Alexandria Engineering Journal 80 (2023) 372–382

Contents lists available at ScienceDirect

Alexandria Engineering Journal

journal homepage: www.elsevier.com/locate/aej

Original Article

Analysis of Leptospirosis transmission dynamics with environmental effects and bifurcation using fractional-order derivative

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ARTICLE INFO

Keywords:
Leptospirosis fractional order differential equation
Boundedness
Positiveness
Global derivative
Lyapunov function
Bifurcation

ABSTRACT

Mathematical formulations are essential tool to show the dynamics that how various diseases spread in the community. Differential equations with fractional or integer order can be utilized to see the effect of the dynamics direct or indirect Leptospirosis transmission, which are analyzed with different aspects. A mathematical description and dynamical sketch of Leptospirosis with environmental effects have been studied as a result of the successful efforts of various writers. In this study, we analyzed the Leptospirosis model described using a nonlinear fractional-order differential equation that takes the environmental effects into consideration. The proposed fractional order system is investigated qualitatively as well as quantitatively to identify its stable position. Local stability of the Leptospirosis system is verified and test the system with flip bifurcation. Also system is investigated for global stability using Lyapunov first and second derivative functions. The existence, boundedness and positivity of the Leptospirosis is checked, which are the key properties for such of type of epidemic problem to identify reliable findings. Effect of global derivative is demonstrated to verify its rate of effects according to their sub-compartments. Solutions for fractional order system are derived with the help of advanced tool fractal fractional operator for different fractional values. Simulation are carried out to see symptomatic as well as a asymptomatic effects of Leptospirosis in the world wide, also show the actual behavior of Leptospirosis which will be helpful to understand the outbreak of Leptospirosis with environmental effects as well as for future prediction and control strategies.

1. Introduction

Leptospirosis is an infectious disease caused by a bacteria called Leptospira. It is considered one of the zoonoses outbreaks with the greatest geographic reach. According to the research, there are an estimate that 1.03 million cases of Leptospirosis worldwide each year, with a death rate of 5–15%. For countless years, Leptospirosis has been around every where. Adolph Weil, a professor of medicine at Heidelberg University, is credited with the first documented case of Leptospirosis in 1886 [2].

The majority of those who contract this sickness are both humans and mammals. Both clinically and in a lab, Leptospirosis is challenging to diagnose. Because, the condition frequently goes unnoticed and is therefore seriously ignored.

Extreme weather conditions and the impact of global warming are important contributors to the prevalence of Leptospirosis. Humans and their environment interact, which leads to the spread of Leptospirosis in humans. Leptospirosis mostly affects those who reside in places with frequent heavy rains, agriculture, unfavorable environmental conditions, flood-affected regions, areas with rapid growth population and, urban areas as a result of urbanization, inadequate waste management and over-population [7]. Most infected homes are those where people walk through unclean water. Infected animals that emit Leptospira

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https://doi.org/10.1016/j.aej.2023.08.063
Received 25 June 2023; Received in revised form 3 August 2023; Accepted 21 August 2023

in their urine which creates infection directly or indirectly. Leptospira pathogen can enter into the human from front and back and the transmission can be directly or indirectly. Other routes of entry into the body include sexual activity, pregnancy, skin cuts, and ingesting when swimming in infected water. The disease spreads quickly, it is an epidemic as a result of the slow diagnosis process and the lack of specialist clinical facilities [21].

To show the dynamics of how various diseases spread, mathematical formulations are essential. Differential equations of fractional or integer-order can be utilized to affect the dynamics of direct or indirect Leptospirosis transmission, and this has been studied, analyzed and described mathematically by a number of academics. These models, each use a different set of assumptions and parameters. A mathematical description and dynamical sketch of Leptospirosis have been produced as a result of the successful efforts of various writers. See for example [1,11,12,17]. The authors in [14] demonstrated the use of the analytical homotopy perturbation approach to resolve a model of the Leptospirosis epidemic. In order to find the solution using conventional and numerical methods, the authors first formulate the problem and then use the methodology. Additionally, they estimate the model parameters for numerical simulations in the last step. They created a mathematical model to explain how the Leptospirosis disease spreads. Humans and rats make up the population (vectors). The behavior of solutions is then examined using the conventional dynamical modeling approach [19]. The authors in [12] improved the Leptospirosis mathematical model by accounting for the number of exposed people, the associated mortality rate and the ratios of susceptible humans to infected vectors. Last, but not least, these authors do not include exposed classes in their assumptions.

