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Approximately Monotone, Hölder, Convex and Affine Functions

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Hereby I declare that I prepared this thesis within the Doctoral Council of Natural Sciences and Information Technology, Doctoral School of Mathematical and Computational Sciences of the University of Debrecen in order to obtain a PhD Degree in Natural Sciences from the University of Debrecen.

I declare that the results published in this thesis are not reported in any other PhD theses.

Debrecen, June 15, 2021.

Angshuman Robin Goswami
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Hereby I confirm that Angshuman Robin Goswami candidate conducted his studies with my supervision within the Mathematical Analysis Program of the Doctoral School of Mathematical and Computational Sciences of the University of Debrecen between 2017 and 2021. The independent studies and research work of the candidate significantly contributed to the results published in this thesis.

I also declare that the results published in the thesis are not reported in any other PhD theses.

I support the acceptance of the dissertation.

Debrecen, June 15, 2021.

Páles Zsolt
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Abstract

In this dissertation, we introduce and investigate approximately monotone, Hölder, convex and affine functions. We focus on characterization, decomposition, building blocks and association of classical inequalities and theorems with these generalized function classes. The dissertation is divided into four chapters as summarized below. Motivations and historical comments will be given in each chapter. In what follows, we briefly describe the content of each chapter now.

In Chapter 1, we broadly study approximately monotone functions. The basic properties about this function class along with envelopes and sandwich type theorems are investigated. We also give a precise formula to obtain the enveloping subadditive function for a given nonnegative function. Finally we discuss about atoms for the approximate monotone functions and characterize them.

In Chapter 2, we deal with generalization of Hölder functions. We start with the initial concepts and later explore a newly defined function property, named as absolute subadditivity. Similarly to the first chapter, we discuss about ideal cover functions and sandwich type theorems. In the last section we go through characterization of approximate Hölder functions along with discussing their building blocks.

Further in the Chapter 3, we discuss about the relationship between approximately monotone and approximately Hölder functions with some extensions of the classical inequalities and results. We introduce a new type of a variation and obtain a generalized Jordan type decomposition theorem. We deduce Hermite–Hadamard and Ostrowski type inequalities for these function classes and then we verify the sharpness and reverse implications. In the last section, we talk about some miscellaneous results such as approximation, characterization etc..

Finally in Chapter 4, we investigate the classes of approximately convex and affine functions. Here we will see how all these function classes are related to each other. We find a new type of nonnegative error function bearing a specific property termed as 'Gamma property'. We briefly study the structural and characteristic properties of all these newly mentioned classes. We will go through the convex minorant and sandwich type theorems. At the end, we characterize the approximately convex and affine functions.

Notation and Symbols

\mathbb{R}	the set of real numbers
\mathbb{R}_+	the set of nonnegative real numbers
I	a nonempty open interval
$\ell(I)$	the length of the interval I
$\mathcal{E}(I)$	the class of error functions on I
$\mathcal{E}_0(I)$	the class of functions $\Phi \in \mathcal{E}(I)$ with $\Phi(0) = 0$
$\mathcal{E}^\Gamma(I)$	the class of error functions with the Γ -property
$\mathcal{E}_0^\Gamma(I)$	the class of functions $\Phi \in \mathcal{E}^\Gamma(I)$ with $\Phi(0) = 0$
Φ, Ψ	symbols for error functions
Φ^σ	the subadditive envelope of Φ
Φ^α	the absolutely subadditive envelope of Φ
$\mathcal{M}_\Phi(I)$	class of Φ -monotone functions
$\mathcal{M}(I)$	class of all approximately monotone functions
$\overline{\mathcal{M}}_\Phi(f)$	the upper Φ -monotone envelop for f
$\underline{\mathcal{M}}_\Phi(f)$	the lower Φ -monotone envelope for f
Φ_f^σ	the individual error function for Φ -monotone function f
$\mathcal{H}_\Phi(I)$	class of Φ -Hölder functions
$\mathcal{H}(I)$	class of all approximately Hölder functions
$\overline{\mathcal{H}}_\Phi(f)$	the upper Φ -Hölder envelope for f
$\underline{\mathcal{H}}_\Phi(f)$	the lower Φ -Hölder envelope for f
Φ_f^α	the individual error function for Φ -Hölder function f
$\mathcal{C}_\Phi(I)$	class of Φ -convex functions
$\underline{\mathcal{C}}_\Phi(f)$	the lower Φ -convex envelope for f
$\mathcal{A}_\Phi(I)$	class of Φ -affine functions
$V^\Phi(f; \tau)$	the Φ -variation of f w.r.t. the partition τ
$V_{[a,b]}^\Phi f$	the total Φ -variation of f on $[a, b]$

CHAPTER 1

Approximately Monotone Functions

1.1. Preliminaries

The main concepts and results of this chapter are distilled from the following elementary observations. Assume that I is a nonempty interval and a function $f : I \rightarrow \mathbb{R}$ satisfies the following inequality

$$(1.1) \quad f(x) \leq f(y) + \varepsilon(y - x)^p \quad (x, y \in I, x < y)$$

for some nonnegative constant ε and real constant $p \in \mathbb{R}$. That is, f is nondecreasing with an error term described in terms of the p th power function. Clearly, if $\varepsilon = 0$, then the above condition is equivalent to the nondecreasingness of f . Conversely, one can notice that every nondecreasing function f satisfies (1.1). On the other hand, if $p = 1$, then (1.1) holds if and only if the function $g(x) := f(x) + \varepsilon x$ is nondecreasing and hence $f(x) = -\varepsilon x$ is a strictly decreasing solution of (1.1). If $p < 1$, then the function $f(x) := -\varepsilon x^p$, ($x > 0$) is a strictly decreasing solution of inequality (1.1) on the interval $I =]0, \infty[$.

Surprisingly, for $p > 1$, the situation is completely different. Fix $a < b$ in I , then choose $n \in \mathbb{N}$ arbitrarily, set $u := (b - a)/n$ and apply inequality (1.1) for the values $x := a + (k - 1)u$ and $y := a + ku$. Then we get

$$f(a + (k - 1)u) \leq f(a + ku) + \varepsilon u^p \quad (k \in \{1, \dots, n\}).$$

Adding up these inequalities side by side for $k \in \{1, \dots, n\}$, after trivial simplifications, we arrive at

$$f(a) = f(a + 0u) \leq f(a + nu) + n\varepsilon u^p = f(b) + \varepsilon n^{1-p}(b - a)^p \quad (n \in \mathbb{N}).$$

Upon taking the limit $n \rightarrow \infty$, it follows that

$$f(a) \leq f(b) \quad (a, b \in I, a < b),$$

which shows that f is nondecreasing. Therefore, for $p > 1$ a function $f : I \rightarrow \mathbb{R}$ satisfies (1.1) for some nonnegative ε if and only if f is nondecreasing.

In the paper [45], the particular case $p = 0$ of inequality (1.1) was considered and the following result was proved: A function $f : I \rightarrow \mathbb{R}$ satisfies (1.1) for some $\varepsilon \geq 0$ with $p = 0$ if and only if there exists a nondecreasing function

$g : I \rightarrow \mathbb{R}$ such that $|f - g| \leq \varepsilon/2$ holds on I . In other words, certain approximately monotone functions can be approximated by nondecreasing functions.

The above described observations and results motivate the investigation of the class of function that obey a more general approximate monotonicity.

Let I be a nonempty open real interval throughout this dissertation and let $\ell(I) \in]0, \infty]$ denote its length.

The class of all functions $\Phi : [0, \ell(I)[\rightarrow \mathbb{R}_+$, called *error functions*, will be denoted by $\mathcal{E}(I)$. Obviously, $\mathcal{E}(I)$ is a convex cone, i.e., it is closed with respect to addition and multiplication by nonnegative scalars. The subset of $\mathcal{E}(I)$ whose elements also satisfy $\Phi(0) = 0$ will be denoted by $\mathcal{E}_0(I)$.

A real valued function f defined on a real open interval I is called *Φ -monotone* if, for all $x, y \in I$ with $x \leq y$ it satisfies the following inequality

$$(1.2) \quad f(x) \leq f(y) + \Phi(y - x).$$

If this inequality is satisfied with the identically zero error function Φ , then we say that f is *monotone (increasing)*. The class of all Φ -monotone functions on I will be denoted by $\mathcal{M}_\Phi(I)$. We also consider the class of all functions that are Φ -monotone for some error function $\Phi \in \mathcal{E}(I)$:

$$\mathcal{M}(I) := \bigcup_{\Phi \in \mathcal{E}(I)} \mathcal{M}_\Phi(I).$$

1.2. Basic Structural Properties

In the beginning, we will describe elementary properties of Φ -monotone functions. We will go through various inclusion and structural properties dealing with supremum and infimum operations.

PROPOSITION 1.2.1. ([13]) *Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$. Then*

$$\alpha_1 \mathcal{M}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{M}_{\Phi_n}(I) \subseteq \mathcal{M}_{\alpha_1 \Phi_1 + \dots + \alpha_n \Phi_n}(I).$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the class $\mathcal{M}_\Phi(I)$ is convex. Furthermore, $\mathcal{M}(I)$ is a convex cone.

PROOF. To prove the first inclusion, let $f \in \alpha_1 \mathcal{M}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{M}_{\Phi_n}(I)$. Then, there exist f_1, \dots, f_n belonging to $\mathcal{M}_{\Phi_1}(I), \dots, \mathcal{M}_{\Phi_n}(I)$, respectively, such that

$$(1.3) \quad f = \alpha_1 f_1 + \dots + \alpha_n f_n.$$

Then, for all $x, y \in I$ with $x \leq y$, we have

$$f_i(x) \leq f_i(y) + \Phi_i(y - x) \quad (i \in \{1, \dots, n\}).$$

Multiplying this inequality by α_i and summing up side by side, we will arrive at

$$f(x) = \sum_{i=1}^n \alpha_i f_i(x) \leq \sum_{i=1}^n \alpha_i f_i(y) + \sum_{i=1}^n \alpha_i \Phi_i(y-x) = f(y) + \Phi(y-x),$$

where $\Phi := \sum_{i=1}^n \alpha_i \Phi_i$. This shows that $f \in \mathcal{M}_\Phi(I)$, which proves statement.

The additional statements are immediate consequences of what we have proved. \square

We say that a family \mathcal{F} of real valued functions is *closed with respect to the pointwise supremum* if $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\}$ is a subfamily of \mathcal{F} and $f : I \rightarrow \mathbb{R}$ such that

$$(1.4) \quad f(x) = \sup_{\gamma \in \Gamma} f_\gamma(x) \quad (x \in I),$$

then $f \in \mathcal{F}$. Similarly, we can define that a family \mathcal{F} of real valued functions which is *closed with respect to the pointwise infimum*.

PROPOSITION 1.2.2. ([13]) *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{M}_\Phi(I)$ is closed under pointwise infimum and supremum. Furthermore, $\mathcal{M}_\Phi(I)$ is closed with respect to the pointwise liminf and limsup operations.*

PROOF. Assume that $\{f_\gamma \mid \gamma \in \Gamma\}$ is a family of Φ -monotone functions with a pointwise supremum $f : I \rightarrow \mathbb{R}$, i.e., (1.4) holds. Let $x, y \in I$ be arbitrary with $x \leq y$. Then, by the Φ -monotonicity property, for all $\gamma \in \Gamma$, we have that

$$f_\gamma(x) \leq f_\gamma(y) + \Phi(y-x) \leq f(y) + \Phi(y-x).$$

Taking the supremum of the left hand side with respect to $\gamma \in \Gamma$, we get

$$f(x) \leq f(y) + \Phi(y-x),$$

which shows that f is Φ -monotone. The proof of the assertion related to the pointwise infimum is similar, therefore it is omitted.

To obtain the statements with respect to the liminf and limsup operations, let $f : I \rightarrow \mathbb{R}$ be the upper limit of a sequence $f_n : I \rightarrow \mathbb{R}$. Then

$$f = \inf_{n \in \mathbb{N}} g_n, \quad \text{where} \quad g_n := \sup_{k \geq n} f_k.$$

If all the functions f_n are Φ -monotone, then for all $n \in \mathbb{N}$, the function g_n is Φ -monotone. On the other hand, the sequence (g_n) is decreasing, therefore f is the pointwise chain infimum of $\{g_n \mid n \in \mathbb{N}\}$, thus f is also Φ -monotone.

In a similar way, one can prove that the class of Φ -monotone functions is closed under the liminf operation. \square

As an immediate consequence, we can see that if f is Φ -monotone, then $f_+ = \max(f, 0)$ is also Φ -monotone.

1.3. Optimal Error Functions

In what follows, a function $\Phi \in \mathcal{E}(I)$ will be called *subadditive* if, for all $u, v \in \mathbb{R}_+$ with $u + v < \ell(I)$, the inequality

$$(1.5) \quad \Phi(u + v) \leq \Phi(u) + \Phi(v)$$

holds. Obviously, a decreasing function $\Phi \in \mathcal{E}(I)$ is automatically subadditive. Indeed, if $u, v \geq 0$ with $u + v < \ell(I)$, then $u \leq u + v$ implies $\Phi(u + v) \leq \Phi(u)$, which, together with the nonnegativity of $\Phi(v)$, yields (1.5).

The simplest but important error functions are of the form

$$\Phi_p(0) := 0, \quad \Phi_p(u) := u^p \quad (u > 0),$$

where $p \in \mathbb{R}$. Their subadditivity is characterized by the following statement.

PROPOSITION 1.3.1. ([13]) *Let $p \in \mathbb{R}$. Then Φ_p is subadditive on \mathbb{R}_+ if and only if $p \in]-\infty, 1]$.*

PROOF. Let $p \leq 1$. To show that Φ_p is subadditive, it is enough to check (1.5) for $\Phi = \Phi_p$ in the case $uv \neq 0$. Then

$$1 = \frac{u}{u+v} + \frac{v}{u+v} \leq \left(\frac{u}{u+v}\right)^p + \left(\frac{v}{u+v}\right)^p.$$

Multiplying this inequality side by side by $(u+v)^p$, we get

$$(1.6) \quad \Phi_p(u+v) = (u+v)^p \leq u^p + v^p = \Phi_p(u) + \Phi_p(v),$$

which shows the subadditivity of Φ_p .

If $p > 1$, then with $u := v > 0$ in (1.6), we get

$$\Phi_p(u+u) = (u+u)^p = (2u)^p = 2^p u^p > 2u^p = \Phi_p(u) + \Phi_p(u),$$

therefore, Φ_p cannot be subadditive (in fact, one can see that Φ_p is superadditive). \square

It is also not difficult to see that the class of subadditive functions is closed with respect to pointwise supremum. Therefore, for any $\Phi \in \mathcal{E}(I)$, there exists a largest subadditive function $\Phi^\sigma \in \mathcal{E}(I)$ which satisfies the inequality $\Phi^\sigma \leq \Phi$. The functions Φ^σ will be called the *subadditive envelope* (or *subadditive minorant*) of the function Φ , respectively. Obviously, the equality $\Phi = \Phi^\sigma$ is valid if and only if Φ is subadditive. More generally, the function Φ^σ can be constructed explicitly from Φ by the following result.

PROPOSITION 1.3.2. ([13]) *Let $\Phi \in \mathcal{E}(I)$ be an arbitrary function. Define the function $\Phi^\sigma : [0, \ell(I)[\rightarrow \mathbb{R}_+$ by*

$$\Phi^\sigma(u) := \inf \left\{ \Phi(u_1) + \cdots + \Phi(u_n) \mid \right. \\ \left. n \in \mathbb{N}, u_1, \dots, u_n \in \mathbb{R}_+ : u_1 + \cdots + u_n = u \right\}.$$

Then Φ^σ is the largest subadditive function which satisfies the inequality $\Phi^\sigma \leq \Phi$ on $[0, \ell(I)[$. Furthermore, $\Phi^\sigma(0) = \Phi(0)$ and, additionally, if Φ is increasing, then Φ^σ is also increasing.

PROOF. First we are going to prove the subadditivity of Φ^σ . Let $u, v \in \mathbb{R}_+$ such that $u + v \in [0, \ell(I)[$. Let $\varepsilon > 0$ be arbitrary. Then there exist $n, m \in \mathbb{N}$ and $u_1, \dots, u_n, v_1, \dots, v_m \in \mathbb{R}_+$ such that

$$u = \sum_{i=1}^n u_i, \quad \sum_{i=1}^n \Phi(u_i) < \Phi^\sigma(u) + \frac{\varepsilon}{2}$$

and

$$v = \sum_{j=1}^m v_j, \quad \sum_{j=1}^m \Phi(v_j) < \Phi^\sigma(v) + \frac{\varepsilon}{2}.$$

We have that $u + v = \sum_{i=1}^n u_i + \sum_{j=1}^m v_j$. Therefore, by the definition of Φ^σ and by the last two inequalities, we get

$$\Phi^\sigma(u + v) \leq \sum_{i=1}^n \Phi(u_i) + \sum_{j=1}^m \Phi(v_j) < \Phi^\sigma(u) + \Phi^\sigma(v) + \varepsilon.$$

Since ε is arbitrary, we conclude that $\Phi^\sigma(u + v) \leq \Phi^\sigma(u) + \Phi^\sigma(v)$, which completes the proof of the subadditivity of Φ^σ . By taking $n = 1$, $u_1 = u$ in the definition of $\Phi^\sigma(u)$, we can see that $\Phi^\sigma(u) \leq \Phi(u)$ also holds for all $u \in [0, \ell(I)[$.

By the definition of Φ^σ and the inequality $\Phi(0) \geq 0$, we have that

$$\Phi^\sigma(0) = \inf_{n \in \mathbb{N}} n\Phi(0) = \Phi(0).$$

Now assume that $\Psi : [0, \ell(I)[\rightarrow \mathbb{R}_+$ is a subadditive function such that $\Psi \leq \Phi$ holds on $[0, \ell(I)[$. To show that $\Psi \leq \Phi^\sigma$, let $u \in [0, \ell(I)[$ and $\varepsilon > 0$ be arbitrary. Then there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathbb{R}_+$ such that

$$(1.7) \quad u = u_1 + \cdots + u_n \quad \text{and} \quad \Phi(u_1) + \cdots + \Phi(u_n) < \Phi^\sigma(u) + \varepsilon.$$

Then, due to the subadditivity of Ψ ,

$$\Psi(u) \leq \Psi(u_1) + \cdots + \Psi(u_n) \leq \Phi(u_1) + \cdots + \Phi(u_n) < \Phi^\sigma(u) + \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, the inequality $\Psi(u) \leq \Phi^\sigma(u)$ follows for all $u \in [0, \ell(I)[$, which was to be proved.

To verify the last assertion, assume that Φ is increasing. Let $u, v \in [0, \ell(I)[$ with $v < u$. Let $\varepsilon > 0$ be arbitrary. Then there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathbb{R}_+$ such that (1.7) is satisfied. Define $v_i := \frac{v}{u}u_i$. Then $v_i \leq u_i$, hence $\Phi(v_i) \leq \Phi(u_i)$. On the other hand, $v_1 + \dots + v_n = \frac{v}{u}(u_1 + \dots + u_n) = v$, which implies that

$$\Phi^\sigma(v) \leq \Phi(v_1) + \dots + \Phi(v_n) \leq \Phi(u_1) + \dots + \Phi(u_n) < \Phi^\sigma(u) + \varepsilon.$$

Passing the limit $\varepsilon \rightarrow 0$, we arrive at the inequality $\Phi^\sigma(v) \leq \Phi^\sigma(u)$, which proves the increasingness of Φ^σ . \square

THEOREM 1.3.3. ([13]) *Let $\Phi \in \mathcal{E}(I)$. Then*

$$\mathcal{M}_\Phi(I) = \mathcal{M}_{\Phi^\sigma}(I).$$

PROOF. The inclusion $\mathcal{M}_{\Phi^\sigma}(I) \subseteq \mathcal{M}_\Phi(I)$ is a trivial consequence of the inequality $\Phi^\sigma \leq \Phi$. To prove the reversed inclusion, let $f \in \mathcal{M}_\Phi(I)$. To show that f is also Φ^σ -monotone, let $x < y$ be arbitrary elements of I and $\varepsilon > 0$ be arbitrary.

For $u := y - x < \ell(I)$, there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathbb{R}_+$ such that (1.7) holds. For the sake of convenience, let $u_0 := 0$ and

$$x_i := x + u_0 + \dots + u_i \quad (i \in \{0, \dots, n\}).$$

Obviously, $x = x_0 \leq x_1 \leq \dots \leq x_n = y$. Applying the Φ -monotonicity of f , we get that

$$f(x_{i-1}) \leq f(x_i) + \Phi(x_i - x_{i-1}) = f(x_i) + \Phi(u_i) \quad (i \in \{1, \dots, n\}).$$

Adding up the above inequalities for $i \in \{1, \dots, n\}$ side by side, we obtain that

$$\begin{aligned} f(x) = f(x_0) &\leq f(x_n) + \Phi(u_1) + \dots + \Phi(u_n) \\ &< f(x_n) + \Phi^\sigma(u) + \varepsilon = f(y) + \Phi^\sigma(y - x) + \varepsilon. \end{aligned}$$

Upon taking the limit $\varepsilon \rightarrow 0$, it follows that

$$f(x) \leq f(y) + \Phi^\sigma(y - x),$$

which completes the proof of the Φ^σ -monotonicity of f . \square

COROLLARY 1.3.4. ([13]) *Let $\Phi \in \mathcal{E}(I)$ such that, for all $0 \leq u < \ell(I)$,*

$$(1.8) \quad \inf_{n \in \mathbb{N}} n\Phi\left(\frac{u}{n}\right) = 0.$$

Then $\Phi^\sigma \equiv 0$. In particular, for $p > 1$, $\Phi_p^\sigma = 0$.

PROOF. If $n \in \mathbb{N}$ and $0 \leq u < \ell(I)$, then, by the construction of Φ^σ ,

$$\Phi^\sigma(u) \leq n\Phi\left(\frac{u}{n}\right).$$

Taking the infimum of the right hand side of this inequality for $n \in \mathbb{N}$, we get that $\Phi^\sigma(u) \leq 0$, which yields $\Phi^\sigma(u) = 0$.

For the particular case, let $\Phi = \Phi_p$ for some $p > 1$. Then $1 - p < 0$, thus

$$\inf_{n \in \mathbb{N}} n\Phi_p\left(\frac{u}{n}\right) = \inf_{n \in \mathbb{N}} n^{1-p}u^p = \lim_{n \rightarrow \infty} n^{1-p}u^p = 0,$$

which shows that $\Phi_p^\sigma = 0$. \square

COROLLARY 1.3.5. ([13]) *Let $\Phi \in \mathcal{E}(I)$ be such that, for all $0 \leq u < \ell(I)$, (1.8) holds. Then $\mathcal{M}_\Phi(I)$ equals the class of increasing functions on I .*

PROOF. In view of Corollary 1.3.4, we have that $\Phi^\sigma = 0$. Combining this with the result of the Theorem 1.3.3, we get that $\mathcal{M}_\Phi(I) = \mathcal{M}_0(I)$, which is equivalent to the statement. \square

The following result demonstrates that the subadditive and increasing error functions optimally determine the corresponding classes of monotone functions.

THEOREM 1.3.6. ([13]) *Let $\Phi \in \mathcal{E}_0(I)$ be an increasing and subadditive function and let $\Psi \in \mathcal{E}(I)$ satisfy the inequality $\Psi \leq \Phi$. Then $\mathcal{M}_\Psi(I) = \mathcal{M}_\Phi(I)$ if and only if $\Psi = \Phi$.*

PROOF. The inclusion $\mathcal{M}_\Psi(I) \subseteq \mathcal{M}_\Phi(I)$ follows from the inequality $\Psi \leq \Phi$. To prove the statement, it suffices to show that if $\Psi(p) < \Phi(p)$ for some $p \in]0, \ell(I)[$ then the inclusion $\mathcal{M}_\Psi(I) \subseteq \mathcal{M}_\Phi(I)$ is proper. For this, we construct a function $f : I \rightarrow \mathbb{R}$ such that $f \in \mathcal{M}_\Phi(I)$ but $f \notin \mathcal{M}_\Psi(I)$.

The inclusion $p \in]0, \ell(I)[$ implies that there exist $u, v \in I$ such that $p = v - u$. Define

$$f(x) := \begin{cases} 0 & \text{if } x \leq u \\ -\Phi(x - u) & \text{if } u < x \end{cases} \quad (x \in I).$$

To prove that $f \in \mathcal{M}_\Phi(I)$, we fix $x, y \in I$ with $x \leq y$ and distinguish three cases.

If $x \leq y \leq u$, then $f(x) = f(y) = 0$, hence the inequality (1.2) is a consequence of the nonnegativity of Φ .

If $x \leq u < y$, then $f(x) = 0$ and $f(y) = -\Phi(y - u)$, therefore the inequality (1.2) is now equivalent to

$$\Phi(y - u) \leq \Phi(y - x),$$

which is a consequence of the increasingness of Φ .

If $u < x \leq y$, then $f(x) = -\Phi(x - u)$ and $f(y) = -\Phi(y - u)$, therefore the inequality (1.2) is now equivalent to

$$\Phi(y - u) \leq \Phi(y - x) + \Phi(x - u),$$

which is a consequence of the subadditivity of Φ .

This completes the proof of the inclusion $f \in \mathcal{M}_\Phi(I)$ and hence shows that $f \in \mathcal{M}_\Phi(I)$. To complete the proof, we have to verify that $f \notin \mathcal{M}_\Psi(I)$. Indeed, we have that

$$f(u) - f(v) = \Phi(v - u) = \Phi(p) > \Psi(p) = \Psi(v - u).$$

This strict inequality shows that f cannot be Ψ -monotone. \square

1.4. Approximately Monotone Envelopes and Sandwich Type Theorems

As we have seen it in Proposition 1.2.2 the class $\mathcal{M}_\Phi(I)$ is closed with respect to pointwise infimum and supremum. Therefore, for any function $f : I \rightarrow \mathbb{R}$, the supremum of all Φ -monotone functions below f (provided that there is at least one such function) is the largest Φ -monotone function which is smaller than or equal to f . Similarly, the infimum of all Φ -monotone functions above f (provided that there is at least one such function) is the smallest Φ -monotone function which is bigger than or equal to f . The next result offers a formula for these enveloping functions.

PROPOSITION 1.4.1. (**[13]**) *Let $\Phi \in \mathcal{E}_0(I)$ and let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -monotone minorant. Then the function $\underline{M}_\Phi(f)$ defined by*

$$\underline{M}_\Phi(f)(x) := \inf_{x \leq y} (f(y) + \Phi^\sigma(y - x)) \quad (x \in I)$$

is real-valued and is the largest Φ -monotone function which is smaller than or equal to f . Analogously, if f admits a Φ -monotone majorant, then the function $\overline{M}_\Phi(f)$ defined by

$$\overline{M}_\Phi(f)(x) := \sup_{y \leq x} (f(y) - \Phi^\sigma(x - y)) \quad (x \in I)$$

is real-valued and is the smallest Φ -monotone function which is bigger than or equal to f .

PROOF. Obviously, $\underline{M}_\Phi(f)$ cannot take the value $+\infty$ at any point in I , i.e., $\underline{M}_\Phi(f)(x) < +\infty$ for all $x \in I$. The condition $\Phi(0) = 0$ implies that $\Phi^\sigma(0) = 0$, therefore, by taking $y = x$ in the defining formula of $\underline{M}_\Phi(f)(x)$, we get that $\underline{M}_\Phi(f)(x) \leq f(x)$ holds for all $x \in I$.

Now suppose g is a Φ -monotone function such that $g \leq f$ holds (by the assumption, there is at least one such function g). Then, by Theorem 1.3.3, g is

also Φ^σ -monotone. In order to show that $g \leq \underline{M}_\Phi(f)$, let $x \in I$ be arbitrarily fixed. Then, for all $y \in I$ with $x \leq y$, we have

$$g(x) \leq g(y) + \Phi^\sigma(y - x) \leq f(y) + \Phi^\sigma(y - x).$$

Upon taking the infimum of the right hand side with respect to $y \geq x$, we get

$$g(x) \leq \inf_{x \leq y} (f(y) + \Phi^\sigma(y - x)) = \underline{M}_\Phi(f)(x),$$

which proves the desired inequality $g(x) \leq \underline{M}_\Phi(f)(x)$ and also that $\underline{M}_\Phi(f)$ cannot take the value $-\infty$ at any point of I .

