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Original Article

A new hybrid special function class and numerical technique for multi-order fractional differential equations

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ABSTRACT

This study aims to investigate the properties of fractional calculus theory (FCT) in the complex domain. We focus on the relationship between the theories of special functions (SFT) and FCT, which have seen recent advancements and have led to various successful applications in fields such as engineering, mathematics, physics, biology, and other allied disciplines. Our main contribution is the development of a special function, specifically the confluent hypergeometric function (CHF) on the complex domain. By deriving various implementations of fractional order derivatives and integral operators using this function, we present a new class of special functions combining certain cases of Mittag-Leffler and confluent hypergeometric functions. Moreover, a new numerical technique for solving linear and nonlinear multi-order fractional differential equations has been developed using the proposed class of functions and the point collocation method. Graphical results are shown to demonstrate the efficacy of this proposed technique and its applicability.

1. Introduction

Fractional calculus theory (FCT) deals with derivatives of fractional (non-integer) order along with integrals [1–3], which is an extension formula of classical (regular) calculus. The origin of this theory dates back to the correspondence between Leibniz and L'Hospital in 1695, where Leibniz wrote a letter to L'Hospital posing the concept of a derivative of non-integer order. Subsequently, Euler and Liouville extensively investigated the computation of fractional order derivatives and integrals. FCT presents a fruitful instrument that has been employed in mathematical (real or complex) analysis, applied mathematics, and other variety of science fields [4–6].

In this context, the pivotal parts of FCT and their significant applications for their appearances are the Riemann–Liouville fractional integral (RLFI) and Riemann–Liouville fractional derivative (RLFD), respectively, were introduced by

$$ {}^{RL}_{\gamma}I^{\rho}_{x} f(x) = \frac{1}{\Gamma(-\rho)} \int_{\gamma}^{x} (x-\tau)^{-\rho-1} f(\tau)\, d\tau, \qquad \Re(\rho) < 0, \tag{1.1} $$

and

$$ {}^{RL}_{\gamma}D^{\rho}_{x} f(x) = \frac{d^{\mu}}{dx^{\mu}} \left({}^{RL}_{\gamma}D^{\rho-\mu}_{x} f(x) \right), \qquad \Re(\rho) \geq 0,\ \mu := \left[\Re(\rho)\right] + 1. \tag{1.2} $$

where γ is a constant of differ-integration and $D^{\rho}\phi$ represents the derivative of f to order β. It is noteworthy that in FCT, the derivatives are based on the choice of a differ-integration constant γ (either $\gamma = 0$ or $\gamma = -\infty$). The following Lemma offers the value γ, which has two differ-integration formulae $\gamma = 0$ and $\gamma = -\infty$.

Lemma 1 (*[7]*). *RL differ-integrals of power functions* (x^{η}, $\Re(\eta) > -1$) *and exponential functions* ($e^{\eta x}$), *with constants of differ-integration* $\gamma = 0$ *and* $\gamma = -\infty$, *respectively, are given as*

$$ {}^{RL}_{0}D^{\rho}_{x} f(x^{\eta}) = {}^{RL}D^{\rho}_{x} f(x^{\eta}) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\rho+1)} x^{\eta-\rho}, \qquad \alpha, \eta \in \mathbb{C},\ \Re(\eta) > -1; \tag{1.3} $$

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$$ {}^{RL}_{-\infty}D^{\rho}_{\varkappa}\left(e^{\eta\varkappa}\right)=\eta^{\rho}e^{\eta\varkappa}, \qquad \rho,\eta\in\mathbb{C},\ \eta\notin\mathbb{R}^{-}_{0}. \tag{1.4}$$

For (1.3) and (1.4), we employ the prime branch with values ranging from $-\pi$ to π to create complex power functions.

The class of Mittag-Leffler functions (M-LFs) is an essential function which is commonly encountered in the field of FCT, [8–11]. M-LF was proposed by Magnus Gösta Mittag-Leffler in 1903, [12]. It is formulated as:

$$\mathcal{E}_{\alpha}(z)=\sum_{\kappa=0}^{\infty}\frac{z^{\kappa}}{\Gamma(\alpha\kappa+1)} \qquad (\rho,z\in\mathbb{C};\ \Re(\rho)>0). \tag{1.5}$$

Later, in [13,14], Wiman provided $\mathcal{E}_{\alpha,\eta}(z)$ as a generalization of $\mathcal{E}_{\alpha}(z)$, called Wiman function and is expressed as:

$$\mathcal{E}_{\alpha,\eta}(z)=\sum_{\kappa=0}^{\infty}\frac{z^{\kappa}}{\Gamma(\alpha\kappa+\eta)} \qquad (\alpha,\eta\in\mathbb{C};\ \Re(\alpha)>0). \tag{1.6}$$

Since then, $\mathcal{E}_{\alpha}(z)$ written by (1.5) and $\mathcal{E}_{\alpha,\eta}(z)$ by (1.6) have been generalized and extended through a variety of methods and procedures.

Some realistic phenomena are better described by fractional models than by integer-order differential equations, as numerous studies have recently demonstrated. This dominance has led to the popularity and significance of fractional calculus in the modeling of real-world applications [15,16], and [17]. The M-LFs and their implementations have evolved in numerous studies such as mathematics, statistics, physics, and engineering [18]. Furthermore, connections between generalized sorts of M-LFs using route models were achieved by Pillai [19].

Confluent hypergeometric functions (CHFs) have been widely used in various branches of physics, for example, to estimate protein molecular weight in ultracentrifuge experiments and in situations involving diffusion and isotope separation.

Indeed, CHF $\Phi(\sigma;\eta;z)$ is the following resolution of 2^{nd}-order linear homogeneous differential equation:

$$z\frac{d^{2}\Phi}{dz^{2}}+(\eta-z)\frac{d\Phi}{dz}-\lambda\Phi=0, \qquad (\sigma,\eta,z\in\mathbb{C}). \tag{1.7}$$

Eq. (1.7) has a normal initial singularity and sporadic singularity at infinity [20].

If η is not integral, the 2^{nd}- resolution of Eq. (1.7) can be clarified as:

$$\Psi(\sigma;\eta;z)=z^{1-\eta}\Phi(\sigma-\eta+1;2-\eta;z) \tag{1.8}$$

If γ is integral, a 2^{nd}- resolution can be explained by

$$\begin{aligned}\Psi(\rho;\eta;z)=\;&\Phi(\rho;\eta;z)\{\ln z+\Lambda(1-\rho)-\Lambda(\gamma)+C\}\\&+\sum_{\kappa=1}^{\infty}\frac{\Gamma(\kappa+\sigma)\Gamma(\eta)A_{\kappa}z^{\kappa}}{\Gamma(\sigma)\Gamma(\kappa+\eta)\kappa!}\\&+(-1)^{\eta}\sum_{\kappa=0}^{\infty}\frac{\Gamma(\eta)\Gamma(\kappa+\sigma-\eta+1)\Gamma(\eta-\kappa-1)(-1)^{\kappa}}{\Gamma(\sigma)\kappa!\,z}\end{aligned} \tag{1.9}$$

where

$$\Lambda(\sigma)=\frac{\Gamma'(\sigma)}{\Gamma(\sigma)},$$

$C\simeq 0.577216$ is Eulers constant, and

$$A_{\kappa}=\left(\frac{1}{\sigma}+\frac{1}{\sigma+1}+\cdots+\frac{1}{\sigma+\kappa-1}\right)-\left(\frac{1}{\eta}+\frac{1}{\eta+1}+\cdots+\frac{1}{\eta+\kappa-1}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{\kappa}\right)$$

In [21], detailed tables of the Λ function are provided.

The series representation of CHF $\Phi(\sigma;\eta;z)$ is

$$\Phi(\sigma;\eta;z)=1+\frac{\sigma}{\eta}z+\frac{\sigma(\sigma+1)}{\eta(\eta+1)}\frac{z^{2}}{2}+\cdots=\sum_{\kappa=0}^{\infty}\frac{\Gamma(\eta)\Gamma(\kappa+\sigma)z^{\kappa}}{\Gamma(\sigma)\Gamma(\kappa+\eta)\kappa!}. \tag{1.10}$$

The prior series clearly converges absolutely.

Both Φ and Ψ affirm the following:

$$\frac{d}{dz}\Phi(\sigma;\eta;z)=\frac{\sigma}{\eta}\Phi(\sigma+1;\eta+1;z) \tag{1.11}$$

$$\sigma\Phi(\sigma+1;\eta+1;z)=(\sigma-\eta)\Phi(\sigma;\eta+1;z)+\eta\Phi(\sigma;\eta;z) \tag{1.12}$$

$$\sigma\Phi(\sigma+1;\eta;z)=(z+2\sigma-\eta)\Phi(\sigma;\eta;z)+(\eta-\sigma)\Phi(\sigma-1;\eta;z). \tag{1.13}$$

A proposed link between $\Phi(\sigma;\eta;z)$ and $\Psi(\rho;\eta;z)$ could be seen in the Wronskian of Eq. (1.7) as well as in the mapping relationship among the number of a Wronskian at every point in a plane and the number at the corresponding point, see [20–22].