It has been demonstrated that, as compared to conventional integer order models, fractional order models more accurately capture the dynamics of complex systems, including biological systems. In recent years, a growing number of biological and medical systems have been modeled using fractional calculus. The authors in [25] has presented a mathematical model of tumor vaccine efficacy in the context of immunotherapy treatment using fractional-order delay differential equations with a control variable. Also the authors in [26] provided a class of fractional-order differential models of biological systems with memory. Several researchers have made attempts on the stability of fractional order systems. See for example [27–29]. Fractional-order models also have been used to predict and analyze Leptospirosis. The various studies that have used fractional order modeling to analyze Leptospirosis will be looked at in this review of the literature. [4] created fractional-order model for Leptospirosis transmission. The fractional order differential equation's nonlocal property gives the epidemic model of Leptospirosis a more realistic than the classical derivative, which lacks this property. Finally, the numerical outcomes prove that the appropriate techniques are dependent on the fractional derivative. In [22], using Matlab software, the authors performed numerical simulation based on optimization approaches with the Caputo derivative to develop MSEIR model. Moreover, the authors in [13] presented a unique positive solution of fractional-order Leptospirosis model using numerical techniques. The authors compared the approximate solution of MGDTM with Fourth-order R-K for the classical derivative and presented the numerical results for the justification of their results. In [23], the authors demonstrated optimal control of the Leptospirosis model with non-linear saturated. The authors in [3] developed the SEIR epidemic model with vertical transmission and a death rate of population that depends upon population density. The authors of [9] addressed the classical and fractional-order SEIR Ebola epidemic models as well as their comparison to actual data taken from periodic reports released by the WHO as of March 27, 2014. Its method provides an accurate representation of the actual data. In an attempt to describe and comprehend the influenza outbreaks, [5] also presented a nonlinear fractional order model. The future state in the fractional model is dependent on both the previous and present states. The authors also showed that the fractional models are better than the integer-order models. For more information regarding the fractional order modeling of Leptospirosis, see [8,13,15,18,20,24] with the use of fractional-order differential equations and accounting for environmental effects. We have analyzed the Leptospirosis mathematical model and its transmission dynamics. The mathematical model has been analyzed both qualitatively and quantitatively.

This paper is organized as follows: Section 2 contains the preliminary definitions and the model's formulation and description. In the Section 3, we analyzed the model's solution by verifying the properties of bifurcation, boundedness and positivity for the proposed system. In Section 4, we verify the effect of global derivative on the proposed disease. In Section 5, the global stability analysis is demonstrated using Lyapunov's approach and Lasalle's invariance principle. Section 6 contains the Computational analysis with fractal fractional operator. Finally, in section 7, the numerical results and conversations are represented visually to highlight how different parameters can affect the condition. At the end, Section 8 concludes our paper.

2. Preliminaries

2.1. Basic concepts

In this part, we will present the basic concepts which we will use in the major part of the article.

Definition 1. For a dynamic system $D_t^\gamma x(t) = f(t,x)$, if its rate of change with respect to time remain fixed or constant i.e. $f(t,x) = 0$, then the solution is known as equilibrium solution.

Lemma 1. *For $\gamma \in (0,1)$, if $D_t^\gamma x(t) \geq 0$ with $x(0) \geq 0 \Rightarrow x(t) \geq 0$.*

Definition 2. The gamma function is defined by:

$$\Gamma(z) = e^{-t} t^{z-1} dt, \tag{1}$$

where Re $(z) > 0$.

Definition 3. The fractional-order integral in the Riemann-Liouville sense is defined as:

$$D_x^\gamma f(y) = \frac{1}{\Gamma(1-\gamma)} \int_x^y f'(y-u)(u-x)^{-\gamma}\, du \tag{2}$$

Definition 4. The Riemann-Liouville integral operator of a function $f \in C[a,b]$ with $\gamma \geq 0$ is defined as:

$$J_a^\gamma f(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \left(\log \frac{x}{t}\right)^{\gamma-1} f(t) \frac{dt}{t}, \tag{3}$$

Definition 5. The Caputo fractional derivative is defined as:

$${}_a D_t^\gamma f(t) = \frac{1}{\Gamma(1-\gamma)} \int_a^t \frac{f'(s)}{(t-s)^\gamma} ds \tag{4}$$

Definition 6. An exponential kernel with a generalized form of fractional operator is given as

$${}_0^{FFE} D_t^{\gamma,\beta} f(t) = \frac{M(\gamma)}{1-\gamma} \frac{d}{dt^\beta} \int_0^t f(\tau) \exp\left[-\frac{\gamma}{1-\gamma}(t- \ tau)\right] d\tau \tag{5}$$

Definition 7. A Mittag Leffler kernel with a generalized form of the fractional operator is given as

$${}_0^{FFM} D_t^{\gamma,\beta} f(t) = \frac{AB(\gamma)}{1-\gamma} \frac{d}{dt^\beta} \int_0^t f(\tau) E_\gamma \left[-\frac{\gamma}{1-\gamma}(t-\tau)^\gamma\right] d\tau \tag{6}$$

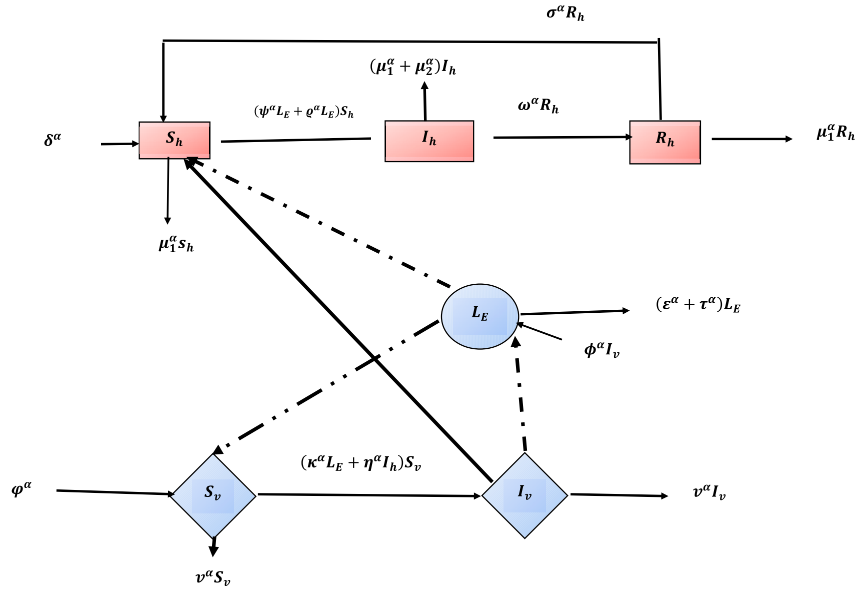

Fig. 1. Flow diagram of Leptospirosis with environmental effects.

