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A comparative analysis of Noyes-Field model for the non-linear Belousov–Zhabotinsky reaction using two reliable techniques

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ABSTRACT

Within the domain of nonlinear dynamics, the Belousov-Zhabotinsky reaction system has consistently captivated researchers, sustaining its position as a vibrant and dynamic area of exploration. As an enduring subject of study, the Belousov-Zhabotinsky system provides continuous opportunities to unravel the foundational tenets of nonlinear dynamics within intricate systems. In our quest to deepen comprehension of this complex system, we introduce an innovative methodology for addressing the time fractional Belousov-Zhabotinsky system. This novel approach leverages both the Natural Transform Iterative Method (NTIM) and the Optimal Homotopy Asymptotic Method (OHAM), aiming to contribute novel insights and methodologies to the ongoing discourse surrounding this intriguing and influential area of research. We obtained a series solution to test the accuracy of the suggested approach. The proposed technique has the advantage of requiring few calculations while producing high-precision results. To help you better understand, 2D and 3D graphical representations are provided to demonstrate the model's behavior and how changing the fractional order derivative in Caputo's sense and time affects the solutions.

1. Introduction

By permitting non integer order integrals and derivatives, fractional calculus (FC) is regarded as a subfield of mathematical analysis that generalizes classical calculus. Leibniz and L'Hopital investigated the notion of creating a derivative of non-integer order in the 17th century, which is when the concept of FC first emerged. However, Liouville, Riemann, and Grunwald's work in the 19th century, when they separately developed fractional derivatives and integrals [1–3], marked the beginning of the contemporary development of FC. The analysis of stochastic and anomalous diffusion processes, for which the standard calculus tools are insufficient, also greatly benefits from FC. Researchers have investigated novel uses and properties of fractional derivatives and integrals as a result of the increased attention that FC has received in recent years. Numerous scientific disciplines and engineering challenges now have access to FC as a valuable tool because to the advancement of numerical methods for FC, which have also created new opportunities for the solution of fractional differential equations (FDEs) [4–6].

The ability of the fractional system of PDEs to model complicated phenomena that cannot be explained by the traditional integer-order models has attracted significant interest in recent years. Numerous disciplines, including geology, biology, economics, physics, and engineering, use the fractional system of PDEs. It offers a potent tool for the modeling and examination of intricate systems with long-range dependencies, memories, and fractal behavior [7–11]. The definition, characteristics, and applications of the fractional system of PDEs are covered in this paper. We also go through the numerical techniques utilized to resolve these equations as well as some unsolved problems in this area [12,13].

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The goal of this study is to use an effective analytical strategy to deal with nonlinear differential equations of any order (ODEs). In doing so, the usage of FDEs can increase the accuracy of analysis in related domains. Many strategies have been developed to deal with this issue; Adomian decomposition [14] is one of the techniques used, variational iteration method [15], the NTDM [16], the iterative transformation technique [17], the fractional natural decomposition method [18], the homotopy perturbation method (HPM) [19], and so on.

A class of chemical reactions known as the Belousov-Zhabotinsky (BZ) reaction oscillates. In these reactions, different reductants typically organic ones are catalyzed by transition metal ions to be oxidized by bromic acid in an acidic water solution. BZ reactions are often homogenous. On a very suitable human time scale of dozens of seconds and space scale of several millimeters, the BZ response allows the observation of the formation of complex patterns in time and space with the naked eye. The BZ reaction allows for the study of chemical waves and patterns without the need for continuous reactant replenishment because it can produce up to several thousand oscillatory cycles in a closed system. The Belousov-Zhabotinsky reaction system has long been a topic of interest for researchers in nonlinear dynamics. Research on the Belousov-Zhabotinsky system is ongoing, providing insights into nonlinear dynamics principles in complex systems. The time fractional Belousov-Zhabotinsky system exhibits a significant application in the realm of bipolar cubic fuzzy graphs, presenting opportunities for chemical applications as highlighted in [20,21]. We take into consideration the Belousov-Zhabotinsky (B-Z) nonlinear oscillatory system with Caputo fractional time derivative. The B-Z family of oscillating chemical processes is particularly interesting because it displays both spatial traveling concentration waves and temporal oscillations, both of which are accompanied by vivid color changes [22]. In a closed system, this reaction has the potential to generate up to several thousand oscillatory cycles, making it possible to analyze chemical waves and patterns without the requirement for constant reactant replenishment [23]. Samir A. El-Tantawy et al. [24] investigated the fractional nonlinear BZ system that describes flow motion in a pipe using the Laplace residual power series approach and a new iterative method. Alsallami et al. [25] also take the B-Z system and the solutions by Caputo and Atangana-Baleanu Caputo fractional derivatives with the double Laplace method.

$$\begin{aligned} D_t^\eta v &= \xi_1 v_{\theta,\theta} + \beta \sigma \psi + v - v^2 - \sigma v \psi, \\ D_t^\eta \psi &= \xi_2 \psi_{\theta,\theta} + \gamma \psi - \lambda v \psi. \end{aligned} \quad (1)$$

Where $0 < \eta \leq 1$ in the Caputo sense and $0 < t < 1$. Studying the numerical solutions of equation (1) with the diffusing constants $\xi_1 = \xi_2 = 1$ is the driving force behind this effort. We take into account Eq. (1) in two separate situations.

- The first scenario is when $\gamma = \beta = 0$.
- The second scenario is when $\gamma = \lambda$ and $\beta = 1$.

Moreover, to visually investigate the effects of the fractional order η on the estimated answers. The two methods recommended for this study's purposes are the natural transform iterative method (NTIM) and the optimum homotopy asymptotic approach (OHAM). The Daftardar-Jafari and Adomian polynomials are swapped out in the natural transform iterative technique (NTIM) [26], which is a variant of the natural transform decomposition method (NTDM). The New Iterative Method (NIM) and natural transformation are combined in the suggested approach. The Optimal Homotopy Asymptotic Method (OHAM) [27] is introduced to tackle second-order boundary value problems in linear and nonlinear systems. In contrast to alternative approaches that could need very massive calculations, these two methods produce approximate and precise solutions to challenging nonlinear problems with great accuracy and minimal processing. The NTIM method that has been used for the solution of BZ equations is the combination of NIM and natural transform method. We observed that the proposed method was easy to implement and give a suitable approximate solution for fractional linear and non-linear differential equations. Nawaz et al. [28] have recently utilized the suggested methods to solve differential equations of noninteger order. Nasir Ali et al. [29] have used the same method for the solution of biological population model (FBPM).

The remaining sections are organized as follows: In Section 2, we present a few key definitions and notations for fractional calculus and natural transform utilized in the current framework. The overall concept of the suggested approaches is briefly addressed in Section 3. Section 4 is dedicated to the application of the suggested approaches to two examples of the time-fractional Belousov-Zhabotinsky system of equations. Numerical experiments and remarks are presented in Section 5. Finally, the conclusion is found in Section 6.

2. Preliminaries

In this section, we discuss some crucial ideas that are essential to the current framework: Natural transform and the fundamentals of fractional calculus.

Definition 1. The following formula represents a function $\varsigma \in C_n, n \geq -1$ Riemann-Liouville fractional integral of order $\alpha (\alpha \geq 0)$

$$D_\tau^\alpha \varsigma(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau - \kappa)^{\alpha-1} \varsigma(\kappa) d\kappa, \quad (2)$$

where Γ represents the traditional gamma function and ${}^0_0\varsigma(\tau) = \varsigma(\tau)$. As an example,

$$D_\tau^\alpha \tau^\nu = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\nu)} \tau^{\alpha+\nu}. \quad (3)$$

Definition 2. The caputo fractional derivative of $\varsigma(\tau)$ (expressed by $D_\tau^\alpha \varsigma(\tau)$) is defined as

$$D_\tau^\alpha \varsigma(\tau) = \begin{cases} \varsigma^{(\phi)}(\tau), & \phi = \alpha; \\ \mathbf{H}_\tau^{\phi-\alpha} \varsigma^{(\phi)}(\tau), & \phi - 1 < \alpha < \phi, \end{cases} \quad (4)$$

where

$$\mathbf{H}_{\tau}^{\phi-\alpha} \zeta^{\phi}(\tau) = \frac{1}{\Gamma(\phi-\alpha)} \int_0^{\tau} (\tau-\kappa)^{\phi-\alpha-1} \zeta^{\phi}(\kappa) d\kappa, \quad \alpha, \tau > 0. \quad (5)$$

Definition 3. The natural transform of $\psi(t)$ is defined as

$$N^+[\psi(t)] = R(\delta, u) = \frac{1}{u} \int_0^{\infty} e^{-\frac{\delta t}{u}} (\psi(t)) dt, \quad (6)$$

where $\delta, u > 0$ are the transform variables.

Definition 4. If $R(\delta, u)$ is $\psi(t)$ natural transform, then $\psi(t)$ inverse natural transform is described as follows:

$$N^-[R(\delta, u)] = \psi(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{\delta t}{u}} (R(\delta, u)) d\delta. \quad (7)$$

The integral is represented as $\delta = a + bi$ in the complex plane along $\delta = c$, where $c \in R$.

Definition 5. Given is the n th derivative of the natural transform of $\psi(t)$

$$N^+[\psi^n(t)] = R_n(\delta, u) = \frac{\delta^n}{u^n} R(\delta, u) - \sum_{k=0}^{n-1} \frac{\delta^{n-k-1}}{u^{n-k}} [\psi^n(0)], \quad n \geq 1. \quad (8)$$

3. Analysis of suggested techniques

Take into account the time-fractional Belousov-Zhabotinsky (TFB-Z) Eq. (1) with the concentration's diffusing constants, $\xi_1 = \xi_2 = 1$.

$$\begin{aligned} D_t^\eta v &= \xi_1 \frac{\partial^2}{\partial \vartheta^2} v + \beta \sigma \psi + v - v^2 - \sigma v \psi, \quad 0 < \eta \leq 1, \\ D_t^\eta \psi &= \xi_2 \frac{\partial^2}{\partial \vartheta^2} \psi + \gamma \psi - \lambda v \psi, \end{aligned} \quad (9)$$

subject to initial conditions

$$\begin{aligned} v(\vartheta, 0) &= g_1(\vartheta), \\ \psi(\vartheta, 0) &= g_2(\vartheta), \quad \vartheta \in \mathbb{R}. \end{aligned} \quad (10)$$

3.1. OHAM

Here, we start by defining OHAM's central idea. Please consider the following ODE.

$$\varpi(v(\vartheta)) + \tau(v(\vartheta)) + \chi(\vartheta) = 0, \quad \vartheta \in \Omega, \quad (11)$$

$$\mathbf{B}\left(v, \frac{\partial v}{\partial \vartheta}\right) = 0, \quad (12)$$

where a linear operator is represented by ϖ and a non-linear operator is represented by τ . $\mathbf{B}$ stands for boundary conditions, $v(\vartheta)$ for unknown function, and $\chi(\vartheta)$ for source function, where ϑ and Ω , respectively, signify the geographic variable and the problem domain.

One can create the ideal homotopy by adhering to the fundamental principle of OHAM

$$\begin{aligned} \psi(\vartheta, \rho) &: \Omega \times [0, 1] \rightarrow \mathfrak{R}, \\ (1-\rho)\left(\varpi(v(\vartheta, \rho)) + \chi(\vartheta)\right) &= \hbar(\rho)\left(\varpi(v(\vartheta)) + \tau(v(\vartheta)) + \chi(\vartheta)\right), \\ \mathbf{B}\left(v(\vartheta, \rho), \frac{\partial v(\vartheta, \rho)}{\partial \vartheta}\right) &= 0, \end{aligned} \quad (13)$$

where an embedding parameter is represented by the letter " ρ ". $\hbar(\rho)$ is a non-zero Auxiliary function for $\rho \neq 0$, $\hbar(0) = 0$. The term "optimal homotopy equation" refers to equation (13). Clearly, we have

$$\rho = 0 \Rightarrow \hbar(v(\vartheta, 0)) = \varpi(v(\vartheta, 0)) + \chi(\vartheta) = 0, \quad (14)$$

$$\rho = 1 \Rightarrow \hbar(v(\vartheta, 1)) = \hbar(1)\varpi(v(\vartheta, 1)) + \tau(v(\vartheta, 1)) + \chi(\vartheta) = 0. \quad (15)$$

It is evident that for $\rho = 0$ and 1 respectively, $v(\vartheta, 0) = v_0(\vartheta)$ and $v(\vartheta, 1) = v_1(\vartheta)$ are true. As ρ varies from 0 to 1, the solution $v(\vartheta, \rho) = v(\vartheta)$ approaches from $v_0(\vartheta)$ to $v(\vartheta)$, where $v_0(\vartheta)$ is found from (11) for $\rho = 0$:

$$\varpi(v_0(\vartheta)) + \chi(\vartheta) = 0, \quad \mathbf{B}\left(v_0, \frac{\partial v_0}{\partial \vartheta}\right) = 0. \quad (16)$$

