On the frequency variogram and on frequency domain methods for the analysis of spatio-temporal data

T. Subba Rao
University of Manchester,U.K. and
C. R. Rao AIMSCS, University of Hyderabad Campus, India

Gy. Terdik
University of Debrecen, Hungary

Abstract

The covariance function and the variogram play very important roles in modelling and in prediction of spatial and spatio-temporal data. The assumption of second order stationarity, in space and time, is often made in the analysis of spatial data and the spatio-temporal data. Several times the assumption of stationarity is considered to be very restrictive in many real situations, and, therefore, a weaker assumption that the data is Intrinsically stationary both in space and time is often made and used, mainly by the geo-statisticians and other environmental scientists. Because of the inclusion of time dimension, estimation and the derivation of sampling properties of various estimators related to spatio-temporal data gets complicated. In this paper our object is to present an alternative way, based on Frequency Domain methods, to study such problems. Here we consider the Discrete Fourier Transforms (DFT) defined for the time series data observed at several locations as our data, and then consider the estimation of the parameters of spatio-temporal covariance function, estimation of Frequency Variogram, tests of independence etc based on the properties of DFT's. We use the well known property that the Discrete Fourier Transforms of stationary time series evaluated over Fourier Frequencies are asymptotically independent and distributed as complex normal in deriving many results considered in this paper. Our object here is to emphasize the usefulness of the Discrete Fourier transforms in the analysis of spatio-temporal data. Under the intrinsic stationarity condition we consider the estimation, discuss the sampling properties of the Frequency Variogram(FV) introduced in an earlier paper by Subba Rao et al. [2014] which was proposed as an alternative to the classical space, time variogram. We show that the FV introduced is a frequency decomposition of the space-time variogram, and can be computed using the Fast Fourier Transform algorithms. Assuming that the DFT's of the intrinsically stationary processes satisfy a Laplacian type of model, an analytic expression for the space-time spectral density function is derived for the intrinsic processes and also an expression for the Frequency Variogram in terms of the spectral density function is also derived.The estimation of the parameters of the spectrum is also considered. A statistical test for spatial independence of spatio-temporal data is also briefly mentioned, and is based on the test proposed earlier by Wahba [1971] for testing independence in multivariate stationary (temporally) time series.

Keywords. Intrinsic stationarity, spatio-temporal random Processes, Frequency Variogram, Laplacian Model, Test for spatial Independence.

Dedication. Professor M. B. Priestley has made many significant contributions to the nonparametric estimation of stationary and nonstationary spectral density functions. He was one of the strong believers and advocated the use of Fourier Transforms, Frequency domain methods in the analysis of time series. This paper is based on the Fourier Transforms and their possible application to spatio-temporal data and is written bearing in mind Professor Priestley's many important contributions in this area. We dedicate this paper to him.

1 Introduction and Summary

Spatio temporal data arises in many areas such as agriculture, geology, environmental sciences, finance, etc. Since the data comes from these areas are functions of both time and space, any statistical method developed must take into account both spatial dependence, temporal dependence and any interaction between these two. In the case of spatial data, the second order spatial dependence is measured by the second order covariance function which is a function of spatial lag only and in the case of spatio-temporal data the dependence is measured by space-time covariance function. These functions are usually estimated under the assumption that the random process is spatially,temporally stationary.

An alternative second order dependence measure is the variogram defined for both spatial processes and spatio-temporal processes. This function is well defined under weaker assumptions such as intrinsic stationarity and it is widely used by geo-statisticians. Its use is strongly advocated by Cressie [1993], Gringarten and Deutsch [2001] and Sherman [2011].

If the process is second order stationary, then there is a one to one correspondence between the variogram and the covariance function. The estimation of the spatial covariance, spatial variogram have been considered by several authors Cressie [1993], Yu et al. [2007], Stein [2012], Gneiting et al. [2001], Huang et al. [2011], Gringarten and Deutsch [2001], Ma [2005] and their sampling properties have been investigated too, and the literature is not very extensive in the case of spatio-temporal random processes. The inclusion of temporal dimension complicates the estimation of spatio- temporal variogram. The estimation and sampling properties of the spatio- temporal covariance function have been considered by Li et al. [2007], Cressie and Huang [1999], Stein [2005].

In this paper our objective is to consider the Discrete Fourier Transforms (DFT) of the time series observed at various locations evaluated at Fourier frequencies and use these complex valued random variables, which are asymptotically independent over Fourier frequencies, as our data. Using the DFT's we model the data. Subba Rao et al. [2014] and Subba Rao and Terdik [2015] have defined the Frequency variogram and considered the estimation of the parameters of spatio-temporal covariance function derived from a complex stochastic partial differential equation.

We show the relationship between spatio temporal variogram and the frequency variogram defined earlier,and also consider the estimation of the frequency variogram. Its sampling properties are discussed. Investigation of the properties of the sample Frequency Variogram is easier in Frequency domain rather than in time domain. The estimation of the variogram spectrum and the sampling properties of the estimator are also considered here. We believe many interesting problems associated with spatio-temporal random processes can be solved using the frequency domain methods. We consider here some of these problems.

Briefly we summarize the contents. In Section 2, the space time covariance function and space time variogram are introduced, and their estimation, under the assumption of stationarity, is considered in Section 3.The properties of Discrete Fourier Transforms of stationary spatial processes, spectral representation of the processes are considered in Section 4. The Frequency Variogram and its relationship to the classical spatio-temporal variogram, and nonparametric estimation of the Frequency variogram are considered in Sections 5 and 6. Assuming the process is intrinsically stationary, and the process is satisfying a Laplacian model, an analytic expression for the spectral density is obtained in Section 7. The estimation of the parameters of the spectral density function of the Intrinsic processes is considered in Section 8. The frequency variogram and its relation to the spectral density function is also considered in Section 8. A test for spatial independence, based on the properties of Complex Wishart distribution, is described in Section 9 and the test is based on the test by Wahba [1971].

2 Space-time Covariance function and the Variogram

Let $\{Y_t(\mathbf{s}), \mathbf{s} \in \mathbb{R}^d, t \in \mathbb{Z}\}$ denote the spatio-temporal random process. Two assumptions are often made which are important for modeling and prediction.

They are that the process is second order stationary in space and time and also that the process is isotropic in space. The assumptions of stationarity and isotropy can be sometimes unrealistic. In view of this, another weaker assumption that is often made is that the process is intrinsically stationary. We note that if the process is second order stationary, then it implies that the process is intrinsically stationary. But the converse is not true. We say the process $\{Y_t(\mathbf{s})\}$ is spatially, temporally second order stationary if

$$\begin{aligned} E[Y_t(\mathbf{s})] &= \mu, \\ Var[Y_t(\mathbf{s})] &= c(0,0) = {\sigma_y}^2 < \infty, \\ Cov\left[(Y_t(\mathbf{s}), Y_{t+u}(\mathbf{s}+\mathbf{h})\right] &= c(\mathbf{h}, u), \mathbf{h} \in \mathbb{R}^d, u \in \mathbb{Z}. \end{aligned}$$

We note $c(\mathbf{h}, 0)$ and $c(0, u)$ correspond to the purely spatial, purely temporal covariances respectively. Without loss of any generality we assume that $\mu = 0$.

The random process is said to be isotropic if

$$c(\mathbf{h}, u) = c\left(\|\mathbf{h}\|; u\right), \mathbf{h} \in \mathbb{R}^d, u \in \mathbb{Z},$$

where $\|\mathbf{h}\|$ is the Euclidean distance. The process $\{Y_t(\mathbf{s})\}$ is intrinsically spatially, temporarily stationary if the incremental process, for a specific $\mathbf{h}$, $((Y_t(\mathbf{s}) - Y_{t+u}(\mathbf{s}+\mathbf{h}))$ satisfies the following conditions:

$$\begin{aligned} E\left[(Y_t(\mathbf{s}) - Y_{t+u}(\mathbf{s}+\mathbf{h})\right] &= \nu, \\ Var\left[(Y_t(\mathbf{s}) - Y_{t+u}(\mathbf{s}+\mathbf{h})\right] &= \gamma(\mathbf{h}, u) < \infty, \text{and} \\ \gamma(\mathbf{h}, u) &= \gamma(-\mathbf{h}, -u). \end{aligned}$$

If it is isotropic, then

$$\gamma(\mathbf{h}, u) = \gamma(\|\mathbf{h}\|, u),$$

where $\gamma(\mathbf{h}, u)$ is also known as the structure function Yaglom [1987]. Without loss of generality we assume $\nu = 0$.