Definition 8. The associated fractal-fractional integral of order (γ, β) for the Mittag-Leffler kernel is given as

$$ {}_0^{FFE}Y_t^{\gamma,\beta} f(t) = \frac{1-\gamma}{AB(\gamma)} t^{1-\beta} f(t) + \frac{\gamma}{AB(\gamma)\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} \tau^{1-\beta} f(\tau)\, d\tau \quad (7) $$

2.2. Description of mathematical model

The interaction between humans, animals and the leptospira present in the environment is shown in the Fig. 1.

In this model, humans are taken as the host population, animals as the vector population, and the Leptospira pathogen present in the environment. The host population (humans) is again divided into three categories: Suspected humans $S_\hbar$, infected humans $I_\hbar$, and the recovered humans $R_\hbar$. The vector population (animals) is again divided into two compartments: suspected vectors S_v and infected vectors I_v, where the number of Leptospirosis in the environment is denoted by L.

Using the assumptions and the parameters description, the model equations are formulated as follows:

$$
\begin{aligned}
\frac{dS_\hbar(t)}{dt} &= \delta^\gamma - \left(\psi^\gamma L + \varrho^\gamma I_v\right) S_\hbar + \sigma^\gamma R_\hbar - \mu_1^\gamma S_\hbar \\
\frac{dI_\hbar(t)}{dt} &= \left(\psi^\gamma L + \varrho^\gamma I_v\right) S_\hbar - W_2 I_\hbar, \\
\frac{dR_\hbar(t)}{dt} &= \omega^\gamma I_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right) R_\hbar, \\
\frac{dS_v(t)}{dt} &= \varphi^\gamma - \left(\eta^\gamma I_\hbar + \kappa^\gamma L\right) S_v - v^\gamma S_v, \\
\frac{dI_v(t)}{dt} &= \left(\eta^\gamma I_\hbar + k^\gamma L\right) S_v - v^\gamma I_v, \\
\frac{dL(t)}{dt} &= \phi^\gamma I_v - (\varepsilon^\gamma + \tau^\gamma) L,
\end{aligned} \quad (8)
$$

with the initial conditions $S_\hbar(0) = S_{\hbar_0}, I_\hbar(0) = I_{\hbar_0}, R_\hbar(0) = R_{\hbar_0}, S_v(0) = S_{v_0}, I_v(0) = I_{v_0}, L(0) = L_0$.

If we apply fractional-order derivative D_t^γ on the model given above, then it takes the form

$$
\begin{aligned}
D_t^\gamma S_\hbar(t) &= \delta^\gamma - \left(\psi^\gamma L + \varrho^\gamma I_v\right) S_\hbar + \sigma^\gamma R_\hbar - \mu_1^\gamma S_\hbar \\
D_t^\gamma I_\hbar(t) &= \left(\psi^\gamma L + \varrho^\gamma I_v\right) S_\hbar - W_2 I_\hbar, \\
D_t^\gamma R_\hbar(t) &= \omega^\gamma I_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right) R_\hbar, \\
D_t^\gamma S_v(t) &= \varphi^\gamma - \left(\eta^\gamma I_\hbar + \kappa^\gamma L\right) S_v - v^\gamma S_v, \\
D_t^\gamma I_v(t) &= \left(\eta^\gamma I_\hbar + k^\gamma L\right) S_v - v^\gamma I_v, \\
D_t^\gamma L(t) &= \phi^\gamma I_v - (\varepsilon^\gamma + \tau^\gamma) L,
\end{aligned} \quad (9)
$$

with the same initial conditions as for classical order system.

Now we give a description of the parameters which are being used in the model:

Parameters	Estimated Value	References
ϱ^γ	Rate of contact of Humans with infected vectors	[4]
ψ^γ	Rate of contact of Humans with contaminated environment	Assumed
η^γ	Rate of contact of Susceptible Vectors with Infected Humans	[4]
κ^γ	Rate of contact of Susceptible Vectors with contaminated environment	Assumed
δ^γ	Rate of increase of Human population	[4]
μ_1^γ	Natural Death rate of Humans	[4]
ω^γ	Pace at which Humans are recovered from disease	[4]
φ^γ	Rate of Growth of Population of Animal Vectors	[4]
v^γ	Death Rate for the Vector Population	[10]
ε^γ	Rate of decay of Leptospira in the Environment	[6]
τ^γ	Sanitation Rate	Assumed
ϕ^γ	Shedding Rate of Leptospira by Vectors	Assumed

3. The model's qualitative and quantitative analysis

Now, we present our main results. We will prove boundedness, uniqueness, and positiveness for the solution of the proposed model which shows that the system is mathematically and biologically well-posed.

