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Modified optimal auxiliary functions method for approximate-analytical solutions in fractional order nonlinear Foam Drainage equations

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ABSTRACT

The study aims to obtain new approximate analytical solutions for the fractional order nonlinear Foam Drainage equation through a novel modification of the well-known Optimal Auxiliary Functions Method (OAFM). This method systematically constructs auxiliary functions, and by optimizing their parameters, it provides approximate solutions closely mirroring the original behavior of the problem. Numerical simulations, presented in tabular form, demonstrate the effectiveness of the proposed approach. Graphical representations of the solutions are utilized to further validate the applicability and accuracy of the suggested technique. Additionally, the results affirm that the modified OAFM stands out as one of the most effective tools for addressing nonlinear problems in the field of science and technology, thereby contributing valuable addition to the field.

1. Introduction

The phenomenon of foam drainage, a naturally occurring process, is discussed in Ref. 1. This process has significant relevance, leading to the development of various technological and industrial applications for foams, such as cleansing, water purification, and mineral extraction, as noted in Ref. 2. Over a decade ago, Verbist and Weaire conducted studies that outlined the primary characteristics of two distinct drainage processes within foams: free drainage, which involves the natural flow of liquid out of foam under the influence of gravity, and forced drainage, which entails the controlled introduction of liquid at the top of a foam column.[3–5] When a constant rate of liquid is injected, the second phase creates a single wave with a constant velocity.[6] Thus, it becomes evident that the utilization of forced foam drainage serves as an exemplary prototype for elucidating a variety of fundamental phenomena governed by nonlinear differential equations. In particular, it proves to be an optimal model for investigating the dynamics of solitary waves, a type of wave that arises in various physical and mathematical contexts.[7,8] The foam drainage model was initially devised by the Belgian physicist Joseph Antoine Ferdinand Plateau in the 19th century. His groundbreaking research in this domain provided the essential groundwork for comprehending the dynamics of thin liquid films, soap films, and foam structures. These findings have had far-reaching applications across diverse scientific and industrial realms.[9]

Lately, there has been a surge in the exploration of fractional order differential equations, driven by their capacity to offer more precise depictions of intricate physical phenomena. Fractional calculus expands the notion of differentiation and integration beyond integer values, emerging as a potent instrument for capturing and representing real-world processes. Authors in Ref. 10 applied both the homotopy perturbation method and the variational iteration method, for solving the system of fractional differential equations (FDE) generated by a multi-order fractional differential equation. D. Avcıand Aylin Yetim[11] explored an advection–diffusion equation of fractional oreder, employing Laplace transformations and the finite sine-Fourier integral transformation methods to ascertain concentration profiles associated with the fundamental solutions. In their study,[12] Xiaoyun Jiang and Mingyu Xu explored the time fractional heat conduction

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equation within both general orthogonal curvilinear coordinate systems and cylindrical coordinate systems. Kumar, et al. (2019) employed the Adomian decomposition approach to get an analytical solution for the nonlinear fractional Jaulent–Miodek oscillator.[13] Fractional-order differential equations provide a more accurate depiction of the fundamental dynamics involving foam drainage.

One of the foremost difficulties encountered when addressing fractional order differential equations, whether they are linear or nonlinear, arises from the scarcity of readily available analytical solutions. Consequently, research associates are very likely to use different digital approaches, each one having their own strengths and weaknesses. Murat Gubes, et al.[14] provided the solutions to the nonlinear, time-dependent FDE processes under different initial conditions. The authors Somayeh Arbabi, et al.[15] used the Haar wavelets method (HWM) namely, in their research to obtain approximations to the foam drainage equation solution. In the last couple of years, some breakthroughs in nonlinear partial differential equations also have been achieved. For illustration, Kaabar et al.[16] presented a novel soliton solution of the (3+1)-dimensional conformable Wazwaz–Benjamin–Bona–Mahony equation. Akbulut and his research team[17] were the first to solve equations with the Mikhailov-Novikov-Wang characteristics, however, Yue and his team[18] further investigated the features of the (2+1)-dimensional Kundu–Mukherjee-Naskar equation. After that, Kaabar et al.[19] presented the analytical solutions for the nonlinear fracture Schrödinger equation too. Many different research studies which are mainly focused on the varied classification of differential equations in the field of fractional calculus have been established, and one can see these studies in the references Refs. 20–24.