To see that $\underline{M}_\Phi(f)$ itself is Φ -monotone, it is sufficient to show that $\underline{M}_\Phi(f)$ is Φ^σ -monotone. Let $u, v \in I$ with $u \leq v$. Then, using the subadditivity of Φ^σ , we obtain

$$\begin{aligned} \underline{M}_\Phi(f)(u) &= \inf_{u \leq y} (f(y) + \Phi^\sigma(y - u)) \\ &\leq \inf_{v \leq y} (f(y) + \Phi^\sigma(y - u)) \\ &\leq \inf_{v \leq y} (f(y) + \Phi^\sigma(y - v)) + \Phi^\sigma(v - u) \\ &= \underline{M}_\Phi(f)(v) + \Phi^\sigma(v - u), \end{aligned}$$

which completes the proof of the Φ -monotonicity of $\underline{M}_\Phi(f)$.

The proof of the second assertion is completely similar. \square

The following result is of a sandwich type one.

COROLLARY 1.4.2. ([13]) *Let $\Phi \in \mathcal{E}_0(I)$ and let $g, h : I \rightarrow \mathbb{R}$. Then in order that there exist a Φ -monotone function $f : I \rightarrow \mathbb{R}$ between g and h it is necessary and sufficient that, for all $x, y \in I$ with $x \leq y$, the inequality*

$$(1.9) \quad g(x) \leq h(y) + \Phi^\sigma(y - x)$$

be valid.

PROOF. Assume first that f is a Φ -monotone function such that $g \leq f \leq h$. Then, f is Φ^σ -monotone and, for all $x, y \in I$ with $x \leq y$, we have

$$g(x) \leq f(x) \leq f(y) + \Phi^\sigma(y - x) \leq h(y) + \Phi^\sigma(y - x),$$

i.e., (1.9) holds.

Conversely, assume that (1.9) holds true for all $x, y \in I$ with $x \leq y$. For a fixed $x \in I$, define

$$f(x) := \underline{M}_\Phi(h)(x) = \inf_{x \leq y} (h(y) + \Phi^\sigma(y - x)).$$

Now, in view of inequality (1.9), we have that $g(x) \leq f(x)$. By taking $y = x$ in the definition of f , the condition $\Phi(0) = 0$ ensures that $f(x) \leq h(x)$ is also

valid. Finally, arguing similarly as at the end of the proof of Proposition 1.4.1, it follows that f is Φ^σ -monotone and hence Φ -monotone as well. \square

Before we formulate and prove the next theorem we shall need the following auxiliary result.

LEMMA 1.4.3. ([13]) *Let $\Phi, \Psi \in \mathcal{E}(I)$ such that $(-\Phi)$ is Ψ -monotone on $]0, \ell(I)[$. Then $(-\Phi^\sigma)$ is also Ψ -monotone on $]0, \ell(I)[$.*

PROOF. To prove this lemma, let $x, y \in]0, \ell(I)[$ with $x < y$ and $\varepsilon > 0$ be arbitrary. Then by definition of Φ^σ , there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathbb{R}_+$ such that $x = u_1 + \dots + u_n$ satisfying

$$(1.10) \quad \Phi(u_1) + \dots + \Phi(u_n) < \Phi^\sigma(x) + \varepsilon.$$

Using the Ψ -monotonicity of $(-\Phi)$, we have

$$(-\Phi)(u_n) \leq (-\Phi)(u_n + (y - x)) + \Psi(y - x),$$

from which we obtain

$$\Phi(u_n + (y - x)) \leq \Phi(u_n) + \Psi(y - x).$$

Observe that $y = u_1 + \dots + u_{n-1} + (u_n + (y - x))$. Thus, using the inequality in (1.10), we arrive at

$$\begin{aligned} \Phi^\sigma(y) &\leq \Phi(u_1) + \dots + \Phi(u_{n-1}) + \Phi(u_n + (y - x)) \\ &\leq \Phi(u_1) + \dots + \Phi(u_{n-1}) + \Phi(u_n) + \Psi(y - x) \\ &< \Phi^\sigma(x) + \Psi(y - x) + \varepsilon. \end{aligned}$$

As ε is an arbitrary positive number, we can conclude that $(-\Phi^\sigma)(x) \leq (-\Phi^\sigma)(y) + \Psi(y - x)$, which completes the proof of the Ψ monotonicity of $(-\Phi^\sigma)$. \square

THEOREM 1.4.4. ([13]) *Let $\Phi, \Psi \in \mathcal{E}(I)$ such that $(-\Phi)$ is Ψ -monotone on $]0, \ell(I)[$ and let $f : I \rightarrow \mathbb{R}$. Then f is Φ -monotone if and only if there exist two Ψ -monotone functions $f_*, f^* : I \rightarrow \mathbb{R}$ such that $f_* \leq f \leq f^*$ hold on I and, for all $x, y \in I$ with $x < y$,*

$$(1.11) \quad f(x) \leq f_*(y) + \Phi^\sigma(y - x) \quad \text{and} \quad f^*(x) \leq f(y) + \Phi^\sigma(y - x).$$

PROOF. First assume that f is Φ -monotone. Then, by Theorem 1.3.3, it is also Φ^σ -monotone. For a fixed point $x \in I$, define

$$f_*(x) := \sup_{u < x} (f(u) - \Phi^\sigma(x - u)) \quad \text{and} \quad f^*(x) := \inf_{x < v} (f(v) + \Phi^\sigma(v - x)).$$

In view of the Φ^σ -monotonicity of f , for all $u < x < v$, we have that

$$f(u) - \Phi^\sigma(x - u) \leq f(x) \quad \text{and} \quad f(x) \leq f(v) + \Phi^\sigma(v - x).$$

Therefore, upon taking the supremum for $u < x$ and the infimum for $x < v$, we get that $f_*(x) \leq f(x)$ and $f(x) \leq f^*(x)$, respectively. That is, we have that f_* and f^* are real valued functions and the inequalities $f_* \leq f \leq f^*$ hold on I .

In the next step, we establish the Ψ -monotonicity of f_* and f^* . By the definition of f^* , for all $x, y \in I$ with $x < y$, we have

$$(1.12) \quad f_*(x) = \sup_{u < x} \left(f(u) - \Phi^\sigma(x - u) \right) \leq \sup_{u < y} \left(f(u) - \Phi^\sigma(x - u) \right).$$

By Lemma 1.4.3, we have the Ψ -monotonicity of $(-\Phi^\sigma)$, which, for all $u \in I$ with $u < x$, implies

$$\begin{aligned} -\Phi^\sigma(x - u) &\leq -\Phi^\sigma(y - u) + \Psi((y - u) - (x - u)) \\ &= -\Phi^\sigma(y - u) + \Psi(y - x). \end{aligned}$$

Applying this inequality to the right most expression of inequality (1.12), we arrive at

$$\begin{aligned} f_*(x) &\leq \sup_{u < y} \left(f(u) - \Phi^\sigma(x - u) \right) \\ &\leq \sup_{u < y} \left(f(u) - \Phi^\sigma(y - u) \right) + \Psi(y - x) \\ &= f_*(y) + \Psi(y - x), \end{aligned}$$

which shows that f_* is also Ψ -monotone.

Take $x, y \in I$ with $x < y$. Then, by the definition of f^* , we have

$$(1.13) \quad f^*(x) = \inf_{x < v} \left(f(v) + \Phi^\sigma(v - x) \right) \leq \inf_{y < v} \left(f(v) + \Phi^\sigma(v - x) \right).$$

By the Ψ -monotonicity of $(-\Phi^\sigma)$, for all $v \in I$ with $y < v$, we obtain

$$\Phi^\sigma(v - x) \leq \Phi^\sigma(v - y) + \Psi((v - x) - (v - y)) = \Phi^\sigma(v - y) + \Psi(y - x).$$

Applying this inequality to the right most expression of inequality (1.13), we arrive at

$$\begin{aligned} f^*(x) &\leq \inf_{y < v} \left(f(v) + \Phi^\sigma(v - x) \right) \\ &\leq \inf_{y < v} \left(f(v) + \Phi^\sigma(v - y) \right) + \Psi(y - x) \\ &= f^*(y) + \Psi(y - x), \end{aligned}$$

which proves that f^* is Ψ -monotone.

Finally, for $x, y \in I$ with $x < y$, from the definitions of f_* and f^* , we obtain the inequalities

$$f(x) - \Phi^\sigma(y - x) \leq f_*(y) \quad \text{and} \quad f^*(x) \leq f(y) + \Phi^\sigma(y - x),$$

respectively, which prove that f_* and f^* satisfy the inequalities stated in (1.11).

Conversely, if the first inequality in (1.11) holds for some function $f_* : I \rightarrow \mathbb{R}$ satisfying $f_* \leq f$, then $f(x) \leq f_*(y) + \Phi^\sigma(y-x) \leq f(y) + \Phi^\sigma(y-x)$, which shows the Φ^σ -monotonicity of f . Similarly, the existence of a function $f^* : I \rightarrow \mathbb{R}$ satisfying $f \leq f^*$ and the second inequality of (1.11), also implies that f is Φ^σ -monotone. \square

By taking the error function $\Psi \equiv 0$, the previous theorem directly implies the following result. Observe that, in this case, Ψ -monotonicity is equivalent to increasingness.

COROLLARY 1.4.5. ([13]) *Let $\Phi \in \mathcal{E}(I)$ such that Φ is decreasing on $]0, \ell(I)[$ and let $f : I \rightarrow \mathbb{R}$. Then, f is Φ -monotone if and only if there exist two increasing functions $f_*, f^* : I \rightarrow \mathbb{R}$ such that $f_* \leq f \leq f^*$ hold on I and, for all $x, y \in I$ with $x < y$, the inequalities in (1.11) are satisfied.*

1.5. Characterization of Approximately Monotone Functions

In the main results of this section, using the notions of upper and lower interpolations, we establish a characterization for approximately monotone functions. This allows one to construct Φ -monotone functions from elementary ones, which could be termed the building blocks for this class.

In what follows, we construct a large class of elementary Φ -monotone functions provided that Φ is a subadditive and nondecreasing error function whose members will turn out to be the building blocks of Φ -monotone functions.

If $\Phi \in \mathcal{E}(I)$, $h : I \rightarrow [-\infty, \infty]$ and $p \in I$, then define the functions $h_p, h^p : I \rightarrow [-\infty, \infty]$ as follows:

$$(1.14) \quad h_p(x) := \begin{cases} h(x) & \text{if } x \leq p, \\ h(p) - \Phi(x-p) & \text{if } p < x, \end{cases}$$

and

$$h^p(x) := \begin{cases} h(p) + \Phi(p-x) & \text{if } x < p, \\ h(x) & \text{if } p \leq x. \end{cases}$$

PROPOSITION 1.5.1. ([14]) *Let $\Phi \in \mathcal{E}(I)$ be subadditive and nondecreasing and $h : I \rightarrow [-\infty, \infty]$ be nondecreasing. Then, for all $p \in I$, the functions h_p and h^p defined by (1.14) are Φ -monotone.*

PROOF. Let $p \in I$ be fixed. To show that h_p is Φ -monotone, let $x, y \in I$ with $x < y$. We now distinguish three subcases according to the position of p

with respect to x and y .

If $x < y \leq p$, then

$$h_p(x) = h(x) \leq h(y) = h_p(y) \leq h_p(y) + \Phi(y - x).$$

If $x \leq p < y$, then the increasingness of Φ implies

$$h_p(x) = h(x) \leq h(p) = h_p(y) + \Phi(y - p) \leq h_p(y) + \Phi(y - x).$$

Finally, if $p < x < y$, then the subadditivity of Φ results

$$\begin{aligned} h_p(x) &= h(p) - \Phi(x - p) = h_p(y) + \Phi(y - p) - \Phi(x - p) \\ &\leq h_p(y) + \Phi(y - x). \end{aligned}$$

The proof of the Φ -monotonicity of h^p is analogous, therefore omitted. \square

If $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ and $p \in I$, then we say that f can be interpolated at p by a Φ -monotone function from below [resp. from above] if there exists a Φ -monotone function $h : I \rightarrow \mathbb{R}$ such that $h(p) = f(p)$ and $h \leq f$ [resp. $f \leq h$].

PROPOSITION 1.5.2. ([14]) *Let $\Phi \in \mathcal{E}(I)$ be a subadditive and nondecreasing function, let $f : I \rightarrow \mathbb{R}$ and $p \in I$ be fixed. Then f can be interpolated at p by a Φ -monotone function from below if and only if, for all $x \in I$,*

$$(1.15) \quad -\infty < \inf_{[x,p]} f \quad \text{if } x \leq p \quad \text{and} \quad f(p) \leq f(x) + \Phi(x - p) \quad \text{if } p < x.$$

Analogously, f can be interpolated at p by a Φ -monotone function from above if and only if, for all $x \in I$,

$$(1.16) \quad f(x) \leq f(p) + \Phi(p - x) \quad \text{if } x < p \quad \text{and} \quad \sup_{[p,x]} f < +\infty \quad \text{if } p \leq x.$$

PROOF. Suppose first that there exists a Φ -monotone function $h : I \rightarrow \mathbb{R}$ such that $h(p) = f(p)$ and $h \leq f$. Let $x \in I$ be arbitrary.

If $x \leq p$ and $t \in [x, p]$, then by the Φ -monotonicity of h and the nondecreasingness of Φ , we get

$$h(x) \leq h(t) + \Phi(t - x) \leq h(t) + \Phi(p - x) \leq f(t) + \Phi(p - x),$$

which implies that

$$\inf_{[x,p]} f \geq h(x) - \Phi(p - x) > -\infty.$$

If $p < x$, then $f(p) = h(p) \leq h(x) + \Phi(x - p) \leq f(x) + \Phi(x - p)$ proving the second part of condition (1.15).

To prove the sufficiency, assume that condition (1.15) holds and define the function $h : I \rightarrow [-\infty, \infty]$ by

$$h(x) := \begin{cases} \inf_{[x,p]} f & \text{if } x < p, \\ f(p) & \text{if } x = p, \\ \sup_{[p,x]} f & \text{if } p < x. \end{cases}$$

Then one can easily check that h is nondecreasing, $h(p) = f(p)$ and, by the first part of (1.15), $h(x)$ is finite if $x \leq p$. Therefore, the function h_p has finite values everywhere. According to Proposition 1.5.1, h_p is Φ -monotone and $h_p(p) = h(p) = f(p)$. Thus, it remains to show that $h_p \leq f$. Indeed, if $x \leq p$, then $h_p(x) = \inf_{[x,p]} f \leq f(x)$. If $p < x$, then, by the second part of (1.15), $h_p(x) = h(p) - \Phi(x - p) = f(p) - \Phi(x - p) \leq f(x)$ and the proof is completed.

The second part of the proposition can be verified analogously. \square

THEOREM 1.5.3. ([14]) *Let $\Phi \in \mathcal{E}(I)$ be a subadditive and nondecreasing function and $f : I \rightarrow \mathbb{R}$. Then the following assertions are equivalent.*

(i) f is Φ -monotone.

(ii) There exists a function $H : I \times I \rightarrow \mathbb{R}$ such that H satisfies the functional equations

$$(1.17) \quad \begin{aligned} \min(H(x, y), H(y, z)) &= H(x, z) && \text{and} \\ \max(H(z, y), H(y, x)) &= H(z, x) \end{aligned}$$

for all $x, y, z \in I$ with $x \leq y \leq z$ and, for all $p \in I$, the function $h := H(\cdot, p)$ is nondecreasing and h_p and h^p are Φ -monotone interpolations for f at p from below and from above, respectively.

(iii) For every $p \in I$, there exists a nondecreasing function $h : I \rightarrow \mathbb{R}$ such that h_p is a Φ -monotone interpolation of f at p from below.

(iv) For every $p \in I$, there exists a nondecreasing function $h : I \rightarrow \mathbb{R}$ such that h^p is a Φ -monotone interpolation of f at p from above.

PROOF. (i) \Rightarrow (ii): Assume that f is Φ -monotone and let's define the function $H : I \times I \rightarrow \mathbb{R}$ by

$$H(x, y) = \begin{cases} \inf_{[x,y]} f & \text{if } x < y, \\ f(x) & \text{if } x = y, \\ \sup_{[y,x]} f & \text{if } x > y. \end{cases}$$

By its Φ -monotonicity, the function f can be interpolated from below and from above by a Φ -monotone function, thus, Proposition 1.5.2 implies that the values of H are finite.

Then, for any $x, y, z \in I$ with $x \leq y \leq z$, we have

$$\min(H(x, y), H(y, z)) = \min \left(\inf_{[x, y]} f, \inf_{[y, z]} f \right) = \inf_{[x, z]} f = H(x, z).$$

Analogously, we can also establish the second equality:

$$\max(H(z, y), H(y, x)) = \max \left(\sup_{[y, z]} f, \sup_{[x, y]} f \right) = \sup_{[x, z]} f = H(z, x).$$

Thus, we have shown that H satisfies the functional equations of assertion (ii).

Let $p \in I$ be fixed. To show the nondecreasingness of $h := H(\cdot, p)$, assume $x, y \in I$ with $x < y$. If $x < y \leq p$, Then, by the defination of h , we have

$$h(x) = H(x, p) = \inf_{[x, p]} f \leq \inf_{[y, p]} f = H(y, p) = h(y).$$

If $x \leq p < y$,

$$h(x) = H(x, p) = \inf_{[x, p]} f \leq f(p) \leq \sup_{[p, y]} f = H(y, p) = h(y).$$

Finally, if $p < x < y$, then

$$h(x) = H(x, p) = \sup_{[p, x]} f \leq \sup_{[p, y]} f = H(y, p) = h(y).$$

This completes the proof of the monotonicity.

Finally, we just need to show that h_p is a Φ -monotone interpolation of f at p from below.

$$h_p \leq f \leq h^p \quad \text{and} \quad h_p(p) = f(p) = h^p(p).$$

For the first inequality, let $x \in I$ be arbitrary. If $x \leq p$, then

$$h_p(x) = h(x) = H(x, p) = \inf_{[x, p]} f \leq f(x).$$

If $p < x$, then the Φ -monotonicity of f yields

$$\begin{aligned} h_p(x) &= h(p) - \Phi(x - p) = H(p, p) - \Phi(x - p) \\ &= f(p) - \Phi(x - p) \leq f(x). \end{aligned}$$

Therefore, $h_p \leq f$ holds on I . The equality $h_p(p) = f(p)$ is obvious.

Analogously, one can see that h^p is a Φ -monotone interpolation of f at p from above.

(ii) \Rightarrow (iii), (iv) : Assume that $H : I \times I \rightarrow \mathbb{R}$ satisfies the conditions of assertion (ii). For any $p \in I$, define $h := H(\cdot, p)$. Then h_p and h^p are Φ -monotone interpolation of f at p from below and above, respectively. Hence assertions (iii) and (iv) are valid.

The proof of the last two implications is based on the result of our previous paper [13] which states that the family of Φ -monotone functions is closed under pointwise supremum and infimum.

(iii) \Rightarrow (i) If we assume that f admits a Φ -monotone interpolation from below at every point $p \in I$, then f is the pointwise supremum of a family of such Φ -monotone functions, and therefore, f itself is a Φ -monotone function.

(iv) \Rightarrow (i) If we assume that f admits a Φ -monotone interpolation from above at every point $p \in I$, then f is the pointwise infimum of a family of such Φ -monotone functions, and therefore, f itself is a Φ -monotone function. \square

In the following chapter, we will discuss a very important subclass of Φ -monotone functions.

CHAPTER 2

Approximately Hölder Functions

2.1. Preliminaries

For a real parameter p , a real valued function f defined on a real open interval I is called p -Hölder if, for all $x, y \in I$, it satisfies

$$(2.1) \quad |f(x) - f(y)| \leq \varepsilon |x - y|^p,$$

for some nonnegative constant ε and real constant $p \in \mathbb{R}$.

When $p = 1$, we simply call it a Hölder function. But when $p > 1$, one can easily verify that f is a constant function. In this chapter we generalize the notion of p -Hölder functions by replacing the right hand side of (2.1) by a more general term.

For an error function $\Phi \in \mathcal{E}(I)$, a map $f : I \rightarrow \mathbb{R}$ is called Φ -Hölder if

$$(2.2) \quad |f(x) - f(y)| \leq \Phi(|x - y|) \quad (x, y \in I).$$

In other words f is Φ -Hölder if both f and $-f$ are Φ -monotone.

In fact, the class of approximate Hölder functions was introduced in the paper [34], but this property was only investigated in the related context of approximate convexity.

The class of all Φ -Hölder functions defined on I will be denoted by $\mathcal{H}_\Phi(I)$. We also consider the class of all functions that are Φ -Hölder for some error function $\Phi \in \mathcal{E}(I)$:

$$\mathcal{H}(I) := \bigcup_{\Phi \in \mathcal{E}(I)} \mathcal{H}_\Phi(I).$$

2.2. Basic Structural Properties

In the following section we will see some of the basic structural properties of Φ -Hölder functions.

PROPOSITION 2.2.1. ([13]) *Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Then*

$$\alpha_1 \mathcal{H}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{H}_{\Phi_n}(I) \subseteq \mathcal{H}_{|\alpha_1| \Phi_1 + \dots + |\alpha_n| \Phi_n}(I).$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the class $\mathcal{H}_\Phi(I)$ is convex and central symmetric, i.e., $\mathcal{H}_\Phi(I)$ is closed with respect to multiplication by (-1) . Furthermore, $\mathcal{H}(I)$ is a linear space.

PROOF. To prove the first inclusion, let $f \in \alpha_1 \mathcal{H}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{H}_{\Phi_n}(I)$. Then, there exist f_1, \dots, f_n belonging to $\mathcal{H}_{\Phi_1}(I), \dots, \mathcal{H}_{\Phi_n}(I)$, respectively, such that (1.3) holds. Then, for all $x, y \in I$, we have

$$|f_i(x) - f_i(y)| \leq \Phi_i(y - x) \quad (i \in \{1, \dots, n\}).$$

Multiplying this inequality by $|\alpha_i|$ and summing up side by side, we will arrive at

$$\begin{aligned} |f(x) - f(y)| &= \left| \sum_{i=1}^n \alpha_i (f_i(x) - f_i(y)) \right| \leq \sum_{i=1}^n |\alpha_i| \cdot |f_i(x) - f_i(y)| \\ &\leq \sum_{i=1}^n |\alpha_i| \Phi_i(y - x) \\ &= \Phi(y - x), \end{aligned}$$

where $\Phi := \sum_{i=1}^n |\alpha_i| \Phi_i$. This shows that $f \in \mathcal{H}_\Phi(I)$, which proves the statement.

The additional statements are immediate consequences of what we have proved. \square

PROPOSITION 2.2.2. ([13]) Let $\Phi \in \mathcal{E}(I)$. Then

$$(2.3) \quad \mathcal{H}_\Phi(I) = \mathcal{M}_\Phi(I) \cap (-\mathcal{M}_\Phi(I)).$$

Furthermore,

$$(2.4) \quad \mathcal{H}(I) = \mathcal{M}(I) \cap (-\mathcal{M}(I)).$$

PROOF. Assume that f is a Φ -Hölder function. Then, for any $x, y \in I$ with $x \leq y$, f will satisfy the inequality (2.2) and hence the inequalities

$$(2.5) \quad f(x) - f(y) \leq \Phi(y - x), \quad f(y) - f(x) \leq \Phi(y - x).$$

Rearranging these two inequalities, we have that both f and $-f$ are Φ -monotone. That is $f \in \mathcal{M}_\Phi(I) \cap (-\mathcal{M}_\Phi(I))$.

To show the inverse inclusion, let $f \in \mathcal{M}_\Phi(I) \cap (-\mathcal{M}_\Phi(I))$. Due to the property of Φ -monotonicity of the two classes of function, f will satisfy the two inequalities in (2.5). Hence, inequality (2.2) holds for $x \leq y$. This inequality being symmetric in x and y , we get that (2.2) is satisfied for all $x, y \in I$.

To verify (2.4), let first f be a member of $\mathcal{H}(I)$. Then there exists $\Phi \in \mathcal{E}(I)$ such that $f \in \mathcal{H}_\Phi(I)$. In view of the first part, this implies that

$$f \in \mathcal{M}_\Phi(I) \cap (-\mathcal{M}_\Phi(I)) \subseteq \mathcal{M}(I) \cap (-\mathcal{M}(I)).$$

Thus, we have shown the inclusion \subseteq for (2.4).

For the reversed inclusion, let $f \in \mathcal{M}(I) \cap (-\mathcal{M}(I))$. Then there exist $\Phi_1, \Phi_2 \in \mathcal{E}(I)$ such that $f \in \mathcal{M}_{\Phi_1}(I)$ and $-f \in \mathcal{M}_{\Phi_2}(I)$. Define $\Phi := \max(\Phi_1, \Phi_2)$. Then, obviously, $f \in \mathcal{M}_\Phi(I)$ and $-f \in \mathcal{M}_\Phi(I)$, therefore, $f \in \mathcal{H}_\Phi(I) \subseteq \mathcal{H}(I)$. This completes the proof. \square

We have already discussed about the notions of pointwise supremum and pointwise infimum in the Chapter 1. Here, we will go through some of the closure properties of approximately Hölder function under these operations.

PROPOSITION 2.2.3. ([13]) *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{H}_\Phi(I)$ is closed under pointwise infimum and pointwise supremum. Consequently, $\mathcal{H}_\Phi(I)$ is closed with respect to the pointwise liminf and limsup operations.*

PROOF. Assume that $f : I \rightarrow \mathbb{R}$ is the pointwise supremum of a family $\{f_\gamma \mid \gamma \in \Gamma\} \subseteq \mathcal{H}_\Phi(I)$. By Proposition 2.2.2, we have that $\pm f_\gamma \in \mathcal{M}_\Phi(I)$ holds for all $\gamma \in \Gamma$. In view of Proposition 1.2.2, this implies that

$$f = \sup_{\gamma \in \Gamma} f_\gamma \in \mathcal{M}_\Phi(I) \quad \text{and} \quad -f = \inf_{\gamma \in \Gamma} (-f_\gamma) \in \mathcal{M}_\Phi(I).$$

Therefore, $f \in \mathcal{M}_\Phi(I) \cap (-\mathcal{M}_\Phi(I)) = \mathcal{H}_\Phi(I)$. The proof of the statement for the pointwise infimum is analogous.

The proof for statement concerning liminf and limsup operations is analogous to the of Proposition 1.2.2 and hence omitted. \square

As a trivial corollary, we obtain that if f is Φ -Hölder, then $|f| = \max(f, -f)$ is also Φ -Hölder.

2.3. The Optimal Error Functions

A stronger property than subadditivity of a function $\Phi \in \mathcal{E}(I)$ is the *absolute subadditivity*, which is possessed by a function $\Phi \in \mathcal{E}(I)$ if, for all $u, v \in \mathbb{R}$ with $|u|, |v|, |u + v| < \ell(I)$, the inequality

$$(2.6) \quad \Phi(|u + v|) \leq \Phi(|u|) + \Phi(|v|)$$

is satisfied. It is clear that absolutely subadditive functions are automatically subadditive.

In this section, we are going to study various properties of absolute subadditive functions. We also give precise formula to obtain the optimal absolute subadditive error function for a given error function.

LEMMA 2.3.1. ([13]) *If $\Phi \in \mathcal{E}(I)$ is increasing and subadditive, then it is absolutely subadditive.*

PROOF. Let $u, v \in \mathbb{R}$ with $|u|, |v|, |u+v| < \ell(I)$. If $uv \geq 0$, then $|u| + |v| = |u + v| < \ell(I)$ and the subadditivity implies

$$\Phi(|u + v|) = \Phi(|u| + |v|) \leq \Phi(|u|) + \Phi(|v|).$$

In the case $uv < 0$, one can easily check that $|u + v| \leq \max(|u|, |v|)$. Therefore, the monotonicity property of Φ yields

$$\Phi(|u + v|) \leq \Phi(\max(|u|, |v|)) = \max(\Phi(|u|), \Phi(|v|)) \leq \Phi(|u|) + \Phi(|v|).$$

Thus, we have proved (2.6) in both cases. \square

PROPOSITION 2.3.2. ([13]) *Let $p \in \mathbb{R}$. Then Φ_p is absolutely subadditive on \mathbb{R}_+ if and only if $p \in [0, 1]$.*

PROOF. If $p \in [0, 1]$, then Φ_p is increasing and also subadditive on \mathbb{R}_+ , hence, by Lemma 2.3.1, it is also absolutely subadditive on \mathbb{R}_+ .