The disease free equilibrium point for the Leptospirosis model (8), (9) are given by [16]:

$E^0 = \left(\frac{\delta^\gamma}{\mu_1^\gamma}, 0, 0, \frac{\varphi^\gamma}{v^\gamma}, 0, 0\right)$ and endemic equilibrium points are

$$
\begin{aligned}
S_h^* &= \frac{\delta^\gamma}{\mu_1^\gamma} + \left(\frac{\sigma^\gamma \omega^\gamma}{\mu_1^\gamma\left(\sigma^\gamma + \mu_1^\gamma\right)} - \frac{W_2}{\mu_1^\gamma}\right) I_h^*, \\
I_h^* &= \frac{\left(\psi^\gamma L + \varrho^\gamma I_v\right) S_\hbar}{W_2}, \\
R_h^* &= \frac{\omega^\gamma}{\sigma^\gamma + \mu_1^\gamma} I_h^*, \\
S_v^* &= \frac{\varphi^\gamma}{v^\gamma} - \frac{W_1}{\psi^\gamma \phi^\gamma + \varrho^\gamma W_1} \left(\frac{\delta^\gamma + \frac{\sigma^\gamma \omega^\gamma}{\sigma^\gamma + \mu_1^\gamma} I_h^*}{\frac{\delta^\gamma}{\mu_1^\gamma} + \left(\frac{\sigma^\gamma \omega^\gamma}{\mu_1^\gamma\left(\sigma^\gamma + \mu_1^\gamma\right)} - \frac{W_2}{\mu_1^\gamma}\right) I_h^*} - \mu_1^\gamma \right), \\
I_v^* &= \frac{W_1}{\psi^\gamma \phi^\gamma + \varrho^\gamma W_1} \left(\frac{\delta^\gamma + \frac{\sigma^\gamma \omega^\gamma}{\sigma^\gamma + \mu_1^\gamma} I_h^*}{\frac{\delta^\gamma}{\mu_1^\gamma} + \left(\frac{\sigma^\gamma \omega^\gamma}{\mu_1^\gamma\left(\sigma^\gamma + \mu_1^\gamma\right)} - \frac{W_2}{\mu_1^\gamma}\right) I_h^*} - \mu_1^\gamma \right), \\
L^* &= \frac{\phi^\gamma}{\sigma^\gamma \psi^\gamma + \varrho^\gamma W_1} \left(\frac{\delta^\gamma + \frac{\sigma^\gamma \omega^\gamma}{\sigma^\gamma + \mu_1^\gamma} I_h^*}{\frac{\delta^\gamma}{\mu_1^\gamma} + \left(\frac{\sigma^\gamma \omega^\gamma}{\mu_1^\gamma\left(\sigma^\gamma + \mu_1^\gamma\right)} - \frac{W_2}{\mu_1^\gamma}\right) I_h^*} - \mu_1^\gamma \right).
\end{aligned}
$$

3.1. Bifurcation

From [16], we see that neither of the eigen values $= 1, -1$, which shows that bifurcation may exist for our model (8) if the set of constants

$$(\delta^\gamma, \psi^\gamma, \eta^\gamma, \mu_1^\gamma, \sigma^\gamma, \kappa^\gamma, \varepsilon^\gamma, \tau^\gamma, \nu^\gamma, \phi^\gamma, \omega^\gamma, \varrho^\gamma)$$

is located in the set:

$$F|_{E(\frac{\delta^\gamma}{\mu_1},0,0,\frac{\varphi^\gamma}{\nu^\gamma},0,0)} = \{(\delta^\gamma, \psi^\gamma, \eta^\gamma, \mu_1^\gamma, \sigma^\gamma, \kappa^\gamma, \varepsilon^\gamma, \tau^\gamma, \nu^\gamma, \phi^\gamma, \omega^\gamma, \varrho^\gamma) : \mu_1^\gamma = -\sigma^\gamma, \nu^\gamma = 0\}. \quad (10)$$

But using the theorem given below, we will show that bifurcation does not exist for our model at

$$E(\frac{\delta^\gamma}{\mu_1}, 0, 0, \frac{\varphi^\gamma}{\nu^\gamma}, 0, 0)$$

if the set of constants

$$(\delta^\gamma, \psi^\gamma, \eta^\gamma, \mu_1^\gamma, \sigma^\gamma, \kappa^\gamma, \varepsilon^\gamma, \tau^\gamma, \nu^\gamma, \phi^\gamma, \omega^\gamma, \varrho^\gamma) \in F|_{E(\frac{\delta^\gamma}{\mu_1},0,0,\frac{\varphi^\gamma}{\nu^\gamma},0,0)}.$$

Theorem 3.1. *Bifurcation does not exist for our model (8) at*

$$E(\frac{\delta^\gamma}{\mu_1}, 0, 0, \frac{\varphi^\gamma}{\nu^\gamma}, 0, 0)$$

if the set of constants

$$(\delta^\gamma, \psi^\gamma, \eta^\gamma, \mu_1^\gamma, \sigma^\gamma, \kappa^\gamma, \varepsilon^\gamma, \tau^\gamma, \nu^\gamma, \phi^\gamma, \omega^\gamma, \varrho^\gamma) \in F|_{E(\frac{\delta^\gamma}{\mu_1},0,0,\frac{\varphi^\gamma}{\nu^\gamma},0,0)}.$$

Proof. Note that our model equation (8) is invariant w.r.t $I_h, R_h, I_\upsilon, L = 0$, therefore to check the existence of bifurcation in our model, we take $I_h, R_h, I_\upsilon, L = 0$ and so we have

$$\begin{aligned} S_h(t) &= h\delta^\gamma + \left(-h\mu_1^\gamma + 1\right) S_h(t), \\ S_\upsilon(t) &= h\varphi^\gamma + (-h\nu^\gamma + 1) S_\upsilon(t). \end{aligned} \quad (11)$$