The spatio-temporal variogram is defined as

$$\gamma(\mathbf{h}, u) = 2\tilde{\gamma}(\mathbf{h}, u) = Var\left[(Y_t(\mathbf{s}) - Y_{t+u}(\mathbf{s}+\mathbf{h})\right],$$

and $\tilde{\gamma}(u, \mathbf{h})$ is defined as the semi spatio-temporal variogram. We note that one can define the variogram under the weaker assumption of intrinsic stationarity. In other words we do not need the assumption of stationarity of the original processes. This phenomenon of differencing to achieve stationarity is similar to what we have in the case of processes with stationary increments for instance the Brownian motion.

Suppose the process is spatially and temporally stationary, then we can show

$$\begin{aligned}\gamma(\mathbf{h},u) &= 2\left[Var(Y_t(\mathbf{s})) - Cov\left(Y_t(\mathbf{s}), Y_{t+u}(\mathbf{s}+\mathbf{h})\right)\right] \\ &= 2\left[c(0,0) - c(\mathbf{h},u)\right] = 2\tilde{\gamma}(\mathbf{h},u),\end{aligned}$$

and we note that there is a one to one correspondence between $\gamma(\mathbf{h},u)$and $c(\mathbf{h},u)$ in the case of stationary processes. One can show that the covariance function $c(\mathbf{h},u)$ is positive semi-definite and $\gamma(\mathbf{h},u)$ is conditionally negative definite.

3 Estimation of $c(\mathbf{h},u)$ and $\gamma\left(\mathbf{h},u\right)$

Let $\{Y_t(\mathbf{s}_i);\ i=1,2,....,m;\ t=1,2,....,n\}$ be a sample from the zero mean, stationary spatio- temporal random process $Y_t(\mathbf{s})$. We define the estimates of $c(\mathbf{h},u)$ and $\gamma(\mathbf{h},u)$ as follows:

$$\hat{c}(\mathbf{h},u) = \frac{1}{|N(\mathbf{h},u)|}\sum_{N(\mathbf{h},u)}[Y_{t_i}(\mathbf{s}_i) - \overline{Y}(\mathbf{s}_i)][Y_{t_j}(\mathbf{s}_j) - \overline{Y}(\mathbf{s}_j)],$$

where

$$\overline{Y}(\mathbf{s}_i) = \frac{1}{n}\sum_{t=1}^{n} Y_t(\mathbf{s}_i),$$

and

$$\hat{\gamma}(\mathbf{h},u) = \frac{1}{|N(\mathbf{h},u)|}\sum_{N(\mathbf{h},u)}\left[Y_{t_i}(s_i) - [Y_{t_j}(\mathbf{s}_j)\right]^2,$$

where $N(\mathbf{h},u) = \{(\mathbf{s}_i,t_i),(\mathbf{s}_j,t_j); \mathbf{s_i} - \mathbf{s_j} = \mathbf{h} \quad \text{and} \quad t_i - t_j = u\}$.

Under certain conditions, Li et al. [2007] have shown that the sample spatio-temporal covariance function defined above is asymptotically normal.

Based on $\hat{\gamma}(\mathbf{h},u)$, Cressie [1993] and Huang et al. [2011] have proposed a weighted least squares criterion for estimating the parameters of the theoretical variogram $\gamma(\mathbf{h},u|\theta)$, and Gneiting [2002] proposed a similar criterion for estimating the parameters based on the space- time covariance function $\hat{c}(\mathbf{h},u)$. Subba Rao et al. [2014] have proposed a frequency domain method for the estimation of the parameters which is robust against departures from Gaussianity and computationally efficient. The method of estimation is similar to Whittle likelihood approach and is based on the frequency variogram which is an analogue of $\gamma(\mathbf{h},u)$, and the proposed criterion which is easy to compute and is based on discrete Fourier transforms. In the following sections we define the frequency variogram and derive the sampling properties of the estimator.

4 Discrete Fourier transforms and spectral representation of the process $\{Y_t(\mathbf{s})\}$

We follow the notation, and use the results obtained in the paper of Subba Rao and Terdik [2015]. Here we briefly highlight and summarize the results we need for our present purposes and for further details we refer to Subba Rao and Terdik [2015]and the books and papers cited in those papers.

We assume the random process $\{Y_t(\mathbf{s})\}$ is second order spatially and temporally stationary. Therefore, the process has the spectral representation given by

$$Y_t(\mathbf{s}) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} e^{i(\mathbf{s}\cdot\boldsymbol{\lambda}+t\omega)} dZ_y(\boldsymbol{\lambda}, \omega),$$

where $\mathbf{s}\cdot\boldsymbol{\lambda} = \sum_{i=1}^{d} s_i\lambda_i$ and $\int_{-\infty}^{\infty}$, represents a d-fold multiple integral, and $Z_y(\boldsymbol{\lambda}, \omega)$ is a zero mean complex valued random process with orthogonal increments and

$$E[dZ_y(\boldsymbol{\lambda}, \omega)] = 0,$$
$$E|dZ_y(\boldsymbol{\lambda}, \omega)|^2 = dF_y(\boldsymbol{\lambda}, \omega),$$

where $dF_y(\boldsymbol{\lambda}, \omega)$ is a spectral measure. If we assume further that $dF_y(\boldsymbol{\lambda}, \omega)$ is absolutely continuous with respect to Lebesgue measure according to the arguments $\boldsymbol{\lambda}$ and ω, then $dF_y(\boldsymbol{\lambda}, \omega) = f(\boldsymbol{\lambda}, \omega)d\boldsymbol{\lambda}d\omega$, where $d\boldsymbol{\lambda} = \prod_{i=1}^{d} d\lambda_i$. Here $f(\boldsymbol{\lambda}, \omega)$ is a strictly positive, real valued function and is defined as the spatio- temporal spectrum of the random process $\{Y_t(\mathbf{s})\}$, and $-\infty < \lambda_1, \lambda_2, ..., \lambda_d < \infty, -\pi \leq \omega \leq \pi$. In view of the orthogonality of the function $Z_y(\boldsymbol{\lambda}, \omega)$, it can be shown that

$$c(\mathbf{h}, u) = \int_{-\infty}^{\infty} \int_{-\pi}^{\pi} e^{i(\mathbf{h}\cdot\boldsymbol{\lambda}+u\omega)} f(\boldsymbol{\lambda}, \omega) d\omega d\boldsymbol{\lambda}, \quad (1)$$

and by inversion we get

$$f(\boldsymbol{\lambda}, \omega) = \frac{1}{(2\pi)^{d+1}} \sum_u \int_{-\infty}^{\infty} e^{-i(\mathbf{h}\cdot\boldsymbol{\lambda}+u\omega)} c(\mathbf{h}, u) d\mathbf{h},$$

and from (1), we have

$$c(\mathbf{0}, u) = \int_{-\pi}^{\pi} e^{iu\omega} g_0(\omega) d\omega,$$

where $g_0(\omega) = \int_{-\infty}^{\infty} f(\boldsymbol{\lambda}, \omega) d\boldsymbol{\lambda}$ is the second order temporal spectral density function of the process $\{Y_t(\mathbf{s})\}$, and in view of our assumption that the process is spatially, temporally stationary $g_0(\omega)$ is same for all the locations $\mathbf{s}$. We note $c(\mathbf{h}, u) = c(-\mathbf{h}, -u)$ and $f(\boldsymbol{\lambda}, \omega) = f(-\boldsymbol{\lambda}, -\omega)$, and $f(\boldsymbol{\lambda}, \omega) > 0$ for all $\boldsymbol{\lambda}$ and ω.

Here $\boldsymbol{\lambda}$ is the spatial frequency associated with the spatial coordinates s_i and is usually called the wave number and ω is the temporal frequency associated with time.

Let $\{Y_t(\mathbf{s_i})\}\,; i = 1, 2, ..., m; t = 1, 2, ...n$ be a sample from the zero mean,stationary spatio-temporal random stationary process $\{Y_t(\mathbf{s})\}$. Consider the time series data at the location $\mathbf{s}_i$ and define the Discrete Fourier transform (DFT)

$$J_{\mathbf{s}_i}(\omega_k) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} Y_t(\mathbf{s}_i) e^{-it\omega_k}; (i = 1, 2, ..., ..., m) \tag{2}$$

where $\omega_k = \frac{2\pi k}{n}, k = 0, 1, 2, ..., \left[\frac{n}{2}\right]$. We note that the Discrete Fourier transforms can be evaluated using the Fast Fourier Transform (FFT) algorithm(FFT), and the number of operations required to calculate FFT from a time series of length n, is of the order $n(\ln n)$. By inversion, we obtain from (2)

$$Y_t(\mathbf{s}) = \sqrt{\frac{n}{2\pi}} \int_{-\pi}^{\pi} e^{it\omega} J_{\mathbf{s}}(\omega) d\omega.$$

The above representation shows that the $\{Y_t(\mathbf{s})\}$ can be decomposed into sine and cosine terms and the complex valued random variables DFT, $J_s(\omega)$ can be considered as the amplitudes corresponding to these sine and cosine functions.