If $p > 1$, then Φ_p is not subadditive, hence, it is also not absolutely subadditive. If $p < 0$, then with $u := n + 1$ and $v := -n$, the absolute subadditivity of Φ_p would imply

$$1 = \Phi_p(1) = \Phi_p(|u + v|) \leq \Phi_p(|u|) + \Phi_p(|v|) = (n + 1)^p + n^p$$

for all $n \in \mathbb{N}$. Upon taking the limit $n \rightarrow \infty$ and using $p < 0$, we arrive at the contradiction $1 \leq 0$. Hence Φ_p cannot be absolutely subadditive. \square

It is also not difficult to see that the class of absolutely subadditive functions is closed with respect to pointwise supremum. Therefore, for any $\Phi \in \mathcal{E}(I)$, there exists a largest absolutely subadditive function $\Phi^\alpha \in \mathcal{E}(I)$ which satisfy the inequality $\Phi^\alpha \leq \Phi$ on $[0, \ell(I)[$. The function Φ^α will be called the *absolutely subadditive envelope* (or *absolutely subadditive minorant*) of the function Φ . Obviously, $\Phi = \Phi^\alpha$ is valid if and only if Φ is absolutely subadditive. More generally, the function Φ^α can be constructed explicitly from Φ by the following results.

LEMMA 2.3.3. ([13]) *Assume that $\Phi \in \mathcal{E}(I)$ is absolutely subadditive. Then, for all $n \in \mathbb{N}$, $u_1, \dots, u_n \in \mathbb{R}$ with $|u_1|, \dots, |u_n|, |u_1 + \dots + u_n| < \ell(I)$,*

$$\Phi(|u_1 + \dots + u_n|) \leq \Phi(|u_1|) + \dots + \Phi(|u_n|).$$

PROOF. The statement is trivial for $n = 1$. For $n \geq 2$, we prove the assertion by induction. If $n = 2$, then it is equivalent to the absolute subadditivity of Φ . Let $n \geq 2$ and assume that the statement holds for n variables. Let $u_1, \dots, u_{n+1} \in]-\ell(I), \ell(I)[$ such that $u_0 := u_1 + \dots + u_{n+1} \in]-\ell(I), \ell(I)[$.

We may assume that $u_0 \geq 0$, (otherwise we can replace u_i by $-u_i$ for all $i \in \{1, \dots, n+1\}$ in the argument). Then, for at least one $i \in \{1, \dots, n+1\}$, we have that $u_i \geq 0$. By the symmetry, we may also assume that $u_{n+1} \geq 0$. Then, using the inequalities $u_0 \geq 0$, $u_{n+1} < \ell(I)$, $u_{n+1} \geq 0$ and $u_0 < \ell(I)$, respectively, we get

$$\begin{aligned} -\ell(I) &\leq u_0 - \ell(I) = u_1 + \dots + u_n + (u_{n+1} - \ell(I)) \\ &< u_1 + \dots + u_n \leq u_1 + \dots + u_n + u_{n+1} = u_0 < \ell(I). \end{aligned}$$

This shows that $u_1 + \dots + u_n \in]-\ell(I), \ell(I)[$. Now, applying the inequality (2.6) with $u := u_1 + \dots + u_n$ and $v := u_{n+1}$ and then the inductive assumption, it follows that

$$\begin{aligned} \Phi(|u_1 + \dots + u_{n+1}|) &\leq \Phi(|u_1 + \dots + u_n|) + \Phi(|u_{n+1}|) \\ &\leq \Phi(|u_1|) + \dots + \Phi(|u_n|) + \Phi(|u_{n+1}|). \end{aligned}$$

Therefore, the statement is valid also for $n+1$ variables. \square

PROPOSITION 2.3.4. ([13]) *Let $\Phi \in \mathcal{E}(I)$ be an arbitrary function. Define the function $\Phi^\alpha : [0, \ell(I)[\rightarrow \mathbb{R}_+$ by*

$$\begin{aligned} \Phi^\alpha(u) &:= \inf \{ \Phi(|u_1|) + \dots + \Phi(|u_n|) \mid n \in \mathbb{N}, \\ &\quad |u_1|, \dots, |u_n| < \ell(I), u_1 + \dots + u_n = u \}. \end{aligned}$$

Then Φ^α is the largest absolutely subadditive function which satisfies the inequality $\Phi^\alpha \leq \Phi$ and hence $\Phi^\alpha \leq \Phi^\sigma$ on $[0, \ell(I)[$.

PROOF. First we are going to prove the absolute subadditivity of Φ^α . Let $u, v \in \mathbb{R}$ such that $|u|, |v|, |u+v| < \ell(I)$. Without loss of generality, we may assume that $u+v$ is nonnegative (otherwise, we replace u and v by $(-u)$ and $(-v)$ in the argument below). Let $\varepsilon > 0$ be arbitrary. Then there exist $n, m \in \mathbb{N}$ and real numbers $u_1, \dots, u_n, v_1, \dots, v_m \in]-\ell(I), \ell(I)[$ such that

$$u = \sum_{i=1}^n u_i, \quad \sum_{i=1}^n \Phi(|u_i|) < \Phi^\alpha(|u|) + \frac{\varepsilon}{2}$$

and

$$v = \sum_{j=1}^m v_j, \quad \sum_{j=1}^m \Phi(|v_j|) < \Phi^\alpha(|v|) + \frac{\varepsilon}{2}.$$

We have that $u+v = \sum_{i=1}^n u_i + \sum_{j=1}^m v_j$. Therefore, by the definition of Φ^α and by the last two inequalities, we get

$$\Phi^\alpha(u+v) \leq \sum_{i=1}^n \Phi(|u_i|) + \sum_{j=1}^m \Phi(|v_j|) < \Phi^\alpha(|u|) + \Phi^\alpha(|v|) + \varepsilon.$$

Since ε is an arbitrary positive number, we conclude that $\Phi^\alpha(u + v) \leq \Phi^\alpha(|u|) + \Phi^\alpha(|v|)$, which completes the proof of the absolute subadditivity of Φ^α . By taking $n = 1$ and $u_1 = u$ in the right hand side expression for Φ^α , we can see that $\Phi^\alpha \leq \Phi$ holds.

Now assume that $\Psi \in \mathcal{E}(I)$ is an absolutely subadditive function such that $\Psi \leq \Phi$ holds on $[0, \ell(I)[$. To show that $\Psi \leq \Phi^\alpha$, let $u \in [0, \ell(I)[$ and $\varepsilon > 0$ be arbitrary. Then there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n \in] - \ell(I), \ell(I)[$ such that

$$(2.7) \quad u = u_1 + \dots + u_n \quad \text{and} \quad \Phi(|u_1|) + \dots + \Phi(|u_n|) < \Phi^\alpha(u) + \varepsilon.$$

Then, due to the absolute subadditivity of Ψ and Lemma 2.3.3, we get

$$\Psi(u) \leq \Psi(|u_1|) + \dots + \Psi(|u_n|) \leq \Phi(|u_1|) + \dots + \Phi(|u_n|) < \Phi^\alpha(u) + \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, the inequality $\Psi(u) \leq \Phi^\alpha(u)$ follows for all $u \in [0, \ell(I)[$, which was to be proved.

The function Φ^α being subadditive, the Proposition 1.3.2 implies that $\Phi^\alpha \leq \Phi^\sigma$ also holds. \square

The following corollaries demonstrate cases when the absolutely subadditive envelopes of an error function are the identically zero functions.

COROLLARY 2.3.5. ([13]) *Let $\Phi \in \mathcal{E}(I)$ such that, for all $0 \leq u < \ell(I)$,*

$$(2.8) \quad \inf_{n \in \mathbb{N}} n\Phi\left(\frac{u}{n}\right) = 0.$$

Then $\Phi^\alpha = 0$. In particular, for $p > 1$, $\Phi_p^\alpha = 0$.

PROOF. If (2.8) holds, then, by Corollary 1.3.4, we have that $\Phi^\sigma = 0$. Hence the inequality $\Phi^\alpha \leq \Phi^\sigma$ implies that $\Phi^\alpha = 0$. The last assertion also follows from Corollary 1.3.4. \square

COROLLARY 2.3.6. ([13]) *Let I be an unbounded interval and $\Phi \in \mathcal{E}(I)$ such that*

$$(2.9) \quad \lim_{v \rightarrow \infty} \Phi(v) = 0.$$

Then $\Phi^\alpha = 0$. In particular, for $p < 0$, $\Phi_p^\alpha = 0$.

PROOF. Let $0 < u$. Then, by the construction of Φ^α , for all $v > 0$, we have the inequality

$$\Phi^\alpha(u) \leq \Phi(|u + v|) + \Phi(|-v|) = \Phi(u + v) + \Phi(v).$$

Upon taking the limit $v \rightarrow \infty$, the equality (2.9) yields that $\Phi^\alpha(u) = 0$. \square

The next result shows that for the notions of Φ -Hölder property, the error function Φ can always be replaced by its subadditive and absolutely subadditive envelope, respectively.

THEOREM 2.3.7. ([13]) Let $\Phi \in \mathcal{E}(I)$. Then

$$\mathcal{H}_\Phi(I) = \mathcal{H}_{\Phi^\sigma}(I).$$

Additionally if, I is an unbounded interval, then

$$\mathcal{H}_\Phi(I) = \mathcal{H}_{\Phi^\alpha}(I).$$

PROOF. Using Proposition 2.2.2 and the above verified equality, we get

$$\mathcal{H}_\Phi(I) = \mathcal{M}_\Phi(I) \cap (-\mathcal{M}_\Phi(I)) = \mathcal{M}_{\Phi^\sigma}(I) \cap (-\mathcal{M}_{\Phi^\sigma}(I)) = \mathcal{H}_{\Phi^\sigma}(I).$$

Now assume that I is unbounded. The inclusion $\mathcal{H}_{\Phi^\alpha}(I) \subseteq \mathcal{H}_\Phi(I)$ is a trivial consequence of the inequality $\Phi^\alpha \leq \Phi$. To prove the reversed inclusion, let $f \in \mathcal{H}_\Phi(I)$. To show that f is also Φ^α -Hölder, let $x, y \in I$ and $\varepsilon > 0$ be arbitrary. We may assume that $x < y$. Then $u := y - x < \ell(I)$ and there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n \in]-\ell(I), \ell(I)[$ such that (2.7) holds.

Now we have to distinguish two cases according to the unboundedness of I . Assume first that I is unbounded from below. Then we may assume that $-\ell(I) := u_0 < u_1 \leq \dots \leq u_n < \ell(I)$. In view of (2.7), we have that $u_n > 0$, therefore there exists a unique $k \in \{1, \dots, n\}$ such that $u_{k-1} \leq 0 < u_k$. For the sake of convenience, let

$$x_0 := x, \quad x_i := x + u_1 + \dots + u_i \quad (i \in \{1, \dots, n\}).$$

By the construction of k , it follows that

$$x = x_0 \geq x_1 \geq \dots \geq x_{k-1} < x_k \leq \dots \leq x_n = y.$$

Therefore, $x_i \leq \max(x, y) =$ for all $i \in \{1, \dots, n\}$. Thus the unboundedness of I from below yields that $x_1, \dots, x_n \in I$ hold. Applying the Φ -Hölder property of f , we get that

$$f(x_{i-1}) \leq f(x_i) + \Phi(|x_i - x_{i-1}|) = f(x_i) + \Phi(|u_i|) \quad (i \in \{1, \dots, n\}).$$

Adding up the above inequalities for $i \in \{1, \dots, n\}$ side by side, we obtain that

$$\begin{aligned} f(x) = f(x_0) &\leq f(x_n) + \Phi(|u_1|) + \dots + \Phi(|u_n|) < f(x_n) + \Phi^\alpha(u) + \varepsilon \\ &= f(y) + \Phi^\alpha(y - x) + \varepsilon. \end{aligned}$$

Upon taking the limit $\varepsilon \rightarrow 0$, it follows that

$$(2.10) \quad f(x) \leq f(y) + \Phi^\alpha(|y - x|).$$

In the case when I is unbounded from above, one should take the ordering $-\ell(I) < u_n \leq \dots \leq u_1 < u_0 := \ell(I)$ and use a completely similar argument to obtain inequality (2.10).

Finally, interchanging the roles of x and y in the above proof, we can get

$$f(y) \leq f(x) + \Phi^\alpha(|y - x|),$$

which, together with (2.10), shows that f is Φ^α -Hölder and hence completes the proof. \square

COROLLARY 2.3.8. ([13]) *Let $\Phi \in \mathcal{E}(I)$ such that, for all $0 \leq u < \ell(I)$, (2.8) holds. Then $\mathcal{H}_\Phi(I)$ consists of constant functions.*

PROOF. In view of Corollary 1.3.4, we have that $\Phi^\sigma = \Phi^\alpha \equiv 0$. Combining this with the result of the Theorem 2.3.7, we get that $\mathcal{H}_\Phi(I) = \mathcal{H}_0(I)$, which is equivalent to the statement. \square

The following result demonstrates that the subadditive and increasing error functions optimally determine the corresponding classes of monotone and Hölder functions.

THEOREM 2.3.9. ([13]) *Let $\Phi \in \mathcal{E}_0(I)$ be an increasing and subadditive function and let $\Psi \in \mathcal{E}(I)$ satisfy the inequality $\Psi \leq \Phi$. Then, $\mathcal{H}_\Psi(I) = \mathcal{H}_\Phi(I)$ if and only if $\Psi = \Phi$.*

PROOF. The theorem is the direct consequence of Theorem 1.3.6 and Proposition 2.2.2 \square

2.4. Approximately Hölder Envelopes and Sandwich Type Theorems

In this section we have given the precise formula for the Φ -Hölder minorant and majorant for a given function defined on a unbounded interval. We also derive a sandwich type theorem.

PROPOSITION 2.4.1. ([13]) *Let I be an unbounded interval, $\Phi \in \mathcal{E}_0(I)$ and let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -Hölder minorant. Then the function $\underline{H}_\Phi(f)$ defined by*

$$\underline{H}_\Phi(f)(x) := \inf_{y \in I} (f(y) + \Phi^\alpha(|y - x|)) \quad (x \in I)$$

is real-valued and is the largest Φ -Hölder function which is smaller than or equal to f . Analogously, if f admits a Φ -Hölder majorant, then the function $\overline{H}_\Phi(f)$ defined by

$$\overline{H}_\Phi(f)(x) := \sup_{y \in I} (f(y) - \Phi^\alpha(|x - y|)) \quad (x \in I)$$

is real-valued and is the smallest Φ -Hölder function which is bigger than or equal to f .

PROOF. Obviously, $\underline{H}_\Phi(f)$ cannot take the value $+\infty$ at any point in I , i.e., $\underline{H}_\Phi(f)(x) < \infty$ for all $x \in I$. The condition $\Phi(0) = 0$ implies that $\Phi^\alpha(0) = 0$, therefore, by taking $y = x$ in the defining formula of $\underline{H}_\Phi(f)(x)$, we get that $\underline{H}_\Phi(f)(x) \leq f(x)$ holds for all $x \in I$.

Now suppose g is a Φ -Hölder function such that $g \leq f$ holds (by the assumption, there is at least one such function g). Then, by Theorem 2.3.7, g is also Φ^α -Hölder. In order to show that $g \leq \underline{H}_\Phi(f)$, let $x \in I$ be arbitrarily fixed. Then, for all $y \in I$, we have

$$g(x) \leq g(y) + \Phi^\alpha(|y - x|) \leq f(y) + \Phi^\alpha(|y - x|),$$

Upon taking the infimum of the right hand side with respect to $y \in I$, we get

$$g(x) \leq \inf_y (f(y) + \Phi^\alpha(|y - x|)) = \underline{H}_\Phi(f)(x),$$

which proves the desired inequality $g(x) \leq \underline{H}_\Phi(f)(x)$ and also that $\underline{H}_\Phi(f)$ cannot take the value $-\infty$ at any point of I .

To see that $\underline{H}_\Phi(f)$ itself is Φ -Hölder, it is sufficient to show that $\underline{H}_\Phi(f)$ is Φ^α -Hölder. For any $u, v \in I$, using the absolute subadditivity of Φ^α , we obtain

$$\begin{aligned} \underline{H}_\Phi(f)(u) &= \inf_y (f(y) + \Phi^\alpha(|y - u|)) \\ &\leq \inf_y \left(f(y) + \Phi^\alpha(|(y - v)|) + \Phi^\alpha(|(v - u)|) \right) \\ &= \inf_y \left(f(y) + \Phi^\alpha(|y - v|) \right) + \Phi^\alpha(|v - u|) \\ &= \underline{H}_\Phi(f)(v) + \Phi^\alpha(|v - u|). \end{aligned}$$

In the same pattern as above, interchanging the roles of u and v in the above equation, we will obtain

$$\underline{H}_\Phi(f)(v) \leq \underline{H}_\Phi(f)(u) + \Phi^\alpha(|v - u|),$$

which shows that $\underline{H}_\Phi(f)$ is Φ -Hölder.

The proof of the second assertion is completely similar. \square

COROLLARY 2.4.2. ([13]) *Let I be an unbounded interval, let $\Phi \in \mathcal{E}_0(I)$ and let $g, h : I \rightarrow \mathbb{R}$. Then in order that there exist a Φ -Hölder function $f : I \rightarrow \mathbb{R}$ between g and h it is necessary and sufficient that, for all $x, y \in I$, the inequality*

$$(2.11) \quad g(x) \leq h(y) + \Phi^\alpha(|y - x|)$$

be valid.

PROOF. Assume first that f is a Φ -Hölder function such that $g \leq f \leq h$. Then, f is Φ^α -Hölder and, for all $x, y \in I$, we have

$$g(x) \leq f(x) \leq f(y) + \Phi^\alpha(|y - x|) \leq h(y) + \Phi^\alpha(|y - x|),$$

i.e., (2.11) holds.

Conversely, assume that (2.11) holds true for all $x, y \in I$. For a fixed $x \in I$, define

$$f(x) := \underline{H}_\Phi(h)(x) = \inf_{y \in I} (h(y) + \Phi^\alpha(|y - x|)).$$

Now, in view of inequality (2.11), we have that $g(x) \leq f(x)$. By taking $y = x$ in the definition of f , the conditions $\Phi(0) = 0$ ensures that $f(x) \leq h(x)$ is also valid. Finally, arguing similarly as at the end of the proof of Proposition 2.4.1, it follows that f is Φ^α -Hölder and hence Φ -Hölder as well. \square

Before we formulate and prove the next theorem we shall need the following auxiliary result.

The analogue of Lemma 1.4.3 for the Ψ -Hölder setting is contained in the following lemma.

LEMMA 2.4.3. ([13]) *Let I be an unbounded interval, let $\Phi, \Psi \in \mathcal{E}(I)$ such that $\Phi \circ |\cdot|$ is Ψ -Hölder on \mathbb{R} . Then $\Phi^\alpha \circ |\cdot|$ is also Ψ -Hölder on \mathbb{R} .*

PROOF. To prove this lemma, let $x, y \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. Then by definition of Φ^α , there exist $n \in \mathbb{N}$ and $u_1, \dots, u_n \in \mathbb{R}$ with $y = u_1 + \dots + u_n$ satisfying

$$(2.12) \quad \Phi(|u_1|) + \dots + \Phi(|u_n|) < \Phi^\alpha(|y|) + \varepsilon.$$

Using the Ψ -Hölder property of $\Phi \circ |\cdot|$, we have

$$\Phi(|u_n + (x - y)|) \leq \Phi(|u_n|) + \Psi(|y - x|).$$

Observe that $x = u_1 + \dots + u_{n-1} + (u_n + (x - y))$. Thus, using the inequality in (2.12), we arrive at

$$\begin{aligned} \Phi^\alpha(|x|) &\leq \Phi(|u_1|) + \dots + \Phi(|u_{n-1}|) + \Phi(|u_n + (x - y)|) \\ &\leq \Phi(|u_1|) + \dots + \Phi(|u_{n-1}|) + \Phi(|u_n|) + \Psi(|y - x|) \\ &< \Phi^\alpha(|y|) + \Psi(|y - x|) + \varepsilon. \end{aligned}$$

As ε is an arbitrary positive number, we can conclude that $\Phi^\alpha \circ |\cdot|$ is Ψ -Hölder. \square

REMARK 2.4.4. The property that $\Phi \circ |\cdot|$ is Ψ -Hölder on \mathbb{R} means that, for all $x, y \in \mathbb{R}$,

$$\Phi(|x|) \leq \Phi(|y|) + \Psi(|y - x|).$$

Denoting $|x|$ and $|y|$ by u and v , respectively, the above inequality implies, for all $u, v \geq 0$,

$$\Phi(u) \leq \Phi(v) + \min(\Psi(|v - u|), \Psi(u + v)).$$

Conversely, one can see that the Ψ -Hölder property of the function $\Phi \circ |\cdot|$ is a consequence of the last inequality. Therefore, if Ψ is increasing, then $\Phi \circ |\cdot|$ is Ψ -Hölder if and only if Φ is Ψ -Hölder.

THEOREM 2.4.5. ([13]) *Let I be an unbounded interval, $\Phi, \Psi \in \mathcal{E}(I)$ such that $\Phi \circ |\cdot|$ is Ψ -Hölder on \mathbb{R} . Then $f : I \rightarrow \mathbb{R}$ is Φ -Hölder if and only if there exist two Ψ -Hölder functions $f_*, f^* : I \rightarrow \mathbb{R}$ such that $f_* \leq f \leq f^*$ holds on I and, for all $x, y \in I$ with $x \neq y$*

(2.13)

$$f(x) \leq f_*(y) + \Phi^\alpha(|y - x|) \quad \text{and} \quad f^*(x) \leq f(y) + \Phi^\alpha(|y - x|).$$

Additionally, for all $x \in I$,

$$(2.14) \quad f^*(x) - f_*(x) \leq \inf_{y \in I} 2\Phi(|y - x|).$$

PROOF. First assume that f is Φ -Hölder. Then, by Theorem 2.3.7, f is also Φ^α -Hölder. For a fixed point $x \in I$, define

$$f_*(x) := \sup_{u \in I} (f(u) - \Phi^\alpha(|u - x|))$$

and

$$f^*(x) := \inf_{u \in I} (f(u) + \Phi^\alpha(|u - x|)).$$

In view of the Φ^α -Hölder property of f , for all $u, x \in I$, we have that

$$f(u) - \Phi^\alpha(|u - x|) \leq f(x) \quad \text{and} \quad f(x) \leq f(u) + \Phi^\alpha(|u - x|).$$

Therefore, upon taking the supremum and infimum for all $u \in I$, we get that $f_*(x) \leq f(x)$ and $f(x) \leq f^*(x)$, respectively. That is, f_* and f^* are real valued functions and $f_* \leq f \leq f^*$ is satisfied on I . In the next step, we establish that f_* and f^* are Ψ -Hölder.

In view of Lemma 2.4.3, the Ψ -Hölder property of $\Phi \circ |\cdot|$ yields that $\Phi^\alpha \circ |\cdot|$ is also Ψ -Hölder. Therefore, for all $u, x, y \in I$, we have

$$\begin{aligned} -\Phi^\alpha(|x - u|) &\leq -\Phi^\alpha(|y - u|) + \Psi(|(y - u) - (x - u)|) \\ &\leq -\Phi^\alpha(|u - y|) + \Psi(|y - x|). \end{aligned}$$

Applying this inequality to the definition of f_* , we arrive at

$$\begin{aligned} f_*(x) &= \sup_{u \in I} (f(u) - \Phi^\alpha(|u - x|)) \\ &\leq \sup_{u \in I} (f(u) - \Phi^\alpha(|u - y|)) + \Psi(|y - x|) \\ &= f_*(y) + \Psi(|y - x|), \end{aligned}$$

which shows that f_* is Ψ -Hölder.

Similarly, by the Ψ -Hölder property of $\Phi^\alpha \circ |\cdot|$, for all $u, x, y \in I$, we obtain

$$\begin{aligned}\Phi^\alpha(|u - x|) &\leq \Phi^\alpha(|u - y|) + \Psi(|(u - y) - (u - x)|) \\ &= \Phi^\alpha(|u - y|) + \Psi(|y - x|).\end{aligned}$$

Applying this inequality to the definition of f^* , it follows that

$$\begin{aligned}f^*(x) &= \inf_{u \in I} \left(f(u) + \Phi^\alpha(|u - x|) \right) \\ &\leq \inf_{u \in I} \left(f(u) + \Phi^\alpha(|u - y|) \right) + \Psi(|y - x|) \\ &= f^*(y) + \Psi(|y - x|),\end{aligned}$$

which proves that f^* is also Ψ -Hölder.

Next we prove that f_* and f^* satisfy the inequalities stated in (2.13). Indeed, for $x, y \in I$, from the definitions of f_* and f^* , we obtain the inequalities

$$f(x) - \Phi^\alpha(|y - x|) \leq f_*(y) \quad \text{and} \quad f^*(x) \leq f(y) + \Phi^\alpha(|y - x|),$$

respectively. Conversely, if the first inequality in (2.13) holds for some function $f_* : I \rightarrow \mathbb{R}$ satisfying $f_* \leq f$, then $f(x) \leq f_*(y) + \Phi^\alpha(|y - x|) \leq f(y) + \Phi^\alpha(|y - x|)$, which shows that f is Φ^α -Hölder. Similarly, the existence of a function $f^* : I \rightarrow \mathbb{R}$ satisfying $f \leq f^*$ and the second inequality of (2.13), also implies that f is Φ^α -Hölder.

Finally, to obtain the last inequality (2.14) of Theorem 2.4.5, we interchange x and y in the first inequality of (2.13) and we obtain

$$-f_*(x) \leq -f(y) + \Phi(|y - x|)$$

By summing up this inequality with the second inequality of (2.13) side by side, we will reach at our desired conclusion. \square

2.5. Characterization of Approximately Hölder Functions

In this section, using the the notions of upper and lower interpolations, we establish a characterization for Φ -Hölder functions. This allows one to construct Φ -Hölder functions from elementary ones, which could be termed the building blocks for those classes.

If $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ and $p \in I$, then we say that f can be interpolated at p by a Φ -Hölder function from below [resp. from above] if there exists a Φ -Hölder function $h : I \rightarrow \mathbb{R}$ such that $h(p) = f(p)$ and $h \leq f$ [resp. $f \leq h$].

In what follows, given an error function $\Phi \in \mathcal{E}(I)$ and $p \in I$, we define the function $\Phi_p : I \rightarrow \mathbb{R}$ by

$$(2.15) \quad \Phi_p(x) := \Phi(|x - p|) \quad (x \in I).$$

PROPOSITION 2.5.1. ([14]) *Let $\Phi \in \mathcal{E}_0(I)$ be an absolutely subadditive function. Then, for all $p \in I$, the function Φ_p is Φ -Hölder on I .*

PROOF. Let $p \in I$ and let $x, y \in I$. Then, using the absolute subadditivity of Φ , it immediately follows that

$$|\Phi_p(x) - \Phi_p(y)| = |\Phi(|x - p|) - \Phi(|y - p|)| \leq \Phi(|y - x|).$$

This shows that Φ_p is Φ -Hölder on I . □

PROPOSITION 2.5.2. ([14]) *Let $\Phi \in \mathcal{E}_0(I)$ be an absolutely subadditive function, let $f : I \rightarrow \mathbb{R}$ and $p \in I$ be fixed. Then f can be interpolated at p by a Φ -Hölder function from below if and only if, for all $x \in I$,*

$$(2.16) \quad f(p) \leq f(x) + \Phi(|x - p|).$$

Analogously, f can be interpolated at p by a Φ -Hölder function from above if and only if, for all $x \in I$,

$$(2.17) \quad f(x) \leq f(p) + \Phi(|x - p|).$$

PROOF. Suppose first that there exists a Φ -Hölder function $h : I \rightarrow \mathbb{R}$ such that $h(p) = f(p)$ and $h \leq f$. Let $x \in I$ be arbitrary. Then

$$f(p) = h(p) \leq h(x) + \Phi(|x - p|) \leq f(x) + \Phi(|x - p|)$$

verifying condition (2.16).

To prove the sufficiency, assume that (2.16) holds for all $x \in I$ and define $h := f(p) - \Phi_p$. According to Proposition 2.5.1, it follows that h is Φ -Hölder. On the other hand, the condition $\Phi(0) = 0$ implies that $h(p) = f(p)$.

It remains to show that $h \leq f$. Indeed, if $x \in I$, then (2.16) implies $h(x) = f(p) - \Phi(|x - p|) \leq f(x)$ and the proof is completed.

