



Riesz logarithmic means of subsequences of partial sums of trigonometric Fourier series

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Abstract

Let $(a(j))$ be an unbounded convex sequence of natural numbers. In 1983, Carleson [7] proved that a necessary and sufficient condition for the $(C, 1)$ means of $(S_{a(j)}f)$ (partial Fourier sums) to converge uniformly to f in the supremum norm is $\sup_j j^{-1/2} \log a(j) < +\infty$. In this paper, we prove that if we consider Riesz logarithmic means instead of $(C, 1)$ means, then almost everywhere convergence holds for any integrable function and for any convex $(a(j))$. That is (see Corollary 1.3): Let $(a(j))$ be any unbounded convex sequence of natural numbers and $f \in L^1$. Then the following convergence holds almost everywhere:

$$\frac{1}{\log n} \sum_{j=1}^n \frac{S_{a(j)}f}{j} \rightarrow f.$$

In fact, we verify a more general statement (see Theorem 1.1).

Keywords Riesz logarithmic and Cesàro means · Subsequences of partial sums · Trigonometric Fourier series · A.e. convergence

Mathematics Subject Classification 42A24

1 Introduction and the main result

It is of central interest in the theory of trigonometric Fourier series how to reconstruct the function from the partial sums of its Fourier series. Du Bois-Reymond proved [8] that the Fourier series of a continuous function can unboundedly diverge at an arbitrary predefined point.

Kolmogorov [16] constructed his famous example of a function $f \in L^1$ such that the partial sums $S_n f(x)$ diverge unboundedly almost everywhere. In a later paper [17] he constructed an everywhere divergent Fourier series. On the other hand, Carleson [6] showed that if $f \in L^2$, then the partial sums converge to the function almost everywhere. The condition $f \in L^2$ in Carleson's theorem was weakened by Hunt [15] (to $f \in L^p$ ($p > 1$)). In 1996

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Antonov [1] proved that if f belongs to the Orlicz-class $L \log^+ L \log^+ \log^+ \log^+ L$, then the partial sums of the Fourier series converge to the function almost everywhere.

A fundamental question is then what to do if all we know about the function under consideration is that it is integrable. That is, $f \in L^1$. In 1904 Fejér proved [9] that, for any continuous function f , the arithmetic means of the partial Fourier sums converge in the supremum norm to the function in question. Inspired by Fejér's results, Lebesgue [19] showed that for each integrable function we have the almost everywhere convergence of these (Fejér) means $\sigma_n f = \frac{1}{n+1} \sum_{j=0}^n S_j f \rightarrow f$.

Another notable summation method is the Riesz logarithmic mean, defined by

$$R_n f = \frac{1}{\log n} \sum_{j=1}^n \frac{S_j f}{j}.$$

In 1931, Hardy gave [14] a necessary and sufficient condition (improving Zygmund's result [27]) for an integrable function $R_n f(x) \rightarrow f(x)$ to be satisfied for a given point x . It follows that $R_n f \rightarrow f$ almost everywhere. Of course, the almost everywhere convergence also follows from Lebesgue's result (shown using an Abel transformation).

Besides, it is also of prior interest what it can be said—concerning this reconstruction issue—if we have only a subsequence of the partial sums. With respect to partial sums in L^1 , Totik [24] showed that for every subsequence $a(j) \subset \mathbb{N}$ there exists an integrable function f for which $\sup_j |S_{a(j)} f| = +\infty$ at every point. Moreover, Konyagin proved [18] that for any increasing sequence $(a(j))$ of positive integers and any nondecreasing function $\phi: [0, +\infty) \rightarrow [0, +\infty)$ satisfying the condition $\phi(u) = o(u \log \log u)$, there is a function $f \in \phi(L)$ such that $\sup_j |S_{a(j)} f| = +\infty$ everywhere. Thus some summation procedure is necessary if we want to obtain positive results for all integrable functions.

This question goes back to Zalcwasser [26]. In 1936, Zalcwasser asked [26] how a sequence of integers $(a(j))$ should be chosen so that

$$\sigma_n^a f = \frac{1}{n} \sum_{j=1}^n S_{a(j)} f \rightarrow f$$

for every function $f \in L^1$. This problem (for convex sequences) with respect to the trigonometric system for continuous functions was completely solved by Carleson [7]. That is, if the sequence $(a(j))$ is convex, then the growth condition $\sup_j j^{-1/2} \log a(j) < +\infty$ is necessary and sufficient for the uniform convergence for every continuous function.

An important question is whether some other summation method might give a much better result than Carleson's. Newman proved [20] that for any regular method of summability associated with a matrix $(a_{n,j})$ there exists a continuous function f such that $\sum_{j=1}^n a_{n,j} S_{2j} f$ does not converge to f in norm.

The main goal of this paper is to show that much better results can be obtained in the question of “almost everywhere convergence for integrable functions” than in the question of “convergence in the supremum norm for continuous functions”.

With respect to almost everywhere convergence and integrable functions much less is known. In 1936, Zalcwasser proved [26] the a.e. relation $\frac{1}{n} \sum_{j=1}^n S_{j^2} f \rightarrow f$ for each integrable function f . In his paper, Salem [21, page 394] writes that this theorem of Zalcwasser is extended to j^3 and j^4 but there is no citation in [21] about it. Belinsky proved [4] the existence of a sequence $a(j) \sim \exp(\sqrt[3]{j})$ such that the relation $\frac{1}{n} \sum_{j=1}^n S_{a(j)} f \rightarrow f$ holds a.e. for every integrable function. In this paper, Belinsky also conjectured that if the sequence $(a(j))$ is convex, then the condition $\sup_j j^{-1/2} \log a(j) < +\infty$ is necessary and sufficient

again. In an article, the author proved [12]—among others—that this is not the case. That is, if $a(j+1) \geq a(j)(1 + 1/j^\delta)$ (for some $\delta < 1/2$), then the $(C, 1)$ means of $(S_{a(j)}f)$ converge a.e. to f for each $f \in L^1$. Unfortunately, this is not true for all convex sequence $(a(j))$.

Furthermore, in a recent article, the author characterized [13] those (not bounded) convex sequences $\mathcal{N} \subset \mathbb{N}$ for which, for any subsequence $a(n) \in \mathcal{N}$, $\sigma_n^a f \rightarrow f$ almost everywhere for any $f \in L^1$. These sequences $\mathcal{N}$ are precisely those convex sequences that are unions of finite many lacunary sequences. Consequently, there is a sequence $(a(n))$ of the natural numbers and a $f \in L^1$ such that $\sigma_n^a f$ diverges almost everywhere. Unfortunately, a subsequence of a convex sequence is not necessarily convex, so this result [13] does not show whether the result of this article concerning Riesz means can be true for Fejér means.