In this investigation, we impose the composition of fractional calculus by developing a new kind of MLF function defined in terms of combined hypergeometric functions (ML-CHf) $\Phi^{\alpha,\sigma}_{\rho,\eta}(z)$ as follows:

$$\Phi^{\alpha,\sigma}_{\rho,\eta}(z)=\sum_{\kappa=0}^{\infty}\frac{\Gamma(\eta)\Gamma(\alpha(\kappa+1)+\sigma)z^{\kappa}}{\Gamma(\sigma)\Gamma(\rho(\kappa+1)+\eta)\kappa!}, \qquad (\rho,\eta,\alpha,\sigma\in C;\ \Re(\rho)>0). \tag{1.14}$$

Interestingly, if α is a mathematical constant, we have

$$\begin{aligned}\Phi^{\alpha,\sigma}_{\rho,\eta}(z)&=\sum_{\kappa=0}^{\infty}\frac{\Gamma(\eta)\Gamma(\alpha(\kappa+1)+\sigma)z^{\kappa}}{\Gamma(\sigma)\Gamma(\rho(\kappa+1)+\eta)\kappa!}\\&=-\frac{(-z)^{-\eta/\rho}\eta\left(\sigma\Gamma\left(\frac{\eta}{\rho}+1,0,-z\right)+\alpha_{F}\Gamma\left(\frac{\eta}{\rho}+1,0,-z\right)-\alpha_{F}\Gamma\left(\frac{\eta}{\rho}+2,0,-z\right)\right)}{z\rho\sigma}\end{aligned} \tag{1.15}$$

where α_F is the Feigenbaum Alpha constant.

If α is a variable, we have

$$\begin{aligned}\Phi^{\alpha,\sigma}_{\rho,\eta}(z)&=\sum_{\kappa=0}^{\infty}\frac{\Gamma(\eta)\Gamma(\alpha(\kappa+1)+\sigma)z^{\kappa}}{\Gamma(\sigma)\Gamma(\rho(\kappa+1)+\eta)\kappa!}\\&=-\frac{(-z)^{-\eta/\rho}\eta\left(\alpha\Gamma\left(\frac{\eta}{\rho}+1,0,-z\right)+\sigma\Gamma\left(\frac{\eta}{\rho}+1,0,-z\right)-\alpha\Gamma\left(\frac{\eta}{\rho}+2,0,-z\right)\right)}{z\rho\sigma}\end{aligned} \tag{1.16}$$

$(\rho,\eta,\alpha,\sigma\in\mathbb{C};\ \mathcal{R}(\rho)>0,\mathcal{R}(\sigma)>0)$, where $\Gamma\left(a,z_0,z_1\right)$ is the generalized incomplete Gamma function $\Gamma\left(a,z_0\right)-\Gamma\left(a,z_1\right)$.

The confluent hypergeometric function, as stated in Eq. (1.10), is a noteworthy consequence of this function if $\alpha=\rho=\frac{\kappa}{\kappa+1}$. If η, α, and σ are all equal to one, we get the following partial sum formula:

$$\begin{aligned}\sum_{\kappa=0}^{n}\frac{(\kappa+1)z^{\kappa}}{\Gamma(\rho\kappa+\rho+1)}=&-\frac{1}{\Gamma\rho^{2}(z-1)}\left(z^{n+1}\psi\left(z,1,-\frac{-\rho-1}{\rho}+n+1\right)-\right.\\&z^{n+2}\psi\left(z,1,-\frac{-\rho-1}{\rho}+n+1\right)+\rho z^{2}\psi\left(z,1,1-\frac{-\rho-1}{\rho}\right)+\\&z^{2}\psi\left(z,1,1-\frac{-\rho-1}{\rho}\right)-\rho z\psi\left(z,1,1-\frac{-\rho-1}{\rho}\right)\\&-z\psi\left(z,1,1-\frac{-\rho-1}{\rho}\right)-\\&\left.\rho z\psi\left(z,1,-\frac{-\rho-1}{\rho}\right)+\rho\psi\left(z,1,-\frac{-\rho-1}{\rho}\right)-\rho z^{n+1}+\rho z\right),\end{aligned}$$

where $\psi(z,s,a)$ is the Lerch transcendent. Furthermore, if $\eta=\sigma=1$, $\alpha=\frac{\kappa}{\kappa+1}$ and $\rho=\frac{\alpha\kappa}{\kappa+1}$, we will get the initial Mittag-Leffler function as stated in Eq. (1.5).

Several researchers (see, [23,24], and [25]) have recently solved convolution equations, which are particular instances of differential equations, using the Laplace transform

$$\int_{0}^{\varkappa}\frac{(\varkappa-\tau)^{\zeta-1}}{\Gamma(\zeta)}{}_{1}\mathcal{F}_{1}(\gamma;\zeta;\delta(\varkappa-\tau))f(\tau)d\tau=g(\varkappa) \qquad \Re(\zeta)>0, \tag{1.17}$$

examined through fractional integration [26–28] . In this study, we investigate an integral equation

$$\int_{\gamma}^{\varkappa}(\tau-\varkappa)^{\eta-1}\Phi^{\alpha,\sigma}_{\rho,\eta}\nu(\tau-\varkappa)^{\rho}f(\tau)d\tau=g(\varkappa) \qquad \Re(\eta)>0, \tag{1.18}$$

for $\rho > 0$ where the function $\Phi^{\alpha,\sigma}_{\rho,\eta}(z)$ provided in (1.14) is an analytic function of order γ involving certain distinctive functions.

In their study [29], Ghanim and Al-Janaby investigated an integral operator $\vartheta^{\alpha,\sigma}_{\rho,\eta}(\nu)$ on a space ω of functions defined by Eq. (1.18). They applied a fractional integration operator $I^{\wp}:\omega\to\omega$ to this operator and examined the results obtained on $\vartheta^{\alpha,\sigma}_{\rho,\eta}(\nu)$. They then used these results to investigate theorems related to the solutions of Eq. (1.18). This approach can be applied to achieve similar results when solving integral equations.

$$\mathfrak{A}^{\alpha,\sigma}_{\rho,\eta}(\nu)\,f(\varkappa)\equiv\int_{\gamma}^{\varkappa}(\tau-\varkappa)^{\eta-1}\Phi^{\alpha,\sigma}_{\rho,\eta}\wp(\tau-\varkappa)^{\rho}f(\tau)\,d\tau=g(\varkappa)\qquad\Re(\eta)>0,\tag{1.19}$$

where the equations mentioned in [30,31], and [32] are included as special cases in this analysis. In the context of this research, ω denotes a set of complex functions f that can be integrated with respect to the ω measure over a finite interval $[\gamma,\zeta],\gamma\geq 0$, with a norm defined as $|f|=\int_{\gamma}^{\zeta}|f(\tau)|\,d\tau$.

The fractional operator $I^{\wp}:\omega\to\omega$ is defined as the fractional integral for complex values of $\wp$ with $\Re(\wp)>0$

$$I^{\wp}f(\varkappa)=\int_{\gamma}^{\varkappa}\frac{(\varkappa-\tau)^{\wp-1}}{\Gamma(\wp)}f(\tau)\,d\tau.\tag{1.20}$$

It has been clearly established that $I^{\wp}$ is bounded, and it is necessary that $I^{\wp}f=0\to f=0$, which shows that the inverse operator exists within the subspace $\omega_{\wp}$ of ω. When $0<\Re(\wp)<\Re(\lambda)$ holds true, it can be demonstrated that $\omega_{\lambda}\subset\omega_{\wp}\subset\omega$ holds, with the inclusion being sufficient.

For $\Re(\wp)<0$, $I^{\wp}$ is expressed as the reciprocal of $I^{-\wp}$. If $\Re(\wp)\neq 0$ and $\Re(\lambda)\neq 0$ then $I^{\wp}I^{\lambda}f=I^{\wp+\lambda}f$ for appropriate functions f. For $\Re(\wp)=0$, $I^{\wp}$ is also given on ω_{ν} when $\Re(\nu)>0$ as $I^{-1}I^{1+\wp}$.

For $\rho,\alpha,\sigma,\eta\in\mathbb{C}$ with $f\in\omega$ and $\Re(\eta)>0$, the fractional operator $\vartheta^{\alpha,\sigma}_{\rho,\eta}$ on ω into itself is defined by

$$\vartheta^{\alpha,\sigma}_{\rho,\eta}(\nu)\,f(\varkappa)=\int_{\gamma}^{\varkappa}(\tau-\varkappa)^{\eta-1}\Phi^{\alpha,\sigma}_{\rho,\eta}\nu(\tau-\varkappa)^{\rho}f(\tau)\,d\tau\qquad\gamma<\varkappa<b.$$

Lemma 2 ([33]). *Given the series $f(\varkappa)=\sum_{\kappa=1}^{\infty}f_{\kappa}(\varkappa)$ uniformly converges on $0\leq|\varkappa-\gamma|\leqslant M$ where $M>0$ and $\gamma\in\mathbb{C}$ are fixed constants, then for $\Re(\rho)<0$*

$${}^{RL}_{\gamma}D^{\rho}_{\varkappa}f(\varkappa)=\sum_{\kappa=1}^{\infty}{}^{RL}_{\gamma}D^{\alpha}_{\varkappa}f_{\kappa}(\varkappa),\qquad|\varkappa-\gamma|\leqslant M.$$

For a fixed order of differ-integration $\rho\in\mathbb{C}$.