The verification of the second assertion is analogous. □

THEOREM 2.5.3. ([14]) *Let $\Phi \in \mathcal{E}_0(I)$ be absolutely subadditive function and let $f : I \rightarrow \mathbb{R}$. Then the following assertions are equivalent to each other:*

- (i) *f is Φ -Hölder.*
- (ii) *For every $p \in I$, the functions $f(p) - \Phi_p$ and $f(p) + \Phi_p$ are Φ -Hölder interpolations of f at p from below and above, respectively.*
- (iii) *For every $p \in I$, f possesses a Φ -Hölder interpolation from below.*
- (iv) *For every $p \in I$, f possesses a Φ -Hölder interpolation from above.*

PROOF. (i) \Rightarrow (ii) Let f be a Φ -Hölder function. Then, trivially, f possesses Φ -Hölder interpolation at p from below and from above. Thus, by Proposition 2.5.2, the inequalities (2.16) and (2.17) are valid. Therefore we

have that $f(p) - \Phi_p \leq f \leq f(p) + \Phi_p$. On the other hand, these two functions are Φ -Hölder and interpolate f at p . This proves (ii).

The implication (ii) \Rightarrow (iii) and (ii) \Rightarrow (iv) are obvious.

The proof of the implications (iii) \Rightarrow (i) and (iv) \Rightarrow (i) is based on a result of our previous paper [13] which states that the family of Φ -Hölder functions is closed under pointwise supremum and infimum. If we assume that f admits a Φ -Hölder interpolation from below (resp. from above) at every point $p \in I$, then f is the pointwise supremum (resp. infimum) of a family of Φ -Hölder functions, and therefore, f itself is a Φ -Hölder function. \square

In the next chapter, we will see the relationship of Φ -monotone and Φ -Hölder functions with various classical results.

CHAPTER 3

Generalization of Some Classical Results

3.1. Preliminaries

The classical Jordan decomposition theorem yields that "Every function of bounded variation defined on a closed interval can be decomposed into the difference of two nondecreasing functions." In the upcoming section we aim to obtain a similar kind of result by defining a new type of variation while dealing with Φ -monotone functions.

Let $f : I \rightarrow \mathbb{R}$ be a convex function. Then for any $a, b \in I$ with $a \neq b$, the following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}$$

holds, which is termed the Hermite–Hadamard inequality (cf. [17, 18]).

On the other hand suppose, f is differentiable in the interior of I with a locally bounded derivative. Then f satisfies Ostrowski's inequality (cf. [43]), i.e., for any $[a, b] \subseteq I$ with arbitrary $x \in [a, b]$, the following inequality holds

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_{[a,b]}.$$

Below we derive analogous inequalities in terms of Φ -monotone and Φ -affine functions.

3.2. Jordan-type Decomposition of Functions With Bounded Φ -Variation

Let $\Phi \in \mathcal{E}(I)$. Then a function $f : I \rightarrow \mathbb{R}$ is called *delta- Φ -monotone* if it is the difference of two Φ -monotone functions. In what follows, we shall extend the celebrated Jordan Decomposition Theorem for delta- Φ -monotone functions. For this purpose, we have to extend the notion of total variation to this more general setting.

Let $[a, b] \subseteq I$ and let $\tau = (t_0, \dots, t_n)$ be a partition of the interval $[a, b]$ (i.e., $a = t_0 < t_1 < \dots < t_n = b$). Then the Φ -variation of f with respect to

τ is defined by

$$V^\Phi(f; \tau) := \sum_{i=1}^n (|f(t_i) - f(t_{i-1})| - \Phi(t_i - t_{i-1})).$$

Finally, the *total Φ -variation of f on the interval $[a, b]$* is defined by

$$V_{[a,b]}^\Phi f := \sup\{V^\Phi(f; \tau) \mid \tau \text{ is a partition of } [a, b]\}.$$

LEMMA 3.2.1. ([13]) *Let $\Phi \in \mathcal{E}(I)$. Then, for all $f : I \rightarrow \mathbb{R}$ and $a < b < c$ in I , we have*

$$(3.1) \quad V_{[a,b]}^\Phi f + V_{[b,c]}^\Phi f \leq V_{[a,c]}^\Phi f.$$

PROOF. Let

$$u < V_{[a,b]}^\Phi f \quad \text{and} \quad v < V_{[b,c]}^\Phi f$$

be arbitrary real numbers. Then there exist a partition $\tau = (t_0, \dots, t_n)$ of $[a, b]$ and partition $\sigma = (s_0, \dots, s_m)$ of $[b, c]$ such that

$$u < \sum_{i=1}^n (|f(t_i) - f(t_{i-1})| - \Phi(t_i - t_{i-1}))$$

and

$$v < \sum_{j=1}^m (|f(s_j) - f(s_{j-1})| - \Phi(s_j - s_{j-1})).$$

Observe that $\tau \cup \sigma := (t_0, \dots, t_n = b = s_0, \dots, s_m)$ is a partition of the interval $[a, c]$. Therefore, adding the above inequalities side by side, we get

$$u + v < V^\Phi(f; \tau \cup \sigma) \leq V_{[a,c]}^\Phi f.$$

Using the arbitrariness of u and v , it follows that (3.1) holds. \square

Our first result characterizes those functions whose total Φ -variation is nonpositive on every subinterval of I .

THEOREM 3.2.2. ([13]) *Let $\Phi \in \mathcal{E}(I)$. Then $V_{[a,b]}^\Phi f \leq 0$ holds for all $a < b$ in I if and only if f is a Φ -Hölder function.*

PROOF. Assume first that f is a Φ -Hölder function and let $a < b$ in I . Then, for any partition $\tau = (t_0, \dots, t_n)$ of $[a, b]$, the Φ -Hölder property of f yields

$$|f(t_i) - f(t_{i-1})| - \Phi(t_i - t_{i-1}) \leq 0 \quad (i \in \{1, \dots, n\}).$$

After summation, this results that $V^\Phi(f; \tau) \leq 0$ for all partition τ and hence $V_{[a,b]}^\Phi f \leq 0$.

Now assume that, for all $a < b$ in I , $V_{[a,b]}^\Phi f \leq 0$. Then $V^\Phi(f; \tau) \leq 0$, where τ is the trivial partition $t_0 = a, t_1 = b$. Therefore,

$$|f(b) - f(a)| - \Phi(b - a) \leq 0.$$

This shows that f is Φ -Hölder, indeed. \square

The main results of this section are as follows.

THEOREM 3.2.3. ([13]) *Let $\Phi, \Psi \in \mathcal{E}(I)$. If $f : I \rightarrow \mathbb{R}$ is the difference of a Φ -monotone and a Ψ -monotone functions, then the total $2 \max(\Phi, \Psi)$ -variation of f is finite on every compact subinterval of I .*

PROOF. Assume that $f = g - h$, where $g : I \rightarrow \mathbb{R}$ is Φ -monotone and $h : I \rightarrow \mathbb{R}$ is Ψ -monotone. Let $[a, b] \subseteq I$ and let $\tau = (t_0, \dots, t_n)$ be a partition of $[a, b]$. Then, by the monotonicity properties of g and h , for all $i \in \{1, \dots, n\}$, we have

$$g(t_i) - g(t_{i-1}) + \Phi(t_i - t_{i-1}) \geq 0$$

and

$$h(t_i) - h(t_{i-1}) + \Psi(t_i - t_{i-1}) \geq 0.$$

Therefore, by the triangle inequality,

$$\begin{aligned} & |f(t_i) - f(t_{i-1})| - 2 \max(\Phi, \Psi)(t_i - t_{i-1}) \\ &= \left| [g(t_i) - g(t_{i-1}) + \Phi(t_i - t_{i-1})] - [h(t_i) - h(t_{i-1}) \right. \\ &\quad \left. + \Psi(t_i - t_{i-1})] + (\Psi - \Phi)(t_i - t_{i-1}) \right| - 2 \max(\Phi, \Psi)(t_i - t_{i-1}) \\ &\leq [g(t_i) - g(t_{i-1}) + \Phi(t_i - t_{i-1})] + [h(t_i) - h(t_{i-1}) \\ &\quad + \Psi(t_i - t_{i-1})] + |\Psi - \Phi|(t_i - t_{i-1}) - 2 \max(\Phi, \Psi)(t_i - t_{i-1}) \\ &= g(t_i) - g(t_{i-1}) + h(t_i) - h(t_{i-1}) + (\Phi + \Psi + |\Psi - \Phi| \\ &\quad - 2 \max(\Phi, \Psi))(t_i - t_{i-1}) \\ &= g(t_i) - g(t_{i-1}) + h(t_i) - h(t_{i-1}). \end{aligned}$$

Summing up these inequalities side by side for $i \in \{1, \dots, n\}$, we obtain

$$V^{2 \max(\Phi, \Psi)}(f; \tau) \leq g(b) - g(a) + h(b) - h(a).$$

Upon taking the supremum with respect to all partitions τ of $[a, b]$, it follows that

$$V_{[a,b]}^{2 \max(\Phi, \Psi)} f \leq g(b) - g(a) + h(b) - h(a) < \infty.$$

Hence f is of bounded $2 \max(\Phi, \Psi)$ -total variation on $[a, b]$. \square

The particular case $\Phi = \Psi$ of the above result yields the following statement.

COROLLARY 3.2.4. ([13]) *Let $\Phi \in \mathcal{E}(I)$. If $f : I \rightarrow \mathbb{R}$ is a delta- Φ -monotone function, then the total 2Φ -variation of f is finite on every compact subinterval of I .*

THEOREM 3.2.5. ([13]) *Let $\Phi \in \mathcal{E}(I)$ let $f : I \rightarrow \mathbb{R}$ such that the total 2Φ -variation of f on is finite on every compact subinterval of I . Then, for all $a \in I$, f is a delta- Φ -monotone function on $I \cap]a, \infty[$.*

PROOF. Assume that the total 2Φ -variation of f on every compact subinterval of I is finite.

Let $a \in I$ be an arbitrarily fixed point and, for $x \in I$, $x > a$, define

$$g(x) := \frac{1}{2}(V_{[a,x]}^{2\Phi}f + f(x)) \quad \text{and} \quad h(x) := \frac{1}{2}(V_{[a,x]}^{2\Phi}f - f(x)).$$

Then, we immediately have that $f = g - h$.

Then, based on the Lemma 3.2.1, for $a < x < y$, we get

$$\begin{aligned} V_{[a,x]}^{2\Phi}f + f(x) - f(y) - 2\Phi(y - x) &\leq V_{[a,x]}^{2\Phi}f + |f(x) - f(y)| - 2\Phi(y - x) \\ &\leq V_{[a,x]}^{2\Phi}f + V_{[x,y]}^{2\Phi}f \leq V_{[a,y]}^{2\Phi}f. \end{aligned}$$

Rearranging this inequality, it follows that

$$g(x) \leq g(y) + \Phi(y - x),$$

which proves that g is Φ -monotone. Similarly, we can see that h is also Φ -monotone. This, together with the identity $f = g - h$ show that f is delta- Φ -monotone function on $I \cap]a, \infty[$. \square

3.3. Hermite–Hadamard-type Inequalities

In the sequel, a function defined on an interval will be called locally integrable if it has a finite Lebesgue integral over every compact subinterval of its domain.

For the description of our subsequent results, we now introduce the following notation and terminology: If $a, b \in I$ then the convex hull of $\{a, b\}$, i.e., the smallest interval containing a and b , will denoted by $\langle a, b \rangle$. If, additionally, $f : \langle a, b \rangle \rightarrow \mathbb{R}$ is Lebesgue integrable, then the integral average of f over $\langle a, b \rangle$ is defined by

$$(3.2) \quad \mathcal{A}(f, \langle a, b \rangle) := \int_0^1 f(ta + (1 - t)b)dt.$$

One can easily see that the following equality holds:

$$(3.3) \quad \langle a, b \rangle := \begin{cases} [a, b] & \text{if } a < b, \\ \{a\} & \text{if } a = b, \\ [b, a] & \text{if } a > b, \end{cases} \quad \text{and } \mathcal{A}(f, \langle a, b \rangle) = \begin{cases} \frac{1}{b-a} \int_a^b f & \text{if } a < b, \\ f(a) & \text{if } a = b, \\ \frac{1}{a-b} \int_b^a f & \text{if } a > b. \end{cases}$$

The inequalities stated in the following results summarize both Hermite–Hadamard and Ostrowski-type inequalities for the Φ -monotone as well as for the Φ -Hölder settings.

THEOREM 3.3.1. ([14]) *Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$ be locally Lebesgue integrable functions. If f is Φ -monotone, then, for all $u, v, w, z \in I$ with $u \leq w$ and $v \leq z$,*

$$(3.4) \quad \mathcal{A}(f, \langle u, v \rangle) \leq \mathcal{A}(f, \langle w, z \rangle) + \mathcal{A}(\Phi, \langle w - u, z - v \rangle).$$

If f is Φ -Hölder, then, for all $u, v, w, z \in I$,

$$(3.5) \quad |\mathcal{A}(f, \langle u, v \rangle) - \mathcal{A}(f, \langle w, z \rangle)| \leq \mathcal{A}(\Phi \circ |\cdot|, \langle w - u, z - v \rangle).$$

PROOF. Assume first that f is Φ -monotone. Then the inequalities $u \leq w$ and $v \leq z$ imply $tu + (1-t)v \leq tw + (1-t)z$ for $t \in [0, 1]$, hence

$$f(tu + (1-t)v) \leq f(tw + (1-t)z) + \Phi(t(w-u) + (1-t)(z-v)).$$

Integrating with respect to t over $[0, 1]$, we get that (3.4) holds.

In the case when f is Φ -Hölder, for $t \in [0, 1]$, we get

$$f(tu + (1-t)v) - f(tw + (1-t)z) \leq \Phi(|t(w-u) + (1-t)(z-v)|),$$

$$f(tw + (1-t)z) - f(tu + (1-t)v) \leq \Phi(|t(w-u) + (1-t)(z-v)|).$$

Integrating both inequalities with respect to t over $[0, 1]$, we get that (3.5) holds. \square

Assuming Φ -monotonicity, we deduce a monotonicity type integral inequality which we will call the *lower and upper Hermite–Hadamard inequalities for Φ -monotone functions*.

THEOREM 3.3.2. ([14]) *Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$ be a Φ -monotone. Assume that both functions are locally Lebesgue integrable. Then, for every $x < y$ in I , the following two inequalities hold:*

$$(3.6) \quad f(x) - \frac{1}{y-x} \int_0^{y-x} \Phi \leq \frac{1}{y-x} \int_x^y f \leq f(y) + \frac{1}{y-x} \int_0^{y-x} \Phi.$$

Furthermore, if Φ is subadditive and nondecreasing, then, for all $x < y$ in I ,

$$(3.7) \quad \begin{aligned} \sup_{f \in \mathcal{M}_{\Phi}(I)} \left(f(x) - \frac{1}{y-x} \int_x^y f \right) &= \sup_{f \in \mathcal{M}_{\Phi}(I)} \left(\frac{1}{y-x} \int_x^y f - f(y) \right) \\ &= \frac{1}{y-x} \int_0^{y-x} \Phi. \end{aligned}$$

PROOF. Let $x < y$. Then the left and right hand side inequalities in (3.6) follow from (3.4) and (3.3) by taking the particular cases $u = v = w = x$, $z = y$ and $u = x$, $v = w = z = y$, respectively.

Now assume that Φ is subadditive and nondecreasing. In view of (3.6), it is clear that both supremums are not bigger than the right hand side of (3.7). To prove the equalities in (3.7), let $x < y$ in I . Then let \underline{f} and \bar{f} be defined by

$$\underline{f}(u) := \begin{cases} 0 & \text{if } u \leq x, \\ -\Phi(u-x) & \text{if } x < u, \end{cases}$$

and

$$\bar{f}(u) := \begin{cases} \Phi(y-u) & \text{if } u < y, \\ 0 & \text{if } y \leq u. \end{cases}$$

By Proposition 1.5.1, we can obtain that both \underline{f} and \bar{f} are Φ -monotone. On the other hand, we have

$$\begin{aligned} \sup_{f \in \mathcal{M}_{\Phi}(I)} \left(f(x) - \frac{1}{y-x} \int_x^y f \right) &\geq \underline{f}(x) - \frac{1}{y-x} \int_x^y \underline{f} \\ &= \frac{1}{y-x} \int_x^y \Phi(u-x) du = \frac{1}{y-x} \int_0^{y-x} \Phi \end{aligned}$$

and

$$\begin{aligned} \sup_{f \in \mathcal{M}_{\Phi}(I)} \left(\frac{1}{y-x} \int_x^y f - f(y) \right) &\geq \frac{1}{y-x} \int_x^y \bar{f} - \bar{f}(y) \\ &= \frac{1}{y-x} \int_x^y \Phi(y-u) du = \frac{1}{y-x} \int_0^{y-x} \Phi. \end{aligned}$$

These two inequalities together with their reverses imply that the equality (3.7) holds. \square

COROLLARY 3.3.3. ([14]) *Let $p \in [0, 1]$, $c \in [0, \infty[$ and $f : I \rightarrow \mathbb{R}$ be a $c(\cdot)^p$ -monotone locally Lebesgue integrable function. Then, for every $x < y$ in I , the following two inequalities hold:*

$$f(x) - \frac{c}{p+1}(y-x)^p \leq \frac{1}{y-x} \int_x^y f \leq f(y) + \frac{c}{p+1}(y-x)^p.$$

Furthermore, for all $x < y$ in I ,

$$\begin{aligned} \sup_{f \in \mathcal{M}_\Phi(I)} \left(f(x) - \frac{1}{y-x} \int_x^y f \right) &= \sup_{f \in \mathcal{M}_\Phi(I)} \left(\frac{1}{y-x} \int_x^y f - f(y) \right) \\ &= \frac{c}{p+1} (y-x)^p. \end{aligned}$$

PROOF. Apply the previous statement for the error function $\Phi \in \mathcal{E}(\mathbb{R}_+)$ given by $\Phi(t) := ct^p$. This error function is subadditive and nondecreasing, therefore, the second part of the theorem can also be applied. \square

LEMMA 3.3.4. ([14]) Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by

$$(3.8) \quad \Phi(u) = \Psi(u) + \int_0^u \frac{\Psi(t)}{t} dt \quad (u \in]0, \ell(I)[).$$

Then Φ is locally integrable and satisfies the following equation:

$$(3.9) \quad \Psi(u) + \frac{1}{u} \int_0^u \Phi = \Phi(u) \quad (u \in]0, \ell(I)[).$$

PROOF. By the assumption, we also have that Ψ is locally integrable and hence Φ defined by (3.8) is also locally integrable. Let $u \in]0, \ell(I)[$ be fixed. Then, by Fubini's theorem, we obtain

$$\begin{aligned} \Psi(u) + \frac{1}{u} \int_0^u \Phi(s) ds &= \Psi(u) + \frac{1}{u} \int_0^u \left(\Psi(s) + \int_0^s \frac{\Psi(t)}{t} dt \right) ds \\ &= \Psi(u) + \frac{1}{u} \int_0^u \Psi(s) ds + \frac{1}{u} \int_0^u \int_0^s \frac{\Psi(t)}{t} dt ds \\ &= \Psi(u) + \frac{1}{u} \int_0^u \Psi(s) ds + \frac{1}{u} \int_0^u \int_t^u \frac{\Psi(t)}{t} ds dt \\ &= \Psi(u) + \frac{1}{u} \int_0^u \Psi(s) ds + \frac{1}{u} \int_0^u (u-t) \frac{\Psi(t)}{t} dt \\ &= \Psi(u) + \frac{1}{u} \int_0^u \Psi(s) ds + \int_0^u \frac{\Psi(t)}{t} dt - \frac{1}{u} \int_0^u \Psi(t) dt \\ &= \Psi(u) + \int_0^u \frac{\Psi(t)}{t} dt = \Phi(u). \end{aligned}$$

This completes the proof of (3.9). \square

THEOREM 3.3.5. ([14]) Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (3.8). If $f : I \rightarrow \mathbb{R}$ is

an upper semicontinuous solution of

$$(3.10) \quad f(u) \leq \frac{1}{v-u} \int_u^v f + \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -monotone on I .

PROOF. Let $f : I \rightarrow \mathbb{R}$ be an upper semicontinuous solution of (3.10). Then f is Lebesgue measurable. Therefore f is upper bounded on any compact subinterval $[u, v]$ of I and hence the Lebesgue integral of f exists and it cannot be $+\infty$. On the other hand, (3.10) shows that $(v-u)(f(u) - \Psi(v-u))$ is a lower bound for the integral of f , thus f has a finite Lebesgue integral over $[u, v]$.

To prove that f is Φ -monotone, let $x, y \in I$ be fixed with $x < y$. Let $USC([x, y])$ denote the family of upper semicontinuous Lebesgue integrable functions and, for $g \in USC([x, y])$, define

$$(Tg)(u) := \begin{cases} \frac{1}{y-u} \int_u^y g + \Psi(y-u) & \text{if } u \in [x, y[, \\ g(y) & \text{if } u = y. \end{cases}$$

Then T is a monotone and affine operator which maps $USC([x, y])$ into itself.

Observe that inequality (3.10) for $v = y$ gives that $f \leq Tf$ holds on $[x, y]$. By the monotonicity of T , it follows that $Tf \leq T(Tf) = T^2f$. By induction, this yields that $T^{n-1}f \leq T^n f$ for all $n \in \mathbb{N}$. Therefore, the sequence $T^n f$ is increasing.

Define $F : [x, y] \rightarrow \mathbb{R}$ by

$$F(u) := \sup_{[u, y]} f.$$

By the upper semicontinuity of f , we have that F is finite.

Let $v \in]x, y[$ be arbitrarily fixed. We prove by induction, for all $n \in \mathbb{N} \cup \{0\}$, that

$$(3.11) \quad T^n f(u) \leq \begin{cases} \left(\frac{v-x}{y-x}\right)^n (F(x) - F(v)) + F(v) + \Phi(y-u) & \text{if } u \in [x, v[, \\ F(v) + \Phi(y-u) & \text{if } u \in [v, y]. \end{cases}$$

If $n = 0$, then $T^n f(u) = f(u)$ and (3.11) simplifies to

$$f(u) \leq \begin{cases} F(x) + \Phi(y-u) & \text{if } u \in [x, v[, \\ F(v) + \Phi(y-u) & \text{if } u \in [v, y], \end{cases}$$

which follows from the definition of F and the nonnegativity of Φ .

Now assume that (3.11) holds for some n . For $u \in [x, v[$, using the first and second inequalities in (3.11) on the intervals $[u, v]$ and $[v, y]$, respectively, and finally the assertion of Lemma 3.3.4, we get

$$\begin{aligned}
(T^{n+1}f)(u) &= \frac{1}{y-u} \int_u^y T^n f + \Psi(y-u) \\
&= \frac{1}{y-u} \left(\int_u^v T^n f + \int_v^y T^n f \right) + \Psi(y-u) \\
&\leq \frac{v-u}{y-u} \left(\left(\frac{v-x}{y-x} \right)^n (F(x) - F(v)) + F(v) \right) + \frac{1}{y-u} \int_u^v \Phi(y-t) dt \\
&\quad + \frac{y-v}{y-u} F(v) + \frac{1}{y-u} \int_v^y \Phi(y-t) dt + \Psi(y-u) \\
&= \frac{v-u}{y-u} \left(\frac{v-x}{y-x} \right)^n (F(x) - F(v)) + F(v) \\
&\quad + \frac{1}{y-u} \int_u^y \Phi(y-t) dt + \Psi(y-u) \\
&\leq \left(\frac{v-x}{y-x} \right)^{n+1} (F(x) - F(v)) + F(v) + \frac{1}{y-u} \int_0^{y-u} \Phi + \Psi(y-u) \\
&= \left(\frac{v-x}{y-x} \right)^{n+1} (F(x) - F(v)) + F(v) + \Phi(y-u).
\end{aligned}$$

If $u \in [v, y[$, then $[u, y] \subseteq [v, y]$ and hence we shall only need the second inequality for $T^n f$ on $[u, y]$. Using this estimate and the assertion of Lemma 3.3.4, we get

$$\begin{aligned}
(T^{n+1}f)(u) &= \frac{1}{y-u} \int_u^y T^n f + \Psi(y-u) \\
&\leq F(v) + \frac{1}{y-u} \int_u^y \Phi(y-t) dt + \Psi(y-u) \\
&= F(v) + \frac{1}{y-u} \int_0^{y-u} \Phi + \Psi(y-u) = F(v) + \Phi(y-u).
\end{aligned}$$

and the trivial inequality $(T^{n+1}f)(y) = f(y) \leq F(v) + \Phi(y-u)$ shows the desired inequality is true for $u = y$.

In view of the monotonicity of the sequence $T^n f$, we have that $f \leq T^n f$ for all $n \in \mathbb{N}$. Thus, (3.11) implies, for all $n \in \mathbb{N}$, that

$$f(u) \leq \begin{cases} \left(\frac{v-x}{y-x}\right)^n (F(x) - F(v)) + F(v) + \Phi(y-u) & \text{if } u \in [x, v[, \\ F(v) + \Phi(y-u) & \text{if } u \in [v, y]. \end{cases}$$

Upon taking the limit $n \rightarrow \infty$, the above inequality yields, for all $v \in [x, y[$, that

$$f(u) \leq F(v) + \Phi(y-u) = \sup_{[v, y]} f + \Phi(y-u) \quad u \in [x, y].$$

Now, taking the limit $v \rightarrow y$, the upper semicontinuity of f yields, that

$$f(u) \leq f(y) + \Phi(y-u) \quad u \in [x, y].$$

In particular, this inequality holds for $u = x$, which completes the proof of the Φ -monotonicity of f . \square

The following result is a counterpart of Theorem 3.3.5. It can be proved directly in an analogous way, however, we will deduce it from this theorem using a sign transformation.

THEOREM 3.3.6. (**[14]**) *Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (3.8). If $f : I \rightarrow \mathbb{R}$ is a lower semicontinuous solution of*

$$(3.12) \quad \frac{1}{v-u} \int_u^v f \leq f(v) + \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -monotone on I .

PROOF. Assume that $f : I \rightarrow \mathbb{R}$ satisfies (3.12). Define $g(x) := -f(-x)$ for $x \in J := -I$. Then $\ell(I) = \ell(J)$ hence $\Phi, \Psi \in \mathcal{E}(J)$ and g is lower semicontinuous over J . On the other hand, for $u, v \in J$ with $u < v$, we have that $-u, -v \in I$ with $-v < -u$. Applying (3.10) for these variables, we obtain

$$\frac{1}{-u - (-v)} \int_{-v}^{-u} f \leq f(-u) + \Psi(-u - (-v)) \quad (u, v \in J, u < v).$$

This, after replacing $f(x)$ by $-g(-x)$, gives

$$-\frac{1}{v-u} \int_u^v g \leq -g(u) + \Psi(v-u) \quad (u, v \in J, u < v).$$

From here, we can see that g satisfies the inequality (3.10) (wherein f is replaced by g). According to Theorem 3.3.5, we can conclude that g is Φ -monotone on J . This, with an analogous substitutions, implies that f is also Φ -monotone. \square

COROLLARY 3.3.7. ([14]) *Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is an upper semicontinuous solution of*

$$f(u) \leq \frac{1}{v-u} \int_u^v f + c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -monotone on I . In particular, f is increasing if $p > 1$.

PROOF. By our assumption, f satisfies (3.10) with Ψ defined by $\Psi(u) := cu^p$. In order to apply the previous theorem, we have to compute Φ which is given by (3.8).

$$\begin{aligned} \Phi(u) &= \Psi(u) + \int_0^u \frac{\Psi(t)}{t} dt = cu^p + \int_0^u ct^{p-1} dt \\ &= cu^p + \frac{c}{p}u^p = \frac{c(p+1)}{p}u^p. \end{aligned}$$

Thus, by Theorem 3.3.5, f is $\frac{c(p+1)}{p}(\cdot)^p$ -monotone on I .

In the case $p > 1$, the $\frac{c(p+1)}{p}(\cdot)^p$ -monotonicity implies that f is in fact increasing. \square

The next assertion is a counterpart of Corollary 3.3.7. Its proof can be obtained from Theorem 3.3.6 exactly in the same way as the previous corollary was deduced from Theorem 3.3.5.

COROLLARY 3.3.8. ([14]) *Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is a lower semicontinuous solution of*

$$\frac{1}{v-u} \int_u^v f \leq f(v) + c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -monotone on I . In particular, f is increasing if $p > 1$.

3.4. Ostrowski-type Inequalities

The result stated in the next theorem will be called an Ostrowski-type inequality for Φ -Hölder functions.