At the end of this paper (Remark 3.6), there is a note explaining why the method used in this article concerning Riesz logarithmic means cannot be applied to prove a similar result for Fejér means (i.e., $\sigma_n^a f \rightarrow f$ a.e. for any integrable f).

The main result of the present paper (Corollary 1.3) is that for any convex index sequence $(a(j))$ and any integrable function f , the Riesz logarithmic means of the sequence $(S_{a(j)}f)$ (a subsequence of the Fourier sums) converge almost everywhere to f . Moreover, we prove more (Theorem 1.1).

The system of functions $e^{i\iota nx}$ ($n = 0, \pm 1, \pm 2, \dots$) ($x \in \mathbb{R}$, $\iota = \sqrt{-1}$) is called the trigonometric system. It is orthogonal over any interval of length 2π , specifically over $T := [-\pi, \pi]$. Let $f \in L^1(T)$, that is, f is an integrable function on T . The k th Fourier coefficient of f is

$$\hat{f}(k) := \frac{1}{2\pi} \int_T f(x) e^{-i\iota kt} dt,$$

where k is any integer number. The n th ($n \in \mathbb{N}$) sum of the Fourier series of f is

$$S_n f(y) := \sum_{k=-n}^n \hat{f}(k) e^{i\iota ky}.$$

The n th ($n \in \mathbb{N}$) Fejér or $(C, 1)$ mean of the function f is defined in the following way:

$$\sigma_n f(y) := \frac{1}{n+1} \sum_{k=0}^n S_k f(y).$$

It is known that

$$\sigma_n f(y) = \frac{1}{\pi} \int_T f(x) K_n(y-x) dx,$$

where the function K_n is known as the n th Fejér kernel; we will now find an appropriate expression for it (see e.g. the book of Bary [2]), namely

$$K_n(u) = \frac{1}{2(n+1)} \left(\frac{\sin(\frac{u}{2}(n+1))}{\sin(\frac{u}{2})} \right)^2.$$

From this expression, one can immediately derive the following properties of the kernel. They will play an essential role later.

$$K_n(u) \geq 0.$$

$$K_n(u) \leq \frac{\pi^2}{2(n+1)u^2} \quad (0 < |u| \leq \pi). \quad (1.1)$$

The n th de la Vallée-Poussin mean of the integrable function f is defined as

$$V_n f := \frac{1}{n} \sum_{j=n}^{2n-1} S_j f.$$

We then state the main result of the paper. After stating the main theorem (Theorem 1.1) and a corollary (with its proof), we will say a few words about the method and the way we will prove Theorem 1.1.

Theorem 1.1 *Let $a : \mathbb{N} \rightarrow \mathbb{N}$ be a monotonically increasing sequence (not necessarily strictly). Assume there exists a $c_0 \in \mathbb{N}$ (which may depend on the sequence a) such that $2a(x) \leq a(c_0 x)$ for all $x \in \mathbb{N}$. Then it holds*

$$\lim_N \frac{1}{\log N} \sum_{j=1}^N \frac{S_{a(j)} f}{j} = f \quad (1.2)$$

a.e. for each integrable function f .

Remark 1.2 We note that it follows from the conditions of the theorem that $a(j) \rightarrow \infty$ as $j \rightarrow \infty$ since $a(c_0^k) \geq 2^k a(1) \geq 2^k$. By Theorem 1.1, the sequence $(a(j))$ does not need to be convex, or even strictly monotone, for convergence almost everywhere to be satisfied. A good example is the following sequence: $a(j) = \lfloor j^\delta \rfloor$ ($j \in \mathbb{N}$) for some fixed $1 > \delta > 0$. However, it is quite obvious that the sequence $(a(j))$ must meet some growth condition. That is, it is not expected that the Theorem 1.1 is satisfied for any monotone sequence $(a(j))$ converging to infinity.

Corollary 1.3 *Let $a : \mathbb{N} \rightarrow \mathbb{N}$ be a convex sequence of positive integers with property $a(+\infty) = +\infty$. Then it holds*

$$\lim_N R_N^a f := \lim_N \frac{1}{\log N} \sum_{j=1}^N \frac{S_{a(j)} f}{j} = f \quad (1.3)$$

a.e. for each integrable function f .

Proof Corollary 1.3 is a trivial consequence of Theorem 1.1. That is, let $a(0) = a(1)$ then the sequence $a : \mathbb{N}_0 \rightarrow \mathbb{N}$ is still convex and consequently $2a(x) \leq a(0) + a(2x)$. It follows that $4a(x) \leq 2a(0) + 2a(2x) \leq 3a(0) + a(4x)$, $2a(x) \leq 3a(0)/2 + a(4x)/2 \leq a(4x)$ for any x such that $3a(0) \leq a(4x)$. This can be assumed, since $a(x) \rightarrow \infty$ and it is obviously sufficient that the inequality $2a(x) \leq a(c_0 x)$ is satisfied if x is large enough. That is, $c_0 = 4$ is a suitable choice. $\square$

We outline the main strategy leading to the proof of Theorem 1.1, as well as the necessary ideas and tools: first informally, then in detail. We use the fact that for any integrable function f , the de la Vallée-Poussin means $(V_n f)$ converge to f almost everywhere. Consequently, it is sufficient to verify the a.e. convergence

$$\frac{1}{\log N} \sum_{j=1}^N \frac{S_{a(j)} f - V_{a(j)} f}{j} \rightarrow 0.$$

Why is this beneficial? Essentially, because if we group the addends of the sum skillfully, some of them will become orthogonal. We will refer to these parts as “first parts” of the

logarithmic means of the sequence $(S_{a(j)}f - V_{a(j)}f)$. At the same time, we will refer to the remaining parts as “last parts”. To be more precise (but still in broad terms): let L be an integer such that $2^{L^2} < N \leq 2^{(L+1)^2}$. (This definition is currently provisional and a little bit approximate.) We have

$$\begin{aligned} & \frac{1}{\log N} \sum_{j=1}^N \frac{S_{a(j)}f - V_{a(j)}f}{j} \\ &= \frac{1}{\log N} \sum_{j=1}^{2^{L_0^2}} \frac{S_{a(j)}f - V_{a(j)}f}{j} + \frac{1}{\log N} \sum_{j=2^{L_0^2}+1}^{2^{L^2}} \frac{S_{a(j)}f - V_{a(j)}f}{j} \\ & \quad + \frac{1}{\log N} \sum_{j=2^{L^2}+1}^N \frac{S_{a(j)}f - V_{a(j)}f}{j} \\ &=: A_1f + A_2f + A_3f. \end{aligned}$$