Then pick an auxiliary function $\hbar(\rho)$ with the structure shown below:

$$\hbar(\rho) = \rho C_1 + \rho^2 C_2 + \dots. \quad (17)$$

It is possible to determine constants $C_1, C_2 \dots$ later. By extending $\varpi(\vartheta, \rho, C_i)$ in Taylor's series around ρ , we obtain the following approximation:

$$\psi(\vartheta, \rho, C_i) = v_0(\vartheta) + \sum v_k(\vartheta, C_i)\rho^k, \quad i = 1, 2, \dots \quad (18)$$

By inserting (18) into (13) and evaluating the coefficients of similar powers of ρ , we derive the zeroth-order equation offered by (16), the first and second order problems provided by (19)(20), respectively, and the general governing equation for $v_k(\vartheta)$ provided by (21).

$$\begin{aligned} \varpi(v_1(\vartheta)) &= C_1 \tau(v_0(\vartheta)), \\ \mathbf{B}\left(v_1, \frac{\partial v_1}{\partial \vartheta}\right) &= 0, \end{aligned} \quad (19)$$

$$\begin{aligned} \varpi(v_2(\vartheta)) - \varpi(v_1(\vartheta)) &= C_2 \tau_0(v_0(\vartheta)) + C_1 [\varpi(v_1(\vartheta, \rho)) + \tau_1(v_0(\vartheta, \rho)v_1(\vartheta, \rho)) + \chi(\vartheta)], \\ \mathbf{B}\left(v_2, \frac{\partial v_2}{\partial \vartheta}\right) &= 0, \end{aligned} \quad (20)$$

$$\begin{aligned} \varpi(v_k(\vartheta)) - \varpi(v_{k-1}(\vartheta)) &= C_k \tau_0(v_0(\vartheta)) + \sum_{i=1}^{k-1} C_i [\varpi(v_{k-i}(\vartheta, \rho)) + \tau_{k-i}(v_0(\vartheta, \rho), v_1(\vartheta, \rho) + \dots \\ &\quad + v_{k-i}(\vartheta, \rho))] \mathbf{B}\left(v_k, \frac{\partial v_k}{\partial \vartheta}\right) = 0, \quad k = 2, 3, \dots, \end{aligned} \quad (21)$$

where $\tau_{k-i}(v_0(\vartheta, \rho), v_1(\vartheta, \rho), v_2(\vartheta, \rho), \dots, v_{k-i}(\vartheta, \rho))$ is the coefficient of ρ^{k-i} in the expansion of $\tau(v(\vartheta))$ about the embedding parameter ρ :

$$\tau(v(\vartheta, \rho, C_i)) = \tau_0(v_0(\vartheta, \rho)) + \sum_{k \geq 1} \tau_k(v_0, v_1, \dots, v_k)\rho^k. \quad (22)$$

For the case when $k \geq 0$, a set of linear equations v_k with linear boundary conditions may be solved rapidly. The auxiliary constants $C_1, C_2, \dots$ affect how quickly the series in Eq. (18) converge. If it converges to $\rho = 1$, then one has:

$$\tilde{v}(\vartheta, C_i) = v_0(\vartheta) + \sum_{k \geq 1} v_k(\vartheta, C_i). \quad (23)$$

The equation for residual is given below when Eq. (23) is substituted into Eq. (11).

$$\mathfrak{R}(\vartheta, C_i) = \varpi(\tilde{v}(\vartheta, C_i)) + \tau(\tilde{v}(\vartheta, C_i)) + \chi(\vartheta). \quad (24)$$

We were able to arrive at the precise solution $v(\vartheta, C_i)$ when $\mathfrak{R}(\vartheta, C_i) = 0$, which denotes that the residual is zero. However, we can minimize as indicated below if $\mathfrak{R}(\vartheta, C_i) \neq 0$, which denotes that the residual is not zero.

$$\mathbf{J} = \int_{\alpha}^{\beta} \mathfrak{R}^2(\vartheta, C_i) d\vartheta. \quad (25)$$

It may be feasible to determine the unknown constants $C_1, C_2, \dots$ and from the conditions where α and β are constants, depending on how the situation is viewed.

$$\frac{\partial \mathbf{J}}{\partial C_i} = 0, \quad i = 1, 2, 3, \dots \quad (26)$$

Once we know the values of these constants, we can get an approximation of the solution from Eq. (23).

3.2. NTIM

Consider FDE of the form

$$D_t^\eta(v(\vartheta, t)) = f(\vartheta, t) + \varpi(v(\vartheta, t)) + \tau(v(\vartheta, t)), \quad \vartheta, t \geq 0, m-1 \leq \beta \leq m, \quad (27)$$

where D_t^η represents caputo fractional derivative of order η , $m \in \mathbb{N}$ and $(\vartheta = y_1, y_2, \dots, y_m)$. The linear and non-linear functions are represented by ϖ and τ , respectively. The known function $f(\vartheta, t)$. The initial condition is given as

$$v(\vartheta, 0) = \varphi(\vartheta). \quad (28)$$

Equation (27) has been transformed naturally using the natural transform.

$$\mathbb{N}^+ \left[D_t^\eta(v(\vartheta, t)) \right] = \mathbb{N}^+ \left[f(\vartheta, t) \right] + \mathbb{N}^+ \left[\varpi(v(\vartheta, t)) + \tau(v(\vartheta, t)) \right]. \quad (29)$$

One way to represent Equation (29), which makes use of the natural transform differentiation feature, is

$$\frac{s^\eta}{v^\eta} \mathbb{N}^+ \left[v(\vartheta, t) \right] - \frac{s^{\eta-1}}{v^\eta} v(\vartheta, 0) = \mathbb{N}^+ \left[f(\vartheta, t) \right] + \mathbb{N}^+ \left[\varpi(v(\vartheta, t)) + \tau(v(\vartheta, t)) \right]. \quad (30)$$

By arranging equation (30), we have

$$\mathbb{N}^+ \left[v(\vartheta, t) \right] = \frac{\varphi(\vartheta)}{s} + \frac{v^\eta}{s^\eta} \left(\mathbb{N}^+ [f(\vartheta, t)] \right) + \frac{v^\eta}{s^\eta} \left(\mathbb{N}^+ \left[\varpi(v(\vartheta, t)) + \tau(v(\vartheta, t)) \right] \right). \quad (31)$$

When calculating the NTIM solution, $v(\vartheta, t)$ is extended as

$$v(\vartheta, t) = \sum_{i=0}^{\infty} v_i(\vartheta, t), \quad (32)$$

and the non-linear term $\tau(v(\vartheta, t))$ is defined as

$$\tau\left(\sum_{m=0}^{\infty} v_m(\vartheta, t)\right) = \tau\left(v_0(\vartheta, t)\right) + \sum_{m=1}^{\infty} \left\{ \tau\left(\sum_{j=0}^i v_j(\vartheta, t)\right) - \tau\left(\sum_{j=0}^{m-1} v_j(\vartheta, t)\right) \right\}. \quad (33)$$

Plugging Equation (32) and Equation (33) in Equation (31), we obtain

$$\mathbb{N}^+ \left[\sum_{i=1}^{\infty} v_i \right] = \frac{\varphi(\vartheta)}{s} + \frac{v^\eta}{s^\eta} \left(\mathbb{N}^+[f(\vartheta, t)] \right) + \frac{v^\eta}{s^\eta} \left[\sum_{m=0}^{\infty} \varpi(v_m) + \tau(v_0) + \sum_{m=1}^{\infty} \left\{ \tau\left(\sum_{j=0}^m v_j\right) - \tau\left(\sum_{j=0}^{m-1} v_j\right) \right\} \right]. \quad (34)$$

Making use of the recursive relation

$$\begin{aligned} \mathbb{N}^+ [v_0(\vartheta, t)] &= \frac{\varphi(\vartheta)}{s} + \frac{v^\eta}{s^\eta} \left(\mathbb{N}^+[f(\vartheta, t)] \right), \\ \mathbb{N}^+ [v_1(\vartheta, t)] &= \frac{v^\eta}{s^\eta} \left[\varpi(v_0(\vartheta, t)) + \tau(v_0(\vartheta, t)) \right], \\ \mathbb{N}^+ [v_2(\vartheta, t)] &= \frac{v^\eta}{s^\eta} \left[\varpi(v_1(\vartheta, t)) + \tau(v_0(\vartheta, t) + v_1(\vartheta, t)) - \tau(v_0(\vartheta, t)) \right], \\ &\vdots \\ \mathbb{N}^+ [v_{i+1}(\vartheta, t)] &= \frac{v^\eta}{s^\eta} \left[\varpi(v_i(\vartheta, t)) + \tau(v_0(\vartheta, t) + v_1(\vartheta, t) + \dots + v_i(\vartheta, t)) \right. \\ &\quad \left. - \tau(v_0(\vartheta, t) + v_1(\vartheta, t) + \dots + v_i(\vartheta, t)) \right], i \geq 0. \end{aligned} \quad (35)$$

The inverse natural transform of equation (35), which gives us

$$\begin{aligned} v_0(\vartheta, t) &= \mathbb{N}^- \left[\frac{\varphi(\vartheta)}{s} + \frac{v^\eta}{s^\eta} \left(\mathbb{N}^+[f(\vartheta, t)] \right) \right], \\ v_1(\vartheta, t) &= \mathbb{N}^- \left[\frac{v^\eta}{s^\eta} \left[\varpi(v_0(\vartheta, t)) + \tau(v_0(\vartheta, t)) \right] \right], \\ v_2(\vartheta, t) &= \mathbb{N}^- \left[\frac{v^\eta}{s^\eta} \left[\varpi(v_1(\vartheta, t)) + \tau(v_0(\vartheta, t) + v_1(\vartheta, t)) - \tau(v_0(\vartheta, t)) \right] \right], \\ &\vdots \\ v_{i+1}(\vartheta, t) &= \mathbb{N}^- \left[\frac{v^\eta}{s^\eta} \left[\varpi(v_i(\vartheta, t)) + \tau(v_0(\vartheta, t) + v_1(\vartheta, t) + \dots + v_i(\vartheta, t)) \right. \right. \\ &\quad \left. \left. - \tau(v_0(\vartheta, t) + v_1(\vartheta, t) + \dots + v_i(\vartheta, t)) \right] \right], i \geq 0. \end{aligned} \quad (36)$$

The approximate answers to equations (27) and (29) using NTIM are provided as after adding all the parts.

$$v(\vartheta, t) = v_0(\vartheta, t) + v_1(\vartheta, t) + \dots + v_{m-1}(\vartheta, t), \quad m \in \mathbb{N}. \quad (37)$$

According to Bhalekar and Daftardar-Gejji [30], convergence of NTIM is equivalent to convergence of NIM.

4. Solution for TFB-z system of equations

Here, two time-fractional Belousov–Zhabotinsky scenarios are used to examine the effectiveness of the two suggested techniques.

4.1. Scenario 1

Take into consideration the nonlinear TFB-Z system where $\gamma = \beta = 0$,

$$D_t^\eta v(\vartheta, t) = \frac{\partial^2}{\partial \vartheta^2} v(\vartheta, t) + v(\vartheta, t) - (v(\vartheta, t))^2 - \sigma v(\vartheta, t) \psi(\vartheta, t), \quad 0 \leq \eta \leq 1. \quad (38)$$

$$D_t^\eta \psi(\vartheta, t) = \frac{\partial^2}{\partial \vartheta^2} \psi(\vartheta, t) - \lambda v(\vartheta, t) \psi(\vartheta, t), \quad (39)$$

along with the initial condition

$$v(\vartheta, 0) = \frac{1}{\left(e^{\sqrt{\frac{\lambda}{6}} \vartheta} + 1 \right)^2}, \quad (40)$$

$$\psi(\vartheta, 0) = \frac{(1 - \lambda) e^{\sqrt{\frac{\lambda}{6}} \vartheta} (e^{\sqrt{\frac{\lambda}{6}} \vartheta} + 2)}{\sigma \left(e^{\sqrt{\frac{\lambda}{6}} \vartheta} + 1 \right)^2}. \quad (41)$$

For $\eta = 1$, the following is the precise solution to equation (39)

$$v(\vartheta, t) = \frac{e^{\frac{5\lambda}{3}t}}{\left(e\sqrt{\frac{\lambda}{6}\vartheta} + e^{\frac{5\lambda}{6}t}\right)^2}, \quad (42)$$

$$\psi(\vartheta, t) = \frac{(1-\lambda)e\sqrt{\frac{\lambda}{6}\vartheta}(e\sqrt{\frac{\lambda}{6}\vartheta} + 2e^{\frac{5\lambda}{6}t})}{\sigma\left(e\sqrt{\frac{\lambda}{6}\vartheta} + e^{\frac{5\lambda}{6}t}\right)^2}. \quad (43)$$

Here σ and $\lambda \neq 1$ are positive parameters.