THEOREM 3.4.1. ([14]) *Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$ be a Φ -Hölder. Assume that both functions are locally Lebesgue integrable. Then, for every $x \leq p \leq y$ in I with $x < y$, the following inequality holds:*

$$(3.13) \quad \left| f(p) - \frac{1}{y-x} \int_x^y f \right| \leq \frac{1}{y-x} \left(\int_0^{p-x} \Phi + \int_0^{y-p} \Phi \right).$$

Furthermore, if Φ is subadditive and nondecreasing with $\Phi(0) = 0$, then, for all $x \leq p \leq y$ in I with $x < y$,

$$(3.14) \quad \sup_{f \in \mathcal{H}_\Phi(I)} \left| f(p) - \frac{1}{y-x} \int_x^y f \right| = \frac{1}{y-x} \left(\int_0^{p-x} \Phi + \int_0^{y-p} \Phi \right).$$

PROOF. Let $p \in [x, y]$ be fixed. Applying the second assertion of Theorem 3.3.1 in the particular case $u = v = p$, $w = x$, and $z = y$, the inequality (3.5) yields

$$\begin{aligned} |\mathcal{A}(f, \{p\}) - \mathcal{A}(f, [x, y])| &\leq \mathcal{A}(\Phi \circ |\cdot|, [x-p, y-p]) \\ &= \frac{1}{y-x} \int_{x-p}^{y-p} \Phi(|t|) dt \\ &= \frac{1}{y-x} \left(\int_0^{p-x} \Phi + \int_0^{y-p} \Phi \right). \end{aligned}$$

which reduces to (3.13).

Assume now that Φ is subadditive, nondecreasing and $\Phi \in \mathcal{E}_0(I)$. Let $x < y$ be fixed. For $p \in [x, y]$, define the function $\Phi_p : I \rightarrow \mathbb{R}$ by (2.15). Then, by Proposition 2.5.1, we have that Φ_p is a Φ -Hölder function. Therefore,

$$\begin{aligned} \sup_{f \in \mathcal{H}_\Phi(I)} \left| f(p) - \frac{1}{y-x} \int_x^y f \right| &\geq \left| \Phi_p(p) - \frac{1}{y-x} \int_x^y \Phi_p \right| \\ &= \frac{1}{y-x} \int_x^y \Phi_p \\ &= \frac{1}{y-x} \left(\int_0^{p-x} \Phi + \int_0^{y-p} \Phi \right). \end{aligned}$$

This inequality together with its reverse imply that the equality in (3.14) is valid. \square

The inequalities obtained in the particular cases $p = x$ and $p = y$ will be called the *lower and upper Hermite–Hadamard-type inequalities for Φ -Hölder functions*.

COROLLARY 3.4.2. ([14]) *Let $q \in [0, 1]$, $c \in [0, \infty[$ and $f : I \rightarrow \mathbb{R}$ be a $c(\cdot)^q$ -Hölder locally Lebesgue integrable function. Then, for every $x < y$ in I ,*

$$\left| f(p) - \frac{1}{y-x} \int_x^y f \right| \leq \frac{c}{q+1} ((p-x)^q + (y-p)^q) \quad (p \in [x, y]).$$

Furthermore, for all $x < y$ in I ,

$$\sup_{f \in \mathcal{H}_{\Phi}(I)} \left| f(p) - \frac{1}{y-x} \int_x^y f \right| = \frac{c}{q+1} ((p-x)^q + (y-p)^q) \quad (p \in [x, y]).$$

PROOF. Apply the previous statement for the error function $\Phi \in \mathcal{E}(\mathbb{R}_+)$ given by $\Phi(t) := ct^p$. \square

THEOREM 3.4.3. ([14]) *Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (3.8). If $f : I \rightarrow \mathbb{R}$ is a continuous solution of*

$$(3.15) \quad \left| f(u) - \frac{1}{v-u} \int_u^v f \right| \leq \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -Hölder on I .

PROOF. If f satisfies (3.15), then f and $-f$ fulfill (3.10). Therefore, in view of Theorem 3.3.5, we obtain that f and $-f$ are Φ -monotone, which implies that f is Φ -Hölder on I . \square

The proofs of the following results are similar to that of the previous theorem, they are left to the reader.

THEOREM 3.4.4. ([14]) *Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (3.8). If $f : I \rightarrow \mathbb{R}$ is a continuous solution of*

$$\left| f(v) - \frac{1}{v-u} \int_u^v f \right| \leq \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -Hölder on I .

COROLLARY 3.4.5. ([14]) *Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is a continuous solution of*

$$\left| f(u) - \frac{1}{v-u} \int_u^v f \right| \leq c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -Hölder on I . In particular, f is constant if $p > 1$.

COROLLARY 3.4.6. ([14]) *Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is a continuous solution of*

$$\left| f(v) - \frac{1}{v-u} \int_u^v f \right| \leq c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -Hölder on I . In particular, f is constant if $p > 1$.

3.5. Individual Error Functions and Other Results

In this section we shall characterize the elements of the classes $\mathcal{M}(I)$ and $\mathcal{H}(I)$. For this purpose, given a function $f : I \rightarrow \mathbb{R}$, we define the following two extended real valued error functions on $[0, \ell(I)[$:

$$\Phi_f^\sigma(u) := \sup_{x \in I \cap (I-u)} (f(x) - f(x+u))_+$$

and

$$\Phi_f^\alpha(u) := \sup_{x \in I \cap (I-u)} |f(x) - f(x+u)|.$$

Here, the positive part of a real number c is defined as $c_+ := \max(c, 0)$.

THEOREM 3.5.1. ([13]) *Let $f : I \rightarrow \mathbb{R}$. Then we have the following two statements.*

- (i) *$f \in \mathcal{M}(I)$ if and only if Φ_f^σ is finite valued on $[0, \ell(I)[$. Additionally, if for some $\Phi \in \mathcal{E}(I)$, we have $f \in \mathcal{M}_\Phi(I)$, then $\Phi_f^\sigma \leq \Phi$.*
- (ii) *$f \in \mathcal{H}(I)$ if and only if Φ_f^α is finite valued on $[0, \ell(I)[$. Additionally, if for some $\Phi \in \mathcal{E}(I)$, we have $f \in \mathcal{H}_\Phi(I)$, then $\Phi_f^\alpha \leq \Phi$.*

PROOF. Assume that $f \in \mathcal{M}(I)$. Then there exists an error function $\Phi \in \mathcal{E}(I)$ such that $f \in \mathcal{M}_\Phi(I)$. Therefore, for all $u \in [0, \ell(I)[$,

$$f(x) - f(x+u) \leq \Phi(u) \quad (x \in I \cap (I-u)).$$

Using that Φ is nonnegative, we get

$$\begin{aligned} (f(x) - f(x+u))_+ &= \max(f(x) - f(x+u), 0) \\ &\leq \Phi(u) \quad (x \in I \cap (I-u)). \end{aligned}$$

Upon taking the supremum with respect to $x \in I \cap (I-u)$, we get that $\Phi_f^\sigma(u) \leq \Phi(u)$, which proves that Φ_f^σ has finite values.

Conversely, assume that Φ_f^σ has finite values. Then $\Phi_f^\sigma \in \mathcal{E}(I)$ and, for all $u \in [0, \ell(I)[$ and $x \in I \cap (I-u)$,

$$f(x) - f(x+u) \leq \Phi_f^\sigma(u).$$

This shows that f is Φ_f^σ -monotone, and hence, $f \in \mathcal{M}(I)$.

The proof about the second assertion is very similar and therefore, it is omitted. \square

THEOREM 3.5.2. ([13]) *Let $f : I \rightarrow \mathbb{R}$. Then we have the following two statements.*

- (i) *If $f \in \mathcal{M}(I)$, then Φ_f^σ is subadditive. Additionally, if for some $\Phi \in \mathcal{E}(I)$, we have $f \in \mathcal{M}_\Phi(I)$, then $\Phi_f^\sigma \leq \Phi^\sigma$.*
- (ii) *Provided that I is unbounded, if $f \in \mathcal{H}(I)$, then Φ_f^α is absolutely subadditive. Additionally, if for some $\Phi \in \mathcal{E}(I)$, $f \in \mathcal{H}_\Phi(I)$, then $\Phi_f^\alpha \leq \Phi^\alpha$.*

PROOF. Let $f \in \mathcal{M}(I)$ and let $u, v \in \mathbb{R}_+$ such that $u + v < \ell(I)$. By the definition of Φ_f^σ and the subadditivity of the function $(\cdot)_+$, we have

$$\begin{aligned}
\Phi_f^\sigma(u + v) &= \sup_{x \in I \cap (I - (u+v))} (f(x) - f(x + (u + v)))_+ \\
&\leq \sup_{x \in I \cap (I - (u+v))} (f(x) - f(x + u))_+ \\
&\quad + (f(x + u) - f(x + (u + v)))_+ \\
&\leq \sup_{x \in I \cap (I - (u+v))} (f(x) - f(x + u))_+ \\
&\quad + \sup_{x \in I \cap (I - (u+v))} (f(x + u) - f(x + (u + v)))_+ \\
&\leq \sup_{x \in I \cap (I - u)} (f(x) - f(x + u))_+ \\
&\quad + \sup_{x \in (I - u) \cap (I - (u+v))} (f(x + u) - f(x + (u + v)))_+ \\
&= \Phi_f^\sigma(u) + \Phi_f^\sigma(v),
\end{aligned}$$

which establishes the subadditivity of Φ_f^σ .

Since, Φ^σ is the largest subadditive function satisfying the inequality $\Phi^\sigma \leq \Phi$. Therefore, the inequality $\Phi_f^\sigma \leq \Phi^\sigma$ follows.

Suppose that the interval I is unbounded from above. Let $f \in \mathcal{H}(I)$ and let $u, v \in \mathbb{R}$ such that $|u|, |v|, |u + v| < \ell(I)$. We may assume that $u + v$ is nonnegative (otherwise we replace u by $(-u)$ and v by $(-v)$). Therefore, by the unboundedness of I , we have that $I \subseteq I - (u + v)$ holds. Then at least one of the value u and v is nonnegative. By symmetry, we may also assume that $u \geq 0$ and thus we have $I \subseteq I - u$. In this case, $x + u \in I$ for all $x \in I$. By

the definition of Φ_f^σ and the subadditivity of the function $|\cdot|$, we have

$$\begin{aligned}
\Phi_f^\sigma(u+v) &= \sup_{x \in I \cap (I-(u+v))} |f(x) - f(x+(u+v))| \\
&= \sup_{x \in I} |f(x) - f(x+(u+v))| \\
&\leq \sup_{x \in I} |f(x) - f(x+u)| + |f(x+u) - f(x+(u+v))| \\
&\leq \sup_{x \in I} |f(x) - f(x+u)| + \sup_{x \in I} |f(x+u) - f(x+(u+v))| \\
&\leq \sup_{x \in I \cap (I-u)} |f(x) - f(x+u)| \\
&\quad + \sup_{x \in (I-u) \cap (I-(u+v))} |f(x+u) - f(x+(u+v))| \\
&= \Phi_f^\sigma(|u|) + \Phi_f^\sigma(|v|),
\end{aligned}$$

which establishes the absolute subadditivity of Φ_f^α if I is unbounded from above. The argument for the remaining case is similar, therefore it is left to the reader.

Since, Φ^α is the largest absolutely subadditive function satisfying the inequality $\Phi^\alpha \leq \Phi$. Therefore, the inequality $\Phi_f^\alpha \leq \Phi^\alpha$ follows. \square

The superadditivity of nonpositive functions can easily be characterized according to the next statement.

PROPOSITION 3.5.3. ([14]) *Let $I \subseteq \mathbb{R}_+$ with $\inf I = 0$. Let $f : I \rightarrow \mathbb{R}$ be a nonpositive function. Then f is superadditive if and only if f is $(-f)$ -monotone.*

PROOF. Assume first that f is $(-f)$ -monotone. Let $x, y \in I$ with $x+y \in I$. Then the $(-f)$ -monotonicity of f implies

$$f(x) \leq f(x+y) + (-f)(y).$$

which yields that f is superadditive.

Conversely, assume that f is superadditive. Then, for any $x, y \in I$ with $x \leq y$,

$$f(y) = f(x + (y-x)) \geq f(x) + f(y-x) = f(x) - (-f)(y-x).$$

Which establishes the $(-f)$ -monotonicity of f . \square

Approximately Convex and Affine Functions

4.1. Preliminaries

The motivation for the notions and results comes from the theory of approximate convexity which has become an active field of research and many important contributions have been made, see for instance [2–12, 16, 19–42, 44–49]. In these papers several aspects of approximate convexity were investigated: stability problems, Bernstein–Doetsch-type theorems, Hermite–Hadamard type inequalities, etc. The notions and results have various applications in nonsmooth and convex analysis and optimization theory, and also in the theory of functional equations and inequalities. The main concepts and results of this chapter are distilled from the following elementary observations; primarily studied in the paper [34].

Assume that I is a nonempty interval and a function $f : I \rightarrow \mathbb{R}$ satisfies the following inequality

$$(4.1) \quad \begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &+ \varepsilon(t((1-t)|x-y|)^p + (1-t)(t|x-y|)^p), \\ &(x, y \in I, t \in [0, 1]) \end{aligned}$$

for some nonnegative constant ε and real constant $p \in \mathbb{R}$. Clearly, if $\varepsilon = 0$, then the above condition is equivalent to the convexity of f . One can easily notice that every convex function f satisfies (4.1). On the other hand, if $p = 0$, then f satisfies (4.1) if and only if it can be decomposed as $f = g + h$, where g is convex and $\|h\|_\infty \leq \varepsilon/2$ (cf. [16, 19]). In the case $p = 1$ then f fulfills (4.1) if and only if $f = g + \ell$, where g is convex and ℓ is Lipschitzian with a Lipschitz modulus $c\varepsilon$ (cf. [45]). Surprisingly, for $p > 2$, the situation is completely different. Then (4.1) holds if and only if f is convex (cf. [2, 46]).

The above described observations and results motivate the investigation of classes of functions that obey a more general approximate convexity and also the related approximately affine property. In fact, the class of Φ -convex functions was introduced in the paper [34], but this property was only characterized therein. In this chapter, we describe the structural properties of this function class and determine the error function which is the most optimal one.

We show that optimal error functions for approximate convexity must possess the so-called Γ property. Then we offer a precise formula for the Φ -convex envelope and also obtain sandwich-type theorems.

A real valued function f defined on a real open interval I is called Φ -convex if, for all $x, y \in I, t \in [0, 1]$ it satisfies

$$(4.2) \quad \begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|). \end{aligned}$$

If the above inequality is satisfied with the identically zero error function Φ , then we say that f is *convex*. The class of Φ -convex functions on I will be denoted by $\mathcal{C}_\Phi(I)$.

Finally, a function $f : I \rightarrow \mathbb{R}$ is said to be Φ -*affine* if, for all $x, y \in I$ and $t \in [0, 1]$, satisfies the following inequality

$$(4.3) \quad \begin{aligned} |f(tx + (1-t)y) - tf(x) - (1-t)f(y)| &\leq t\Phi((1-t)|x-y|) \\ &\quad + (1-t)\Phi(t|x-y|). \end{aligned}$$

In other word, If f and $-f$ are simultaneously Φ -convex, then f is said to be a Φ -affine function. In particular, the above inequality is satisfied with the identically zero error function Φ , then we say that f is *affine*. The class of Φ -affine functions on I will be denoted by $\mathcal{A}_\Phi(I)$.

4.2. Basic properties

Below we are going to establish the structural properties of the classes of Φ -convex and Φ -affine functions. The statements of the below mentioned Proposition 4.2.1 and Proposition 4.2.2 are analogous to the Proposition 1.2.1 and Proposition 2.2.1, respectively.

PROPOSITION 4.2.1. ([15]) *Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$. Then*

$$\alpha_1\mathcal{C}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{C}_{\Phi_n}(I) \subseteq \mathcal{C}_{\alpha_1\Phi_1 + \dots + \alpha_n\Phi_n}(I).$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the class $\mathcal{C}_\Phi(I)$ is convex.

PROOF. To prove the inclusion, let $f \in \alpha_1\mathcal{C}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{C}_{\Phi_n}(I)$. Then, f is of the form

$$(4.4) \quad f = \alpha_1 f_1 + \dots + \alpha_n f_n$$

for some elements f_1, \dots, f_n belonging to $\mathcal{C}_{\Phi_1}(I), \dots, \mathcal{C}_{\Phi_n}(I)$, respectively. Then, for all $x, y \in I, t \in [0, 1]$, and $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} f_i(tx + (1-t)y) &\leq tf_i(x) + (1-t)f_i(y) \\ &\quad + t\Phi_i((1-t)|x-y|) + (1-t)\Phi_i(t|x-y|). \end{aligned}$$

Multiplying this inequality by α_i and summing up side by side, we will arrive at the inequality (4.2), where $\Phi := \sum_{i=1}^n \alpha_i \Phi_i$. This proves that $f \in \mathcal{C}_\Phi(I)$ and completes the proof of inclusion. \square

The following result is the counterpart of the previous statement.

PROPOSITION 4.2.2. ([15]) *Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Then*

$$\alpha_1 \mathcal{A}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{A}_{\Phi_n}(I) \subseteq \mathcal{A}_{|\alpha_1|\Phi_1 + \dots + |\alpha_n|\Phi_n}(I).$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the class $\mathcal{A}_\Phi(I)$ is convex and central symmetric, i.e., $\mathcal{A}_\Phi(I)$ is closed with respect to multiplication by (-1) .

PROOF. To prove the inclusion, let $f \in \alpha_1 \mathcal{A}_{\Phi_1}(I) + \dots + \alpha_n \mathcal{A}_{\Phi_n}(I)$. Then, f is of the form (4.4) for some elements f_1, \dots, f_n belonging to $\mathcal{A}_{\Phi_1}(I), \dots, \mathcal{A}_{\Phi_n}(I)$, respectively. Then, for all $x, y \in I$, $t \in [0, 1]$, and $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} |f_i(tx + (1-t)y) - tf_i(x) - (1-t)f_i(y)| &\leq t\Phi_i((1-t)|x-y|) \\ &\quad + (1-t)\Phi_i(t|x-y|). \end{aligned}$$

Multiplying this inequality by $|\alpha_i|$ and summing up side by side, we will arrive at the inequality

$$\begin{aligned} &|f(tx + (1-t)y) - tf(x) - (1-t)f(y)| \\ &= \left| \sum_{i=1}^n \alpha_i (f_i(tx + (1-t)y) - tf_i(x) - (1-t)f_i(y)) \right| \\ &\leq \sum_{i=1}^n |\alpha_i| |f_i(tx + (1-t)y) - tf_i(x) - (1-t)f_i(y)| \\ &\leq \sum_{i=1}^n |\alpha_i| \left(t\Phi_i((1-t)|x-y|) + (1-t)\Phi_i(t|x-y|) \right) \\ &= t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|), \end{aligned}$$

where $\Phi := \sum_{i=1}^n |\alpha_i| \Phi_i$. This proves that (4.3) holds, i.e., $f \in \mathcal{A}_\Phi(I)$ and completes the proof of inclusion.

The additional statements are immediate consequences of what we have proved. \square

PROPOSITION 4.2.3. ([15]) *Let $\Phi \in \mathcal{E}(I)$. Then*

$$(4.5) \quad \mathcal{A}_\Phi(I) = \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I)) \quad \text{and} \quad \mathcal{H}_\Phi(I) \subseteq \mathcal{A}_\Phi(I)$$

PROOF. To show the equality assertion, assume that f is a Φ -affine function. Then, for any $x, y \in I$ and $t \in [0, 1]$, f will satisfy the inequality (4.3) and hence the inequalities

$$(4.6) \quad \begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|). \end{aligned}$$

and

$$\begin{aligned} tf(x) + (1-t)f(y) &\leq f(tx + (1-t)y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|). \end{aligned}$$

hold. Rearranging these two inequalities, we have that both f and $-f$ are Φ -convex. That is $f \in \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I))$.

To show the inverse inclusion, let $f \in \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I))$. Due to the property of Φ -convexity of the two classes of function, f will satisfy the two inequalities in (4.6). Hence, inequality (4.3) holds for any $x, y \in I$ and $t \in [0, 1]$. This inequality being symmetric in x and y , we get that (4.3) is satisfied for all $x, y \in I$.

To prove the inclusion in (4.5), let $f \in \mathcal{H}_\Phi(I)$. Let $x, y \in I$ and $t \in [0, 1]$. By the Φ -Hölder property of f , we have

$$\begin{aligned} f(tx + (1-t)y) &\leq f(x) + \Phi((1-t)|x-y|), \\ f(tx + (1-t)y) &\leq f(y) + \Phi(t|x-y|). \end{aligned}$$

Multiplying the above inequalities by t and $1-t$, then summing up the inequalities so obtained side by side, we get

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|), \end{aligned}$$

which shows that $f \in \mathcal{C}_\Phi(I)$. Repeating the same argument with $(-f)$ instead of f , it follows that $-f \in \mathcal{C}_\Phi(I)$. Therefore, $f \in \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I)) = \mathcal{A}_\Phi(I)$. Which completes the proof. \square

COROLLARY 4.2.4. ([15]) Let $\Phi, \Psi \in \mathcal{E}(I)$. Then

$$\mathcal{H}_\Phi(I) + \mathcal{C}_\Psi(I) \subseteq \mathcal{C}_{\Phi+\Psi}(I) \quad \text{and} \quad \mathcal{H}_\Phi(I) + \mathcal{A}_\Psi(I) \subseteq \mathcal{A}_{\Phi+\Psi}(I)$$

PROOF. By Proposition 4.2.3, we have $\mathcal{H}_\Phi(I) \subseteq \mathcal{A}_\Phi(I) \subseteq \mathcal{C}_\Phi(I)$. Therefore, the statements follow from Proposition 4.2.1 and Proposition 4.2.2 respectively. \square

The closedness under pointwise supremum and infimum are already defined in Chapter 1. A family $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\}$ is called a *chain* if, for

all $\alpha, \beta \in \Gamma$, either $f_\alpha \leq f_\beta$ or $f_\beta \leq f_\alpha$ holds on I . We say that a family \mathcal{F} of real valued functions is *closed with respect to the pointwise chain supremum (chain infimum)* if $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\} \subseteq \mathcal{F}$ is a chain with a pointwise supremum (infimum) $f : I \rightarrow \mathbb{R}$, then $f \in \mathcal{F}$.

PROPOSITION 4.2.5. ([34]) *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{C}_\Phi(I)$ is closed under pointwise supremum and pointwise chain infimum.*

PROOF. Assume that $\{f_\gamma \mid \gamma \in \Gamma\}$ is a family of Φ -convex functions with a pointwise supremum $f : I \rightarrow \mathbb{R}$, i.e., assume that (1.4) holds. Let $x, y \in I$ be arbitrary with $x \neq y$ and $t \in]0, 1[$. Then, by the Φ -convexity property, for all $\gamma \in \Gamma$, we have that

$$\begin{aligned} f_\gamma(tx + (1-t)y) &\leq tf_\gamma(x) + (1-t)f_\gamma(y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|) \\ &\leq tf(x) + (1-t)f(y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|). \end{aligned}$$

Taking the supremum of the left hand side with respect to $\gamma \in \Gamma$, we get

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|), \end{aligned}$$

which shows that f is Φ -convex.

Assume that $\{f_\gamma \mid \gamma \in \Gamma\}$ is a chain of Φ -convex functions with a pointwise infimum $f : I \rightarrow \mathbb{R}$. Let $x, y \in I$ be arbitrary with $x \neq y$ and $t \in]0, 1[$. Then, by the Φ -convexity property and by the inequality $f \leq f_\gamma$, for all $\gamma \in \Gamma$, we have that

$$(4.7) \quad \begin{aligned} f(tx + (1-t)y) &\leq tf_\gamma(x) + (1-t)f_\gamma(y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|). \end{aligned}$$

Let $\alpha, \beta \in \Gamma$ be arbitrary. Then either $f_\alpha \leq f_\beta$ or $f_\beta \leq f_\alpha$ holds on I . In the first case, $f_\alpha(y) \leq f_\beta(y)$, therefore, (4.7) for $\gamma := \alpha$ implies

$$(4.8) \quad \begin{aligned} f(tx + (1-t)y) &\leq tf_\alpha(x) + (1-t)f_\beta(y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|). \end{aligned}$$

In the other case $f_\beta(x) \leq f_\alpha(x)$ holds, therefore (4.7) $\gamma := \beta$ implies (4.8). Now taking the infimum of the right hand side of (4.8) with respect to α , it follows for all $\beta \in \Gamma$ that

$$\begin{aligned} f(tx + (1-t)y) &\leq tf(x) + (1-t)f_\beta(y) \\ &\quad + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|). \end{aligned}$$

Finally, computing the infimum of the right hand side with respect to β , we get that f is Φ -convex. \square

COROLLARY 4.2.6. ([15]) *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{C}_\Phi(I)$ is closed with respect to the limsup operation.*

PROOF. Assume that $f : I \rightarrow \mathbb{R}$ is the upper limit of a sequence $f_n : I \rightarrow \mathbb{R}$. Then

$$f = \inf_{n \in \mathbb{N}} g_n, \quad \text{where} \quad g_n := \sup_{k \geq n} f_k.$$

If all the functions f_n are Φ -convex, then for all $n \in \mathbb{N}$, the function g_n is Φ -convex. (Because these classes are closed with respect to the pointwise supremum.) On the other hand, the sequence (g_n) is decreasing, therefore f is the pointwise chain infimum of $\{g_n \mid n \in \mathbb{N}\}$, thus f is also Φ -convex. \square

PROPOSITION 4.2.7. ([15]) *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{A}_\Phi(I)$ is closed under pointwise chain infimum and pointwise chain supremum. Consequently $\mathcal{A}_\Phi(I)$ is closed with respect to the pointwise limit operation.*

PROOF. The statement easily follows from Proposition 4.2.5. The statement concerning the pointwise limit operation is obvious. \square

THEOREM 4.2.8. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$, let $f : I \rightarrow \mathbb{R}$ be a Φ -convex function and $g : \mathbb{R} \rightarrow \mathbb{R}_+$ be an increasing and sublinear function. Then $(g \circ f)$ is a $(g \circ \Phi)$ -convex function.*

PROOF. Using the Φ -convexity of f , the nondecreasingness and sublinearity and properties of g , we arrive at

$$\begin{aligned} (g \circ f)(tx + (1-t)y) &\leq g\left(t(f(x) + \Phi((1-t)|y-x|))\right. \\ &\quad \left.+ (1-t)(f(y) + \Phi(t|y-x|))\right) \\ &\leq t(g \circ f(x) + g \circ \Phi((1-t)|y-x|)) \\ &\quad + (1-t)(g \circ f(y) + g \circ \Phi(t|y-x|)). \end{aligned}$$

Therefore, we arrive at our desired conclusion. \square

4.3. Optimal Error Functions

We say that an error function $\Phi \in \mathcal{E}(I)$ possesses the property Γ if it satisfies the inequality

$$(4.9) \quad \Phi(x+y) \leq \Phi(x) + \frac{2x+y}{y}\Phi(y) \quad (x \geq 0, y > 0, x+y < \ell(I)).$$

The subclass of error functions in $\mathcal{E}(I)$ with the property Γ will be denoted by $\mathcal{E}^\Gamma(I)$. The subset of $\mathcal{E}^\Gamma(I)$ whose elements also satisfy $\Phi(0) = 0$ will be denoted by $\mathcal{E}_0^\Gamma(I)$.

One can easily see that any subadditive error function satisfies property Γ , however, as we will see later, the reversed implication is not true.

The next result establishes some necessary and some sufficient conditions for the Γ property.