We will see that the operators $\sup_N |A_i|$ ($i = 1, 2, 3$) are of weak type (L^1, L^1) . The case $i = 1$ is trivial. We will treat the expression $\sup_N |A_3f|$ (“last parts”) with the following inequality.

$$\int_{T \setminus \gamma F} \sup_L \left| \frac{1}{L^2} \sum_{s=L^2}^{(L+1)^2-1} \sum_{j=2^{s+1}}^{2^{s+1}} \left| \frac{S_{a(j)}f - V_{a(j)}f}{j} \right| \right|^2 \leq C \|f\|_1 \lambda.$$

This is precisely the statement of Lemma 3.2. The set γF is a set that depends on the function f and the number $\lambda > 0$. This set is essentially where the Hardy–Littlewood maximal function of the function f is at most $C\lambda$. Besides, its measure is no greater than $C\|f\|_1/\lambda$.

That is, we will prove (Lemma 3.2) that the L^2 -norm on a somehow perforated version of the interval $T = [-\pi, \pi)$ of the maximal function of the “last parts” of the logarithmic means of the sequence $(S_{a(j)}f - V_{a(j)}f)$ can be estimated by the L^1 norm of the function f .

We then define the set sequences βF_{β_j} ($j \in \mathbb{N}$). We will verify (Lemma 3.4) that the L^2 -norm of the maximum function of the “first parts” of the logarithmic means of the sequence $(S_{a(j)}f - V_{a(j)}f)\sigma_{m_j} 1_{\beta F_{\beta_j}}$ ($j \in \mathbb{N}$) can be estimated by the L^1 norm of the function f .

We prove (Lemma 3.5) that the logarithmic means of the parts of the sequence $(S_{a(j)}f - V_{a(j)}f)\sigma_{m_j} 1_{\beta F_{\beta_j}}$ of this type (“first parts”) are almost everywhere equiconvergent with the logarithmic means of the parts of the sequence $(S_{a(j)}f - V_{a(j)}f)$ of the same type, so it is essentially irrelevant which one we consider. The last step in proving the main theorem (Theorem 1.1) follows from the well-known fact that $V_n f \rightarrow f$ a.e. for each integrable function f .

The norm inequalities (Lemmas 2.2, 2.3) in the author’s article [12] play a crucial role in proving the main lemmas of the present paper. This essentially says that if we assume that the function f is integrable, then the L^2 -norm of the function $S_n f$ is not necessarily bounded, however, if we drop a “small” subset of T , then the integral of $|S_n f|^2$ on the remaining set is bounded.

2 Tools needed from paper [12]

The dyadic subintervals of T are defined in the following way.

$$\begin{aligned}\mathcal{J}_0 &:= \{T\}, \quad \mathcal{J}_1 := \{[-\pi, 0), [0, \pi)\}, \\ \mathcal{J}_2 &:= \{[-\pi, -\pi/2), [-\pi/2, 0), [0, \pi/2), [\pi/2, \pi)\}, \dots \\ \mathcal{J} &:= \bigcup_{n=0}^{\infty} \mathcal{J}_n.\end{aligned}$$

The elements of $\mathcal{J}$ are said to be dyadic intervals. If $F \in \mathcal{J}$, then there exists a unique $n \in \mathbb{N}_0$ such that $F \in \mathcal{J}_n$, and consequently $\text{mes}(F) = |F| = \frac{2\pi}{2^n}$. Each $\mathcal{J}_n$ has 2^n disjoint elements ($n \in \mathbb{N}_0$).

The following Calderon–Zygmund type decomposition lemma can be found for instance in [12] or [23, page 17] or [22, page 90] (more precisely, in a slightly different way) or in [10] (with an elementary proof). This will play a prominent role in the proof of the main theorem of this paper.

Lemma 2.1 *Let $f \in L^1(T)$, and $\lambda > \|f\|_1/(2\pi)$. Then there exists a sequence of integrable functions (f_i) such that*

$$\begin{aligned}f &= \sum_{i=0}^{\infty} f_i \quad \text{a.e.}, \\ \|f_0\|_{\infty} &\leq 2\lambda, \quad \|f_0\|_1 \leq 2\|f\|_1, \quad \text{and} \\ \text{supp } f_i &\subset I^i, \quad \text{where}\end{aligned}$$

$I^i \in \mathcal{J}$ are disjoint dyadic intervals depending only on $|f|$ (and λ),

$$\text{mes}(I^i) = \frac{2\pi}{2^{k_i}} \quad \text{for some}$$

$k_i \geq 1$ ($i \geq 1$). Moreover, $\int_T f_i(x)dx = \int_{I^i} f_i(x)dx = 0$ ($i \geq 1$),

$$\lambda < \frac{1}{\text{mes}(I^i)} \int_{I^i} |f| \leq 2\lambda, \quad \frac{1}{\text{mes}(I^i)} \int_{I^i} |f_i| \leq 4\lambda$$

and for the union

$$F := \bigcup_{i=1}^{\infty} I^i$$

of the disjoint dyadic intervals I^i ($i \geq 1$) we have $\text{mes}(F) = |F| \leq \|f\|_1/\lambda$.

Also using the notation with respect to Lemma 2.1 we define $\mathcal{F} := \{I^i : i = 1, \dots\}$. That is, $\mathcal{F}$ is the set of dyadic intervals, whose union is the set F . Moreover, for any dyadic interval I , $I \in \mathcal{F}$ if and only if $|I|^{-1} \int_I |f| > \lambda$ and $|J|^{-1} \int_J |f| \leq \lambda$ for every dyadic interval $J \supsetneq I$. For any positive number $\beta > 0$ let $\mathcal{F}_{\beta} := \{I \in \mathcal{F} : |I| > \beta\}$, $F_{\beta} := \bigcup_{I \in \mathcal{F}_{\beta}} I$. Besides, remark that $f_i = (f - |I^i|^{-1} \int_{I^i} f)1_{I^i}$ ($i \geq 1$), $f_0 = f1_{\bar{F}} + \sum_{i=1}^{\infty} (|I^i|^{-1} \int_{I^i} f)1_{I^i}$ ($1_{\bar{F}}$ is the characteristic function of the complement of F).