4.1.1. NTIM solution

By applying RL-Integral on equation (38) and (39), we obtain the following form

$$v(\vartheta, t) = \frac{v(\vartheta, 0)}{s} + \mathbb{N}^- \left[\frac{u^\eta}{s^\eta} \mathbb{N}^+ \left[\frac{\partial^2}{\partial \vartheta^2} v(\vartheta, t) + v(\vartheta, t) - (v(\vartheta, t))^2 - \sigma v(\vartheta, t) \psi(\vartheta, t) \right] \right], \quad (44)$$

$$\psi(\vartheta, t) = \frac{\psi(\vartheta, 0)}{s} + \mathbb{N}^- \left[\frac{u^\eta}{s^\eta} \mathbb{N}^+ \left[\frac{\partial^2}{\partial \vartheta^2} \psi(\vartheta, t) - \lambda v(\vartheta, t) \psi(\vartheta, t) \right] \right]. \quad (45)$$

Using the NTIM formulation as discussed in the section 3, we get zero component solution

$$v_0(\vartheta, t) = \frac{1}{\left(e\sqrt{\frac{\lambda}{6}\vartheta} + 1\right)^2}, \quad (46)$$

$$\psi_0(\vartheta, t) = \frac{(1-\lambda)e\sqrt{\frac{\lambda}{6}\vartheta}(e\sqrt{\frac{\lambda}{6}\vartheta} + 2)}{\sigma\left(e\sqrt{\frac{\lambda}{6}\vartheta} + 1\right)^2}, \quad (47)$$

first component solution

$$v_1(\vartheta, t) = \frac{5\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{3\Gamma(\eta+1)\left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^3}, \quad (48)$$

$$\psi_1(\vartheta, t) = \frac{5(\lambda-1)\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{3\sigma\Gamma(\eta+1)\left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^3}, \quad (49)$$

second component solution

$$v_2(\vartheta, t) = \frac{25\lambda^2 e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} \left(\Gamma(\eta+1)^2 t^{2\eta} \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^2 \left(2e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} - 1 \right) - 2\lambda\Gamma(2\eta+1)t^{3\eta} e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} \right)}{18\Gamma(\eta+1)^2\Gamma(2\eta+1)\Gamma(3\eta+1)\left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^6}, \quad (50)$$

$$\psi_2(\vartheta, t) = \frac{25(\lambda-1)\lambda^2 e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} \left(\Gamma(\eta+1)^2 t^{2\eta} \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^2 \left(2e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} - 1 \right) - 2\lambda\Gamma(2\eta+1)t^{3\eta} e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} \right)}{18\sigma\Gamma(\eta+1)^2\Gamma(2\eta+1)\Gamma(3\eta+1)\left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^6}. \quad (51)$$

Now adding all the components

$$v(\vartheta, t) = v_0(\vartheta, t) + v_1(\vartheta, t) + v_2(\vartheta, t), \quad (52)$$

and

$$\psi(\vartheta, t) = \psi_0(\vartheta, t) + \psi_1(\vartheta, t) + \psi_2(\vartheta, t), \quad (53)$$

we get the solution upto second component

$$v(\vartheta, t) = \frac{5\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}}}{3\Gamma(\eta+1)\left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1\right)^3} + \frac{25\lambda^2 t^{2\eta} e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} \left(\Gamma(\eta+1)^2 \left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1 \right)^2 \left(2e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} - 1 \right) - 2\lambda\Gamma(2\eta+1)t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} \right)}{18\Gamma(\eta+1)^2\Gamma(2\eta+1)\Gamma(3\beta+1)\left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1\right)^6}$$

$$+ \frac{1}{\left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1\right)^2},$$

$$\psi(\vartheta, t) = \frac{25(\lambda-1)\lambda^2 e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} \left(\Gamma(\eta+1)^2 t^{2\eta} \left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1 \right)^2 \left(2e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} - 1 \right) - 2\lambda\Gamma(2\eta+1)t^{3\eta} e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} \right)}{18\sigma\Gamma(\eta+1)^2\Gamma(2\eta+1)\Gamma(3\eta+1)\left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1\right)^6}$$

$$+ \frac{5(\lambda-1)\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}}}{3\sigma\Gamma(\eta+1)\left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1\right)^3} + \frac{(1-\lambda)e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} \left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 2 \right)}{\sigma \left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1 \right)^2}.$$

4.1.2. OHAM solution

Here we have solved equation (38) and (39) by utilizing OHAM we get the following components as follows:

Zero component solution

$$v_0(\vartheta, t) = \frac{1}{\left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + 1\right)^2}, \quad (56)$$

$$\psi_0(\vartheta, t) = \frac{(1-\lambda)e^{\sqrt{\frac{\lambda}{6}}\vartheta}(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + 2)}{\sigma \left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + 1\right)^2}, \quad (57)$$

first component solution

$$v_1(\vartheta, t) = -\frac{5c_1\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}}}{3\beta\Gamma(\eta)\left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1\right)^3}, \quad (58)$$

$$\psi_1(\vartheta, t) = -\frac{5c_4(\lambda-1)\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}}}{3\eta\sigma\Gamma(\eta)\left(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1\right)^3}, \quad (59)$$

second component solution

$$v_2(\vartheta, t) = -\frac{1}{18\eta^2\Gamma(\eta)^2(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1)^5} \left(5\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} (c_1^2(6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1)^2 - t^\eta(\lambda + 5\lambda e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} \right.$$

$$\left. + 10\lambda e^{\sqrt{\frac{\lambda}{3}}\vartheta} - 6)) + 6c_1(c_4(\lambda-1)t^\beta + \eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1)^2) + 6\beta c_2\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1)^2 \right), \quad (60)$$

$$\psi_2(\vartheta, t) = -\frac{1}{18\eta^2\sigma\Gamma(\eta)^2(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1)^5} \left(5(\lambda-1)\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} (c_4^2(6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1)^2 - \lambda t^\eta(4e^{\sqrt{\frac{\lambda}{3}}\vartheta} \right.$$

$$\left. - 7e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} - 5)) + 6c_4(\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1)^2 - c_1\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}}(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 2)) \right.$$

$$\left. + 6\eta c_5\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}}{\sqrt{6}} + 1)^2 \right).$$

By adding all the components

$$v(\vartheta, t) = v_0(\vartheta, t) + v_1(\vartheta, t) + v_2(\vartheta, t), \quad (62)$$

and

$$\psi(\vartheta, t) = \psi_0(\vartheta, t) + \psi_1(\vartheta, t) + \psi_2(\vartheta, t). \quad (63)$$

Their solutions are

$$v(\vartheta, t) = \frac{1}{(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2} - \frac{5c_1\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{3\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^3} - \frac{1}{18\eta^2\Gamma(\eta)^2(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^5} \left(5\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} (c_1^2(6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2 - t^\eta(\lambda + 5\lambda e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 10\lambda e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} - 6)) + 6c_1(c_4(\lambda - 1)t^\eta + \eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2) + 6\eta c_2\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2) \right), \quad (64)$$

$$\psi(\vartheta, t) = -\frac{1}{18\eta^2\sigma\Gamma(\eta)^2(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^5} \left((\lambda - 1)e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} (-5c_4^2\lambda t^\eta(\lambda t^\eta(4e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} - 7e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} - 5) - 6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2) - 30c_4\lambda t^\eta(c_1\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} (e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 2) - 2\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2) + 6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2(5c_5\lambda t^\eta + 3\eta\Gamma(\eta)(e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} + 3e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 2))) \right). \quad (65)$$

The values of c_1, c_2, c_4, c_5 have been produced for $v(\vartheta, t)$ using the least square approach

c_1	c_2	c_4	c_5
-0.8929183833553361	0.04371588774059267	-0.09993710726412326	1.5328084611240569

By putting all the values of c_i in eq. (64) we get the following equation

$$v(\vartheta, t) = \frac{1}{\eta^2\Gamma(\eta)^2(1 + e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}})^5} \left(\lambda t^{2\eta} e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} (\lambda(0.072747 + 2.21473e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} + 1.10737e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}) - 1.18011) + 1.5747\eta\lambda\Gamma(\eta)t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} (1 + e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}})^2 + \eta^2\Gamma(\eta)^2(1 + e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}})^3 \right). \quad (66)$$

We produced the values of c_1, c_2, c_4, c_5 for $\psi(\vartheta, t)$ using the same methodology

c_1	c_2	c_4	c_5
-0.061772090444812945	0.08440063881769243	0.5619624748829146	-2.43881245454972

By inserting all the values of c_i in eq. (65) we get the following equation

$$\psi(\vartheta, t) = -\frac{1}{\eta^2\sigma\Gamma(\eta)^2(1 + e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}})^5} \left((\lambda - 1)e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} (\lambda^2 t^{2\eta}(0.438614 - 0.293035e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} + 0.729771e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}) - 1.66514\eta\lambda\Gamma(\eta)t^\eta(1 + e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}})^2 + \eta^2\Gamma(\eta)^2(1 + e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}})^3(2 + e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}})) \right). \quad (67)$$

4.2. Scenario 2

Take into account the nonlinear TFB-Z system at $\gamma = \lambda$ and $\beta = 1$,

$$D_t^\eta v(\vartheta, t) = \frac{\partial^2}{\partial \vartheta^2} v(\vartheta, t) + \sigma \psi(\vartheta, t) + v(\vartheta, t) - (v(\vartheta, t))^2 - \sigma v(\vartheta, t) \psi(\vartheta, t), \quad 0 \leq \eta \leq 1, \quad (68)$$

$$D_t^\eta \psi(\vartheta, t) = \frac{\partial^2}{\partial \vartheta^2} \psi(\vartheta, t) + \lambda \psi(\vartheta, t) - \lambda v(\vartheta, t) \psi(\vartheta, t), \quad (69)$$

along with the initial condition

$$v(\vartheta, 0) = \frac{1}{\left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + 1\right)^2}, \quad (70)$$

$$\psi(\vartheta, 0) = \frac{(\lambda - 1)}{\sigma \left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + 1\right)^2}. \quad (71)$$

The exact solution of equation (68) and (69) for $\eta = 1$ is given as follows

$$v(\vartheta, t) = \frac{e^{\frac{5\lambda}{3}t}}{\left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + e^{\frac{5\lambda}{6}t}\right)^2}, \quad (72)$$

$$\psi(\vartheta, t) = \frac{(\lambda - 1)e^{\frac{5\lambda}{3}t}}{\sigma\left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + e^{\frac{5\lambda}{6}t}\right)^2}. \quad (73)$$

Here $\lambda \neq 1$ are positive parameters.

4.2.1. NTIM solution

We derive the following equivalent form by using RL-Integral on both sides of equations (68) and (69)

$$v(\vartheta, t) = \frac{v(\vartheta, 0)}{s} + \mathbb{N}^- \left[\frac{u^\eta}{s^\eta} \mathbb{N}^+ \left[\frac{\partial^2}{\partial \vartheta^2} v(\vartheta, t) + \sigma \psi(\vartheta, t) + v(\vartheta, t) - (v(\vartheta, t))^2 - \sigma v(\vartheta, t) \psi(\vartheta, t) \right] \right], \quad (74)$$

$$\psi(\vartheta, t) = \frac{\psi(\vartheta, 0)}{s} + \mathbb{N}^- \left[\frac{u^\eta}{s^\eta} \mathbb{N}^+ \left[\frac{\partial^2}{\partial \vartheta^2} \psi(\vartheta, t) + \lambda \psi(\vartheta, t) - \lambda v(\vartheta, t) \psi(\vartheta, t) \right] \right]. \quad (75)$$

We get using the NTIM formulation mentioned in Section 3 the following:

Zero component solution

$$v_0(\vartheta, t) = \frac{1}{\left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + 1\right)^2}, \quad (76)$$

$$\psi_0(\vartheta, t) = \frac{\lambda - 1}{\sigma \left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + 1\right)^2}, \quad (77)$$

first component solution

$$v_1(\vartheta, t) = \frac{5\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{3\Gamma(\eta + 1) \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^3}, \quad (78)$$

$$\psi_1(\vartheta, t) = \frac{5(\lambda - 1)\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{3\sigma\Gamma(\eta + 1) \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^3}, \quad (79)$$

second component solution

$$v_2(\vartheta, t) = \frac{\left(25\lambda^2 e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}\right) \left(\frac{\Gamma(\eta + 1)^2 t^{2\eta} \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^2 \left(2e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} - 1\right)}{\Gamma(2\eta + 1)} - \frac{2\lambda\Gamma(2\eta + 1)t^{3\eta} e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{\Gamma(3\eta + 1)} \right)}{18\Gamma(\eta + 1)^2 \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^6}, \quad (80)$$

$$\psi_2(\vartheta, t) = \frac{\left(25(\lambda - 1)\lambda^2 e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}\right) \left(\frac{\Gamma(\eta + 1)^2 t^{2\eta} \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^2 \left(2e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} - 1\right)}{\Gamma(2\eta + 1)} - \frac{2\lambda\Gamma(2\eta + 1)t^{3\eta} e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{\Gamma(3\eta + 1)} \right)}{18\sigma\Gamma(\eta + 1)^2 \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1\right)^6}. \quad (81)$$

Now adding all the components

$$v(\vartheta, t) = v_0(\vartheta, t) + v_1(\vartheta, t) + v_2(\vartheta, t), \quad (82)$$

and

$$\psi(\vartheta, t) = \psi_0(\vartheta, t) + \psi_1(\vartheta, t) + \psi_2(\vartheta, t). \quad (83)$$