Furthermore, if $f(\varkappa)=\sum_{\kappa=1}^{\infty}f_{\kappa}(\varkappa)$ also uniformly converge for $\Re(\rho)\geqslant 0$, then term wise differ-integration is also valid for $\Re(\rho)\geqslant 0$.

Multiple studies on special functions, engineering modeling and control of biological systems by using fractional-order differential equations have been published, including those by [8,11,15–17,34–37], and many other scholars. To verify the first part of our findings, we will utilize the same techniques as [7] as well as those of numerous other studies in this research.

2. Fractional outcomes

Theorem 1. *Let $\rho,\alpha,\eta,\sigma\in\mathbb{C}$ with $\Re(\rho),\Re(\eta)>0$, then*

$$\mathcal{M}^{\alpha,\sigma}_{\rho,\eta}(z)=\frac{\Gamma(\eta)}{\Gamma(\sigma)}{}^{RL}_{0}D^{(\kappa+1)\alpha+\sigma-\kappa-1}_{z}\left[z^{\alpha+\sigma-1}\mathcal{E}_{\rho,\rho+\eta}(z^{\alpha})\right],\qquad z\in\mathbb{C}\tag{2.1}$$

Proof. It follows from Lemma 1 that a fractional power function differ-integral is usually used to describe the starting point of a fraction of gamma functions. The expression $\frac{\Gamma(\alpha\kappa+\sigma)}{\kappa!}$ that exists in the series equation coefficients of (1.14) results in the following:

$$\mathcal{M}^{\alpha,\sigma}_{\rho,\eta}(z)=\sum_{\kappa=0}^{\infty}\frac{\Gamma(\eta)\,\Gamma(\alpha(\kappa+1)+\sigma)\,z^{\kappa}}{\Gamma(\sigma)\,\Gamma(\rho(\kappa+1)+\eta)\,\kappa!}$$

$$=\sum_{\kappa=0}^{\infty}\frac{\Gamma(\eta)}{\Gamma(\sigma)\,\Gamma(\rho(\kappa+1)+\eta)}\frac{\Gamma(\alpha(\kappa+1)+\sigma)}{\Gamma(\kappa+1)}z^{\kappa}$$

$$=\sum_{\kappa=0}^{\infty}\frac{\Gamma(\eta)}{\Gamma(\sigma)\,\Gamma(\rho(\kappa+1)+\eta)}{}^{RL}_{0}D^{(\kappa+1)\alpha+\sigma-\kappa-1}_{z}\left[z^{(\kappa+1)\alpha+\sigma-1}\right]$$

$$=\sum_{\kappa=0}^{\infty}\frac{\Gamma(\eta)}{\Gamma(\sigma)}{}^{RL}_{0}D^{(\kappa+1)\alpha+\sigma-\kappa-1}_{z}\left[\frac{z^{(\kappa+1)\alpha+\sigma-1}}{\Gamma(\rho(\kappa+1)+\eta)}\right].$$

As the sequence here is uniformly convergent, and by using Lemma 2, the result is to substitute fractional differ-integration for the summation. Then we yield

$$\mathcal{M}^{\alpha,\sigma}_{\rho,\eta}(z)=\frac{\Gamma(\eta)}{\Gamma(\sigma)}\sum_{\kappa=0}^{\infty}{}^{RL}_{0}D^{(\kappa+1)\alpha+\sigma-\kappa-1}_{z}\left[\frac{z^{(\kappa+1)\alpha+\sigma-1}}{\Gamma(\rho(\kappa+1)+\eta)}\right]$$

$$=\frac{\Gamma(\eta)}{\Gamma(\sigma)}{}^{RL}_{0}D^{(\kappa+1)\alpha+\sigma-\kappa-1}_{z}\left[\sum_{\kappa=0}^{\infty}\frac{z^{(\kappa+1)\alpha+\sigma-1}}{\Gamma(\rho(\kappa+1)+\eta)}\right]$$

$$=\frac{\Gamma(\eta)}{\Gamma(\sigma)}{}^{RL}_{0}D^{(\kappa+1)\alpha+\sigma-\kappa-1}_{z}\left[z^{\alpha+\sigma-1}\sum_{\kappa=0}^{\infty}\frac{z^{\alpha\kappa}}{\Gamma(\rho(\kappa+1)+\eta)}\right]$$

$$=\frac{\Gamma(\eta)}{\Gamma(\sigma)}{}^{RL}_{0}D^{(\kappa+1)\alpha+\sigma-\kappa-1}_{z}\left[z^{\alpha+\sigma-1}\mathcal{E}_{\rho,\rho+\eta}(z^{\alpha})\right],$$

As a result, the proof is achieved. □

Corollary 1. *Given that $\rho,\sigma\in\mathbb{C}$ and $\Re(\rho)>0$, we obtain:*

$$\mathcal{M}^{\alpha,\sigma}_{\rho,1}(z)=\frac{1}{\Gamma(\sigma)}{}^{RL}_{0}D^{(\kappa+1)\alpha+\sigma-\kappa-1}_{z}\left[z^{\sigma-1}\mathcal{E}_{\rho,\rho+1}(z^{\alpha})\right]\tag{2.2}$$

Proof. The proof is immediately followed by substituting η in Theorem 1 with 1. □

Using exactly the same argument as in Theorem 1, we yield

Corollary 2. *Let $\rho,\alpha,\eta,\sigma,\xi\in\mathbb{C}$ with $\Re(\rho)>0$, $\Re(\eta)>0$. We get*

$$M^{\alpha,\sigma}_{\rho,\eta}(\xi z)=\frac{\Gamma(\eta)}{\Gamma(\sigma)}{}^{RL}_{0}D^{(\kappa+1)\alpha+\sigma-\kappa-1}_{z}\left[z^{\alpha+\sigma-1}\mathcal{E}_{\rho,\rho+\eta}(\xi z^{\alpha})\right],\qquad z\in\mathbb{C}.\tag{2.3}$$

Proof. The demonstration of this result is similar to that of Theorem 1; however, each term of the sums has an additional component of ξ^{κ}. □

Theorem 2. *With $\rho,\alpha,\eta,\xi\in\mathbb{C}$ and the condition of $\sigma=1$ and $\Re(\rho)>0,\Re(\eta)>0$ holds, we obtain:*

$$\mathcal{M}^{\alpha}_{\rho,\eta}(\xi z^{\rho})=\frac{z^{1-\eta}}{\Gamma(\eta)}{}^{RL}_{0}D^{(\kappa+1)\alpha-\rho\kappa-\eta+1}_{z}\left[\mathcal{E}_{\rho,\rho+\eta}(\xi z^{\rho\alpha})\right],\qquad z\in\mathbb{C}.\tag{2.4}$$

Proof. The first step in this analysis is to study the right-hand side of the corresponding identity and then employ the definition of $\mathcal{E}_{\rho,\rho+\eta}$ provided in Eq. (1.6):

$$\frac{z^{1-\gamma}}{\Gamma(\gamma)}{}^{RL}_{0}D^{(\kappa+1)(\alpha-\rho)-\eta+1}_{z}\left[\mathcal{E}_{\rho,\rho+\eta}(\xi z^{\rho\alpha})\right]$$

$$=\frac{z^{1-\eta}}{\Gamma(\eta)}{}^{RL}_{0}D^{(\kappa+1)(\alpha-\rho)-\eta+1}_{z}\left[\sum_{\kappa=0}^{\infty}\frac{(\xi z^{\rho\alpha})^{\kappa}}{\Gamma(\rho(\kappa+1)+1)}\right]$$

$$=\frac{z^{1-\eta}}{\Gamma(\eta)}{}^{RL}_{0}D^{(\kappa+1)(\alpha-\rho)-\eta+1}_{z}\left[\sum_{\kappa=0}^{\infty}\frac{\xi^{\kappa}z^{\kappa\rho\alpha}}{\Gamma(\rho(\kappa+1)+1)}\right].$$

We then switch between summation and fractional differ-integration using Lemma 2, since the series is uniformly convergent and uniform convergence will always accrue to the resultant series (at least in the situation of $1>\Re(\eta)>0$). As we move through the steps, we conclude with the appropriate illustration.

