Extensions of loops and quasigroups

Doctoral (PhD) dissertation

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a jelölt aláírása

Tanusítom, hogy Stuhl Izabella doktorjelölt 2005-2008 között a fent megnevezett Doktori Iskola Differenciálgeometria és alkalmazásai programjának keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

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Chapter 1

Introduction

In the sense of the well-known Schreier extension, beginning with two given groups a new one is constructed which contains the first group as a normal subgroup such that the quotient group with respect to this normal subgroup is isomorphic to the second one. This construction is carried out by defining an appropriate operation on the cartesian product of the given structures and described by the following exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \mathfrak{G} \longrightarrow G \longrightarrow 1.$$

Let G and Γ be two groups, the group $\mathfrak{G}$ is an extension of Γ by G if on the set $G \times \Gamma$ is defined a multiplication by

$$(a, \alpha)(b, \beta) = (ab, f(a, b) \cdot \alpha^{T(b)} \beta),$$

where $T : G \longrightarrow Aut(\Gamma)$ and $f : G \times G \longrightarrow \Gamma$ is the factor system with

1. $f(1, a) = f(a, 1) = 1$;
2. $T(a)^{-1} T(b)^{-1} f(a, b) T(ab) f(a, b)^{-1} = 1$;

3. $f(b,c)f(a,bc) = f(a,b)^{T(c)}f(ab,c)$;

for all $a,b,c \in G$.

The simplest case occurs when T and f are both trivial maps, in this case $\mathfrak{G}$ is the direct product of Γ and G. If the map f is trivial and T is non-trivial, then $\mathfrak{G}$ is a semidirect product of Γ and G. If T is trivial and f is non-trivial, then we have for the multiplication

$$(a,\alpha)(b,\beta) = (ab, f(a,b) \cdot \alpha \cdot \beta),$$

which is called f-extension.

Similar construction can be given for loops and quasigroups with appropriate modification, but the lack of associativity changes the situation so drastically, that a general extension theory of these structures does not exist. Nowadays, many authors apply extensions of loops and quasigroups in different interpretations for example [3], [11], [14], [16], [17], [18]. Loop extension of groups by loops is studied by Péter T. Nagy and Karl Strambach in [22] very systematically. The goal of this dissertation is to show some further investigations of f-extensions of loops and quasigroups in sense of the above mentioned authors, described by exact sequences

$$1 \longrightarrow \Lambda \longrightarrow \mathfrak{L} \longrightarrow L \longrightarrow 1,$$

respectively

$$Q \longrightarrow \mathfrak{Q} \longrightarrow K,$$

where Λ, L are loops and K, Q are quasigroups.

1.1 Preliminaries

A *quasigroup* $(Q, \cdot)$ is a set Q with a binary operation $(x, y) \mapsto x \cdot y$, where the equations $a \cdot y = b$ and $x \cdot a = b$ have precisely one solution in Q which we denote by $y = a \backslash b$ and $x = b/a$. The elements $e_l(a) = a/a$ respectively $e_r(a) = a \backslash a$ are the left respectively the right local identity element of the element a. If the left (right) local identity elements coincide for all elements of $(Q, \cdot)$, then the element $e_l = e_l(a)$, respectively $e_r = e_r(a)$, is called the left, respectively right, unit element of $(Q, \cdot)$. If a quasigroup $(Q, \cdot)$ has both left and right unit elements, then they coincide $e = e_l = e_r$, in this case $(Q, \cdot)$ is called a *loop*. A quasigroup (loop) L satisfies the *left inverse property*, respectively the *right inverse property* if for any $x \in L$ there exists an element x^{-1} such that $x^{-1} \cdot (x \cdot y) = y$, respectively $(y \cdot x) \cdot x^{-1} = y$ holds for all $x, y \in L$. A quasigroup (loop) L has the *automorphic inverse property* if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in L$. Another type of inverse property is the *cross inverse property* $(xy)x' = y$ for all $x, y \in L$, where $x' \in L$ depends only on x.

A quasigroup (loop) L is *left alternative*, respectively *right alternative* if $x \cdot (x \cdot y) = x^2 \cdot y$, respectively $(y \cdot x) \cdot x = y \cdot x^2$ for all $x, y \in L$.

A quasigroup (loop) L is *flexible* if $x \cdot (y \cdot x) = (x \cdot y) \cdot x$ for all $x, y \in L$.

A quasigroup (loop) L is *diassociative* if any two elements generate a subgroup i.e., if $x, y \in L$, then $< x, y >$ is a group.

A quasigroup (loop) L is a *left*, respectively *right Bol quasigroup (loop)* if it satisfies the identity

$$(x \cdot (y \cdot x)) \cdot z = x \cdot (y \cdot (x \cdot z)), \quad \text{respectively} \quad z \cdot (x \cdot (y \cdot x)) = ((z \cdot x) \cdot y) \cdot x.$$

A quasigroup (loop) which has both the left and the right Bol identity is called a *Moufang quasigroup (loop)*. Each of the Moufang identities in a quasigroup implies that the quasigroup is a loop [19], p. 232. A quasigroup Q is called *totally symmetric* if $x \cdot y = y \cdot x$, and $x \cdot (x \cdot y) = y$ hold for all $x, y \in Q$. If $x^2 = x$ for all $x \in Q$, then Q is called a *Steiner* *quasigroup*, whereas if $x^2 = e$ for all $x \in Q$, then Q is called a *Steiner loop.* If Q is a Steiner quasigroup or loop, then for the translations we have $\lambda_a = \rho_a$ it means they are involutions. A loop $(Q, \cdot)$ is called *involutorial* if $x^2 = e$ for all $x \in Q$.

The *left, right* respectively *middle nucleus* of Q are the subgroups of Q which are defined in the following way:

$$N_l(Q) = \{u;\ (u \cdot x) \cdot y = u \cdot (x \cdot y),\ x, y \in Q\},$$

$$N_r(Q) = \{u;\ (x \cdot y) \cdot u = x \cdot (y \cdot u),\ x, y \in Q\},$$

$$N_m(Q) = \{u;\ (x \cdot u) \cdot y = x \cdot (u \cdot y),\ x, y \in Q\}.$$

The intersection $N(Q) = N_l(Q) \cap N_r(Q) \cap N_m(Q)$ is called the *nucleus* of Q. Any subgroup of $N(Q)$ is called a *nuclear subgroup* of Q.
The *commutant* $C(Q)$ of $(Q, \cdot)$ is the subset consisting of all elements $c \in Q$ such that $c \cdot x = x \cdot c$ for all $x \in Q$.
The *center* $C(Q)$ of Q is the largest subgroup of the nucleus $N(Q)$ such that $z \cdot x = x \cdot z$ for all $x \in Q, z \in C(Q)$. Any subgroup of $C(Q)$ is a normal subgroup of Q, called a *central subgroup* of Q.
For any $a \in Q$ the maps $L_a : y \mapsto a \cdot y$ and $R_a : y \mapsto y \cdot a$ are the *left* and the *right translation*s, respectively. The left translations of $(Q, \cdot)$ generate the group $G_l(Q)$, the right translations of $(Q, \cdot)$ generate the group $G_r(Q)$.

A bijection $\lambda : Q \to Q$ is a *left-regular permutation* of $(Q, \cdot)$, if $\lambda(xy) = \lambda(x) \cdot y$ for all $x, y \in Q$. A bijection $\rho : Q \to Q$ is a *right-regular permutation* of $(Q, \cdot)$, if $\rho(xy) = x \cdot \rho(y)$ for all $x, y \in Q$. If λ is a left-regular permutation, then $\lambda = \lambda_{\lambda(x)}\lambda_x^{-1}$, similarly if ρ is a right-regular permutation, then $\rho = \rho_{\rho(x)}\rho_x^{-1}$ for all $x \in Q$. Hence $\Lambda(Q) \subset G_l(Q)$ and $\mathrm{R}(Q) \subset G_r(Q)$. Moreover, if $(Q, \cdot)$ is a left loop, then $\lambda = \lambda_{\lambda(e_l)}$ and hence $\lambda_{\lambda(e_l)}$ belongs to the left nucleus N_l of $(Q, \cdot)$. Similarly, if $(Q, \cdot)$ is a right loop, then $\rho = \rho_{\rho(1_r)}$ and hence $\rho_{\rho(1_r)}$ belongs to the right nucleus N_r of $(Q, \cdot)$. Conversely, for any element

$n \in N_l$ in a left loop $(Q, \cdot)$ the permutation λ_n is left-regular and hence $\Lambda(Q)$ is isomorphic to the left nucleus N_l of $(Q, \cdot)$. Analogously, for any element $n \in N_r$ in a right loop $(Q, \cdot)$ the permutation ρ_n is right-regular and hence $\mathrm{R}(Q)$ is isomorphic to the right nucleus N_r of $(Q, \cdot)$.

The group of left-regular permutations $\Lambda(Q)$ of a quasigroup $(Q, \cdot)$ give an equivalence relation on the quasigroup, two elements a and b are equivalent if there is a left-regular permutation $\lambda \in \Lambda(Q)$ such that $b = \lambda(a)$. An equivalence class has the shape $\Lambda(Q)q = \{\lambda q, \lambda \in \Lambda(Q)\}$, where q is an arbitrary element of Q. Such a class $\Lambda(Q)q$ is called the *left q-nucleus* with regard to the element $q \in Q$. The *right q-nucleus* $\mathrm{R}(Q)q$ with regard to the element $q \in Q$ of the quasigroup $(Q, \cdot)$ can be defined analogously. These results for regular permutations issue from V. D Belousov [1] p. 22-25. and although can be found in [25] Chap.III, Sec.3, p. 71-73.

Chapter 2

Central extension

In this chapter we investigate central extensions of groups by loops which have combinatorial background. In the first section we consider these combinatorial structures, give them an algebraic face and determine their groups of translations. Since for a loop L the knowledge of the group G generated by the set $\mathfrak{L}$ of its left translations is essential. Namely, if H is the stabilizer of the identity of L in G, then L is isomorphic to the loop defined on $\mathfrak{L}$ by $(x, y) \mapsto x * y = \pi(x, y) : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$, where π assigns to the element $xy \in G$ the left translation of L contained in xyH. (cf. [21] Section 1.2). These results can be fined in [26].

2.1 Translation groups of Steiner loops

A *Steiner triple system* $\mathfrak{S}$ is an incidence structure consisting of points and blocks such that every two distinct points are contained in precisely one block and any block has precisely three points. A finite Steiner triple system with n points exists if and only if $n \equiv 1$ or 3

(*mod* 6) (cf. [25] V.1.9 Definition, p. 124).

For turning a Steiner triple system $\mathfrak{S}$ into a Steiner quasigroup $(Q(\mathfrak{S}), *)$, we define as $a * b$ the third point of the line determined by a, b and put $a*a = a^2 = a$ for all $a \in \mathfrak{S}$. Moreover, with a Steiner triple system is associated a Steiner loop $(S(\mathfrak{S}), \circ)$ such that the elements of $S(\mathfrak{S}) \setminus \{e\}$, where e is the identity of $S(\mathfrak{S})$, are the points of the Steiner triple system $\mathfrak{S}$ and $a \circ b := a * b$ for $a \neq b$, $a, b \in S(\mathfrak{S}) \setminus \{e\}$, whereas $a \circ a = a^2 = e$. A Steiner loop is a commutative loop of exponent 2 with the inverse property [25], Chap.V.

Very often the identities which hold in a Steiner quasigroup associated with a Steiner triple system $\mathfrak{S}$ do not hold in the Steiner loop $S(\mathfrak{S})$ associated with $\mathfrak{S}$. Examples are the Steiner triple systems, called *Hall systems*, in which every three non-collinear points generate an affine plane of order 3. In the associated Steiner quasigroup $Q(\mathfrak{S})$ one has $(xy)z = (xz)(yz)$ for all $x, y, z \in Q(\mathfrak{S})$ but this identity fails to hold in the associated Steiner loop $S(\mathfrak{S})$.

The automorphism group of a Steiner loop coincides with the automorphism group of the corresponding Steiner triple system. In [7], Corollary 1, p. 251 it is proved that a Steiner loop $S(\mathfrak{S})$ is an elementary abelian group of order $n+1 = 2^m$ if and only if the Steiner triple system $\mathfrak{S}$ corresponding to $S(\mathfrak{S})$ is isomorphic to the projective space of dimension $m-1$ over the field $GF(2)$. Hence the group of translations of a Steiner loop corresponding to a projective space is an elementary abelian 2-group.

For Steiner loops corresponding to Steiner triple systems which are not projective spaces the situation changes drastically.

Proposition 2.1.1. *Let $\mathfrak{S}$ be a Steiner triple system of order n, let $Q(\mathfrak{S})$ be the Steiner quasigroup and let $S(\mathfrak{S})$ be the Steiner loop corresponding to $\mathfrak{S}$. If the product ${\lambda^*}_a {\lambda^*}_b$ of different translations ${\lambda^*}_a$ and ${\lambda^*}_b$ of $Q(\mathfrak{S})$ has odd order, then in the group generated by the* *translations of* $S(\mathfrak{S})$ *there is an involution* $\delta_{a,b}$ *interchanging* a *and* b *as well as* e *with* $a \circ b$ *and fixing all other elements of* $S(\mathfrak{S})$*.*

Proof. The translation $\lambda^*{}_a$ of the Steiner quasigroup $Q(\mathfrak{S})$ fixes a and maps x on the third point of the block a, x. The product $\lambda^*{}_a\lambda^*{}_b$ leaves the block $\{a, b, a \circ b\}$ invariant. The translation λ_a of the Steiner loop $S(\mathfrak{S})$ acts on the point set $S(\mathfrak{S}) \setminus \{e, a\}$ in the same manner as $\lambda^*{}_a$ however it interchanges e with a. The product $\lambda_a\lambda_b$ operates on $S(\mathfrak{S}) \setminus \{e, a, b, a \circ b\}$ in the same manner as the product $\lambda^*{}_a\lambda^*{}_b$ but interchanges e with $a \circ b$, respectively a with b. Since $\lambda^*{}_a\lambda^*{}_b$ has odd order, say m, and $\lambda_a\lambda_b$ induces on $\{e, a, b, a \circ b\}$ an involution it follows that the element $\delta_{a,b} := (\lambda_a\lambda_b)^m$ is an involution fixing the set $S(\mathfrak{S}) \setminus \{e, a, b, a \circ b\}$ element-wise and interchanging e with $a \circ b$, respectively a with b. ☐

Theorem 2.1.2. *Let* $S(\mathfrak{S})$ *be a proper Steiner loop of order* n *and let* $Q(\mathfrak{S})$ *be a Steiner quasigroup both corresponding to the Steiner triple system* $\mathfrak{S}$*. If the product* $\lambda^*{}_a\lambda^*{}_b$ *of any two distinct translations of* $Q(\mathfrak{S})$ *has odd order, then the group* G *generated by the translations of* $S(\mathfrak{S})$ *is the alternating group* A_n *or the symmetric group* S_n *depending whether* n *is divisible by* 4 *or not.*

Proof. Since $S(\mathfrak{S})$ is a proper Steiner loop the order n of $S(\mathfrak{S})$ is at least 8. As the element e of $S(\mathfrak{S})$ is no fixed point of G the group G acts 2-transitively precisely if the stabilizer G_e is transitive on $S(\mathfrak{S}) \setminus \{e\}$. Let a, b be different elements of $S(\mathfrak{S}) \setminus \{e\}$ and let B be a block

$\{r, s, a \circ b\}$, where $r, s \notin \{a, b\}$, but $r \circ s = a \circ b$. Let $\delta_{r,s}$ and $\delta_{a,b}$ be involutions as in the previous Proposition. Then $\delta_{r,s}\delta_{a,b}(a) = \delta_{r,s}(b) = b$ and $\delta_{r,s}\delta_{a,b}(e) = \delta_{r,s}(a \circ b) = e$, and G is 2-transitive.

Let a and b be distinct elements of $S(\mathfrak{S}) \setminus \{e\}$ and let $G_{a,b}$ be the stabilizer of G fixing a and b. If c is any point of $S(\mathfrak{S}) \setminus \{a, b\}$ and H is a block $\{r, s, r \circ s\}$ with $r, s \notin \{a, b\}$ and $r \circ s = c$, then for the involution $\delta_{r,s}$ of Proposition 2.1.1 one has $\delta_{r,s}(e) = c$ which shows that G is a 3-transitive group.

Let a, b be different points of $S(\mathfrak{S}) \setminus \{e\}$. We consider the stabilizer $G_{a,b,a\circ b}$ which fixes each of the points a, b and $a \circ b$. Let r be any point of $S(\mathfrak{S}) \setminus \{e, a, b, a \circ b\}$. If any block of $\mathfrak{S}$ through r intersects the block $D = \{a, b, a \circ b\}$, then the Steiner triple system $\mathfrak{S}$ would be the projective plane of order 2 and the Steiner loop $S(\mathfrak{S})$ would be the elementary abelian 2-group of order 8 (cf. [7], Corollary 1, p. 251). Hence there is a block $B = \{s, t, r\}$ of $\mathfrak{S}$ such that $B \cap D$ is empty. But then the involution $\delta_{s,t}$ of Proposition 2.1.1 maps e onto r and fixes each of the points a, b and $a \circ b$. From this it follows that G is a 4-transitive permutation group on $S(\mathfrak{S})$.

Now from [2], Theorem 5.2, p. 625. it follows that a finite 4-transitive group G on the set $S(\mathfrak{S})$ belongs to one of the following classes

(i) G is the alternating group A_n or the symmetric group S_n,

(ii) G is either a Mathieu group acting on 11, 12, 22, 23 or 24 points or the automorphism group $Aut(M_{22})$ of the Mathieu group M_{22}.

If G is a Mathieu group, then G is isomorphic to the group M_{22} since for the order n of $S(\mathfrak{S})$ one has $n \equiv 2$ or 4 $(mod\ 6)$. If G is M_{22} or $Aut(M_{22})$, then G is the automorphism group of the unique Steiner system $\mathfrak{S}(3, 6, 22)$ with 22 points such that three distinct points determines precisely one block and any block has 6 different points (see [8], Theorem 6.6D, p. 200). Since the stabilizer of a point of $\mathfrak{S}(3, 6, 22)$ in the automorphism group of $\mathfrak{S}(3, 6, 22)$ is the group $PSL_3(2)$ ([8], Theorem 6.5B, p. 196) no element of G different from identity can have 18 fixed points.

Hence the group G is isomorphic to A_n or S_n. The translations λ_a of $S(\mathfrak{S})$ are involutions and have no fixed points. Therefore λ_a is a product of even or odd number of transpositions depending whether n is divisible by 4 or not. Since G is generated by the translations of $S(\mathfrak{S})$ the group G is isomorphic either to A_n or to S_n depending whether n is divisible by 4 or not. □

Remark 2.1.3. *Any Steiner quasigroup $Q(\mathfrak{S})$ corresponding to a Hall system satisfies the conditions of Theorem 2.1.2.*

This follows from the fact that the product of any two distinct translations of any such Steiner quasigroup $Q(\mathfrak{S})$ has order three ([6], p. 152). In particular, the group G generated by the translations of the Steiner loop corresponding to the affine space of order 3^n over the field $GF(3)$ is the symmetric group S_{3^n+1} if n is even and the alternating group A_{3^n+1} if n is odd.

There are also Steiner triple systems which are not Hall systems but for which the products of any two distinct translations of the associated Steiner quasigroup have odd order. We illustrate this for

Steiner triple systems constructed in [9], 2.1. p. 291.
Let C be a cyclic group of order $k = \frac{1}{3}(4^n - 1)$ such that $n > 1$ is odd. Let $\mathfrak{S}$ be the disjoined union $\mathfrak{S} = C_0 \cup C_1 \cup C_2$ such that C_0, C_1, C_2 are three exemplars of C. If the element $x \in C$ is contained in the exemplar C_i, then we denote this element with x_i. The blocks of a Steiner triple system $\mathfrak{S}_C$ on the point set $\mathfrak{S}$ of size $4^n - 1$ are the following triples:

(i) all subsets $\{x_0, x_1, x_2\}$, with $x_0 = x_1 = x_2 = x \in C$,

(ii) all subsets $\{x_0, y_0, z_1\}, \{x_1, y_1, z_2\}, \{x_2, y_2, z_0\}$ of $\mathfrak{S}$ with $x, y, z \in C$ such that $x \neq y$ and $xy = z^2$.

The group G generated by the products $\lambda_{a_i}\lambda_{b_j}$, $a_i, b_j \in \mathfrak{S}_C$, of translations of the Steiner quasigroup $Q(\mathfrak{S}_C)$ is a subgroup of the semidirect product of a normal group N by the cyclic group of order 3 acting on the indices $\{i, j, k\}$. The group N is generated by the elements $\tau_{a,b}(x) \mapsto a^{-2}b^{-4}x^{-4}$, $a, b \in C$, which are the projections of $\lambda_{a_i}\lambda_{b_j}$ to N. Since ${\tau_{a,b}}^n : x \mapsto a^{\frac{-2(4^n-1)}{3}} b^{\frac{-4(4^n-1)}{3}} x^{-4n}$ and C has the order $\frac{4^n-1}{3}$ one has ${\tau_{a,b}}^n = 1$. Hence the product of any two distinct translations of $Q(\mathfrak{S}_C)$ has odd order and we obtain

Remark 2.1.4. *Let C be a cyclic group of order $\frac{4^n-1}{3}$ with odd $n > 1$ and let $Q(\mathfrak{S}_C)$ be the Steiner quasigroup corresponding to the Steiner triple system constructed from C as above. Then the translation group of the Steiner loop $S(\mathfrak{S}_C)$ of order 4^n corresponding to $Q(\mathfrak{S}_C)$ is the alternating group A_{4^n}.*

If $\frac{2^{2n}-1}{3}$ has only two prime divisors, then there are three non-collinear points generating $\mathfrak{S}_C$ as well as triples of non-collinear points which do not generate $\mathfrak{S}_C$ ([9], 2.6.Theoreme 1, p. 293). Since in a Hall system any three non-collinear point generate the affine plane of size 9 no of these Steiner triple systems $\mathfrak{S}_C$ is a Hall system.

2.2 Oriented Steiner loops

The loop extensions of a group by a loop has been treated along the ideas of Schreier in [22], but this theory does not allow a deeper insight into the structure of such extensions. Even the simplest case of non-associative extensions of the group of order 2 by an abelian 2-group yields a rich variety of loops which are distinguish among them by different weak associativity conditions. For loops of order 8 we illustrate this fact collecting in the first subsection striking examples.

To obtain more homogeneous classes of extensions it is necessary to restrict the possibilities for the factor systems in an efficient manner. The best known examples in this direction are the code loops (see [13], [15]) and C-loops (cf. [17]). Since one essential source for construction of loops are incidence geometries we treat in this section loop extensions L which are strictly related to the oriented Steiner triple systems; they are extensions of the group of order 2 by a Steiner loop, where the factor systems are given by the orientations of the associated Steiner triple system. As usual for loops related to incidence structures the oriented Steiner loops L have in general only weak associative properties. If L has exponent 2, then L is flexible, has trivial center and satisfies the cross inverse property, but it has neither the right nor the left inverse property. If L does not have exponent 2, then it has exponent 4, the center has order 2 and L satisfies the inverse property. The only oriented Steiner loop of exponent 4 satisfying the alternative laws are the quaternion group of order 8 and the proper Moufang loop of order 16.

The automorphism group Γ of an oriented Steiner loop L is a semidirect product of an elementary abelian 2-group Δ by the automorphism group Σ of the corresponding oriented Steiner triple system $(\mathfrak{S}, T)$. The normal subgroup Δ of Γ consists of automorphisms of L which induce on $\mathfrak{S}$ the identity and the group Σ has odd order. Moreover, the elements of Δ different from the identity are one-to-one correspondence with the proper subsystems $\mathfrak{K}$ of the Steiner triple system $\mathfrak{S}$ having with any block of $\mathfrak{S}$ a non-void intersection.

If K is the automorphism group of an oriented Steiner loop L and if α is an automorphism of the non-oriented Steiner triple system $\mathfrak{S}$ belonging to L, then there exists an oriented Steiner loop isomorphic to L and having the conjugate group K^α as its automorphism group.