So using equation (11), we can write

$$\begin{aligned} f(S_h(t)) &= h\delta^\gamma + \left(-h\mu_1^\gamma + 1\right) S_h(t), \\ g(S_\upsilon(t)) &= h\varphi^\gamma + (-h\nu^\gamma + 1) S_\upsilon(t). \end{aligned} \quad (12)$$

Now if $\mu_1^\gamma = -\sigma^\gamma, S_h(t) = \frac{\delta^\gamma}{\mu_1^\gamma}, S_\upsilon(t) = \frac{\varphi^\gamma}{\nu^\gamma}, \nu^\gamma = 0$, then using equation (12), we obtain

$$\begin{aligned} \frac{\partial f(S_h(t))}{\partial S_h(t)}\Big|_{\mu_1=-\sigma^\gamma, S_h(t)=\frac{\delta^\gamma}{\mu_1^\gamma}} &= h\sigma^\gamma + 1, \\ \frac{\partial g(S_\upsilon(t))}{\partial S_\upsilon(t)}\Big|_{S_\upsilon(t)=\frac{\varphi^\gamma}{\nu^\gamma}, \nu^\gamma=0} &= 1. \end{aligned} \quad (13)$$

Now by taking partial derivative of $f(S_h(t))$ w.r.t μ_1^γ and using values of $S_h(t)$ and μ_1^γ, we get $-h\frac{\delta^\gamma}{\mu_1^\gamma} \neq 0$, and by taking partial derivative of $f(S_h(t))$ w.r.t ν^γ and using values of $S_\upsilon(t)$ and ν^γ, we get $-h\frac{\varphi^\gamma}{\nu^\gamma} \neq 0$.
Now taking second order partial derivative of equation (13), we have

$$\begin{aligned} \frac{\partial^2 f(S_h(t))}{\partial S_h^2(t)} &= 0, \\ \frac{\partial^2 g(S_\upsilon(t))}{\partial S_\upsilon^2(t)} &= 0. \end{aligned} \quad (14)$$

The above results show that bifurcation does not exist for our model (8) because the condition (14) does not satisfy the condition for bifurcation existence if the set of constants

$$(\delta^\gamma, \psi^\gamma, \eta^\gamma, \mu_1^\gamma, \sigma^\gamma, \kappa^\gamma, \varepsilon^\gamma, \tau^\gamma, \nu^\gamma, \phi^\gamma, \omega^\gamma, \varrho^\gamma) \in F|_{E(\frac{\delta^\gamma}{\mu_1},0,0,\frac{\varphi^\gamma}{\nu^\gamma},0,0)}.$$

3.2. Boundedness and positivity of model

Theorem 2. *In straight line conditions, the model suggested solution is distinct and constrained in $\mathbb{R}_+^6$.*

Proof. We have

$$\begin{aligned} D_t^\gamma S_h(t) &= \delta^\gamma + \sigma^\gamma R_h \geq 0, \\ D_t^\gamma I_h(t) &= \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_h \geq 0, \\ D_t^\gamma R_h(t) &= \omega^\gamma I_h \geq 0, \\ D_t^\gamma S_\upsilon(t) &= \phi^\gamma \geq 0 \\ D_t^\gamma I_\upsilon(t) &= \left(\eta^\gamma I_h + k^\gamma L\right) S_\upsilon \geq 0 \\ D_t^\gamma L(t) &= \phi^\gamma I_\upsilon \geq 0 \end{aligned} \quad (15)$$

Theorem 3. *If the solutions obtained from the system of equation (8), (9) are positive $\forall\ t > 0$, if $S_h(0), I_h(0), R_h(0), S_\upsilon(0), I_\upsilon(0)$ and $L(0)$ are also positive.*

Proof. If the solutions to the system of equations are positive $\forall\ t > 0$, then

$$\frac{dS_h}{dt} = \delta^\gamma - \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_h + \sigma^\gamma R_h - \mu_1^\gamma S_h$$

After solving this, we obtain

$$S_h\left(t_*\right) = S_h(0)\, e^{-\left(\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) + \mu_1^\gamma\right)t} + e^{-\left(\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) + \mu_1^\gamma\right)t} \int_0^{t_*} \left(\delta^\gamma + \sigma^\gamma R_h\right) e^{\left(\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) + \mu_1^\gamma\right)t} d\tau$$

So if $S_h(0) > 0$, then $S_h\left(t_*\right) > 0$.
Similarly, we have

$$I_h\left(t_*\right) = I_h(0)\, e^{-W_2 t} + e^{-W_2 t} \int_0^{t_*} \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) e^{W_2 t} d\tau,$$

So when $I_h(0) > 0$, then $I_h\left(t_*\right) > 0$

$$R_h\left(t_*\right) = R_h(0)\, e^{-\left(\sigma^\gamma + \mu_1^\gamma\right)t} + e^{-\left(\sigma^\gamma + \mu_1^\gamma\right)t} \int_0^{t_*} \omega^\gamma I_h e^{-\left(\sigma^\gamma + \mu_1^\gamma\right)t} d\tau,$$

So when $R_h(0) > 0$, then $R_h\left(t_*\right) > 0$

$$S_\upsilon\left(t_*\right) = S_\upsilon(0)\, e^{-\left(\left(\eta^\gamma I_h + k^\gamma L\right) + \upsilon^\gamma\right)t} + e^{-\left(\left(\eta^\gamma I_h + k^\gamma L\right) + \upsilon^\gamma\right)t} \int_0^{t_*} \varphi^\gamma e^{\left(\left(k^\gamma L + \eta^\gamma I_h\right) + \upsilon^\gamma\right)t} d\tau,$$