Let $I_n(x)$ be the dyadic interval of measure $2\pi/2^n$ with $x \in I_n(x)$. For any dyadic interval $I \in \mathcal{J}$ and integer i let $I^{(i)} := I + |I|i$ be the i th neighbour of I (the addition $I + |I|i$ is done modulo $T = [-\pi, \pi)$, that is a circle represents T). For instance $I^0 = I$ and for

$I = I_n(x) \in \mathcal{J}$, $I^{(1)} = I_n^{(1)}(x) := I_n(x) + 2\pi/2^n$, $I^{(-1)} = I_n^{(-1)}(x) := I_n(x) - 2\pi/2^n$ are the right and left adjacent (modulo T) dyadic intervals of $I_n(x)$ with the same measure.

For any $I_n(x) \in \mathcal{J}$ let $3I_n(x) := \bigcup_{i=-1}^1 I_n^{(i)}(x)$ be the tripling of $I_n(x)$. That is, $3I_n(x)$ is an interval with the same centre as $I_n(x)$ and with its measure being the measure of $I_n(x)$ tripled. Similarly, we define

$$5I_n(x) := \bigcup_{i=-2}^2 I_n^{(i)}(x)$$

and so on. That is, we can define $7I_n(x), 9I_n(x), \dots$. Moreover, use the notation for any odd integer γ :

$$\gamma F_\beta := \bigcup_{I \in \mathcal{F}_\beta} \gamma I, \quad \gamma F := \bigcup_{I \in \mathcal{F}} \gamma I.$$

From this point on, we assume throughout the paper that $\alpha > \beta > \gamma > 5$ are fixed odd integers. (Say 11, 9, and 7.) Furthermore, C will denote an absolute constant, which may or may not be different from line to line. C may depend on the value of α, β, γ , but since we have fixed them (once and for all), we do not denote this. The forthcoming two lemmas (more precisely they are named as corollaries in the cited paper) can be found in [12]:

Lemma 2.2 [12] *Let $\gamma > 5$ be an odd integer, $n \in \mathbb{N}$. Let $f \in L^1(T)$, $\lambda > \|f\|_1/(2\pi)$. Then the inequality*

$$\int_{T \setminus \gamma F} |S_n f(y) - V_n f(y)|^2 dy \leq C \|f\|_1 \lambda$$

holds. The constant C is uniform in f, n and λ .

Recall that $\bar{A} = T \setminus A$ denotes the complement of the set $A \subset T$ and $1_A : T \rightarrow \{0, 1\}$ denotes its characteristic function. That is, $1_A(x) = 1$ if $x \in A$ and 0 if $x \notin A$. $\sigma_m 1_A$ is the m -th Fejér mean of 1_A .

Lemma 2.3 [12] *Let $\beta > 7$ be an odd integer, $N \in \mathbb{N}$. Let $f \in L^1(T)$, $\lambda > \|f\|_1/(2\pi)$, $(a(j))$ be a lacunary (with parameter not less than 2, $a(j+1) \geq 2a(j)$) sequence of natural numbers and $a(j)\beta_j = 20(j+1)\log^2(j+1)$, $m_j = \lfloor a(j)/10 \rfloor$ for $j = 1, \dots, N$. Then the inequality*

$$\sum_{j=1}^N \int_T |S_{a(j)} f(y) - V_{a(j)} f(y)|^2 \left| \sigma_{m_j} 1_{\overline{\beta F_{\beta_j}}}(y) \right|^2 dy \leq CN \log^5(N+1) \|f\|_1 \lambda$$

holds. The constant C is uniform in f, λ, N and $(a(j))$.

3 The proof of the main theorem

For the monotone increasing and unbounded sequence $a : \mathbb{N} \rightarrow \mathbb{N}$ introduce its distribution function as:

$$D_a(x) := \max \{n \in \mathbb{N} : a(n) < x\}.$$

If it is clear what the sequence a is, we simply write $D(x)$ instead of the longer $D_a(x)$. That is,

$$a(D(l)) < l \leq a(D(l) + 1).$$

Besides, introduce the notation:

$$b_l := \min \left\{ n \in \mathbb{N} : D(2^n) \geq 2^l \right\}.$$

The following lemma will be used quite often in the sequel.

Lemma 3.1 *Suppose that for the monotone increasing sequence $a : \mathbb{N} \rightarrow \mathbb{N}$ we have a constant $c_0 \in \mathbb{N}$ such that for every $x \in \mathbb{N}$ it holds that $2a(x) \leq a(c_0x)$. Then*

$$2^l \leq D(2^{b_l}) \leq c_0 2^{l+1}$$

for every $l \in \mathbb{N}$.

Proof Since for any positive integer t , we have that $D(t)$ is also a positive integer, then $2D(t) \geq D(t) + 1$. The monotonicity of the sequence a and the definition of the distribution function D imply that

$$a(2D(t)) \geq a(D(t) + 1) \geq t.$$

Using the property $2a(x) \leq a(c_0x)$ of the sequence a , it follows that

$$2t \leq 2a(2D(t)) \leq a(c_0 2D(t)).$$

Since at $c_0 2D(t)$ the sequence a is at least $2t$, then $D(2t)$ cannot be larger than this (even if it is smaller). That is,

$$D(2t) \leq 2c_0 D(t).$$

Apply this latter inequality to $t = 2^{b_l-1}$.

$$D(2^{b_l}) \leq 2c_0 D(2^{b_l-1}) < 2c_0 2^l,$$

where the last inequality came from the definition of the sequence (b_l) . That definition—of course—also gives the inequality $2^l \leq D(2^{b_l})$. This completes the proof of Lemma 3.1. $\square$

In the proof of the main theorem, the following lemma will play a prominent role.

Lemma 3.2 *Let $\gamma > 5$ be an odd integer, $n \in \mathbb{N}$. Let $f \in L^1(T)$, $\lambda > \|f\|_1/(2\pi)$ and suppose that a is any sequence of positive integers. Then the inequality*

$$\int_{T \setminus \gamma F} \sup_L \left| \frac{1}{L^2} \sum_{s=L^2}^{(L+1)^2-1} \sum_{j=2^{s+1}}^{2^{s+1}} \left| \frac{S_{a(j)}f - V_{a(j)}f}{j} \right| \right|^2 \leq C \|f\|_1 \lambda$$

holds. The constant C is uniform in f and λ (and F is the set from Lemma 2.1 regarding f and λ).

The constant C on the right-hand side of the statement is of course independent of γ , since, if we increase the value of γ , the set we are integrating over becomes smaller. (Furthermore, before Lemma 2.2, we have assumed throughout the whole paper that $\alpha > \beta > \gamma > 5$ are fixed odd integers.)