Their solutions are

$$v(\vartheta, t) = \frac{25\lambda^2 e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} \left(\frac{\Gamma(\eta+1)^2 t^{2\eta} \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^2 \left(2e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} - 1 \right)}{\Gamma(2\eta+1)} - \frac{2\lambda\Gamma(2\eta+1)t^{3\eta} e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{\Gamma(3\eta+1)} \right)}{18\Gamma(\eta+1)^2 \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^6} \quad (84)$$

$$+ \frac{5\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{3\Gamma(\eta+1) \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^3} + \frac{1}{\left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^2},$$

$$\psi(\vartheta, t) = \frac{25(\lambda-1)\lambda^2 e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} \left(\frac{\Gamma(\eta+1)^2 t^{2\eta} \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^2 \left(2e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} - 1 \right)}{\Gamma(2\eta+1)} - \frac{2\lambda\Gamma(2\eta+1)t^{3\eta} e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{\Gamma(3\eta+1)} \right)}{18\sigma\Gamma(\eta+1)^2 \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^6} \quad (85)$$

$$+ \frac{5(\lambda-1)\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{3\sigma\Gamma(\eta+1) \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^3} + \frac{\lambda-1}{\sigma \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^2}.$$

4.2.2. OHAM solution

Here equations (68) and (69) are solved using OHAM, we get the following solution:
Zero component

$$v_0(\vartheta, t) = \frac{1}{\left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + 1 \right)^2}, \quad (86)$$

$$\psi_0(\vartheta, t) = \frac{\lambda-1}{\sigma \left(e^{\sqrt{\frac{\lambda}{6}}\vartheta} + 1 \right)^2}, \quad (87)$$

first component

$$v_1(\vartheta, t) = - \frac{5c_1 \lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{3\eta\Gamma(\eta) \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^3}, \quad (88)$$

$$\psi_1(\vartheta, t) = - \frac{5c_4(\lambda-1)\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}}{3\eta\sigma\Gamma(\eta) \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^3}, \quad (89)$$

second component

$$v_2(\vartheta, t) = - \frac{1}{18\beta^2\Gamma(\eta)^2 \left(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1 \right)^5} \left(5\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} (c_1^2(t^\eta(5\lambda-2(2\lambda+3))e^{\sqrt{\frac{2}{3}}\vartheta} + (7\lambda \right.$$

$$- 12)e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}}) + 6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2 + 6c_1(\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2 - c_4(\lambda-1)t^\eta e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} (e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 2)) + 6\eta c_2\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}{\sqrt{6}}} + 1)^2 \Big), \quad (90)$$

$$\psi_2(\vartheta, t) = \frac{-1}{18\eta^2\sigma\Gamma(\eta)^2(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^5} \left(5(\lambda - 1)\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}} (c_4^2(6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2 - \lambda t^\eta(10e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} + 5e^{\frac{\sqrt{\lambda}\vartheta}} + 1)) + 6c_4(c_1\lambda t^\eta + \eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2) + 6\eta c_5\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2) \right). \quad (91)$$

Now adding all the components

$$v(\vartheta, t) = v_0(\vartheta, t) + v_1(\vartheta, t) + v_2(\vartheta, t), \quad (92)$$

and

$$\psi(\vartheta, t) = \psi_0(\vartheta, t) + \psi_1(\vartheta, t) + \psi_2(\vartheta, t). \quad (93)$$

Their solutions are

$$v(\vartheta, t) = \frac{1}{(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2} - \frac{5c_1\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}}{3\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^3} - \frac{1}{18\eta^2\Gamma(\eta)^2(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^5} \left(5\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}} (c_1^2 t^\eta (5\lambda - 2(2\lambda + 3)e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} + (7\lambda - 12)e^{\frac{\sqrt{\lambda}\vartheta}}) + 6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2) + 6c_1(\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2 - c_4(\lambda - 1)t^\eta e^{\frac{\sqrt{\lambda}\vartheta}}(e^{\frac{\sqrt{\lambda}\vartheta}} + 2)) + 6\eta c_2\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2) \right), \quad (94)$$

$$\psi(\vartheta, t) = \frac{1}{18\eta^2\sigma\Gamma(\eta)^2(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^5} \left((\lambda - 1)(5c_4^2\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}} (\lambda t^\eta(10e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} + 5e^{\frac{\sqrt{\lambda}\vartheta}} + 1) - 6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2) - 30c_4\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}} (c_1\lambda t^\eta + 2\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2) + 6\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1)^2(3\eta\Gamma(\eta)(e^{\frac{\sqrt{\lambda}\vartheta}} + 1) - 5c_5\lambda t^\eta e^{\frac{\sqrt{\lambda}\vartheta}})) \right). \quad (95)$$

The values of c_1, c_2, c_4, c_5 have been produced for $v(\vartheta, t)$ using the least square approach.

c_1	c_2	c_4	c_5
-0.6549450409408456	-0.1125825136714605	0.5071412826008064	-0.46415009791557854

By putting all the values of c_i in eq. (94) we get the following equation

$$v(\vartheta, t) = \frac{1}{\eta^2\Gamma(\eta)^2(1 + e^{\frac{\sqrt{\lambda}\vartheta}})^5} \left(\lambda t^{2\eta} e^{\frac{\sqrt{\lambda}\vartheta}} ((1.2685 - 0.0769683\lambda)e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} + (2.53701 - 1.94124\lambda)e^{\frac{\sqrt{\lambda}\vartheta}} - 0.595768\lambda) + 1.65587\eta\lambda\Gamma(\eta)t^\eta e^{\frac{\sqrt{\lambda}\vartheta}} (1 + e^{\frac{\sqrt{\lambda}\vartheta}})^2 + \eta^2\Gamma(\eta)^2(1 + e^{\frac{\sqrt{\lambda}\vartheta}})^3 \right). \quad (96)$$

The values of c_1, c_2, c_4, c_5 for $\psi(\vartheta, t)$ were obtained using the least square approach

c_1	c_2	c_4	c_5
0.57779598177464977	-1.373548635995297	0.8842990200060423	-3.506149609659647

By inserting all the values of c_i in eq. (95) we get the following equation

$$\psi(\vartheta, t) = \frac{1}{\eta^2\sigma\Gamma(\eta)^2(1 + e^{\frac{\sqrt{\lambda}\vartheta}})^5} \left(2.17218(\lambda - 1)(0.733186\eta\lambda\Gamma(\eta)t^\eta e^{\frac{\sqrt{\lambda}\vartheta}} (1 + e^{\frac{\sqrt{\lambda}\vartheta}})^2 + 0.460367\eta^2\Gamma(\eta)^2(1 + e^{\frac{\sqrt{\lambda}\vartheta}})^3 + \lambda^2 t^{2\eta} e^{\frac{\sqrt{\lambda}\vartheta}} (1.e^{\sqrt{\frac{2}{3}}\sqrt{\lambda}\vartheta} + 0.5e^{\frac{\sqrt{\lambda}\vartheta}} - 0.292148)) \right). \quad (97)$$

5. Results and discussion

In this article, the two trustworthy methods for solving the TFB-Z system of equations are numerically simulated. Figs. 1-4, 11-14 demonstrate that there is virtually no difference between the sets of numerical values produced using the two proposed methodologies and the precise values, with the exception of the improbable case where $\eta = 1$. Figs. 5, 6, 7, 8, 9, 15, 16, 17, 18, 19 show the solution profiles that explained the impact and behavior of the fractional order. Figs. 10 and 20 depicts the convergence of both cases vividly. Understanding the nature of the model under consideration is made easier by these solution profiles, which show fluctuated behavior for various fractional orders η ($\eta = 0.5, 0.6, 0.7, 0.8, 0.9$ & 1). These graphs show that the solutions produced by the two approaches tend to the integer-order solution, which is asymptotically continuously

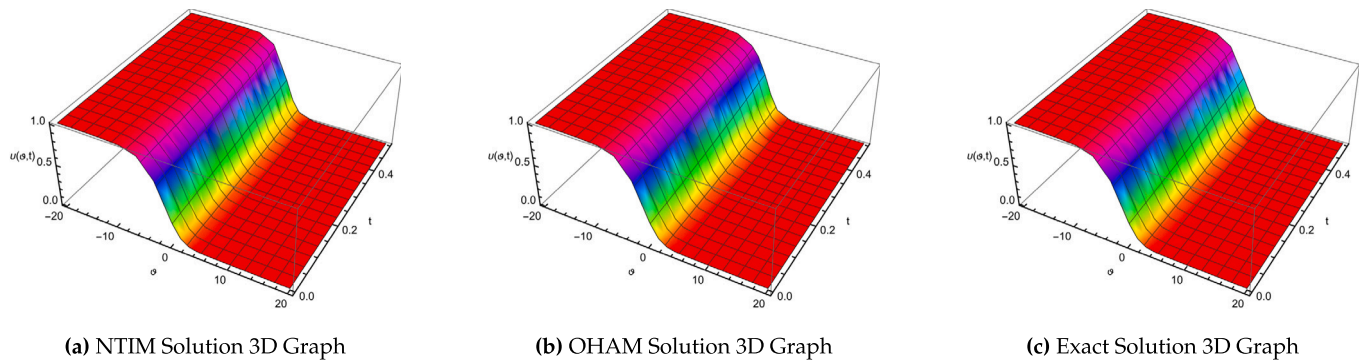


Fig. 1. The 3D comparison of NTIM, OHAM and exact solution of $v(\theta, t)$ in Fig. 1a-1c when $\eta = 1$, $\sigma = \lambda = 2$ for Case 4.1.

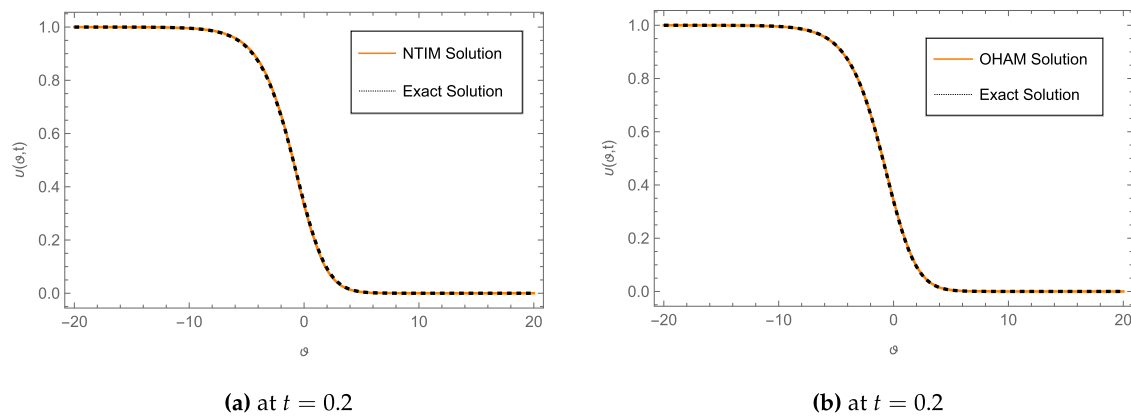


Fig. 2. The 2D comparison of NTIM, OHAM and exact solution of $v(\theta, t)$ is shown in Fig. 2a-2b for $\eta = 1$ and $\sigma = \lambda = 2$ for Case 4.1.

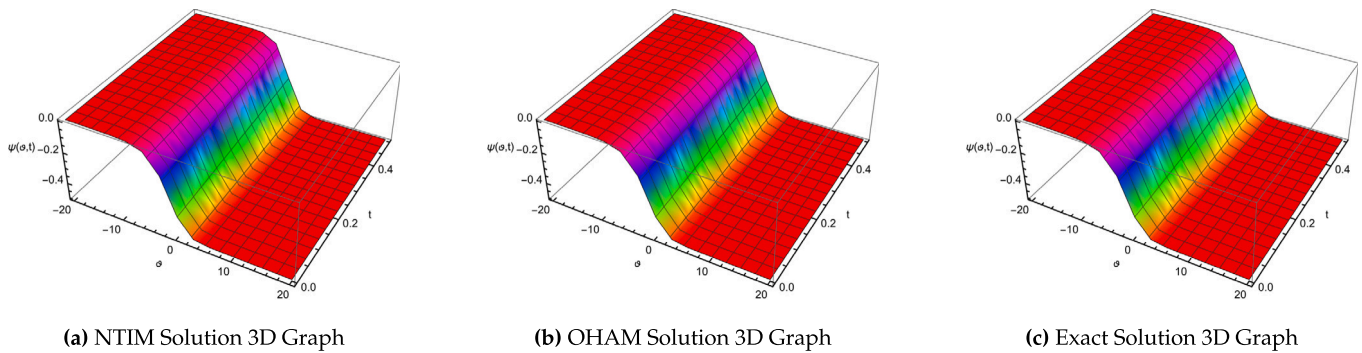


Fig. 3. The 3D comparison of NTIM, OHAM and exact solution of $\psi(\theta, t)$ in Fig. 3a-3c when $\eta = 1$, $\sigma = \lambda = 2$ for Case 4.1.