We will briefly summarize some well known results associated with DFT's (see Appendix) which will be required later. It is well known that the discrete Fourier transforms $\{J_{\mathbf{s}}(\omega_k)\}$ evaluated at discrete Fourier frequencies ω_k are asymptotically independent, and is distributed as complex normal (see for details Brillinger [2001])

For example, for large n, and for a specific ω_k, $\{J_{\mathbf{s}}(\omega_k)\}$ is approximately distributed as complex normal with mean zero and variance $\{g_{\mathbf{s}}(\omega_k)\}$ which is the second order temporal spectrum of the process at the location $\mathbf{s}$. In view of the spatial stationarity assumption, $\{g_{\mathbf{s}}(\omega_k)\}$ is same for all locations, and we denote this common temporal spectrum by $\{g_0(\omega_k)\}$.

Let $I_{\mathbf{s}}(\omega_k) = |J_{\mathbf{s}}(\omega_k)|^2$ be the periodogram, and let $I_{\mathbf{s}_i,\mathbf{s}_j}(\omega_k) = J_{\mathbf{s}_i}(\omega_k) J^*_{\mathbf{s}_j}(\omega_k)$ be the cross periodogram between the two time series $\{Y_t(\mathbf{s}_i)\}$ and $\{Y_t(\mathbf{s}_j)\}$ In the appendix we summarize the results associated with periodograms (see also Subba Rao and Terdik [2015]). In the following section, we use the discrete Fourier transforms to define the Frequency Variogram and consider its estimation and also consider its asymptotic sampling properties.

5 Frequency Variogram (FV), Properties and its estimation

As stated earlier, Variogram is used as an alternative measure of second order dependence. It can be defined under weaker conditions and as such it is widely used. Though the statistical properties of the sample variogram are well studied in the case of spatial processes, the estimation and the asymptotic properties of various estimators defined for spatio-temporal processes, such as $\hat{\gamma}(\mathbf{h}, u)$ defined earlier are not well investigated and this could be due to the inclusion of the time dimension in the processes. To circumvent such problems, Subba Rao et al. [2014] have considered frequency domain approach for the analysis, model construction and estimation.

These authors have introduced frequency variogram as an alternative to spatio-temporal variogram defined earlier and was found to be very useful in the estimation of parameters of spatio-temporal spectrum. As no inversion of high dimensional matrices are required in the estimation suggested,the computation of the minimizing criterion is easy. In this paper we consider further properties of the Frequency Variogram and also discuss its nonparametric estimation. We use the FV as a tool for estimating the parameters of the spatio-temporal spectrum of the intrinsic processes.

Let $\{J_{\mathbf{s}}(\omega_k)\}$ be the DFT evaluated at the Fourier frequency $\omega_k = \frac{2\pi k}{n}, k = 0, 1, 2, ..., \left[\frac{n}{2}\right]$ calculated using the time series data $\{Y_t(\mathbf{s})\}$.

The frequency variogram is defined, for a fixed spatial lag $\mathbf{h}$ and at the location $\mathbf{s}$, as follows. Let $X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t) = Y_t(\mathbf{s}) - Y_t(\mathbf{s} + \mathbf{h})$ and

$$\begin{aligned} G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega_k) &= 2\tilde{G}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega_k) \\ &= E|J_{\mathbf{s}}(\omega_k) - J_{\mathbf{s}+\mathbf{h}}(\omega_k)|^2 \\ &= E|J_{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}}(\omega_k)|^2, \end{aligned}$$

where

$$J_{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}}(\omega_k) = \frac{1}{\sqrt{2\pi n}} \sum_t X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t) e^{-it\omega_k},$$

and

$$\begin{aligned} &E\{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t)\} = 0, \\ &Var\{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t)\} = \gamma(\mathbf{h}, 0) \quad \mathbf{h} \in \mathbb{R}^d. \end{aligned}$$

We note $J_{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}}(\omega_k)$ is the DFT of the incremental random process $\{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t)\}$. If the incremental process defined is spatially intrinsically stationary, and also temporally stationary, then the discrete Fourier transforms $\{J_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega_k)\}$ are asymptotically independent, and distributed as Complex Gaussian (Brillinger [2001]). These functions are well defined and no assumptions of spatial, temporal stationarity of $\{Y_t(\mathbf{s})\}$ is required. The FV $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ can be used as a measure of dissimilarity between the two random process $\{Y_t(\mathbf{s})\}$ and $\{Y_t(\mathbf{s}+\mathbf{h})\}$ at the frequency ω. As one would expect this measure to increase as the spatial lag $\|\mathbf{h}\|$ increases. In the following we show the relationship between the spatio temporal variogram $\gamma(\mathbf{h}, u)$ and the FV. We make use of fact that the intrinsic process $\{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t)\}$ is temporally stationary.

Proposition 1 *Let*

$$G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = E|J_{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}}(\omega)|^2,$$

then

$$\int_{-\pi}^{\pi} G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)d\omega = \gamma(\mathbf{h}, 0), \tag{3}$$

Proof. An application of Parseval's theorem gives the above result. ■

In the derivation of the above results we used the assumption that the incremental process $\{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t)\}$ is stationary temporally and spatially even though the original process $\{Y_t(\mathbf{s})\}$ may not be spatially, temporally stationary.

The above result (3) shows that the FV $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is the frequency decomposition of the classical spatio temporal variogram $\gamma(\mathbf{h}, u)$ when $u = 0$, similar to the frequency decomposition we have for the power of the stationary random process in terms of the power spectral density function. Since $\gamma(\mathbf{h}, u)$ is a measure of dissimilarity between two spatial processes separated by lag $\mathbf{h}$, $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is also a measure of dissimilarity of the two process at the frequency ω. By plotting this function as a function of ω, one can observe in which frequency band there is a large amount of lack of similarity. This information could be useful in prediction where one can a predict a time series using the time series data from other neighborhood locations.

Proposition 2 *Let $\{Y_t(\mathbf{s})\}$ be a zero-mean second order stationary process in space and time and let $\{J_{s_i}(\omega)\}$ $(i = 1, 2, ..., m)$ be the DFT of $\{Y_t(\mathbf{s}_i), i = 1, 2, ...m\}$. Let $G_{\mathbf{s}_i,\mathbf{s}_j}(\omega)$ be the frequency variogram. Then*

1. *The covariance function $g_{\mathbf{s}_i,\mathbf{s}_j}(\omega) = cov(J_{\mathbf{s}_i}(\omega), J_{\mathbf{s}_j}(\omega))$ is a positive semi-definite function.*

2. *The FV $G_{\mathbf{s}_i,\mathbf{s}_j}(\omega)$ is conditionally negative definite.*

Proof. Consider the sum $I_1(\omega) = \sum_{i=1}^{m} a_i J_{\mathbf{s}_i}(\omega)$ where $\{a_i\}$ can be complex. Then

$$VarI_1(\omega) = \sum\sum a_i a_j{}^* Cov(J_{\mathbf{s}_i}(\omega), J_{\mathbf{s}_j}(\omega)) \geq 0.$$

Hence the result (1).
To prove the second result, assume $\sum a_i = 0$. we assume the process to be isotropic then we can show that

$$\begin{aligned}\sum\sum a_i a_j{}^* G_{\mathbf{s}_i,\mathbf{s}_j}(\omega) &= \sum\sum a_i a_j{}^* E|J_{\mathbf{s}_i}(\omega) - J_{\mathbf{s}_j}(\omega)|^2 \\ &= -2\sum\sum a_i a_j g_{\mathbf{s}_i-\mathbf{s}_j}(\omega).\end{aligned}$$

Since $g_{\mathbf{s}_i,\mathbf{s}_j}(\omega) = g_{\|\mathbf{s}_i-\mathbf{s}_j\|}(\omega)$ because of isotropy assumption. Hence is the result (2). ■

5.1 Frequency Variogram and Nugget effect

For illustration purposes we consider the case $d = 2$. Suppose instead of observing the process $\{Y_t(\mathbf{s}), \mathbf{s} \in \mathbb{R}^2, t \in \mathbb{Z}\}$, we observe a corrupted random process $\left\{\tilde{Y}_t(\mathbf{s}), \mathbf{s} \in \mathbb{R}^2, t \in \mathbb{Z}\right\}$, where for each $\mathbf{s}$ and t,

$$\widetilde{Y}_t(\mathbf{s}) = Y_t(\mathbf{s}) + \eta_t(\mathbf{s}),$$

and $\{Y_t(\mathbf{s})\}$ and $\{\eta_t(\mathbf{s})\}$ are zero mean spatially, temporally stationary processes and $\{Y_t(\mathbf{s})\}$ and $\{\eta_t(\mathbf{s})\}$ are independent for all t and $\mathbf{s}$. Further, we assume that $\{\eta_t(\mathbf{s})\}$ is a white noise with a second order space-time spectrum $g_\eta(\boldsymbol{\lambda}, \omega) = \frac{\sigma_\eta{}^2}{(2\pi)^3}$ for all $\boldsymbol{\lambda}$ and ω. Define the DFT of the incremental random process $(\tilde{Y}_t(\mathbf{s}) - \tilde{Y}_t(\mathbf{s}+\mathbf{h}))$,

$$\tilde{J}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = J^Y_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) + J^\eta_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega), \quad |\omega| \leq \pi,$$

where

$$\begin{aligned}\tilde{J}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) &= \frac{1}{\sqrt{2\pi n}}\sum(\tilde{Y}_t(s) - \tilde{Y}_t(s+\mathbf{h}))e^{-i\omega t}, \\ J^Y_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) &= \frac{1}{\sqrt{2\pi n}}\sum(Y_t(s) - Y_t(s+\mathbf{h}))e^{-i\omega t},\end{aligned}$$

$J^\eta{}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is defined similarly.