$$\frac{z^{1-\eta}}{\Gamma(\eta)}{}^{RL}_{0}D^{(\kappa+1)(\alpha-\rho)-\eta+1}_{z}\left[\mathcal{E}_{\rho,\rho+\eta}(\xi z^{\rho\alpha})\right]$$

$$=\frac{z^{1-\eta}}{\Gamma(\eta)}\sum_{n=0}^{\infty}{}^{RL}_{a}D^{(\kappa+1)(\alpha-\rho)-\eta+1}_{z}\left[\frac{\xi^{\kappa}z^{\kappa\rho\alpha}}{\Gamma(\alpha n+\gamma)}\right]$$

$$= \frac{z^{1-\eta}}{\Gamma(\eta)} \sum_{n=0}^{\infty} \frac{\Gamma(\rho(\kappa+1)+1)}{\Gamma(\rho(\kappa+1)+\eta)} \left[\frac{\xi^{\kappa} z^{\kappa\rho\alpha+\eta-1}}{\Gamma(\rho(\kappa+1)+1)} \right]$$

$$= \frac{1}{\Gamma(\eta)} \sum_{n=0}^{\infty} \frac{\xi^{\kappa} z^{\kappa\rho\alpha+\eta-1}}{\Gamma(\rho(\kappa+1)+\eta)}.$$

Consequently, a uniform convergence accrues to the series in (1.6) and the series expression for $\mathcal{E}_{\rho,\rho+\eta}(\xi z^{\alpha})$ is achieved. As a result, our previously indicated change of operations was correct, and the evidence is now clear. □

Corollary 3. *Given that $\rho, \alpha, \xi \in \mathbb{C}$ and the constraints of $\eta = \sigma = 1$ and $\Re(\rho) > 0$, we can deduce:*

$$\mathcal{M}_{\rho}^{\alpha}(\xi z) = z^{1-\eta}\, {}_{0}^{RL}D_{z}^{(\kappa+1)(\alpha-\rho)} \left[\mathcal{E}_{\rho,\rho+1}(\xi z^{\rho\alpha})\right] \tag{2.5}$$

Proof. The proof is followed immediately by the substitution of η for 1 as in Theorem 2. □

Theorem 3. *Suppose that $\alpha, \sigma, \rho, \eta, \xi \in \mathbb{C}$ and $\Re(\rho) > 0$, $\Re(\eta) > 0$ with $\Re(\sigma) < 1$, we get:*

$$\mathcal{M}_{\rho,\eta}^{\alpha,\sigma}(\xi z^{\rho}) = \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)} \tag{2.6}$$
$$\times \int_0^z (z^{\rho} - \upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-\gamma}\, {}_{0}^{RL}D_{\upsilon}^{(\kappa+1)(\alpha-\rho)-\eta+1} \left[\mathcal{E}_{\rho,\rho+\eta}(\xi \upsilon^{\rho\alpha})\right] d\upsilon,$$

where $\mathcal{B}$ is beta function.

Proof. In both Corollary 1 as well as Theorem 1, the fractional differ-integrals are of integral type, so that Eq. (2.3) may be altered in this manner:

$$\mathcal{M}_{\rho,\eta}^{\alpha,\sigma}(\xi z) = \frac{\Gamma(\eta)}{\Gamma(\sigma)}\, {}_{0}^{RL}D_{z}^{(\kappa+1)(\alpha-1)+\sigma} \left[z^{\sigma-1} \mathcal{E}_{\rho,\rho+\eta}(\xi z^{\alpha})\right]$$
$$= \frac{\Gamma(\eta)}{\Gamma(\sigma)\, \Gamma((1-\alpha)(\kappa+1)-\sigma)} \int_0^z (z-y)^{-(\kappa+1)\alpha+\kappa-\sigma} y^{(\kappa+1)(\alpha-1)+\sigma} \left[\mathcal{E}_{\rho,\rho+\eta}(\xi y^{\alpha})\right] dy$$

By utilizing the formula for the beta function and the reflection equation for the gamma function, we change the variable y to υ^{ρ} and obtain

$$\upsilon_{\rho,\eta}^{\alpha,\sigma}(\xi z) = \frac{\Gamma(\eta) \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)}$$
$$\times \int_0^{z^{1/\rho}} (z-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)(\alpha-1)+\sigma)} \left[\mathcal{E}_{\rho,\rho+\eta}(\xi \upsilon^{\rho\alpha})\right] \rho \upsilon^{\rho-1} d\upsilon$$
$$= \frac{\rho \Gamma(\eta) \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(n-(n+1)\beta)}$$
$$\times \int_0^{z^{1/\rho}} (z-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-1} \left[\mathcal{E}_{\rho,\rho+\eta}(\xi \upsilon^{\rho\alpha})\right] d\upsilon.$$

Applying the outcome of Theorem 2, we have

$$\mathcal{M}_{\rho,\eta}^{\alpha,\sigma}(\xi z) = \frac{\rho \Gamma(\eta) \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)}$$
$$\times \int_0^{z^{1/\rho}} (z-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-1}$$
$$\times \left[\frac{\upsilon^{1-\eta}}{\Gamma(\eta)}\, {}_{0}^{RL}D_{z}^{(\kappa+1)(\alpha-\rho)-\eta+1} \left[\mathcal{E}_{\rho,\rho+\eta}(\xi \upsilon^{\rho\alpha})\right] \right] d\upsilon$$
$$= \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)}$$
$$\times \int_0^{z^{1/\rho}} (z-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-\eta}\, {}_{0}^{RL}D_{z}^{(\kappa+1)(\alpha-\rho)-\eta+1}$$
$$\times \left[\mathcal{E}_{\rho,\rho+\eta}(\xi \upsilon^{\rho\alpha})\right] d\upsilon.$$

Substituting z^{ρ} for z, we get

$$\mathcal{M}_{\rho,\eta}^{\alpha,\sigma}(\xi z^{\rho}) = \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)}$$
$$\times \int_0^z (z^{\rho}-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-\eta}\, {}_{0}^{RL}D_{\upsilon}^{(\kappa+1)(\alpha-\rho)-\eta+1} \left[\mathcal{E}_{\rho,\rho+\eta}(\xi \upsilon^{\rho\alpha})\right] d\upsilon,$$

In conclusion, the above outcome is achieved. □

Corollary 4. *Let $\rho, \alpha, \sigma, \xi \in \mathbb{C}$ and $\Re(\rho) > 0$ with $\Re(\sigma) < 1$, we get:*

$$\mathcal{M}_{\rho,1}^{\alpha,\sigma}(\xi z^{\rho}) = \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)} \tag{2.7}$$
$$\times \int_0^z (z^{\rho}-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-1}\, {}_{0}^{RL}D_{\upsilon}^{(\kappa+1)(\alpha-\rho)} \left[\mathcal{E}_{\rho,\rho+1}(\xi \upsilon^{\rho\alpha})\right] d\upsilon,$$

where $\mathcal{B}$ represents the classical beta function.

Proof. Following Theorem 3, the proof is followed by substituting η with 1. □

Corollary 5. *Let $\rho, \alpha, \eta, \sigma, \xi \in \mathbb{C}$ and $\Re(\rho) > 0$, $\Re(\eta) > 0$ with $\Re(\sigma) < 1$, we obtain:*

$$\mathcal{M}_{\rho,\eta}^{\alpha,\sigma}(\xi z^{\rho}) = \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)\, \Gamma(\rho\kappa+\eta-(\kappa+1)\alpha)} \tag{2.8}$$
$$\times \int_0^z (z^{\rho}-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-\eta}$$
$$\times \int_0^{\upsilon} (\upsilon-\varkappa)^{(\kappa+1)(\rho-\alpha)+\eta-1} \mathcal{E}_{\rho,\rho+\eta}(\xi \varkappa^{\rho\alpha}) d\varkappa\, d\upsilon,$$

$z \in \mathbb{C}$.

Proof. Since $\Re(1-\eta) < 0$, the differ-integral operator of a fractional type in (2.6), which appears in the mentioned equation, would be a component of integral type. Consequently, one would have that

$${}_{0}^{RL}D_{\upsilon}^{(\kappa+1)(\rho-\alpha)+\eta-1} \left[\mathcal{E}_{\rho,\rho+\eta}(\xi \upsilon^{\rho\alpha})\right]$$
$$= \frac{1}{\Gamma((\kappa+1)(\rho-\alpha)+\eta)} \int_0^{\upsilon} (\upsilon-\varkappa)^{(\kappa+1)(\rho-\alpha)+\eta-1} \mathcal{E}_{\rho,\rho+\eta}(\xi \varkappa^{\rho\alpha}) d\varkappa.$$

This may be substituted into Eq. (2.6) to get:

$$\mathcal{M}_{\rho,\eta}^{\alpha,\sigma}(\xi z^{\rho}) = \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)}$$
$$\times \int_0^z (z^{\rho}-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-\eta}$$
$$\times \left[\frac{1}{\Gamma(\kappa+1)(\rho-\alpha)+\eta} \int_0^{\upsilon} (\upsilon-\varkappa)^{(\kappa+1)(\rho-\alpha)+\eta-1} \mathcal{E}_{\rho,\rho+\eta}(\xi \varkappa^{\rho\alpha}) d\varkappa \right] du,$$

which gives the required proof. □

Theorem 4. *ML-CHf in Eq. (1.14) could be expressed as an integral transformation using the format shown below:*

$$\mathcal{M}_{\rho,\eta}^{\alpha,\sigma}(\xi z^{\rho})$$
$$= \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)\, \Gamma((\kappa+1)(\rho-\alpha)+\eta)} \int_0^z \Lambda_{\rho,\eta,\sigma}(\varkappa; z)\, \mathcal{E}_{\rho,\rho+\eta}(\xi \varkappa^{\rho\alpha}) d\varkappa, \tag{2.9}$$