Since the introduction of OAFM (Optimal Auxiliary Functions Method), the method was known to work with a lot of equations that are for fractional order as 1, 2,… till different number of differential equations we want to consider. Developed later than it, OAFM breaks down into a kind of method which aims for treatment of nonlinear differential equations. This method demonstrates robustness and efficiency, offering analytical solutions to intricate nonlinear problems that frequently challenge conventional solving methodologies. The initial introduction of OAFM for dealing with nonlinear differential equations can be attributed to Herisanu and Marinca (2016)[25]. Several related developments and applications have been done in a series of papers, see Refs. 26–29. Later, in 2021, a new modification of the OAFM was established by Nawaz, et al. in Ref. 30. Their work showcased the precision and reliability of the method by successfully applying it to solve several types of fractional order partial differential equations. After that, several researchers investigated the applicability of OAFM and other related methods to various integer-order and fractional-order partial differential equations, see Refs. 31–33.

In this paper, we focus on the solution of a specific fractional order nonlinear foam drainage equation using the optimal auxiliary functions method, while considering the fractional derivative in the Caputo sense. Furthermore, this study investigates the physical attributes of the solution it offers through 3D and contour plots, examining different fractional order values in three separate scenarios. To ensure accuracy, the approximate solution obtained from the proposed model is compared to the exact solution. The results highlight the effectiveness of the OAFM in tackling both linear and nonlinear fractional order physical problems.

The rest of the article is organized in the following manner: Section 2 provides fundamental definitions pertaining to fractional calculus. Section 3 outlines the general procedure of the modified optimal auxiliary functions method. Section 4 examines various problems and discusses the corresponding results. The final section offers the conclusions and summarizes the key findings.

2. Preliminaries

In this section, we recall some definitions and results from previously published work that well be needed in our methodology.

Definition 2.1 (*Ref. 34*)**.** The Caputo derivative operator is a mathematical tool used to work with functions $\delta(\zeta,\tau)$ that involve fractional orders ϱ greater than zero. It is absolutely different sort to expand our comprehension of the way of doing fractional calculus, a branch in mathematics which inspects differential and integrating operations other than integer order. In other words, what the Caputo derivative operator does, is to enable us to do differentiation to functions with respect to time or any other independent variable always while fractional orders consideration.

$$D^{\varrho}_{\tau}\delta(\zeta,\tau)=\frac{1}{\Gamma(k-\varrho)}\int_0^{\tau}(\tau-\mu)^{k-\varrho-1}\delta^{k}(\zeta,\mu)d\mu,\quad k-1<\varrho<k, k\in Z^{+}. \tag{2.1}$$

Definition 2.2 (*Ref. 35*)**.** The basic definition of integration is generalized, and Riemann-Liouville is allowing us to manage function analyzing and manipulating through orders that can be non-integers. Basically, in the same way that integer order calculus, the calculus of real numbers, is used to handle, study and understand the various problems of mathematics and several fields of science. The Riemann-Liouville integral operator for a function denoted as $\delta(\zeta,\tau)$ of a fractional order ϱ greater than 0 is defined as follows:

$$J^{\varrho}_{\tau}\delta(\zeta,\tau)=\frac{1}{\Gamma(\varrho)}\int_0^{\tau}(\tau-r)^{\varrho-1}\delta(\zeta,r)dr. \tag{2.2}$$

Lemma 2.1 (*Ref. 34*)**.** *For $n-1<\varrho\le n$, $p>-1$, $\tau\ge 0$ and $\lambda\in R$, we have:*

1. $D^{\varrho}_{\tau}\tau^{p}=\frac{\Gamma(\varrho+1)}{\Gamma(p-\varrho+1)}\tau^{p-\varrho}$,
2. $D^{\varrho}_{\tau}\lambda=0$,
3. $D^{\varrho}_{\tau}I^{\varrho}_{\tau}\delta(\zeta,\tau)=\delta(\zeta,\tau)$,
4. $I^{\varrho}_{\tau}D^{\varrho}_{\tau}\delta(\zeta,\tau)=\delta(\varrho,\tau)-\sum_{i=0}^{n-1}\partial^{i}\delta(\zeta,0)\frac{\tau^{i}}{i!}$.

3. General procedure of OAFM

Consider the following nonlinear differential equation, as shown in Eq. (3.1)

$$\frac{\partial^{\varrho}\delta(\zeta,\varphi)}{\partial\varphi^{\varrho}}=\xi(\zeta,\varphi)+NL(\delta(\zeta,\varphi)), \tag{3.1}$$

with the given initial condition as

$$\frac{\partial^{\varrho}\delta(\zeta,0)}{\partial\varphi^{\varrho}}=\varphi_{\varrho}(\zeta). \tag{3.2}$$