Define the FV for the process $\left\{\tilde{Y}_t(s)\right\}$,

$$\begin{aligned}\tilde{G}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) &= E|\tilde{J}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)|^2 \\ &= E|J^Y{}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)|^2 + E|J^\eta{}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)|^2 \\ &\approx G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) + \frac{\sigma_e^2}{(2\pi)^3} \qquad (4)\end{aligned}$$

.

The above result follows because of our assumption that the random process $\{\eta_t(\mathbf{s})\}$ is a white noise. From (4), we observe that is $\|\mathbf{h}\| \to 0$, $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) \to 0$ for all ω and, therefore, $\tilde{G}_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) \to \frac{\sigma_e^2}{(2k)^3}$ as $\|\mathbf{h}\| \to 0$.

If we plot $\int G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)d\omega$ as a function of $\|\mathbf{h}\|$ and if we observe a jump near the origin $\|\mathbf{h}\| \sim 0$, then this could be due to the presence of noise in the process. This effect is usually called the "Nugget effect" in geo-mining literature. In the following section we consider the estimation of $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$. In practice one uses the Fast Fourier algorithm for computing DFT's.

6 Estimation of the Frequency variogram

Let $\{Y_t(\mathbf{s}_i); i = 1, 2, 3, ..., m; t = 1, 2, ..., n\}$ be a sample form the spatio temporal random process $\{Y_t(\mathbf{s}_i)\}$. Here we consider the estimation of FV under the assumption of the Intrinsic stationarity of the process. We assume the process $\{Y_t(\mathbf{s}_i)\}$ we observe is not corrupted by noise.

Let $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = E|J_s(\omega) - J_{\mathbf{s}+\mathbf{h}}(\omega)|^2, |\omega| \leq \pi$. We consider its estimation under the assumption that the process is intrinsically stationary. We noted earlier that the FV $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is expected value of the periodogram of the incremental process $X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t) = Y_t(\mathbf{s}) - Y_t(\mathbf{s}+\mathbf{h}), (t = 1, 2, ...)$. The process $X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t)$ is spatially, temporally stationary when $\mathbf{h}$ is fixed. Therefore for large n, it is well known that the periodogram is an unbiased estimator of the second order spectral density function of the stationary process $X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t)$ though this is not a consistent estimator. Therefore, our object here is to obtain a consistent estimator of the spectrum of the incremental process $X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t)$ for a given $\mathbf{h}$, using the entire sample of discrete of Fourier transforms $\{J_{\mathbf{s}_i}(\omega_k); i = 1, 2, ..., m\}$, where $\omega_k = \frac{2\pi k}{n}, (k = 0, 1, ..., \left[\frac{n}{2}\right])$.

Let $X_{\mathbf{s}_i,\mathbf{s}_i+h}(t) = Y_t(\mathbf{s}_i) - Y_t(\mathbf{s}_i+\mathbf{h}), t = 1, 2, ..., n$, and let

$$I_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\omega) = |J_{\mathbf{s}_i,\mathbf{s}_i+h}(\omega)|^2,$$

where

$$J_{\mathbf{s}_i,\mathbf{s}_i+h}(\omega) = J_{\mathbf{s}_i}(\omega) - J_{\mathbf{s}_i+h}(\omega).$$

Let $g_{\mathbf{s}_i,\mathbf{h}}(\omega)$ be the second order spectrum of the incremental process process $\{X_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(t)\}$. Since the intrinsic process is spatially stationary $g_{\mathbf{s}_i}(\omega)$ does not depend $\mathbf{s}_i$. We denote such a stationary spectrum of the intrinsic process by $g_{\mathbf{h}}(\omega)$.

Let Ω denote the set of all location $\mathbf{s}_1, \mathbf{s}_2, ...\mathbf{s}_m$, and let $N(\mathbf{h})$ denote the subset of pairs of locations $(\mathbf{s}_i, \mathbf{s}_j)$, such that
$N(\mathbf{h}) = \{(\mathbf{s}_i, \mathbf{s}_j); i, j = 1, 2, ..., m; (\mathbf{s}_i, \mathbf{s}_j \in \Omega), \mathbf{s}_i - \mathbf{s}_j = \mathbf{h}\}$. $|N(\mathbf{h})|$ be the number

of distinct elements in the set $N(\mathbf{h})$. The estimation of stationary spectrum of a time series is well known and, therefore, we discuss the estimation of $g_{\mathbf{h}}(\omega)$ only briefly. For details, we refer to Priestley [1981], Brillinger [2001], Brockwell and Davis [1987].

Consider the estimator,

$$\hat{g}_{\mathbf{h}}(\omega) = \int_{-\pi}^{\pi} W_n(\omega - \theta) \left(\frac{1}{|N(\mathbf{h})|} \sum_i I_{\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}}(\theta) d\theta \right), \tag{5}$$

where the sum is taken over the set $N(\mathbf{h})$, and the weight function $W_n(\theta)$, which is a real valued even function of θ, satisfies the following conditions (Priestley [1981], Brillinger [2001])

1. $W_n(\theta) \geq 0$ for all n and θ,
2. $\int W_n(\theta) d\theta = 1$, all n,
3. $\int W_n^2(\theta) d\theta < \infty$, all n,
4. For any $\varepsilon\ (> 0)$, $W_n(\theta) \to 0$, uniformly as $n \to \infty$, for $|\theta| > \varepsilon$.

Theorem 1 *Let $g_{\mathbf{h}}(\omega)$ be the spectral density function of the process $\{X_{\mathbf{s}_i, \mathbf{s}_i + h}(t)\}$ for all $\mathbf{s}_i$ and let $g_{\mathbf{s}_i, \mathbf{s}_j}(h, \omega)$ be the cross spectral density function of the process $\{X_{\mathbf{s}_i, \mathbf{s}_i + h}(t)\}$ and $\{X_{\mathbf{s}_j, \mathbf{s}_j + h}(t)\}$. Then*

1. $E(\hat{g}_h(\omega)) = g_h(\omega) + O(\frac{\ln n}{n})$,
2. $Var(\hat{g}_h(\omega)) \approx \frac{1}{|N(\mathbf{h})|^2} \frac{2\pi}{n} \int W_n^2(\omega - \theta) \left[\sum_{i,j} |g_{\mathbf{s}_i, \mathbf{s}_j}(\mathbf{h}, \theta)|^2 \right] d\theta.$

Proof. Take expectations both sides of (5),

$$E\left(\hat{g}_h(\omega)\right) = \int W_n(\omega - \theta) (\frac{1}{|N(\mathbf{h})|}) \sum_i E(I_{\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}}(\theta)) d\theta,$$

we note that

$$E(I_{\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}}(\theta)) = g_{\mathbf{h}}(\theta) + O\left(\frac{\ln n}{n}\right),$$

therefore we have

$$E\left(\hat{g}_{\mathbf{h}}(\omega)\right) = g_{\mathbf{h}}(\theta) + O\left(\frac{\ln n}{n}\right) \approx g_{\mathbf{h}}(\omega),$$

in view of assumption 2. and the fact that $W_n(\theta)$ is approaching the Dirac-Delta function; concentrating its mass at $\theta = 0$. Therefore, $\hat{g}_{\mathbf{h}}(\omega)$ is asymptotically an unbiased estimator of $g_{\mathbf{h}}(\omega)$. As we have noted earlier, estimating the frequency variogram is equivalent to (for large n) estimating $g_{\mathbf{h}}(\omega)$. To obtain an expression for the variance, we consider a discrete approximation of $\widehat{g}_{\mathbf{h}}(\omega)$. Our derivation here is heuristic , and to obtain an expression for the covariance we assume the intrinsic process is Gaussian , even though this assumption is not essential for proving normality or consistency (see Brillinger [2001]). ■