Any oriented Steiner loop corresponds to an oriented Steiner triple system $(\mathfrak{S}, T)$, where T is an orientation of the Steiner triple system $\mathfrak{S}$. Clearly, if the oriented Steiner loops L_1 and L_2 correspond to isomorphic oriented Steiner triple systems $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$, then they are isomorphic. But also in the simplest case if the automorphism groups of $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$ coincide the relations between the oriented Steiner loops L_i, $(i = 1, 2)$ remain intricate. They depend heavily on the restrictions of the orientations T_i, $i \in \{1, 2\}$, to the orbits of the automorphism group Γ of $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$. Only if the group Γ acts transitively on the blocks of $\mathfrak{S}$, then the orientations T_1 and T_2 either coincide or are opposite and L_i are isomorphic. If the oriented Steiner loops L_1 and L_2 are isomorphic, then there exists an isomorphism from L_1 onto L_2 which induces on $\mathfrak{S}$ the identity.

The groups $\hat{G}_l$ and $\hat{G}_r$ generated by the left, respectively the right translations of an oriented Steiner loop L of order n are isomorphic; they are extensions of an elementary abelian 2-group with the group generated by the translations of the Steiner loop S belonging to L. If L does not have exponent 2, then these extensions do not split. The group $\hat{G}$ generated by all translations of L contains a normal elementary abelian 2-subgroup Θ such that $\hat{G}/\Theta$ is isomorphic to $\hat{G}_l$ as well as to $\hat{G}_r$. The group Θ has order 2^{n-1} or 2^{n-2} depending whether n is odd or even.

In the section special attention is given to oriented Steiner loops for which the corresponding Steiner triple systems $\mathfrak{S}$ are projective spaces over the field $GF(2)$ or affine spaces over the field $GF(3)$. These loops are highly non-associative also in the case that $\mathfrak{S}$ is a projective space over GF(2) of dimension greater than 3.

In the final subsection a classification of oriented Steiner loops corresponding to the affine plane of order 3 or to the projective plane of order 2 is given.

Results obtained in this section are in [27].

2.2.1 Extensions by Steiner loops

We consider the loop extensions

$$(*) \qquad 1 \longrightarrow A \longrightarrow L \longrightarrow S \longrightarrow 1,$$

where A is the group of order 2 and S is a Steiner loop.

The loop L is realized on the set $S \times A = \{(a, \epsilon) : a \in S, \epsilon \in A\}$ such that the multiplication is given by

$$(a_1, \epsilon_1) \circ (a_2, \epsilon_2) = (a_1 a_2, \epsilon_1 \epsilon_2 f(a_1, a_2)),$$

where f is a function $f : S \times S \longrightarrow A$ with $f(a, e) = f(e, a) = 1$ for all $a \in S$ and the unit element e of S. The identity of L is the element $(e, 1)$ and the left and right inverse of any element coincide. The extension L is determined by the function f which is called a *factor system* of L. If L is not the direct product of A and S, then L does not split.

In a Steiner loop S any two different elements generate the elementary abelian 2-group of order 4 the elements of which we denote always by $\{e, x, y, xy\}$. Hence in an extension L of the type $(*)$ the subloop generated by different elements $(x, \epsilon_1), (y, \epsilon_2)$ of $L \backslash \{(e, 1), (e, -1)\}$ has order 8 and exponent 2 or 4 depending on whether $f(x, x) = 1$ for all $x \in S$ or not.

For a loop L which is a central extension of a group by a loop in [22] necessary and sufficient conditions for the function f in the form of identities are given that L is flexible, left as well as right alternative, has the automorphic inverse property, respectively is a Bol loop (cf. in [22] formulas (14) – (21) p. 771-772). From these identities it follows that an extension L of the form $(*)$ is left alternative if and only if it has the left inverse property, and it is right alternative if and only if it has the right inverse property. Moreover, examples in [22], section 4.1.1, show that there are many extensions of type $(*)$ which have the right or the left inverse property but not the inverse property. Since in the formula (14) in [22] enter only values $f(x,x)$ $x \in S$, the automorphic inverse property is independent on almost all properties discussed in this section.

Since in this section the cross inverse property plays an important role we note

Remark 2.2.1. *An extension L of the shape $(*)$ has the cross inverse property if and only if $f(x,y)f(xy,x) = f(x,x)$ $x, y \in S$.*

Using the identity $(xy) \cdot x' = y$ for $x, y \in S$ it follows that a loop L satisfies the cross inverse property if and only if
$[(x,\epsilon_1)(y,\epsilon_2)](x,\eta(x,\epsilon_1)) = (y, \epsilon_1\epsilon_2\eta(x,\epsilon_1)f(x,y)f(xy,x)) = (y,\epsilon_2)$
for all $x, y \in S$. This yields for $x = y$ and $\epsilon_1 = \epsilon_2$ that $\eta(x,\epsilon_1) = \epsilon_1 f(x,x)$ for all x.

Any commutative extension L of type $(*)$ is flexible. Many examples of such loops can be constructed in a simple way already from the Steiner triple system $\mathfrak{S}$. Divide the set of lines of $\mathfrak{S}$ into two non-void disjoint subsets $\mathfrak{A}_+$ and $\mathfrak{A}_-$. We obtain a commutative extension L the type $(*)$ such that the Steiner loop S belongs to $\mathfrak{S}$ defining the function f in such a way that for $x \neq y$ $x, y \in S \setminus \{e\}$ one has $f(x,y) = 1$ if the line l joining x and y belongs to $\mathfrak{A}_+$ whereas $f(x,y) = -1$ if l belongs to $\mathfrak{A}_-$. If L has exponent 2, i.e. $f(x,x) = 1$ for all $x \in S$, then it follows from [22] formulas (14) – (19) and Remark 2.2.1 that L is flexible, alternative and has the inverse, the automorphic inverse as well as the cross inverse property. If in $\mathfrak{S}$ there exists a triangle x, y, z such that the lines determined by $\{x, y\}$, $\{x, z\}$ and $\{y, z\}$ belong to the same subset and the line determined by $\{xy, xz\}$ belongs to the other subset, then it follows from formula (20) in [22] that L is not a left Bol loop. This is not surprising since if a commutative loop L of type $(*)$ is a Bol loop, then it is an elementary abelian 2-group (cf. [4], Proposition 2, p. 35).

An interesting example of a commutative loop L of type $(*)$ is the C-loop of order 8 constructed in [17] Proposition 4.1 in the following way: Take for S the elementary abelian 2-group of order 4 such that for the symmetric factor system f one has $f(x,x) = f(xy,xy) = 1, f(y,y) = -1, f(y,xy) = d, f(x,y) = f(x,xy) = d$ with $d \in \{1,-1\}$. This loop has neither the inverse nor the automorphic inverse property since $f(y,x)f(y,xy) \neq f(y,y)$, $f(xy,xy) \neq f((x,x)f(y,y)$ (cf. [22] formulas $(16), (14)$).

The best known examples among the non-commutative proper loops of type $(*)$ are the code loops (for definition see e.g. [13] p. 225-226, or Definition 24 in [15]); these loops are Moufang loops of exponent 4.

Another well known examples of non-commutative loops of type $(*)$ are the proper Bol loops of order 8. The remake of Burn's classification of these loops in [10], pp. 790-791 by means of the factor system f admits using formulas $(13), (19)$ in [22] p. 771 and Remark 2.2.1 to decide which common properties they have. None of them is flexible.

Three of them have the automorphic inverse property, but not the cross inverse property. For the loop among them with the simplest factor system one has $f(xy,x) = f(xy,y) = -1$, whereas all other values of f equals 1.

Two of the six proper Bol loops of order 8 have neither the automorphic nor the cross inverse property. For the one with the simplest factor system one has $f(xy,y) = f(xy,xy) = -1$, whereas all other values of f are 1.

There is precisely one proper left Bol loop of order 8 having the cross inverse property, but not the automorphic inverse property. For its factor system one has
$f(x,x) = f(y,y) = f(xy,xy) = f(x,y) = f(xy,y) = f(y,xy) = -1$ and the other values of f are 1.

Already for order 8 there are loops L of type $(*)$ which are highly non-associative. Taking for S the elementary abelian 2-group of order 4 and defining for the factor system f the relations $f(xy,x) = f(xy,y) = 1$, but $f(x,y) = f(x,xy) = f(y,x) = f(y,xy) = -1$ and $f(x,x) = -1$ for all $x, y \in S \setminus \{e\}$, we obtain a loop L having the cross inverse property but which is neither flexible nor left or right alternative, and has neither inverse nor automorphic inverse property.

In contrast to the above non-commutative examples we deal here with extensions of type $(*)$ in which $f(x,y) = -f(y,x)$ holds for all different $x, y \in S \setminus \{e\}$ and all elements of $L \setminus \{(e,1)\}$ have either order 2 or 4, i. e., $f(x,x) = 1$, respectively $f(x,x) = -1$ for all $x \in S \setminus \{e\}$. It follows immediately from [22] formula (19) that such loops are flexible and can be classified if they have order 8.

Proposition 2.2.2. *If L is an extension of type $(*)$ has order 8 such that $f(x,y) = -f(y,x)$ for all elements $x \neq y$ of $S \setminus \{e\}$, then we have the following cases:*

(a) If all elements of $L \setminus \{(e,1)\}$ have order 2, then there are exactly two non-isomorphic loops. One has the cross inverse property but the other one not. Both loops are neither left alternative nor right alternative but they have the automorphic inverse property.

(b) If all elements of $L \setminus \{(e,1),(e,-1)\}$ have order 4, then there exist exactly 2 non-isomorphic loops. One is the quaternion group and

the other one is neither left alternative, nor right alternative nor has the cross inverse property nor the automorphic inverse property.

Proof. There are the following possibilities for the factor system f:

(1) $f(a,b) = f(b,ab) = f(ab,a) = 1$

(2) $f(a,b) = f(a,ab) = f(ab,b) = 1$

(3) $f(a,b) = f(ab,a) = f(ab,b) = 1$

(4) $f(a,b) = f(a,ab) = f(b,ab) = 1.$

The loops belonging to (2), (3) or (4) are isomorphic in both cases (a) and (b). The automorphism of $S = \{e, x, y, xy\}$ given by the involution $x \mapsto xy$ and $y \mapsto y$, respectively $x \mapsto x$ and $y \mapsto xy$ transfers the factor system (2) into the factor system (3), respectively (4).
The loop belonging to the factor system (1) satisfies in the case (a) the identity $f(x,y)f(xy,x) = f(x,x)$ $x, y \in S$ and hence it has the cross inverse property, whereas in case (b) it is the quaternion group. $\square$

From this proposition it follows immediately that any loop of type $(*)$ in which every element not contained in the center of L has order 4 is diassociative if and only if any two non-commuting elements generate the quaternion group of order 8.

2.2.2 Oriented Steiner loops

Definition 2.2.3. *An oriented Steiner triple system* $(\mathfrak{S}, T)$ *is a Steiner triple system* $\mathfrak{S}$ *such that on the set* T *of blocks for each block is given a cyclic order.*

Clearly from a Steiner triple system $\mathfrak{S}$ with n points and $b = \frac{n(n-1)}{6}$ blocks we can construct 2^b oriented Steiner triple systems ([30]).

If $t \in T$ consists of points a_1, a_2, a_3, then (a_1, a_2, a_3) means the orientation with respect to the permutation (123).

An automorphism of an oriented Steiner triple system is an automorphism ϕ of the Steiner triple system $\mathfrak{S}$ such that for all oriented blocks $(a_1, a_2, a_3)^\phi = ({a_1}^\phi, {a_2}^\phi, {a_3}^\phi)$ holds, i.e., ϕ preserves the orientation on blocks.

To any oriented Steiner triple system $(\mathfrak{S}, T)$ we may associate a function $f^* : (\mathfrak{S}, T) \times (\mathfrak{S}, T) \longrightarrow A$ called the *orientation function* of $(\mathfrak{S}, T)$. If a_1, a_2 are distinct points determining the oriented block (a_1, a_2, a_3), then $f^*(a_1, a_2) = 1$ and $f^*(a_2, a_1) = -1$.

Definition 2.2.4. *An oriented Steiner loop* L *is an extension* $(*)$ *for which there exists an oriented Steiner triple system* $(\mathfrak{S}, T)$ *such that the elements different from the identity of the Steiner loop* S *are the points of* $\mathfrak{S}$ *and the restriction of the factor system of* L *to* $(\mathfrak{S} \times \mathfrak{S}) \setminus \{(x, x), x \in \mathfrak{S}\}$ *coincides with the orientation function of* $(\mathfrak{S}, T)$ *and* $f(x, x) = -1$*, respectively* $f(x, x) = 1$ *for all* $x \in S \setminus \{e\}$ *holds.*

The center of an oriented Steiner loop L has order 2 if the exponent of L is 4 and it is trivial if the exponent of L is 2. The oriented Steiner triple system $(\mathfrak{S}, T)$ of an oriented Steiner loop L is uniquely determined by L and conversely, an oriented Steiner triple system

$(\mathfrak{S}, T)$ determines a unique oriented Steiner loop L of exponent $d \in \{2, 4\}$.

Theorem 2.2.5. *Any oriented Steiner loop L is flexible.*

The loop L satisfies the inverse property if and only if every element of L not contained in the center has order 4.

The loop L has the cross inverse and the automorphic inverse properties but satisfies neither the left nor the right inverse property if and only if L has exponent 2.

Proof. The first claim follows from formula (19) in [22] p. 771. The left and the right inverse property in an oriented Steiner loop L is satisfied if and only if $f(x, x) = -1$ for all $x \in S \setminus \{e\}$ (cf. formulas $(16), (15)$ [22] p. 771.). The loop L fulfills the cross inverse property and the automorphic inverse property if and only if $f(x, x) = 1$ for all $x \in S$ (see formula (13) in [22] p. 771. and Remark 2.2.1). □

Theorem 2.2.6. *An oriented Steiner loop L is a group if and only if it is the quaternion group of order* 8.

Proof. The quaternion group of order 8 is an oriented Steiner loop (Proposition 2.2.2).

If L is a group, then by previous Theorem every element not contained in the center Z of L has order 4 and Z has order 2. The Steiner loop S corresponding to L is a projective space over the field $GF(2)$ (cf. [7], Corollary 1, p. 251). If L has order 8, then L is the quaternion group. If the order of L is greater than 8, then S contains a projective plane P as a subspace. Let L_1 be the subgroup of L of order 16 corresponding to P. The group L_1 has also only one element of order 2, but there is no group of order 16 having only one involution (e.g. [29]). $\square$

2.2.3 Groups generated by translations of an oriented Steiner loop

Let L be an extension of type $(*)$ and let G be the group generated by the translations of the associated Steiner loop S. Let $\hat{G}_l$ $(\hat{G}_r)$ be the group generated by the left translations $\lambda_{(a,\epsilon)}$ (the right translations $\rho_{(a,\epsilon)}$) of L. The mappings $\lambda_{(a,\epsilon)} \mapsto \lambda_a = \rho_a$ and $\rho_{(a,\epsilon)} \mapsto \rho_a = \lambda_a$ from the set of left, respectively right translations of L onto the set of translations of S can be extended to an epimorphism $\omega_l : \hat{G}_l \longrightarrow G$ or $\omega_r : \hat{G}_r \longrightarrow G$. The kernels of the epimorphisms ω_l, respectively ω_r are elementary abelian 2-groups. The same holds for the kernel of the epimorphism ω from the group $\hat{G}$ generated by all translations of L onto G. Hence the groups $\hat{G}_l$, $\hat{G}_r$ and $\hat{G}$ are extensions of elementary abelian 2-groups by G. If L does not have exponent 2, then these extensions do not split.

If S is a proper Steiner loop of order n and $Q(\mathfrak{S})$ is a Steiner quasigroup both corresponding to the Steiner triple system $\mathfrak{S}$ such that the product of any two distinct translations of $Q(\mathfrak{S})$ has odd order, then the group G is the alternating group A_n or the symmetric group S_n depending on whether n is divisible by 4 or not (see [26], Theorem).

The left translation $\lambda_{(a,\epsilon)}$ of L is given by
$\lambda_{(a,\epsilon)} : (y, \delta) \mapsto (ay, \epsilon\delta f(a, y))$, the right translation $\rho_{(a,\epsilon)}$ has the form

$\rho_{(a,\epsilon)} : (y,\delta) \mapsto (ya, \epsilon\delta f(y,a))$. One has
$[\lambda_{(a,\epsilon)}\rho_{(a,\epsilon)}](x,\delta) = (x, \delta f(x,a) f(a,ax))$,
$[\rho_{(a,\epsilon)}\lambda_{(a,\epsilon)}](x,\delta) = (x, \delta f(a,x) f(ax,a))$.
If L is an oriented Steiner loop, then one has $\lambda_{(a,\epsilon)}\rho_{(a,\epsilon)} = \rho_{(a,\epsilon)}\lambda_{(a,\epsilon)}$. Moreover, $\rho_{(a,\epsilon)} = \lambda_{(a,\epsilon)}\tau_a$, with $a \in S$, where the map $\tau_a : L \longrightarrow L$ with $a \in S \setminus \{e\}$ is given by $\tau_a : (x,\epsilon) \mapsto (x,-\epsilon)$ for $x \in S \setminus \{a,e\}$, $\epsilon \in \{1,-1\}$ and τ_a is the identity for $x \in \{a,e\}$. Since $\rho^{-1}_{(a,\epsilon)}\tau_b\rho_{(a,\epsilon)} = \lambda^{-1}_{(a,\epsilon)}\tau_b\lambda_{(a,\epsilon)} = \tau_{ab}$ and $\tau_a\tau_b = \tau_b\tau_a$ the group Θ generated by the elements $\tau_a, a \in S$, is a normal elementary abelian 2-subgroup of $\hat{G}$.

Let L be an oriented Steiner loop of order n. For $x \in S$ let ${D_\epsilon}^x = \{(x,\epsilon), \epsilon \in \{1,-1\}\}$. Since the group Θ fixes ${D_\epsilon}^a$ elementwise we consider the action of Θ on the set $\{{D_\epsilon}^x,\ x \in S^* := S \setminus \{e\}\}$ which has cardinality $n-1$.

Let $n-1$ be even and let b be an element of S^*. Then $\tau_{a_1} ... \tau_{a_{n-2}}$, where a_i are all different elements of $S^* \setminus \{b\}$, fixes any ${D_\epsilon}^{a_i}$ elementwise and induces on ${D_\epsilon}^b$ an involution. From this it follows that Θ has the order 2^{n-1}.

Let $n-1$ be odd. Then $\tau_{a_1} ... \tau_{a_{n-3}}$, where a_i are all different elements of $S^* \setminus \{b_1, b_2\}$ with $b_1 \neq b_2 \in S^*$, fixes each ${D_\epsilon}^{a_i}$ elementwise and induces an involution on ${D_\epsilon}^{b_1}$ and ${D_\epsilon}^{b_2}$. The abelian group Θ has order 2^{n-2}.

Theorem 2.2.7. *If the oriented Steiner loop L corresponds to a Steiner loop S of order n, then the normal subgroup $\Theta =< \tau_a, a \in S >$ of the group $\hat{G}$ generated by all translations of L is the elementary abelian 2-group of order 2^{n-1} or 2^{n-2} depending on whether n is odd or even and the factor group $\hat{G}/\Theta$ is isomorphic to $\hat{G}_l$ as well as to $\hat{G}_r$. Hence the groups $\hat{G}_l$ and $\hat{G}_r$ are isomorphic.*

2.2.4 Automorphisms of oriented Steiner triple systems

Proposition 2.2.8. *The automorphism group* Γ *of an oriented Steiner triple system has odd order.*

Proof. If Γ has even order, then it contains an involution α. Let a_1 and a_2 be an orbit of α. If a_3 is the third point of the block determined by a_1, a_2, then ${a_3}^{\alpha} = a_3$. From $(a_1, a_2, a_3)^{\alpha} = ({a_1}^{\alpha}, {a_2}^{\alpha}, {a_3}^{\alpha})$ a contradiction follows. $\square$

Proposition 2.2.9. *Let* U *be a group of automorphisms of a Steiner triple system* $\mathfrak{S}$ *such that* U *has odd order. Then there is an oriented Steiner triple system having* U *as a group of automorphisms.*

Proof. Choose in any orbit B a block t containing the points a_1, a_2, a_3 and put (a_1, a_2, a_3) as well as $({a_1}^{\phi}, {a_2}^{\phi}, {a_3}^{\phi})$ for all $\phi \in U$. Since the stabilizer of the block t has odd order we obtain an oriented Steiner triple system. $\square$

Let $(\mathfrak{S}, T)$ and $(\mathfrak{S}, T')$ be oriented Steiner triple systems over the Steiner triple system $\mathfrak{S}$. Then $(\mathfrak{S}, T)$ and $(\mathfrak{S}, T')$ are isomorphic if and only if there is an automorphism ϕ of $\mathfrak{S}$ such that (a_1, a_2, a_3) is equivalent to $({a_1}^{\phi}, {a_2}^{\phi}, {a_3}^{\phi})$ for any block. The automorphism groups of $(\mathfrak{S}, T)$ and $(\mathfrak{S}, T')$ are conjugate subgroups under ϕ in the automorphism group of $\mathfrak{S}$.

We call a subgroup U of odd order in a finite group G *maximal odd subgroup* of G if there is no subgroup of G having odd order and containing U properly. Maximal odd subgroups of a finite group G

are in general not conjugate. This occurs already in the alternating group A_5 of order 60. Since A_5 has no subgroup of order 15 the Sylow 3-subgroups and the Sylow 5-subgroups are maximal odd subgroups of A_5. Since any finite group G is the automorphism group of a Steiner triple system ([20], Theorem 8, p. 103.) the automorphism groups of a Steiner triple system can have different conjugate classes of maximal odd subgroups.

Remark 2.2.10. *Let $(\mathfrak{S}, T)$ be a finite oriented Steiner triple system having K as a group of automorphisms and let K_0 be a proper subgroup of K such that a block orbit B of K splits into different orbits $B_1, .., B_m$ of K_0. Then there exists an oriented Steiner triple system $(\mathfrak{S}, T_0)$ such that K_0 consists of automorphisms, but K is not a group of automorphisms of $(\mathfrak{S}, T_0)$.*

Taking for any block in B_1 the opposite orientation and leaving the orientation of all blocks not contained in B_1 unchanged we obtain an oriented Steiner triple system $(\mathfrak{S}, T_0)$ having K_0 but not K as a group of automorphisms.

2.2.5 Automorphisms of oriented Steiner loops

Let L be an oriented Steiner loop with the factor system f and $(\mathfrak{S}, T)$ be the corresponding oriented Steiner triple system. Since the automorphism group Γ of $(\mathfrak{S}, T)$ preserves the orientation T of $(\mathfrak{S}, T)$ any element $\tilde{\gamma} \in \Gamma$ induces on the Steiner loop S belonging to L an automorphism γ such that $f(a, b) = f(a^\gamma, b^\gamma)$ for all $a, b \in S$.

Proposition 2.2.11. *A mapping $\hat{\gamma} : L \longrightarrow L$ satisfying*

$$(a, \epsilon)^{\hat{\gamma}} = (a, 1)^{\hat{\gamma}}(e, \epsilon)^{\hat{\gamma}} = (a^\gamma, v^{\hat{\gamma}}(a)\epsilon),$$

where γ is an automorphism of S induced by the orientd Steiner triple system $(\mathfrak{S}, T)$ corresponding with L and $v^{\hat{\gamma}}$ is a map from S into $\mathbb{Z}_2$, is an automorphism of L if and only if $v^{\hat{\gamma}}(a)$ is a homomorphism from S into $\mathbb{Z}_2$.