The unknown function and the nonlinear function are represented by $\xi(\zeta,\varphi)$ and $NL(\delta(\zeta,\varphi))$ respectively in Eq. (3.1). Further, the fractional derivative of order ϱ is given by $\frac{\partial^{\varrho}\delta(\zeta,\varphi)}{\partial\varphi^{\varrho}}$. For solving Eq. (3.1), we will use two components of approximation as follows:

$$\delta^{*}(\zeta,\varphi,C_i)=\delta_0(\zeta,\varphi)+\delta_1(\zeta,\varphi,C_n),\quad n=1,2,3,4\ldots s. \tag{3.3}$$

To get first and zero order approximation we put Eq. (3.3) in Eq. (3.1) to get

$$\begin{aligned}&\frac{\partial^{\varrho}\delta_0(\zeta,\varphi)}{\partial\varphi^{\varrho}}+\frac{\partial^{\varrho}\delta_1(\zeta,\varphi)}{\partial\varphi^{\varrho}}+\xi(\zeta,\varphi)\\&\quad+NL\Big[\Big(\frac{\partial^{\varrho}\delta_0(\zeta,\varphi)}{\partial\varphi^{\varrho}}\Big)+\Big(\frac{\partial^{\varrho}\delta_1(\zeta,\varphi,C_i)}{\partial\varphi^{\varrho}}\Big)\Big]=0.\end{aligned} \tag{3.4}$$

The initial approximation $\delta_0(\zeta,\varphi)$ is obtained by utilizing the linear equation shown below;

$$\frac{\partial^{\varrho}\delta_0(\zeta,\varphi)}{\partial\varphi^{\varrho}}+\xi(\zeta,\varphi)=0. \tag{3.5}$$

The initial approximation $\delta_0(\zeta,\varphi)$ is calculated by applying the inverse operator shown in the following expression;

$$\delta_0(\zeta,\varphi) = \omega(\zeta,\varphi). \tag{3.6}$$

The nonlinear terms obtained by expanding Eq. (3.4) are

$$NL\left[\frac{\partial^{\varrho}\delta_0(\zeta,\varphi)}{\partial\varphi^{\varrho}} + \frac{\partial\delta_1^{\varrho}(\zeta,\varphi,C_i)}{\partial\varphi^{\varrho}}\right] = NL\left[\delta_0(\zeta,\varphi)\right] + \sum_{k=1}^{\infty}\frac{\delta_1^k}{k!}NL^{(k)}\left[\delta_0(\zeta,\varphi)\right]. \tag{3.7}$$

To rapidly solve Eq. (3.7) and speed up the convergence of the first order approximation $\delta_1(\zeta,\varphi,C_n)$, we offer the replacement equation shown below.

$$\frac{\partial^{\varrho}\delta_1(\zeta,\varphi,C_i)}{\partial\varphi^{\varrho}} = -A_1\left[\delta_0(\zeta,\varphi)\right]N\left[\delta_0(\zeta,\varphi)\right] - A_2\left[\delta_0(\zeta,\varphi),C_j\right]. \tag{3.8}$$

After putting the auxiliary functions into Eq. (3.8) we obtain first-order approximation $\delta_1(\zeta,\varphi)$ via the inverse operator.

To minimize error, the least squares approach is used to calculate the values of the convergence control parameters C_i. Some other methods are also used, like Galerkin's, Ritz, and collocation to calculate the convergence control parameters.

$$J(c_i,c_j) = \int_0^{\varphi}\int_{\Omega} R^2(\zeta,\varphi;c_i,c_j)\,d\zeta\,d\varphi. \tag{3.9}$$

R denotes the Residual.

$$R(\zeta,\varphi,c_i,c_j) = \frac{\partial\tilde{\delta}(\zeta,\varphi,c_i,c_j)}{\partial\varphi} + \xi(\zeta,\varphi) + N[\tilde{\delta}(\zeta,\varphi,c_i,c_j)], \quad i = 1,2,\ldots,r,\ j = r+1,r+2,\ldots,t \tag{3.10}$$

4. Results and discussion

In this section, we apply our new modified method to two problems and illustrate the results in figures and tables to validate the proposed work.