Consider the discrete approximation, and take variance both sides, we get

$$var(\hat{g}_{\mathbf{h}}(\omega) = \frac{1}{|N(\mathbf{h})|^2}\left(\frac{2\pi}{n}\right)^2 \sum_P \sum_{P'} W_n(\omega - \theta_P) W_n(\omega - \theta_{P'})$$
$$Cov\left(\sum_i I_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\theta_P), \sum_j I_{\mathbf{s}_j,\mathbf{s}_{j+\mathbf{h}}}(\theta_{P'})\right)$$

and we have

$$Cov\left(\sum_i I_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\theta_P), \sum_j I_{\mathbf{s}_j,\mathbf{s}_j+\mathbf{h}}(\theta_{P'})\right)$$
$$= \eta(\theta_p - \theta_{p'}) \sum_i \sum_j |g_{\mathbf{s}_i,\mathbf{s}_j}(\mathbf{h}, \theta_p)|^2 + \eta(\theta_p + \theta_{p'}) \sum_i \sum_j |g_{\mathbf{s}_i,\mathbf{s}_j}(\mathbf{h}, \theta_p)|^2$$

where $\eta(\theta) = \sum_{-\infty}^{\infty} \delta(\theta - 2\pi j)$ is a Dirac comb (Brillinger [2001] Corollary 7.22) and to obtain the above expression we used the results concerning the covariance between two periodogram ordinates (see Brillinger [2001]), and after substitution of this expression and simplification, we get

$$Var(\hat{g}_{\mathbf{h}}(\omega)) \approx \frac{1}{|N(\mathbf{h})|^2} \frac{2\pi}{n} \int W_n^2(\omega - \theta) \left[\sum \sum (g_{\mathbf{s}_i,\mathbf{s}_j}(\mathbf{h}, \theta)^2)\right] d\theta.$$

The above shows $\hat{g}_h(\omega)$ is a mean square consistent estimator of $g_h(\omega)$ which is the frequency variogram.

Remark 1 *In the derivation of the above results, we have only assumed the that the intrinsic process is Gaussian. The assumption of the Gaussianity is made only to obtain a simple expression for the variance. The result that the estimator $\widehat{g}_{\mathbf{h}}(\omega)$ is a consistent estimator is still valid under non Gaussianity assumption (see Brillinger [2001]). We noted earlier that estimating the Frequency variogram $G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega)$ is (asymptotically) equivalent to estimating $g_{\mathbf{h}}(\omega)$. To obtain an expression for the variance, we consider a discrete approximation of $\tilde{g}_{\mathbf{h}}(\omega)$. Our* *derivation here is heuristic and for convenience, we assume the incremental process* $\{X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t)\}$ *is Gaussian for all* $\mathbf{s}$*. We note the estimator is consistent even if the process is not Gaussian, and for this result to hold we need absolute summability of higher order cumulants (see Brillinger [2001]).*

We can show by following similar lines that as $n \to \infty$ (see Priestley [1981]).

$$Cov\left(\hat{g}_{\mathbf{h}}(\omega_1), \hat{g}_{\mathbf{h}}(\omega_2)\right) = 0 \quad \text{for} \quad \omega_1 \neq \omega_2$$

The asymptotic normality of $\hat{g}_{\mathbf{h}}(\omega)$ can be shown using the results of Hannan [1973], Taniguchi [1980], Deo and Chen [2000].

7 Complex stochastic partial differential equation for the intrinsic process and spectrum for the FV

In a recent paper, Subba Rao and Terdik [2015] defined a complex stochastic partial differential equation for the spatio temporal process and obtained an analytic expression for the spectrum of the spatio-temporal process. This spectrum is non-separable. As we mentioned earlier, stationarity assumption may not be realistic always, and therefore, a weaker assumption that the process is intrinsically stationary is made. Here our object is to define a model for such an intrinsic process, and obtain an analytic parametric expression for the spectrum. In a later Section, we consider the estimation of the parameters of such a function. We may note that Yaglom [1987] and Huang et al. [2011] and others have obtained spectra for the variogram in the case of spatial process. Yu et al. [2007], Huang et al. [2011], have considered non-parametric estimation of the variogram. Consider the incremental random process $X_{\mathbf{s}}^{(\mathbf{h})}(t) = X_{\mathbf{s},\mathbf{s}+\mathbf{h}}(t) = Y_t(\mathbf{s}) - Y_t(\mathbf{s}+\mathbf{h})$, $\mathbf{s} \in \mathbb{R}^d, t \in \mathbb{Z}$ for a fixed lag $\mathbf{h}$, the incremental process is a function of the spatial location $\mathbf{s} \in \mathbb{R}^d$, and time $t \in \mathbb{R}$.

We consider the process $\{X_{\mathbf{s}}^{(\mathbf{h})}(t)\}$ which is zero mean and stationary in space and time. We now define DFT of the time series $\{X_{\mathbf{s}}^{(\mathbf{h})}(t)\}$,

$$J_{X_{\mathbf{s}}}^{(\mathbf{h})}(\omega_k) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_{\mathbf{s}}^{(\mathbf{h})}(t) e^{it\omega_k},$$

where

$$\omega_k = \frac{2\pi k}{n}, (k = 0, 1, 2, ..., \left[\frac{n}{2}\right]).$$

We note the frequency variogram is defined as

$$G_{\mathbf{s},\mathbf{s}+\mathbf{h}}(\omega) = Var(J_{X_{\mathbf{s}}}^{(\mathbf{h})}(\omega)) = E|J_{X_s}^{(\mathbf{h})}(\omega)|^2.$$

Similarly we can define the covariance between two distinct Fourier Transforms

$$g^{(\mathbf{h})}_{\mathbf{s},\mathbf{s}+\mathbf{L}}(\omega) = Cov(J^{(\mathbf{h})}_{X_\mathbf{s}}(\omega), J^{(\mathbf{h})}_{X_{\mathbf{s}+\mathbf{L}}}(\omega)),$$

where $J^{(\mathbf{h})}_{X_\mathbf{s}}(\omega)$, $J^{(\mathbf{h})}_{X_{\mathbf{s}+\mathbf{L}}}(\omega)$ are discrete Fourier Transforms of the incremental processes

$$X^{(\mathbf{h})}_{\mathbf{s}}(t) = Y_t(\mathbf{s}) - Y_t(\mathbf{s}+\mathbf{h}), \quad \text{and} \quad X^{(\mathbf{h})}_{\mathbf{s}+\mathbf{L}}(t) = Y_t(\mathbf{s}+\mathbf{L}) - Y_t(\mathbf{s}+\mathbf{L}+\mathbf{h}),$$

for $t = 1, ..., n, \quad \mathbf{s} = \mathbf{s}_1, ..., \mathbf{s}_m$ and $\mathbf{L} \in \mathbf{R}^d$. We note in computing the above, we fix $\mathbf{h}$ and consider$\{X_\mathbf{s}{}^{(\mathbf{h})}(t)\}$ as one spatio-temporal series.
Since the process $\{X_\mathbf{s}{}^{(\mathbf{h})}(t)\}$ is a zero mean second order spatially, temporally stationary, it has the spectral representation.

$$X_s{}^{(\mathbf{h})}(t) = \int\int e^{i(\mathbf{s}\cdot\boldsymbol{\lambda}+t\omega)} d\xi^{(\mathbf{h})}_X(\boldsymbol{\lambda}, \omega).$$

where $d\xi^{(\mathbf{h})}_X(\boldsymbol{\lambda}, \omega)$ is a zero mean complex random process with orthogonal increments with

$$\begin{aligned} &E[d\xi^{(\mathbf{h})}_X(\boldsymbol{\lambda}, \omega)] = 0, \\ &E|d\xi^{(\mathbf{h})}_X(\boldsymbol{\lambda}, \omega)|^2 = dF_X{}^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega) = f^{(\ \mathbf{h})}_X\ (\boldsymbol{\lambda}, \omega) d\boldsymbol{\lambda} d\omega \end{aligned}$$

We define $f_X{}^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega)$, as the spectral density function of the stationary intrinsic process $\{X_\mathbf{s}{}^{(\mathbf{h})}(t)\}$ we have the following spectral representation for the DFT.