Proof Apply the well-known Cauchy–Bunyakovsky inequality twice:

$$\begin{aligned}
 A_L &:= \left| \frac{1}{L^2} \sum_{s=L^2}^{(L+1)^2-1} \sum_{j=2^s+1}^{2^{s+1}} \left| \frac{S_{a(j)}f - V_{a(j)}f}{j} \right| \right|^2 \\
 &\leq C \frac{1}{L^3} \sum_{s=L^2}^{(L+1)^2-1} \left| \sum_{j=2^s+1}^{2^{s+1}} \frac{S_{a(j)}f - V_{a(j)}f}{j} \right|^2 \\
 &\leq C \frac{1}{L^3} \sum_{s=L^2}^{(L+1)^2-1} \sum_{j=2^s+1}^{2^{s+1}} \frac{1}{j^2} \sum_{j=2^s+1}^{2^{s+1}} |S_{a(j)}f - V_{a(j)}f|^2
 \end{aligned} \tag{3.1}$$

Lemma 2.2 immediately gives:

$$\int_{T \setminus \gamma F} A_L \leq C \|f\|_1 \lambda \frac{1}{L^3} \sum_{s=L^2}^{(L+1)^2-1} \sum_{j=2^s+1}^{2^{s+1}} \frac{1}{j^2} 2^s \leq C \frac{1}{L^2} \|f\|_1 \lambda.$$

Thus, we have

$$\int_{T \setminus \gamma F} \sup_L A_L \leq C \|f\|_1 \lambda.$$

The proof of Lemma 3.2 is complete $\square$

In the proof of the main theorem, the operator defined as follows will also be essential. Recall that $\beta > \gamma > 5$ are fixed odd positive integers. Let us introduce the following notations. Let K_0 be the upper integer part of $2 + 3 \log_2(c_0)$ and

$$I_l^{i_0} := (D(2^{b_{2lK_0+i_0K_0}}), D(2^{b_{(2l+1)K_0+i_0K_0}})]$$

(interval), where $l \in \mathbb{N}$, $i_0 \in \{0, 1\}$. Furthermore, let us introduce the sequences of numbers (m_j) , (β_j) and the sequence of functions (ξ_j) as:

$$m_j := \lfloor a(j)/10 \rfloor, \quad a(j)\beta_j := 20(l+1) \log^2(l+1),$$

where $j \in I_l^{i_0}$ ($i_0 = 0, 1$) and

$$\xi_j := \sigma_{m_j} 1_{\overline{\beta F \beta_j}}$$

($1_{\overline{B}}$ is the characteristic function of the complement of set B).

Let i_0 is either 0 or 1 and be fixed. Besides, let $f \in L^1$. Then we define operators $T_L^{i_0} f$, $\tilde{T}_L^{i_0} f$ as:

$$\begin{aligned}
 T_L^{i_0} f &:= \frac{1}{L^2} \sum_{l=1}^{L^2-1} \sum_{j \in I_l^{i_0}} \frac{S_{a(j)}f - V_{a(j)}f}{j}, \\
 \tilde{T}_L^{i_0} f &:= \frac{1}{L^2} \sum_{l=1}^{L^2-1} \sum_{j \in I_l^{i_0}} \frac{S_{a(j)}f - V_{a(j)}f}{j} \xi_j.
 \end{aligned} \tag{3.2}$$

We call functions $\tilde{T}_L^{i_0} f$ the “first parts” of the logarithmic means of the sequence $(S_{a(j)}f - V_{a(j)}f)\xi_j$ ($j \in \mathbb{N}$, $i_0 = 0, 1$).

Before we state one important lemma, we first confirm another simple one. Let i_0 be either 0 or 1 be fixed.

Lemma 3.3 Suppose that $j_1 \in I_{l_1}^{i_0}$, $j_2 \in I_{l_2}^{i_0}$ and $l_1 \neq l_2$. Then $\frac{S_{a(j_1)}f - V_{a(j_1)}f}{j_1}\xi_{j_1}$ and $\frac{S_{a(j_2)}f - V_{a(j_2)}f}{j_2}\xi_{j_2}$ are orthogonal.

Proof The proof is quite simple and based on the fact that the trigonometric polynomials $(S_{a(j)}f - V_{a(j)}f) \left(\sigma_{m_j} \frac{1}{\beta F_{\beta j}} \right)$ for different j 's (described in (3.2)) are orthogonal because for a j the set of k 's for which the k th Fourier coefficient is different from zero is a subset of

$$\{k \in \mathbb{Z} : k \in [a(j) - m_j, 2a(j) + m_j] \cup [-2a(j) - m_j, -a(j) + m_j]\} \\ \subset \{k \in \mathbb{Z} : k \in [0.9a(j), 2.1a(j)] \cup [-2.1a(j), -0.9a(j)]\}.$$

We will only examine the case where $i_0 = 0$, because the case where $i_0 = 1$ can be discussed in exactly the same way.

For example if we suppose that $l_1 < l_2$, then for any $j_i \in I_{l_i}^0 = (D(2^{b_{2l_i}K_0}), D(2^{b_{(2l_i+1)K_0}})]$, $i = 1, 2$ by Lemma 3.1 we have

$$4a(j_1) \leq 4a \left(D(2^{b_{(2l_1+1)K_0}}) \right) \leq a \left(c_0^3 2^{(2l_1+1)K_0+1} \right) \leq a \left(2^{(2l_1+2)K_0} \right) \leq a(j_2)$$

and consequently the sets

$$[0.9 \cdot a(j_1), 2.1 \cdot a(j_1)] \cup [-2.1a(j_1), -0.9a(j_1)]$$

and

$$[0.9 \cdot a(j_2), 2.1 \cdot a(j_2)] \cup [-2.1a(j_2), -0.9a(j_2)]$$

are disjoint. This completes the proof of Lemma 3.3. $\square$

The following lemma is also crucial in proving the main theorem. We can say that it is the second of the three most important lemmas. The first was Lemma 3.1.

Lemma 3.4 Suppose that for the sequence a we have $2a(x) \leq a(c_0x)$ for some $c_0 \in \mathbb{N}$ and for all x . Let $f \in L^1(T)$, $\lambda > \|f\|_1/(2\pi)$. Then the inequality

$$\left\| \sup_{L \in \mathbb{N}} \left| \tilde{T}_L^{i_0} f \right| \right\|_2^2 \leq C \|f\|_1 \lambda \quad (i_0 = 0, 1)$$

holds. (The definition of the operator $\tilde{T}_L^{i_0}$ is given in (3.2).) The constant C is uniform in f and λ (see Lemma 2.1 regarding f and λ) and may depend on i_0, c_0 .