$z \in \mathbb{C}$, where $\rho, \alpha, \eta, \sigma, \xi \in \mathbb{C}$ and $\Re(\rho) > 0$, $\Re(\eta) > 1$ with $\Re(\sigma) < 1$ and Λ is a function given by

$$\Lambda_{\rho,\eta,\sigma}(\varkappa; z) = \int_{\varkappa}^z (z^{\rho}-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-\eta} (\upsilon-\varkappa)^{(\kappa+1)(\rho-\alpha)+\eta-1} d\upsilon. \tag{2.10}$$

Proof. According to Fubini's Theorem, the integrals' order in (2.8) may be reversed. After exchanging, we get $0 \leqslant \upsilon \leqslant z$ and $0 \leqslant \varkappa \leqslant u$, that is similar to $0 \leqslant \varkappa \leqslant z$ as well as $\varkappa \leqslant \upsilon \leqslant z$. As a result, one gets (2.8):

$$\mathcal{M}_{\rho,\eta}^{\alpha,\sigma}(\xi z^{\rho}) = \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)\, \Gamma((\kappa+1)(\rho-\alpha)+\eta)}$$
$$\times \int_0^z \int_0^{\upsilon} (z^{\rho}-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-\eta} (\upsilon-\varkappa)^{(\kappa+1)(\rho-\alpha)+\eta-1}$$
$$\times \mathcal{E}_{\rho,\rho+\eta}(\xi \upsilon^{\rho\alpha})\, d\varkappa\, d\upsilon$$
$$= \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)\, \Gamma((\kappa+1)(\eta-\alpha)+\eta)}$$
$$\times \int_0^z \int_{\varkappa}^z (z^{\rho}-\upsilon^{\rho})^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-\eta} (\upsilon-\varkappa)^{(\kappa+1)(\rho-\alpha)+\eta-1}$$

$$\times\, \mathcal{E}_{\rho,\rho+\eta}(\xi\upsilon^{\rho\alpha})\, d\upsilon d\varkappa$$
$$= \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)\, \Gamma((\kappa+1)(\rho-\beta)+\eta)}$$
$$\times \int_0^z \mathcal{E}_{\rho,\rho+\eta}(\xi\upsilon^{\rho\alpha}) \int_\varkappa^z (z^\rho - \upsilon^\rho)^{-(\kappa+1)\alpha+\kappa-\sigma} \upsilon^{\rho((\kappa+1)\alpha+\sigma-\kappa)-\eta}$$
$$\times\, (\upsilon-\varkappa)^{(\kappa+1)(\rho-\alpha)+\eta-1} d\upsilon d\varkappa$$
$$= \frac{\rho \sin(\pi\sigma)\, \mathcal{B}(1-\sigma, \kappa-(\kappa+1)\alpha)}{\pi\, \Gamma(\kappa-(\kappa+1)\alpha)\, \Gamma((\kappa+1)(\rho-\beta)+\eta)}$$
$$\times \int_0^z \mathcal{E}_{\rho,\rho+\eta}(\xi\varkappa^{\rho\alpha})\, \Lambda_{\rho,\gamma,\sigma}(\varkappa; z) d\varkappa,$$

as required. □

3. ML-CHF point collocation numerical method

In this section, a new method will be developed based on the function given in Eq. (1.14), and the collocation method used here is the ML-CHF collocation method. This numerical method is based on an ML-CHF series solution with some unknown coefficients. Hence, the objective is to obtain those unknown coefficients using the point collocation method and to find the approximated series for fractional differential equations.

3.1. Methodology for linear fractional differential equation

Consider a general linear boundary value problem as

$$L\Big(D^\beta f(z) = g(z)\Big), \qquad z \in [p_0, p_1]$$
$$f(p_0) = p_a, \quad f(p_1) = p_b \tag{3.1}$$

here, where L stands for the term linear. Assume the solution to this problem to be

$$f(z) = \sum_{\kappa=0}^{n} c_\kappa \Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(z) = \sum_{\kappa=0}^{n} c_\kappa \frac{\Gamma(\eta)\, \Gamma(\alpha(\kappa+1)+\sigma)\, z^\kappa}{\Gamma(\sigma)\, \Gamma(\rho(\kappa+1)+\eta)\, \kappa!} \qquad \kappa = 0, \dots, n$$

where $c_\kappa \quad \kappa = 1, \dots, n$ are the unknown coefficients. Since this is the case for linear fractional differential equations, therefore

$$L\Big(f(z)\Big) = \sum_{\kappa=0}^{n} c_\kappa L\left(\frac{\Gamma(\eta)\, \Gamma(\alpha(\kappa+1)+\sigma)\, z^\kappa}{\Gamma(\sigma)\, \Gamma(\rho(\kappa+1)+\eta)\, \kappa!}\right) \qquad \kappa = 0, \dots, n. \tag{3.2}$$

Based on the solution $f(z)$, Eqs. (3.1) and (3.2) become

$$L\left(\sum_{\kappa=0}^{n} c_\kappa D^\beta \Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(z) = g(z)\right), \qquad z \in [p_0, p_1] \tag{3.3}$$

To calculate the unknown coefficients c_0 and c_1, let us solve the following conditions:

$$\sum_{\kappa=0}^{n} c_\kappa L\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(p_0) = p_a\right)$$
$$\sum_{\kappa=0}^{n} c_\kappa L\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(p_1) = p_b\right). \tag{3.4}$$

For the remaining $n-2$ equations, apply the collocation method by considering $z_2, z_3, \dots, z_{n-1}$ as the collocation points and discretizing the differential equation as

$$\sum_{\kappa=0}^{n} c_\kappa L\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(p_0) = p_a\right)$$
$$\sum_{\kappa=0}^{n} c_\kappa L\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(z_m) = g(z_m)\right) \qquad m = 2, \dots, n-1$$
$$\sum_{\kappa=0}^{n} c_\kappa L\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(p_1) = p_b\right) \tag{3.5}$$

To solve the linear equation system for the conditions $z_0 = p_a$ and $z_1 = p_b$, its matrix form can be written as

$$\begin{pmatrix} \Phi_0(z_0) & \Phi_1(z_0) & \Phi_2(z_0) & \cdots & \Phi_n(z_0) \\ L\Phi_0(z_1) & L\Phi_1(z_1) & L\Phi_2(z_1) & \cdots & L\Phi_n(z_1) \\ \vdots & & & \vdots & \\ L\Phi_0(z_{n-1}) & L\Phi_1(z_{n-1}) & L\Phi_2(z_{n-1}) & \cdots & L\Phi_n(z_{n-1}) \\ \Phi_0(z_n) & \Phi_1(z_n) & \Phi_2(z_n) & \cdots & \Phi_n(z_n) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} p_a \\ g(z_1) \\ \vdots \\ g(z_{n-1}) \\ p_b \end{pmatrix} \tag{3.6}$$

If the matrix in Eq. (3.6) is non-singular for the chosen collocation points $z_m \quad m = 1, \dots, n-1$, then the solution of the linear fractional differential equation is written as

$$f(z) = \sum_{\kappa=0}^{n} c_\kappa \frac{\Gamma(\eta)\, \Gamma(\alpha(\kappa+1)+\sigma)\, z^\kappa}{\Gamma(\sigma)\, \Gamma(\rho(\kappa+1)+\eta)\, \kappa!} \qquad \kappa = 0, \dots, n. \tag{3.7}$$

3.2. Methodology for non-linear fractional differential equation

Consider a general non-linear boundary value problem as

$$N\Big(D^\beta f(z) = g(z)\Big), \qquad z \in [p_0, p_1]$$
$$f(p_0) = p_a, \quad f(p_1) = p_b \tag{3.8}$$

where N stands for the term non-linear. Assume the solution to this problem to be

$$f(z) = \sum_{\kappa=0}^{n} c_\kappa \Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(z) = \sum_{\kappa=0}^{n} c_\kappa \frac{\Gamma(\eta)\, \Gamma(\alpha(\kappa+1)+\sigma)\, z^\kappa}{\Gamma(\sigma)\, \Gamma(\rho(\kappa+1)+\eta)\, \kappa!} \qquad \kappa = 0, \dots, n$$

where $c_\kappa \quad \kappa = 1, \dots, n$ are the unknown coefficients. Since this is the case for non-linear fractional differential equations, therefore

$$N\Big(f(z)\Big) = \sum_{\kappa=0}^{n} c_\kappa L\left(\frac{\Gamma(\eta)\, \Gamma(\alpha(\kappa+1)+\sigma)\, z^\kappa}{\Gamma(\sigma)\, \Gamma(\rho(\kappa+1)+\eta)\, \kappa!}\right) \qquad \kappa = 0, \dots, n \tag{3.9}$$

Based on the solution $f(z)$, Eqs. (3.8) and (3.9) become

$$N\left(\sum_{\kappa=0}^{n} c_\kappa D^\beta \Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(z) = g(z)\right), \qquad z \in [p_0, p_1] \tag{3.10}$$