Proposition 3 *Let $J_{X_\mathbf{s}}{}^{(\mathbf{h})}(\omega)$ be the DFT of the stationary time since $\{X^{(\mathbf{h})}_\mathbf{s}(t)\}$. Then.*

$$J_{X_\mathbf{s}}{}^{(\mathbf{h})}(\omega) \approx \int e^{i\mathbf{s}\cdot\boldsymbol{\lambda}} \sqrt{\frac{n}{2\pi}} d\xi_X{}^{(\mathbf{h})}(\boldsymbol{\lambda}, \omega)$$

Proof. The proof is similar to the proof given in Proposition 2 of Subba Rao and Terdik [2015] and hence the details are omitted. ■

Theorem 2 *Let $\{J_{X_{\mathbf{s}_i}}{}^{(\mathbf{h})}(\omega); i = 1, 2, ..., m\}$ be the discrete Fourier transforms of the incremental process $\{J_{X_{\mathbf{s}_i}}{}^{(h)}(t)\}$. Let*

$$\left[\sum_{i=1}^{d} \frac{\partial^2}{\partial \mathbf{s}_i{}^2} - |P_\mathbf{h}(\omega, \boldsymbol{\psi})|^2\right]^{\nu} J^{(\mathbf{h})}_{X_\mathbf{s}}(\omega) = J^{(\mathbf{h})}_{\eta_\mathbf{s}}(\omega), \quad |\omega| \leq \pi,$$

where $\nu > 0$,and $J^{(\mathbf{h})}_{\eta_{\mathbf{s}}}(\omega)$ is the DFT of the white noise process $\{\eta_{\mathbf{s}}(t)\}$ and $P(\omega, \boldsymbol{\psi})$ is a polynomial and it is a function of some parameter vector $\boldsymbol{\psi}$. The second order space-time spectrum is given by

$$f^{(\mathbf{h})}_X(\boldsymbol{\lambda}, \omega) = \frac{{\sigma_\eta}^2}{(2\pi)^d} \frac{1}{\left(\sum_{i=1}^d \lambda_i^2 + P_{\mathbf{h}}|(\omega, \boldsymbol{\psi})|^2\right)^{2\nu}}$$

and the covariance between the periodograms (which is a spectrum dependent on spatial distance $\mathbf{L}$, and the temporal frequency ω) is given by..

$$\begin{aligned} g^{(\mathbf{h})}_{\mathbf{s},\mathbf{s}+\mathbf{L}}(\omega) &= Cov\left(J^{(\mathbf{h})}_{X_{\mathbf{s}}}(\omega), J^{(\mathbf{h})}_{X_{\mathbf{s},\mathbf{s}+\mathbf{L}}}(\omega)\right) \\ &= \frac{{\sigma_\eta}^2}{(2\pi)^d 2^{2\nu-1}\Gamma(2\nu)} \left(\frac{||\mathbf{L}||}{|P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|}\right)^{2\nu-\frac{d}{2}} K_{2\nu-\frac{d}{2}}(||\mathbf{L}|||P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|). \end{aligned}$$

where $K_\nu(x)$ is the modified Bessel function of the second kind of order ν. We note that in view if spatial stationarity the right hand side expression doesn't depend on $\mathbf{s}$, and depends on the Euclidean spatial distance $||\mathbf{L}||$. Further, as $||\mathbf{L}|| \to 0$, the spectrum is given by

$$g^{(\mathbf{h})}_0(\omega) = Var(J^{(\mathbf{h})}_{X_s}(\omega)) = \frac{{\sigma_\eta}^2}{(2\pi)^{\frac{d}{2}} 2^{\frac{d}{2}} \left(|P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|^2\right)^{2v-\frac{d}{2}}} \frac{\Gamma(2v - \frac{d}{2})}{\Gamma(2v)}$$

Proof. The proof is similar to the proof of Theorem 1 of the paper by Subba Rao and Terdik [2015] and hence omitted. We note that both covariance function ,and the variance given above depend on $\mathbf{h}$ since the polynomial $P_{\mathbf{h}}(\omega, \boldsymbol{\psi})$ is related to the second order spectral density function of the intrinsic process $X^{(\mathbf{h})}_{\mathbf{s}}(t)$. ■

Proposition 4 (Special case 1) *Let $d = 2$, $\nu = 1$ and assume $\mathbf{h}$ is fixed. Then*

$$g^{(\mathbf{h})}_0(\omega) = \frac{{\sigma_\eta}^2}{4\pi} |P_{\mathbf{h}}(\omega, \boldsymbol{\psi})|^{-2}$$

The above result show that the function $|P(\omega, \boldsymbol{\psi})|^2$ is related to stationary temporal spectrum of the process $\left\{X^{(\mathbf{h})}_{\mathbf{s}}(t)\right\}$. We note further that $f_X(\boldsymbol{\lambda}, \omega)$ is the spatio-temporal spectrum and $g^{(\mathbf{h})}_0(\omega)$ is the stationary temporal spectrum of the process $\left\{X^{(\mathbf{h})}_{\mathbf{s}}(t)\right\}$. For large n and for a fixed $\mathbf{h}$, $Var(J^{\mathbf{h}}_{X_{\mathbf{s}}}(\omega)) \approx g^{(\mathbf{h})}_0(\omega)$, $|\omega| \leq \pi$.

In the above, we have shown that we can obtain an analytic expression for the spectral density function for an intrinsic stationary process similar to the stationary spatio temporal process obtained earlier by Subba Rao and Terdik [2015]. The spectral density function $g_0^{(\mathbf{h})}(\omega)$ is a function of some unknown parameters characterizing the statistical properties of the process $X_{\mathbf{s}}{}^{(\mathbf{h})}(t)$, and let us denote these parameters by $\underline{\boldsymbol{\psi}}$. In the following section, we consider the estimation of parameter vector $\underline{\boldsymbol{\psi}}$ using the discrete Fourier transform of the process $X_{\mathbf{s}}^{(\mathbf{h})}(t)$.

8 Estimation of the parameters of the frequency variogram of the intrinsic process

Matheron [1963], Cressie [1993], Stein [2012], Yu et al. [2007], and many others have stressed the importance of the variogram in Kriging and in view of this, several method of estimation of the variogram in the case of spatial processes have been proposed. Yu et al. [2007] have proposed nonparametric estimation of the variogram, Huang et al. [2011] proposed the estimation of the variogram and its spectrum. For example ,if we are assuming that the intrinsic process is satisfying the model stated in Theorem 2, we have seen the spectral density function of the intrinsic process depends on the variance of white noise , the parameter vector $\boldsymbol{\psi}$, and other parameters , and these parameters need to be estimated . We need to consider the estimation of the parameters from the data $\left\{X_{\mathbf{s}}^{(\mathbf{h})}(t)\right\}$. It is important to estimate the variogram under the weaker conditions. The above authors considered the estimation of variogram under the intrinsic stationarity assumption which is weaker.

In this section we consider the estimation of the frequency variogram parameters under the assumption that the spatio-temporal process is intrinsically spatially-temporarily stationary. We further assume that the incremental process is isotropic in space.

The Frequency Variogram is defined as

$$G_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\omega) = E|J_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\omega)|^2 = E[I_{\mathbf{s_i}}{}^{(\mathbf{h})}(\omega)],$$

where

$$J_{\mathbf{s_i},\mathbf{s}_i+\mathbf{h}}(\omega) = \frac{1}{\sqrt{2\pi n}} \sum X_{\mathbf{s_i},\mathbf{s}_i+\mathbf{h}}(t) e^{-it\omega},$$
$$X_{\mathbf{s_i},\mathbf{s}_i+\mathbf{h}}(t) = X_{\mathbf{s}_i}^{(\mathbf{h})}(t) = Y_t(\mathbf{s_i}) - Y_t(\mathbf{s_i}+\mathbf{h}).$$

We assume the process $X_{\mathbf{s_i}}^{(\mathbf{h})}(t)$ is spatially and temporally stationary with

$$E[X_{\mathbf{s}_i}^{(\mathbf{h})}(t)] = 0,$$
$$Var[X_{\mathbf{s}_i}^{(\mathbf{h})}(t)] = \gamma(\mathbf{h}, 0).$$

We have noted earlier that for large n, $G_{\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}}(\omega)$ is approximately equivalent to the second order spectral density function $g_{\mathbf{s}_i,\mathbf{h}}(\omega)$ of the intrinsic process $\left\{X_{\mathbf{s}_i}^{(\mathbf{h})}(t)\right\}$ and in Section 6 we have considered the estimation of $g_{\mathbf{s}_i,\mathbf{h}}(\omega)$ following the classical non-parametric methods of spectral estimation. Here in this section our object is to consider the estimation of the parameters which characterize the frequency variogram, as this is often required in Kriging.