THEOREM 4.3.1. ([15]) *Let $\Phi \in \mathcal{E}^\Gamma(I)$. Then $\sqrt{\Phi}$ and the map $t \mapsto t^{-1}\Phi(t)$ is subadditive on $]0, \ell(I)[$. If, in addition, $\varphi : [0, \ell(I)[\rightarrow \mathbb{R}_+$ is decreasing on $]0, \ell(I)[$, then $\varphi \cdot \Phi \in \mathcal{E}^\Gamma(I)$. In particular, if $\Psi \in \mathcal{E}(I)$ and $t \mapsto t^{-2}\Psi(t)$ is decreasing on $]0, \ell(I)[$, then $\Psi \in \mathcal{E}^\Gamma(I)$.*

PROOF. To prove the subadditivity of $\sqrt{\Phi}$ on $]0, \ell(I)[$, let $x, y > 0$ with $x + y < \ell(I)$. We have

$$\Phi(x + y) \leq \Phi(x) + \frac{2x + y}{y}\Phi(y) = \Phi(x) + \Phi(y) + 2\frac{x}{y}\Phi(y).$$

and, interchanging the roles of x and y , we also have

$$\Phi(x + y) \leq \Phi(x) + \Phi(y) + 2\frac{y}{x}\Phi(x).$$

These two inequalities imply that

$$(4.10) \quad \Phi(x + y) \leq \Phi(x) + \Phi(y) + 2 \min\left(\frac{x}{y}\Phi(y), \frac{y}{x}\Phi(x)\right).$$

By using that the geometric mean of two numbers exceeds their minimum, we get

$$\Phi(x + y) \leq \Phi(x) + \Phi(y) + 2\sqrt{\Phi(x)\Phi(y)} = (\sqrt{\Phi(x)} + \sqrt{\Phi(y)})^2.$$

Taking square root of this inequality side by side, it follows that

$$\sqrt{\Phi(x + y)} \leq \sqrt{\Phi(x)} + \sqrt{\Phi(y)},$$

which completes the proof of the subadditivity of $\sqrt{\Phi}$.

To show the subadditivity of $t \mapsto t^{-1}\Phi(t)$, we use that the minimum of two numbers is smaller than their arithmetic mean. Thus, from (4.10) we arrive at

$$\begin{aligned} \Phi(x + y) &\leq \Phi(x) + \Phi(y) + \left(\frac{x}{y}\Phi(y) + \frac{y}{x}\Phi(x)\right) \\ &= (x + y)\frac{\Phi(x)}{x} + (x + y)\frac{\Phi(y)}{y}. \end{aligned}$$

Dividing both sides by $x + y$, we can reach at our desired conclusion.

To prove the next assertion, denote $\Psi := \varphi \cdot \Phi$ and let $x > 0, y > 0$ with $x + y < \ell(I)$. Using the Γ property of Φ and then the decreasingness of φ , we get

$$\begin{aligned} \Psi(x + y) &= \varphi(x + y)\Phi(x + y) \leq \varphi(x + y)\Phi(x) + \frac{2x + y}{y}\varphi(x + y)\Phi(y) \\ &\leq \varphi(x)\Phi(x) + \frac{2x + y}{y}\varphi(y)\Phi(y) \\ &= \Psi(x) + \frac{2x + y}{y}\Psi(y), \end{aligned}$$

which establishes the Γ property of Ψ .

Finally, let $\Psi \in \mathcal{E}(I)$ be arbitrary such that $t \mapsto t^{-2}\Psi(t) =: \varphi(t)$ is decreasing on $]0, \ell(I)[$. Since $\Phi(t) := t^2$ possesses the Γ property, it follows that $\Psi = \varphi \cdot \Phi$ belongs to $\mathcal{E}^\Gamma(I)$. \square

COROLLARY 4.3.2. ([15]) *Let $p \in \mathbb{R}$ and define the function $\Phi_p : [0, \infty[\rightarrow \mathbb{R}_+$ by $\Phi_p(t) := t^p$ for $t > 0$ and $\Phi_p(0) := 0$. Then $\Phi_p \in \mathcal{E}^\Gamma(\mathbb{R}_+)$ if and only if $p \leq 2$.*

PROOF. Assume that Φ_p possesses the Γ property. Then, according to the first part of Theorem 4.3.1, $t \mapsto t^{-1}\Phi_p(t) = t^{p-1}$ is subadditive on \mathbb{R}_+ . Therefore, $p - 1 \leq 1$, which yields $p \leq 2$.

On the other hand, if $p \leq 2$, then $t \mapsto t^{-2}\Phi_p(t) = t^{p-2}$ is decreasing, therefore the last part of Theorem 4.3.1 yields that $\Phi_p \in \mathcal{E}^\Gamma(\mathbb{R}_+)$. \square

PROPOSITION 4.3.3. ([15]) *Let $\Phi \in \mathcal{E}^\Gamma(I)$. Then, for all $n \in \mathbb{N}$ and for all $u_1 \geq 0, u_2, \dots, u_n > 0$ with $u_1 + \dots + u_n < \ell(I)$, the following inequality holds*

(4.11)

$$\begin{aligned} \Phi(u_1 + \dots + u_n) &\leq \Phi(u_1) + \frac{2u_1 + u_2}{u_2}\Phi(u_2) + \dots \\ &\quad \dots + \frac{2(u_1 + \dots + u_{n-1}) + u_n}{u_n}\Phi(u_n). \end{aligned}$$

PROOF. Assume that Φ possesses the Γ property, i.e., (4.9) is satisfied.

If $n = 1$, then the statement trivially holds with equality. If $n = 2$, then (4.9) with $x := u_1$ and $y := u_2$ is equivalent to (4.11). Assume that (4.11) has been proved for some $n \geq 2$ and let $u_1 \geq 0, u_2, \dots, u_{n+1} > 0$ with $u_1 + \dots + u_{n+1} < \ell(I)$. Then, by the Γ property with $x := u_1 + \dots + u_n$,

$y := u_{n+1}$ and then by inductive assumption, we get

$$\begin{aligned} & \Phi(u_1 + \cdots + u_{n+1}) \\ & \leq \Phi(u_1 + \cdots + u_n) + \frac{2(u_1 + \cdots + u_n) + u_{n+1}}{u_{n+1}} \Phi(u_{n+1}) \\ & \leq \Phi(u_1) + \cdots + \frac{2(u_1 + \cdots + u_{n-1}) + u_n}{u_n} \Phi(u_n) \\ & \quad + \frac{2(u_1 + \cdots + u_n) + u_{n+1}}{u_{n+1}} \Phi(u_{n+1}), \end{aligned}$$

which is exactly the inequality to be proved for $n + 1$ variables. \square

PROPOSITION 4.3.4. ([15]) *The classes $\mathcal{E}^\Gamma(I)$ and $\mathcal{E}_0^\Gamma(I)$ are closed with respect to addition, multiplication by nonnegative scalars, pointwise supremum, pointwise chain infimum and the limsup operation.*

PROOF. The proof is similar to that of Proposition 4.2.1 and Proposition 4.2.5, therefore it is left to the reader. \square

In view of the above proposition, we can see that the supremum of all error functions with the Γ property which are smaller than a given error function $\Phi \in \mathcal{E}(I)$ is the largest error function below Φ with the Γ property, which will be denoted as Φ^Γ and called the Γ -envelope of Φ . In what follows, we describe a construction for Φ^Γ in the class $\mathcal{E}_0^\Gamma(I)$. For this aim, for any error function $\Phi \in \mathcal{E}_0(I)$, we define its γ -transform $\Phi^\gamma : [0, \ell(I)[\rightarrow \mathbb{R}_+$ of Φ by

$$\Phi^\gamma(0) := 0,$$

and

$$\Phi^\gamma(u) := \inf \left\{ \Phi(x) + \frac{2x + y}{y} \Phi(y) \mid x \geq 0, y > 0: x + y = u \right\} \quad (u > 0).$$

Obviously, taking $x = 0$ in the above definition, it follows from $\Phi(0) = 0$ that $\Phi^\gamma \leq \Phi$ on $[0, \ell(I)[$.

THEOREM 4.3.5. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$ and define the sequence $(\Phi_n)_{n=0}^\infty$ by*

$$(4.12) \quad \Phi_1 := \Phi, \quad \Phi_{n+1} := \Phi_n^\gamma \quad (n \in \mathbb{N}).$$

Then (Φ_n) is a pointwise decreasing sequence on $[0, \ell(I)[$ whose pointwise limit function equals Φ^Γ on $[0, \ell(I)[$.

PROOF. The inequality $\Phi_2 \leq \Phi_1$ is the consequence of $\Phi^\gamma \leq \Phi$. Now, using the obvious monotonicity of the mapping $\Phi \mapsto \Phi^\gamma$, by induction on n , it follows that $\Phi_{n+1} \leq \Phi_n$ holds on $[0, \ell(I)[$, which proves the decreasingness

of the sequence (Φ_n) . This implies that the pointwise limit function of this sequence, denoted as Φ_0 , exists and is not greater than Φ .

By the construction, for all $n \in \mathbb{N}$, we have that $\Phi_n(0) = 0$ and

$$\Phi_{n+1}(u) = \Phi_n^\gamma(u) = \inf \left\{ \Phi_n(x) + \frac{2x+y}{y} \Phi_n(y) \mid \begin{array}{l} x \geq 0, y > 0: x+y = u \end{array} \right\} \quad (0 < u < \ell(I)).$$

This implies that

$$\Phi_{n+1}(x+y) \leq \Phi_n(x) + \frac{2x+y}{y} \Phi_n(y) \quad (x \geq 0, y > 0, x+y < \ell(I)).$$

Upon taking the limit as $n \rightarrow \infty$, it follows that Φ_0 possesses the Γ property.

It remains to show that Φ_0 is nonsmaller than any error function $\Psi \in \mathcal{E}(I)$ with $\Psi \leq \Phi$ which possesses the Γ property. We prove, by induction on n , that $\Psi \leq \Phi_n$ on $[0, \ell(I)[$. The inequality $\Psi \leq \Phi_1 = \Phi$ holds by assumption. Assume that $\Psi \leq \Phi_n$ is valid for some $n \in \mathbb{N}$. Let $u \in]0, \ell(I)[$ and let $x \geq 0$, $y > 0$ such that $u = x + y$. Then

$$\Psi(u) = \Psi(x+y) \leq \Psi(x) + \frac{2x+y}{y} \Psi(y) \leq \Phi_n(x) + \frac{2x+y}{y} \Phi_n(y).$$

Upon taking the infimum for all $x \geq 0$, $y > 0$ with $u = x + y$, it follows that

$$\Psi(u) \leq \Phi_n^\gamma(u) = \Phi_{n+1}(u).$$

This shows that $\Psi \leq \Phi_{n+1}$ holds on $[0, \ell(I)[$, which was to be verified.

Finally, taking the limit $n \rightarrow \infty$, we can conclude that $\Psi \leq \Phi_0$ is valid on $[0, \ell(I)[$, which demonstrates that Φ_0 is the largest error function below Φ possessing the Γ property. Therefore, $\Phi_0 = \Phi^\Gamma$ holds. \square

The next result establishes a sufficient condition that ensures the Γ envelope to be zero.

THEOREM 4.3.6. ([15]) *Let $\Phi \in \mathcal{E}(I)$ and assume that*

$$(4.13) \quad \lim_{t \rightarrow 0^+} t^{-2} \Phi(t) = 0.$$

Then $\Phi^\Gamma(u) = 0$ holds for $u \in]0, \ell(I)[$.

PROOF. Let $u \in]0, \ell(I)[$ and let $n \in \mathbb{N}$. Then, with $u_1 := \cdots := u_n := \frac{u}{n}$, the inequality (4.11) applied for Φ^Γ (instead of Φ) and the inequality $\Phi^\Gamma \leq \Phi$

yield that

$$\begin{aligned}\Phi^\Gamma(u) &\leq (1 + 3 + \cdots + (2n - 1))\Phi^\Gamma\left(\frac{u}{n}\right) \\ &= n^2\Phi^\Gamma\left(\frac{u}{n}\right) \leq n^2\Phi\left(\frac{u}{n}\right) = u^2\left(\frac{n}{u}\right)^2\Phi\left(\frac{u}{n}\right).\end{aligned}$$

Upon taking the limit $n \rightarrow \infty$ and using the relation (4.13), we get that $\Phi^\Gamma(u) = 0$, which proves the statement. \square

COROLLARY 4.3.7. ([15]) *Let $p \in \mathbb{R}$ and define Φ_p as in Corollary 4.3.2. Then*

$$\Phi_p^\Gamma = \begin{cases} \Phi_p & \text{if } p \leq 2, \\ 0 & \text{if } p > 2. \end{cases}$$

PROOF. If $p \leq 2$, then, by Corollary 4.3.2, Φ_p possesses the Γ property, and hence, $\Phi_p^\Gamma = \Phi_p$. In the case $p > 2$, we have that

$$\lim_{t \rightarrow 0^+} t^{-2}\Phi_p(t) = \lim_{t \rightarrow 0^+} t^{p-2} = 0.$$

Therefore, by Theorem 4.3.6, it follows that $\Phi_p^\Gamma = 0$. \square

The following theorem is one of the main results of this section.

THEOREM 4.3.8. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$. Then*

$$\mathcal{C}_\Phi(I) = \mathcal{C}_{\Phi^\Gamma}(I) \quad \text{and} \quad \mathcal{A}_\Phi(I) = \mathcal{A}_{\Phi^\Gamma}(I).$$

PROOF. Due to the inequality $\Phi^\Gamma \leq \Phi$, it follows that

$$\mathcal{C}_\Phi(I) \supseteq \mathcal{C}_{\Phi^\Gamma}(I) \quad \text{and} \quad \mathcal{A}_\Phi(I) \supseteq \mathcal{A}_{\Phi^\Gamma}(I).$$

To prove the reversed inclusions, we will first verify that $\mathcal{C}_\Phi(I) \subseteq \mathcal{C}_{\Phi^\Gamma}(I)$.

Let f be a Φ -convex function and let $x, y \in I$ with $x < y$ and $t \in]0, 1[$. Define $u := tx + (1-t)y$ and then choose $v \in [x, u[$ and $w \in]u, y]$ arbitrarily. Applying the Φ -convexity of f in the interval $[v, w]$, we obtain

$$(4.14) \quad f(u) \leq \frac{w-u}{w-v}(f(v) + \Phi(u-v)) + \frac{u-v}{w-v}(f(w) + \Phi(w-u))$$

Again upon considering the intervals $[x, u[$ and $]u, y]$, respectively and applying Φ -convexity, we get the following inequalities

$$f(v) \leq \frac{u-v}{u-x}(f(x) + \Phi(v-x)) + \frac{v-x}{u-x}(f(u) + \Phi(u-v))$$

and

$$f(w) \leq \frac{y-w}{y-u}(f(u) + \Phi(w-u)) + \frac{w-u}{y-u}(f(y) + \Phi(y-w))$$

After replacing $f(v)$ and $f(w)$ in the right of side of (4.14) and conducting a simple calculation, this reduces to the inequality

$$(4.15) \quad f(u) \leq \frac{y-u}{y-x} \left(f(x) + \Phi(v-x) + \frac{u+v-2x}{u-v} \Phi(u-v) \right) + \frac{u-x}{y-x} \left(f(y) + \Phi(y-w) + \frac{2y-u-w}{w-u} \Phi(w-u) \right).$$

Observe, that by the definition of Φ^γ , we have that

$$\inf_{v \in [x, u[} \left(\Phi(v-x) + \frac{u+v-2x}{u-v} \Phi(u-v) \right) = \Phi^\gamma(u-x)$$

and

$$\inf_{w \in]u, y]} \left(\Phi(y-w) + \frac{2y-u-w}{w-u} \Phi(w-u) \right) = \Phi^\gamma(y-u).$$

Therefore, upon taking the infimum with respect to $v \in [x, u[$ and $w \in]u, y]$ in (4.15), it follows that

$$f(u) \leq \frac{y-u}{y-x} (f(x) + \Phi^\gamma(u-x)) + \frac{u-x}{y-x} (f(y) + \Phi^\gamma(y-u)),$$

which proves that f is Φ^γ -convex.

Now define the sequence of error function Φ_n by the iteration (4.12). By the assumption, f is $\Phi_1 = \Phi$ -convex. On the other hand, using what we have proved above, it follows that if f is Φ_n -convex, then it is also $\Phi_{n+1} = \Phi_n^\gamma$ -convex. Therefore, f is Φ_n -convex for all $n \in \mathbb{N}$. Upon taking the limit $n \rightarrow \infty$ and using Theorem 4.3.5, we obtain that f is Φ^Γ -convex.

For the second part of the assertion, observe that if f is Φ -affine, then by Proposition 4.2.3, f and $(-f)$ are Φ -convex. Thus, by the first part, we get that f and $(-f)$ are Φ^Γ -convex, which according to Proposition 4.2.3 again implies that f is Φ^Γ -affine. Therefore, the reversed inclusion $\mathcal{A}_\Phi(I) \subseteq \mathcal{A}_{\Phi^\Gamma}(I)$ also holds. \square

COROLLARY 4.3.9. ([15]) *Let $p > 2$ and define Φ_p as in Corollary 4.3.2. Then*

$$\mathcal{C}_{\Phi_p}(I) = \mathcal{C}_0(I) \quad \text{and} \quad \mathcal{A}_{\Phi_p}(I) = \mathcal{A}_0(I).$$

PROOF. The statement directly follows from Theorem 4.3.8 and Corollary 4.3.7. \square

4.4. The Approximately Convex Envelope and Sandwich Type Theorems

For $\Phi \in \mathcal{E}_0(I)$ and $f : I \rightarrow \mathbb{R}$, define $\underline{C}_\Phi(f) : I \rightarrow [-\infty, \infty)$ by

$$(4.16) \quad \underline{C}_\Phi(f)(u) := \inf_{\substack{x, y \in I, t \in [0, 1] \\ u = tx + (1-t)y}} t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|)).$$

THEOREM 4.4.1. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$. Then the operator \underline{C}_Φ is monotone and concave in the pointwise sense, i.e., for $f, g : I \rightarrow \mathbb{R}$ and $\lambda \in [0, 1]$, we have*

$$\lambda \underline{C}_\Phi(f) + (1-\lambda) \underline{C}_\Phi(g) \leq \underline{C}_\Phi(\lambda f + (1-\lambda)g).$$

Furthermore, for all $f : I \rightarrow \mathbb{R}$, the inequality $\underline{C}_\Phi(f) \leq f$ holds and here equality is valid if and only if f is Φ -convex. In other words, the fixed points of \underline{C}_Φ are exactly the Φ -convex functions.

PROOF. The monotonicity of \underline{C}_Φ is obvious. To see its subadditivity, observe that, by the definition, for all $u \in I$ and $x, y \in I, t \in [0, 1]$ with $u = tx + (1-t)y$, we have

$$\begin{aligned} & \lambda \underline{C}_\Phi(f)(u) + (1-\lambda) \underline{C}_\Phi(g)(u) \\ & \leq \lambda(t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|))) \\ & \quad + (1-\lambda)(t(g(x) + \Phi((1-t)|y-x|)) + (1-t)(g(y) + \Phi(t|y-x|))) \\ & = t((\lambda f + (1-\lambda)g)(x) + \Phi((1-t)|y-x|)) \\ & \quad + (1-t)((\lambda f + (1-\lambda)g)(y) + \Phi(t|y-x|)). \end{aligned}$$

Upon taking the infimum for $x, y \in I, t \in [0, 1]$ with $u = tx + (1-t)y$, it follows that

$$\lambda \underline{C}_\Phi(f)(u) + (1-\lambda) \underline{C}_\Phi(g)(u) \leq \underline{C}_\Phi(\lambda f + (1-\lambda)g)(u).$$

Taking $x := y := u$ in the definition of $\underline{C}_\Phi(f)(u)$ and using $\Phi(0) = 0$, we can see that $\underline{C}_\Phi(f)(u) \leq f(u)$ holds for all $u \in I$.

On the other hand, if $f = \underline{C}_\Phi(f)$ holds for some $f : I \rightarrow \mathbb{R}$, then, for all $x, y \in I$ and $t \in [0, 1]$ with $u := tx + (1-t)y$, we get

$$\begin{aligned} f(tx + (1-t)y) &= f(u) = \underline{C}_\Phi(f)(u) \\ &\leq t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|)), \end{aligned}$$

which shows that f is Φ -convex.

Conversely, if f is Φ -convex, then, for all $x, y \in I$ and $t \in [0, 1]$ with $u = tx + (1 - t)y$,

$$\begin{aligned} f(u) &= f(tx + (1 - t)y) \\ &\leq t(f(x) + \Phi((1 - t)|y - x|)) + (1 - t)(f(y) + \Phi(t|y - x|)). \end{aligned}$$

Now, taking the infimum for $x, y \in I$, $t \in [0, 1]$ with $u = tx + (1 - t)y$, it follows that $f(u) \leq \underline{C}_\Phi(f)(u)$. This implies that, in fact $f(u) = \underline{C}_\Phi(f)(u)$ and completes the proof of the equality $f = \underline{C}_\Phi(f)$. \square

In what follows, we construct the Φ -convex envelope of any function which admits a Φ -convex minorant in terms of the operator \underline{C}_Φ . The following auxiliary result will be useful.

LEMMA 4.4.2. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$ be increasing and subadditive. Then $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$. In particular, for $p \in [0, 1]$, $\Psi_p(u) := -\Phi_p(|u|)$ is Φ_p -convex on \mathbb{R} .*

PROOF. Let $x, y \in J$ and $t \in [0, 1]$. Then

$$\begin{aligned} -\Psi(x) &= \Phi(|x|) \leq \Phi(|tx + (1 - t)y| + |(1 - t)(x - y)|) \\ &\leq \Phi(|tx + (1 - t)y|) + \Phi(|(1 - t)(x - y)|) \\ &= -\Psi(tx + (1 - t)y) + \Phi(|(1 - t)(x - y)|) \end{aligned}$$

and

$$\begin{aligned} -\Psi(y) &= \Phi(|y|) \leq \Phi(|tx + (1 - t)y| + |t(y - x)|) \\ &\leq \Phi(|tx + (1 - t)y|) + \Phi(|t(x - y)|) \\ &= -\Psi(tx + (1 - t)y) + \Phi(|t(x - y)|). \end{aligned}$$

Multiplying the first inequality by t and the second one by $(1 - t)$ and then adding up the inequalities so obtained side by side, we obtain that

$$\begin{aligned} -t\Psi(x) - (1 - t)\Psi(y) &\leq -\Psi(tx + (1 - t)y) \\ &\quad + t\Phi(|(1 - t)(x - y)|) + (1 - t)\Phi(|t(x - y)|), \end{aligned}$$

which proves that Ψ is Φ -convex.

The last assertion of the lemma follows from the fact the Φ_p is nondecreasing and subadditive provided that $p \in [0, 1]$. \square

THEOREM 4.4.3. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$ such that $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant. Then the function $\underline{C}_\Phi(f)$ is the largest Φ -convex function which is smaller than or equal to f .*

PROOF. The inequality $\underline{C}_\Phi(f) \leq f$ follows from the previous theorem.

Now suppose g is a Φ -convex function such that $g \leq f$ holds (by the assumption, there is at least one such function g). Then, according to Theorem 4.4.1, $g = \underline{C}_\Phi(g) \leq \underline{C}_\Phi(f)$, which proves that $g \leq \underline{C}_\Phi(f)$.

To complete the proof, it will be sufficient to show that $g := \underline{C}_\Phi(f)$ is Φ -convex. Let $u \in I$ be fixed arbitrarily. Then, for all $x < u < y$, with $t := \frac{y-u}{y-x}$, we have $u = tx + (1-t)y$. Thus, the definition of $g(u) = \underline{C}_\Phi(f)(u)$ yields

$$g(u) \leq \frac{y-u}{y-x}(f(x) + \Phi(u-x)) + \frac{u-x}{y-x}(f(y) + \Phi(y-u)).$$

This inequality implies

$$\varphi(u) := \sup_{x < u} \frac{g(u) - f(x) - \Phi(u-x)}{u-x} \leq \inf_{u < y} \frac{f(y) + \Phi(y-u) - g(u)}{y-u}.$$

Therefore, for every $z \in I$, we have

$$(4.17) \quad g(u) + \varphi(u)(z-u) \leq f(z) + \Phi(|z-u|).$$

Let $v \in I$ and $\varepsilon > 0$ be arbitrary. Then, by the definition of $g(v) = \underline{C}_\Phi(f)(v)$, there exist $x, y \in I, t \in [0, 1]$ with $v = tx + (1-t)y$ such that

$$t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|)) < g(v) + \varepsilon.$$

Now, applying the inequality (4.17) for $z := x$ and $z := y$, and in the last step using the Φ -convexity of Ψ , it follows that

$$\begin{aligned} & g(u) + \varphi(u)(v-u) \\ &= t(g(u) + \varphi(u)(x-u)) + (1-t)(g(u) + \varphi(u)(y-u)) \\ &\leq t(f(x) + \Phi(|x-u|)) + (1-t)(f(y) + \Phi(|y-u|)) \\ &< g(v) + \varepsilon + t(\Phi(|x-u|) - \Phi((1-t)|y-x|)) \\ &\quad + (1-t)(\Phi(|y-u|) - \Phi(t|y-x|)) \\ &= g(v) + \varepsilon + t(-\Psi(x-u) - \Phi((1-t)|y-x|)) \\ &\quad + (1-t)(-\Psi(y-u) - \Phi(t|y-x|)) \\ &\leq g(v) + \varepsilon - \Psi((tx + (1-t)y) - u) \\ &= g(v) + \varepsilon + \Phi(|v-u|). \end{aligned}$$

Upon taking the limit $\varepsilon \rightarrow 0^+$, we obtain that g satisfies the inequality

$$g(u) + \varphi(u)(v-u) \leq g(v) + \Phi(|v-u|) \quad (u, v \in I).$$

Therefore, according to the characterization theorem of Φ -convexity, this implies that g is Φ -convex. \square

Combining the assertions of Lemma 4.4.2 and Theorem 4.4.3, we immediately obtain the following consequence.

COROLLARY 4.4.4. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$ be nondecreasing and subadditive. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant. Then the function $\underline{C}_\Phi(f)$ is the largest Φ -convex function which is smaller than or equal to f .*

The next result is an extension of a classical result of Baron, Matkowski and Nikodem [1].

THEOREM 4.4.5. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$ and let $f, g : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ for some Φ -convex function $h : I \rightarrow \mathbb{R}$. Then, for all $x, y \in I$ and $t \in [0, 1]$, the functional inequality*

$$(4.18) \quad \begin{aligned} g(tx + (1-t)y) &\leq tf(x) + (1-t)f(y) \\ &\quad + t(\Phi(1-t)|y-x|) + (1-t)\Phi(t|y-x|) \end{aligned}$$

holds. Conversely, if $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$ and (4.18) is valid for all $x, y \in I$ and $t \in [0, 1]$, then there exists a Φ -convex function $h : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ holds on I .

PROOF. Let $f, g : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ holds for some Φ -convex function h . Then, for all $x, y \in I$ and $t \in [0, 1]$, we have

$$\begin{aligned} g(tx + (1-t)y) &\leq h(tx + (1-t)y) \\ &\leq th(x) + (1-t)h(y) \\ &\quad + t(\Phi(1-t)|y-x|) + (1-t)\Phi(t|y-x|) \\ &\leq tf(x) + (1-t)f(y) \\ &\quad + t(\Phi(1-t)|y-x|) + (1-t)\Phi(t|y-x|). \end{aligned}$$

This shows the validity of (4.18) on the indicated domain.

Assume that $f, g : I \rightarrow \mathbb{R}$ satisfy (4.18). Then, this inequality yields that $g \leq \underline{C}_\Phi(f)$. In view of Theorem 4.4.3 and the Φ -convexity of Ψ , it follows that $\underline{C}_\Phi(f)$ is Φ -convex. Therefore, the statement is fulfilled with $h := \underline{C}_\Phi(f)$. \square

Combining the assertions of Lemma 4.4.2 and Theorem 4.4.5, we immediately obtain the following consequence.

COROLLARY 4.4.6. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$ be nondecreasing, subadditive and let $f, g : I \rightarrow \mathbb{R}$. Then the inequalities $g \leq h \leq f$ hold for some Φ -convex function $h : I \rightarrow \mathbb{R}$ if and only if, for all $x, y \in I$ and $t \in [0, 1]$, the functional inequality (4.18) is satisfied.*

THEOREM 4.4.7. ([15]) *Let $\Phi \in \mathcal{E}_0(I)$. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant and define the sequence $f_n : I \rightarrow \mathbb{R}$ by*

$$f_1 := f, \quad f_{n+1} := \underline{C}_\Phi(f_n) \quad (n \in \mathbb{N}).$$

Then the sequence (f_n) is pointwise decreasing and its limit function is the largest Φ -convex function which is smaller than or equal to f .

PROOF. The pointwise monotonicity of the sequence follows from the property $\underline{C}_\Phi(f) \leq f$ established in Theorem 4.4.1. Let f_0 denote the pointwise limit function of the sequence (f_n) . Now suppose g is an arbitrary Φ -convex function such that $g \leq f$. Then $g \leq \underline{C}_\Phi(g) \leq \underline{C}_\Phi(f) = \underline{C}_\Phi(f_1) = f_2$. Using the same iterative argument, we can show that $g \leq f_n$ for all $n \in \mathbb{N}$. Upon taking the limit, we get that $g \leq f_0$ holds.