Proof We will only examine the case where $i_0 = 0$, because the case where $i_0 = 1$ can be discussed in exactly the same way.

For any fixed L by Lemma 3.3 we have

$$\|\tilde{T}_L^0 f\|_2^2 = \frac{1}{L^4} \sum_{l=1}^{L^2-1} \left\| \sum_{j \in I_l^0} \frac{S_{a_j} f - V_{a_j} f}{j} \xi_j \right\|_2^2.$$

By the well-known Cauchy–Bunyakovsky inequality (for any fixed l), we get

$$\begin{aligned} & \left\| \sum_{j \in I_l^0} \frac{S_{a_j} f - V_{a_j} f}{j} \xi_j \right\|_2^2 \\ & \leq \sum_{j=D(2^{b_{2l}K_0})+1}^{D(2^{b_{(2l+1)}K_0})} \frac{1}{j^2} \sum_{j=D(2^{b_{2l}K_0})+1}^{D(2^{b_{(2l+1)}K_0})} \|(S_{a(j)} f - V_{a(j)} f) \xi_j\|_2^2 \\ & \leq C \frac{1}{2^{2lK_0}} \sum_{j=D(2^{b_{2l}K_0})+1}^{D(2^{b_{(2l+1)}K_0})} \|(S_{a(j)} f - V_{a(j)} f) \xi_j\|_2^2. \end{aligned} \quad (3.3)$$

The last inequality in (3.3) is given by Lemma 3.1 and the following estimation:

$$\sum_{j=D(2^{b_{2l}K_0})+1}^{D(2^{b_{(2l+1)}K_0})-1} \frac{1}{j^2} \leq \frac{1}{4^{2lK_0}} \left(D(2^{b_{(2l+1)}K_0}) - D(2^{b_{2l}K_0}) \right) \leq \frac{c_0 2^{(2l+1)K_0+1} - 2^{2lK_0}}{4^{2lK_0}} \leq C \frac{1}{2^{2lK_0}}.$$

Henceforth, until the opposite is stated, let L and l be fixed. Define the integer $j_l \in I_l^0 = (D(2^{b_{2l}K_0}), D(2^{b_{(2l+1)}K_0})]$ in a way that the following inequality is satisfied. (Such j_l obviously exists.)

$$\frac{1}{D(2^{b_{(2l+1)}K_0}) - D(2^{b_{2l}K_0})} \sum_{j \in I_l^0} \|(S_{a(j)} f - V_{a(j)} f) \xi_j\|_2^2 \leq \|(S_{a(j_l)} f - V_{a(j_l)} f) \xi_{j_l}\|_2^2.$$

Therefore, by (3.3) we have

$$\left\| \sum_{j \in I_l^0} \frac{S_{a_j} f - V_{a_j} f}{j} \xi_j \right\|_2^2 \leq C \|(S_{a(j_l)} f - V_{a(j_l)} f) \xi_{j_l}\|_2^2, \quad (3.4)$$

where the constant C may depend on c_0 . Recall that $m_j := \lfloor a(j)/10 \rfloor$, $a(j)\beta_j := 20(l+1)\log^2(l+1)$ (see at (3.2)) for every $l \in \{1, \dots, L^2 - 1\}$.

Then by Lemmas 3.3 and 2.3 (the applicability condition of Lemma 2.3, that $a(j_{l+1}) \geq 2a(j_l)$ be, is obviously satisfied) we have

$$\begin{aligned} & \|\tilde{T}_L^0 f\|_2^2 \\ & \leq C \frac{1}{L^4} \sum_{l=1}^{L^2-1} \left\| (S_{a(j_l)} f - V_{a(j_l)} f) \sigma_{m_{j_l}} \frac{1}{\beta_{F\beta_{j_l}}} \right\|_2^2 \\ & \leq C \frac{1}{L^4} \sum_{l=1}^{L^2-1} \left\| (S_{a(j_l)} f - V_{a(j_l)} f) \sigma_{m_{j_l}} \frac{1}{\beta_{F 20(l+1)\log^2(l+1)/a(j_l)}} \right\|_2^2 \\ & \leq C \|f\|_{1\lambda} \frac{1}{L^4} L^2 \log^5(L+1) \end{aligned} \quad (3.5)$$

for every fixed $L \in \mathbb{N}$. Finally, by (3.5)

$$\|\sup_{L \in \mathbb{N}} |\tilde{T}_L^0 f|\|_2^2 \leq C \|f\|_{1\lambda} \sum_{L=1}^{\infty} \frac{1}{L^4} L^2 \log^5(L+1) \leq C \|f\|_{1\lambda}$$

and the proof of Lemma 3.4 is complete. $\square$

The last important tool on the road to proving the main theorem is the next lemma (Lemma 3.5). Recall the definition of operators $T_L^{i_0}, \tilde{T}_L^{i_0} f$ ($i_0 = 0, 1$) in (3.2).

Lemma 3.5 *Let $f \in L^1(T)$, $\lambda > \|f\|_1/(2\pi)$. Let $(a(j))$ be a monotone increasing sequence of natural numbers with property $2a(x) \leq a(c_0x)$ for all $x \in \mathbb{N}$ with some constant c_0 . We will prove that the maximal operators $\sup_{L \in \mathbb{N}} |T_L^{i_0}|$ and $\sup_{L \in \mathbb{N}} |\tilde{T}_L^{i_0}|$ ($i_0 = 0, 1$) are essentially equivalent in a weak (L^1, L^1) sense. That is, exactly the following:*

$$\text{mes} \left(y \in T : \sup_{L \in \mathbb{N}} |T_L^{i_0} f(y) - \tilde{T}_L^{i_0} f(y)| > \lambda \right) \leq C \sqrt{\frac{\|f\|_1}{\lambda}}.$$

That is, the maximal operator of $\sup_{L \in \mathbb{N}} |T_L^{i_0} f - \tilde{T}_L^{i_0} f|$ is a kind of weak type (L^1, L^1) . The constant C is uniform in f, λ and may depend on the constant c_0 belonging to the sequence $(a(j))$.