Proof. $\hat{\gamma}$ is an automorphism of L if and only if

$$((a,\epsilon_1)(b,\epsilon_2))^{\hat{\gamma}} = (ab^{\gamma}, v^{\gamma}(ab)\epsilon_1\epsilon_2 f(a,b)) =$$

$$(a^{\gamma}b^{\gamma}, v^{\hat{\gamma}}(a)v^{\hat{\gamma}}(b)f(a^{\gamma},b^{\gamma})) = (a,\epsilon_1)^{\hat{\gamma}}(b,\epsilon_2)^{\hat{\gamma}}.$$

Since $f(a,b) = f(a^{\gamma}, b^{\gamma})$ this is true if and only if and v^{γ} is a homomorphism. □

The elements $(a,\epsilon) \mapsto (a^{\gamma},\epsilon)$, where γ is an element of the group induced by the automorphism group Γ of $(\mathfrak{S}, T)$ on the Steiner loop S, form a subgroup Σ of the automorphism group $\hat{\Gamma}$ of L.

The automorphisms of L of the type $(a,\epsilon) \mapsto (a, v(a)\epsilon)$ form a normal subgroup Δ inducing on the factor loop L/A the identity. The group Δ is an elementary abelian 2-group since any two automorphisms of Δ commute. This yields the following

Theorem 2.2.12. *The automorphism group $\hat{\Gamma}$ of an oriented Steiner loop L is a semidirect product of a normal elementary abelian 2-group Δ inducing on the factor loop L/A the identity and corresponding to the set of homomorphisms from S into $\mathbb{Z}_2$ by the group Σ which is isomorphic to the automorphism group of the oriented Steiner triple system corresponding to L.*

Definition 2.2.13. *Let $\mathfrak{S}$ be a Steiner triple system. We call a proper subsystem $\mathfrak{K}$ a blockade subsystem if each block of $\mathfrak{S}$ not contained in $\mathfrak{K}$ meets $\mathfrak{K}$ in a point.*

Proposition 2.2.14. *Let γ be an epimorphism from a Steiner loop S onto $\mathbb{Z}_2$. Then S contains a normal Steiner subloop K such that S is the union $S = K \cup Ka$, where a is an element of S not contained in K.*
If $\mathfrak{S}$ and $\mathfrak{K}$ are Steiner triple systems corresponding to S, respectively to K, then $\mathfrak{K}$ is a blockade subsystem of $\mathfrak{S}$.

Proof. The kernel K of γ is a normal Steiner subloop of S and the factor loop S/K has order 2. If a is an element not contained in K, then precisely ka, $k \in K$ is not contained in K. □

Theorem 2.2.15. *There is a bijection between the non-trivial automorphisms of an oriented Steiner loop L, which induce on the corresponding Steiner triple system $\mathfrak{S}$ of S the identity, and the blockade subsystems of $\mathfrak{S}$.*

Proof. Any non-trivial automorphism of L which induces on $\mathfrak{S}$ the identity is determined by an epimorphism from the Steiner loop S onto $\mathbb{Z}_2$. As the previous Proposition shows the kernel of this epimorphism corresponds to a blockade subsystem of $\mathfrak{S}$.
Conversely, a blockade subsystem $\mathfrak{K}$ of $\mathfrak{S}$ determines an epimorphism from S onto $\mathbb{Z}_2$ if we put $v(a) = 1$ for $a = e$ and for all elements

of S which correspond to elements of $\mathfrak{K}$ and $v(b) = -1$ for all elements corresponding to $\mathfrak{S} \setminus \mathfrak{K}$. This epimorphism yields a non-trivial automorphism of L inducing on $\mathfrak{S}$ the identity. □

Theorem 2.2.16. *Let $\hat{\Gamma}$ be the automorphism group of the oriented Steiner loop L corresponding to an oriented Steiner triple system $(\mathfrak{S}, T)$. If $\mathfrak{S}$ is an affine space over the field $GF(3)$, then the group Δ consists only of the identity and $\hat{\Gamma}$ is isomorphic to the automorphism group Γ of $(\mathfrak{S}, T)$.*
If $\mathfrak{S}$ is a projective space of dimension n over the field $GF(2)$, then Δ is the elementary abelian group of order 2^n.

Proof. If $\mathfrak{S}$ is an affine space over $GF(3)$, then $\mathfrak{S}$ contains no blockade subsystem. If $\mathfrak{S}$ is the n-dimensional projective space over $GF(2)$, then $\mathfrak{S}$ contains $2^n - 1$ hyperplanes and these are the blockade subsystems of $\mathfrak{S}$. □

Corollary 2.2.17. *The automorphism group of an oriented Steiner loop of order 8 is isomorphic to the alternating group A_4.*

Proof. The oriented Steiner triple system $(\mathfrak{S}, T)$ corresponding to L is a line consisting of three points and each of these points is a blockade subsystem of $\mathfrak{S}$. Hence the normal subgroup Δ of automorphisms of L inducing on $(\mathfrak{S}, T)$ the identity is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\hat{\Gamma}$ is the semidirect product of Δ by the automorphism group Γ of $(\mathfrak{S}, T)$ of order 3 acting faithfully on Δ. □

2.2.6 Isomorphisms of oriented Steiner loops

Let S be a Steiner loop corresponding to the Steiner triple system $\mathfrak{S}$. A *valuation* v of S is a mapping $S \longrightarrow \{1, -1\}$ such that $v(e) = 1$ for the identity e of S.

Let L_1 and L_2 be oriented Steiner loops corresponding to the same Steiner triple system $\mathfrak{S}$ and having as factor systems f_1 or f_2 respectively, such that $f_1(a,a) = f_2(a,a)$ for all a contained in the Steiner loop S which corresponds to $\mathfrak{S}$.

We suppose that L_1 and L_2 correspond to oriented Steiner triple systems $(\mathfrak{S}, T_1)$, respectively $(\mathfrak{S}, T_2)$ over $\mathfrak{S}$ such that the automorphism groups of $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$ coincide. If Γ is this automorphism group, then any element $\tilde{\gamma}$ of Γ induces on the Steiner loop S an automorphism γ such that $f_1(a,b) = f_1(a^\gamma, b^\gamma)$ and $f_2(a,b) = f_2(a^\gamma, b^\gamma)$.

A valuation v is *compatible* with the pair (f_1, f_2) if $v(a)v(b)f_2(a,b) = v(ab)f_1(a,b)$ for all $a, b \in S$. The valuations v compatible with the pair (f_1, f_2) correspond in a unique way with isomorphisms from L_1 onto L_2 which induce on S the identity.

Lemma 2.2.18. *The loops L_{f_i} $(i = 1,2)$ such that for the factor systems f_i $(i = 1,2)$ one has $f_1(x,y) = -f_2(x,y)$ for distinct $x, y \in S \backslash \{e\}$ and $f_1(x,x) = f_2(x,x)$, $x \in S$ are isomorphic. An isomorphism ϕ is given by the mapping $(x, \epsilon) \mapsto (x, -\epsilon)$.*

Theorem 2.2.19. *Let L_1 and L_2 be oriented Steiner loops of the same exponent with the factor systems f_1 and f_2 respectively, such that the factor loops L_1/A and L_2/A correspond to the same Steiner triple system $\mathfrak{S}$. If the automorphism groups of the oriented Steiner triple systems $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$ corresponding to L_1, respectively L_2 coincide,* *then L_1 and L_2 are isomorphic if and only if there exists a valuation v of the Steiner loop S belonging to $\mathfrak{S}$ compatible with (f_1, f_2).*

Proof. If v is a valuation compatible with the pair (f_1, f_2), then there is an isomorphism from L_1 onto L_2 which induces the identity on S.

Conversely, let $\hat{\omega}$ be an isomorphism from L_1 onto L_2. Then $\hat{\omega}$ is determined by an automorphism $\tilde{\omega}$ of $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$. Since we may assume that (e, ϵ) with $\epsilon \in \{1, -1\}$ is an element of L_1 as well of L_2 one has $(e, \epsilon)^{\hat{\omega}} = (e, \epsilon)$. Hence $\hat{\omega}$ is determined by the images $(a, 1)^{\hat{\omega}} = (a^{\omega}, v^{\hat{\omega}}(a))$, $a \in S$, where ω is the isomorphism of S induced by $\tilde{\omega}$ and $v^{\hat{\omega}}$ is a valuation. We have

$$[(a,1)(b,1)]^{\hat{\omega}} = (ab, f_1(a,b))^{\hat{\omega}} = ((ab)^{\omega}, v^{\hat{\omega}}(ab) f_1(a,b)) = (a,1)^{\hat{\omega}}(b,1)^{\hat{\omega}} =$$

$$= (a^{\omega}, v^{\hat{\omega}}(a))(b^{\omega}, v^{\hat{\omega}}(b)) = (a^{\omega} b^{\omega}, v^{\hat{\omega}}(a) v^{\hat{\omega}}(b) f_2(a^{\omega}, b^{\omega})).$$

This is equivalent to $v^{\hat{\omega}}(ab) f_1(a,b) = v^{\hat{\omega}}(a) v^{\hat{\omega}}(b) f_2(a^{\omega}, b^{\omega})$. Since $f_2(a^{\omega}, b^{\omega}) = f_2(a, b)$ the valuation $v^{\hat{\omega}}$ is compatible with (f_1, f_2). □

Since to isomorphism which induces on S the identity there exists a compatible valuation with factor systems f_1 of L_1 and f_2 of L_2 it follows from the previous Theorem

Theorem 2.2.20. *Let L_1 be an oriented Steiner loop having K as the automorphism group. If α is an automorphism of the Steiner loop S corresponding to L_1, then there exists an oriented Steiner loop L_2 isomorphic to L_1 and having the conjugate group K^{α} as its automorphism group.*

Proof. Let $(\mathfrak{S}, T_1)$ be the oriented Steiner triple system corresponding to L_1 and let $C = \{(a_1{}^{\gamma}, a_2{}^{\gamma}, a_3{}^{\gamma}); \gamma \in K\}$ be the block orbit of the oriented block (a_1, a_2, a_3). We define now an orientation T_2 on the blocks of $\mathfrak{S}$ by $C^{\alpha} = \{(a_1{}^{\gamma\alpha}, a_2{}^{\gamma\alpha}, a_3{}^{\gamma\alpha}); \gamma \in K\}$ for any orbit C^{α}. Since
$(a_1{}^{\alpha}, a_2{}^{\alpha}, a_3{}^{\alpha})^{\alpha^{-1}\gamma\alpha} = (a_1{}^{\gamma\alpha}, a_2{}^{\gamma\alpha}, a_3{}^{\gamma\alpha})$ for all $\gamma \in K$ the oriented Steiner triple systems $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$ are isomorphic. Hence there exists a loop L_2 corresponding to $(\mathfrak{S}, T_2)$ and isomorphic to L_1. □

Proposition 2.2.21. *Oriented Steiner loops L_1 and L_2 of the same exponent associated with the Steiner triple system $\mathfrak{S}$ and having block transitive automorphism groups are isomorphic.*

Proof. The loops L_1 and L_2 correspond to oriented Steiner triple systems $(\mathfrak{S}, T_1)$, respectively $(\mathfrak{S}, T_2)$ over $\mathfrak{S}$. Both orientation functions f_i^* of $(\mathfrak{S}, T_i)$ $i = 1, 2$ are determined by the orientation of a block $B \in \mathfrak{S}$. Since for the restrictions $f_i^*|B$ of f_i^* to the block B one has either $f_1^*|B = f_2^*|B$ or $f_1^*|B = -f_2^*|B$ one has $f_1^* = \epsilon f_2^*$ and the assertion follows (Lemma 2.2.18). □

2.2.7 Oriented projective Steiner loops

We call an oriented Steiner loop L *projective* if the associated Steiner triple system $\mathfrak{S}$ is isomorphic to the point-line design of a projective geometry over $GF(2)$. If L has the order 2^{n+1}, then the automorphism group of L is a subgroup U of odd order in $SL(n, 2)$, see Proposition 2.2.8. Therefore U is a solvable subgroup of $SL(n, 2)$.

If U is a solvable group of automorphisms of the vector space V of an even dimension n over $GF(2)$ such that U has odd order and acts transitively on the vectors different from zero, then V may be identified with the additive group of the field $GF(2^n)$ and U is as a permutation group isomorphic to the sharply transitive group $\Sigma := \{x \mapsto ax; 0 \neq a \in GF(2^n)\}$ since the automorphism group of $GF(2^n)$ is a cyclic group of even order (cf. [24], Theorem 19.9, p. 246.).

Let U be the automorphism group of an oriented projective Steiner loop L. Since U acts transitively on the points of $\mathfrak{S}$ for any given point a there is in any line orbit of U a line incident with a. As the stabilizer U_a of a in U consists only of the identity the orientation function f^* of the oriented Steiner triple system $(\mathfrak{S}, T)$ belonging to a loop of order 2^{n+1} and having U as the automorphism group is determined by the restriction f^*_a of f^* to the lines trough a since U leaves the orientation of lines invariant.

Proposition 2.2.22. *Let L be an oriented projective Steiner loop of order 16 such that its automorphism group Γ of L induces a transitive group on the lines of the corresponding oriented projective plane $\mathfrak{S}$ of order 2. Then the group Γ is the semidirect product of the normal subgroup Δ isomorphic to ${\mathbb{Z}_2}^3$ by the group of order 21 acting on Δ faithfully.*

Proof. As any line of the projective plane $\mathfrak{S}$ is a blockade subsystem of $\mathfrak{S}$ the normal subgroup Δ of all automorphisms of L inducing on $\mathfrak{S}$ the identity is the elementary abelian group of order 8. The automorphism group Γ of L is the semidirect product of Δ by a line transitive subgroup Ω of $SL_3(2)$ of odd order (Proposition 2.2.8). If a line transitive subgroup of odd order is the automorphism group of the oriented projective plane of order 3, then it is a maximal odd subgroup of $SL_3(2)$. Hence Ω has order 21 and the assertion follows from Theorem 2.2.16. $\square$

Proposition 2.2.23. *Any oriented projective Steiner loop* L *of order* 16 *and of exponent* 4 *such that the order of its automorphism group* Γ *of* L *is divisible by* 7 *is the Moufang loop of order* 16.

Proof. The automorphism group Γ of L induces on the corresponding projective plane a line transitive group of automorphisms. Since in view of Proposition 2.2.21 there is only one oriented Steiner loop of order 16 and exponent 4. The assertion follows. $\square$

Theorem 2.2.24. *Any oriented projective Steiner loop* L *of order* ≥ 32 *is not Moufang.*

Proof. According to Theorem 2.2.5 we may assume that L has exponent 4. The loop L cannot be a group (cf. Theorem 2.2.6) and the Steiner triple system corresponding to L is a projective space of dimension at least 3. If all subloops of order 16 in L were Moufang loops, then M would be Moufang and would have center of order 2. But there is no Moufang loop of order 32 having only one element of order 2 (see [5] Table 14, p. 74.). $\square$

Since any oriented Steiner loop L satisfying a Bol condition is a proper Moufang loop L ([25] IV.6.9, p. 116., [22] formulas (20), (21) p. 772. and Theorem 2.2.5) we have

Corollary 2.2.25. *An oriented projective proper Steiner loop which is a Bol loop is isomorphic to the Moufang loop of order* 16.

2.3 Small examples

Let L_1 and L_2 be oriented Steiner loops of the same exponent over the Steiner triple system $\mathfrak{S}$ such that the automorphism groups of $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$ coincide. Then in view of Theorem 2.2.19 the loop L_1 and L_2 are isomorphic if and only if there exists a valuation v compatible with the pair (f_1, f_2), where f_i is the factor system of L_i $(i = 1, 2)$. The decision whether a valuation v with $v(a)v(b)f_2(a, b) = v(ab)f_1(a, b)$ for all $a, b \in S$ exists requires a detailed discussion of the relations among the restrictions of the orientation functions of $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$ to the orbits of the automorphism group Γ of $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$. We illustrate this situation for oriented Steiner loops with 16, respectively 20 elements. In the following we identify the the factor system f of an oriented Steiner loop L on the pairs of distinct elements with the orientation function of the oriented Steiner triple system $(\mathfrak{S}, T)$ belonging to L.

The affine plane $\mathfrak{A}$ of order 3

If Γ has order 27, then Γ has precisely two orbits $\mathcal{O}$ and $\mathcal{H}$ of lines. The orbit $\mathcal{O}$ consists of three parallel classes, the orbit $\mathcal{H}$ consists of the fourth parallel class of lines. According to Lemma 2.2.18 we may assume that the orientation functions f_1 and f_2 coincide precisely on $\mathcal{O}$ and that v is a compatible valuation (f_1, f_2).

Let a be a point of $\mathfrak{A}$ such that $v(a) = 1$ and let H be the line trough a not contained in $\mathcal{O}$. Let H_1 and H_2 be the other two lines contained in $\mathcal{H}$. Then either the value for all points of H_1 equals -1 or precisely two points of H_1 have the value 1. Since any point of H_2 is the point of the intersection of a line belonging to $\mathcal{O}$ through a and a point of H_1 the points of H_2 have the same values as the points of H_1. In the first case for all points outside of H has the value -1. Since H is contained in $\mathcal{H}$ the line H contains precisely one point z with $v(z) = -1$. If C is a line through z which is parallel to a line E with $a \in E \neq H$, then all points of C have values -1. Since C belongs to $\mathcal{O}$ we have a contradiction.

In the second case let z be the point of H_1 with $v(z) = -1$. Then the intersection of H_2 with the line K joining a and z is the only point u on H_2 with $v(u) = -1$. Let C be the line trough the point of H which has the value -1 and through a point of H_1 with value 1. This line belongs to $\mathcal{O}$, but intersects H_2 in a point with value 1. This is a contradiction. Hence *there are two non-isomorphic loops of exponent* $d \in \{2, 4\}$ *arising from the oriented affine plane of order* 3 *the automorphism group of which has order* 27.

Now we assume that Γ has order 9. Then Γ is either the translation group of $\mathfrak{S}$ or Γ leaves any line of a parallel class $\mathcal{A}$ invariant and acts transitively on the lines not contained in $\mathcal{A}$.

First we consider the case that the group Γ has four block orbits consisting of parallel classes of lines.

If the loops L_1 and L_2 are isomorphic, but for the orientation functions f_1 and f_2 of L_1, respectively L_2 one has $f_1 \neq \pm f_2$ (cf. Lemma 2.2.18), then we may assume that f_1 and f_2 coincide precisely on two block orbits $\mathcal{A}$ and $\mathcal{B}$ of parallel lines since otherwise Γ would have order 27.

Let a be a point of $\mathfrak{A}$ with $v(a) = -1$ and let K be the line of $\mathcal{A}$ through a. Let M be the line of $\mathcal{B}$ through a. Then there is a further point b on M with the value -1. Let Q be the parallel line to K consisting of the points b, b_1, b_2. Since Q belongs to $\mathcal{A}$ we have $v(b_1) = 1$ and $v(b_2) = -1$. If $W \in \mathcal{A}$ is the third line parallel to K and Q, then the line M intersects W in a point with value 1. The line joining b_1 and a intersects the line W in a point with value 1 and the line joining b_2 and a intersects W in a point with value -1 since these lines belong to $\mathcal{C}$ or $\mathcal{D}$. This contradiction shows that *there are two non-isomorphic loops of exponent* $d \in \{2, 4\}$ *arising from the oriented affine plane of order* 3 *the automorphism group of which is the translation group of order* 9.

Finally, we treat the case that the group Γ has four line orbits such that three of them are the three lines of a parallel class $\mathcal{A}$ and the fourth orbit $\mathcal{B}$ consists of all other lines. In view of Lemma 2.2.18 we may assume that f_1 and f_2 coincide precisely on $\mathcal{B}$ and either on one or on two lines of $\mathcal{A}$ since otherwise Γ would have only one line orbit.

First we consider the case that f_1 and f_2 coincide precisely on $\mathcal{B}$ and on one line K of $\mathcal{A}$. We give for all points of K the value 1 and for all points of the other two parallel lines in $\mathcal{A}$ the value -1. In this way we obtain a valuation compatible with (f_1, f_2).

Now we treat the case that f_1 and f_2 coincide precisely on $\mathcal{B}$ and on two lines of $\mathcal{A}$. Let K be the line in $\mathcal{A}$ on which f_1 and f_2 are opposite. Let M and Q be the other two parallel lines of $\mathcal{A}$ on which f_1 and f_2 coincide. The line K contains a point a with $v(a) = -1$. Let P_1, P_2, P_3 be the three lines trough a different from K. Since f_1 and f_2 coincide on these lines we may assume, that $M \cap P_1$ has value -1. Then $P_1 \cap Q$ has value 1. Since also on M the functions f_1 and f_2 coincide we may assume that $M \cap P_2$ has value -1. Then $M \cap P_3$ has value 1. Moreover, $P_2 \cap Q$ has value 1 and $P_3 \cap Q$ has value -1 which is a contradiction. Hence *there are precisely two non-isomorphic loops of exponent* $d \in \{2, 4\}$ *having the automorphism group of order* 9 *which leaves any line of a parallel class invariant and acts transitively on all other lines.*

The projective plane of order 2

Let $\mathfrak{S}$ be the projective plane over the field $GF(2)$. Let L_1 and

L_2 be oriented Steiner loops of the same exponent $d \in \{2, 4\}$ with the orientation functions f_1 and f_2 respectively, such that their automorphism group induce on $\mathfrak{S}$ the same group Γ of automorphisms. Since Γ has odd order (cf. Proposition 2.2.8) and the automorphism group of $\mathfrak{S}$ has the order 168 the order of Γ divides 21.

If Γ has order 21 or 7, then it acts transitively on the lines of $(\mathfrak{S}, T)$ and from Proposition 2.2.21 it follows that the loops L_1 and L_2 of exponent $d \in \{2, 4\}$ are isomorphic. The full automorphism group of L_i $(i = 1, 2)$ has order 21. If the exponent of such a loop is 4, then this loop is isomorphic to the Moufang loop of order 16 (cf. Corollary 2.2.25).

Now we assume that the group Γ has order 3. Then Γ has three orbits: one of them consists only of one line l, whereas the other two orbits consist each with three lines. Moreover, Γ has a fixed point $a \notin l$. We denote by $\mathfrak{A}$ the orbit of Γ consisting of lines through a and by $\mathfrak{B}$ the orbit of lines different from l and not incident with a.

In view of Lemma 2.2.18 we have to treat for the orientation functions f_1 and f_2 the following three cases:
(*i*) f_1 and f_2 coincide on $\mathcal{A}$ and are opposite on $\mathcal{B} \cup l$,
(*ii*) f_1 and f_2 coincide on $\mathcal{B}$ and are opposite on $\mathcal{A} \cup l$,
(*iii*) f_1 and f_2 coincide on l and are opposite on $\mathcal{A} \cup \mathcal{B}$.

In case (*i*) we put $v(a) = 1$ and $v(x) = -1$ for all $x \neq a$ and obtain a valuation compatible with (f_1, f_2). Hence the loops L_1 and L_2 corresponding to the orientation functions f_1 and f_2 are isomorphic.

If (*ii*) holds, then either all points of the line $l = \{c_1, c_2, c_3\}$ have value -1 or precisely one point, say c_1, of l has value -1.

In the first case all points outside l have the value 1 since otherwise $f_1 = -f_2$. The line joining two points outside of l and different from a belongs to $\mathcal{B}$ and intersects l in a point with value -1. This is a contradiction.

In the second case let $v(a) = \varepsilon$ with $\varepsilon \in \{1, -1\}$. Then the third point d_1 of the line A_1 joining a and c_2 as well as the third point d_2 of the line A_2 joining a and c_3 has the value $-\varepsilon$. Since the line D joining d_1 and d_2 belongs to $\mathcal{B}$ but intersects l in the point c_1 with $v(c) = -1$ we obtain a contradiction. Hence the loop L_2 with orientation function f_2 which coincide with f_1 only on the orbit $\mathcal{B}$ is not isomorphic to L_1.

In case (iii) holds, then either all points or only one point of the line $l = \{c_1, c_2, c_3\}$ have the value 1. If $v(c_i) = 1$ for $i = 1, 2, 3$ and $v(a) = \varepsilon$, then the third point d_i of the line D_i joining a with c_i has the value $-\varepsilon$, $(i = 1, 2, 3)$. Since f_1 and f_2 are opposite on any line joining d_i and d_j for $i \neq j$ we have a contradiction.