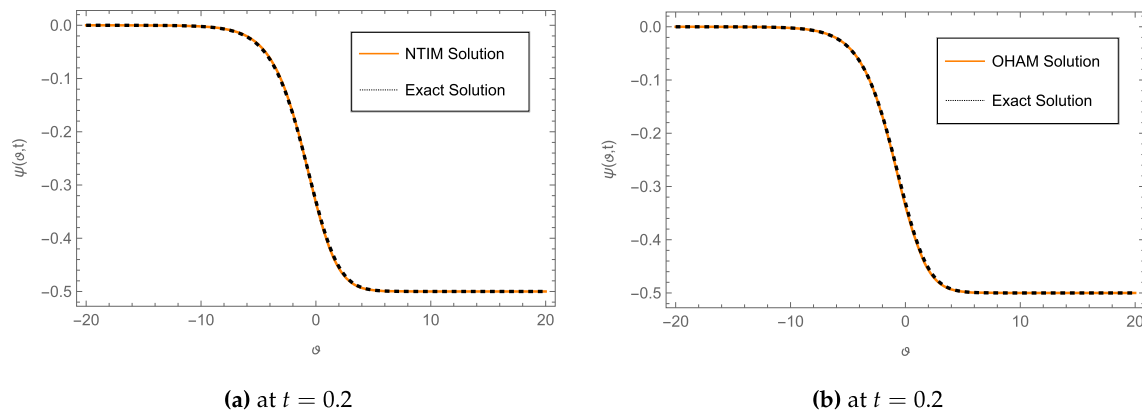


Fig. 4. The 2D comparison of NTIM, OHAM and exact solution of $\psi(\theta, t)$ is shown in Fig. 4a-4b for $\eta = 1$ and $\sigma = \lambda = 2$ for Case 4.1.

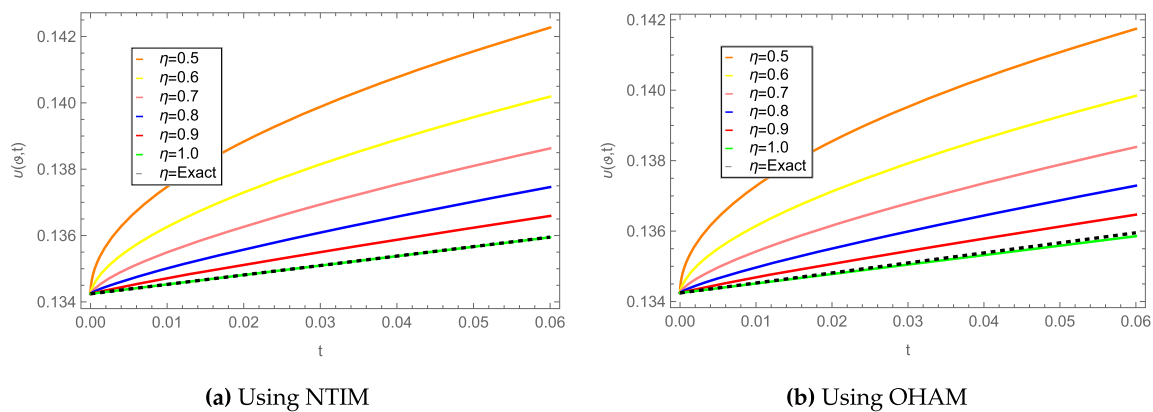


Fig. 5. Solution profiles in Fig. 5a-5b with different η values when $\vartheta = 3$, $\lambda = 0.2$ and $\sigma = 2$ for Case 4.1.

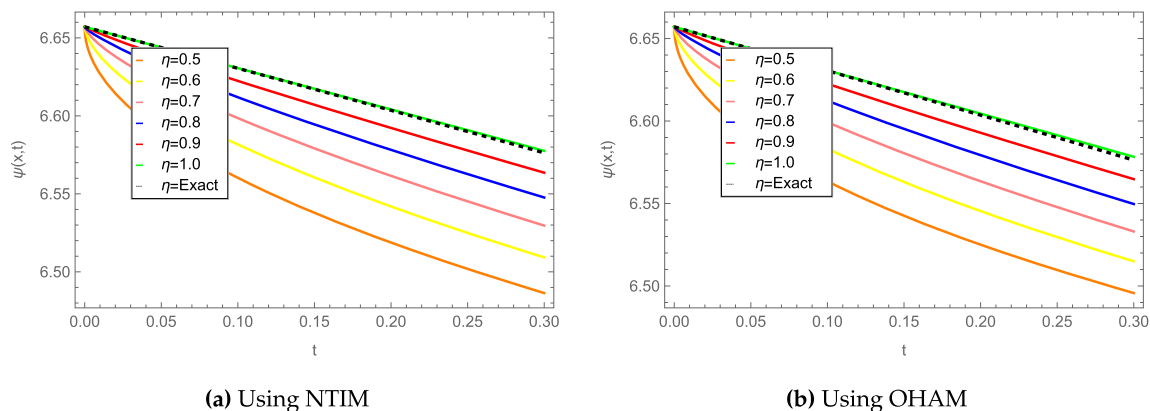


Fig. 6. Solution profiles in Fig. 6a-6b with different η values when $\vartheta = 3$, $\lambda = 0.2$ and $\sigma = 0.1$ for Case 4.1.

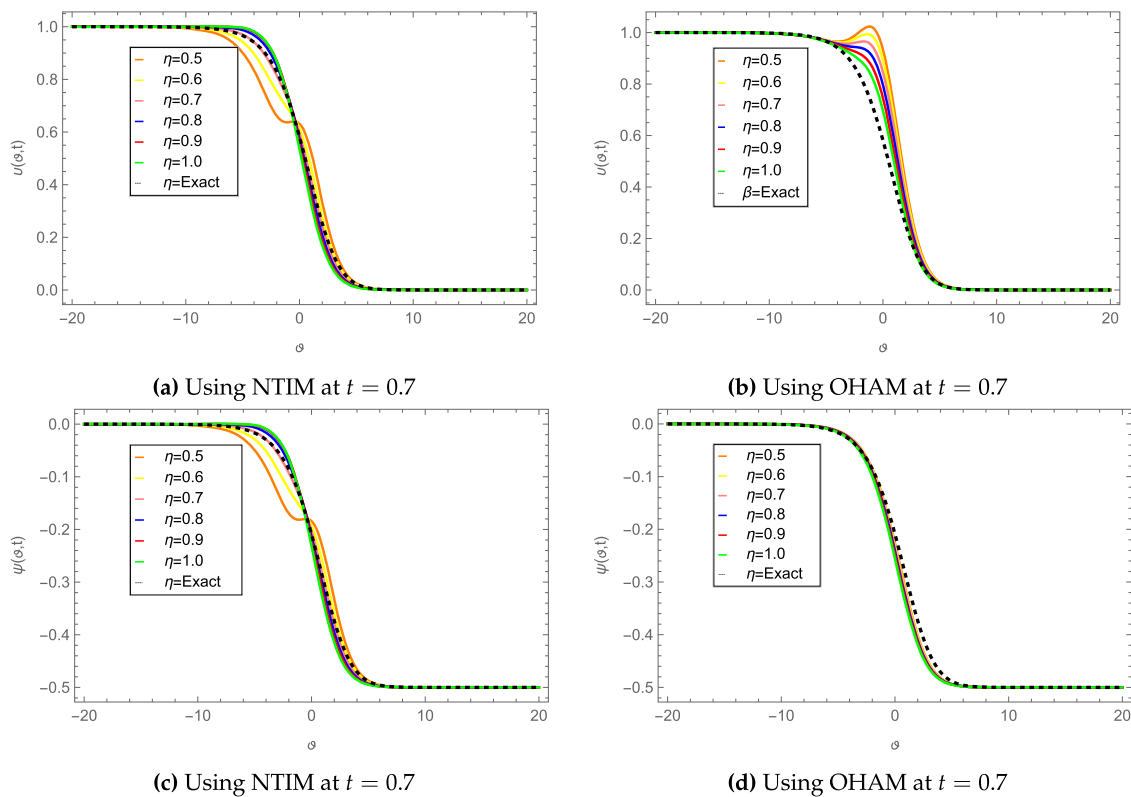


Fig. 7. Solution profiles in Fig. 7a-7d with different η values when $\lambda = 2$ and $\sigma = 2$ for Case 4.1.

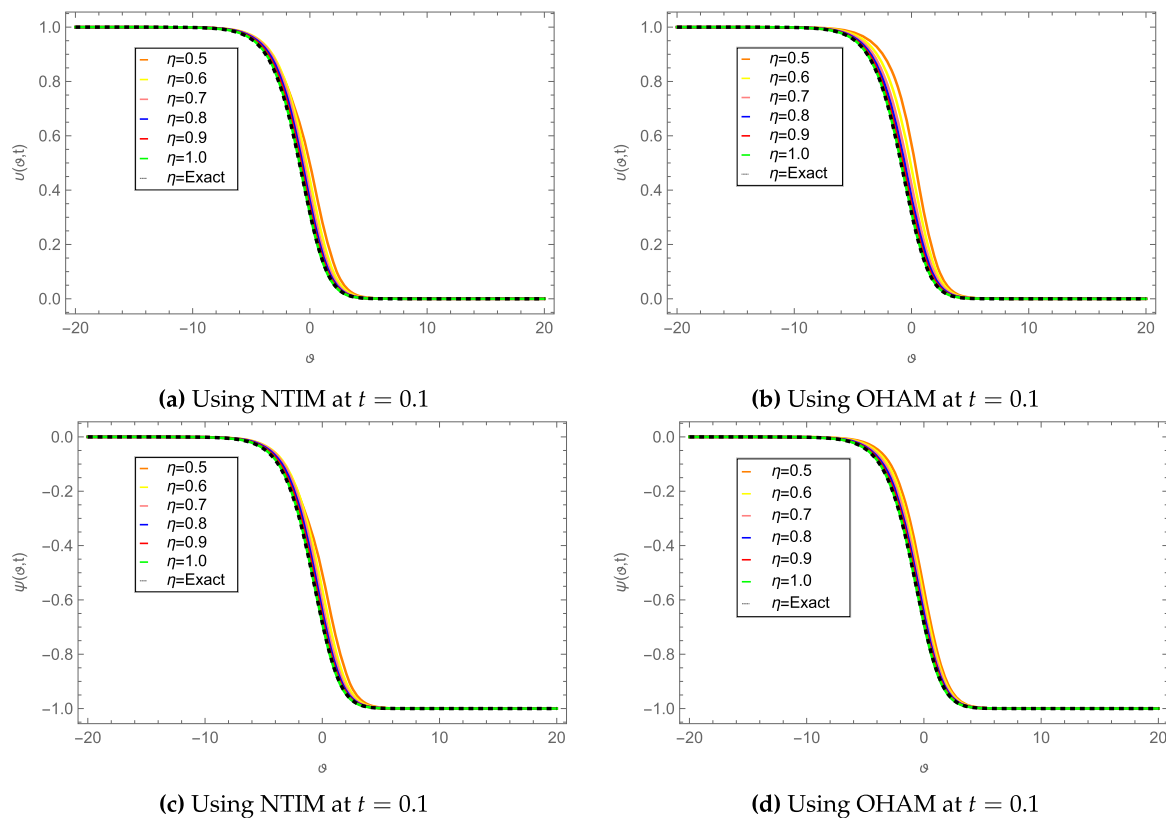


Fig. 8. Solution profiles in Fig. 8a-8d with different η values when $\lambda = 3$ and $\sigma = 2$ for Case 4.1.