To calculate the unknown coefficients c_0 and c_1, let us solve the following conditions:

$$\sum_{\kappa=0}^{n} c_\kappa N\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(p_0) = p_a\right)$$
$$\sum_{\kappa=0}^{n} c_\kappa N\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(p_1) = p_b\right) \tag{3.11}$$

For the remaining $n-2$ equations, apply the collocation method by considering $z_2, z_3, \dots, z_{n-1}$ as collocation points and discretizing the differential equation as

$$\sum_{\kappa=0}^{n} c_\kappa N\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(p_0) = p_a\right)$$
$$\sum_{\kappa=0}^{n} c_\kappa N\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(z_m) = g(z_m)\right) \qquad m = 2, \dots, n-1$$
$$\sum_{\kappa=0}^{n} c_\kappa N\left(\Phi^{\alpha,\sigma}_{\rho,\eta,\kappa}(p_1) = p_b\right) \tag{3.12}$$

To solve the non-linear equation system for the conditions as $z_0 = p_a$ and $z_1 = p_b$, its matrix form can be written as

$$\begin{pmatrix} \Phi_0(z_0) & \Phi_1(z_0) & \Phi_2(z_0) & \cdots & \Phi_n(z_0) \\ N\Phi_0(z_1) & N\Phi_1(z_1) & N\Phi_2(z_1) & \cdots & N\Phi_n(z_1) \\ \vdots & & & \vdots & \\ N\Phi_0(z_{n-1}) & N\Phi_1(z_{n-1}) & N\Phi_2(z_{n-1}) & \cdots & N\Phi_n(z_{n-1}) \\ \Phi_0(z_n) & \Phi_1(z_n) & \Phi_2(z_n) & \cdots & \Phi_n(z_n) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} p_a \\ g(z_1) \\ \vdots \\ g(z_{n-1}) \\ p_b \end{pmatrix} \tag{3.13}$$

If the matrix in Eq. (3.13) is non-singular for the chosen collocation points $z_m \quad m = 1, \ldots, n-1$, then the solution of the linear fractional differential equation is written as

$$f(z) = \sum_{\kappa=0}^{n} c_\kappa \frac{\Gamma(\eta)\,\Gamma(\alpha(\kappa+1)+\sigma)\, z^\kappa}{\Gamma(\sigma)\,\Gamma(\rho(\kappa+1)+\eta)\,\kappa!} \qquad \kappa = 0, \ldots, n \tag{3.14}$$

3.3. Example 1

Let us consider the example for a linear fractional order differential equation as

$$D^\beta y(z) + 2y(z) = 0 \quad 0 < \beta \le 1 \quad y(0) = 1 \tag{3.15}$$

Using Eqs. (3.2) and (3.4) for this example

$$\sum_{k=0}^{3} \frac{c_k z^k(\Gamma(\eta)\Gamma((k+1)\alpha+\sigma))}{k!\,\Gamma(\sigma)\Gamma(\eta+(k+1)\rho)} = \frac{c_0\Gamma(\eta)\Gamma(\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+\rho)} + \frac{c_3 z^3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_2 z^2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_1 z\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} \tag{3.16}$$

and utilizing the initial condition, i.e. $y = 0, z = 0$ implies

$$c_0 = \frac{\Gamma(\sigma)\Gamma(\eta+\rho)}{\Gamma(\eta)\Gamma(\alpha+\sigma)}. \tag{3.17}$$

As for the collocation points using Eq. (3.7) in Eq. (3.15), we obtain the following equations:

$$\frac{c_1\Gamma(\eta)z^{1-\beta}\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{c_2\Gamma(\eta)z^{2-\beta}\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_3\Gamma(\eta)z^{3-\beta}\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + 2\left(\frac{c_3 z^3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_2 z^2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_1 z\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + 1\right) = 0. \tag{3.18}$$

At $z = 1, 0.5, 0.25, 0.75$ the following equations can be obtained from Eq. (3.18)

$$\frac{c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)} + 2\left(\frac{c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{24\Gamma(\sigma)\Gamma(\eta+5\rho)} + 1\right) = 0 \tag{3.19}$$

$$\frac{0.5^{1-\beta}c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.5^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.5^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.5^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)} + 2\left(\frac{0.5c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.0208333c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.00260417c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)} + 1\right) = 0 \tag{3.20}$$

$$\frac{0.25^{1-\beta}c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.25^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.25^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.25^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)} + 2\left(\frac{0.25c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.03125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.00260417c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.00016276c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)} + 1\right) = 0 \tag{3.21}$$

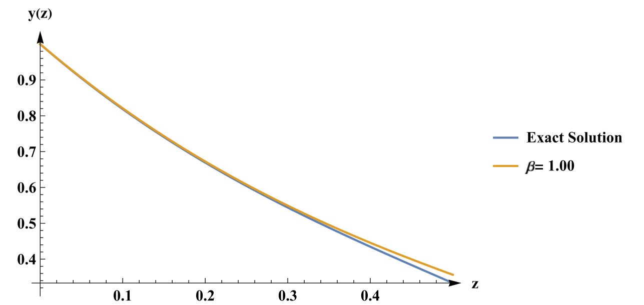

Fig. 1. Comparison of exact solution and numerical solution at $\beta = 1$ by solving Ex. (3.15) at $\alpha = \rho = 0$ and $\eta = \sigma = \beta = 1$.

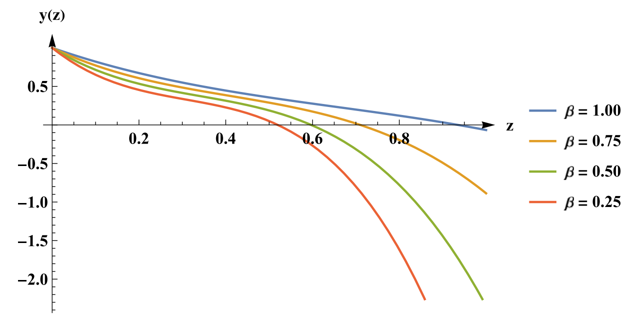

Fig. 2. Numerical solution obtained by solving Ex. (3.15) at $\beta = 1, 0.75, 0.5, 0.25, \alpha = \rho = 0$ and $\eta = \sigma = 1$.

$$\frac{0.75^{1-\beta}c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.75^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.75^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.75^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)} + 2\left(\frac{0.75c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.28125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.0703125c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.0131836c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)} + 1\right) = 0. \tag{3.22}$$

We use Eq. (3.6) to solve this linear equation in order to obtain $c_1 = -1.96215, c_2 = 3.60883, c_3 = -5.4511, c_4 = 4.84543$ at $\alpha = \rho = 0 and \eta = \sigma = \beta = 1$. Then, the series solution becomes

$$y_{approx}(z) = -0.908517z^3 + 1.80442z^2 - 1.96215z + 1 \tag{3.23}$$

which is approximately the series of its exact solution $y(z) = e^{-2z}$.

Upon solving Eq. (3.15) with the ML-CHF point collocation method, an approximate numerical solution as Eq. (3.23) is obtained, which is a very good approximation as can be seen in Fig. 1. If we put $\alpha = \rho = 0$ and $\eta = \sigma = \beta = 1$, it becomes the ordinary differential equation, and hence the series solution obtained from the exact solution is the same as that in Eq. (3.23). Fig. 2 shows the fractional behavior for the numerical solution obtained in Eq. (3.23) for different values of β. Hence, Figs. 1 and 2 show that this method is very effective in solving a fractional differential equation.

3.4. Example 2

Let us consider the example of a linear multi-fractional order differential equation as

$$D^{\beta}y(z)+D^{p}y(z)+y(z)=1+z \quad 0<p\le 1 \quad 1<\beta\le 2 \quad y(0)=1,\, y'(0)=1 \tag{3.24}$$

Using Eqs. (3.2) and (3.4),

$$\frac{c_0\Gamma(\eta)\Gamma(\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+\rho)}+\frac{c_5 z^5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{120\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{c_4 z^4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{24\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_3 z^3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_2 z^2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{c_1 z\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} \tag{3.25}$$

for this example, from the initial conditions, we obtain

$$c_0=\frac{\Gamma(\sigma)\Gamma(\eta+\rho)}{\Gamma(\eta)\Gamma(\alpha+\sigma)} \qquad c_1=\frac{\Gamma(\sigma)\Gamma(\eta+2\rho)}{\Gamma(\eta)\Gamma(2\alpha+\sigma)} \tag{3.26}$$

For the collocation points using Eq. (3.7) in Eq. (3.24), we obtain the following equation

$$\frac{c_2\Gamma(\eta)z^{2-p}\Gamma(3\alpha+\sigma)}{\Gamma(3-p)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{c_3\Gamma(\eta)z^{3-p}\Gamma(4\alpha+\sigma)}{\Gamma(4-p)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_4\Gamma(\eta)z^{4-p}\Gamma(5\alpha+\sigma)}{\Gamma(5-p)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_5\Gamma(\eta)z^{5-p}\Gamma(6\alpha+\sigma)}{\Gamma(6-p)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{c_2\Gamma(\eta)z^{2-\beta}\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{c_3\Gamma(\eta)z^{3-\beta}\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_4\Gamma(\eta)z^{4-\beta}\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_5\Gamma(\eta)z^{5-\beta}\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{c_5 z^5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{120\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{c_4 z^4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{24\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_3 z^3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_2 z^2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{z^{1-p}}{\Gamma(2-p)}+\frac{z^{1-\beta}}{\Gamma(2-\beta)}+\frac{z^{-\beta}}{\Gamma(1-\beta)}+z+1=z+1 \tag{3.27}$$