4.1. Problem 1

Consider the fractional-order Foam Drainage equation, which is represented by

$$D_t^{\alpha}\psi(y,t) - \frac{1}{2}\psi(y,t)\frac{\partial^2\psi(y,t)}{\partial y^2} - \psi^2(y,t)\frac{\partial\psi(y,t)}{\partial y} - \left(\frac{\partial\psi(y,t)}{\partial y}\right)^2 = 0, \tag{4.1}$$

subject to initial condition

$$\psi(y,0) = -\sqrt{c}.Tanh(\sqrt{c}y). \tag{4.2}$$

Consider the linear and nonlinear terms from Eq. (4.1)

$$\begin{aligned} L(\psi) &= \frac{\partial^{\rho}\psi_0(y,t)}{\partial t^{\rho}}, \\ N(\psi) &= -\psi_0(y,t)\frac{\partial^2\psi_0(y,t)}{\partial y^2} - 2\psi_0^2(y,t)\frac{\partial\psi_0(y,t)}{\partial y} - \left(\frac{\partial\psi_0(y,t)}{\partial y}\right)^2. \end{aligned} \tag{4.3}$$

the zeroth order of approximation is given by

$$\frac{\partial^{\rho}\psi_0(y,t)}{\partial t^{\rho}} = 0. \tag{4.4}$$

By making use of the inverse operator, we get the following solution:

$$\psi_0(y,t) = -\sqrt{c}.Tanh(\sqrt{c}y). \tag{4.5}$$

By making use of Eq. (4.5) in Eq. (4.3), the system of nonlinear term becomes

$$N[\psi_0(y,t)] = -c^2Sech^4(\sqrt{c}y) + 3c^2Sech^2(\sqrt{c}y)Tanh^2(\sqrt{c}y). \tag{4.6}$$

Table 1
Numerical solutions of Problem 1 at various values of α.

y	$\psi_{\alpha=0.4}(y,t)$	$\psi_{\alpha=0.6}(y,t)$	$\psi_{\alpha=0.8}(y,t)$	$\psi_{\alpha=1.0}(y,t)$
0.1	−0.00325643	−0.00429716	−0.00520736	−0.00595502
0.2	−0.0112911	−0.0123209	−0.0132216	−0.0139615
0.3	−0.0193255	−0.0203389	−0.0212252	−0.0219533
0.4	−0.0273406	−0.0283335	−0.0292019	−0.0299152
0.5	−0.0353172	−0.036287	−0.0371352	−0.0378319
0.6	−0.0432338	−0.0441799	−0.0450075	−0.0456872
0.7	−0.0510637	−0.0519891	−0.0527984	−0.0534632
0.8	−0.0587712	−0.059684	−0.0604823	−0.061138
0.9	−0.0663061	−0.0672232	−0.0680252	−0.068684
1	−0.0735981	−0.0745495	−0.0753817	−0.0760652

We choose the auxiliary functions λ_1 and λ_2 as :

$$\begin{aligned} \lambda_1 &= \epsilon_1 + \epsilon_2\left(-\sqrt{c}Tanh(\sqrt{c}y)\right)^3, \\ \lambda_2 &= \epsilon_3\left(-\sqrt{c}Tanh(\sqrt{c}y)\right)^5 + \epsilon_4\left(-\sqrt{c}Tanh(\sqrt{c}y)\right)^7. \end{aligned} \tag{4.7}$$

and the first order approximation according to OAFM procedure which we discussed in the previous section is:

$$\frac{\partial^{\rho}\psi_1(y,t)}{\partial t^{\rho}} = -\Big(\lambda_1 N[\psi_0(y,t)] + \lambda_2\Big). \tag{4.8}$$

Now, using Eqs. (4.6) and (4.7) in Eq. (4.8)

$$\begin{aligned} \frac{\partial^{\rho}\psi_1(y,t)}{\partial t^{\rho}} &= \epsilon_3Tanh^5(\sqrt{c}y) + \epsilon_4Tanh^7(\sqrt{c}y) - \Big(-c^2Sech^4(\sqrt{c}y) \\ &+ 3c^2Sech^2(\sqrt{c}y)Tanh^2(\sqrt{c}y)\Big)\Big((\epsilon_1-\epsilon_2)Tanh^2(\sqrt{c}y)\Big) \end{aligned} \tag{4.9}$$

applying inverse operator to Eq. (4.9) we get

$$\begin{aligned} \psi_1(y,t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)}\Big(&\epsilon_3Tanh^5(\sqrt{c}y) + \epsilon_4Tanh^7(\sqrt{c}y) - \Big(-c^2Sech^4(\sqrt{c}y) \\ &+ 3c^2Sech^2(\sqrt{c}y)Tanh^2(\sqrt{c}y)\Big)\Big((\epsilon_1-\epsilon_2)Tanh^2(\sqrt{c}y)\Big)\Big) \end{aligned} \tag{4.10}$$

According to the OAFM procedure

$$\psi(y,t) = \psi_0(y,t) + \psi_1(y,t) \tag{4.11}$$

By making use of Eqs. (4.5) and (4.10).