If $v(c_1) = v(c_2) = -1$ and $v(c_3) = 1$ and $v(a) = \varepsilon$, then for the third points d_1 and d_2 of the line D_1, respectively D_2 joining a with c_1, respectively c_2 one has $v(d_1) = v(d_2) = \varepsilon$. The third point d_3 of the line D_3 joining a with c_3 has value $-\varepsilon$. Since on the line $\{d_1, d_3, c_2\}$ the orientation functions coincide we have again a contradiction. Hence the loop L_2' with orientation function f_2' which coincide with f_1 only on the orbit l is not isomorphic to L_1.

Since the orientation functions f_2 and f_2' coincide only on the orbit $\mathcal{A}$ the loops L_2 and L_2' are isomorphic.

Summarizing this discussion we see that *there exist two isomorphism classes of oriented Steiner loops of exponent* $d \in \{2, 4\}$ *such that the corresponding Steiner triple system is the projective plane of order* 2 *and the automorphism group* Γ *has order* 3.

Chapter 3

Nuclear extension

A loop extension is usually called (right) nuclear, if the kernel of the corresponding homomorphism is contained in the (right) nucleus of the extension. (Right) nuclear extensions are very natural generalizations of central extensions, they have been investigated by many authors and used for different constructions in loop theory (e. g. [11], [14], [16], [17], [18], [21]). Most of the examples treated in these papers are central extensions, but only a few examples are known for non-central (right) nuclear extensions.

The aim of this chapter is the systematic study of right nuclei of quasigroups obtained by an extension process in the category of quasigroups with right unit. The investigated extensions of quasigroups are defined by a slight modification of non-associative Schreier-type extensions of groups or loops (c.f. [18], [22], [26]). These extensions will be determined by a triple (L, K, f), where L is a loop, K is a quasigroup with right unit and $f : K \times K \to L$ is a function. The main results of this chapter give characterizations of quasigroup extensions satisfying particular nuclear conditions. We apply the results to the description of constructions of quasigroups with right inverse property, having a prescribed right nucleus. Achievements which are contained in this chapter can be found in [23].

3.1 Right nuclei of f-extensions

Now, let $\mathfrak{Q}_f = (K \times L, \circ)$ be an f-extension of a loop $(L, \cdot)$ by a quasigroup $(K, \cdot)$ and assume that $\mathfrak{Q}_f$ has non-trivial right nucleus. The homomorphic image of the right unit of $\mathfrak{Q}_f$ in $(K, \cdot)$ is a right unit of $(K, \cdot)$, which will be denoted by $e_r \in K$. Let ϵ denote the unit of $(L, \cdot)$. Let us denote $\xi^\lambda = \epsilon/\xi$ and $\xi^\rho = \xi\backslash\epsilon$ for any $\xi \in L$. The element $\alpha' \in L$ is called the *cross inverse* of $\alpha \in L$ if $\alpha\xi \cdot \alpha' = \xi$ for any $\xi \in L$. In this case necessarily $\alpha' = \alpha^\rho$ holds. Clearly, if $\alpha \in Z(Q)$, then α^ρ is its cross inverse.

Definition 3.1.1. An f-extension $\mathfrak{Q}_f = (K \times L, \circ)$ is called *(right) nuclear* if the kernel $\phi^{-1}(e_r)$ of the extension is a (right) nuclear subgroup of $\mathfrak{Q}_f$.

Lemma 3.1.2. *The right unit of $\mathfrak{Q}_f$ has the shape $(e_r, f(e_r, e_r)^\rho)$. The function $f : K \times K \to L$ satisfies*

(a) *$f(x, e_r) = f(e_r, e_r)$ for any $x \in K$,*

(b) *$f(e_r, e_r)^\rho$ is the cross inverse of $f(e_r, e_r)$.*

Proof. The element $(e_r, \delta) \in K \times L$ is a right unit of $\mathfrak{Q}_f$ if and only if $f(x, e_r) \cdot \xi\delta = \xi$ for any $x \in K$ and $\xi \in L$. Hence $f(x, e_r) = f(e_r, e_r)$ is constant, $\delta = f(e_r, e_r)^\rho$ and $\lambda^{-1}_{f(e_r,e_r)} = \rho_{f(e_r,e_r)^\rho}$. It follows, that the equation $f(e_r, e_r)\xi \cdot f(e_r, e_r)^\rho = \xi$ holds for any $\xi \in L$. □

In the following we will consider f-extensions $\mathfrak{Q}_f$ satisfying the conditions (a) and (b) of Lemma 3.1.2.

Theorem 3.1.3. *An element* $(n,\nu)\in\mathfrak{Q}_f$ *belongs to the right nucleus* $N_r(\mathfrak{Q}_f)$ *of* $\mathfrak{Q}_f$ *if and only if the following conditions hold:*

(i) $n\in N_r(K)$ *and* $f(e_r,n)=f(e_r,e_r)$,

(ii) $f(e_r,e_r)\nu\in N_r(L)$ *and* $\rho_{f(e_r,e_r)\nu}=\lambda_{f(e_r,e_r)}\circ\rho_\nu$,

(iii) $f(y,n)\nu\in N_r(L)$ *and* $\rho_{f(y,n)\nu}=\lambda_{f(y,n)}\circ\rho_\nu$ *for any* $y\in K$,

(iv) $\lambda_{f(x,y)}\circ\lambda_{f(xy,n)}=\lambda_{f(x,yn)}\circ\lambda_{f(y,n)}$ *for any* $x,y\in K$.

Proof. The element $(n,\nu)\in K\times L$ belongs to $N_r(\mathfrak{Q}_f)$ if and only if $n\in N_r(K)$ and for any $x,y\in K$ and $\xi,\eta\in L$ the equation

$$f(xy,n)\cdot\big(f(x,y)\cdot\xi\eta\big)\nu=f(x,yn)\cdot\xi\big(f(y,n)\cdot\eta\nu\big)\tag{3.1}$$

holds. For $y=e_r$ we get $\big(f(e_r,e_r)\cdot\xi\eta\big)\nu=\xi\big(f(e_r,n)\cdot\eta\nu\big)$, since for any $x\in K$ we have $f(x,e_r)=f(e_r,e_r)$ and $e_rn=n$ in the group $N_r(K)$. It follows that $f(e_r,n)=f(e_r,e_r)$, giving assertion (i).
Putting $\xi=\epsilon$ into the identity (3.1), we obtain

$$f(xy,n)\big(f(x,y)\eta\cdot\nu\big)=f(x,yn)\cdot\big(f(y,n)\cdot\eta\nu\big).\tag{3.2}$$

Replacing η by $\xi\eta$ we get

$$f(xy,n)\cdot\big(f(x,y)\cdot\xi\eta\big)\nu=f(x,yn)\cdot\big(f(y,n)(\xi\eta\cdot\nu)\big).\tag{3.3}$$

Now, we obtain from (3.1) and (3.3) the identity

$$\xi\big(f(y,n)\cdot\eta\nu\big)=f(y,n)\big(\xi\eta\cdot\nu\big) \tag{3.4}$$

for $y\in K$ and $\xi,\eta\in L$. Putting $\eta=\epsilon$ this gives the equation

$$\xi\cdot f(y,n)\nu=f(y,n)\cdot\xi\nu. \tag{3.5}$$

Applying this to both sides of the equation (3.4) we obtain

$$\xi\big(\eta\cdot f(y,n)\nu\big)=\xi\big(f(y,n)\cdot\eta\nu\big)=f(y,n)\big(\xi\eta\cdot\nu\big)=\xi\eta\cdot f(y,n)\nu. \tag{3.6}$$

Equations (3.5) and (3.6) yield assertion (iii).
Now, using first $\lambda_{f(xy,n)}\circ\rho_\nu=\rho_{f(xy,n)\nu}$, then $f(y,n)\nu\in N_r(L)$, we get

$$f(x,y)\big(f(xy,n)\cdot\xi\nu\big)=f(x,y)\big(\xi\cdot f(xy,n)\nu\big)=f(x,y)\xi\cdot f(xy,n)\nu.$$

Using $\lambda_{f(xy,n)}\circ\rho_\nu=\rho_{f(xy,n)\nu}$ once more, we obtain that the right hand side of the previous equation is equal to $f(xy,n)\big(f(x,y)\xi\cdot\nu\big)$. Putting $\eta=\epsilon$ into equation (3.1) and using this equation we obtain

$$f(xy,n)\big(f(x,y)\xi\cdot\nu\big)=f(x,yn)\big(\xi\cdot f(y,n)\nu\big).$$

Lastly, applying once more (3.5) to this expression, we obtain

$$f(x,y)\big(f(xy,n)\cdot\xi\nu\big)=f(x,yn)\big(f(y,n)\cdot\xi\nu\big),$$

which means that assertion (iv) is true.
Replacing $\eta=\epsilon$ and $y=e_r$ in equation (3.1) and using Lemma 3.1.2 and condition (i) we get

$$f(e_r,e_r)\xi\cdot\nu=f(x,e_r)\xi\cdot\nu=\xi\cdot f(e_r,n)\nu=\xi\cdot f(e_r,e_r)\nu,$$

which proves the second assertion of (ii). Hence (3.1) yields with $y = e_r$ the identity $\xi\eta \cdot f(e_r, e_r)\nu = \xi\big(\eta \cdot f(e_r, n)\nu\big)$, from which follows the first assertion of (ii).

Conversely, from conditions (i)-(iv) we have to prove identity (3.1). Using $\lambda_{f(xy,n)} \circ \rho_\nu = \rho_{f(xy,n)\nu}$ and $f(y, n)\nu \in N_r(L)$, the left hand side can be written as

$$f(xy, n) \cdot \big(f(x, y) \cdot \xi\eta\big) = \big(f(x, y) \cdot \xi\eta\big) \cdot f(xy, n)\nu =$$

$$= f(x, y)\big(\xi\eta \cdot f(xy, n)\nu\big) = f(x, y) \cdot f(xy, n)\big(\xi\eta \cdot \nu\big).$$

Applying $\lambda_{f(y,n)} \circ \rho_\nu = \rho_{f(y,n)\nu}$ to the right hand side of (3.1) and using $f(y, n)\nu \in N_r(L)$, we have

$$f(x, yn) \cdot \xi\big(f(y, n) \cdot \eta\nu\big) = f(x, yn) \cdot \xi\big(\eta \cdot f(y, n)\nu\big).$$

Using condition (iv), we obtain from the last equations identity (3.1).

□

Putting $n = e_r$ in Theorem 3.1.3 we get

Corollary 3.1.4. *An element* (e_r, ν) *belongs to the right nucleus* $N_r(\mathfrak{Q}_f)$ *if and only if* $f(e_r, e_r)\nu \in N_r(L)$ *and* $\rho_{f(e_r,e_r)\nu} = \lambda_{f(e_r,e_r)} \circ \rho_\nu$.

Corollary 3.1.5. *Assume that* $f(e_r, e_r) \in N_r(L)$. *An element* (n, ν) *belongs to the right nucleus* $N_r(\mathfrak{Q}_f)$ *if and only if the following conditions hold:*

(i) $n \in N_r(K)$ *and* $f(e_r, n) = f(e_r, e_r)$,

(ii) *$\nu \in N_r(L)$ and $f(e_r, e_r) \in C(L)$,*

(iii) *$f(y, n) \in N_r(L) \cap C(L)$ for any $y \in K$,*

(iv) *$\lambda_{f(x,y)} \circ \lambda_{f(xy,n)} = \lambda_{f(x,yn)} \circ \lambda_{f(y,n)}$ for any $x, y \in K$.*

Particularly, $(e_r, \nu) \in N_r(\mathfrak{Q}_f)$ if and only if (ii) holds.

3.2 Nuclear properties of f-extensions

Let $\tilde{\mathfrak{Q}}_{\tilde{f}} = (H \times M, \circ) \subset \mathfrak{Q}_f$ be an f-subextension of the subloop $M \subset L$ by the subquasigroup $H \subset K$.

Theorem 3.2.1. *The subquasigroup $\tilde{\mathfrak{Q}}_{\tilde{f}} = (H \times M, \circ) \subset \mathfrak{Q}_f$ of $\mathfrak{Q}_f = (K \times L, \circ)$ is right nuclear if and only if*

(i) *$H \subset N_r(K)$ and $f(e_r, h) = f(e_r, e_r)$ for all $h \in H$,*

(ii) *$M \subset N_r(L)$ and $f(y, h) \in N_r(L) \cap C(L)$ for any $y \in K$, $h \in H$,*

(iii) *$\lambda_{f(x,y)} \circ \lambda_{f(xy,h)} = \lambda_{f(x,yh)} \circ \lambda_{f(y,h)}$ for any $x, y \in K$ and $h \in H$.*

Proof. (i) follows from condition (i) of Theorem 3.1.3. The first part of the condition (ii) of Theorem 3.1.3 implies $M \subset N_r(L)$ and $f(y, h) \in N_r(L)$, the second part yields $f(y, h) \in C(L)$ for any $y \in K$, $h \in H$, hence (ii) is proved. Assertion (iv) follows from condition (iv) of Theorem 3.1.3.
Conversely, if (i), (ii), (iii) are satisfied then it follows from Theorem 3.1.3 that any element of $\tilde{\mathfrak{Q}}_{\tilde{f}}$ is contained in $N_r(\mathfrak{Q}_f)$. □

Since $e_r \in K$ is an idempotent element the subset $\mathfrak{N} = \{(e_r, \nu);\ \nu \in N_r(L)\}$ is a subquasigroup of $\mathfrak{Q}_f$. Applying the previous theorem to the trivial subgroup $H = \{e_r\} \subset K$ and to the right nucleus $M = N_r(L)$ we get the following

Corollary 3.2.2. *The subquasigroup $\mathfrak{N}$ is right nuclear in $\mathfrak{Q}_f$ if and only if $f(e_r, e_r) \in N_r(L) \cap C(L)$.*

Let $(H, \cdot)$ be a subquasigroup of $(K, \cdot)$ and consider the f-subextension $\tilde{\mathfrak{Q}}_{\tilde{f}} = (\varphi_f^{-1}(H), \circ) = (H \times L, \circ) \subset \mathfrak{Q}_f$ of the loop $(L, \cdot)$ by the subquasigroup $(H, \cdot)$.

Corollary 3.2.3. *The subquasigroup $\varphi_f^{-1}(H)$ is right nuclear in $\mathfrak{Q}_f$ if and only if*

(i) *$H \subset N_r(K)$ and $f(e_r, h) = f(e_r, e_r)$ for all $h \in H$,*

(ii) *$(L, \cdot)$ is a group and $f(y, h)$ is contained in the center $Z(L)$ for any $y \in K,\ h \in H$,*

(iii) *$f(x, y)f(xy, h) = f(x, yh)f(y, h)$ for any $x, y \in K$ and $h \in H$.*

This assertion implies for trivial subquasigroup $H = \{e_r\}$ the following

Theorem 3.2.4. *An f-extension $\mathfrak{Q}_f$ is right nuclear if and only if $(L, \cdot)$ is a group and $f(e_r, e_r)$ is contained in the center $Z(L)$.*

If the nucleus $N(\mathfrak{Q}_f)$ of $\mathfrak{Q}_f$ is non-empty, then $\mathfrak{Q}_f$ has a unit and hence it is a loop. Consequently its homomorphic image $(K, \cdot)$ is also a loop.

Corollary 3.2.5. *An f-extension $\mathfrak{Q}_f$ of a loop $(L, \cdot)$ by a quasigroup $(K, \cdot)$ is nuclear if and only if*

(A) *$(K, \cdot)$ is a loop (with unit $e \in K$) and $(L, \cdot)$ is a group,*

(B) *$f(e, x) = f(x, e) = f(e, e)$ for all $x \in K$ and this element is contained in the center $Z(L)$.*

Proof. According to Theorem 3.2.4 if $\mathfrak{Q}_f$ is right nuclear then $(L, \cdot)$ is a group and $f(x, e_r) = f(e_r, e_r)$ for any $x \in K$ is contained in the center $Z(L)$. If it is nuclear, then $(e_r, f(e_r, e_r)^\rho)$ is the unit of $\mathfrak{Q}_f$ and hence it is a loop. Consequently, its homomorphic image $(K, \cdot)$ is also a loop and $e = e_r \in K$ is its unit. It follows $(e, f(e, e)^\rho)(x, \xi) = (x, f(e, x)f(e, e)^\rho \xi) = (x, \xi)$ for any $(x, \xi) \in K \times Q$, i. e. $f(e, x) = f(e, e)$ for all $x \in K$.
Conversely, according to Proposition 3.2. (i) in [22], conditions (A) and (B) imply that the f-extension $\mathfrak{Q}_f$ is nuclear. $\square$

Remark 3.2.6. *A nuclear f-extension $\mathfrak{Q}_f$ is central extension if and only if the group $(L, \cdot)$ is abelian.*

3.3 f-extensions with right inverse property

Let $\mathfrak{Q}_f$ be a right nuclear f-extension, i. e. $(L, \cdot)$ is a group and $f(e_r, e_r)$ is contained in the center $Z(L)$ of $(L, \cdot)$. Let $\mathfrak{Q}_f^*$ be the f^*-extension defined by the function $f^*(x, y) = f(x, y)f(e_r, e_r)^{-1}$ and the corresponding multiplication $(x,\xi)*(y,\eta)=(xy, f(x,y)f(e_r,e_r)^{-1}\xi\eta)$ on $K\times L$. Clearly, the quasigroups $\mathfrak{Q}_f$ and $\mathfrak{Q}_f^*$ are isomorphic with respect to the isomorphism $(x,\xi)\mapsto(x, f(e_r,e_r)^{-1}\xi)$. In the following we will assume that the function $f: K\times K\to L$ is normalized by the assumption $f(x,e_r)=\epsilon$ for all $x\in K$.

Proposition 3.3.1. *Let $\mathfrak{Q}_f$ be an f-extension of a group $(L,\cdot)$ by a quasigroup $(K,\cdot)$ with right unit e_r satisfying $f(x,e_r)=\epsilon$ for all $x\in K$. $\mathfrak{Q}_f$ has the right inverse property if and only if*

(i) *the quasigroup $(K,\cdot)$ has the right inverse property,*

(ii) *there exists a function $\mu: K\to Z(L)$ such that for any $x,y\in K$ $\mu(y)=f(xy,\iota(y))f(x,y)$ holds, where $\rho_{\iota(x)}=\rho_x^{-1}$.*

Proof. The quasigroup $\mathfrak{Q}_f$ has the right inverse property if and only if there exists a bijective map $J: L\times K\to L\times K$ satisfying the identity $(x,\xi)(y,\eta)\cdot J(y,\eta)=(x,\xi)$ for any $(x,\xi),(y,\eta)\in K\times L$. It follows that $(K,\cdot)$ has also the right inverse property and the bijection $J: L\times K\to L\times K$ has the shape $J(y,\eta)=(\iota(y),\eta'(y,\eta))$. Hence one has $f(xy,\iota(y))f(x,y)\xi\eta\eta'(y,\eta)=\xi$ for any $(x,\xi)\in K\times L$. Putting $\xi=\epsilon$ we get $\eta'(y,\eta)=\eta^{-1}\left(f(xy,\iota(y))f(x,y)\right)^{-1}$. This means that $f(xy,\iota(y))f(x,y)$ does not depend on $x\in K$ and is contained in the center $Z(L)$. Hence the right inverse property of $\mathfrak{Q}_f$ is equivalent to the right inverse property of $(K,\cdot)$ and to the existence of a function $\mu: K\to Z(L)$ such that $\mu(y)=f(xy,\iota(y))f(x,y)$ and $J(y,\eta)=(\iota(y),\eta^{-1}\mu(y)^{-1})$ hold for any $x,y\in K$. ☐

Now, we assume that the quasigroup $\mathfrak{Q}_f$ has the right inverse property and hence conditions (i) and (ii) of Proposition 3.3.1 are satisfied. Let Γ be the permutation group acting on the set $K \times K$ generated by the bijection $\phi : (x, y) \mapsto (xy, \iota(y)) : K \times K \to K \times K$. Since the map ϕ is an involution, the group Γ has order 2. The orbit $\Gamma(x, y) = \{(x, y), (xy, \iota(y)\}$ consists of one point if and only if $y = e_r$, in the case $y \neq e_r$ the orbit $\Gamma(x, y)$ consists of two different points. We denote by $(K \times K)/\Gamma$ the set of orbits of Γ in $K \times K$.
Proposition 3.3.1 yields the following

Corollary 3.3.2. *Let $(L, \cdot)$ be a group and $(K, \cdot)$ be a quasigroup with right inverse property. Let be given a map $\mu : K \to Z(L)$, a choice function $c : (K \times K)/\Gamma \to K \times K : \gamma \mapsto c(\gamma) \in \gamma$ of orbits of Γ in $K \times K$ and a function $\nu : (K \times K)/\Gamma \to L$ satisfying $\nu(\{(x, e_r)\}) = \epsilon$ for all $x \in K$. Then the function $f : K \times K \to L$ is determined by the conditions*

(i) $f(c(\gamma)) = \nu(\gamma)$ *for any* $\gamma \in (K \times K)/\Gamma$,

(ii) $f(\phi(x, y)) = \mu(y) f(x, y)^{-1}$ *for any* $x, y \in K$

yields an f-extension $\mathfrak{Q}_f$ with right inverse property. Conversely, any right nuclear f-extension with right inverse property can be obtained by the previous construction.

Now, let $(K, \cdot)$ be an involutorial right Bol loop with unit $e \in K$. Then we have

Theorem 3.3.3. *Let $\mathfrak{Q}_f$ be an f-extension of a group $(L, \cdot)$ by an involutorial right Bol loop $(K, \cdot)$ such that $f(x, e) = \epsilon$ for any $x \in K$.*

The f-extension $\mathfrak{Q}_f$ is right alternative if and only if $\mathfrak{Q}_f$ is a loop with right inverse property. In this case the f-extension $\mathfrak{Q}_f$ is nuclear.

Proof. The quasigroup $\mathfrak{Q}_f$ is right alternative if and only if the identity $(x,\xi)(y,\eta)\cdot(y,\eta) = (x,\xi)\cdot(y,\eta)(y,\eta)$ holds for any $(x,\xi),(y,\eta) \in K \times L$. Since the involutorial right Bol loop $(K,\cdot)$ is right alternative, $\mathfrak{Q}_f$ is right alternative if and only if $f(xy,y)f(x,y)\xi = \xi f(y,y)$ for any $x, y \in K$ and $\xi \in L$. Since $\iota(x) = x$ holds for any $x \in K$, according to Proposition 3.3.1, the right alternative quasigroup $\mathfrak{Q}_f$ has the right inverse property . Moreover, putting $x = y$ into this equation we obtain $f(e,y)f(y,y) = f(y,y)$, from which follows $f(e,y) = \epsilon$ for any $y \in K$. Hence Corollary 3.2.5 yields that the f-extension $\mathfrak{Q}_f$ is nuclear and $\mathfrak{Q}_f$ is a loop. Conversely, if $\mathfrak{Q}_f$ is a loop with right inverse property, then Corollary 3.2.5 shows that the equation $f(xy,y)f(x,y)\xi = \xi\mu(y)$ is satisfied. Putting $x = e$ and $\xi = \epsilon$ we get the equation $f(y,y)f(e,y) = \mu(y)$. Since $\mathfrak{Q}_f$ is a loop, one has $f(e,y) = \epsilon$, and hence $\mathfrak{Q}_f$ is right alternative. □

Chapter 4

Regular permutations

If a quasigroup has non-empty right or left nucleus, then it has right or left unit element, respectively. V.D. Belousov introduced the notions of right and left regular permutations which can be used for the measure of the near-associativity of quasigroups having neither right nor left unit element. In the case if the quasigroup has right or left unit element, then the right or left regular permutations coincide with right or left translations by elements of the right or left nuclei, respectively. Hence the notions of regular permutations can be considered as natural generalizations of the notions of nuclei.

For the investigation of groups of right or left regular permutations we use the methods of extension theory. In [23] P. T. Nagy and I. Stuhl investigated nuclear extensions of quasigroups having left or right unit element. In this chapter we investigate quasigroup extensions having empty nuclei and describe their groups of left or right regular permutations. We give conditions under which the orbits of the groups of right or left regular permutations are contained in the kernels of the homomorphism associated with the extension. This construction alerts us quasigroups with prescribed groups of right or left regular permutations of different sizes. These are published in [28].