At $z=1,0.75,0.5,0.25$ the following equations can be obtained from Eq. (3.27)

$$\frac{1}{\Gamma(1-\beta)}+\frac{1}{\Gamma(2-\beta)}+\frac{c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{24\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{120\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-p)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-p)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-p)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-p)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{1}{\Gamma(2-p)}+2=2 \tag{3.28}$$

$$\frac{0.75^{-\beta}}{\Gamma(1-\beta)}+\frac{0.75^{1-\beta}}{\Gamma(2-\beta)}+\frac{0.75^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{0.75^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{0.75^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{0.75^{5-\beta}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{0.28125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{0.0703125c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{0.0131836c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{0.00197754c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{c_2 0.75^{2-p}\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-p)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{c_3 0.75^{3-p}\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-p)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_4 0.75^{4-p}\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-p)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_5 0.75^{5-p}\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-p)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{0.75^{1-p}}{\Gamma(2-p)}+1.75=1.75 \tag{3.29}$$

$$\frac{0.5^{-\beta}}{\Gamma(1-\beta)}+\frac{0.5^{1-\beta}}{\Gamma(2-\beta)}+\frac{0.5^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{0.5^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{0.5^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{0.5^{5-\beta}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{0.125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{0.0208333c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{0.00260417c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{0.000260417c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{c_2 0.5^{2-p}\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-p)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{c_3 0.5^{3-p}\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-p)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_4 0.5^{4-p}\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-p)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_5 0.5^{5-p}\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-p)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{0.5^{1-p}}{\Gamma(2-p)}+1.5=1.5 \tag{3.30}$$

$$\frac{0.25^{-\beta}}{\Gamma(1-\beta)}+\frac{0.25^{1-\beta}}{\Gamma(2-\beta)}+\frac{0.25^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{0.25^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{0.25^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{0.25^{5-\beta}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{0.03125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{0.00260417c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{0.00016276c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{8.13802083\times 10^{-6}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{c_2 0.25^{2-p}\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-p)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{c_3 0.25^{3-p}\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-p)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{c_4 0.25^{4-p}\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-p)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{c_5 0.25^{5-p}\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-p)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{0.25^{1-p}}{\Gamma(2-p)}+1.25=1.25 \tag{3.31}$$

We use Eq. (3.6) to solve this linear equation to obtain $c_1=-1.003083$, $c_2=1.028449$, $c_3=-0.163810, c_4=-0.416960$ at $\alpha=\rho=0$ and $\eta=\sigma=p=1, \beta=2$. Then, the series solution becomes

$$y_{approx}(z)=0.171408z^3-0.501542z^2+z+1 \tag{3.32}$$

which is approximately the series of its exact solution $y(z)=\frac{1}{3}e^{-\frac{z}{2}}\left(3e^{z/2}z+\sqrt{3}\sin\left(\frac{\sqrt{3}z}{2}\right)+3\cos\left(\frac{\sqrt{3}z}{2}\right)\right)$.

Upon solving Eq. (3.24) with the ML-CHF point collocation method, an approximate numerical solution as Eq. (3.32) is obtained, which is a very good approximation as can be seen in Fig. 3 for $\beta=2, p=1$. If we put $\alpha=\rho=0$, $\beta=2$ and $\eta=\sigma=p=1$, it becomes the ordinary differential equation, and hence the series solution obtained

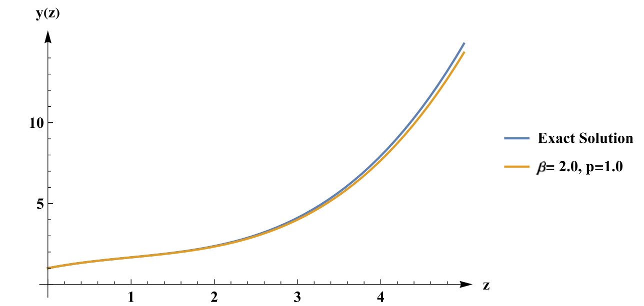

Fig. 3. Comparison of exact solution and numerical solution at $\beta = 2$ and $p = 1$ by solving Ex. (3.24) at $\alpha = \rho = 0$ and $\eta = \sigma = 1$.

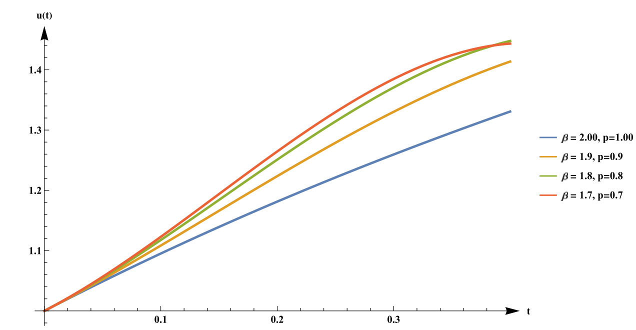

Fig. 4. Numerical solution obtained by solving Ex. (3.24) at $p = 1, 0.75, 0.5, 0.25, \beta = 2, 1.75, 1.5, 1.25 \alpha = \rho = 0$ and $\eta = \sigma = 1$.

from the exact solution is the same as that in Eq. (3.32). Fig. 4 shows the fractional behavior for the numerical solution obtained in Eq. (3.32) for different values of β and p. Hence, Figs. 3 and 4 show that this method is very effective in solving a multi-order fractional differential equation. While solving this particular problem, it has been observed that if we need more accuracy in the series solution, then the number of terms can be increased in Eq. (3.2) and (3.9), and the problem can be solved accordingly. As the number of terms increases, more coefficients need to be obtained, hence the accuracy of the series solution.

3.5. Example 3

Let us consider the example of a fractional order differential equation as

$$D^{\beta} y(z) + y(z) - (z+1)\sin(z) - z\cos(z) = 0, \quad 0 < \beta \leq 1 \quad y(0) = 1 \tag{3.33}$$

Using Eqs. (3.2) and (3.4),

$$\sum_{k=0}^{5} \frac{c_k z^k (\Gamma(\eta)\Gamma((k+1)\alpha+\sigma))}{k!\Gamma(\sigma)\Gamma(\eta+(k+1)\rho)} = \frac{c_0\Gamma(\eta)\Gamma(\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+\rho)} + \frac{c_5 z^5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{120\Gamma(\sigma)\Gamma(\eta+6\rho)} + \frac{c_4 z^4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{24\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{c_3 z^3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_2 z^2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_1 z\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} \tag{3.34}$$

for this example, from initial conditions, we obtain

$$c_0 = \frac{\Gamma(\sigma)\Gamma(\eta+\rho)}{\Gamma(\eta)\Gamma(\alpha+\sigma)}$$

As for the collocation points using Eq. (3.7) in Eq. (3.33), we obtain the following equation

$$\frac{c_1\Gamma(\eta)z^{1-\beta}\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{c_2\Gamma(\eta)z^{2-\beta}\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_3\Gamma(\eta)z^{3-\beta}\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_4\Gamma(\eta)z^{4-\beta}\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{c_5\Gamma(\eta)z^{5-\beta}\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)} + \frac{c_5 z^5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{120\Gamma(\sigma)\Gamma(\eta+6\rho)} + \frac{c_4 z^4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{24\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{c_3 z^3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_2 z^2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_1 z\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} - z\sin(z) - \sin(z) - z\cos(z) + 1 = 0 \tag{3.35}$$

At $z = 1, 0.75, 0.5, 0.25, 0.15$, the following equations can be obtained from Eq. (3.35)

$$\frac{c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)} + \frac{c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{24\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{120\Gamma(\sigma)\Gamma(\eta+6\rho)} + 1 - 2\sin(1) - \cos(1) = 0 \tag{3.36}$$

$$\frac{0.75^{1-\beta}c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.75^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.75^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.75^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{0.75^{5-\beta}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)} + \frac{0.75c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.28125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.0703125c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.0131836c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{0.00197754c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+6\rho)} - 0.741634 = 0 \tag{3.37}$$

$$\frac{0.5^{1-\beta}c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.5^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.5^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.5^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{0.5^{5-\beta}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)} + \frac{0.5c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.0208333c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.00260417c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{0.000260417c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+6\rho)} - 0.15793 = 0 \tag{3.38}$$

$$\frac{0.25^{1-\beta}c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.25^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.25^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.25^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{0.25^{5-\beta}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)} + \frac{0.25c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.03125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.00260417c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.00016276c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)} + \frac{8.13802083\times10^{-6}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+6\rho)} + 0.448517 = 0 \tag{3.39}$$

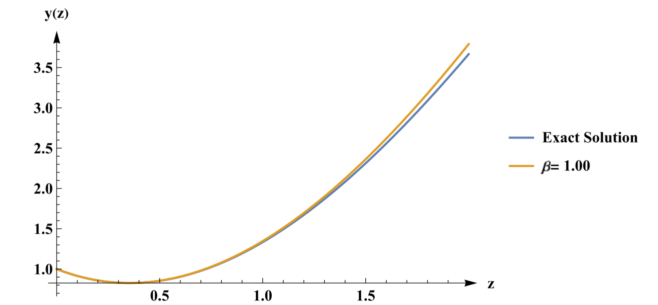

Fig. 5. Comparison of exact solution and numerical solution at $\beta = 1$ by solving Ex. (3.33) at $\alpha = \rho = 0$ and $\eta = \sigma = 1$.