So when $S_\upsilon(0) > 0$, then $S_\upsilon\left(t_*\right) > 0$

$$I_\upsilon\left(t_*\right) = I_\upsilon(0)\, e^{-\upsilon^\gamma t} + e^{-\upsilon^\gamma t} \int_0^{t_*} \left(\eta^\gamma I_h + k^\gamma L\right) S_\upsilon e^{\upsilon^\gamma t} d\tau,$$

So when $I_\upsilon(0) > 0$, then $I_\upsilon\left(t_*\right) > 0$

$$L\left(t_*\right) = L(0)\, e^{-(\varepsilon^\gamma + \tau^\gamma)t} + e^{-(\varepsilon^\gamma + \tau^\gamma)t} \int_0^{t_*} \phi^\gamma I_\upsilon e^{(\varepsilon^\gamma + \tau^\gamma)t} d\tau,$$

So when $L(0) > 0$, then $L\left(t_*\right) > 0$

4. Effect of global derivative

The Riemann Stieltjes integration

$$\int f(x)\, dx = F(x)$$

is the most famous definition of integration. Geometrically it gives the area under the curve of any function $f(x)$. The Riemann Stieltjes integral for any function $g(x)$ w.r.t function f is given by

$$F_g(x) = \int f(x)\, dg(x)$$

R-S integral is related to global derivative. The global derivative of a function f w.r.t a function g is given by

$$D_g f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{g(x+h) - g(x)}.$$

So if we take the classical derivative of both functions, then we have the expression

$$D_g f(x) = \frac{f'(x)}{g'(x)},$$

where $g'(x) \neq 0$, $\forall\, x \in D_{g'}$.

Now, we will use the global derivative in place of the conventional derivative

$$\begin{aligned}
D_g S_\hbar(t) &= \delta^\gamma - \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar + \sigma^\gamma R_\hbar - \mu_1^\gamma S_\hbar \\
D_g I_\hbar(t) &= \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar - W_2 I_\hbar, \\
D_g R_\hbar(t) &= \omega^\gamma I_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right) R_\hbar, \\
D_g S_\upsilon(t) &= \varphi^\gamma - \left(\eta^\gamma I_\hbar + \kappa^\gamma L\right) S_\upsilon - \upsilon^\gamma S_\upsilon, \\
D_g I_\upsilon(t) &= \left(\eta^\gamma I_\hbar + k^\gamma L\right) S_\upsilon - \upsilon^\gamma I_\upsilon, \\
D_g L(t) &= \phi^\gamma I_\upsilon - \left(\varepsilon^\gamma + \tau^\gamma\right) L.
\end{aligned}$$

For simplicity, we assume that g is differentiable. Thus, we can write

$$\begin{aligned}
S_\hbar(t) &= g'\left(\delta^\gamma - \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar + \sigma^\gamma R_\hbar - \mu_1^\gamma S_\hbar\right) &&= J_1(t,\xi) \\
I_\hbar(t) &= g'\left(\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar - W_2 I_\hbar\right) &&= J_2(t,\xi), \\
R_\hbar(t) &= g'\left(\omega^\gamma I_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right) R_\hbar\right) &&= J_3(t,\xi), \\
S_\upsilon(t) &= g'\left(\varphi^\gamma - \left(\eta^\gamma I_\hbar + \kappa^\gamma L\right) S_\upsilon - \upsilon^\gamma S_\upsilon\right) &&= J_4(t,\xi), \\
I_\upsilon(t) &= g'\left(\left(\eta^\gamma I_\hbar + k^\gamma L\right) S_\upsilon - \upsilon^\gamma I_\upsilon\right) &&= J_5(t,\xi), \\
L(t) &= g'\left(\phi^\gamma I_\upsilon - \left(\varepsilon^\gamma + \tau^\gamma\right) L\right) &&= J_6(t,\xi.),
\end{aligned}$$

where

$$\xi = S_\hbar, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L.$$

If we select the function g in a suitable manner, it will take us to a particular procedure. For example, if we choose $g(t) = t^\gamma, \gamma \in \mathbb{R}$, then we will observe the fractal behavior. But the condition is that

$$\left\|g'\right\|_\infty = \sup_{t\in D_{g'}} \left|g'(t)\right| < N \tag{16}$$

Equation (16) shows that the system of equations can accept only one solution. If we verify the two conditions given by:

$$\begin{aligned}
&J\left(t, S_\hbar, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right) < \kappa\left(1 + \left|S_\hbar\right|^2\right) \\
&\quad\left\|J\left(t, \left(S_\hbar\right)_1, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right) - J\left(t, \left(S_\hbar\right)_2, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right)\right\| \\
&< \kappa\left\|\left(S_\hbar\right)_1 - \left(S_\hbar\right)_2\right\|_\infty^2, \ \forall \left(S_\hbar\right)_1, \left(S_\hbar\right)_2,
\end{aligned}$$