Since the intrinsic process is $X_{\mathbf{s_i}}{}^{(\mathbf{h})}(t)$, is for a given $\mathbf{h}$, is spatially stationary, the spectral density function $g_{\mathbf{s_i},\mathbf{h}}(\omega)$ does not depend on the location $\mathbf{s}_i(i = 1, 2, ...m)$. For convenience, we denote the common spectral density function by $g_0{}^{(\mathbf{h})}(\omega)$ and we denote the parameter vector of the spectrum $\boldsymbol{\psi}$. Our object here is to estimate $\boldsymbol{\psi}$ of $g_0^{(\mathbf{h})}(\omega, \boldsymbol{\psi})$ given that the sample $\{X_{\mathbf{s}_i}{}^{(\mathbf{h})}(t); t = 1, 2, ..., n\}$ and the set $N(\mathbf{h}) = \{(\mathbf{s}_i, \mathbf{s}_j); i = 1, 2, ..., m,$ such that, $\mathbf{s}_i - \mathbf{s}_j = \mathbf{h},$ $(\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}) \in \Omega\}$. For example , if we are assuming the intrinsic process satisfies the model stated in the Theorem 2 , Section 7, then the parameters we have to consider for the estimation are $\boldsymbol{\psi}$ of the Polynomial $(P(\omega, \boldsymbol{\psi})$ related to the spectrum $g_0{}^{(\mathbf{h})}(\omega)$ and $\sigma_n{}^2$. These parameters need to be estimated from the data $X_{\mathbf{s}_i}{}^{(\mathbf{h})}(t)$. Here we construct a likelihood function using the DFT's, and the approach is similar to the method described in Subba Rao et al. [2014]. We refer to the above paper for details.

Consider the Discrete Fourier Transforms $\{J_{X_{\mathbf{s}_i}}^{(\mathbf{h})}(\omega_k)\}$ corresponding to the time series $\{Y_t(\mathbf{s}_i)\}$, $\{Y_t(\mathbf{s}_i + \mathbf{h})\}$. We note that for large n, the random variable $\{J_{X_{\mathbf{s}_i}}^{(\mathbf{h})}(\omega_k)\}$ is distributed as complex normal with mean zero and variance $g_0{}^{(\mathbf{h})}(\omega_k, \boldsymbol{\psi})$ (see Brillinger [2001]). Let $M = \left[\frac{n}{2}\right]$. Consider the M dimensional complex valued random vector

$$\chi_{\|\mathbf{h}\|}(\omega) = \{J_{X_{\mathbf{s}_i}}^{(\mathbf{h})}(\omega_k), J_{X_{\mathbf{s}_i}}^{(\mathbf{h})}(\omega_k), ..., J_{X_{\mathbf{s}_i}}^{(\mathbf{h})}(\omega_k)\},$$

which is distributed asymptotically as complex multivariate normal with mean zero and variance covariance matrix with diagonal elements

$$\left[g_0^{\|\mathbf{h}\|}(\omega_1, \boldsymbol{\psi}), g_0^{\|\mathbf{h}\|}(\omega_2, \boldsymbol{\psi}), ..., g_0{}^{\|\mathbf{h}\|}(\omega_M, \boldsymbol{\psi})\right].$$

We note that off diagonal elements are zero. Proceeding as in Subba Rao et al. [2014], we can show that the log likelihood function of $\chi_{\|\mathbf{h}\|}(\omega)$ is proportional to

$$Q_{n,i}^{(\mathbf{h})}(\boldsymbol{\psi}) = \sum_{k=1}^{M} \left[\ln g_0{}^{\|\mathbf{h}\|}(\omega_k, \boldsymbol{\psi}) + \frac{I_{\mathbf{s}_i}{}^{(\mathbf{h})}(\omega_k)}{g_0{}^{(\mathbf{h})}(\omega_k, \psi)}\right].$$

Now consider all the locations $(\mathbf{s}_i, \mathbf{s}_i + \mathbf{h})$; $i = 1, 2, ..., m$ belonging to the set $N(\mathbf{h}) = \{(\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}); i = 1, 2..., m.; (\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}) \in \Omega\}$. Then we have the pooled criterion

$$Q_{n,N(\mathbf{h})}(\boldsymbol{\psi}) = \frac{1}{|N(\mathbf{h})|} \sum_{(\mathbf{s}_i, \mathbf{s}_i \in N(\mathbf{h}))} Q_{n,i}^{(\mathbf{h})}(\boldsymbol{\psi}). \tag{6}$$

We minimize 6 with respect to $\boldsymbol{\psi}$. The asymptotic normality of the estimator of $\boldsymbol{\psi}$ can be proved using the methodology described in Subba Rao et al. [2014]. For large n, we can show

$$\sqrt{n}(\tilde{\boldsymbol{\psi}} - \boldsymbol{\psi}) \xrightarrow{D} N(0, \left[\nabla^2 Q_{n,N(\mathbf{h})}(\boldsymbol{\psi})\right]^{-1}) V \left[\nabla^2 Q_{n,N(\mathbf{h})}(\boldsymbol{\psi})\right],$$

where $V = \lim_{n\to\infty} Var\left[\frac{1}{\sqrt{n}} \nabla Q_{n,N(\mathbf{h})}(\boldsymbol{\psi})\right]$, and $\nabla Q_{n,N(\mathbf{h})}(\boldsymbol{\psi})$ is a Jacobian vector of first order partial derivatives, and $\left[\nabla^2 Q_{n,N(\mathbf{h})}(\boldsymbol{\psi})\right]$ is a Hessian matrix of second order partial derivatives.

9 Test for independence of m spatial time series

So far we have considered the analysis of spatio-temporal data using various frequency domain methods. We assumed that there is second order dependence in space and time. It is important to test for Independence over space and time before the analysis of the data Henebry [1995] proposed a test statistic for testing for spatio temporal Independence; the statistics is as an extension of Moran's test. Cressie and Wikle [2011] briefly discussed the test. In this section, we propose a test for spatial independence using the Discrete Fourier transforms and the test is based on the test proposed by Wahba [1971]. The test proposed by Wahba [1971] is an extension of the classical test for independence used in multivariate analysis. Here we briefly describe the test. Let.

$$\underline{Y}_t' = (Y_t(\mathbf{s}_1), Y_t(\mathbf{s}_2), ..., Y_t(\mathbf{s}_m)).$$

We say, the multivariate time series $\{\underline{Y}_t\}$ is second order stationary if (see Brockwell and Davis [1987])

1. $E(\underline{Y}_t) = \underline{\mu}$,
2. $E(\underline{Y}_t - \underline{\mu})(\underline{Y}_{t+p} - \underline{\mu})' = \underline{\Gamma}(p)$,where

$$\begin{aligned}
\underline{\mu}' &= (\mu_1, \mu_2, ..., \mu_m), \\
\Gamma(p) &= (\sigma_{ij}(p)), \\
\sigma_{ij}(p) &= E\left(Y_t(\mathbf{s}_i) - \mu_i\right)\left(Y_{t+p}(\mathbf{s}_i) - \mu_j\right), (i, j = 1, 2, ..., m), \\
\sigma_{ij}(p) &= \sigma_{ji}(-p),
\end{aligned}$$

Here we are assuming that the spatio-temporal data is temporarily stationary only, no assumption of spatial stationarity is assumed. We assume further that $\underline{Y}_t$ is Gaussian. Define the complex valued random vector

$$\underline{J}'(\omega_k) = (J_{\mathbf{s}_1}(\omega_k), J_{\mathbf{s}_2}(\omega_k), ..., J_{\mathbf{s}_m}(\omega_k)),$$

where $J_{\mathbf{s}_i}(\omega_k)$ is the DFT of $\{Y_t(\mathbf{s}_i)\}$, and $\omega_k = \frac{2\pi k}{n}$, $(k = 0, 1, ..., \left[\frac{n}{2}\right])$. We know that the random vector $\underline{J}(\omega_k)$ is distributed as complex normal with mean $\mathbf{0}$ and variance covariance matrix $\underline{F}(\omega_k)$, where $\underline{F}(\omega_k) = \left[E(J_{\mathbf{s}_i}(\omega_k)J^*_{\mathbf{s}_j}(\omega_k)\right]$. We note that $\underline{F}(\omega_k)$ is Hermitian, with elements

$$f_{\mathbf{s}_i,\mathbf{s}_j}(\omega_k) = E(J_{\mathbf{s}_i}(\omega_k)J^*_{\mathbf{s}_j}(\omega_k)) = f_{\mathbf{s}_j,\mathbf{s}_i}(-\omega_k).$$

In the above $f_{\mathbf{s}_i,\mathbf{s}_i}(\omega_k)$ is the second order spectral density function of the process $\{Y_t(\mathbf{s}_i)\}$, and $f_{\mathbf{s}_i,\mathbf{s}_j}(\omega_k)$ is the cross spectral density function of the process $\{Y_t(\mathbf{s}_i)\}$ and $\{Y_t(\mathbf{s}_j)\}$. The cross spectral density function is usually a complex valued function.