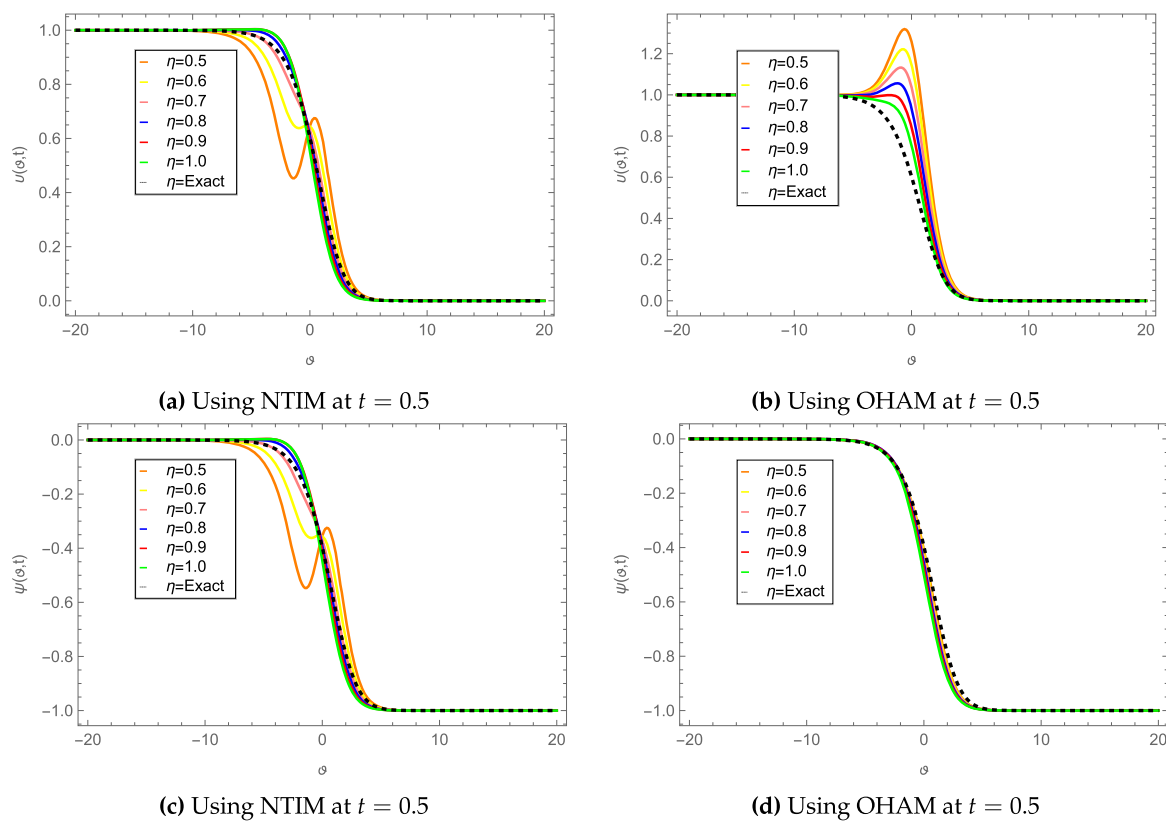
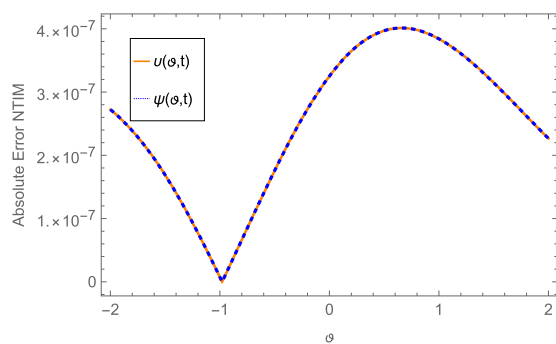
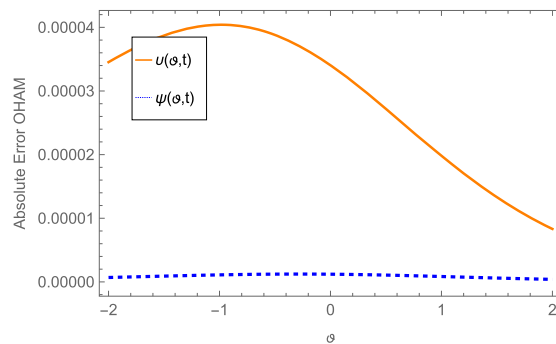


Fig. 9. Solution profiles in Fig. 9a-9d with different η values when $\lambda = 3$ and $\sigma = 2$ for Case 4.1.

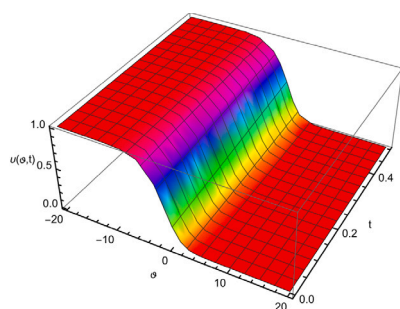


(a) Absolute Error of 2nd order NTIM solution

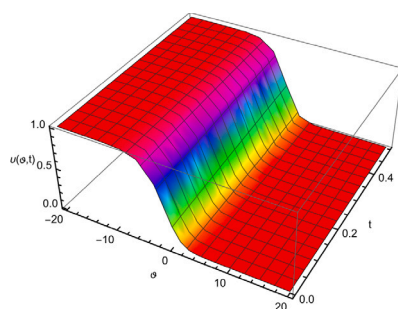


(b) Absolute Error of 2nd order OHAM solution

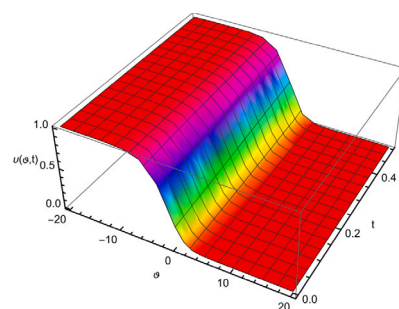
Fig. 10. Comparison of the absolute errors of $v(\theta, t)$ and $\psi(\theta, t)$ at $t = 0.001$, $\lambda = 3$, $\sigma = 2$ and $\beta = 1$ for Case 4.1.



(a) NTIM Solution 3D Graph



(b) OHAM Solution 3D Graph



(c) Exact Solution 3D Graph

Fig. 11. The 3D comparison of NTIM, OHAM and exact solution of $v(\theta, t)$ in Fig. 11a–11c when $\eta = 1$, $\sigma = \lambda = 2$ for Case 4.2.

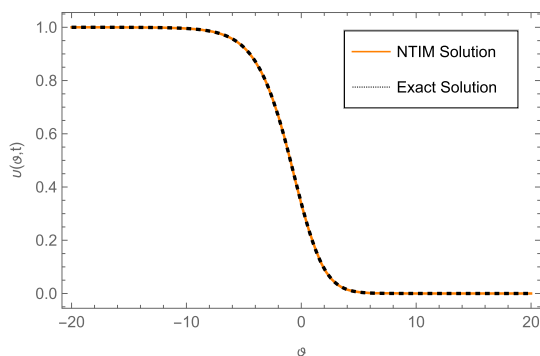
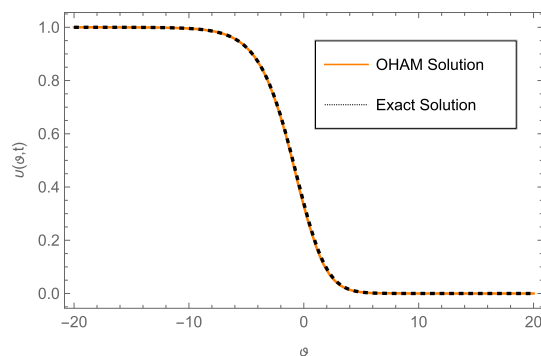
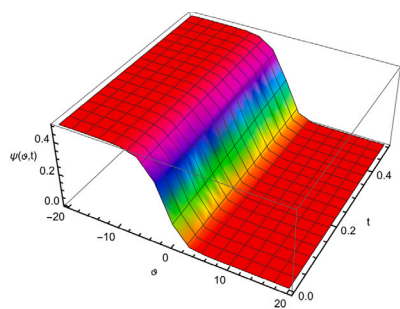
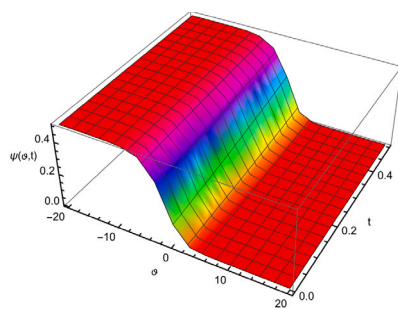
(a) at $t = 0.2$ (b) at $t = 0.2$

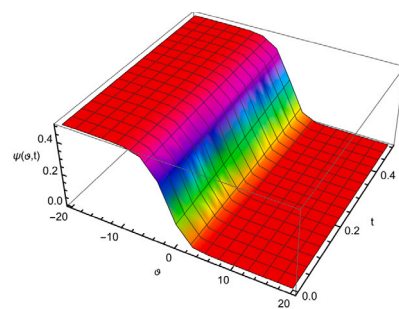
Fig. 12. The 2D comparison of NTIM, OHAM and exact solution of $v(\theta, t)$ is shown in Fig. 12a–12b for $\eta = 1$ and $\sigma = \lambda = 2$ for Case 4.2.



(a) NTIM Solution 3D Graph



(b) OHAM Solution 3D Graph



(c) Exact Solution 3D Graph

Fig. 13. The 3D comparison of NTIM, OHAM and exact solution of $\psi(\theta, t)$ in Fig. 13a–13c when $\eta = 1$, $\sigma = \lambda = 2$ for Case 4.2.

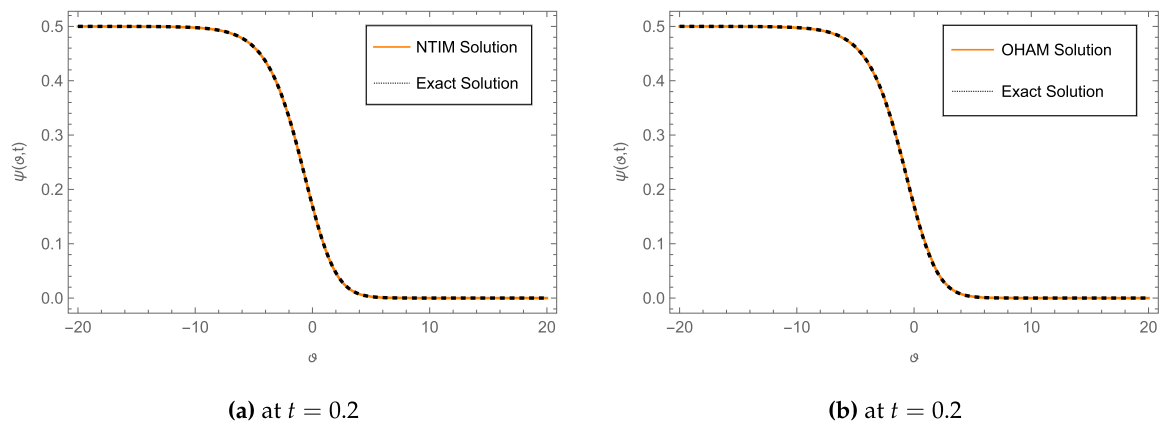


Fig. 14. The 2D comparison of NTIM, OHAM and exact solution of $\psi(\theta, t)$ is shown in Fig. 14a-14b for $\eta = 1$ and $\sigma = \lambda = 2$ for Case 4.2.

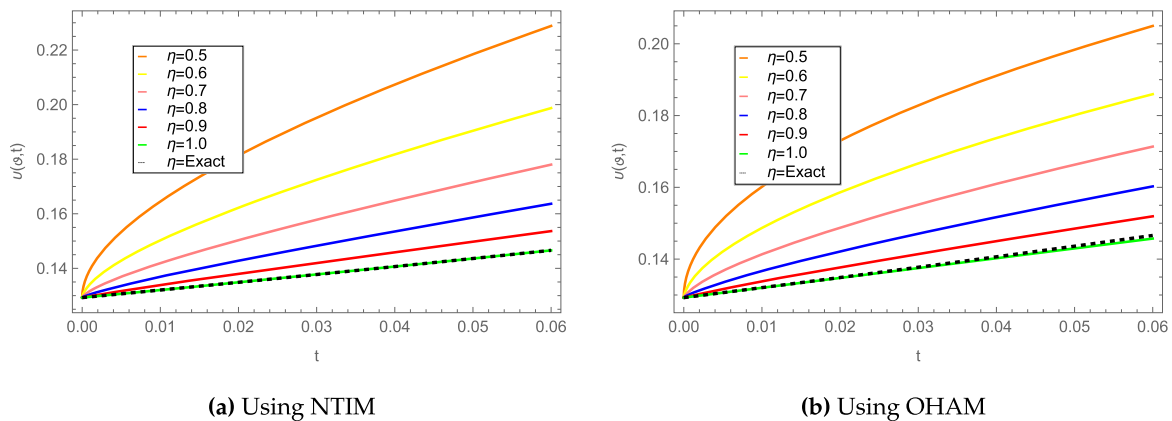


Fig. 15. Solution profiles of $v(\theta, t)$ in Fig. 15a-15b with different η values when $\theta = 1$, $\lambda = 2$ and $\sigma = 2$ for Case 4.2.

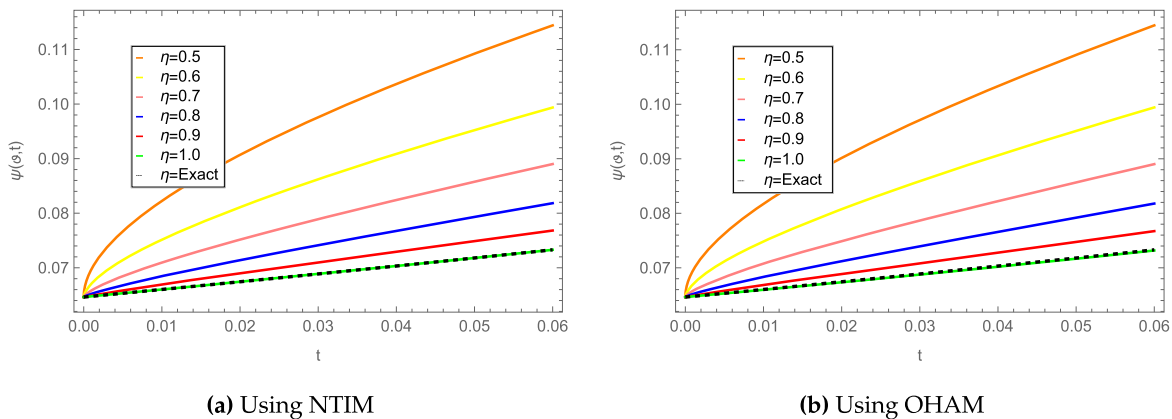


Fig. 16. Solution profiles of $\psi(\theta, t)$ in Fig. 16a-16b with different η values when $\theta = 1$, $\lambda = 2$ and $\sigma = 2$ for Case 4.2.

convergent to the exact solution ($\eta = 1$) as η grows from 0.5 to 1. We compare the NTIM, OHAM, and exact solution in Tables 1, 2, 3, 4, 5, 6, 7 and 8. The findings show that, for the scenario where $\eta = 1$, the results from the two techniques are quite accurate and agree with the exact solutions. The absolute errors in the cited tables are also shown.