$$\begin{aligned} \psi(y,t) = -\sqrt{c}Tanh(\sqrt{c}y) + \frac{t^{\alpha}}{\Gamma(1+\alpha)}\Big(&\epsilon_3Tanh^5(\sqrt{c}y) + \epsilon_4Tanh^7(\sqrt{c}y) - \\ &\Big(-c^2Sech^4(\sqrt{c}y) + 3c^2Sech^2(\sqrt{c}y)Tanh^2(\sqrt{c}y)\Big)\Big((\epsilon_1-\epsilon_2)Tanh^2(\sqrt{c}y)\Big)\Big) \end{aligned} \tag{4.12}$$

Fig. 1 illustrates a comparative analysis of the results obtained from both the approximate OAFM solution and the exact solution for Problem 1, with $\alpha = 1$ and $c = 0.08$. The 3D plot provides clear evidence of the rapid convergence of the OAFM approximate solution to the exact solution.

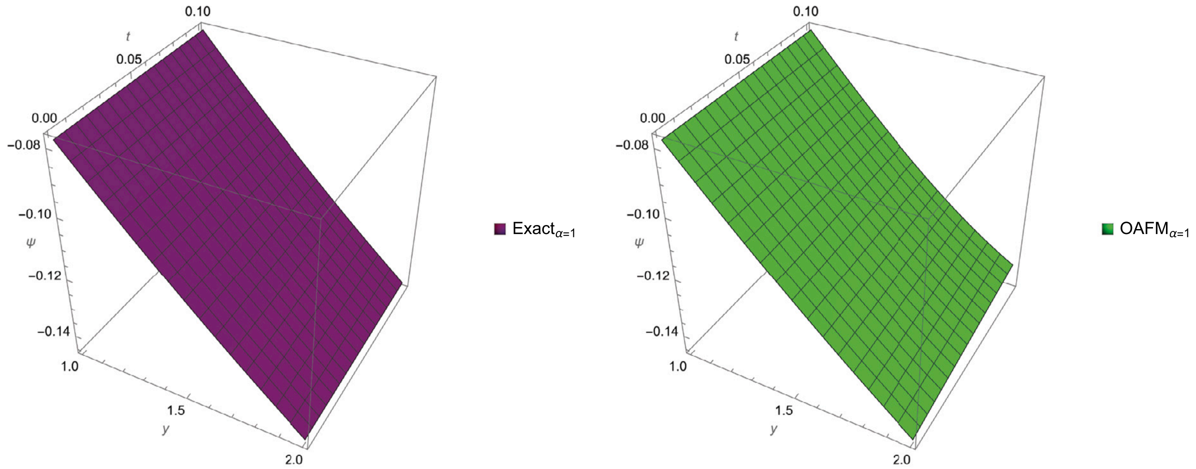
Fig. 2 presents 3D and 2D graphs of Problem 1 for various fractional order values at $t = 0.03$ using the OAFM solution. These graphs reveal that as the fractional order α approaches the integer order solution, the graph also converges toward it.

Table 1 shows the OAFM solution of Problem 1 for various values of α. The table contains the numerical values of $\psi(y,t)$ using the OAFM solution at $\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$, and $\alpha = 1.0$.

4.2. Problem 2

Consider the fractional order nonlinear Foam Drainage equation, which is represented by

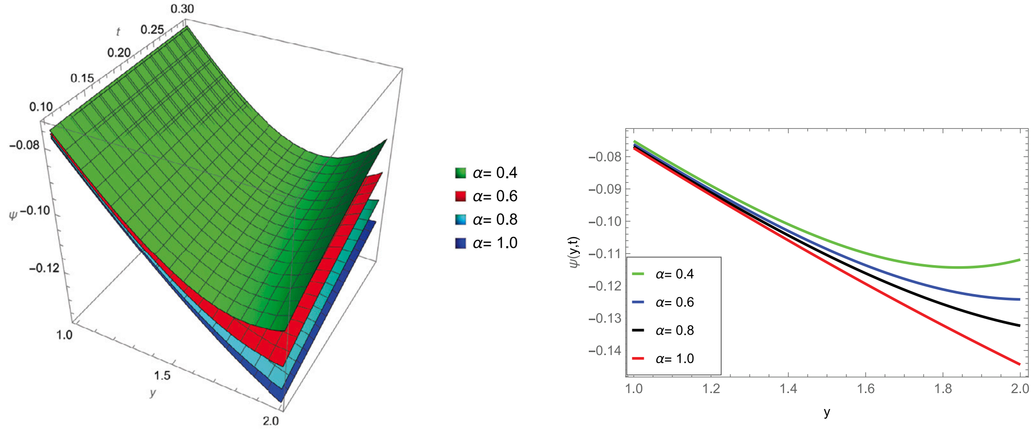
$$D_t^{\alpha}\psi(y,t) - \frac{1}{2}\psi(y,t)\frac{\partial^2\psi(y,t)}{\partial y^2} - \psi^2(y,t)\frac{\partial\psi(y,t)}{\partial y} - \left(\frac{\partial\psi(y,t)}{\partial y}\right)^2 = 0 \tag{4.13}$$



(a) 3D-plots of $\psi(y,t)$ using exact solution

(b) 3D-plots of $\psi(y,t)$ using OAFM solution

Fig. 1. 3D-plots of $\psi(y,t)$ of Problem 1 for exact and OAFM solutions.