Proof Since the cases $i_0 = 0$ and $i_0 = 1$ are discussed in exactly the same way, we will only examine the case $i_0 = 0$. The main idea of the proof of this lemma follows some of the steps of the proof of Lemma 6.2 in a recent article by the author [12]. Let $y \in T \setminus \alpha F$, where set F comes from Lemma 2.1 and $\alpha > \beta > \gamma > 5$ fixed odd naturals. Investigate the inner sum in $T_L^0 f(y) - \tilde{T}_L^0 f(y)$. More precisely its j -th addend. Which is,

$$\frac{S_{a(j)}f - V_{a(j)}f}{j} \left(1 - \sigma_{m_j} 1_{\beta F_{\beta_j}} \right).$$

Since $\sigma_{m_j} 1 = 1$ everywhere, by (1.1) we have

$$\begin{aligned} 1 - \sigma_{m_j} 1_{\beta F_{\beta_j}}(y) &= \sigma_{m_j} 1_{\beta F_{\beta_j}}(y) = \frac{1}{\pi} \int_{-\pi}^{\pi} 1_{\beta F_{\beta_j}}(x) K_{m_j}(y-x) dx \\ &= \frac{1}{\pi} \int_{\beta F_{\beta_j}} K_{m_j}(y-x) dx \leq \frac{C}{m_j} \int_{\beta F_{\beta_j}} \frac{1}{|y-x \pmod{T}|^2} dx, \end{aligned}$$

where $y-x \pmod{T} \in T$ and $y-x \pmod{T} = y-x+u2\pi$ for a $u \in \{-1, 0, 1\}$. That is, if $y-x$ is not in the interval T , then it is shifted by 2π . This can be done, since $\sin^2((y-x)/2) = \sin^2((y-x+u2\pi)/2)$. Besides, notice that $0 \leq |(y-x \pmod{T})/2| \leq \pi/2$. Since $\alpha > \beta > \gamma > 5$ are odd (and fixed) integers and $y \in T \setminus \alpha F$ (this can be assumed, and we do assume it) then for every $j \in \mathbb{N}$ we have $y \in T \setminus \alpha F_{\beta_j}$. If $x \in \beta F_{\beta_j}$, then there is an $I \in \mathcal{F}_{\beta_j}$ (that is, $I \in \mathcal{F}$, $|I| > \beta_j$) such that $x \in \beta I$. This gives $|y-x \pmod{T}| > (\alpha-\beta)|I|/2 > (\alpha-\beta)\beta_j/2$. Thus, for $j \in (D(2^{b_{2l\kappa_0}}), D(2^{b_{(2l+1)\kappa_0}}))$ by

$$m_j := \lfloor a(j)/10 \rfloor, \quad a(j)\beta_j = 20(l+1)\log^2(l+1)$$

we have

$$\begin{aligned} 0 &\leq 1 - \sigma_{m_j} 1_{\beta F_{\beta_j}}(y) \\ &\leq \frac{C}{m_j} \int_{\{z: z > (\alpha-\beta)\beta_j/2\}} \frac{1}{z^2} dz \\ &\leq \frac{C}{\alpha-\beta} \frac{1}{m_j \beta_j} \\ &\leq \frac{C}{\alpha-\beta} \frac{1}{(l+1)\log^2(l+1)}. \end{aligned}$$

We then use Lemma 2.2, the well-known Cauchy–Bunyakovsky inequality and

$$\sum_{j \in I_l^0} \frac{1}{j} = \sum_{j=D(2^{b(2l+1)K_0})+1}^{D(2^{b(2l+1)K_0})} \frac{1}{j} \leq C$$

(since $2a(x) \leq a(c_0x)$ we can apply Lemma 3.1). That is,

$$\begin{aligned} \text{mes} \left(y \in T : \sup_{L \in \mathbb{N}} |T_L^0 f(y) - \tilde{T}_L^0 f(y)| > \lambda \right) \\ \leq \text{mes}(\alpha F) + \text{mes} \left(y \in T \setminus \alpha F : \sup_{L \in \mathbb{N}} |T_L^0 f(y) - \tilde{T}_L^0 f(y)| > \lambda \right) \\ \leq C \frac{\|f\|_1}{\lambda} + C \frac{1}{\lambda} \int_{T \setminus \alpha F} \sup_L |T_L^0 f(y) - \tilde{T}_L^0 f(y)| dy \\ \leq C \frac{\|f\|_1}{\lambda} + C \frac{1}{\lambda} \int_{T \setminus \alpha F} \sup_L \frac{1}{L^2} \sum_{l=1}^{L^2-1} \sum_{j=D(2^{b(2l+1)K_0})+1}^{D(2^{b(2l+1)K_0})} \left| \frac{S_{a(j)}f - V_{a(j)}f}{j} (1 - \sigma_{m_j} 1_{\overline{\beta F_{\beta_j}}}) \right| \\ \leq C \frac{\|f\|_1}{\lambda} + C \frac{1}{\lambda} \int_{T \setminus \alpha F} \sum_{l=1}^{\infty} \frac{1}{l^2} \frac{1}{(l+1) \log^2(l+1)} \sum_{j \in I_l^0} \frac{1}{j} |S_{a(j)}f - V_{a(j)}f| \\ \leq C \frac{\|f\|_1}{\lambda} + C \frac{1}{\lambda} \sum_{l=1}^{\infty} \frac{1}{l^2(l+1) \log^2(l+1)} \sum_{j \in I_l^0} \frac{1}{j} \int_{T \setminus \alpha F} |S_{a(j)}f - V_{a(j)}f| \\ \leq C \frac{\|f\|_1}{\lambda} + C \sum_{l=1}^{\infty} \frac{1}{l^2(l+1) \log^2(l+1)} \sum_{j \in I_l^0} \frac{1}{j} \sqrt{\frac{\|f\|_1}{\lambda}} \\ \leq C \left(\frac{\|f\|_1}{\lambda} + \sqrt{\frac{\|f\|_1}{\lambda}} \right). \end{aligned} \quad (3.6)$$

Since $\lambda > \|f\|_1/(2\pi)$, then by the above written we proved the required inequality for the operator $\sup_L |T_L^0 - \tilde{T}_L^0|$. The case $i_0 = 1$ can be discussed in exactly the same way. This completes the proof of Lemma 3.5. $\square$