Finally, we show the Φ -convexity of f_0 . Let $x, y \in I$ and $t \in [0, 1]$ be arbitrary. From definition of f_{n+1} it follows that

$$\begin{aligned} f_{n+1}(tx + (1-t)y) &\leq t(f_n(x) + \Phi((1-t)|y-x|)) \\ &\quad + (1-t)(f_n(y) + \Phi(t|y-x|)). \end{aligned}$$

Upon taking the limit as $n \rightarrow \infty$, we get that

$$\begin{aligned} f_0(tx + (1-t)y) &\leq tf_0(x) + (1-t)f_0(y) \\ &\quad + t(\Phi(1-t)|y-x|) + (1-t)(\Phi t|y-x|), \end{aligned}$$

which shows that f_0 is Φ -convex, indeed. □

4.5. Characterizations of Approximately Convex and Affine Functions

In this section, we have characterized the Φ -convex and Φ -affine functions. We also establish the relationship of a Φ -convex function with an approximate monotone function.

THEOREM 4.5.1. ([34]) *Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$. Then the following conditions are equivalent to each other:*

- (i) f is Φ -convex;
- (ii) For all $x, u, y \in I$ with $x < u < y$,

$$\frac{f(u) - f(x) - \Phi(u-x)}{u-x} \leq \frac{f(y) - f(u) + \Phi(y-u)}{y-u};$$

- (iii) There exists a function $\varphi : I \rightarrow \mathbb{R}$ such that, for all $x, u \in I$,

$$(4.19) \quad f(u) + (x-u)\varphi(u) \leq f(x) + \Phi(|u-x|);$$

(iv) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$,

$$f(t_1x_1 + \dots + t_nx_n) \leq \sum_{i=1}^n t_i (f(x_i) + \Phi(|(t_1x_1 + \dots + t_nx_n) - x_i|)).$$

PROOF. (i) \Rightarrow (ii): Assume that f is Φ convex and let $x < u < y$ be arbitrary elements of I . Choose $t \in [0, 1]$ such that $u = tx + (1 - t)y$. That is let $t = \frac{y - u}{y - x}$. Then (4.2) can be rewritten as

$$f(u) \leq \frac{y - u}{y - x} f(x) + \frac{u - x}{y - x} f(y) + \frac{y - u}{y - x} \Phi(u - x) + \frac{u - x}{y - x} \Phi(y - u).$$

Therefore,

$$\begin{aligned} (y - u + u - x)f(u) &\leq (y - u)f(x) + (u - x)f(y) \\ &\quad + (y - u)\Phi(u - x) + (u - x)\Phi(y - u). \end{aligned}$$

Rearranging this inequality, it follows that

$$\frac{f(u) - f(x) - \Phi(u - x)}{u - x} \leq \frac{f(y) - f(u) + \Phi(y - u)}{y - u}.$$

(ii) \Rightarrow (iii): Assume that (ii) holds and define the function φ on I by

$$\varphi(u) := \sup_{x < u} \left(\frac{f(u) - f(x) - \Phi(u - x)}{u - x} \right) \quad (u \in I).$$

In view of condition (ii), for all $x < u < y$ in I , we have

$$(4.20) \quad \frac{f(u) - f(x) - \Phi(u - x)}{u - x} \leq \varphi(u) \leq \frac{f(y) - f(u) + \Phi(y - u)}{y - u}.$$

From the left hand side inequality in (4.20), we get

$$(4.21) \quad f(u) + (x - u)\varphi(u) \leq f(x) + \Phi(u - x) \quad (x \in I, x < u).$$

Similarly, from the right hand side inequality in (4.20) (replacing y by x , it follows that

$$(4.22) \quad f(u) + (x - u)\varphi(u) \leq f(x) + \Phi(x - u) \quad (x \in I, u < x).$$

Now, combining inequalities (4.21) and (4.22), the condition (iii) follows (also in the case $x = u$).

(iii) \Rightarrow (iv): To deduce (iv) from (iii), let $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$ and $u := t_1x_1 + \dots + t_nx_n$. Substituting x by x_i in

the inequality of condition (iii), then multiplying this inequality by t_i , finally adding up the inequalities so obtained side by side, we get

$$\sum_{i=1}^n t_i(f(u) + (x_i - u)\varphi(u)) \leq \sum_{i=1}^n t_i(f(x_i) + \Phi(|u - x_i|)).$$

Using that $\sum_{i=1}^n t_i(x_i - u) = 0$, the above inequality simplifies to the inequality of condition (iv).

(iv) \Rightarrow (i): To deduce the Φ -convexity of f from condition (iv), let $x, y \in I$ and $t \in [0, 1]$. Taking $n = 2$, $x_1 := x$, $x_2 := y$, $t_1 := t$ and $t_2 := 1 - t$ in condition (iv), it is immediate to see that the inequality reduces to the defining inequality of Φ -convexity. \square

Motivated by the condition (iii) of Theorem 4.5.1, we say that $\varphi : I \rightarrow \mathbb{R}$ is a Φ -slope function for f if it satisfies inequality (4.19) for all $x, u \in I$. Then Theorem 4.5.1 states that the Φ -convexity of a function f is equivalent to the existence of a Φ -slope function for f .

THEOREM 4.5.2. ([15]) *Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$. Then the following conditions are equivalent to each other:*

- (i) f is Φ -affine;
- (ii) For all $x, u, y \in I$ with $x < u < y$,

$$\left| \frac{f(u) - f(x)}{u - x} - \frac{f(y) - f(u)}{y - u} \right| \leq \frac{\Phi(u - x)}{u - x} + \frac{\Phi(y - u)}{y - u};$$

- (iii) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$,

$$\begin{aligned} \left| f(t_1x_1 + \dots + t_nx_n) - \sum_{i=1}^n t_i(f(x_i)) \right| \\ \leq \sum_{i=1}^n t_i\Phi(|(t_1x_1 + \dots + t_nx_n) - x_i|). \end{aligned}$$

Provided that Φ is increasing, each of the properties (i), (ii) and (iii) is also equivalent to the following condition:

- (iv) There exists a function $\varphi : I \rightarrow \mathbb{R}$ such that, for all $x, u \in I$,

$$(4.23) \quad |f(u) - f(x) - (u - x)\varphi(u)| \leq \Phi(|u - x|).$$

PROOF. Assertions (i), (ii) and (iii) are equivalent to each other by Theorem 4.5.1.

Assume that (ii) holds and Φ is increasing. First we are going to show the following inequality:

$$(4.24) \quad \max \left(\sup_{x < u} \frac{f(u) - f(x) - \Phi(u - x)}{u - x}, \sup_{u < y} \frac{f(y) - f(u) - \Phi(y - u)}{y - u} \right) \\ \leq \min \left(\inf_{x < u} \frac{f(u) - f(x) + \Phi(u - x)}{u - x}, \inf_{u < y} \frac{f(y) - f(u) + \Phi(y - u)}{y - u} \right).$$

By condition (ii), for all $x, u, y \in I$ with $x < u < y$, we have the following two inequalities

$$(4.25) \quad \frac{f(u) - f(x)}{u - x} - \frac{f(y) - f(u)}{y - u} \leq \frac{\Phi(u - x)}{u - x} + \frac{\Phi(y - u)}{y - u}, \\ -\frac{f(u) - f(x)}{u - x} + \frac{f(y) - f(u)}{y - u} \leq \frac{\Phi(u - x)}{u - x} + \frac{\Phi(y - u)}{y - u}.$$

Then, one can easily see that the first and the second of the above inequalities imply

$$\sup_{x < u} \frac{f(u) - f(x) - \Phi(u - x)}{u - x} \leq \inf_{u < y} \frac{f(y) - f(u) + \Phi(y - u)}{y - u}, \\ \sup_{u < y} \frac{f(y) - f(u) - \Phi(y - u)}{y - u} \leq \inf_{x < u} \frac{f(u) - f(x) + \Phi(u - x)}{u - x}$$

for all $u \in I$, respectively. In order to show that (4.24) holds, it remains to prove that

$$(4.26) \quad \sup_{x < u} \frac{f(u) - f(x) - \Phi(u - x)}{u - x} \leq \inf_{x < u} \frac{f(u) - f(x) + \Phi(u - x)}{u - x}, \\ \sup_{u < y} \frac{f(y) - f(u) - \Phi(y - u)}{y - u} \leq \inf_{u < y} \frac{f(y) - f(u) + \Phi(y - u)}{y - u}.$$

For the first inequality, we need to prove that, for all $x_1, x_2 < u$,

$$(4.27) \quad \frac{f(u) - f(x_1) - \Phi(u - x_1)}{u - x_1} \leq \frac{f(u) - f(x_2) + \Phi(u - x_2)}{u - x_2}.$$

If $x_1 = x_2$, this follows from the nonnegativity of Φ . We have to consider two cases according to the possibilities $x_1 < x_2$ or $x_2 < x_1$. In the first case, using

(4.25) for the triplet $x_1 < x_2 < u$, then the monotonicity of Φ , we get

$$\begin{aligned} & \frac{f(u) - f(x_1)}{u - x_1} - \frac{f(u) - f(x_2)}{u - x_2} \\ &= \frac{x_2 - x_1}{u - x_1} \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(u) - f(x_2)}{u - x_2} \right) \\ &\leq \frac{x_2 - x_1}{u - x_1} \left(\frac{\Phi(x_2 - x_1)}{x_2 - x_1} + \frac{\Phi(u - x_2)}{u - x_2} \right) \\ &\leq \frac{\Phi(x_2 - x_1)}{u - x_1} + \frac{x_2 - x_1}{u - x_1} \frac{\Phi(u - x_2)}{u - x_2} \\ &\leq \frac{\Phi(u - x_1)}{u - x_1} + \frac{\Phi(u - x_2)}{u - x_2}, \end{aligned}$$

which shows (4.27) in the case $x_1 < x_2$. In the case $x_2 < x_1$, the proof is analogous, and hence the first inequality in (4.26) has been verified. The second inequality in (4.26) can be shown in a similar manner. Thus, inequality (4.24) has been proved, too. Define now the function $\varphi : I \rightarrow \mathbb{R}$ by

$$\varphi(u) := \max \left(\sup_{x < u} \frac{f(u) - f(x) - \Phi(u - x)}{u - x}, \sup_{u < y} \frac{f(y) - f(u) - \Phi(y - u)}{y - u} \right).$$

Then, for all $x < u < y$, we get

$$\begin{aligned} & f(u) - f(x) - (u - x)\varphi(u) \leq \Phi(u - x), \\ (4.28) \quad & \text{and} \\ & f(y) - f(u) + (u - y)\varphi(u) \leq \Phi(y - u). \end{aligned}$$

Based on inequality (4.24), we also have

$$\varphi(u) \leq \min \left(\inf_{x < u} \frac{f(u) - f(x) + \Phi(u - x)}{u - x}, \inf_{u < y} \frac{f(y) - f(u) + \Phi(y - u)}{y - u} \right),$$

which implies, for all $x < u < y$, that

$$\begin{aligned} & f(x) - f(u) + (u - x)\varphi(u) \leq \Phi(u - x) \\ (4.29) \quad & \text{and} \\ & f(u) - f(y) - (u - y)\varphi(u) \leq \Phi(y - u). \end{aligned}$$

The first inequalities in (4.28) and (4.29) yield, for all $x < u$, that

$$|f(u) - f(x) - (u - x)\varphi(u)| \leq \Phi(u - x).$$

On the other hand, the second inequalities in (4.28) and (4.29) show, for all $u < y$, that

$$|f(u) - f(y) - (u - y)\varphi(u)| \leq \Phi(y - u).$$

Combining these two inequalities, we can see that assertion (iv) must be valid.

Conversely, if assertion (iv) holds, then φ is a Φ -slope function for f , whence it follows that f is Φ -convex. Similarly, (iv) also implies that $(-\varphi)$ is a Φ -slope function for $(-f)$, hence $(-f)$ is also Φ -convex. These two Φ -convexity properties yield that f is Φ -affine. \square

Motivated by the condition (iv) of Theorem 4.5.2, we say that $\varphi : I \rightarrow \mathbb{R}$ is an *absolute Φ -slope function for f* if it satisfies inequality (4.23) for all $x, u \in I$. Then Theorem 4.5.2 states that a function f is Φ -affine (provided that Φ is increasing) if and only if there exists an absolute Φ -slope function for f .

The next two propositions state that the Φ -slope function of a Φ -convex (resp. absolute Φ -slope function of a Φ -affine) function is approximately monotone (resp. Hölder) with respect to a transformation Φ^* of the error function Φ .

PROPOSITION 4.5.3. ([15]) *Let $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ be a Φ -convex function and $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f . Then φ is Φ^* -monotone, where*

$$(4.30) \quad \Phi^*(t) := \begin{cases} 2\frac{\Phi(t)}{t} & \text{if } 0 < t < \ell(I), \\ 0 & \text{if } t = 0. \end{cases}$$

PROOF. Let $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f and let $x, y \in I$ with $x < y$ be arbitrary. By obvious substitution into the inequality (4.19), we get

$$\begin{aligned} f(x) + (y - x)\varphi(x) &\leq f(y) + \Phi(y - x), \\ f(y) + (x - y)\varphi(y) &\leq f(x) + \Phi(y - x). \end{aligned}$$

Adding these inequalities side by side, the values $f(x)$ and $f(y)$ cancel out, hence we arrive at

$$(y - x)(\varphi(x) - \varphi(y)) \leq 2\Phi(y - x).$$

Dividing by $y - x$, this inequality implies that φ is Φ^* -monotone, which was to be proved. \square

PROPOSITION 4.5.4. ([15]) *Let $\Phi \in \mathcal{E}(I)$ be increasing, $f : I \rightarrow \mathbb{R}$ be a Φ -affine function and $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f . Then φ is Φ^* -Hölder, where Φ^* is defined by (4.30).*

PROOF. Let $\varphi : I \rightarrow \mathbb{R}$ be an absolute Φ -slope function for f and let $x, y \in I$ with $x \neq y$ be arbitrary. By obvious substitution into the inequality (4.23), we get

$$\begin{aligned} |f(x) - f(y) - (x - y)\varphi(x)| &\leq \Phi(|x - y|), \\ |f(y) - f(x) - (y - x)\varphi(y)| &\leq \Phi(|y - x|). \end{aligned}$$

Therefore, by the triangle inequality,

$$\begin{aligned} |(y-x)(\varphi(x) - \varphi(y))| \\ \leq |f(x) - f(y) - (x-y)\varphi(x)| + |f(y) - f(x) - (y-x)\varphi(y)| \\ \leq 2\Phi(|y-x|). \end{aligned}$$

Dividing by $|y-x|$, this inequality shows that φ is Φ^* -Hölder, which was to be proved. \square

In the following sections, we provide a brief summary of our research findings. There are still scope for further studies; we also discuss about it along with some open problems.

Summary

In this section we summarize the most important results of this PhD dissertation. We mention some of the interesting lemmas, propositions, theorems, and corollaries based on our research which can be found in full details in the papers [13–15].

We start with the definition of the four approximate classes of functions (monotone, Hölder, convex and affine). Next we will go through some of the basic structural and inclusion properties of these function classes. Then we will obtain precise formulas for the optimal error functions for each of the pre-mentioned function classes. We also give formulas for the minorants and majorants. Besides, a few sandwich type theorems are also presented. Towards the end, we characterize these approximate function classes and also we are able to find the building blocks for some of them. In the application part we primarily focus on associating our results with *Jordan-type decompositions*, *Hermite–Hadamard-type inequalities* and *Ostrowski-type inequalities*.

Let I be a nonempty open real interval throughout this summary and let $\ell(I) \in]0, \infty]$ denote its length. The symbols \mathbb{R} and \mathbb{R}_+ denote the sets of real and nonnegative real numbers, respectively.

The class of all functions $\Phi : [0, \ell(I)[\rightarrow \mathbb{R}_+$, called *error functions*, will be denoted by $\mathcal{E}(I)$. Obviously, $\mathcal{E}(I)$ is a convex cone, i.e., it is closed with respect to addition and multiplication by nonnegative scalars. The subset of $\mathcal{E}(I)$ whose elements also satisfy $\Phi(0) = 0$ will be denoted by $\mathcal{E}_0(I)$. In what follows, we are going to define four concepts related to an error function $\Phi \in \mathcal{E}(I)$.

A function $f : I \rightarrow \mathbb{R}$ will be called Φ -*monotone* if, for all $x, y \in I$ with $x \leq y$,

$$f(x) \leq f(y) + \Phi(y - x).$$

If this inequality is satisfied with the identically zero error function Φ , then we say that f is *monotone (increasing)*. The class of Φ -monotone functions on I will be denoted by $\mathcal{M}_\Phi(I)$.

A function $f : I \rightarrow \mathbb{R}$ will be called Φ -*Hölder* if, for all $x, y \in I$,

$$|f(x) - f(y)| \leq \Phi(|x - y|).$$

The class of Φ -Hölder functions on I will be denoted by $\mathcal{H}_\Phi(I)$.

We say that a function $f : I \rightarrow \mathbb{R}$ is Φ -convex if, for all $x, y \in I$ and $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|).$$

If the above inequality is satisfied with the identically zero error function Φ , then we say that f is convex. The class of Φ -convex functions on I will be denoted by $\mathcal{C}_\Phi(I)$.

A function $f : I \rightarrow \mathbb{R}$ is said to be Φ -affine if, for all $x, y \in I$ and $t \in [0, 1]$,

$$|f(tx + (1-t)y) - tf(x) - (1-t)f(y)| \leq t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|).$$

If, in particular, the above inequality is satisfied with the identically zero error function Φ , then we say that f is affine. The class of Φ -affine functions on I will be denoted by $\mathcal{A}_\Phi(I)$.

Below we describe some of the basic structural properties by dealing with various operations such as sup, limsup, inf, liminf etc.

PROPOSITION. Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$. Then

$$\begin{aligned} \alpha_1\mathcal{M}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{M}_{\Phi_n}(I) &\subseteq \mathcal{M}_{\alpha_1\Phi_1 + \dots + \alpha_n\Phi_n}(I), \\ \alpha_1\mathcal{C}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{C}_{\Phi_n}(I) &\subseteq \mathcal{C}_{\alpha_1\Phi_1 + \dots + \alpha_n\Phi_n}(I). \end{aligned}$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the classes $\mathcal{M}_\Phi(I)$ and $\mathcal{C}_\Phi(I)$ are convex.

PROPOSITION. Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Then

$$\begin{aligned} \alpha_1\mathcal{H}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{H}_{\Phi_n}(I) &\subseteq \mathcal{H}_{|\alpha_1|\Phi_1 + \dots + |\alpha_n|\Phi_n}(I), \\ \alpha_1\mathcal{A}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{A}_{\Phi_n}(I) &\subseteq \mathcal{A}_{|\alpha_1|\Phi_1 + \dots + |\alpha_n|\Phi_n}(I). \end{aligned}$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the classes $\mathcal{H}_\Phi(I)$ and $\mathcal{A}_\Phi(I)$ are convex and central symmetric, i.e., $\mathcal{H}_\Phi(I)$ and $\mathcal{A}_\Phi(I)$ are closed with respect to multiplication by (-1) .

We say that a family \mathcal{F} of real valued functions is closed with respect to the pointwise supremum if $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\}$ is a subfamily of \mathcal{F} with a pointwise supremum $f : I \rightarrow \mathbb{R}$, i.e.,

$$f(x) = \sup_{\gamma \in \Gamma} f_\gamma(x) \quad (x \in I),$$

then $f \in \mathcal{F}$. Similarly, we can define that a family \mathcal{F} of real valued functions is closed with respect to the pointwise infimum.

A family $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\}$ is called a *chain* if, for all $\alpha, \beta \in \Gamma$, either $f_\alpha \leq f_\beta$ or $f_\beta \leq f_\alpha$ holds on I . We say that a family \mathcal{F} of real valued functions is *closed with respect to the pointwise chain supremum (chain infimum)* if $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\} \subseteq \mathcal{F}$ is a chain with a pointwise supremum (infimum) $f : I \rightarrow \mathbb{R}$, then $f \in \mathcal{F}$.

PROPOSITION. *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{M}_\Phi(I)$ is closed under pointwise infimum and supremum and the class $\mathcal{C}_\Phi(I)$ is closed under pointwise supremum and pointwise chain infimum. Furthermore, $\mathcal{M}_\Phi(I)$ is closed with respect to the pointwise liminf and limsup operations, and $\mathcal{C}_\Phi(I)$ is closed with respect to the limsup operation.*

PROPOSITION. *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{H}_\Phi(I)$ is closed under pointwise infimum and pointwise supremum, furthermore the class $\mathcal{A}_\Phi(I)$ is closed under pointwise chain infimum and pointwise chain supremum. Consequently, $\mathcal{H}_\Phi(I)$ is closed with respect to the pointwise liminf and limsup operations, and $\mathcal{A}_\Phi(I)$ is closed with respect to the pointwise limit operation.*

The upcoming proposition indicates the relations and inclusions among these classes of functions.

PROPOSITION. *Let $\Phi \in \mathcal{E}(I)$. Then*

$$\mathcal{H}_\Phi(I) = \mathcal{M}_\Phi(I) \cap (-\mathcal{M}_\Phi(I)) \quad \text{and} \quad \mathcal{A}_\Phi(I) = \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I)).$$

Furthermore,

$$\mathcal{H}_\Phi(I) \subseteq \mathcal{A}_\Phi(I).$$

Now we are going to briefly study the class $\mathcal{E}(I)$ of error functions. We begin with subadditivity and describe the optimal subadditive minorant for a given error function. Next, we introduce a new class of error function, termed as absolute subadditive functions. A similar type of study is conducted on it as well. Finally, we also define a new type of functional property called the Γ -property. We will also look at some of the basic behavioral properties for this subclass of error function.

In what follows, a function $\Phi \in \mathcal{E}(I)$ will be called *subadditive* if, for all $u, v \in \mathbb{R}_+$ with $u + v < \ell(I)$, the inequality

$$\Phi(u + v) \leq \Phi(u) + \Phi(v)$$

holds. It is easy to see that a decreasing function $\Phi \in \mathcal{E}(I)$ is automatically subadditive.

The simplest but important error functions are of the form

$$(1) \quad \Phi_p(0) := 0, \quad \Phi_p(u) := u^p \quad (u > 0),$$

where $p \in \mathbb{R}$. Their subadditivity is characterized by the following statement.

PROPOSITION. Let $p \in \mathbb{R}$. Then Φ_p is subadditive on \mathbb{R}_+ if and only if $p \in]-\infty, 1]$.

PROPOSITION. Let $\Phi \in \mathcal{E}(I)$ be an arbitrary function. Define the function $\Phi^\sigma : [0, \ell(I)[\rightarrow \mathbb{R}_+$ by

$$\Phi^\sigma(u) := \inf \left\{ \Phi(u_1) + \cdots + \Phi(u_n) \mid \right. \\ \left. n \in \mathbb{N}, u_1, \dots, u_n \in \mathbb{R}_+ : u_1 + \cdots + u_n = u \right\}.$$

Then Φ^σ is the largest subadditive function which satisfies the inequality $\Phi^\sigma \leq \Phi$ on $[0, \ell(I)[$. Furthermore, $\Phi^\sigma(0) = \Phi(0)$ and, additionally, if Φ is increasing, then Φ^σ is also increasing.

A stronger property of a function $\Phi \in \mathcal{E}(I)$ is its *absolute subadditivity*, which is defined as follows: for all $u, v \in \mathbb{R}$ with $|u|, |v|, |u+v| < \ell(I)$, the inequality

$$\Phi(|u+v|) \leq \Phi(|u|) + \Phi(|v|)$$

is satisfied. It is clear that absolutely subadditive functions are automatically subadditive. For a reversed implication we have the following

LEMMA. If $\Phi \in \mathcal{E}(I)$ is increasing and subadditive, then it is absolutely subadditive.

PROPOSITION. Let $\Phi \in \mathcal{E}(I)$ be an arbitrary function. Define the function $\Phi^\alpha : [0, \ell(I)[\rightarrow \mathbb{R}_+$ by

$$\Phi^\alpha(u) := \inf \left\{ \Phi(|u_1|) + \cdots + \Phi(|u_n|) \mid n \in \mathbb{N}, \right. \\ \left. |u_1|, \dots, |u_n| < \ell(I), u_1 + \cdots + u_n = u \right\}.$$

Then Φ^α is the largest absolutely subadditive function which satisfies the inequality $\Phi^\alpha \leq \Phi$ and hence $\Phi^\alpha \leq \Phi^\sigma$ on $[0, \ell(I)[$.

We say that an error function $\Phi \in \mathcal{E}(I)$ possesses the property Γ if it satisfies the inequality

$$\Phi(x+y) \leq \Phi(x) + \frac{2x+y}{y} \Phi(y) \quad (x \geq 0, y > 0, x+y < \ell(I)).$$

The subclass of error functions in $\mathcal{E}(I)$ with the property Γ will be denoted by $\mathcal{E}^\Gamma(I)$. The subset of $\mathcal{E}^\Gamma(I)$ whose elements also satisfy $\Phi(0) = 0$ will be denoted by $\mathcal{E}_0^\Gamma(I)$.

One can easily see that any subadditive error function possesses property Γ , however, as we will see later, the reversed implication is not true.

The next result establishes some necessary and some sufficient conditions for the Γ property.

THEOREM. Let $\Phi \in \mathcal{E}^\Gamma(I)$. Then $\sqrt{\Phi}$ and the map $t \mapsto t^{-1}\Phi(t)$ is subadditive on $]0, \ell(I)[$. If, in addition, $\varphi : [0, \ell(I)[\rightarrow \mathbb{R}_+$ is decreasing on $]0, \ell(I)[$, then $\varphi \cdot \Phi \in \mathcal{E}^\Gamma(I)$. In particular, if $\Psi \in \mathcal{E}(I)$ and $t \mapsto t^{-2}\Psi(t)$ is decreasing on $]0, \ell(I)[$, then $\Psi \in \mathcal{E}^\Gamma(I)$.

COROLLARY. Let $p \in \mathbb{R}$. Then $\Phi_p \in \mathcal{E}^\Gamma(\mathbb{R}_+)$ if and only if $p \leq 2$.

PROPOSITION. Let $\Phi \in \mathcal{E}^\Gamma(I)$. Then, for all $n \in \mathbb{N}$ and for all $u_1 \geq 0$, $u_2, \dots, u_n > 0$ with $u_1 + \dots + u_n < \ell(I)$, the following inequality holds

$$\begin{aligned} \Phi(u_1 + \dots + u_n) \leq & \Phi(u_1) + \frac{2u_1 + u_2}{u_2} \Phi(u_2) + \dots \\ & \dots + \frac{2(u_1 + \dots + u_{n-1}) + u_n}{u_n} \Phi(u_n). \end{aligned}$$

PROPOSITION. The classes $\mathcal{E}^\Gamma(I)$ and $\mathcal{E}_0^\Gamma(I)$ are closed with respect to addition, multiplication by nonnegative scalars, pointwise maximum, pointwise chain infimum and the limsup operation.

In view of this proposition, for every error function $\Phi \in \mathcal{E}(I)$ there exists a largest error function below Φ which possesses the Γ property. This will be denoted by Φ^Γ . To give a formula for this function, for any error function $\Phi \in \mathcal{E}_0(I)$, we define its γ -transform $\Phi^\gamma : [0, \ell(I)[\rightarrow \mathbb{R}_+$ of Φ by

$$\Phi^\gamma(0) := 0,$$

and

$$\Phi^\gamma(u) := \inf \left\{ \Phi(x) + \frac{2x + y}{y} \Phi(y) \mid x \geq 0, y > 0: x + y = u \right\} \quad (u > 0).$$

It follows from $\Phi(0) = 0$ that $\Phi^\gamma \leq \Phi$ on $[0, \ell(I)[$.

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$ and define the sequence $(\Phi_n)_{n=0}^\infty$ by

$$\Phi_1 := \Phi, \quad \Phi_{n+1} := \Phi_n^\gamma \quad (n \in \mathbb{N}).$$

Then (Φ_n) is a pointwise decreasing sequence on $[0, \ell(I)[$ whose pointwise limit function equals Φ^Γ on $[0, \ell(I)[$.

We are now ready to present the optimal error functions for our approximate classes of functions.