Initially,

$$\begin{aligned}
&\left|J_1\left(t, \left(S_\hbar\right)_1, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right)\right|^2 \\
&= \left|g'\left(\delta^\gamma - \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar + \sigma^\gamma R_\hbar - \mu_1^\gamma S_\hbar\right)\right|^2 \\
&= \left|g'\right|^2 \left|\left(\delta^\gamma - \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar + \sigma^\gamma R_\hbar - \mu_1^\gamma S_\hbar\right)\right|^2 \\
&\le 2\left[\left|g'\right|^2 |\delta^\gamma|^2 + \left|\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) - \mu_1^\gamma\right|^2 \left|S_\hbar\right|^2 + \left|\sigma^\gamma R_\hbar\right|^2\right] \\
&= 2\left|g'\right|^2 \left(\left|\sigma^\gamma R_\hbar\right|^2 + |\delta^\gamma|^2\right)\left[1 + \frac{\left|\left(\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) - \mu_1^\gamma\right)\right|^2 \left|S_\hbar\right|^2}{|\delta^\gamma|^2 + \left|\sigma^\gamma R_\hbar\right|^2}\right] \\
&< \kappa_1\left(1 + \left|S_\hbar\right|^2\right)
\end{aligned}$$

under the condition

$$\frac{\left|\left(\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) - \mu_1^\gamma\right)\right|^2 \left|S_\hbar\right|^2}{|\delta^\gamma|^2 + \left|\sigma^\gamma R_\hbar\right|^2} < 1,$$

where

$$\kappa_1 = 2\left|g'\right|^2 \left(\left|\sigma^\gamma R_\hbar\right|^2 + |\delta^\gamma|^2\right)$$

Now,

$$\begin{aligned}
&\left|J_2\left(t, \left(S_\hbar\right)_1, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right)\right|^2 \\
&= \left|g'\left(\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar - W_2 I_\hbar\right)\right|^2 \\
&= \left|g'\right|^2 \left|\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar - W_2 I_\hbar\right|^2 \\
&\le 2\left[\left|g'\right|^2 \left|\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar\right|^2 + \left|W_2\right|^2 \left|I_\hbar\right|^2\right] \\
&= 2\left|g'\right|^2 \left|\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar\right|^2 \left[1 + \frac{\left|W_2\right|^2 \left|I_\hbar\right|^2}{\left|\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar\right|^2}\right] \\
&< \kappa_2\left(1 + \left|I_\hbar\right|^2\right)
\end{aligned}$$

under the condition

$$\frac{\left|W_2\right|^2 \left|I_\hbar\right|^2}{\left|\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar\right|^2} < 1,$$

where

$$\kappa_2 = 2\left|g'\right|^2 \left|\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar\right|^2$$

Now,

$$\begin{aligned}
&\left|J_3\left(t, \left(S_\hbar\right)_1, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right)\right|^2 \\
&= \left|g'\left(\omega^\gamma I_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right) R_\hbar\right)\right|^2 \\
&= \left|g'\right|^2 \left|\left(\omega^\gamma I_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right) R_\hbar\right)\right|^2 \\
&\le 2\left[\left|g'\right|^2 \left|\omega^\gamma I_\hbar\right|^2 + \left|\left(\sigma^\gamma + \mu_1^\gamma\right)\right|^2 \left|R_\hbar\right|^2\right] \\
&= 2\left|g'\right|^2 \left|\omega^\gamma I_\hbar\right|^2 \left[1 + \frac{\left|\left(\sigma^\gamma + \mu_1^\gamma\right)\right|^2 \left|R_\hbar\right|^2}{\left|\omega^\gamma I_\hbar\right|^2}\right] \\
&< \kappa_3\left(1 + \left|R_\hbar\right|^2\right)
\end{aligned}$$

under the condition

$$\frac{\left|\left(\sigma^\gamma + \mu_1^\gamma\right)\right|^2 \left|R_\hbar\right|^2}{\left|\omega^\gamma I_\hbar\right|^2} < 1,$$

where

$$\kappa_3 = 2\left|g'\right|^2 \left|\omega^\gamma I_\hbar\right|^2$$

Now,

$$\begin{aligned}
&\left|J_4\left(t, \left(S_\hbar\right)_1, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right)\right|^2 \\
&= \left|g'\left(\varphi^\gamma - \left(\eta^\gamma I_\hbar + \kappa^\gamma L\right) S_\upsilon - \upsilon^\gamma S_\upsilon\right)\right|^2 \\
&= \left|g'\right|^2 \left|\left(\varphi^\gamma - \left(\eta^\gamma I_\hbar + \kappa^\gamma L + \upsilon^\gamma\right) S_\upsilon\right)\right|^2 \\
&\le 2\left[\left|g'\right|^2 \left|\varphi^\gamma\right|^2 + \left|\left(\eta^\gamma I_\hbar + \kappa^\gamma L + \upsilon^\gamma\right)\right|^2 \left|S_\upsilon\right|^2\right] \\
&= 2\left|g'\right|^2 \left|\varphi^\gamma\right|^2 \left[1 + \frac{\left|\left(\eta^\gamma I_\hbar + \kappa^\gamma L + \upsilon^\gamma\right)\right|^2 \left|S_\upsilon\right|^2}{\left|\varphi^\gamma\right|^2}\right] \\
&< \kappa_4\left(1 + \left|S_\upsilon\right|^2\right)
\end{aligned}$$

under the condition

$$\frac{\left|\left(\eta^\gamma I_\hbar + \kappa^\gamma L + \upsilon^\gamma\right)\right|^2 \left|S_\upsilon\right|^2}{\left|\varphi^\gamma\right|^2} < 1,$$