6. Concluding remarks

The time-fractional Belousov-Zhabotinsky system in this article was solved using the Natural transform iterative method and the Optimal homotopy asymptotic method. Both techniques were employed and evaluated on two different versions of the model. The effect of the fractional operator on various arbitrary orders was observed, describing more interesting physical behavior of the model. This system simulates the behavior of reacting chemicals, specifically the Belousov-Zhabotinsky reaction, over time by using fractional derivatives to capture non-local and memory-dependent effects. The time-fractional aspect accounts for complex dynamics and anomalous diffusion, providing a more accurate portrayal of the intricate

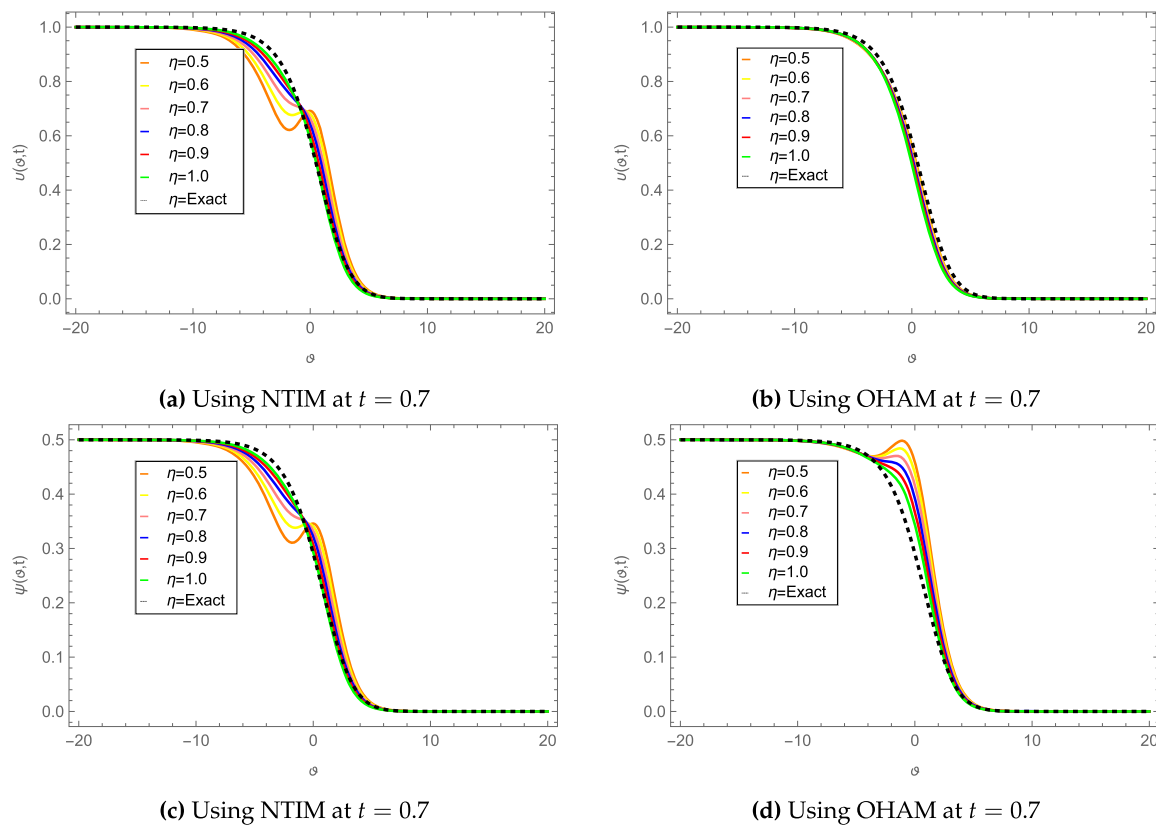


Fig. 17. Solution profiles of $v(\vartheta, t)$ in Fig. 17a-17d with different η values when $\lambda = 2$ and $\sigma = 2$ for Case 4.2.

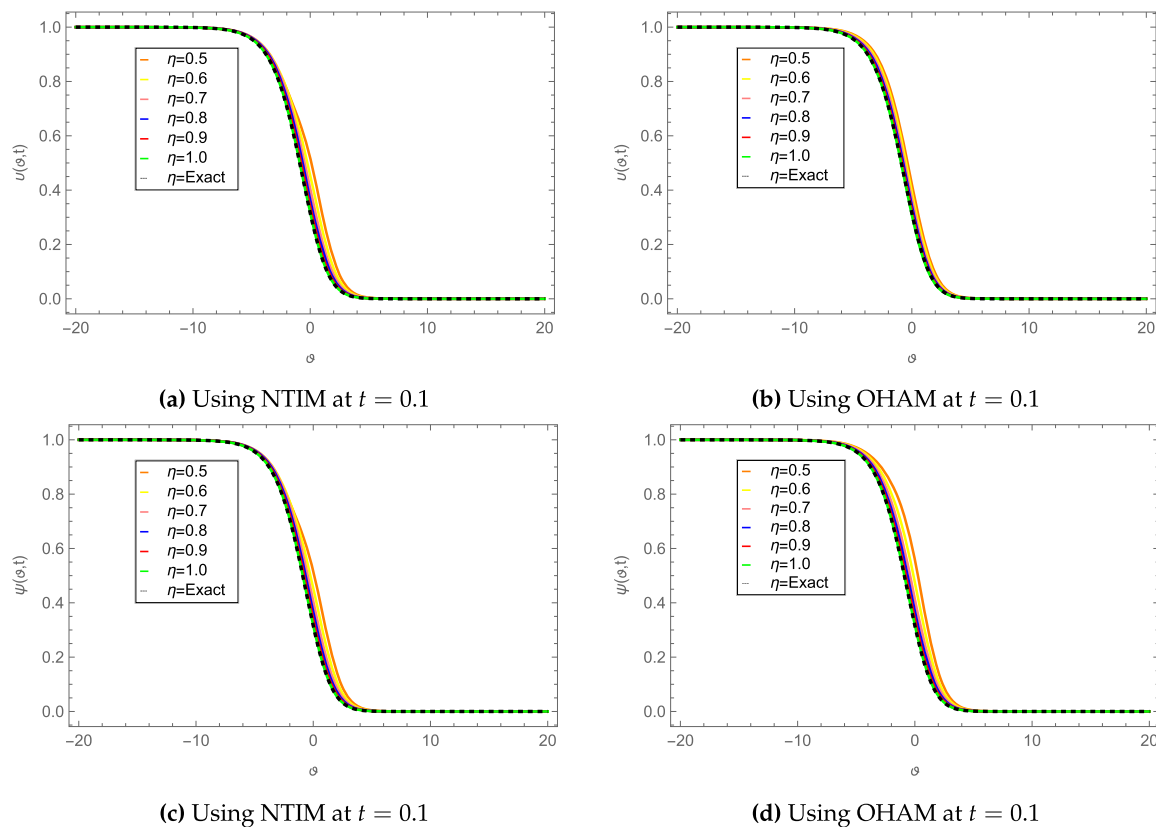


Fig. 18. Solution profiles of $v(\vartheta, t)$ in Fig. 18a-18d with different η values when $\lambda = 3$ and $\sigma = 2$ for Case 4.2.

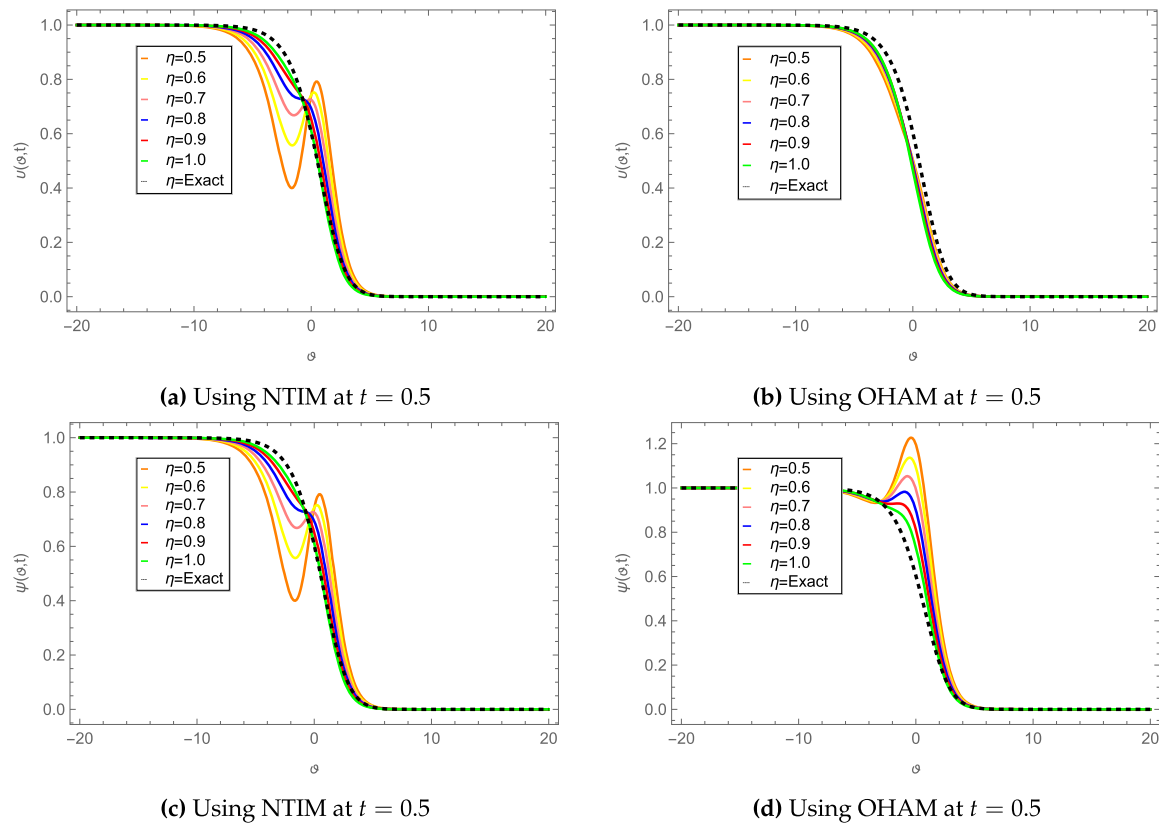


Fig. 19. Solution profiles of $v(\theta, t)$ in Fig. 19a-19d with different η values when $\lambda = 3$ and $\sigma = 2$ for Case 4.2.

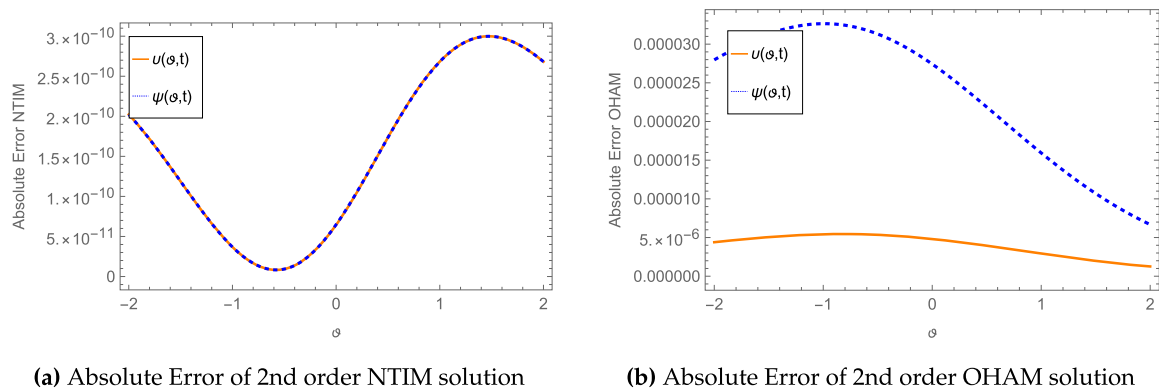


Fig. 20. Comparison of the absolute errors of $v(\theta, t)$ and $\psi(\theta, t)$ at $t = 0.001$, $\lambda = 3$, $\sigma = 2$ and $\beta = 1$ for Case 4.2.