If we assume that the spatio-temporal process $\{Y_t(\mathbf{s})\}$ is stationary in space and time, and further assume that the process is isotropic in space, then

$$\begin{aligned} f_{\mathbf{s}_i,\mathbf{s}_i}(\omega) &= f_0(\omega), \\ f_{\mathbf{s}_i,\mathbf{s}_j}(\omega) &= f_{||\mathbf{s}_i-\mathbf{s}_j||}(\omega). \end{aligned}$$

In this case the matrix $\underline{F}(\omega)$ is real and symmetric, and all the diagonal elements are equal to $f_0(\omega)$.

As pointed out earlier, for testing spatial independence we do not need the assumption of spatial stationarity. Below we assume that the process is Gaussian under the null hypothesis that the spatial process is spatially independent, the spectral matrix $F(\omega)$ is a diagonal matrix for all $|\omega| \leq \pi$. For constructing the test, we proceed as in Wahba [1971]. Consider the discrete Fourier transforms defined earlier. For each location s_i, let the Fourier transform be given by $(J_{s_i}(\omega_l))$ where $\omega_l = \frac{2\pi j_l}{n}$, $j_l = (l-1)(2k+1) + (k+1)$; $l = 1, 2, ..., M_1$ where M_1 is chosen such that $2(k+1)M_1 = \frac{n-1}{2}$. (Here we assume that the number of observations n, is odd.) As in Wahba [1971] we define the cross spectral estimator of $f_{s_i,\mathbf{s}_j}(\omega)$ by

$$\hat{f}_{s_i,\mathbf{s}_j}(\omega_l) = \frac{1}{2k+1}\sum_{j=-k}^{k} I_{i,j}(\omega_l + \frac{2\pi j}{n}), \quad (l = 1, 2, ..., M_1),$$

where the cross periodogram $I_{ij}(\omega_l) = J_{s_i}(\omega_l)J^*_{\mathbf{s}_j}(\omega_l)$.

Let $\hat{F}(\omega_l) = (\hat{f}_{s_i,\mathbf{s}_j}(\omega_l)) \quad (l = 1, 2, ..., M_1)$.

We note that the random matrices $\hat{F}(\omega_l); l = 1, 2, ..., M_1$, for large k, are approximately distributed as random matrices $\tilde{F}(\omega_l), (l = 1, 2, ..., M)$ which are distributed as complex Wishart, usually denoted by $W_c(F, m, 2k+1)$. (Wahba [1971]) has shown that the likelihood ratio test for testing the null hypothesis that the matrices $F(w_l)$ are diagonal for all $\{\omega_l\}$ leads to the test statistic, for each w_l,

$$\tilde{\boldsymbol{\lambda}}_l = \frac{|\tilde{F}(\omega_l)|}{\prod_{j=1}^{m} \tilde{f}_{\mathbf{s}_j,\mathbf{s}_j}(\omega_l)} \quad (l = 1, 2, ..., M_1),$$

and the over-all test statistic to consider is $\mathbf{\Lambda} = -\frac{1}{M_1}\sum \ln \tilde{\boldsymbol{\lambda}}_l$. For large k and M_1, under the null hypothesis, the statistic $\mathbf{\Lambda}$ is asymptotically distributed as normal with mean

$$E(\mathbf{\Lambda}) = \sum_{j=1}^{m-1} \frac{m-j}{k'-j}$$

and variance

$$Var(\mathbf{\Lambda}) = \frac{1}{M_1}\sum_{j=1}^{m-1} \frac{m-j}{(k'-j)^2}$$

where $k' = 2k+1$. Under the null hypotheses of spatial independence, for large k and M, the statistic $S = \frac{\mathbf{\Lambda}-E(\mathbf{\Lambda})}{\sqrt{Var(\mathbf{\Lambda})}}$ is distributed as standard normal. We note that if for each $\mathbf{s}_i$, $\{Y_t(\mathbf{s}_i)\}$ is a Gaussian white noise, then the spectral density function is given by $f_{\mathbf{s}_i,\mathbf{s}_i}(\omega) = \frac{{\sigma_{s_i}}^2}{2\pi}$, where ${\sigma_{\mathbf{s}_i}}^2$ is the variance of the white noise. If the null hypothesis is both spatially and temporally independent then the diagonal elements of the matrix $F(\omega_l)$ will be proportional to $({\sigma_{\mathbf{s}_1}}^2, {\sigma_{\mathbf{s}_2}}^2, {\sigma_{\mathbf{s}_3}}^2, , ..., {\sigma_{\mathbf{s}_m}}^2)$, and all off diagonal elements will be zero.

10 Appendix: Discrete Fourier Transforms

In this section, we will briefly summarize some results related to the Discrete Fourier Transforms, further details, we refer to Subba Rao and Terdik [2015].

Let $\{Y_t(\mathbf{s})\}$, where $\left\{\mathbf{s} \in \mathbb{R}^d; t \in Z\right\}$ denote a zero mean second order spatially,temporally stationary process with spectral representation

$$Y_t(\mathbf{s}) = \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} e^{i(\mathbf{s}\cdot\boldsymbol{\lambda}+t\omega)} dZ_y(\boldsymbol{\lambda},\omega), \tag{7}$$

and let $\{Y_t(\mathbf{s_i})); i = 1,2,...,m; t = 1,2,...,n\}$ be a sample. We note that $Z_y(\boldsymbol{\lambda},\omega)$ is a zero mean complex valued function with orthogonal increments and

$$E[dZ_y(\boldsymbol{\lambda},\omega)] = 0,$$
$$E|dZ_y(\boldsymbol{\lambda},\omega)|^2 = dF_y(\boldsymbol{\lambda},\omega),$$

where $dF_y(\boldsymbol{\lambda},\omega)$ is a spectral measure, Let $dF(\boldsymbol{\lambda},\omega) = f(\boldsymbol{\lambda},\omega)d\boldsymbol{\lambda}d\omega$, where $f(\boldsymbol{\lambda},\omega)$ is the spatio-temporal spectral density function of the process $\{Y_t(\mathbf{s})\}$. Define the Discrete Fourier Transform

$$J_{\mathbf{s}}(\omega) = \frac{1}{\sqrt{2\pi n}}\sum_{t=1}^{n} Y_t(\mathbf{s})e^{it\omega}, |\omega| \leq \pi, .$$

Proposition 5 *Let the spectral representation of the process* $\{Y_t(\mathbf{s}_i)\}$ *be given by 7, and let* $J_{\mathbf{s}}(\omega)$ *be the DFT of the sample* $\{Y_t(\mathbf{s}); t = 1, 2, ...n\}$. *Then we have*

1. $Y_t(\mathbf{s}) = \sqrt{\frac{n}{2\pi}} \int J_{\mathbf{s}}(\omega) e^{it\omega} d\omega$,
2. $J_{\mathbf{s}}(\omega) \approx \int e^{is\boldsymbol{\lambda}} \sqrt{\frac{n}{2\pi}} dZ_y(\boldsymbol{\lambda}, \omega)$.

Proof. By substitution and using the properties of Dirac Delta function, one can show (2). (1) follows by inversion. For details, refer to Subba Rao and Terdik [2015]. ■

Let $I_{\mathbf{s}}(\omega_k) = |J_{\mathbf{s}}(\omega_k)|^2$ be the periodogram. The following results are well known (Priestley [1981], Brillinger [2001])

1. $E(I_{\mathbf{s}}(\omega_k)) = g_{\mathbf{s}}(\omega_k) + O(n^{-1})$
2. $Var(I_{\mathbf{s}}(\omega_k)) = g^2_{\mathbf{s}}(\omega_k) + O(n^{-1}), \quad \omega_k \neq 0, \pi,$
3. $Cov(I_{\mathbf{s}}(\omega_k), I_{\mathbf{s}}(\omega_l) = O(n^{-1}) \quad if \quad \omega_k + \omega_l \neq 0 \quad (mod \quad 2\pi)$, In view of spatial stationarity, $g_{\mathbf{s}}(\omega) = g_0(\omega)$ for all $\mathbf{s}$, and
$$g_{\mathbf{s}}(\omega) = \frac{1}{2\pi} \sum Cov(Y_t(\mathbf{s}), Y_{t+k}(\mathbf{s})) e^{-i\omega k}, |\omega| \leq \pi$$
4. $Cov(J_{\mathbf{s}_i}(\omega_k), J_{\mathbf{s}_j}(\omega_k)) = O(n^{-1}), \quad if \quad \omega_k + \omega_l \neq 0 \quad (mod \quad 2\pi)$
5. $Cov(J_{\mathbf{s}_i}(\omega_k), J_{\mathbf{s}_j}(\omega_k)) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c(\mathbf{s}_i - \mathbf{s}_j, n) e^{-in\omega_k} = g_{\mathbf{s}_i - \mathbf{s}_j}(\omega_k) + O(n^{-1})$, where the spectral density function $g_{||\mathbf{s}_i - \mathbf{s}_j||}(\omega_k, \theta_L)$ is real valued, and depends on the parameter vector θ_L.

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