4.1 Regular permutations of quasigroups

Let we consider the f-extension of the proper quasigroup $(Q, \cdot)$ by the proper quasigroup $(K, \cdot)$ and let be denoted by $(\mathfrak{Q}_f, \circ)$.

Let $\varphi : \mathfrak{Q}_f \to K : (a, \alpha) \mapsto a$ be the canonical homomorphism of the f-extension $((\mathfrak{Q}_f, \circ)$. The kernel θ of $\varphi : \mathfrak{Q}_f \to K : (a, \alpha) \mapsto a$ is a normal equivalence relation on $\mathfrak{Q}_f$ given by

$$(a, \alpha)\theta(b, \beta) \iff \varphi(a, \alpha) = \varphi(b, \beta).$$

The equivalence classes have the shape $\{(a, \alpha); \alpha \in Q\}$ for $a \in K$. We call θ *the equivalence relation of the extension* and the equivalence classes of θ the *equivalence classes of the extension.* If the quasigroup K is idempotent, then these equivalence classes are normal subquasigroups of $(\mathfrak{Q}_f, \circ)$, which are called *kernel quasigroups* of the extension.

Lemma 4.1.1. *A bijection $\rho : (x, \xi) \mapsto (\rho_1(x, \xi), \rho_2(x, \xi)) : K \times Q \to K \times Q$ is a right-regular permutation of an f-extension $(\mathfrak{Q}_f, \circ)$ if and only if*

(i) *ρ_1 is constant on the equivalence classes of the extension and induces a right-regular permutation $\rho_1 : K \to K$ of the quasigroup $(K, \cdot)$,*

(ii) *ρ_2 satisfies*

$$\rho_2(xy, f(x, y) \cdot \xi\eta) = f(x, \rho_1(y, \eta)) \cdot \xi\rho_2(y, \eta)$$

for all $x, y \in K,\ \xi, \eta \in Q$.

Proof. If $(x,\xi) \mapsto (\rho_1(x,\xi),\rho_2(x,\xi))$ is a right-regular permutation, then

$$\rho\left((x,\xi)\circ(y,\eta)\right) = (\rho_1(xy, f(x,y)\cdot\xi\eta), \rho_2(xy, f(x,y)\cdot\xi\eta)) =$$

$$= (x,\xi)\circ\rho(y,\eta) = (x\rho_1(y,\eta), f(x,\rho_1(y,\eta))\cdot\xi\rho_2(y,\eta)).$$

It follows that that the condition (ii) is satisfied and $\rho_1(xy, f(x,y)\cdot\xi\eta) = x\rho_1(y,\eta)$ for all $x, y \in K,\ \xi,\eta \in Q$. Consequently $\rho_1(xy, f(x,y)\cdot\xi\eta)$ is independent on ξ, hence ρ_1 is constant on the equivalence classes of the extension and the induced map $\rho_1 : K \to K$ satisfies $\rho_1(xy) = x\rho_1(y)$. ☐

4.2 f-extensions by quasigroups $(K,\cdot)$ with trivial $\mathbf{R}(K)$

The following assertion follows from the previous theorem:

Remark 4.2.1. *If the group of right-regular permutations of a quasigroup $(K,\cdot)$ is trivial, then the orbits of the group of right-regular permutations of the f-extension $(\mathfrak{Q}_f,\circ)$ are contained in the congruence classes of the extension.*

This result motivates the investigation of extensions by quasigroups which have trivial group of right-regular permutations.
Quasigroups with trivial right-regular permutation groups form a wide class. For example, idempotent quasigroups have this property, since if a quasigroup $(K,\cdot)$ is idempotent and a map $\phi : K \to K$ satisfies

$x\phi(x) = \phi(x^2)$ for any $x \in K$, then $x\phi(x) = \phi(x)^2$ and hence $\phi(x) = x$. Many constructions of idempotent quasigroups are given in [21], Sections 9 and 10 by the study of the core of Bol loops.

Theorem 4.2.2. *Assume that the group of right-regular permutations of the quasigroup $(K, \cdot)$ is trivial and $(Q, \cdot)$ is a loop. A map $\rho : \mathfrak{Q}_f \to \mathfrak{Q}_f$ is a right-regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$ if and only if it has the shape $\rho = (id, \rho_\nu)$, where $\nu \in N_r(Q)$.*

Proof. Using Lemma 4.1.1 we prove that a map $\rho_2 : K \times Q \to Q$ satisfying the identity

$$\rho_2(xy, f(x,y) \cdot \xi\eta) = f(x,y) \cdot \xi\rho_2(y,\eta)$$

is a right translation of $(Q, \cdot)$ by an element $\nu \in N_r(Q)$. Putting $\xi = \varepsilon$ into this identity, where ε is the unit element of $(Q, \cdot)$, we get

$$\rho_2(xy, f(x,y)\eta) = f(x,y)\rho_2(y,\eta). \tag{4.1}$$

It follows, that

$$f(x,y) \cdot \xi\rho_2(y,\eta) = \rho_2(xy, f(x,y) \cdot \xi\eta) = f(x,y) \cdot \rho_2(y, \xi\eta),$$

and hence $\rho_2(y, \xi\eta) = \xi\rho_2(y,\eta)$ for any $y \in K$. Applying this to equation (4.1) we obtain

$$f(x,y)\rho_2(xy,\eta) = \rho_2(xy, f(x,y)\eta) = f(x,y)\rho_2(y,\eta).$$

Hence the map $(x,\xi) \mapsto \rho_2(x,\xi)$ does not depend on $x \in K$ and $\xi \mapsto \rho_2(\xi)$ is a right-regular permutation of $(Q, \cdot)$. Consequently ρ_2

is a right translation ρ_ν of $(Q,\cdot)$, where $\nu \in N_r(Q)$. Conversely, it is clear, that for any $\nu \in N_r(Q)$ the map $\rho = (id, \rho_\nu)$ is a right regular permutation of $(\mathfrak{Q}_f, \circ)$. $\square$

Corollary 4.2.3. *The group of the right-regular permutations of the quasigroup* $(\mathfrak{Q}_f, \circ)$ *is isomorphic with the right nucleus of* $(Q,\cdot)$.

Corollary 4.2.4. *If the right nucleus* $N_r(Q)$ *is a normal subgroup, then the equivalence relation induced by the orbits of the right-regular permutation group* $R(\mathfrak{Q}_f)$ *is a normal congruence.*

Corollary 4.2.5. *If* $(K,\cdot)$ *is an idempotent quasigroup and the right nucleus of* $(Q,\cdot)$ *is a normal subgroup, then the orbits of the right-regular permutation group are normal subquasigroups of the kernel quasigroups of the extension.*

4.3 Left-regular permutations of f-extensions

Similarly to the right-regular case we have

Lemma 4.3.1. *A bijection* $\lambda : (x,\xi) \mapsto (\lambda_1(x,\xi), \lambda_2(x,\xi)) : K \times Q \to K \times Q$ *is a left-regular permutation of an* f*-extension* $(\mathfrak{Q}_f, \circ)$ *if and only if*

(i) λ_1 *is constant on the equivalence classes of the extension and induces a left-regular permutation* $\lambda_1 : K \to K$ *of the quasigroup* $(K,\cdot)$,

(ii) λ_2 *satisfies*

$$\lambda_2(xy, f(x,y)\cdot\xi\eta) = f(\lambda_1(x,\xi), y)\cdot\lambda_2(x,\xi)\eta$$

for all $x, y \in K,\ \xi, \eta \in Q$.

If the group of left-regular permutations of the quasigroup $(K,\cdot)$ is trivial, then the orbits of the group of left-regular permutations are contained in the congruence classes of the extension.

Theorem 4.3.2. *Assume that the group of left-regular permutations of the quasigroup* $(K,\cdot)$ *is trivial and* $(Q,\cdot)$ *is a loop. A map* $\lambda : K\times Q \to K\times Q$ *is a left-regular permutation of the* f*-extension* $(\mathfrak{Q}_f, \circ)$ *if and only if it has the shape* $(x,\xi)\mapsto(x,\nu(x)\xi)$*, where* ν *is a mapping* $\nu : K \to N_l(Q)$ *satisfying*

$$(4.2)\qquad \nu(xy)f(x,y) = f(x,y)\nu(x) \quad \textit{and} \quad \lambda_{f(x,y)\nu(x)} = \lambda_{f(x,y)}\lambda_{\nu(x)}$$

for any $x, y \in K$.

Proof. According to Lemma 4.3.1 the component λ_2 of a left-regular permutation $\lambda : \mathfrak{Q}_f \to \mathfrak{Q}_f$ satisfies

$$(4.3)\qquad \lambda_2(xy, f(x,y)\cdot\xi\eta) = f(x,y)\cdot\lambda_2(x,\xi)\eta.$$

Putting $\eta = \varepsilon$ we get

$$\lambda_2(xy, f(x,y)\xi) = f(x,y)\lambda_2(x,\xi).$$

Applying this to the previous identity we have

$$f(x,y)\lambda_2(x,\xi\eta) = \lambda_2(xy, f(x,y)\cdot\xi\eta) = f(x,y)\cdot\lambda_2(x,\xi)\eta.$$

Consequently for any $x \in K$ the map $\xi \mapsto \lambda_2(x, \xi)$ is a left-regular permutation of the loop $(Q, \cdot)$, and hence it is a left multiplication by an element $\nu(x) \in N_l(Q)$. Now, equation (4.3) gives

$$\nu(xy)\,(f(x,y) \cdot \xi\eta) = f(x,y)\,(\nu(x)\xi \cdot \eta)\,.$$

Since $\nu(x) \in N_l(Q)$ we obtain the equivalent identity

$$\nu(xy) \cdot f(x,y)\xi = f(x,y) \cdot \nu(x)\xi. \tag{4.4}$$

With $\xi = \varepsilon$ we obtain the first part of condition (4.2). Using $\nu(xy) \in N_l(Q)$ equation (4.4) implies the second part of (4.2).
Obviously, a map $(x, \xi) \mapsto (x, \nu(x)\xi)$ satisfying $\nu(x) \in N_l(Q)$ and condition (4.2) for any $x, y \in K$ is a left-regular permutation. □

Let F denote the subloop of $(Q, \cdot)$ the element of which commute with all $f(x,y)$; $x, y \in K$. The previous theorem yields the following

Corollary 4.3.3. *The map $\lambda = (id, \lambda_\nu)$ is a left regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$ if $\nu \in N_l(Q) \cap F$ and $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ for any $x, y \in K$.*

Theorem 4.3.4. *Assume that the group of left-regular permutations of the quasigroup $(K, \cdot)$ is trivial, $(Q, \cdot)$ is a loop and there exists $k \in K$ such that $f(k, y) = \kappa$ is constant for any $y \in K$ with $\kappa \in F$. A map $\lambda : K \times Q \to K \times Q$ is a left-regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$ if and only if $\lambda = (id, \lambda_\nu)$, where $\nu \in N_l(Q) \cap F$ and $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ for any $x, y \in K$.*

Proof. We prove the first condition in (4.2) putting $x = k$ and $y = k\backslash t$ in the first condition in (4.2) one has $\nu(t)f(k, k\backslash t) = f(k, k\backslash t)\nu(k)$ for any $t \in K$ and we get that $\nu(t) = \nu(k)$ is constant for all $t \in K$. The element $\nu(k)$ has to commute with any element of F, hence $\nu(k) \in N_l(Q) \cap F$. The second condition in (4.2) implies $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ for any $x, y \in K$. Conversely, any element $\nu \in N_l(Q) \cap F$ satisfying $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ for any $x, y \in K$ fulfils the conditions (4.2) and hence $\lambda = (id, \lambda_\nu)$ is a left-regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$. □

Theorem 4.3.5. *Assume that the group of left-regular permutations of the quasigroup $(K, \cdot)$ is trivial and the loop $(Q, \cdot)$ has the cross inverse property. A map $\lambda : K \times Q \to K \times Q$ is a left-regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$ if and only if $\lambda = (id, \lambda_\nu)$, where $\nu \in N_l(Q)$ and $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ for any $x, y \in K$.*

Proof. We multiply from the right the first identity of (4.2) with the cross inverse $f(x, y)^{-1}$ of the element $f(x, y)$. Since $\nu(x) \in N_l(Q)$ for any $x \in K$ we obtain

$$\nu(xy) = \nu(xy) \cdot f(x, y)f(x, y)^{-1} = \nu(xy)f(x, y) \cdot f(x, y)^{-1} =$$

$$= f(x, y)\nu(x) \cdot f(x, y)^{-1} = \nu(x),$$

i. e. $\nu : K \to Q$ is a constant function and hence the assertion is proved. □

Chapter 5

Összefoglaló

A csoportokra jól ismert Schreier-féle bővítés elmélet értelmében, két adott csoportból kiindulva egy új csoport készíthető olymódon, hogy a kapott csoport az elsőt normálosztóként tartalmazza, illetve amelyre vonatkozó faktorcsoport izomorf a másik csoporttal. Ez a konstrukció az alábbi egzakt sorozattal irható le:

$$1 \longrightarrow \Gamma \longrightarrow \mathfrak{G} \longrightarrow G \longrightarrow 1.$$

Legyenek G és Γ adott csoportok, a $\mathfrak{G}$ csoport a Γ bővítése G-vel, ha a $G \times \Gamma$ halmazon egy művelet a következő módon van definiálva

$$(a,\alpha)(b,\beta) = (ab, f(a,b)\cdot\alpha^{T(b)}\cdot\beta),$$

ahol $T : G \longrightarrow Aut(\Gamma)$ és $f : G \times G \longrightarrow \Gamma$ leképezésekre teljesül

1. $f(1,a) = f(a,1) = 1;$
2. $T(a)^{-1}T(b)^{-1}f(a,b)T(ab)f(a,b)^{-1} = 1;$
3. $f(b,c)f(a,bc) = f(a,b)^{T(c)}f(ab,c);$

minden $a, b, c \in G$ esetén.

A legegyszerűbb esetben, amikor mind T mind f triviális, a $\mathfrak{G}$ csoport a Γ és G csoportok direktszorzata. Ha az f faktor rendszer triviális és a T leképezés nem triviális, akkor $\mathfrak{G}$ a Γ és G csopoprtok szemidirektszorzata. Ha T triviális és f nemtriviális, akkor a művelet a következő alakot ölti

$$(a, \alpha)(b, \beta) = (ab, f(a, b) \cdot \alpha \cdot \beta),$$

ezt az esetet f-bővítésnek fogjuk nevezni.

Hasonló konstrukció adható meg loopok és kvázicsoportok bővítéseire a megfelelő változtatásokkal, ám az asszociativítás hiánya folytán olyan drasztikusan változás áll be, hogy általános bővítéselmélet nem létezik. Az utóbbi időben gyakran alkalmazzák ezen struktúrák bővítéseit eltérő interpretációban [3], [11], [14], [16], [17], [18].

Csoportok loopokkal történő loop bővítéseit áttekintően, rendszerezve taglalja Nagy Péter és Karl Strambach [22] munkája, melynek eszméjét követve loopok és kvázicsoportok f-bővítéseit vizsgáljuk ezen disszertációban az alábbi egzakt sorozatok szerint

$$1 \longrightarrow \Lambda \longrightarrow \mathfrak{L} \longrightarrow L \longrightarrow 1$$

illetve

$$Q \longrightarrow \mathfrak{Q} \longrightarrow K,$$

ahol Λ, L loop és K, Q kvázicsoport.

Ahogy a csoport bővítések között kitüntetett figyelmet élveznek a centrális bővítések, úgy a loop- és kvázicsoport bővítései között is gyakran vizsgáltak a centrális és természetes általánosításuk, a nukleáris bővítések.

Az első fejezet a disszertáció megértéséhez szükséges fogalmakat és összefüggéseket tartalmazza.

A második fejezetben egy centrális loop bővítést vizsgálunk, amelynek kombinatorikai háttere van. Egy bővítést centrálisnak nevezünk, ha a hozzá tartozó homomorfizmus magja centrális részcsoport. Mielőtt a vizsgálatára térnénk, tekintsük ezt a kombinatorikai hátteret és meghatározzuk az eltolásai által generált csoportot. A fejezet első részének eredményei megtalálhatóak a [26]-ben.

Legyen $\mathfrak{S}$ egy Steiner-hármasrendszer, azaz egy $(n, 3, 1)$-illeszkedési rendszer. A pontjainak halmazán bevezetünk egy műveletet a következő képpen: $a * b = c$ akkor és csak akkor, ha (a, b, c) egy blokk és $a * a = a$. Erre a műveletre vonatkozóan az $a * y = c$ és $x * b = c$ egyenletek egyértelműen megoldhatóak ami azt jelenti, hogy $(\mathfrak{S}, *)$ kvázicsoport. A pontok szerepe a bokkon belül szimmetrikus, ezért $a * b = b * a$, $a * (a * b) = b$ is teljesül és ezzel együtt $Q(\mathfrak{S}) := (\mathfrak{S}, *)$ totálisan szimmetrikus idempotens kvázicsoport, *Steiner kvázicsoport.*

Ha a Steiner-hármasrendszer pontjainak halmazát kibővítjük egy e-vel jelölt szimbólummal $S := \mathfrak{S} \cup \{e\}$ és az előző műveletet annyiban módosítjuk, hogy $a * a = e$, $a * e = e * a = a$, akkor $S(\mathfrak{S}) := (S, *)$ egy totálisan szimmetrikus, 2 exponensű, egységelemes kvázicsoport, azaz egy *Steiner loop* [25], Chap.V.

A Steiner-kvázicsoport és a Steiner-loop ugyanannak a struktúrának két arca és egyformán viselkednek mindaddig, amíg az egységelem nem jut lényeges szerephez például a Steiner-hármasrendszer automorfizmus csoportja izomorf a belőle származtatott Steiner-kvázicsoport és Steiner-loop automorfizmus csoportjával: $Aut(\mathfrak{S}) \cong Aut(Q(\mathfrak{S})) \cong Aut(S(\mathfrak{S}))$. Mihelyt megnő az egységelem szerepe a helyzet drasztikusan megváltozik, például ha $\mathfrak{S}$ egy Hall-rendszer (olyan Steiner-hármasrendszer, amelyben bármely három nemkollineáris pont az affín síkot generálja), akkor $Q(\mathfrak{S})$-ban teljesül az $(xy)z = (xz)(yz)$ azonosság, ami egy záródási tulajdonság az affín síkon, míg $S(\mathfrak{S})$-ban nem ami rögtön adódik ha az x helyére e-t írunk. Egy másik példa, hogy az eltolások által generált csoport Hall-rendszerek esetén a kvázicsoportnál feloldható [12] és [3], 86 o., míg a loopnál nemfeloldható.

Egy $S(\mathfrak{S})$ Steiner loop csoport, akkor és csak akkor, ha bármely 3 nemkollineáris pontja a Fano-síkot generálja [7], Corollary 1, 251. o., így egy projektív térből származó Steiner-loop eltolás csoportja egy elemi Abel-2-csoport. Arra az esetre amikor a Steiner-loop nem projektív térből származik a következő eredményt kaptuk:

Tétel. *Legyen $\mathfrak{S}$ $n-1$ elemű, $S(\mathfrak{S})$ illetve $Q(\mathfrak{S})$ a belőle származtatott Steiner-loop illetve Steiner-kvázicsoport. Ha $Q(\mathfrak{S})$ két különböző eltolásának szorzata páratlan rendű, akkor a $S(\mathfrak{S})$ eltolásai által generált csoport az A_n vagy az S_n attól függően, hogy n osztható-e 4-gyel vagy sem.*

Megjegyzés. *Minden Hall-rendszerből származó Steiner-kvázicsoport eleget tesz a Tétel feltételének.*

A tétel alkalmazható további nem Hall-rendszerekből származó Steiner-loopokra, például tekintsük a következő konstrukciót [9], 2.1., 291. o.: Legyen C $\frac{4^n-1}{3}$-rendű ciklikus csoport, legyen $\mathfrak{S} := C_0 \cup C_1 \cup C_2$, ahol $C_i = C$ $(i = 0, 1, 2)$ és ha $x \in C$ a C_i példányban van, akkor jelöljük x_i-vel, ezek alkossák a Steiner-hármas rendszer pontjait és a blokkjai legyenek

- $\{x_0, x_1, x_2\}$ halmazok $x_0 = x_1 = x_2 = x \in C$
- $\{x_0, y_0, z_1\}, \{x_1, y_1, z_2\}, \{x_2, y_2, z_0\}$,
  ahol $x, y, z \in C : x \neq y, xy = z^2$.

Megjegyzés. *Legyen C $\frac{4^n-1}{3}$ $(n > 1)$ rendű ciklikus csoport és legyen $Q(\mathfrak{S}_C)$ a C-ből a fenti konstrukció által készített Steiner-hármasrendszerhez tartózó Steiner-kvázicsoport. Ekkor a $Q(\mathfrak{S}_C)$-hez tartozó $S(\mathfrak{S}_C)$ Steiner-loop eltolásai által generált csoport az A_{4^n} alternáló csoprt.*

A fejezet második részében tekintjük a másodrendű ciklikus csoport alábbi f-bővítését Steiner-loopokkal:

$$1 \longrightarrow C_2 \longrightarrow L \longrightarrow S \longrightarrow 1$$

a következőképpen az

$$S \times C_2 = \{(a, \epsilon) : a \in S, \epsilon \in C_2$$

halmazon definiáljunk egy műveletet az alábbi módon

$$(a_1, \epsilon_1) \circ (a_2, \epsilon_2) = (a_1 a_2, \epsilon_1 \epsilon_2 f(a_1, a_2)),$$

ahol az $f : S \times S \longrightarrow C_2$ leképezés a bővítés faktor rendszere. Ebben a bővítésben az automorfizmusok identitások, így a bővítést az f függvény határozza meg, ezért ezeket f-bővítéseknek nevezzük. Az $(a, \alpha) \circ (x, \xi) = (b, \beta)$ és $(x, \xi) \circ (a, \alpha) = (b, \beta)$ egyenletek egyértelműen megoldhatóak, így $(L, \circ)$ egy kvázicsoport. Ha a faktor rendszerre előírjuk, hogy $f(a, e) = f(e, a) = 1$ minden $a \in S$, akkor $(e, 1)$ az egységeleme lesz, azaz egy loopot kapunk. A kapott L loop minden további tulajdonságát az f faktor rendszer határozza meg. Már a legegyszerűbb esetben, amikor a másodrendű csoportot azzal a négyelemű Steiner-looppal bővítjük, amelyik a negyedrendű elemi Abel-2-csoport, loopok egy széles osztályát kapjuk különböző gyenge asszociativitási tulajdonságokkal. A legismertebb példák a kód loopok ([13], [15]) és a C-loopok ([17]). A további vizsgálatainkat azokra az L f-bővítésekre szorítjuk, amelyek faktor rendszerére $f(x, y) = -f(y, x)$ $\forall x \neq y \in S \setminus \{e\}$ és $f(x, x) = 1$ ill -1 minden $y \in S \setminus \{e\}$-ra teljesül. Ugyanis az így kapott loopok kombinatorikai tartalommal bírnak. Az erre vonatkozó eredmények [27]-ban vannak ismertetve.

Ha egy Steiner-hármasrendszer minden blokkján adott egy ciklikus rendezés, akkor irányított Steiner-hármasrendszerről beszélünk. Minden irányított Steiner-hármasrendszerhez rendelhetünk egy f :

$(\mathfrak{S}, T) \times (\mathfrak{S}, T) \longrightarrow \{\pm 1\}$ irányítás függvényt olymódon, hogy ha (a_1, a_2, a_3) egy irányított blokk, akkor $f(a_1, a_2) = 1$ és $f(a_2, a_1) = -1$. Ezek ismeretében definiálhatjuk a következő fogalmat

Definíció. *Irányított Steiner-loopnak nevezünk egy olyan L bővítést, amelyhez létezik egy $(\mathfrak{S}, T)$ irányított Steiner-hármasrendszer úgy, hogy a bővítésbeli S Steiner-loop egységelemtől különböző elemei az $\mathfrak{S}$ pontjai és a bővítés faktor rendszerének leszűkítése az $(\mathfrak{S} \times \mathfrak{S}) \setminus \{(x, x), x \in \mathfrak{S}\}$-ra megegyezik az $(\mathfrak{S}, T)$ irányítás függvényével, továbbá $f(x, x) = -1$ vagy $f(x, x) = 1$ minden $x \in S \setminus \{e\}$-re.*

Ha az L irányított Steiner-loop exponense 4 illetve 2, akkor a centruma másodrendű illetve triviális.