(a) 3D-plots of $\psi(y,t)$ using OAFM solution

(b) 2D-plots of $\psi(y,t)$ using OAFM solution

Fig. 2. 3D-plots and 2D-plots of $\psi(y,t)$ of Problem 1.

Subject to initial condition

$$\psi(y,0) = -\frac{1}{2} + \frac{1}{(1+e^y)}. \tag{4.14}$$

Consider linear and nonlinear terms from Eq. (4.13)

$$\begin{aligned} L(\psi) &= \frac{\partial^\rho \psi_0(y,t)}{\partial t^\rho}, \\ N(\psi) &= -\psi_0(y,t)\frac{\partial^2\psi_0(y,t)}{\partial y^2} - 2\psi_0^2(y,t)\frac{\partial\psi_0(y,t)}{\partial y} - \Big(\frac{\partial\psi_0(y,t)}{\partial y}\Big)^2. \end{aligned} \tag{4.15}$$

The zeroth order of approximation is

$$\frac{\partial^\rho\psi_0(y,t)}{\partial t^\rho} = 0. \tag{4.16}$$

By making use of the inverse operator, we get the following solution:

$$\psi_0(y,t) = -\frac{1}{2} + \frac{1}{1+e^y}. \tag{4.17}$$

By making use of Eq. (4.17) in Eq. (4.15), we get

$$N[\psi_0(y,t)] = -\frac{e^{2y}}{(1+e^y)^4} - \frac{1}{2}\Big(\frac{2e^{2y}}{(1+e^y)^3} - \frac{e^y}{(1+e^y)^2}\Big)\Big(-\frac{1}{2} + \frac{1}{(1+e^y)}\Big) + \frac{1}{(1+e^y)^2}\Big(2e^y\Big(\frac{1}{2} - \frac{1}{(1+e^y)}\Big)^2\Big). \tag{4.18}$$

We choose the auxiliary functions λ_1 and λ_2 as :

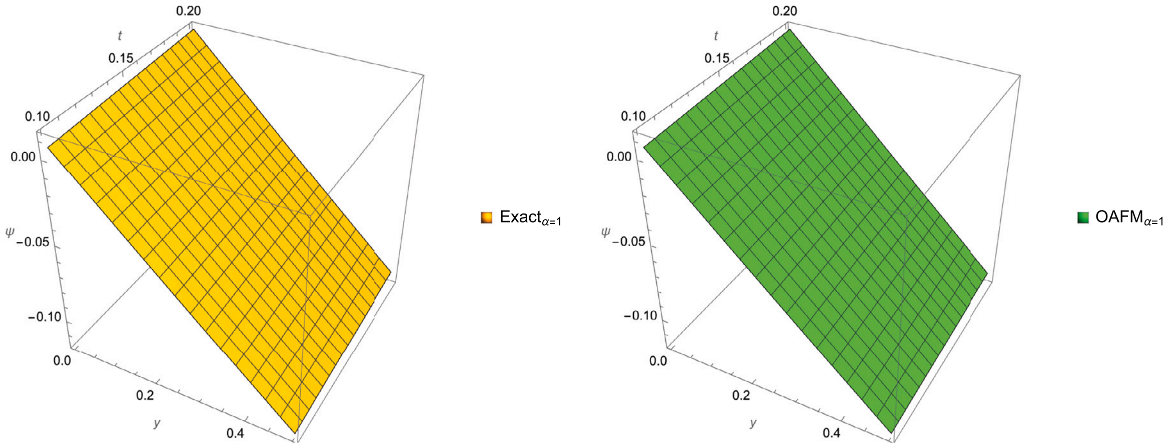
$$\begin{aligned} \lambda_1 &= \epsilon_1 + \epsilon_2\Big(-\frac{1}{2} + \frac{1}{1+e^y}\Big)^4, \\ \lambda_2 &= \epsilon_3\Big(-\frac{1}{2} + \frac{1}{1+e^y}\Big)^5 + \epsilon_4\Big(-\frac{1}{2} + \frac{1}{1+e^y}\Big)^7. \end{aligned} \tag{4.19}$$

