in \mathbb{Z}_M^2$ *be a solution (2). Assume*

$$|y| > 19.5671.$$

Then

$$x_2 y_1 = x_1 y_2.$$

Further,

$$\begin{array}{ll}
\text{IIA1.} & \text{if } y_1=0, \text{ then } x_1=0 \text{ and } |F_t^{(6)}(x_2,y_2)|\le 1,\\
\text{IIA2.} & \text{if } |y_1|\ge 1.9685, \text{ then } |F_t^{(6)}(x_1,y_1)|\le 1.9772,\\
\text{IIB1.} & \text{if } y_2=0, \text{ then } x_2=0 \text{ and } |F_t^{(6)}(x_1,y_1)|\le 1,\\
\text{IIB2.} & \text{if } |y_2|\ge 1.9865, \text{ then } |F_t^{(6)}(x_2,y_2)|\le 1.9772.
\end{array}$$

a) Assume $|y|>19.5671$.

If $y_2=0$, then by IIB1 of the above Corollary we have $|F_t^{(6)}(x_1,y_1)|\le 1$. By Lemma 10 the possible values of y_1 and $y_2=0$ contradict $|y|>19.5671$.

If $|y_2|\ge 2$, then by IIB2 $|F_t^{(6)}(x_2,y_2)|\le 1$ which implies by Lemma 10 $|y_2|\le 1$, a contradiction.

Therefore only $|y_2|=1$ is possible.

IIA1 and IIA2 similarly implies that only $|y_1|=1$ is possible. But $|y_1|=1$, $|y_2|=1$ contradict $|y|>19.5671$.

b) Therefore only $|y|\le 19.5671$ is possible. If (x,y) is a solution, then by Lemma 11 so also is $(-x-y,x)$, hence we also have $|x|\le 19.5671$. Enumerating all possible values of x,y with $|x|\le 19.5671$, $|y|\le 19.5671$ up to sign we get the solutions

$$(x,y)=(0,0),(0,1),(1,0),(1,-1),(0,i),(i,0),(i,-i).$$

According to Remark 2 these are all solutions for $m=1$ and all t with $t\neq -8,-3,0,5$.

5.3 Proof of Theorem 2 for m = 3

Assume $t\ge 89$. Then we have $A>0.4986$ and $B>101.83$. Set

$$\varepsilon=0.41, \qquad \eta=0.02\ .$$

Corollary 14 *Let $(x,y)\in\mathbb{Z}_M^2$ be a solution of (2). Assume that*

$$|y|>4.8917.$$

Then

$$x_2y_1=x_1y_2.$$

Further,

$$\begin{array}{ll}
\text{IA1.} & \text{if } 2y_1+y_2=0, \text{ then } 2x_1+x_2=0 \text{ and } |F_t^{(6)}(x_2,y_2)|\le 2.3703,\\
\text{IA2.} & \text{if } |2y_1+y_2|\ge 3.0965, \text{ then } |F_t^{(6)}(2x_1+x_2,2y_1+y_2)|\le 988.372,\\
\text{IB1.} & \text{if } y_2=0, \text{ then } x_2=0 \text{ and} |F_t^{(6)}(x_1,y_1)|\le 1,\\
\text{IB2.} & \text{if } |y_2|\ge 1.7877, \text{ then } |F_t^{(6)}(x_2,y_2)|\le 36.606.
\end{array}$$

a) Assume $|y| > 4.8917$.
If $2y_1 + y_2 = 0$, then by IA1 $2x_1 + x_2 = 0$ and $|F_t^{(6)}(x_2, y_2)| \leq 2$. By Lemma 11 we have the solutions of this inequality. Only $x = 0, y = 0$ is possible, contradicting $|y| > 4.8917$.
If $|2y_1 + y_2| \geq 4$ then by IA2 $|F_t^{(6)}(2x_1 + x_2, 2y_1 + y_2)| \leq 988.372$. Considering the possible primitive and non-primitive solutions of this inequality, Lemma 11 implies $|2y_1 + y_2| \leq 3$.
Therefore only $|2y_1 + y_2| = 1, 2, 3$ is possible.
If $y_2 = 0$, then by IB1 $x_2 = 0$. The possible values of x_1, y_1 we obtain from $|F_t^{(6)}(x_1, y_1)| \leq 1$ by Lemma 10. These contradict $|y| > 4.8917$.
If $|y_2| \geq 2$, then by IB2 $|F_t^{(6)}(x_2, y_2)| \leq 36$. Using Lemma 11 we consider the primitive and non-primitive solutions of this inequality and we obtain $|y_2| \leq 2$.
Therefore only $|y_2| = 1, 2$ is possible.

b) If $|x| > 4.8917$ then we similarly obtain $|2x_1 + x_2| = 1, 2, 3$, $|x_2| = 1, 2$, since if (x, y) is a solution, then by Remark 2 so also is $(-x - y, x)$.

c1) If $|x| > 4.8917$, $|y| > 4.8917$ then we test the finite set $|2x_1 + x_2| = 1, 2, 3$, $|x_2| = 1, 2$, $|2y_1 + y_2| = 1, 2, 3$, $|y_2| = 1, 2$.

c2) If $|x| > 4.8917$, $|y| \leq 4.8917$ then we test the finite set $|2x_1 + x_2| = 1, 2, 3$, $|x_2| = 1, 2$, $|y| \leq 4.8917$.

c3) If $|x| \leq 4.8917$, $|y| > 4.8917$ then we test the finite set $|x| \leq 4.8917$, $|2y_1 + y_2| = 1, 2, 3$, $|y_2| = 1, 2$.

c4) If $|x| \leq 4.8917$, $|y| \leq 4.8917$ then we test this finite set.