Tétel. *Minden L irányított Steiner-loop flexibilis.*

Az L irányított Steiner-loop inverz tulajdonságú akkor és csak akkor, ha a centrumán kívüli elemek rendje 4.

Az L irányított Steiner-loop cross és automorf inverz tulajdonságú de sem bal- sem jobbinverz tulajdonságú pontosan akkor, ha az exponense 2.

Ezek után egy irányított Steiner-loop asszociativítására az alábbi eredményt kapjuk

Tétel. *Egy irányított Steiner-loop pontosan akkor csoport, ha a 8-ad rendű kvaternió csoport.*

A továbbiakban jelölje $\hat{G}$, $\hat{G}_l$ és $\hat{G}_r$ rendre az L loop összes-, bal- és jobb- eltolásai által generált csoportot. Ahhoz, hogy ezeket le tudjuk írni, be kell vezetnünk egy leképezést, legyen $\tau_a : L \longrightarrow L$ $(a \in S \setminus \{e\})$,

$\tau_a : (y,\epsilon) \mapsto (y,\epsilon)$ $y \in \{e,a\}$, $\epsilon \in \{1,-1\}$; $\tau_a : (x,\epsilon) \mapsto (x,-\epsilon)$ $a \neq x \in S \setminus \{a,e\}$, $\epsilon \in \{1,-1\}$.

Tétel. *Ha az L irányított Steiner-loop egy n elemű S Steiner-loophoz tartozik, akkor $\hat{G}$ normális részcsoportja $\Theta =< \tau_a, a \in S >$, a 2^{n-1}-ed vagy a 2^{n-2}-ed rendű elemi Abel-2-csoport attól függően, hogy n páratlan vagy páros és a $\hat{G}/\Theta$ faktorcsoport izomorf $\hat{G}_l$-al és $\hat{G}_r$-al.*

Következmény. *A $\hat{G}_l$ és $\hat{G}_r$ csoportok izomorfak.*

Az irányított Steiner-hármasrendszer automorfizmusaira vonatkozóan

Állítás. *Irányított Steiner-hármasrendszerek automorfizmus csoportja páratlan rendű.*

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Állítás. *Legyen U egy $\mathfrak{S}$ Steiner-hármasrendszer páratlan rendű automorfizmus csoportja. Ekkor létezik irányított Steiner-hármasrendszer, amelynek U automorfizmus csoportja.*

Megjegyzés. *Legyen $(\mathfrak{S}, T)$ egy véges irányított Steiner-hármasrendszer K automorfizmus csoporttal és legyen K_0 olyan valódi részcsoportja K-nak, hogy a K egy B blokk orbitja a K_0 $B_1, .., B_m$ különböző orbitjaira bomlik fel. Ekkor létezik egy $(\mathfrak{S}, T_0)$ irányított Steiner-hármasrendszer úgy, hogy K_0 automorfizmusokból áll, de K nem automorfizmus csoportja.*

Az irányított Steiner-hármasrendszer automorfizmus csoportjának ismeretében leírhatjuk a hozzá tartozó irányított Steiner-loop automorfizmus csoportját.

Tétel. *Egy L irányított Steiner-loop automorfizmus csoportja, az irányított Steiner-hármasrendszer automorfizmus csoportjának szemidirekt szorzata egy olyan normális elemi Abel-2-csoporttal, amely az L/A faktorloopon az identitást indukálja és az $S \longrightarrow \mathbb{Z}_2$ homomorfizmusok halmazához tartozik.*

A továbbiakhoz be kell vezetnünk egy újabb fogalmat:

Definíció. *Legyen $\mathfrak{S}$ egy Steiner-hármasrendszer. Egy $\mathfrak{K}$ valódi részrendszerét blokád részrendszernek nevezzük, ha $\mathfrak{S}$ minden nem $\mathfrak{K}$-beli blokkja $\mathfrak{K}$-t egy pontban metszi.*

Tétel. *Az $\mathfrak{S}$ Steiner-hármasrendszer blokád részrendszerei és az L irányított Steiner-loop azon automorfizmusai között, amelyek $\mathfrak{S}$-en az identitást indukálják, bijekció létezik.*

Tétel. *Legyen Γ az $(\mathfrak{S}, T)$ irányított Steiner-hármasrendszerhez tartozó L irányított Steiner-loop automorfizmus csoportja.*
Ha $\mathfrak{S}$ egy a $GF(3)$ test feletti affin tér, akkor a Δ csoport triviális és Γ izomorf $(\mathfrak{S}, T)$ automorfizmus csoportjával.
Ha $\mathfrak{S}$ egy n dimenziós projektív tér a $GF(2)$ test felett, akkor Δ a 2^n-ed rendű elemi Abel-2-csoport.

Következmény. *Egy nyolcadrendű irányított Steiner-loop automorfizmus csoportja izomorf az A_4 alternáló csoporttal.*

Az irányított Steiner-loopok izomorfiáinak vizsgálatánál előbukkan egy leképezés, aminek az izomorfia eldöntésében nagy szerepe van, tekintsük ezt a leképezést.

Az L irányított Steiner-loop v értékelése egy $S \longrightarrow \{1, -1\}$ leképezés, amelyre $v(e) = 1$, ahol e az S loop egységeleme. Egyazon S Steiner-loophoz tartozó f_1 illetve f_2 függvény által meghatározott L_1 illetve L_2 irányított Steiner-loopok esetén egy v értékelést kompatibilisnek mondunk az (f_1, f_2) párral, ha $v(a)v(b)f_1(a,b) = v(ab)f_2(a,b)$ minden $a, b \in S$. Az (f_1, f_2) párral kompatibilis értékelések, illetve az L_1 és L_2 loopok azon izomorfizmusai között, melyek az $\mathfrak{S}$-n az identitást indukálják, egyértelmű megfeleltetés van.

Ennek ismeretében az izomorfia kérdésére a következőket kapjuk:

Lemma. *Az L_{f_i} $(i = 1, 2)$ loopok, amelyek f_i $(i = 1, 2)$ faktor rendszereire $f_1(x, y) = -f_2(x, y)$ minden különböző $x, y \in S \setminus \{e\}$ és $f_1(x, x) = f_2(x, x)$, izomorfak. Az $(x, \epsilon) \mapsto (x, -\epsilon)$ leképezés izomorfiát definiál.*

Tétel. *Legyenek L_1 és L_2 olyan azonos exponensű f_1 és f_2 faktor rendszerű irányított Steiner-loopok, hogy az L_1/A és L_2/A faktor loopok ugyanahhoz az $\mathfrak{S}$ Steiner-hármasrendszerhez tartoznak. Ha az L_1 és L_2 loopokhoz tartozó $(\mathfrak{S}, T_1)$ és $(\mathfrak{S}, T_2)$ irányított Steiner-hármasrendszerek automorfizmus csoportjai megegyeznek, akkor L_1 és L_2 pontosan akkor izomorfak, ha S rendelkezik az (f_1, f_2) párral kompatibilis értékeléssel.*

Tétel. *Legyen L_1 egy K automorfizmus csoportú irányított Steiner-loop. Ha α az L_1-hez tartozó S Steiner-loop automorfizmusa, akkor* *létezik egy az* L_1*-gyel izomorf* L_2 *irányított Steiner-loop, amelynek a* K^α *konjugált csoport az automorfizmus csoportja.*

Állítás. *Az* $\mathfrak{S}$ *Steiner-hármasrendszerhez tartozó azonos exponensű blokk tranzitív automorfizmus csoportú* L_1 *és* L_2 *irányított Steiner-loopok izomorfak.*

Ezek után vizsgáljuk, azokat az irányított Steiner-loopokat, amelyekhez tartozó $\mathfrak{S}$ Steiner-hármasrendszer pont-egyenes sémája izomorf egy a $GF(2)$ test feletti projektív geometriával, ezeket hívjuk projektív irányított Steiner-loopoknak. Ha az L rendje 2^{n+1}, akkor az S és így az L automorfizmus csoportja egy páratlan rendű részcsoportja az $SL(n,2)$ csoportnak. Ennek folytán U feloldhato részcsoportja $SL(n,2)$-nek.

Ha U a $GF(2)$ test feletti n páros dimenziójú V vektortér automorfizmusainak feloldható csoportja úgy, hogy U páratlan rendű és tranzitívan hat a nem-nulla vektorokon, akkor V beazonosítható $GF(2^n)$ additív csoportjával és U mint permutáció csoport izomorf a $\Sigma := \{x \mapsto ax; 0 \neq a \in GF(2^n)\}$ erősen tranzitív csoporttal, mivel $GF(2^n)$ automorfizmus csoportja egy páros rendű ciklikus csoport (lsd. [24], Theorem 19.9, 246. o.).

Állítás. *Legyen* L *egy olyan* 16*-od rendű irányított projektív Steiner-loop, amely* Γ *automorfizmus csoportja tranzitívan hat a hozzá tartozó* 2*-od rendű* $\mathfrak{S}$ *projektív sík egyenesein. Ekkor a* Γ *csoport szemidirekt szorzata, a* ${\mathbb{Z}_2}^3$*-al izomorf* Δ *normális részcsoportnak, a* Δ*-n hűen ható* 21*-ed rendű csoporttal.*

Következmény. *Tetszőleges* 4 *exponensű* 16*-od rendű olyan irányított projektív Steiner-loop, amely automorfizmus csoportjának a rendje osztható* 7*-tel a* 16*-od rendű Moufang-loop.*

Tétel. *Egyetlen* ≥ 32*-ed rendű és* 4 *exponensű irányított projektív Steiner-loop sem lehet Moufang-loop.*

Következmény. *Tetszőleges irányított projektív Steiner-loop, amelyik Bol-loop, izomorf a* 16*-od rendű Moufang-looppal.*

A fejezet utolsó részében az izomorfia kérdésének összetettségére világítunk rá abban az esetben, ahol az L_1 illetve L_2 irányított Steiner-loopok az f_1 illetve f_2 irányítás függvényű $(\mathfrak{S}, T_1)$, illetve $(\mathfrak{S}, T_2)$ irányított Steiner-hármasrendszerből származnak. Továbbá automorfizmus csoportjuk konjugált csoportok $\mathfrak{S}$ automorfizmus csoportjában. Ekkor feltehető, hogy L_1, L_2 automorfizmus csoportja ugyanazt a Γ automorfizmus csoportot indukálják. Az L_1 és L_2 loopok izomorfiájának eldöntése az f_1 és f_2 függvények leszűkítéseinek vizsgálatát igényli a Γ csoport orbitjaira. Ezt a vizsgálatot illusztráljuk a 3-ad rendű affín illetve a 2-od rendű projektív síkból származó irányított Steiner-loopokra.

Egy loop bővítést (jobb) nukleárisnak nevezünk, ha a bővítéshez tartozó homomorfizmus magja a bővítés (jobb) nukleuszában fekszik. A (jobb) nukleáris bővítéseket a loopok elméletében az elmúlt időszakban sokan vizsgálják és különböző konstrukciókhoz használják ([11], [14], [16], [17], [18], [21]). Az előforduló nukeáris bővítések nagyrészt centrálisak, kevés olyan példa ismert, amely nem centrális nukleáris bővítést ír le.

A harmadik fejezetben tetszőleges L loop f-bővítését, $\mathfrak{Q}_f$-t, vizsgáljuk tetszőleges K kvázicsoporttal és keressük azokat a feltételeket, amelyek esetén a bővítésünk nukleáris lesz. A kapott eredmények segítségével olyan konstrukiót adunk meg, amely előre megadott jobb nukleuszú jobb inverz tulajdonságú kvázicsoportot eredményez. Mindezek megtalálhatóak [23]-ben.

A kapott $\mathfrak{Q}_f$ f bővítés egy kvázicsoport, amelynek ha az $N_r(\mathfrak{Q}_f)$ jobb nukleusza nem üres, akkor ennek a részcsoportnak az egységeleme a $\mathfrak{Q}_f$ kvávicsoport jobb egységeleme.

Bevezetjük a következő jelöléseket $\xi^\lambda = \epsilon/\xi$ és $\xi^\rho = \xi\backslash\epsilon$ bármely $\xi \in L$. Az $\alpha' \in L$ elem az $\alpha \in L$ cross inverze, azaz $\alpha\xi \cdot \alpha' = \xi$ minden $\xi \in L$ esetén, így tetszőleges $\alpha \in Z(Q)$ elemre az α^ρ a cross inverz eleme α-nak.

Lemma. *A* $\mathfrak{Q}_f = (K \times L, \circ)$ *bővítés jobb egységeleme* $(e_r, f(e_r, e_r)^\rho)$ *alakú. Az* $f : K \times K \to L$ *függvényre:*

(a) $f(x, e_r) = f(e_r, e_r)$ *teljesül minden* $x \in K$*-ra,*

(b) $f(e_r, e_r)^\rho$ *az* $f(e_r, e_r)$ *cross inverze.*

A következőkben olyan $\mathfrak{Q}_f$ f-bővítéseket tekintünk, amelyekre az előző lemma (a) és (b) feltétele teljesül.

Állítás. $(n, \nu) \in \mathfrak{Q}_f$ *a* $\mathfrak{Q}_f$ *jobb nukleuszának* $N_r(\mathfrak{Q}_f)$*-nek egy eleme, akkor és csak akkor, ha a következő feltételek teljesülnek:*

(i) $n \in N_r(K)$ *és* $f(e_r, n) = f(e_r, e_r)$,

(ii) $f(y, n)\nu$ *az* $(L, \cdot)$ *loop jobb nukleuszában van minden* $y \in K$ *esetén és* $\rho_{f(y,n)\nu} = \lambda_{f(y,n)} \circ \rho_\nu$ *bármely* $y \in K$*-ra,*

(iii) $\lambda_{f(x,y)} \circ \lambda_{f(xy,n)} = \lambda_{f(x,yn)} \circ \lambda_{f(y,n)}$ *minden* $x, y \in K$,

(iv) *bármely* $(n, \nu) \in N_r(\mathfrak{Q}_f)$ *esetén* $(e_r, \nu) \in N_r(\mathfrak{Q}_f)$.

Következmény. *Minden* (e_r, ν) *elem akkor és csak akkor fekszik* $\varphi_f^{-1}(e_r) \cap N_r(\mathfrak{Q}_f)$*-ben, ha* $f(e_r, e_r)\nu \in N_r(L)$ *és* $\rho_{f(e_r,e_r)\nu} = \lambda_{f(e_r,e_r)} \circ \rho_\nu$.

Következmény. *Tegyük fel, hogy* $f(e_r, e_r) \in N_r(L)$. *Egy* (n, ν) *elem pontosan akkor eleme az* $N_r(\mathfrak{Q}_f)$ *jobb nukleusznak, ha a következő feltételek teljesülnek:*

(i) $n \in N_r(K)$ *és* $f(e_r, n) = f(e_r, e_r)$,

(ii) $\nu \in N_r(L)$ *és* $f(e_r, e_r) \in C(L)$,

(iii) $f(y, n) \in N_r(L) \cap C(L)$ *bármely* $y \in K$*-ra,*

(iv) $\lambda_{f(x,y)} \circ \lambda_{f(xy,n)} = \lambda_{f(x,yn)} \circ \lambda_{f(y,n)}$ *minden* $x, y \in K$ *esetén.*

Speciálisan, $(e_r, \nu) \in N_r(\mathfrak{Q}_f)$ *akkor és csak akkor, ha (ii) teljesül.*

A jobb nukleusz struktúrájára, nukleáris tulajdonságaira vonatkozóan szükségünk van a részbővítés fogalmára.

Legyen $(H, \cdot)$ részkvázicsoportja $(K, \cdot)$-nak és legyen $(M, \cdot)$ részloopja $(L, \cdot)$-nek. Ha az $f : K \times K \to L$ függvényre teljesül, hogy $f(h_1, h_2) \in M$ minden $h_1, h_2 \in H$-ra és $\tilde{f} : H \times H \to M$ az $f : K \times K \to L$ leszűkítése $H \times H$-ra, akkor az $\tilde{f}$-bővítését $\tilde{\mathfrak{Q}}_{\tilde{f}} = (H \times M, \circ)$ $(M, \cdot)$-nek $(H, \cdot)$-val $\mathfrak{Q}_f = (K \times L, \circ)$ f-részbővítésének nevezzük.

Tétel. *Legyen* $\tilde{\mathfrak{Q}}_{\tilde{f}} = (H \times M, \circ) \subset \mathfrak{Q}_f$ *egy* f*-részbővítése az* $M \subset L$ *részloopnak a* $H \subset K$ *részkvázicsoporttal.* $\mathfrak{Q}_f = (K \times L, \circ)$*-nek a* $\tilde{\mathfrak{Q}}_{\tilde{f}} = (H \times M, \circ) \subset \mathfrak{Q}_f$ *részkvázicsoportja jobb nukleáris akkor és csak akkor, ha*

(i) $H \subset N_r(K)$ *és* $f(e_r, h) = f(e_r, e_r)$ *minden* $h \in H$ *esetén,*

(ii) $M \subset N_r(L)$ *és* $f(y, h) \in N_r(L) \cap C(L)$ *minden* $y \in K$, $h \in H$,

(iii) $\lambda_{f(x,y)} \circ \lambda_{f(xy,h)} = \lambda_{f(x,yh)} \circ \lambda_{f(y,h)}$ *bármely* $x, y \in K$ *és* $h \in H$.

Mivel $e_r \in K$ idempotens, az $\mathfrak{N} = \{(e_r, \nu);\ \nu \in N_r(L)\}$ részkvázicsoportja $\mathfrak{Q}_f$-nek. Az előző tételt alkalmazva a $H = \{e_r\} \subset K$ triviális részcsoportra és az $M = N_r(L)$ jobb nukleuszra kapjuk, hogy

Következmény. *Az* $\mathfrak{N}$ *részcsoport jobb nukleáris* $\mathfrak{Q}_f$*-ban akkor és csak akkor, ha* $f(e_r, e_r) \in N_r(L) \cap C(L)$.

Legyen $(H, \cdot)$részkvázicsoportja $(K, \cdot)$-nak és tekintsük az $(L, \cdot)$ loop f-részbővítését $\tilde{\mathfrak{Q}}_{\tilde{f}} = (\varphi_f^{-1}(H), \circ) = (H \times L, \circ) \subset \mathfrak{Q}_f$-t a $(H, \cdot)$ részkvázicsoporttal.

Következmény. *A* $\varphi_f^{-1}(H)$ *részkvázicsoport pontosan akkor jobb nukleáris* $\mathfrak{Q}_f$*-ban, ha*

(i) $H \subset N_r(K)$ *és* $f(e_r, h) = f(e_r, e_r)$ *minden* $h \in H$,

(ii) $(L, \cdot)$ *csoport és* $f(y, h)$ *a* $Z(L)$ *centrumban fekszik bármely* $y \in K,\ h \in H$ *esetén,*

(iii) $f(x, y)f(xy, h) = f(x, yh)f(y, h)$ *minden* $x, y \in K$ *és* $h \in H$.

Ez az állítás a $H = \{e_r\}$ triviális részkvázicsoportra a következőt adja

Tétel. *Egy* $\mathfrak{Q}_f$ f*-bővítés pontosan akkor jobb nukleáris, ha* $(L, \cdot)$ *csoport és* $f(e_r, e_r)$ *a* $Z(L)$ *centrumban fekszik.*

Ha $\mathfrak{Q}_f$-nek a nukleusza $N(\mathfrak{Q}_f)$ nemüres, akkor $\mathfrak{Q}_f$-nak létezik egységeleme és így egy loop. Következésképpen a homomorf képe $(K, \cdot)$ szintén loop.

Következmény. *Az* $(L,\cdot)$ *loop* $\mathfrak{Q}_f$ f*-bővítése a* $(K,\cdot)$ *kvázicsoporttal nukleáris, akkor és csak akkor, ha*

(A) $(K,\cdot)$ *loop (*$e\in K$ *egységelemmel) és* $(L,\cdot)$ *csoport,*

(B) $f(e,x)=f(x,e)=f(e,e)$ *bármely* $x\in K$ *és ez az elem a* $Z(L)$ *centrumban fekszik.*

Megjegyzés. *Egy nukleáris* $\mathfrak{Q}_f$ f*-bővítés pontosan akkor centrális, ha az* $(L,\cdot)$ *csoport kommutatív.*

Legyen $\mathfrak{Q}_f$ jobb nukleáris f-bővítés, azaz $(L,\cdot)$ egy csoport és $f(e_r,e_r)\in Z(L)$. Legyen $\mathfrak{Q}_f^*$ az az f^*-bővítés, amelyet az $f^*(x,y)=f(x,y)f(e_r,e_r)^{-1}$ függvény és a

$$(x,\xi)*(y,\eta)=(xy,f(x,y)f(e_r,e_r)^{-1}\xi\eta)$$

művelet definiál. A $\mathfrak{Q}_f$ és $\mathfrak{Q}_f^*$ kvázicsoportok izomorfak, a köztük lévő izomorfia $(x,\xi)\mapsto(x,f(e_r,e_r)^{-1}\xi)$. Így az $f:K\times K\to L$ függvényt normalizálhatjuk az $f(x,e_r)=f(e_r,e_r)=\epsilon$ feltétellel.

Állítás. *Egy* $(L,\cdot)$ *csoport* $\mathfrak{Q}_f$ f*-bővítése egy* e_r *jobb egységelemes* $(K,\cdot)$ *kvázicsoporttal, amelyre* $f(x,e_r)=f(e_r,e_r)=\epsilon$ *jobbinverz tulajdonságú akkor és csak akkor, ha*

(i) $(K,\cdot)$ *jobbinverz tulajdonságú,*

(ii) *létezik egy* $\mu:K\to Z(L)$ *függvény úgy, hogy minden* $x,y\in K$ *esetén* $\mu(y)=f(xy,\iota(y))f(x,y)$ *teljesül, ahol* $\rho_{\iota(x)}=\rho_x^{-1}$.

Legyen Γ a $K \times K$ halmazon ható és a $\gamma : (x,y) \mapsto (xy, \iota(y)) : K \times K \to K \times K$ bijekció által generált permutáció csoport. Mivel $(xy \cdot \iota(y))\, y = y$ teljesül minden $x, y \in K$-ra, az $\iota : K \to K$ és $\phi : K \times K \to K \times K$ bijekciók involúciók és a Γ csoport rendje 2. A ϕ leképezés fixpontjainak halmaza $\Sigma = \{(x, e_r);\ x \in K\}$ és tetszőleges $(x,y) \in (K \times K) \setminus \Sigma$ orbitja két elemből áll.
Legyen $\mathfrak{Q}_f$ f-bővítése egy $(L, \cdot)$ csoportnak egy e_r jobb egységelemes $(K, \cdot)$ kvázicsoporttal. Tegyük fel, hogy $f(x, e_r) = f(e_r, e_r) = \epsilon$ minden $x \in K$ esetén és $(K, \cdot)$ jobbinverz tulajdonságú. Ekkor, az előző állítás alapján

Következmény. *Legyen $(L, \cdot)$ csoport és $(K, \cdot)$ egy jobbinverz tulajdonságú kvázicsoport. Legyen adott egy $\mu : K \to Z(L)$ leképezés, egy $c : (K \times K)/\Gamma \to K \times K : \gamma \mapsto c(\gamma) \in \gamma$ kiválasztási függvénye Γ orbitjainak $K \times K$-ban és egy $\nu : (K \times K)/\Gamma \to L$ leképezés, amelyre $\nu(\{(x, e_r)\}) = \epsilon$ teljesül minden $x \in K$ esetén. Ekkor az alábbi feltételekel megadott $f : K \times K \to L$ függvény*

(i) *$f(c(\gamma)) = \nu(\gamma)$ minden $\gamma \in (K \times K)/\Gamma$-ra,*

(ii) *$f(\phi(x,y)) = \mu(y) f(x,y)^{-1}$ minden $x, y \in K$-ra*

egy jobb inverz tulajdonságú $\mathfrak{Q}_f$ f-bővítést eredményez.
Megfordítva, tetszőleges jobbinverz tulajdonságú jobb nukleáris f bővítés megkapható a fenti konstrukcióval.