The first order approximation according to OAFM procedure is:

$$\frac{\partial^\rho\psi_1(y,t)}{\partial t^\rho} = -\Big(\lambda_1 N[\psi_0(y,t)] + \lambda_2\Big). \tag{4.20}$$

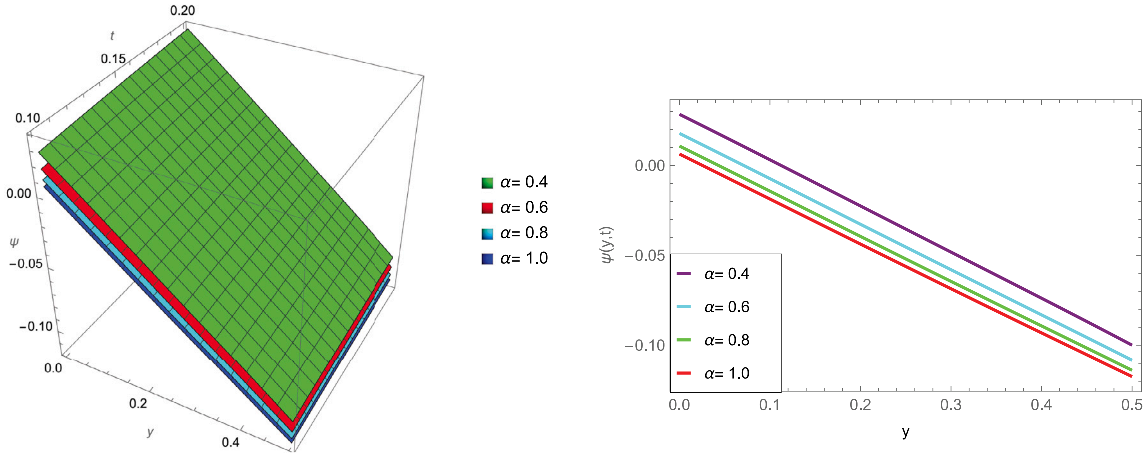
Using Eqs. (4.18) and (4.19) in Eq. (4.20), we get

$$\frac{\partial^\rho\psi_1(y,t)}{\partial t^\rho} = -\Big(-\frac{e^{2y}}{(1+e^y)^4} - \frac{1}{2}\Big(\frac{2e^{2y}}{(1+e^y)^3} - \frac{e^y}{(1+e^y)^2}\Big)\Big(-\frac{1}{2} + \frac{1}{(1+e^y)}\Big) + \frac{1}{(1+e^y)^2}\Big(2e^y\Big(\frac{1}{2} - \frac{1}{(1+e^y)}\Big)^2\Big)\Big)\Big(\epsilon_1 + \epsilon_2\Big(-\frac{1}{2} + \frac{1}{1+e^y}\Big)^4\Big) - \epsilon_3\Big(-\frac{1}{2} + \frac{1}{1+e^y}\Big)^5 - \epsilon_4\Big(-\frac{1}{2} + \frac{1}{1+e^y}\Big)^7. \tag{4.21}$$



(a) 3D-plot of $\psi(y,t)$ using exact solution (b) 3D-plot of $\psi(y,t)$ using OAFM solution

Fig. 3. 3D-plots of $\psi(y,t)$ of Problem 2 for exact and OAFM solutions.



(a) 3D-plots of $\psi(y,t)$ using OAFM solution (b) 2D-plots of $\psi(y,t)$ using OAFM solution

Fig. 4. 3D-plots and 2D-plots of $\psi(y,t)$ of Problem 2.

By making use of Riemann–Liouville integral to Eq. (4.21) we get

$$\begin{aligned}\psi_1(y,t) = \frac{t^\alpha}{\Gamma(1+\alpha)}\Bigg(-\Bigg(-\frac{e^{2y}}{(1+e^y)^4} - \frac{1}{2}\Bigg(\frac{2e^{2y}}{(1+e^y)^3} \\ - \frac{e^y}{(1+e^y)^2}\Bigg)\Big(-\frac{1}{2} + \frac{1}{(1+e^y)}\Big) \\ + \frac{1}{(1+e^y)^2}\Big(2e^y\Big(\frac{1}{2} - \frac{1}{(1+e^y)}\Big)^2\Big)\Bigg)\Bigg(\epsilon_1 + \epsilon_2\Big(-\frac{1}{2} + \frac{1}{(1+e^y)}\Big)^4\Bigg) - \\ \epsilon_3\Big(-\frac{1}{2} + \frac{1}{(1+e^y)}\Big)^5 - \epsilon_4\Big(-\frac{1}{2} + \frac{1}{(1+e^y)}\Big)^7\Bigg).\end{aligned} \tag{4.22}$$