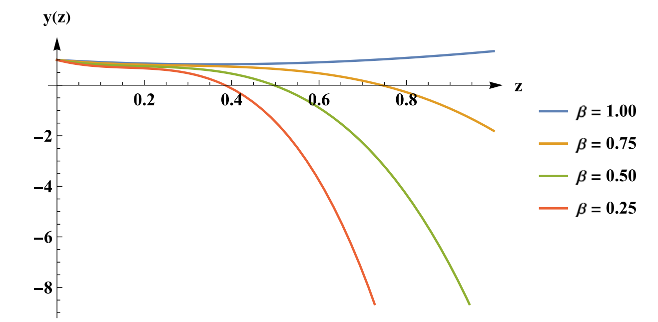

Fig. 6. Numerical solution obtained by solving Ex. (3.33) at $\beta = 1, 0.75, 0.5, 0.25, \alpha = \rho = 0$ and $\eta = \sigma = 1$.

$$\frac{0.15^{1-\beta}c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)}+\frac{0.15^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{0.15^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{0.15^{4-\beta}c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(5-\beta)\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{0.15^{5-\beta}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(6-\beta)\Gamma(\sigma)\Gamma(\eta+6\rho)}+\frac{0.15c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)}+\frac{0.01125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)}+\frac{0.0005625c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)}+\frac{0.0000210938c_4\Gamma(\eta)\Gamma(5\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+5\rho)}+\frac{6.3281249\times10^{-7}c_5\Gamma(\eta)\Gamma(6\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+6\rho)}+0.67983=0 \tag{3.40}$$

We use Eq. (3.6) to solve this linear equation in order to obtain $c_0 = 1, c_1 = -0.999374, c_2 = 2.990039, c_3 = -0.8927, c_4 = -3.76682, c_5 = 2.3368$ at $\alpha = \rho = 0$ and $\eta = \sigma = 1, \beta = 1$. Then, the series solution becomes

$$y_{approx}(z) = 0.148783z^3 + 1.49501z^2 - 0.999374z + 1 \tag{3.41}$$

which is approximately the series of its exact solution $y(z) = e^{-z}(e^z z\sin(z) + 1)$.

Upon solving Eq. (3.33) with the ML-CHF point collocation method, an approximate numerical solution as Eq. (3.41) is obtained, which is a very good approximation as can be seen in Fig. 5 for $\beta = 1$. If we put $\alpha = \rho = 0$, $\beta = 1$ and $\eta = \sigma = 1$, it becomes the ordinary differential equation, and hence the series solution obtained from the exact solution is the same as that in Eq. (3.41). Fig. 6 shows the fractional behavior for the numerical solution obtained in Eq. (3.41) for different values of β. Hence, Figs. 5 and 6 show that this method is very effective in solving a fractional order differential equation.

3.6. Example 4

Let us consider another fractional order differential equation as

$$D^\beta y(z) + y(z) + \frac{z^{1-\beta}}{\Gamma(2-\beta)} - \frac{2z^{2-\beta}}{\Gamma(3-\beta)} - z^2 + z = 0 \quad 0 < \beta \le 1 \quad y(0) = 0 \tag{3.42}$$

Using Eqs. (3.2) and (3.4),

$$\sum_{k=0}^{3}\frac{c_k z^k(\Gamma(\eta)\Gamma((k+1)\alpha+\sigma))}{k!\Gamma(\sigma)\Gamma(\eta+(k+1)\rho)} = \frac{c_0\Gamma(\eta)\Gamma(\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+\rho)} + \frac{c_3z^3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_2z^2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_1z\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} \tag{3.43}$$

for this example, from the initial conditions, we obtain

$$c_0 = 0 \tag{3.44}$$

For the collocation points, using Eq. (3.7) in Eq. (3.42), we obtain the following equation

$$\frac{c_1\Gamma(\eta)z^{1-\beta}\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{c_2\Gamma(\eta)z^{2-\beta}\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_3\Gamma(\eta)z^{3-\beta}\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_3z^3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_2z^2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_1z\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{z^{1-\beta}}{\Gamma(2-\beta)} - \frac{2z^{2-\beta}}{\Gamma(3-\beta)} - z^2 + z = 0 \tag{3.45}$$

At $z = 1, 0.75, 0.5$ the following equations can be obtained from Eq. (3.45)

$$\frac{1}{\Gamma(2-\beta)} - \frac{2}{\Gamma(3-\beta)} + \frac{c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{2\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{6\Gamma(\sigma)\Gamma(\eta+4\rho)} = 0 \tag{3.46}$$

$$\frac{0.75^{1-\beta}}{\Gamma(2-\beta)} - \frac{20.75^{2-\beta}}{\Gamma(3-\beta)} + \frac{0.75^{1-\beta}c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.75^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.75^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.75c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.28125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.0703125c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)} + 0.1875 = 0 \tag{3.47}$$

$$\frac{0.5^{1-\beta}}{\Gamma(2-\beta)} - \frac{20.5^{2-\beta}}{\Gamma(3-\beta)} + \frac{0.5^{1-\beta}c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(2-\beta)\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.5^{2-\beta}c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(3-\beta)\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.5^{3-\beta}c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(4-\beta)\Gamma(\sigma)\Gamma(\eta+4\rho)} + \frac{0.5c_1\Gamma(\eta)\Gamma(2\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+2\rho)} + \frac{0.125c_2\Gamma(\eta)\Gamma(3\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+3\rho)} + \frac{0.0208333c_3\Gamma(\eta)\Gamma(4\alpha+\sigma)}{\Gamma(\sigma)\Gamma(\eta+4\rho)} + 0.25 = 0 \tag{3.48}$$

We use Eq. (3.6) to solve this linear equation to obtain $c_1 = -1, c_2 = 2, c_3 = 0$ at $\alpha = \rho = 0$ and $\eta = \sigma = 1, \beta = 1$. Then, the series solution becomes

$$y_{approx}(z) = z^2 - z \tag{3.49}$$

which is approximately the series of its exact solution, $y(z) = -z + z^2$.

Upon solving Eq. (3.42) with the ML-CHF point collocation method, an approximate numerical solution as Eq. (3.49) is obtained, which is

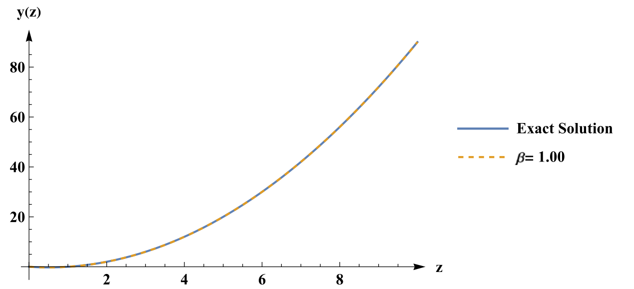

Fig. 7. Comparison of exact solution and numerical solution at $\beta = 1$ by solving Ex. (3.42) at $\alpha = \rho = 0$ and $\eta = \sigma = 1$.

a very good approximation as can be seen in Fig. 7 for $\beta = 1$. If we put $\alpha = \rho = 0$, $\beta = 1$ and $\eta = \sigma = p = 1$, it becomes the ordinary differential equation, and hence the series solution obtained from the exact solution is the same as that in Eq. (3.49). Hence, Fig. 7 shows that this method is very effective in solving a fractional order differential equation. This specific problem demonstrates the effectiveness of the method, as it can also be used to obtain the analytical solution of a fractional order differential equation in certain situations.

4. Conclusions

To conclude, this study presented the introduction and examination of a new class of special functions, which includes the Mittag-Leffler and confluent hypergeometric functions ML-CHF. The combination of fractional calculus with this new class was explored, and several applications were discovered. A new ML-CHF point collocation method has been developed using this special class of functions. This paper uses a numerical method to solve some problems and demonstrate its effectiveness for specific examples. The results demonstrate the high efficiency of this method. Further research on these outcomes could lead to the exploration of connections between the Mittag-Leffler function and the new fractional operator, potentially simplifying significant physical models and applications in probability theory.

CRediT authorship contribution statement

F. Ghanim: Writing – original draft, Methodology, Conceptualization. **Fareeha Sami Khan:** Writing – original draft, Validation, Methodology, Data curation. **Hiba F. Al-Janaby:** Writing – review & editing, Supervision, Resources, Investigation, Formal analysis. **Ali Hasan Ali:** Writing – review & editing, Visualization, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data were used to support the study.

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