THEOREM. Let $\Phi \in \mathcal{E}(I)$. Then

$$\mathcal{M}_\Phi(I) = \mathcal{M}_{\Phi^\sigma}(I) \quad \text{and} \quad \mathcal{H}_\Phi(I) = \mathcal{H}_{\Phi^\sigma}(I)$$

Additionally if I is unbounded, then

$$\mathcal{H}_\Phi(I) = \mathcal{H}_{\Phi^\alpha}(I)$$

THEOREM. *Let $\Phi \in \mathcal{E}_0(I)$. Then*

$$\mathcal{C}_\Phi(I) = \mathcal{C}_{\Phi^\Gamma}(I) \quad \text{and} \quad \mathcal{A}_\Phi(I) = \mathcal{A}_{\Phi^\Gamma}(I).$$

COROLLARY. *Let $p > 2$ and define Φ_p as in (1). Then*

$$\mathcal{C}_{\Phi_p}(I) = \mathcal{C}_0(I) \quad \text{and} \quad \mathcal{A}_{\Phi_p}(I) = \mathcal{A}_0(I).$$

Now, in this part of summary, for a given function we focus on the construction of the best minorants and majorants with respect to the one of the approximate classes of functions. Moreover, a few interesting sandwich types theorems are also presented.

PROPOSITION. *Let $\Phi \in \mathcal{E}_0(I)$ and let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -monotone minorant. Then the function $\underline{M}_\Phi(f)$ defined by*

$$\underline{M}_\Phi(f)(x) := \inf_{x \leq y} (f(y) + \Phi^\sigma(y - x)) \quad (x \in I)$$

is real-valued and is the largest Φ -monotone function which is smaller than or equal to f . Analogously, if f admits a Φ -monotone majorant, then the function $\overline{M}_\Phi(f)$ defined by

$$\overline{M}_\Phi(f)(x) := \sup_{y \leq x} (f(y) - \Phi^\sigma(x - y)) \quad (x \in I)$$

is real-valued and is the smallest Φ -monotone function which is bigger than or equal to f .

COROLLARY. *Let $\Phi \in \mathcal{E}_0(I)$ and let $g, h : I \rightarrow \mathbb{R}$. Then in order that there exist a Φ -monotone function $f : I \rightarrow \mathbb{R}$ between g and h it is necessary and sufficient that, for all $x, y \in I$ with $x \leq y$, the inequality*

$$g(x) \leq h(y) + \Phi^\sigma(y - x)$$

be valid.

PROPOSITION. *Let I be an unbounded interval, $\Phi \in \mathcal{E}_0(I)$ and let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -Hölder minorant. Then the function $\underline{H}_\Phi(f)$ defined by*

$$\underline{H}_\Phi(f)(x) := \inf_{y \in I} (f(y) + \Phi^\alpha(|y - x|)) \quad (x \in I)$$

is real-valued and is the largest Φ -Hölder function which is smaller than or equal to f . Analogously, if f admits a Φ -Hölder majorant, then the function $\overline{H}_\Phi(f)$ defined by

$$\overline{H}_\Phi(f)(x) := \sup_{y \in I} (f(y) - \Phi^\alpha(|x - y|)) \quad (x \in I)$$

is real-valued and is the smallest Φ -Hölder function which is bigger than or equal to f .

COROLLARY. Let I be an unbounded interval, let $\Phi \in \mathcal{E}_0(I)$ and let $g, h : I \rightarrow \mathbb{R}$. Then in order that there exist a Φ -Hölder function $f : I \rightarrow \mathbb{R}$ between g and h it is necessary and sufficient that, for all $x, y \in I$, the inequality

$$g(x) \leq h(y) + \Phi^\alpha(|y - x|)$$

be valid.

For $\Phi \in \mathcal{E}_0(I)$ and $f : I \rightarrow \mathbb{R}$, define $\underline{C}_\Phi(f) : I \rightarrow [-\infty, \infty)$ by

$$\underline{C}_\Phi(f)(u) := \inf_{\substack{x, y \in I, t \in [0, 1] \\ u = tx + (1-t)y}} t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|)).$$

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$. Then the operator \underline{C}_Φ is monotone and concave in the pointwise sense, i.e., for $f, g : I \rightarrow \mathbb{R}$ and $\lambda \in [0, 1]$, we have

$$\lambda \underline{C}_\Phi(f) + (1-\lambda) \underline{C}_\Phi(g) \leq \underline{C}_\Phi(\lambda f + (1-\lambda)g).$$

Furthermore, for all $f : I \rightarrow \mathbb{R}$, the inequality $\underline{C}_\Phi(f) \leq f$ holds and here equality is valid if and only if f is Φ -convex. In other words, the fixed points of \underline{C}_Φ are exactly the Φ -convex functions.

LEMMA. Let $\Phi \in \mathcal{E}_0(I)$ be increasing and subadditive. Then $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$. In particular, for $p \in [0, 1]$, $\Psi_p(u) := -\Phi_p(|u|)$ is Φ_p -convex on \mathbb{R} .

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$ such that $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant. Then the function $\underline{C}_\Phi(f)$ is the largest Φ -convex function which is smaller than or equal to f .

COROLLARY. Let $\Phi \in \mathcal{E}_0(I)$ be nondecreasing and subadditive. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant. Then the function $\underline{C}_\Phi(f)$ is the largest Φ -convex function which is smaller than or equal to f .

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$ and let $f, g : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ for some Φ -convex function $h : I \rightarrow \mathbb{R}$. Then, for all $x, y \in I$ and $t \in [0, 1]$, the functional inequality

$$(2) \quad g(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t(\Phi(1-t)|y-x|) + (1-t)\Phi(t|y-x|)$$

holds. Conversely, if $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$ and (2) is valid for all $x, y \in I$ and $t \in [0, 1]$, then there exists a Φ -convex function $h : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ holds on I .

COROLLARY. Let $\Phi \in \mathcal{E}_0(I)$ be nondecreasing, subadditive and let $f, g : I \rightarrow \mathbb{R}$. Then the inequalities $g \leq h \leq f$ hold for some Φ -convex function $h : I \rightarrow \mathbb{R}$ if and only if, for all $x, y \in I$ and $t \in [0, 1]$, the functional inequality (2) is satisfied.

The net result establishes a formula for the Φ -convex envelope of real function.

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant and define the sequence $f_n : I \rightarrow \mathbb{R}$ by

$$f_1 := f, \quad f_{n+1} := \underline{C}_\Phi(f_n) \quad (n \in \mathbb{N}).$$

Then the sequence (f_n) is pointwise decreasing and its limit function is the largest Φ -convex function which is smaller than or equal to f .

In this part of the summary, we describe how using the notions of upper and lower interpolations, we can establish a characterization for approximately monotone and Hölder functions. We also see the construction of Φ -monotone and Φ -Hölder functions from elementary ones, which could be termed the building blocks for this class.

If $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ and $p \in I$, then define the three functions $h, h_p, h^p : I \rightarrow [-\infty, \infty]$ as follows:

$$h(x) := \begin{cases} \inf_{[x,p]} f & \text{if } x < p, \\ f(p) & \text{if } x = p, \\ \sup_{[p,x]} f & \text{if } p < x. \end{cases}$$

$$h_p(x) := \begin{cases} h(x) & \text{if } x \leq p, \\ h(p) - \Phi(x - p) & \text{if } p < x, \end{cases}$$

and

$$h^p(x) := \begin{cases} h(p) + \Phi(p - x) & \text{if } x < p, \\ h(x) & \text{if } p \leq x. \end{cases}$$

PROPOSITION. Let $\Phi \in \mathcal{E}(I)$ be subadditive and nondecreasing and $h : I \rightarrow [-\infty, \infty]$ be nondecreasing. Then, for all $p \in I$, the functions h_p and h^p defined above are Φ -monotone.

If $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ and $p \in I$, then we say that f can be interpolated at p by a Φ -monotone function from below [resp. from above] if there exists a Φ -monotone function $h : I \rightarrow \mathbb{R}$ such that $h(p) = f(p)$ and $h \leq f$ [resp. $f \leq h$].

PROPOSITION. Let $\Phi \in \mathcal{E}(I)$ be a subadditive and nondecreasing function, let $f : I \rightarrow \mathbb{R}$ and $p \in I$ be fixed. Then f can be interpolated at p by a Φ -monotone function from below if and only if, for all $x \in I$,

$$-\infty < \inf_{[x,p]} f \quad \text{if } x \leq p \quad \text{and} \quad f(p) \leq f(x) + \Phi(x - p) \quad \text{if } p < x.$$

Analogously, f can be interpolated at p by a Φ -monotone function from above if and only if, for all $x \in I$,

$$f(x) \leq f(p) + \Phi(p - x) \quad \text{if } x < p \quad \text{and} \quad \sup_{[p,x]} f < +\infty \quad \text{if } p \leq x.$$

THEOREM. Let $\Phi \in \mathcal{E}(I)$ be a subadditive and nondecreasing function and $f : I \rightarrow \mathbb{R}$. Then the following assertions are equivalent.

- (i) f is Φ -monotone.
- (ii) There exists a function $H : I \times I \rightarrow \mathbb{R}$ such that H satisfies the functional equations

$$\begin{aligned} \min(H(x, y), H(y, z)) &= H(x, z) & \text{and} \\ \max(H(z, y), H(y, x)) &= H(z, x) \end{aligned}$$

for all $x, y, z \in I$ with $x \leq y \leq z$ and, for all $p \in I$, the function $h := H(\cdot, p)$ is nondecreasing and h_p and h^p are Φ -monotone interpolations for f at p from below and from above, respectively.

- (iii) For every $p \in I$, there exists a nondecreasing function $h : I \rightarrow \mathbb{R}$ such that h_p is a Φ -monotone interpolation of f at p from below.
- (iv) For every $p \in I$, there exists a nondecreasing function $h : I \rightarrow \mathbb{R}$ such that h^p is a Φ -monotone interpolation of f at p from above.

If $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ and $p \in I$, then we say that f can be interpolated at p by a Φ -Hölder function from below [resp. from above] if there exists a Φ -Hölder function $h : I \rightarrow \mathbb{R}$ such that $h(p) = f(p)$ and $h \leq f$ [resp. $f \leq h$].

In what follows, given an error function $\Phi \in \mathcal{E}(I)$ and $p \in I$, we define the function $\Phi_p : I \rightarrow \mathbb{R}$ by

$$\Phi_p(x) := \Phi(|x - p|) \quad (x \in I).$$

PROPOSITION. Let $\Phi \in \mathcal{E}_0(I)$ be an absolutely subadditive function. Then, for all $p \in I$, the function Φ_p is Φ -Hölder on I .

PROPOSITION. Let $\Phi \in \mathcal{E}_0(I)$ be an absolutely subadditive function and let $f : I \rightarrow \mathbb{R}$ and $p \in I$ be fixed. Then f can be interpolated at p by a Φ -Hölder function from below if and only if, for all $x \in I$,

$$f(p) \leq f(x) + \Phi(|x - p|).$$

Analogously, f can be interpolated at p by a Φ -Hölder function from above if and only if, for all $x \in I$,

$$f(x) \leq f(p) + \Phi(|x - p|).$$

THEOREM. Let $\Phi \in \mathcal{E}_0(I)$ be absolutely subadditive function and $f : I \rightarrow \mathbb{R}$. Then the following assertions are equivalent to each other:

- (i) f is Φ -Hölder.
- (ii) For every $p \in I$, the functions $f(p) - \Phi_p$ and $f(p) + \Phi_p$ are Φ -Hölder interpolations of f at p from below and above, respectively.
- (iii) For every $p \in I$, f possesses a Φ -Hölder interpolation from below.
- (iv) For every $p \in I$, f possesses a Φ -Hölder interpolation from above.

The theorems mentioned below characterize the Φ -convex and Φ -affine functions, respectively. The first one was partly established in the paper [34].

THEOREM. Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$. Then the following conditions are equivalent to each other:

- (i) f is Φ -convex;
- (ii) For all $x, u, y \in I$ with $x < u < y$,

$$\frac{f(u) - f(x) - \Phi(u - x)}{u - x} \leq \frac{f(y) - f(u) + \Phi(y - u)}{y - u};$$

- (iii) There exists a function $\varphi : I \rightarrow \mathbb{R}$ such that, for all $x, u \in I$,

$$(3) \quad f(u) + (x - u)\varphi(u) \leq f(x) + \Phi(|u - x|);$$

- (iv) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$,

$$f(t_1x_1 + \dots + t_nx_n) \leq \sum_{i=1}^n t_i (f(x_i) + \Phi(|(t_1x_1 + \dots + t_nx_n) - x_i|)).$$

THEOREM. Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$. Then the following conditions are equivalent to each other:

- (i) f is Φ -affine;
- (ii) For all $x, u, y \in I$ with $x < u < y$,

$$\left| \frac{f(u) - f(x)}{u - x} - \frac{f(y) - f(u)}{y - u} \right| \leq \frac{\Phi(u - x)}{u - x} + \frac{\Phi(y - u)}{y - u};$$

(iii) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$,

$$\begin{aligned} & \left| f(t_1x_1 + \dots + t_nx_n) - \sum_{i=1}^n t_i(f(x_i)) \right| \\ & \leq \sum_{i=1}^n t_i \Phi(|(t_1x_1 + \dots + t_nx_n) - x_i|). \end{aligned}$$

Provided that Φ is increasing, each of the properties (i), (ii) and (iii) is also equivalent to the following condition:

(iv) There exists a function $\varphi : I \rightarrow \mathbb{R}$ such that, for all $x, u \in I$,

$$(4) \quad |f(u) - f(x) - (u - x)\varphi(u)| \leq \Phi(|u - x|).$$

The following results of slope functions tell us about the connections of Φ -convexity with Φ -monotone function and Φ -affinity with Φ -Hölder function.

PROPOSITION. Let $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ be a Φ -convex function and $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f , i.e., it satisfies (3) Then φ is Φ^* -monotone, where

$$\Phi^*(t) := \begin{cases} 2\frac{\Phi(t)}{t} & \text{if } 0 < t < \ell(I), \\ 0 & \text{if } t = 0. \end{cases}$$

PROPOSITION. Let $\Phi \in \mathcal{E}(I)$ be increasing, $f : I \rightarrow \mathbb{R}$ be a Φ -affine function and $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f , i.e., it satisfies (4) Then φ is Φ^* -Hölder, where

$$\Phi^*(t) := \begin{cases} 2\frac{\Phi(t)}{t} & \text{if } 0 < t < \ell(I), \\ 0 & \text{if } t = 0. \end{cases}$$

Now we are ready to discuss about some of the application of these approximate monotone classes.

Let $\Phi \in \mathcal{E}(I)$. Then a function $f : I \rightarrow \mathbb{R}$ is called *delta- Φ -monotone* if it is the difference of two Φ -monotone functions. In what follows, we shall extend the celebrated Jordan Decomposition Theorem for delta- Φ -monotone functions. For this purpose, we extend the notion of total variation to this more general setting.

Let $[a, b] \subseteq I$ and let $\tau = (t_0, \dots, t_n)$ be a partition of the interval $[a, b]$ (i.e., $a = t_0 < t_1 < \dots < t_n = b$). Then the Φ -variation of f with respect to τ is defined by

$$V^\Phi(f; \tau) := \sum_{i=1}^n (|f(t_i) - f(t_{i-1})| - \Phi(t_i - t_{i-1})).$$

Finally, the total Φ -variation of f on the interval $[a, b]$ is defined by

$$V_{[a,b]}^{\Phi}f := \sup\{V^{\Phi}(f; \tau) \mid \tau \text{ is a partition of } [a, b]\}.$$

LEMMA. Let $\Phi \in \mathcal{E}(I)$. Then, for all $f : I \rightarrow \mathbb{R}$ and $a < b < c$ in I , we have

$$V_{[a,b]}^{\Phi}f + V_{[b,c]}^{\Phi}f \leq V_{[a,c]}^{\Phi}f.$$

THEOREM. Let $\Phi \in \mathcal{E}(I)$. Then $V_{[a,b]}^{\Phi}f \leq 0$ holds for all $a < b$ in I if and only if f is a Φ -Hölder function.

THEOREM. Let $\Phi, \Psi \in \mathcal{E}(I)$. If $f : I \rightarrow \mathbb{R}$ is the difference of a Φ -monotone and a Ψ -monotone functions, then the total $2 \max(\Phi, \Psi)$ -variation of f is finite on every compact subinterval of I .

The particular case $\Phi = \Psi$ of the above result yields the following statement.

COROLLARY. Let $\Phi \in \mathcal{E}(I)$. If $f : I \rightarrow \mathbb{R}$ is a delta- Φ -monotone function, then the total 2Φ -variation of f is finite on every compact subinterval of I .

THEOREM. Let $\Phi \in \mathcal{E}(I)$ let $f : I \rightarrow \mathbb{R}$ such that the total 2Φ -variation of f on is finite on every compact subinterval of I . Then, for all $a \in I$, f is a delta- Φ -monotone function on $I \cap]a, \infty[$.

The classical Hermite–Hadamard inequality states that "The integral mean of a convex function in any subinterval of its domain is sandwich between the functional value of the midpoint and the average of the two functional values at the extreme points." Likewise, we can deal with locally integrable Φ -monotone functions and give a proper bound to their integral mean.

In the sequel, a function defined on an interval will be called locally integrable if it has a finite Lebesgue integral over every compact subinterval of its domain.

For the description of our subsequent results, we now introduce the following notation and terminology: If $a, b \in I$ then the convex hull of $\{a, b\}$, i.e., the smallest interval containing a and b , will denoted by $\langle a, b \rangle$. If, additionally, $f : \langle a, b \rangle \rightarrow \mathbb{R}$ is Lebesgue integrable, then the integral average of f over $\langle a, b \rangle$ is defined by

$$\mathcal{A}(f, \langle a, b \rangle) := \int_0^1 f(ta + (1-t)b)dt.$$

One can easily see that the following equality holds:

$$\langle a, b \rangle := \begin{cases} [a, b] & \text{if } a < b, \\ \{a\} & \text{if } a = b, \\ [b, a] & \text{if } a > b, \end{cases} \quad \text{and } \mathcal{A}(f, \langle a, b \rangle) = \begin{cases} \frac{1}{b-a} \int_a^b f & \text{if } a < b, \\ f(a) & \text{if } a = b, \\ \frac{1}{a-b} \int_b^a f & \text{if } a > b. \end{cases}$$

The inequalities stated in the following results summarize both Hermite–Hadamard and Ostrowski-type inequalities for the Φ -monotone as well as for the Φ -Hölder settings.

THEOREM. *Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$ be locally Lebesgue integrable functions. If f is Φ -monotone, then, for all $u, v, w, z \in I$ with $u \leq w$ and $v \leq z$,*

$$\mathcal{A}(f, \langle u, v \rangle) \leq \mathcal{A}(f, \langle w, z \rangle) + \mathcal{A}(\Phi, \langle w - u, z - v \rangle).$$

If f is Φ -Hölder, then, for all $u, v, w, z \in I$,

$$|\mathcal{A}(f, \langle u, v \rangle) - \mathcal{A}(f, \langle w, z \rangle)| \leq \mathcal{A}(\Phi \circ |\cdot|, \langle w - u, z - v \rangle).$$

Assuming Φ -monotonicity, we deduce a monotonicity type integral inequality which we will call the *lower and upper Hermite–Hadamard inequalities for Φ -monotone functions*.

THEOREM. *Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$ be a Φ -monotone. Assume that both functions are locally Lebesgue integrable. Then, for every $x < y$ in I , the following two inequalities hold:*

$$f(x) - \frac{1}{y-x} \int_0^{y-x} \Phi \leq \frac{1}{y-x} \int_x^y f \leq f(y) + \frac{1}{y-x} \int_0^{y-x} \Phi.$$

Furthermore, if Φ is subadditive and nondecreasing, then, for all $x < y$ in I ,

$$\begin{aligned} \sup_{f \in \mathcal{M}_\Phi(I)} \left(f(x) - \frac{1}{y-x} \int_x^y f \right) &= \sup_{f \in \mathcal{M}_\Phi(I)} \left(\frac{1}{y-x} \int_x^y f - f(y) \right) \\ &= \frac{1}{y-x} \int_0^{y-x} \Phi. \end{aligned}$$

COROLLARY. *Let $p \in [0, 1]$, $c \in [0, \infty[$ and $f : I \rightarrow \mathbb{R}$ be a $c(\cdot)^p$ -monotone locally Lebesgue integrable function. Then, for every $x < y$ in I , the following two inequalities hold:*

$$f(x) - \frac{c}{p+1} (y-x)^p \leq \frac{1}{y-x} \int_x^y f \leq f(y) + \frac{c}{p+1} (y-x)^p.$$

Furthermore, for all $x < y$ in I ,

$$\begin{aligned} \sup_{f \in \mathcal{M}_{\Phi}(I)} \left(f(x) - \frac{1}{y-x} \int_x^y f \right) &= \sup_{f \in \mathcal{M}_{\Phi}(I)} \left(\frac{1}{y-x} \int_x^y f - f(y) \right) \\ &= \frac{c}{p+1} (y-x)^p. \end{aligned}$$

LEMMA. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $]0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by

$$(5) \quad \Phi(u) = \Psi(u) + \int_0^u \frac{\Psi(t)}{t} dt \quad (u \in]0, \ell(I)[).$$

Then Φ is locally integrable and satisfies the following equation:

$$\Psi(u) + \frac{1}{u} \int_0^u \Phi = \Phi(u) \quad (u \in]0, \ell(I)[).$$

THEOREM. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $]0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (5). If $f : I \rightarrow \mathbb{R}$ is an upper semicontinuous solution of

$$f(u) \leq \frac{1}{v-u} \int_u^v f + \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -monotone on I .

The following result is a counterpart of previous theorem. It can be proved directly in an analogous way, however, in our dissertation we deduce it from this theorem using a sign transformation.

THEOREM. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $]0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (5). If $f : I \rightarrow \mathbb{R}$ is a lower semicontinuous solution of

$$\frac{1}{v-u} \int_u^v f \leq f(v) + \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -monotone on I .

COROLLARY. Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is an upper semicontinuous solution of

$$f(u) \leq \frac{1}{v-u} \int_u^v f + c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -monotone on I . In particular, f is increasing if $p > 1$.

COROLLARY. Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is a lower semicontinuous solution of

$$\frac{1}{v-u} \int_u^v f \leq f(v) + c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -monotone on I . In particular, f is increasing if $p > 1$

The result in the next theorem will be called an Ostrowski-type inequality for Φ -Hölder functions. The Ostrowski's inequality provides a sharp upper bound between the difference of integral mean of a bounded differentiable function and functional value at any point in that interval.

THEOREM. Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$ be a Φ -Hölder. Assume that both functions are locally Lebesgue integrable. Then, for every $x < y$ in I , the following inequality hold:

$$\left| f(p) - \frac{1}{y-x} \int_x^y f \right| \leq \frac{1}{y-x} \left(\int_0^{p-x} \Phi + \int_0^{y-p} \Phi \right) \quad (p \in [x, y]).$$

Furthermore, if Φ is subadditive and nondecreasing with $\Phi(0) = 0$, then, for all $x < y$ in I ,

$$\sup_{f \in \mathcal{H}_\Phi(I)} \left| f(p) - \frac{1}{y-x} \int_x^y f \right| = \frac{1}{y-x} \left(\int_0^{p-x} \Phi + \int_0^{y-p} \Phi \right) \quad (p \in [x, y]).$$

The inequalities obtained in the particular cases $p = x$ and $p = y$ will be called the *lower and upper Hermite-Hadamard-type inequalities for Φ -Hölder functions*.

COROLLARY. Let $q \in [0, 1]$, $c \in [0, \infty[$ and $f : I \rightarrow \mathbb{R}$ be a $c(\cdot)^q$ -Hölder locally Lebesgue integrable function. Then, for every $x < y$ in I ,

$$\left| f(p) - \frac{1}{y-x} \int_x^y f \right| \leq \frac{c}{q+1} ((p-x)^q + (y-p)^q) \quad (p \in [x, y]).$$

Furthermore, for all $x < y$ in I ,

$$\sup_{f \in \mathcal{H}_\Phi(I)} \left| f(p) - \frac{1}{y-x} \int_x^y f \right| = \frac{c}{q+1} ((p-x)^q + (y-p)^q) \quad (p \in [x, y]).$$

THEOREM. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (5). If $f : I \rightarrow \mathbb{R}$ is a continuous solution of

$$\left| f(u) - \frac{1}{v-u} \int_u^v f \right| \leq \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -Hölder on I .

THEOREM. Let $\Psi \in \mathcal{E}(I)$ and assume that the map $t \mapsto \Psi(t)/t$ is locally integrable on $[0, \ell(I)[$ and define $\Phi \in \mathcal{E}(I)$ by (5). If $f : I \rightarrow \mathbb{R}$ is a continuous solution of

$$\left| f(v) - \frac{1}{v-u} \int_u^v f \right| \leq \Psi(v-u) \quad (u, v \in I, u < v),$$

then f is Φ -Hölder on I .

COROLLARY. Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is a continuous solution of

$$\left| f(u) - \frac{1}{v-u} \int_u^v f \right| \leq c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -Hölder on I . In particular, f is constant if $p > 1$.

COROLLARY. Let $p \in]0, 1]$, $c \in [0, \infty[$. If $f : I \rightarrow \mathbb{R}$ is a continuous solution of

$$\left| f(v) - \frac{1}{v-u} \int_u^v f \right| \leq c(v-u)^p \quad (u, v \in I, u < v),$$

then f is $\frac{c(p+1)}{p}(\cdot)^p$ -Hölder on I . In particular, f is constant if $p > 1$.

Further Possible Studies and Open Problems

So far we have broadly studied the classes of approximately monotone, Hölder, convex and affine functions. On the other hand, still there are many tempting and challenging open questions in these fields.

One of the most recent interesting branches of Mathematics is stochastic calculus. Due to its application in financial industry, in the last few decades many renowned mathematicians work in this field. One of the concepts in stochastic analysis is Wiener process or Brownian motion. In simple mathematical words, it can be considered as a continuous, stationary function which has normally distributed graph. This function can be decomposed into two parts, one part consists of regularity properties and the other part of it give randomness to it, makes it almost impossible to perform some of the basic mathematical operations including differentiation and integration. In the similar way we can consider our defined approximate functions to model financial scenarios where the associated error term can be used to provide necessary randomness.

During our study of approximate monotone, Hölder, convex and affine functions, we avoid regularity properties except the application part. It gives us a future opportunity to investigate these classes of functions under some conditions. One can consider a function equipped continuity, differentiation and integrability properties and reconsider research on it under those assumptions.

In several branches of mathematics, functional decomposition is the process of resolving a functional relationship into its constituent parts in such a way that the original function can be reconstructed (i.e., recomposed) from those parts by a function composition. For instance, a function of bounded variation in a closed interval can be decomposed as the difference of two monotone functions. Proposition 1.2.1 states that algebraic sum of a monotone and a Φ -Hölder function is Φ -monotone. Similarly, in Corollary 4.2.4, we proved that algebraic sum of a convex and a Φ -Hölder function is Φ -convex. These findings can lead us to study the possible decomposition for the above mentioned approximate functions.

While studying the optimal error function for Φ -convexity, we developed a new functional property, termed as Γ property (more precisely Γ_2 property). It has been established that the function t^p satisfies Γ property for any $p \leq 2$ (Corollary 4.3.2). We can think about the ways to generalize the idea of the Γ_2 property to Γ_n , such that with each improvement we expand the classes of function till t^n where $n \in \mathbb{N}$.

In Chapters 1 and 2, while dealing with Φ -monotone and Φ -Hölder characterization; we find out the building blocks of them in the form of upper (lower) envelopes each interpolating the function at a specific point. The collective pointwise supremum (infimum) reconstruct back the original function (Theorem 1.5.3, Theorem 2.5.3). But, the atoms for the Φ -convex and Φ -affine functions are still unknown.

In Chapter 3, in the sections 'Hermite–Hadamard' and Ostrowski's type inequalities we pre-assumed that the associated error function is Lebesgue integrable along with other regularity properties (Theorem 3.3.1, Theorem 3.3.2, Theorem 3.3.5, Theorem 3.4.1, Theorem 3.4.3 etc.). However, from the application point of view, one can think about minimizing the assumed condition by introducing new mathematical techniques such as numerical integration.

In Chapter 4 Theorem 4.4.3 and Theorem 4.4.7, we have formulated the best Φ -convex minorant for any arbitrary function provided it admits at least one Φ -convex envelope from below. Based upon these formulas, we can also derive a sandwich type theorem (Theorem 4.4.5) as well. But, until now the exact formula describing the minorant or majorant of Φ -affine function is still unknown.

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