According to the OAFM procedure

$$\psi(y,t) = \psi_0(y,t) + \psi_1(y,t). \tag{4.23}$$

By making use of Eqs. (4.17) and (4.22).

$$\begin{aligned}\psi(y,t) = -\frac{1}{2} + \frac{1}{(1+e^y)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\Bigg(-\Bigg(-\frac{e^{2y}}{(1+e^y)^4} - \frac{1}{2}\Bigg(\frac{2e^{2y}}{(1+e^y)^3} \\ - \frac{e^y}{(1+e^y)^2}\Bigg)\Big(-\frac{1}{2} + \frac{1}{(1+e^y)}\Big) + \frac{1}{(1+e^y)^2}\Big(2e^y\Big(\frac{1}{2} - \frac{1}{(1+e^y)}\Big)^2\Big)\Bigg)\Bigg(\epsilon_1 \\ + \epsilon_2\Big(-\frac{1}{2} + \frac{1}{1+e^y}\Big)^4\Bigg) - \epsilon_3\Big(-\frac{1}{2} + \frac{1}{1+e^y}\Big)^5 - \epsilon_4\Big(-\frac{1}{2} + \frac{1}{1+e^y}\Big)^7\Bigg).\end{aligned} \tag{4.24}$$

Fig. 3 presents a comparative analysis showcasing the outcomes of Problem 2 with $\alpha = 1$, contrasting the approximate OAFM solution with the exact one. The 3D plot vividly demonstrates the swift convergence of the OAFM approximate solution to the exact solution.

Moreover, Fig. 4 illustrates both 3D and 2D plots of Problem 2 at $t = 0.03$, showcasing various fractional order values. The utilization of the OAFM method reveals a notable trend: as the fractional order α approaches 1, the approximate solution graph converges more closely with the graph of the integer order solution.

The approximate solutions obtained by the OAFM for Problem 2 with different values of α are presented in Table 2 . The table contains the numerical values of $\psi(y,t)$ using the OAFM solution at $\alpha = 0.4$, $\alpha = 0.6$, $\alpha = 0.8$, and $\alpha = 1.0$.

Table 2
Numerical solutions of Problem 2 at various values of α.

y	$\psi_{\alpha=0.4}(y,t)$	$\psi_{\alpha=0.6}(y,t)$	$\psi_{\alpha=0.8}(y,t)$	$\psi_{\alpha=1.0}(y,t)$
0.1	0.0031123	−0.00737867	−0.0143255	−0.0187184
0.2	−0.0225622	−0.0327471	−0.0394912	−0.0437559
0.3	−0.0482248	−0.058016	−0.0644995	−0.0685994
0.4	−0.0738719	−0.0831395	−0.0892763	−0.093157
0.5	−0.0999881	−0.10838	−0.113937	−0.117451
0.6	−0.12718	−0.13408	−0.138649	−0.141539
0.7	−0.155338	−0.160137	−0.163315	−0.165324
0.8	−0.182499	−0.185291	−0.187139	−0.188308
0.9	−0.203704	−0.20641	−0.208202	−0.209335
1	−0.210168	−0.21797	−0.223136	−0.226403

5. Conclusion

In this study, we have proposed and investigated the modified Optimal Auxiliary Functions Method (OAFM) as an effective numerical approach for addressing complex differential equations, particularly when dealing with fractional-order derivatives. Our investigation focused on comparing the numerical solution obtained using the OAFM with the exact solution of the fractional-order nonlinear Foam Drainage problem. To provide a comprehensive understanding of the solution's behavior across various fractional order values, we have presented 2D and 3D plots. The obtained results in this study bear the truth of the proposal of the modified OAFM to help solving the nonlinear Foam Drainage equation. The main purpose of the study is to evaluate the accuracy and efficiency of this technique with the proposed tool. Moreover, we attempted to create pictures that show the solution's nature in 2D and 3D twice as the fractional order parameter changes. The generated graphs helped us to see the evolution of the solution in the (changing the fractional order) process, demonstrating the precise effect which the fractional calculus has on the dynamics of the foam drainage equation. The latest results emphasize the adaptability and potency of this modified OAFM in solving differential equations with mixed orders. Additionally, they highlight that fractional calculus comes in handy to describe non-classical diffusion processes in physical systems that are governed by the presence of phase separation. To sum up, the investigation on numerical methods for solving mathematically complex differential equations, including for fractional orders of derivatives, and their essence in physics in reality is the main one of this study.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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