Legyen most $(K, \cdot)$ egy involutorikus jobb Bol loop $e \in K$ egységelemmel. Ekkor

Tétel. *Legyen $\mathfrak{Q}_f$ az $(L, \cdot)$ csoport olyan f-bővítése egy $(K, \cdot)$ involutorikus jobb Bol loopppal, hogy $f(x, e) = \epsilon$ minden $x \in K$ esetén. A*

$\mathfrak{Q}_f$ f-bővítés pontosan akkor jobb alternatív, ha $\mathfrak{Q}_f$ jobbinverz tulajdonságú loop és ebben az esetben a $\mathfrak{Q}_f$ f-bővítés nukleáris.

A negyedik fejezetben ([28]) az előző kvázicsoport bővítést vizsgáljuk abban az esetben, amikor az valódi kvázicsoportot eredményez. Ilyen esetben, ahogy azt már korábban láttuk, a nukleuszok üres halmazok és nem megfelelően mérik az asszociativítást. Belouszov vezette be a reguláris permutációk fogalmát, bizonyos értelemben általánosítva a nukleusz definícióját valódi kvázicsoportokra.

Egy $\lambda : Q \to Q$ illetve $\rho : Q \to Q$ bijekció bal-reguláris permutáció illetve jobb-reguláris permutáció, ha minden $x, y \in Q$ esetén $\lambda(xy) = \lambda(x) \cdot y$ illetve $\rho(xy) = x \cdot \rho(y)$ teljesül. A jobb- illetve bal-reguláris permutációk csoportot alkotnak, amelyek a bal- illetve jobb eltolások által generált csoport részcsoportjai. Amennyiben a kvázicsoportnak van jobb egységeleme, akkor a jobb-reguláris permutációk csoportja izomorf a jobb nukleusszal, míg ha bal egységeleme van, akkor a bal-reguláris permutáció csoport izomorf a bal nukleusszal. Mind a jobb-, mind a bal-reguláris permutáció csoport indukál egy ekvivalenciarelációt a kvázicsoporton, és az ilymódon kapott osztályok az őket reprezentáló elemekhez tartozó elem-nukleuszok. Így az osztályon belüli elemek asszociálnak egymással.

Lemma. *Egy $\rho : (x, \xi) \mapsto (\rho_1(x, \xi), \rho_2(x, \xi)) : K \times Q \to K \times Q$ bijekció a $(\mathfrak{Q}_f, \circ)$ f-bővítés jobb-reguláris permutációja, akkor és csak akkor, ha*

(i) *ρ_1 konstans a bővítés ekvivalenciaosztályain és egy $\rho_1 : K \to K$ jobb-reguláris permutációt indukál K-n,*

(ii) ρ_2 *kielégíti az alábbi azonosságot*

$$\rho_2(xy, f(x,y)\cdot\xi\eta) = f(x, \rho_1(y,\eta))\cdot\xi\rho_2(y,\eta)$$

minden $x, y \in K,\ \xi, \eta \in Q$.

Azok a kvázicsoportok, amelyeknek csak triviális jobb-reguláris permutációi vannak, széles osztályt képeznek. Ilyenek például a Steiner-kvázicsoportok, vagy azok a kvázicsoportok, amelyek egy ciklikus csoport corját képezik. Ha a bővítésünkben ilyen a K kvázicsoport, akkor

Megjegyzés. *Ha a K kvázicsoport jobb-reguláris permutációinak csoportja triviális, akkor $R(\mathfrak{Q}_f)$ orbitjait tartalmazzák a bővítés kongruencia osztályai.*

Itt a nukleáris bővítés feltételével ellentétes tartalmazást kapunk az elem-nukleuszokra.

Tétel. *Tegyük fel, hogy $R(K)$ triviális és Q loop. Egy $\rho : \mathfrak{Q}_f \to \mathfrak{Q}_f$ leképezés pontosan akkor jobb-reguláris permutációja $\mathfrak{Q}_f$-nek, ha $\rho = (id, \rho_\nu)$ alakú, ahol $\nu \in N_r(Q)$.*

Következmény. $R(\mathfrak{Q}_f) \cong N_r(Q)$.

Abban az esetben, mikor a Q kvázicsoport csoport, akkor $\mathrm{R}(\mathfrak{Q}_f)$ izomorf Q-val és ilyenkor az elemnukleuszok egybeesnek a bővítés kongruencia osztályaival, így nukleáris bővítést kapunk ebben az általánosított értelemben.

Következmény. *Ha az $N_r(Q)$ jobb nukleusz normális részcsoport, akkor a $R(\mathfrak{Q}_f)$ csoport által indukált ekvivalencia reláció normális kongruencia.*

Következmény. *Ha K idempotens kvázicsoport és $N_r(Q)$ normális részcsoport Q-ban, akkor $R(\mathfrak{Q}_f)$ orbitjai normális részkvázicsoportjai a bővítés magkvázicsoportjainak.*

Hasonló eredmények érvényesek a bal-reguláris permutációkra, illetve az azok által alkotott csoportra.

Tétel. *Tegyük fel, hogy $\Lambda(K)$ triviális és Q loop. Egy $\lambda : K \times Q \to K \times Q$ leképezés pontosan akkor bal-reguláris permutációja $\mathfrak{Q}_f$-nek, ha $(x, \xi) \mapsto (x, \nu(x)\xi)$ alakú, ahol a $\nu : K \to N_l(Q)$ leképezés teljesíti a*

$$\nu(xy)f(x,y) = f(x,y)\nu(x) \quad \text{\textit{és}} \quad \lambda_{f(x,y)\nu(x)} = \lambda_{f(x,y)}\lambda_{\nu(x)}$$

feltételeket bármely $x, y \in K$.

Jelölje F azt a részloopját Q-nak, amelynek az elemei kommutálnak minden $f(x,y)$; $x, y \in K$-el.

Következmény. *A $\lambda = (id, \lambda_\nu)$ leképezés bal-reguláris permutációja a $(\mathfrak{Q}_f, \circ)$ f-bővítésnek, ha $\nu \in N_l(Q) \cap F$ és $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ $\forall x, y \in K$.*

Tétel. *Tegyük fel, hogy $\Lambda(K)$ triviális, Q loop és létezik olyan $k \in K$, hogy $f(k,y) = \kappa$ konstans minden $y \in K$-ra, $\kappa \in F$-ra. Egy $\lambda : K \times Q \to K \times Q$ leképezés pontosan akkor bal-reguláris permutációja $\mathfrak{Q}_f$-nak, ha $\lambda = (id, \lambda_\nu)$, ahol $\nu \in N_l(Q) \cap F$ és $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ minden $x, y \in K$.*

Tétel. *Tegyük fel, hogy $\Lambda(K)$ triviális és Q egy cross inverz tulajdonságú loop. Egy $\lambda : K \times Q \to K \times Q$ leképezés, akkor és csak akkor bal-reguláris permutációja $\mathfrak{Q}_f$-nak, ha $\lambda = (id, \lambda_\nu)$, ahol $\nu \in N_l(Q)$.*

Ezen eredmények arra adnak lehetőséget, hogy előre megadott reguláris permutáció csoportú valódi kvázicsoportot állítsuk elő, így előre tudhatjuk, hogy mely részhalmazok elemei asszociálnak egymással.

Chapter 6

Summary

In the sense of the well-known Schreier extension, beginning with two given groups a new one is constructed which contains the first group as a normal subgroup such that the quotient group with respect to this normal subgroup is isomorphic to the second one. This construction is carried out by defining an appropriate operation on the cartesian product of the given structures and described by the following exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \mathfrak{G} \longrightarrow G \longrightarrow 1.$$

Let G and Γ be two groups, the group $\mathfrak{G}$ is an extension of Γ by G if on the set $G \times \Gamma$ is defined a multiplication by

$$(a, \alpha)(b, \beta) = (ab, f(a, b) \cdot \alpha^{T(b)}\beta),$$

where $T : G \longrightarrow Aut(\Gamma)$ and $f : G \times G \longrightarrow \Gamma$ is the factor system with

1. $f(1, a) = f(a, 1) = 1;$
2. $T(a)^{-1}T(b)^{-1}f(a, b)T(ab)f(a, b)^{-1} = 1;$

3. $f(b,c)f(a,bc) = f(a,b)^{T(c)} f(ab,c)$;

for all $a, b, c \in G$.

The simplest case occurs when T and f are both trivial maps, in this case $\mathfrak{G}$ is the direct product of Γ and G. If the map f is trivial and T is non-trivial, then $\mathfrak{G}$ is a semidirect product of Γ and G. If T is trivial and f is non-trivial, then we have for the multiplication

$$(a,\alpha)(b,\beta) = (ab, f(a,b) \cdot \alpha \cdot \beta),$$

which is called f-extension.

Similar construction can be given for loops and quasigroups with appropriate modification, but the lack of associativity changes the situation so drastically, that a general extension theory of these structures does not exist. Nowadays, many authors apply extensions of loops and quasigroups in different interpretations for example [3], [11], [14], [16], [17], [18]. Loop extension of groups by loops is studied by Péter T. Nagy and Karl Strambach in [22] very systematically. The goal of this dissertation is to show some further investigations of f-extensions of loops and quasigroups in the sense of the above mentioned authors, described by exact sequences

$$1 \longrightarrow \Lambda \longrightarrow \mathfrak{L} \longrightarrow L \longrightarrow 1,$$

respectively

$$Q \longrightarrow \mathfrak{Q} \longrightarrow K,$$

where Λ, L are loops and K, Q are quasigroups.

In the first chapter are collected the basic facts and notions which we need in this dissertation.

In the second chapter are investigated central extensions of groups by loops which have combinatorial background. In the first section we consider these combinatorial structures, give them an algebraic face and determine their groups of translations. Since for a loop L the knowledge of the group G generated by the set $\mathfrak{L}$ of its left translations is essential. Namely, if H is the stabilizer of the identity of L in G, then L is isomorphic to the loop defined on $\mathfrak{L}$ by $(x,y) \mapsto x * y = \pi(x,y) : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$, where π assigns to the element $xy \in G$ the left translation of L contained in xyH. (cf. [21] Section 1.2). Results of the first section can be fined in [26].

For turning a Steiner triple system $\mathfrak{S}$ into a Steiner quasigroup $(Q(\mathfrak{S}), *)$, we define as $a * b$ the third point of the line determined by a, b and put $a*a = a^2 = a$ for all $a \in \mathfrak{S}$. Moreover, with a Steiner triple system is associated a Steiner loop $(S(\mathfrak{S}), \circ)$ such that the elements of $S(\mathfrak{S}) \setminus \{e\}$, where e is the identity of $S(\mathfrak{S})$, are the points of the Steiner system $\mathfrak{S}$ and $a \circ b := a*b$ for $a \neq b$, $a, b \in S(\mathfrak{S}) \setminus \{e\}$, whereas $a \circ a = a^2 = e$ [25], Chap.V.

Very often the identities which hold in a Steiner quasigroup associated with a Steiner triple system $\mathfrak{S}$ do not hold in the Steiner loop $S(\mathfrak{S})$ associated with $\mathfrak{S}$. Examples are the Steiner triple systems, called *Hall systems*, in which every three non-collinear points generate an affine plane of order 3. In the associated Steiner quasigroup $Q(\mathfrak{S})$ one has $(xy)z = (xz)(yz)$ for all $x, y, z \in Q(\mathfrak{S})$ but this identity fails to hold in the associated Steiner loop $S(\mathfrak{S})$. Another convincing example for this is the group Φ generated by the translations of $Q(\mathfrak{S})$ and the group G generated by the translations of $S(\mathfrak{S})$, if $\mathfrak{S}$ is a finite Hall triple system of size n. Namely, in this case the group Φ is solvable (see [12] and [3] p. 86), but G does not.

The automorphism group of a Steiner loop coincides with the automorphism group of the corresponding Steiner triple system. In [7], Corollary 1, p. 251 it is proved that a Steiner loop $S(\mathfrak{S})$ is an elementary abelian group of order $n+1 = 2^m$ if and only if the Steiner triple system $\mathfrak{S}$ corresponding to $S(\mathfrak{S})$ is isomorphic to the projective space of dimension $m-1$ over the field $GF(2)$. Hence the group of translations of a Steiner loop corresponding to a projective space is an elementary abelian 2-group.

For Steiner loops corresponding to Steiner triple systems which are not projective spaces the situation changes drastically.

Theorem. *Let* $S(\mathfrak{S})$ *be a proper Steiner loop of order* n *and let* $Q(\mathfrak{S})$ *be a Steiner quasigroup both corresponding to the Steiner triple system* $\mathfrak{S}$*. If the product* ${\lambda^*}_a {\lambda^*}_b$ *of any two distinct translations of* $Q(\mathfrak{S})$ *has odd order, then the group* G *generated by the translations of* $S(\mathfrak{S})$ *is the alternating group* A_n *or the symmetric group* S_n *depending whether* n *is divisible by* 4 *or not.*

Remark. *Any Steiner quasigroup* $Q(\mathfrak{S})$ *corresponding to a Hall system satisfies the conditions of Theorem.*

There are also Steiner triple systems which are not Hall systems but for which the products of any two distinct translations of the associated Steiner quasigroup have odd order. We illustrate this for Steiner triple systems constructed in [9], 2.1. p. 291.
Let C be a cyclic group of order $k = \frac{1}{3}(4^n - 1)$ such that $n > 1$ is odd. Let $\mathfrak{S}$ be the disjoined union $\mathfrak{S} = C_0 \cup C_1 \cup C_2$ such that C_0, C_1, C_2 are three exemplars of C. If the element $x \in C$ is contained in the exemplar C_i then we denote this element with x_i. The blocks of a Steiner triple system $\mathfrak{S}_C$ on the point set $\mathfrak{S}$ of size $4^n - 1$ are the following triples:

(i) all subsets $\{x_0, x_1, x_2\}$, with $x_0 = x_1 = x_2 = x \in C$,

(ii) all subsets $\{x_0, y_0, z_1\}, \{x_1, y_1, z_2\}, \{x_2, y_2, z_0\}$ of $\mathfrak{S}$ with $x, y, z \in C$ such that $x \neq y$ and $xy = z^2$.

Remark. *Let* C *be a cyclic group of order* $\frac{4^n-1}{3}$ *with odd* $n > 1$ *and let* $Q(\mathfrak{S}_C)$ *be the Steiner quasigroup corresponding to the Steiner triple* *system constructed from C as above. Then the translation group of the Steiner loop $S(\mathfrak{S}_C)$ of order 4^n corresponding to $Q(\mathfrak{S}_C)$ is the alternating group A_{4^n}.*

In the second section of this chapter we consider loop extensions

$$(*) \qquad 1 \longrightarrow A \longrightarrow L \longrightarrow S \longrightarrow 1,$$

where A is the group of order 2 and S is a Steiner loop.

The loop L is realized on the set $S \times A = \{(a,\epsilon) : a \in S, \epsilon \in A\}$ such that the multiplication is given by

$$(a_1,\epsilon_1) \circ (a_2,\epsilon_2) = (a_1 a_2, \epsilon_1\epsilon_2 f(a_1,a_2)),$$

where f is a function $f : S \times S \longrightarrow A$ with $f(a,e) = f(e,a) = 1$ for all $a \in S$ and the unit element e of S. The identity of L is the element $(e,1)$ and the left and right inverse of any element coincide.

Even the simplest case of non-associative extensions of the group of order 2 by an abelian 2-group yields a rich variety of loops which are distinguish among them by different weak associativity conditions. To obtain more homogeneous classes of extensions it is necessary to restrict the possibilities for the factor systems f. The best known examples in this direction are the code loops (see [13], [15]) and C-loops (cf. [17]). We deal here with extensions of type $(*)$ in which $f(x,y) = -f(y,x)$ holds for all different $x, y \in S \setminus \{e\}$ and all elements of $L \setminus \{(e,1)\}$ have either order 2 or 4, i. e., $f(x,x) = 1$, respectively $f(x,x) = -1$ for all $x \in S \setminus \{e\}$. Results obtained in this section are in [27].

Definition. *An oriented Steiner triple system $(\mathfrak{S},T)$ is a Steiner triple system $\mathfrak{S}$ such that on the set T of blocks for each block is given a cyclic order.*

Definition. *An oriented Steiner loop L is an extension $(*)$ for which there exists an oriented Steiner triple system $(\mathfrak{S}, T)$ such that the elements different from the identity of the Steiner loop S are the points of $\mathfrak{S}$ and the restriction of the factor system of L to $(\mathfrak{S} \times \mathfrak{S}) \setminus \{(x,x), x \in \mathfrak{S}\}$ coincides with the orientation function of $(\mathfrak{S}, T)$ and $f(x,x) = -1$, respectively $f(x,x) = 1$ for all $x \in S \setminus \{e\}$ holds.*

The center of an oriented Steiner loop L has order 2 if the exponent of L is 4 and it is trivial if the exponent of L is 2. The oriented Steiner triple system $(\mathfrak{S}, T)$ of an oriented Steiner loop L is uniquely determined by L and conversely, an oriented Steiner triple system $(\mathfrak{S}, T)$ determines a unique oriented Steiner loop L of exponent $d \in \{2, 4\}$. .

Theorem. *Any oriented Steiner loop L is flexible.*

The loop L satisfies the inverse property if and only if every element of L not contained in the center has order 4.

The loop L has the cross inverse and the automorphic inverse properties but satisfies neither the left nor the right inverse property if and only if L has exponent 2.

Theorem. *An oriented Steiner loop L is a group if and only if it is the quaternion group of order* 8.

Let $\hat{G}_l$, $\hat{G}_r$ and $\hat{G}$ be the group generated by left, right and all translations of the Steiner loop L. To describe these groups we need the following mapping, let $\tau_a : L \longrightarrow L$ $(a \in S \setminus \{e\})$, $\tau_a : (y, \epsilon) \mapsto (y, \epsilon)$, $y \in \{e, a\}$, $\epsilon \in \{1, -1\}$, $\tau_a : (x, \epsilon) \mapsto (x, -\epsilon)$, $a \neq x \in S \setminus \{a, e\}$, $\epsilon \in \{1, -1\}$. Then we have

Theorem. *If the oriented Steiner loop L corresponds to a Steiner loop S of order n, then the normal subgroup $\Theta =< \tau_a, a \in S >$ of $\hat{G}$ is the elementary abelian 2-group of order 2^{n-1} or 2^{n-2} depending on whether n is odd or even and the factor group $\hat{G}/\Theta$ is isomorphic to $\hat{G}_l$ as well as to $\hat{G}_r$.*

Corollary. *The groups $\hat{G}_l$ and $\hat{G}_r$ are isomorphic.*

We obtain for the group of automorphisms of an oriented Steiner triple system

Proposition. *The automorphism group Γ of an oriented Steiner triple system has odd order.*

Proposition. *Let U be a group of automorphisms of a Steiner triple system $\mathfrak{S}$ such that U has odd order. Then there is an oriented Steiner triple system having U as a group of automorphisms.*

Remark. *Let $(\mathfrak{S}, T)$ be a finite oriented Steiner triple system having K as a group of automorphisms and let K_0 be a proper subgroup of K such that a block orbit B of K splits into different orbits $B_1, .., B_m$ of K_0. Then there exists an oriented Steiner triple system $(\mathfrak{S}, T_0)$ such that K_0 consists of automorphisms, but K is not a group of automorphisms of $(\mathfrak{S}, T_0)$.*

Now we can determine the groups of automorphisms of oriented Steiner loops

Theorem. *The automorphism group Γ of an oriented Steiner loop L is a semidirect product of a normal elementary abelian 2-group Δ*

inducing on the factor loop L/A *the identity and corresponding to the set of homomorphisms from* S *into* $\mathbb{Z}_2$ *by the automorphism group* Σ *of the oriented Steiner loop.*

Definition. *Let* $\mathfrak{S}$ *be a Steiner triple system. We call a proper subsystem* $\mathfrak{K}$ *a blockade subsystem if each block of* $\mathfrak{S}$ *not contained in* $\mathfrak{K}$ *meets* $\mathfrak{K}$ *in a point.*

Theorem. *There is a bijection between the non-trivial automorphisms of an oriented Steiner loop* L*, which induce on the corresponding Steiner triple system* $\mathfrak{S}$ *of* S *the identity, and the blockade subsystems of* $\mathfrak{S}$*.*

Proposition. *Let* γ *be an epimorphism from a Steiner loop* S *onto* $\mathbb{Z}_2$*. Then* S *contains a normal Steiner subloop* K *such that* S *is the union* $S = K \cup Ka$*, where* a *is an element of* S *not contained in* K*.*
If $\mathfrak{S}$ *and* $\mathfrak{K}$ *are Steiner triple systems corresponding to* S*, respectively to* K*, then* $\mathfrak{K}$ *is a blockade subsystem of* $\mathfrak{S}$*.*

Theorem. *Let* Γ *be the automorphism group the oriented Steiner loop* L*. Corresponding to an oriented Steiner triple system* $(\mathfrak{S}, T)$*.*
If $\mathfrak{S}$ *is an affine space over the field* $GF(3)$*, then the group* Δ *consists only of the identity and* Γ *is isomorphic to the automorphism group* Σ *of* $(\mathfrak{S}, T)$*.*
If $\mathfrak{S}$ *is a projective space of dimension* n *over the field* $GF(2)$*, then* Δ *is the elementary abelian group of order* 2^n*.*

Corollary. *The automorphism group of an oriented Steiner loop of order* 8 *is isomorphic to the alternating group* A_4*.*

Let S be a Steiner loop corresponding to the Steiner triple system $\mathfrak{S}$. A *valuation* v of S is a mapping $S \longrightarrow \{1, -1\}$ such that $v(e) = 1$ for the identity e of S.

Let L_1 and L_2 be oriented Steiner loops corresponding to the same Steiner triple system $\mathfrak{S}$ and having as factor systems f_1 or f_2, respectively such that $f_1(a, a) = f_2(a, a)$ for all a contained in the Steiner loop S which corresponds to $\mathfrak{S}$.

We suppose that L_1 and L_2 correspond to oriented Steiner triple systems $(\mathfrak{S}, T_1)$, respectively $(\mathfrak{S}, T_2)$ over $\mathfrak{S}$ such that the automorphism groups of $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$ coincide. If Γ is this automorphism group, then any element $\tilde{\gamma}$ of Γ induces on the Steiner loop S an automorphism γ such that $f_1(a, b) = f_1(a^\gamma, b^\gamma)$ and $f_2(a, b) = f_2(a^\gamma, b^\gamma)$.

A valuation v is *compatible* with the pair (f_1, f_2) if $v(a)v(b)f_2(a, b) = v(ab)f_1(a, b)$ for all $a, b \in S$. The valuations v compatible with the pair (f_1, f_2) correspond in a unique way with isomorphisms from L_1 onto L_2 which induce on S the identity.

Lemma. *The loops* L_{f_i} $(i = 1, 2)$ *such that for the factor systems* f_i $(i = 1, 2)$ *one has* $f_1(x, y) = -f_2(x, y)$ *for distinct* $x, y \in S \setminus \{e\}$ *and* $f_1(x, x) = f_2(x, x)$, $x \in S$ *are isomorphic. An isomorphism* ϕ *is given by the mapping* $(x, \epsilon) \mapsto (x, -\epsilon)$*.*

Theorem. *Let* L_1 *and* L_2 *be oriented Steiner loops of the same exponent with factor systems* f_1 *and* f_2 *respectively, such that the factor loops* L_1/A *and* L_2/A *correspond to the same Steiner triple system* $\mathfrak{S}$*. If the automorphism groups of the oriented Steiner triple systems*

$(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$ corresponding to L_1, respectively L_2 coincide, then L_1 and L_2 are isomorphic if and only if there exists a valuation v of the Steiner loop S belonging to $\mathfrak{S}$ compatible with (f_1, f_2).