Proof of Theorem 1.1 The proof of the main theorem is a combination of the following lemmas: 3.2, 3.4 and 3.5. Let $N \in \mathbb{N}$, $f \in L^1(T)$ and $L := \max\{l \in \mathbb{N} : N > D(2^{b_{2l^2}K_0})\}$. That is, $N \in I_{L^2}^0 \cup I_{L^2}^1 \cup \dots \cup I_{(L+1)^2}^0 \cup I_{(L+1)^2}^1$. Consequently, we have

$$\begin{aligned} R_N^a f - \frac{1}{\log N} \sum_{j=1}^N \frac{V_{a(j)}f}{j} &= \frac{1}{\log N} \sum_{j=1}^N \frac{S_{a(j)}f - V_{a(j)}f}{j} \\ &= \frac{1}{\log N} \sum_{j=1}^{D(2^{b_{2K_0}})} \frac{S_{a(j)}f - V_{a(j)}f}{j} + \frac{1}{\log N} \sum_{j=D(2^{b_{2L^2}K_0})+1}^{D(2^{b_{2L^2}K_0})} \frac{S_{a(j)}f - V_{a(j)}f}{j} \\ &\quad + \frac{1}{\log N} \sum_{j=D(2^{b_{2L^2}K_0})+1}^N \frac{S_{a(j)}f - V_{a(j)}f}{j} \end{aligned}$$

$$=: A_1 + A_2 + A_3.$$

Recall that,

$$2^{2L^2K_0} \leq D \left(2^{b_{2L^2K_0}} \right) < N \leq D \left(2^{b_{2(L+1)^2K_0}} \right) < c_0 2^{2(L+1)^2K_0+1}.$$

Since

$$|A_3| \leq C \frac{1}{L^2} \sum_{j=D \left(2^{b_{2L^2K_0}} \right) + 1}^{D \left(2^{b_{2(L+1)^2K_0}} \right)} \left| \frac{S_{a(j)}f - V_{a(j)}f}{j} \right| \leq C \frac{1}{L^2} \sum_{s=2}^{2(L+1)^2K_0+K_0} \sum_{j=2^s+1}^{2^{s+1}} \left| \frac{S_{a(j)}f - V_{a(j)}f}{j} \right|$$

therefore by Lemma 3.2 and $\text{mes}(\gamma F) \leq C \|f\|_1/\lambda$ we have that for any $\lambda > 0$ the inequality

$$\text{mes}(\sup_N |A_3| > \lambda) \leq C \|f\|_1/\lambda$$

holds. Besides, it is trivial that the very same also holds for $\sup_N |A_1|$. What about $\sup_N |A_2|$? This is a more complex task, but not very. We have for A_2 :

$$\begin{aligned} A_2 &= \frac{1}{\log N} \sum_{l=1}^{L^2-1} \sum_{j=D(2^{b_{2l(l+1)K_0}})}^{D(2^{b_{2(l+1)K_0}})} \frac{S_{a(j)}f - V_{a(j)}f}{j} \\ &= \frac{1}{\log N} \sum_{l=1}^{L^2-1} \left(\sum_{j \in I_l^0} \frac{S_{a(j)}f - V_{a(j)}f}{j} + \sum_{j \in I_l^1} \frac{S_{a(j)}f - V_{a(j)}f}{j} \right) \\ &= \frac{L^2}{\log N} (T_L^0 f + T_L^1) \end{aligned}$$

and thus

$$|A_2| \leq C |T_L^0 f + T_L^1 f|.$$

This immediately gives

$$\begin{aligned} \sup_N |A_2| &\leq C \left(\sup_{L \in \mathbb{N}} |T_L^0 f - \tilde{T}_L^0 f| + \sup_{L \in \mathbb{N}} |T_L^1 f - \tilde{T}_L^1 f| \right. \\ &\quad \left. + \sup_{L \in \mathbb{N}} |\tilde{T}_L^0 f| + \sup_{L \in \mathbb{N}} |\tilde{T}_{L, K_0, i_0}^1 f| \right). \end{aligned}$$

Finally, Lemmas 3.4 and 3.5 imply that

$$\text{mes} \left(y \in T : \sup_N \left| \frac{1}{\log N} \sum_{j=1}^N \frac{S_{a(j)}f - V_{a(j)}f}{j} \right| > \lambda \right) \leq C \sqrt{\frac{\|f\|_1}{\lambda}}$$

for any positive λ . (Actually, the whole procedure is done in the case of $\lambda > \|f\|_1/(2\pi)$. The case $\lambda \leq \|f\|_1/(2\pi)$ requires no further consideration since the left-hand side of the inequality above is trivially not more than 2π .) That is, the maximal operator

$$T_*^a f := \sup_N |T_N^a f| := \sup_N \left| \frac{1}{\log N} \sum_{j=1}^N \frac{S_{a(j)}f - V_{a(j)}f}{j} \right|$$

is in a sense like a weak (L^1, L^1) type one. Furthermore, since for every trigonometric polynomial P it is self-evident that $\lim_{N \rightarrow \infty} T_N^a P = 0$ everywhere, using the standard density principle, it can be seen that $\lim_{N \rightarrow \infty} T_N^a f = 0$ almost everywhere for any integrable function f . Since for the sequence of the de la Vallée-Poussin means $(V_n f)$ we have the a.e. relation $\lim_{N \rightarrow \infty} V_N f = f$ for any integrable function, then the proof of Theorem 1.1 is complete. $\square$

Remark 3.6 Why cannot the methods of this paper be applied to verify the a.e. convergence of $\sigma_n^a f \rightarrow f$? We explain the obstruction informally; for clarity we sometimes use informal wording to convey the main idea, though not always with full technical accuracy.

Let $2^L < N \leq 2^{L+1}$. It is true that the sum $\sum_{j=1}^{2^L} (S_{a(j)} f - V_{a(j)} f) \xi_j / N$ can be broken down into the sum of partial sums $\sum_{j=2^{s+1}}^{2^{s+1}} (S_{a(j)} f - V_{a(j)} f) \xi_j / N$ that are orthogonal. These individual partial sums can (essentially) be replaced with terms of the form $2^{s-L} (S_{a(j_s)} f - V_{a(j_s)} f) \xi_{j_s}$. Where $(a(j_s))$ will be lacunary (as s varies) and thus Lemma 2.3 can be applied. However, we cannot handle the supremum taken according to L because there is no “weight” that would “compress” these terms.

This can be illustrated well by the fact that if the Lemma 3.4 were fulfilled, then the L^2 -norm of $\sup_L 2^{-L} |\sum_{j=2^{L-1}+1}^{2^L} (S_{a(j)} f - V_{a(j)} f) \xi_j|$ would also be estimable (case $s = L-1$) by $C \|f\|_1 \lambda$. With the method described in this article, it is certainly not possible to see if it is true at all.

Thus, we cannot make a statement similar to Lemma 3.4. Similarly, it is not possible to prove a statement corresponding to Lemma 3.2 in this case (operator σ_n^a). It is possible that the main conclusion of this article is not true in the case of means σ_n^a . (The method described in article [13] could serve as a starting point for constructing a possible counterexample.) In the case of Riesz logarithmic means, we take the arithmetic mean of orthogonal (and essentially normal) slices of Fourier partial sums corresponding to lacunary indices. This is a significant difference.

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