Finally, for $t \geq 89$, $m = 3$, all solutions of $|F_t^{(6)}(x, y)| \leq 1$ up to sign are
$(x, y) = (0, 0), (1, 0), (0, 1), (1, -1), (\omega, 0), (0, \omega), (\omega, -\omega), (1 - \omega, 0), (0, \omega - 1), (\omega - 1, -\omega + 1)$.
According to Remark 2 these are valid for all values of t, for $t \leq -92$, as well.

Assume now $-1 \leq t < 89$. Then we have $A > 0.4646$ and $B > 3.3121$. Set

$$\varepsilon = 0.1124, \qquad \eta = 0.0195 .$$

Corollary 15 *Assume that*

$$|y| > 19.149 .$$

Then

$$x_2 y_1 = x_1 y_2.$$

Further,

$$\begin{array}{ll}
\text{IA1.} & \text{if } 2y_1 + y_2 = 0, \text{ then } 2x_1 + x_2 = 0 \text{ and } |F_t^{(6)}(x_2, y_2)| \leq 2.371, \\
\text{IA2.} & \text{if } |2y_1 + y_2| \geq 3.962, \text{ then } |F_t^{(6)}(2x_1 + x_2, 2y_1 + y_2)| \leq 127.946, \\
\text{IB1.} & \text{if } y_2 = 0, \text{ then } x_2 = 0 \text{ and } |F_t^{(6)}(x_1, y_1)| \leq 1, \\
\text{IB2.} & \text{if } |y_2| \geq 2.287, \text{ then } |F_t^{(6)}(x_2, y_2)| \leq 4.739.
\end{array}$$

a) Assume $|y| > 19.149$.

If $y_2 = 0$, then by IB1 $|F_6(x_1, y_1)| \leq 1$. We consider its solutions by Lemma 10 and find that the possible y_1, y_2 are in contradiction with $|y| > 19.149$.

If $|y_2| \geq 3$, then by IB2 $|F_6(x_2, y_2)| \leq 4$. Using Magma [2] we solve $F_t^{(6)}(x_2, y_2) = d$ for $-1 \leq t < 89$ and $|d| \leq 4$ and find that only $y_2 = 0, \pm 1$ are possible, contradicting $|y_2| \geq 3$.

Hence only $|y_2| = 1, 2$ is possible.

b) If $|x| > 19.149$ then similarly we obtain $|x_2| = 1, 2$, since if (x, y) is a solution then so also is $(-x - y, x)$.

c1) If $|x| > 19.149$ and $|y| > 19.149$, then we have $|x_2| = 1, 2$ and $|y_2| = 1, 2$. For any possible pair x_2, y_2 we parametrize x_1, y_1 with a single parameter, say z, using $x_2 y_1 = x_1 y_2$ (e.g. if $x_2 = 1, y_2 = 2$, then $x_1 = z, y_1 = 2z$). For $-1 \leq t < 89$ and for the possible right hand sides we substitute x_1, x_2, y_1, y_2 into our original equation (2) to determine the parameter z. We do not obtain any solutions this way.

c2) If $|x| \leq 19.149$ and $|y| > 19.149$, then $|y_2| = 1, 2$ and we can enumerate all possible x_1, x_2. For all $-1 \leq t < 89$ we determine y_1 from our original inequality (2) using x_1, x_2 and $|y_2| = 1, 2$. We do not find any solutions.

c3) If $|x| > 19.149$ and $|y| \leq 19.149$, then we proceed similarly.

c4) If $|x| \leq 19.149$ and $|y| \leq 19.149$, then we test this finite set.

Finally, for $-1 \leq t < 89$, $m = 3$, all solutions of $|F_t^{(6)}(x, y)| \leq 1$ up to sign are
$(x, y) = (0, 0), (1, 0), (0, 1), (1, -1), (\omega, 0), (0, \omega), (\omega, -\omega), (1 - \omega, 0), (0, \omega - 1), (\omega - 1, -\omega + 1)$.
These are valid for all values of t, for $-92 < t \leq -2$, as well. Therefore we have proved Theorem 2 for $m = 3$.

6 Computational aspects

All auxiliary calculations were made by using Maple [3]. Testing a great number of possible solutions took a few hours.

The resolution of Thue equations was performed by using Magma [2]. In the quartic case we solved $F_t^{(4)}(2x_1 + x_2, 2y_1 + y_2) = d$ for all $t \leq 58$ and $|d| \leq 17$. This took a few minutes. In the sextic case we solved $F_t^{(6)}(x_2, y_2) = d$ for $-1 \leq t < 89$ and $|d| \leq 4$. This took about 30 minutes.

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