Theorem. *Let L_1 be an oriented Steiner loop having K as the automorphism group. If α is an automorphism of the Steiner loop S corresponding to L_1, then there exists an oriented Steiner loop L_2 isomorphic to L_1 and having the conjugate group K^α as its automorphism group.*

Proposition. *Oriented Steiner loops L_1 and L_2 of the same exponent associated with the Steiner triple system $\mathfrak{S}$ and having block transitive automorphism groups are isomorphic.*

We call an oriented Steiner loop L *projective* if the associated Steiner triple system $\mathfrak{S}$ is isomorphic to the point-line design of a projective geometry over $GF(2)$. If L has order 2^{n+1}, then the automorphism group of L is a subgroup U of odd order in $SL(n, 2)$. Therefore U is a solvable subgroup of $SL(n, 2)$.

If U is a solvable group of automorphisms of the vector space V of an even dimension n over $GF(2)$ such that U has odd order and acts transitively on the vectors different from zero, then V may be identified with the additive group of the field $GF(2^n)$ and U is as a permutation group isomorphic to the sharply transitive group $\Sigma := \{x \mapsto ax; 0 \neq a \in GF(2^n)\}$ since the automorphism group of $GF(2^n)$ is a cyclic group of even order (cf. [24], Theorem 19.9, p. 246.).

Proposition. *Let L be an oriented projective Steiner loop of order 16 such that its automorphism group Γ acts transitively on the lines of the corresponding oriented projective plane $\mathfrak{S}$ of order 2. Then the group*

Γ *is the semidirect product of the normal subgroup* Δ *isomorphic to* ${\mathbb{Z}_2}^3$ *by the group of order 21 acting on* Δ *faithfully.*

Corollary. *Any oriented projective Steiner loop* L *of order* 16 *and of exponent* 4 *such that the order of its automorphism group* Γ *is divisible by* 7 *is the Moufang loop of order* 16*.*

Theorem. *Any oriented projective Steiner loop* L *of order* ≥ 32 *and exponent* 4 *is not Moufang.*

Theorem. *Any oriented projective Steiner loop* L *of order* ≥ 32 *and exponent* 4 *is not Moufang.*

Let L_1 and L_2 be oriented Steiner loops arising from the oriented Steiner triple systems $(\mathfrak{S}, T_1)$, respectively $(\mathfrak{S}, T_2)$ with the orientation functions f_1, respectively f_2 such that the automorphism groups of L_i $(i = 1, 2)$ are conjugate subgroups in the automorphism group of $\mathfrak{S}$. Then we may assume that the automorphism groups of L_i $(i = 1, 2)$ induce on $\mathfrak{S}$ the same group Γ of automorphisms. The decision when L_1 and L_2 are isomorphic requires a detailed discussion on the relations among the restrictions of the orientation functions of $(\mathfrak{S}, T_1)$ and $(\mathfrak{S}, T_2)$ to the orbits of the group Γ. In the last subsection is illustrated this situation for oriented Steiner loops with 16 respectively 20 elements seeking a valuation v of $\mathfrak{S}$ such that $v(a)v(b)f_1(a, b) = v(ab)f_2(a, b)$ for all $a, b \in \mathfrak{S}$.

In the third chapter we deal with nuclear extensions of quasigroups our results are in [23]. A loop extension is usually called (right) nuclear, if the kernel of the corresponding homomorphism is contained in the (right) nucleus of the extension. (Right) nuclear extensions are very natural generalizations of central extensions, they have been investigated by many authors and used for different constructions in loop theory (e. g. [11], [14], [16], [17], [18], [21]). Most of the examples treated in these papers are central extensions, but only a few examples are known for non-central (right) nuclear extensions.

The aim of the third chapter is the systematic study of right nuclei of quasigroups obtained by an extension process in the category of quasigroups with right unit. The investigated extensions of quasigroups are defined by a slight modification of non-associative Schreier-type extensions of groups or loops (c.f. [18], [22], [26]). These extensions will be determined by a triple (L, K, f), where L is a loop, K is a quasigroup with right unit and $f : K \times K \to L$ is a function. The main results give characterizations of quasigroup extensions satisfying particular nuclear conditions. We apply the results to the description of constructions of quasigroups with right inverse property, having a prescribed right nucleus.

Now, let $\mathfrak{Q}_f = (K \times L, \circ)$ be an f-extension of a loop $(L, \cdot)$ by a quasigroup $(K, \cdot)$ and assume that $\mathfrak{Q}_f$ has non-trivial right nucleus. The homomorphic image of the right unit of $\mathfrak{Q}_f$ in $(K, \cdot)$ is a right unit of $(K, \cdot)$, which will be denoted by $e_r \in K$. Let ϵ denote the unit of $(L, \cdot)$. Let us denote $\xi^\lambda = \epsilon/\xi$ and $\xi^\rho = \xi\backslash\epsilon$ for any $\xi \in L$. The element $\alpha' \in L$ is called the *cross inverse* of $\alpha \in L$ if $\alpha\xi \cdot \alpha' = \xi$ for any $\xi \in L$. In this case necessarily $\alpha' = \alpha^\rho$ holds. Clearly, for any $\alpha \in Z(Q)$ the element α^ρ is the cross inverse of α.

Lemma. *The right unit of $\mathfrak{Q}_f$ has the shape $(e_r, f(e_r, e_r)^\rho)$. The function $f : K \times K \to L$ satisfies*

(a) *$f(x, e_r) = f(e_r, e_r)$ for any $x \in K$,*

(b) *$f(e_r, e_r)^\rho$ is the cross inverse of $f(e_r, e_r)$.*

In the following we will consider f-extensions $\mathfrak{Q}_f$ satisfying conditions (a) and (b) of the previous Lemma.

Proposition. *An element* $(n,\nu) \in \mathfrak{Q}_f$ *belongs to the right nucleus* $N_r(\mathfrak{Q}_f)$ *of* $\mathfrak{Q}_f$ *if and only if the following conditions hold:*

(i) $n \in N_r(K)$ *and* $f(e_r,n) = f(e_r,e_r)$,

(ii) $f(y,n)\nu$ *belongs to the right nucleus of the loop* $(L,\cdot)$ *for any* $y \in K$ *and* $\rho_{f(y,n)\nu} = \lambda_{f(y,n)} \circ \rho_\nu$ *holds for any* $y \in K$,

(iii) *the equality* $\lambda_{f(x,y)} \circ \lambda_{f(xy,n)} = \lambda_{f(x,yn)} \circ \lambda_{f(y,n)}$ *holds any* $x, y \in K$,

(iv) *for any* $(n,\nu) \in N_r(\mathfrak{Q}_f)$ *the element* (e_r,ν) *belongs to* $N_r(\mathfrak{Q}_f)$, *too.*

Putting $n = e_r$ in the previous Proposition we get

Corollary. *An element* (e_r,ν) *is contained in* $\varphi_f^{-1}(e_r) \cap N_r(\mathfrak{Q}_f)$ *if and only if* $f(e_r,e_r)\nu \in N_r(L)$ *and* $\rho_{f(e_r,e_r)\nu} = \lambda_{f(e_r,e_r)} \circ \rho_\nu$ *hold.*

Corollary. *Assume that* $f(e_r,e_r) \in N_r(L)$. *An element* (n,ν) *belongs to the right nucleus* $N_r(\mathfrak{Q}_f)$ *if and only if the following conditions hold:*

(i) $n \in N_r(K)$ *and* $f(e_r,n) = f(e_r,e_r)$,

(ii) $\nu \in N_r(L)$ *and* $f(e_r,e_r) \in C(L)$,

(iii) $f(y,n) \in N_r(L) \cap C(L)$ *for any* $y \in K$,

(iv) $\lambda_{f(x,y)} \circ \lambda_{f(xy,n)} = \lambda_{f(x,yn)} \circ \lambda_{f(y,n)}$ *for any* $x, y \in K$.

Particularly, $(e_r,\nu) \in N_r(\mathfrak{Q}_f)$ *if and only if (ii) holds.*

Now we study f-extensions with different nuclear properties. Let $\mathfrak{Q}_f = (K \times L, \circ)$ be an f-extension of a loop $(L, \cdot)$ by a quasigroup $(K, \cdot)$ and let $\tilde{\mathfrak{Q}}_{\tilde{f}} = (H \times M, \circ) \subset \mathfrak{Q}_f$ be an f-subextension of the subloop $M \subset L$ by the subquasigroup $H \subset K$.

Theorem. *Let $\tilde{\mathfrak{Q}}_{\tilde{f}} = (H \times M, \circ) \subset \mathfrak{Q}_f$ be an f-subextension of the subloop $M \subset L$ by the subquasigroup $H \subset K$. The subquasigroup $\tilde{\mathfrak{Q}}_{\tilde{f}} = (H \times M, \circ) \subset \mathfrak{Q}_f$ of $\mathfrak{Q}_f = (K \times L, \circ)$ is right nuclear if and only if*

(i) *$H \subset N_r(K)$ and $f(e_r, h) = f(e_r, e_r)$ for all $h \in H$,*

(ii) *$M \subset N_r(L)$ and $f(y, h) \in N_r(L) \cap C(L)$ for any $y \in K,\ h \in H$,*

(iii) *$\lambda_{f(x,y)} \circ \lambda_{f(xy,h)} = \lambda_{f(x,yh)} \circ \lambda_{f(y,h)}$ for any $x, y \in K$ and $h \in H$.*

Since $e_r \in K$ is an idempotent element the subset $\mathfrak{N} = \{(e_r, \nu);\ \nu \in N_r(L)\}$ is a subquasigroup of $\mathfrak{Q}_f$. Applying the previous theorem to the trivial subgroup $H = \{e_r\} \subset K$ and to the right nucleus $M = N_r(L)$ we get the following

Corollary. *The subquasigroup $\mathfrak{N}$ is right nuclear in $\mathfrak{Q}_f$ if and only if $f(e_r, e_r) \in N_r(L) \cap C(L)$.*

Let $(H, \cdot)$ be a subquasigroup of $(K, \cdot)$ and consider the f-subextension $\tilde{\mathfrak{Q}}_{\tilde{f}} = (\varphi_f^{-1}(H), \circ) = (H \times L, \circ) \subset \mathfrak{Q}_f$ of the loop $(L, \cdot)$ by the subquasigroup $(H, \cdot)$.

Corollary. *The subquasigroup $\varphi_f^{-1}(H)$ is right nuclear in $\mathfrak{Q}_f$ if and only if*

(i) *$H \subset N_r(K)$ and $f(e_r, h) = f(e_r, e_r)$ for all $h \in H$,*

(ii) *$(L,\cdot)$ is a group and $f(y,h)$ is contained in the center $Z(L)$ for any $y \in K,\ h \in H$,*

(iii) *$f(x,y)f(xy,h) = f(x,yh)f(y,h)$ for any $x,y \in K$ and $h \in H$.*

This assertion implies for trivial subquasigroup $H = \{e_r\}$ the following

Theorem. *An f-extension $\mathfrak{Q}_f$ is right nuclear if and only if $(L,\cdot)$ is a group and $f(e_r,e_r)$ is contained in the center $Z(L)$.*

If the nucleus $N(\mathfrak{Q}_f)$ of $\mathfrak{Q}_f$ is non-empty, then $\mathfrak{Q}_f$ has a unit and hence it is a loop. Consequently, its homomorphic image $(K,\cdot)$ is also a loop.

Corollary. *An f-extension $\mathfrak{Q}_f$ of a loop $(L,\cdot)$ by a quasigroup $(K,\cdot)$ is*
nuclear if and only if

(A) *$(K,\cdot)$ is a loop (with unit $e \in K$) and $(L,\cdot)$ is a group,*

(B) *$f(e,x) = f(x,e) = f(e,e)$ for all $x \in K$ and this element is contained in the center $Z(L)$.*

Remark. *A nuclear f-extension $\mathfrak{Q}_f$ is a central extension if and only if the group $(L,\cdot)$ is abelian.*

Let $\mathfrak{Q}_f$ be a right nuclear f-extension, i. e. $(L,\cdot)$ is a group and $f(e_r,e_r)$ is contained in the center $Z(L)$ of $(L,\cdot)$. Let $\mathfrak{Q}^*_f$ be the f^*-extension defined by the function $f^*(x,y) = f(x,y)f(e_r,e_r)^{-1}$ and the corresponding multiplication $(x,\xi) * (y,\eta) = (xy, f(x,y)f(e_r,e_r)^{-1}\xi\eta)$

on $K \times L$. Clearly, the quasigroups $\mathfrak{Q}_f$ and $\mathfrak{Q}_f^*$ are isomorphic with respect to the isomorphism $(x, \xi) \mapsto (x, f(e_r, e_r)^{-1}\xi)$. Hence we may normalize the function $f : K \times K \to L$ by the assumption $f(x, e_r) = f(e_r, e_r) = \epsilon$.

Proposition. *Let $\mathfrak{Q}_f$ be an f-extension of a group $(L, \cdot)$ by a quasigroup $(K, \cdot)$ with right unit e_r satisfying $f(x, e_r) = \epsilon$ for all $x \in K$. $\mathfrak{Q}_f$ has the right inverse property if and only if*

(i) *the quasigroup $(K, \cdot)$ has the right inverse property,*

(ii) *there exists a function $\mu : K \to Z(L)$ such that for any $x, y \in K$ $\mu(y) = f(xy, \iota(y))f(x, y)$ holds, where $\rho_{\iota(x)} = \rho_x^{-1}$.*

Let Γ be the permutation group acting on the set $K \times K$ generated by the bijection $\phi : (x, y) \mapsto (xy, \iota(y)) : K \times K \to K \times K$. Since the map ϕ is an involution, the group Γ has order 2. The orbit $\Gamma(x, y) = \{(x, y), (xy, \iota(y)\}$ consists of one point if and only if $y = e_r$, in the case $y \neq e_r$ the orbit $\Gamma(x, y)$ consists of two different points. We denote by $(K \times K)/\Gamma$ the set of orbits of Γ in $K \times K$.

Corollary. *Let $(L, \cdot)$ be a group and $(K, \cdot)$ be a quasigroup with right inverse property. Let be given a map $\mu : K \to Z(L)$, a choice function $c : (K \times K)/\Gamma \to K \times K : \gamma \mapsto c(\gamma) \in \gamma$ of orbits of Γ in $K \times K$ and a function $\nu : (K \times K)/\Gamma \to L$ satisfying $\nu(\{(x, e_r)\}) = \epsilon$ for all $x \in K$. Then the function $f : K \times K \to L$ is determined by the conditions*

(i) *$f(c(\gamma)) = \nu(\gamma)$ for any $\gamma \in (K \times K)/\Gamma$,*

(ii) *$f(\phi(x, y)) = \mu(y)f(x, y)^{-1}$ for any $x, y \in K$*

yields an f-extension $\mathfrak{Q}_f$ with right inverse property. Conversely, any right nuclear f-extension with right inverse property can be obtained by the previous construction.

Now, let $(K, \cdot)$ be an involutorial right Bol loop with unit $e \in K$. Then we have

Theorem. *Let $\mathfrak{Q}_f$ be an f-extension of a group $(L, \cdot)$ by an involutorial right Bol loop $(K, \cdot)$ such that $f(x, e) = \epsilon$ for any $x \in K$. The f-extension $\mathfrak{Q}_f$ is right alternative if and only if $\mathfrak{Q}_f$ is a loop with right inverse property. In this case the f-extension $\mathfrak{Q}_f$ is nuclear.*

In the fourth chapter we investigate f-extension of proper quasigroups. If a quasigroup has non-empty right or left nucleus, then it has right or left unit element, respectively. V.D. Belousov introduced the notions of right and left regular permutations which can be used for the measure of the near-associativity of quasigroups having neither right nor left unit element. In the case if the quasigroup has right or left unit element, then the right or left regular permutations coincide with right or left translations by elements of the right or left nuclei, respectively. Hence the notions of regular permutations can be considered as natural generalizations of the notions of nuclei. These are published in [28].

For the investigation of groups of right or left regular permutations we use the methods of extension theory. Now, we investigate quasigroup extensions having empty nuclei and describe their groups of left or right regular permutations. We give conditions under which the orbits of the groups of right or left regular permutations are contained in the kernels of the homomorphism associated with the extension. This construction alerts us quasigroups with prescribed groups of right or left regular permutations of different sizes.

Lemma. *A bijection* $\rho : (x, \xi) \mapsto (\rho_1(x, \xi), \rho_2(x, \xi)) : K \times Q \to K \times Q$ *is a right-regular permutation of an* f*-extension* $(\mathfrak{Q}_f, \circ)$ *if and only if*

(i) ρ_1 *is constant on the equivalence classes of the extension and induces a right-regular permutation* $\rho_1 : K \to K$ *of the quasigroup* $(K, \cdot)$,

(ii) ρ_2 *satisfies*

$$\rho_2(xy, f(x, y) \cdot \xi\eta) = f(x, \rho_1(y, \eta)) \cdot \xi\rho_2(y, \eta)$$

for all $x, y \in K$, $\xi, \eta \in Q$.

Remark. *If the group of right-regular permutations of a quasigroup* $(K, \cdot)$ *is trivial, then the orbits of the group of right-regular permutations of the* f*-extension* $(\mathfrak{Q}_f, \circ)$ *are contained in the congruence classes of the extension.*

This result motivates the investigation of extensions by quasigroups which have trivial group of right-regular permutations. Quasigroups with trivial right-regular permutation groups form a wide class. For example, idempotent quasigroups have this property, since if a quasigroup $(K, \cdot)$ is idempotent and a map $\phi : K \to K$ satisfies $x\phi(x) = \phi(x^2)$ for any $x \in K$, then $x\phi(x) = \phi(x)^2$ and hence $\phi(x) = x$. Many constructions of idempotent quasigroups are given in [21], Sections 9 and 10 by the study of the core of Bol loops.

Theorem. *Assume that the group of right-regular permutations of the quasigroup* $(K, \cdot)$ *is trivial and* $(Q, \cdot)$ *is a loop. A map* $\rho : \mathfrak{Q}_f \to \mathfrak{Q}_f$ *is a right-regular permutation of the* f*-extension* $(\mathfrak{Q}_f, \circ)$ *if and only if it has the shape* $\rho = (id, \rho_\nu)$, *where* $\nu \in N_r(Q)$.

Corollary. *The group of the right-regular permutations of the quasigroup* $(\mathfrak{Q}_f, \circ)$ *is isomorphic with the right nucleus of* $(Q, \cdot)$.

Corollary. *If the right nucleus* $N_r(Q)$ *is a normal subgroup, then the equivalence relation induced by the orbits of the right-regular permutation group* $R(\mathfrak{Q}_f)$ *is a normal congruence.*

Corollary. *If* $(K, \cdot)$ *is an idempotent quasigroup and the right nucleus of* $(Q, \cdot)$ *is a normal subgroup, then the orbits of the right-regular permutation group are normal subquasigroups of the kernel quasigroups of the extension.*

Similarly to the right regular case we have

Lemma. *A bijection* $\lambda : (x, \xi) \mapsto (\lambda_1(x, \xi), \lambda_2(x, \xi)) : K \times Q \to K \times Q$ *is a left-regular permutation of an* f*-extension* $(\mathfrak{Q}_f, \circ)$ *if and only if*

(i) λ_1 *is constant on the equivalence classes of the extension and induces a left-regular permutation* $\lambda_1 : K \to K$ *of the quasigroup* $(K, \cdot)$,

(ii) λ_2 *satisfies*

$$\lambda_2(xy, f(x, y) \cdot \xi\eta) = f(\lambda_1(x, \xi), y) \cdot \lambda_2(x, \xi)\eta$$

for all $x, y \in K,\ \xi, \eta \in Q$.

Theorem. *Assume that the group of left-regular permutations of the quasigroup* $(K, \cdot)$ *is trivial and* $(Q, \cdot)$ *is a loop. A map* $\lambda : K \times Q \to$

$K \times Q$ is a left-regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$ if and only if it has the shape $(x, \xi) \mapsto (x, \nu(x)\xi)$, where ν is a mapping $\nu : K \to N_l(Q)$ satisfying

$$\nu(xy)f(x,y) = f(x,y)\nu(x) \quad \text{and} \quad \lambda_{f(x,y)\nu(x)} = \lambda_{f(x,y)}\lambda_{\nu(x)} \tag{6.1}$$

for any $x, y \in K$.

Let F denote the subloop of $(Q, \cdot)$ the element of which commute with all $f(x,y)$; $x, y \in K$. The previous theorem yields the following

Corollary. *The map $\lambda = (id, \lambda_\nu)$ is a left regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$ if $\nu \in N_l(Q) \cap F$ and $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ for any $x, y \in K$.*

Theorem. *Assume that the group of left-regular permutations of the quasigroup $(K, \cdot)$ is trivial, $(Q, \cdot)$ is a loop and there exists $k \in K$ such that $f(k, y) = \kappa$ is constant for any $y \in K$ with $\kappa \in F$. A map $\lambda : K \times Q \to K \times Q$ is a left-regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$ if and only if $\lambda = (id, \lambda_\nu)$, where $\nu \in N_l(Q) \cap F$ and $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ for any $x, y \in K$.*

Theorem. *Assume that the group of left-regular permutations of the quasigroup $(K, \cdot)$ is trivial and the loop $(Q, \cdot)$ has the cross inverse property. A map $\lambda : K \times Q \to K \times Q$ is a left-regular permutation of the f-extension $(\mathfrak{Q}_f, \circ)$ if and only if $\lambda = (id, \lambda_\nu)$, where $\nu \in N_l(Q)$ and $\lambda_{f(x,y)\nu} = \lambda_{f(x,y)}\lambda_\nu$ for any $x, y \in K$.*

Chapter 7

List of talks

1. *Topologikus algebrák fedő homomorfizmusai*, Debreceni Egyetem Tehetséggondozó Programjának I. Konferenciája, Debrecen (2002),
2. *Folytonos loopok*, Tudományos Diákköri Konferencia, Debrecen (2003),
3. *Differentierbare 1 dimensionale reale Loops*, Algebra und Geometrie Gemeinschaft, Erlangen, Germany (2005),
4. *Die einheitsgruppe des Gruppenringes über den Restklassring*, Algebra Seminar, Erlangen, Germany (2005),
5. *Karakterisierung der Translationengruppe der Steiner loops*, Algebra und Geometrie Gemeinschaft, Würzburg, Germany (2006),
6. *Steiner loopok kis bővítései*, Véges geometria szeminárium, Eötvös Loránd Egyetem, Budapest (2008),

7. *Oriented Steiner loops*, The Sixth Conference of PhD Students in Computer Science, Szeged (2008),
8. *Quasigroup extensions*, The 5th International Students' Conference on Analysis, Szare, Poland (2009),
9. *Nuclei of quasigroup extensions*, Groups and topological groups, Milan, Italy, (2009),
10. *Loopok és kvázicsoportok bővítéseiről*, Országos algebra szeminárium, MTA Rényi Alfréd Matematikai Kutatóintézet, Budapest, (2009).

Chapter 8

List of publications

1. Peter T. Nagy - Izabella Stuhl, Differentiable loops on the real line, Publ. Math. Debrecen, 70/3-4 (2007), 361-370
2. Karl Strambach - Izabella Stuhl, Translation groups of Steiner loops, Discrete Mathematics, 309 (2009), 4225-4227
3. Izabella Stuhl, Regular permutations of quasigroups, Journal of Mathematical Studies, 3 (2010), 111-116
4. Peter T. Nagy - Izabella Stuhl, Right nuclei of quasigroup extensions, (submitted 9 p.)
5. Karl Strambach - Izabella Stuhl, Oriented Steiner loops, (submitted 20 p.),
6. Stuhl Izabella, Topologikus algebrák fedő homomorfizmusai, Debreceni Egyetem Tehetségkutató program I. Konferenciája, konferencia kiadványban (2002).

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Extensions of loops and quasigroups

Értekezés a doktori (PhD) fokozat megszerzése érdekében
a matematika tudományágban.

Írta: Stuhl Izabella okleveles matematikus.

Készült a Debreceni Egyetem Matematika- és Számítástudományok Doktori Iskola
Differenciálgeometria és alkalmazásai alprogramja keretében.

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