Table 1
The comparison between NTIM, OHAM and the exact solution for $v(\theta, t)$ when $\eta = 1$, $\lambda = 2$, $\sigma = 2$ for Case 4.1.

t	θ	$v_{NTIM}(\theta, t)$	$v_{OHAM}(\theta, t)$	$v_{Exact}(\theta, t)$	Abs Diff NTIM	Abs Diff OHAM
0.001	0.1	0.233244	0.233212	0.233245	3.46778×10^{-7}	3.27824×10^{-5}
	0.3	0.200509	0.20048	0.20051	3.79106×10^{-7}	3.00729×10^{-5}
	0.5	0.170674	0.170647	0.170674	3.97046×10^{-7}	2.718×10^{-5}
0.003	0.1	0.234449	0.234356	0.234452	3.11923×10^{-6}	9.56651×10^{-5}
	0.3	0.201615	0.201531	0.201619	3.41148×10^{-6}	8.76522×10^{-5}
	0.5	0.171674	0.171598	0.171677	3.57426×10^{-6}	7.91159×10^{-5}
0.005	0.1	0.235654	0.235508	0.235663	8.65953×10^{-6}	1.54961×10^{-4}
	0.3	0.202722	0.20259	0.202731	9.47491×10^{-6}	1.41802×10^{-4}
	0.5	0.172675	0.172557	0.172685	9.93078×10^{-6}	1.27816×10^{-4}
0.007	0.1	0.23686	0.236667	0.236877	1.69627×10^{-5}	2.10655×10^{-4}
	0.3	0.203829	0.203655	0.203848	1.85679×10^{-5}	1.92514×10^{-4}
	0.5	0.173676	0.173523	0.173696	1.94686×10^{-5}	1.73277×10^{-4}

Table 2

The comparison between NTIM, OHAM and the exact solution for $\psi(\vartheta, t)$ when $\eta = 1$, $\lambda = 3$, $\sigma = 2$ for Case 4.1.

t	ϑ	$\psi_{NTIM}(\vartheta, t)$	$\psi_{OHAM}(\vartheta, t)$	$\psi_{Exact}(\vartheta, t)$	Abs Diff NTIM	Abs Diff OHAM
0.001	0.1	-0.766756	-0.766757	-0.766755	3.46778×10^{-7}	1.18952×10^{-6}
	0.3	-0.799491	-0.799491	-0.79949	3.79106×10^{-7}	1.13777×10^{-6}
	0.5	-0.829326	-0.829327	-0.829326	3.97046×10^{-7}	1.06865×10^{-6}
0.003	0.1	-0.765551	-0.765555	-0.765548	3.11923×10^{-6}	7.39727×10^{-6}
	0.3	-0.798385	-0.798388	-0.798381	3.41148×10^{-6}	7.20459×10^{-6}
	0.5	-0.828326	-0.828329	-0.828323	3.57426×10^{-6}	6.87421×10^{-6}
0.005	0.1	-0.764346	-0.764356	-0.764337	8.65953×10^{-6}	1.86987×10^{-5}
	0.3	-0.797278	-0.797287	-0.797269	9.47491×10^{-6}	1.83192×10^{-5}
	0.5	-0.827325	-0.827333	-0.827315	9.93078×10^{-6}	1.75676×10^{-5}
0.007	0.1	-0.76314	-0.763158	-0.763123	1.69627×10^{-5}	3.50801×10^{-5}
	0.3	-0.796171	-0.796187	-0.796152	1.85679×10^{-5}	3.44727×10^{-5}
	0.5	-0.826324	-0.826337	-0.826304	1.94686×10^{-5}	3.31448×10^{-5}

Table 3

The comparison between NTIM, OHAM and the exact solution for $v(\vartheta, t)$ when $\eta = 1$, $\lambda = 3$, $\sigma = 2$ for Case 4.1.

t	ϑ	$v_{NTIM}(\vartheta, t)$	$v_{OHAM}(\vartheta, t)$	$v_{Exact}(\vartheta, t)$	Abs Diff NTIM	Abs Diff OHAM	Abs Diff NIM [24]
0.095	1.	0.144388	0.148978	0.148001	0.00361278	0.000976691	0.009557
	1.2	0.120338	0.124671	0.123781	0.00344237	0.000890299	0.008622
	1.4	0.0993842	0.10338	0.102584	0.00320006	0.000796201	0.00767
	1.6	0.0813717	0.0849803	0.0842809	0.00290914	0.000699451	0.006738
	1.8	0.0660834	0.0692795	0.0686751	0.00259168	0.000604377	0.005855
	2.	0.053261	0.0560423	0.0555279	0.0022669	0.000514347	0.00504
0.010	1.	0.112717	0.112587	0.112756	3.85951×10^{-5}	1.68628×10^{-4}	0.009557
	1.2	0.092998	0.0928903	0.0930343	3.63084×10^{-5}	1.44038×10^{-4}	0.008622
	1.4	0.076081	0.0759929	0.0761144	3.33523×10^{-5}	1.21484×10^{-4}	0.00767
	1.6	0.0617462	0.0616749	0.0617762	2.99845×10^{-5}	1.01259×10^{-4}	0.006738
	1.8	0.0497394	0.0496823	0.0497658	2.64376×10^{-5}	8.34837×10^{-5}	0.005855
	2.	0.0397906	0.0397454	0.0398136	2.29045×10^{-5}	6.81435×10^{-5}	0.00504

Table 4

The comparison between NTIM, OHAM and the exact solution for $\psi(\vartheta, t)$ when $\eta = 1$, $\lambda = 3$, $\sigma = 2$ for Case 4.1.

t	ϑ	$\psi_{NTIM}(\vartheta, t)$	$\psi_{OHAM}(\vartheta, t)$	$\psi_{Exact}(\vartheta, t)$	Abs Diff NTIM	Abs Diff OHAM	Abs Diff NIM [24]
0.095	2.	-0.946739	-0.947006	-0.944472	0.0022669	0.00253386	0.001571810
0.095	2.2	-0.957375	-0.957561	-0.955425	0.00195017	0.00213665	0.001343914
0.095	2.4	-0.966107	-0.966234	-0.964454	0.00165273	0.00177965	0.001140890
0.095	2.6	-0.97321	-0.973294	-0.971828	0.00138187	0.00146577	0.000962688
0.095	2.8	-0.978938	-0.978992	-0.977796	0.00114148	0.00119507	0.000808172
0.095	3.	-0.983522	-0.983555	-0.982589	0.000932724	0.000965494	0.000675545
0.01	1.	-0.887283	-0.887299	-0.887244	3.85951×10^{-5}	5.4661×10^{-5}	0.009557
0.01	1.4	-0.923919	-0.923929	-0.923886	3.33523×10^{-5}	4.33719×10^{-5}	0.00469676
0.01	1.8	-0.950261	-0.950266	-0.950234	2.64376×10^{-5}	3.22487×10^{-5}	0.00401319
0.01	2.2	-0.968371	-0.968374	-0.968351	1.95318×10^{-5}	2.26907×10^{-5}	0.00340511
0.01	2.6	-0.980352	-0.980354	-0.980339	1.36289×10^{-5}	1.52512×10^{-5}	0.00287199
0.01	3.	-0.988029	-0.988029	-0.98802	9.08379×10^{-6}	9.87652×10^{-6}	0.00201405

spatiotemporal patterns seen in chemical systems. The two approaches showed stable series form solutions as the values of the fractional order approached integer order 1. Numerical examples demonstrated the close match between the findings and the exact solutions. The accuracy of the offered approaches was showcased with reference to actual examples. These highly effective methods required fewer calculations. The presented techniques could also solve a variety of nonlinear partial differential equation systems with fractional orders. Ultimately, the methodologies provided insights into studying non-linear phenomena in plasma physics, such as soliton waves, rogue waves, shock waves, etc.

Table 5

The comparison between NTIM, OHAM and the exact solution for $v(\vartheta, t)$ when $\eta = 1, \lambda = 2, \sigma = 2$ for Case 4.2.

t	ϑ	$v_{NTIM}(\vartheta, t)$	$v_{OHAM}(\vartheta, t)$	$v_{Exact}(\vartheta, t)$	Abs Diff NTIM	Abs Diff OHAM
0.001	0.1	0.236183	0.23618	0.236183	2.38719×10^{-11}	2.88115×10^{-6}
	0.3	0.20905	0.209048	0.20905	3.42036×10^{-11}	2.70417×10^{-6}
	0.5	0.183815	0.183812	0.183815	4.52451×10^{-11}	2.51242×10^{-6}
0.003	0.1	0.236993	0.236983	0.236993	6.39987×10^{-10}	1.02084×10^{-5}
	0.3	0.209808	0.209798	0.209808	9.18695×10^{-10}	9.64541×10^{-6}
	0.5	0.184516	0.184507	0.184516	1.21683×10^{-9}	9.0176×10^{-6}
0.005	0.1	0.237805	0.237785	0.237805	2.94183×10^{-9}	1.9619×10^{-5}
	0.3	0.210566	0.210548	0.210566	4.23099×10^{-9}	1.86281×10^{-5}
	0.5	0.185218	0.185201	0.185218	5.61126×10^{-9}	1.74951×10^{-5}
0.007	0.1	0.238618	0.238586	0.238618	8.01463×10^{-9}	3.11086×10^{-5}
	0.3	0.211327	0.211297	0.211327	1.15488×10^{-8}	2.96491×10^{-5}
	0.5	0.185923	0.185895	0.185923	1.53364×10^{-8}	2.79431×10^{-5}

Table 6

The comparison between NTIM, OHAM and the exact solution for $\psi(\vartheta, t)$ when $\eta = 1, \lambda = 2, \sigma = 2$ for Case 4.2.

t	ϑ	$\psi_{NTIM}(\vartheta, t)$	$\psi_{OHAM}(\vartheta, t)$	$\psi_{Exact}(\vartheta, t)$	Abs Diff NTIM	Abs Diff OHAM
0.001	0.1	0.118091	0.118083	0.118091	1.1936×10^{-11}	8.90479×10^{-6}
	0.3	0.104525	0.104517	0.104525	1.71018×10^{-11}	8.31684×10^{-6}
	0.5	0.0919075	0.0918998	0.0919075	2.26226×10^{-11}	7.69092×10^{-6}
0.003	0.1	0.118497	0.11847	0.118497	3.19993×10^{-10}	2.62487×10^{-5}
	0.3	0.104904	0.104879	0.104904	4.59347×10^{-10}	2.44863×10^{-5}
	0.5	0.0922579	0.0922353	0.0922579	6.08413×10^{-10}	2.26156×10^{-5}
0.005	0.1	0.118902	0.118859	0.118902	1.47092×10^{-9}	4.29701×10^{-5}
	0.3	0.105283	0.105243	0.105283	2.11549×10^{-9}	4.00354×10^{-5}
	0.5	0.0926092	0.0925722	0.0926092	2.80563×10^{-9}	3.69301×10^{-5}
0.007	0.1	0.119309	0.11925	0.119309	4.00732×10^{-9}	5.90667×10^{-5}
	0.3	0.105663	0.105608	0.105663	5.77442×10^{-9}	5.49628×10^{-5}
	0.5	0.0929613	0.0929106	0.0929613	7.6682×10^{-9}	5.06334×10^{-5}

Table 7

The comparison of NTIM and OHAM solutions with the absolute errors for $\psi(\vartheta, t)$, when $\eta = 1, \lambda = 3, \sigma = 2$ for Case 4.2.

t	ϑ	$\psi_{NTIM}(\vartheta, t)$	$\psi_{OHAM}(\vartheta, t)$	$\psi_{Exact}(\vartheta, t)$	Abs Diff NTIM	Abs Diff OHAM	Abs Diff DLM [25]
0.01	-10	0.496956	0.496954	0.496957	2.33645×10^{-9}	2.24584×10^{-6}	1.01806×10^{-6}
	-6	0.470582	0.470561	0.470582	1.92246×10^{-8}	2.07581×10^{-5}	8.93003×10^{-6}
	-2	0.291384	0.291285	0.291384	2.14267×10^{-8}	9.83681×10^{-5}	2.67739×10^{-5}
	2	0.0294471	0.0294194	0.0294472	4.38975×10^{-8}	2.77636×10^{-5}	7.98941×10^{-7}
	6	4.75718×10^{-4}	4.7519×10^{-4}	4.75721×10^{-4}	2.47207×10^{-9}	5.30318×10^{-7}	2.19483×10^{-9}
	10	4.9648×10^{-6}	4.9592×10^{-6}	4.96483×10^{-6}	2.94538×10^{-11}	5.62742×10^{-9}	2.94228×10^{-11}

Table 8

The comparison of NTIM and OHAM solutions with the absolute errors for $v(\vartheta, t)$, when $\eta = 1, \lambda = 3, \sigma = 2$ for Case 4.2.

t	ϑ	$v_{NTIM}(\vartheta, t)$	$v_{OHAM}(\vartheta, t)$	$v_{Exact}(\vartheta, t)$	Abs Diff NTIM	Abs Diff OHAM	Abs Diff DLM [25]
0.01	-10	0.993913	0.993912	0.993913	4.67291×10^{-9}	5.54328×10^{-7}	8.43019×10^{-7}
	-6	0.941163	0.941157	0.941163	3.84491×10^{-8}	5.60573×10^{-6}	7.17341×10^{-6}
	-2	0.582767	0.582724	0.582767	4.28534×10^{-8}	4.31185×10^{-5}	1.09066×10^{-5}
	2	0.0588942	0.0588722	0.0588943	8.7795×10^{-8}	2.21545×10^{-5}	1.56091×10^{-5}
	6	0.000951437	0.00095095	0.000951442	4.94413×10^{-9}	4.91595×10^{-7}	4.78581×10^{-7}
	10	9.9296×10^{-6}	9.92435×10^{-6}	9.92966×10^{-6}	5.89075×10^{-11}	5.30895×10^{-9}	5.35382×10^{-9}

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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