where

$$\kappa_4 = 2\left|g'\right|^2 |\varphi^\gamma|^2$$

Now,

$$\left|J_5\left(t,(S_\hbar)_1, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right)\right|^2$$
$$= \left|g'\left(\left(\eta^\gamma I_\hbar + k^\gamma L\right)S_\upsilon - \upsilon^\gamma I_\upsilon\right)\right|^2$$
$$= \left|g'\right|^2\left|\left(\left(\eta^\gamma I_\hbar + k^\gamma L\right)S_\upsilon - \upsilon^\gamma I_\upsilon\right)\right|^2$$
$$\le 2\left[\left|g'\right|^2\left|\left(\eta^\gamma I_\hbar + k^\gamma L\right)S_\upsilon\right|^2 + |\upsilon^\gamma|^2\left|I_\upsilon\right|^2\right]$$
$$= 2\left|g'\right|^2\left|\left(\eta^\gamma I_\hbar + k^\gamma L\right)S_\upsilon\right|^2\left[1+\frac{|\upsilon^\gamma|^2\left|I_\upsilon\right|^2}{\left|\left(\eta^\gamma I_\hbar + k^\gamma L\right)S_\upsilon\right|^2}\right]$$
$$< \kappa_5\left(1+\left|I_\upsilon\right|^2\right)$$

under the condition

$$\frac{|\upsilon^\gamma|^2\left|I_\upsilon\right|^2}{\left|\left(\eta^\gamma I_\hbar + k^\gamma L\right)S_\upsilon\right|^2} < 1,$$

where

$$\kappa_5 = 2\left|g'\right|^2\left|\left(\eta^\gamma I_\hbar + k^\gamma L\right)S_\upsilon\right|^2$$

Now,

$$\left|J_6\left(t,(S_\hbar)_1, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right)\right|^2$$
$$= \left|g'\left(\phi^\gamma I_\upsilon - (\epsilon^\gamma + \tau^\gamma)L\right)\right|^2$$
$$= \left|g'\right|^2\left|\left(\phi^\gamma I_\upsilon - (\epsilon^\gamma + \tau^\gamma)L\right)\right|^2$$
$$\le 2\left[\left|g'\right|^2\left|\phi^\gamma I_\upsilon\right|^2 + |(\epsilon^\gamma + \tau^\gamma)|^2|L|^2\right]$$
$$= 2\left|g'\right|^2\left|\phi^\gamma I_\upsilon\right|^2\left[1+\frac{|(\epsilon^\gamma + \tau^\gamma)|^2|L|^2}{\left|\phi^\gamma I_\upsilon\right|^2}\right]$$
$$< \kappa_6\left(1+|L|^2\right)$$

under the condition

$$\frac{|(\epsilon^\gamma + \tau^\gamma)|^2|L|^2}{\left|\phi^\gamma I_\upsilon\right|^2} < 1,$$

where

$$\kappa_6 = 2\left|g'\right|^2\left|\phi^\gamma I_\upsilon\right|^2$$

Hence it is proved that it is defined for linear growth condition. Further, we validate the Lipschitz condition. If

$$\left|J_1\left(t,(S_\hbar)_1, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right) - J_1\left(t,(S_\hbar)_2, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, L\right)\right|^2$$
$$= \left|\psi^\gamma L + \varrho^\gamma I_\upsilon - \mu_1^\gamma\right|^2\left|(S_\hbar)_1 - (S_\hbar)_2\right|^2$$
$$\le 2\left[|\psi^\gamma|^2|L|^2 + |\varrho^\gamma|^2\left|I_\upsilon\right|^2 + \left|\mu_1^\gamma\right|^2\right]\left|(S_\hbar)_1 - (S_\hbar)_2\right|^2$$
$$\le 2\left[|\psi^\gamma|^2\sup_{t\in D_L}|L|^2 + |\varrho^\gamma|^2\sup_{t\in D_{I_\upsilon}}\left|I_\upsilon\right|^2 + \left|\mu_1^\gamma\right|^2\right]\sup_{t\in D_{S_\hbar}}\left|(S_\hbar)_1 - (S_\hbar)_2\right|^2$$
$$\le 2\left[|\psi^\gamma|^2\|L\|_\infty^2 + |\varrho^\gamma|^2\left\|I_\upsilon\right\|^2 + \left|\mu_1^\gamma\right|^2\right]\left\|(S_\hbar)_1 - (S_\hbar)_2\right\|_\infty^2$$
$$\le \overline{\kappa}_1\left\|(S_\hbar)_1 - (S_\hbar)_2\right\|_{\inf}^2$$

where

$$\overline{\kappa}_1 = 2\left[|\psi^\gamma|^2\|L\|_\infty^2 + |\varrho^\gamma|^2\left\|I_\upsilon\right\|^2 + \left|\mu_1^\gamma\right|^2\right]$$

Now

$$\left|J_2\left(t,S_\hbar,(I_\hbar)_1, R_\hbar, S_\upsilon, I_\upsilon, L\right) - J_2\left(t,S_\hbar,(I_\hbar)_2, R_\hbar, S_\upsilon, I_\upsilon, L\right)\right|^2$$
$$= |-W_2|^2\left|(I_\hbar)_1 - (I_\hbar)_2\right|^2$$
$$\le \left(W_2^2+\varepsilon\right)\left|(I_\hbar)_1 - (I_\hbar)_2\right|^2$$
$$\le \left(W_2^2+\varepsilon\right)\sup_{t\in D_{I_\hbar}}\left|(I_\hbar)_1 - (I_\hbar)_2\right|^2$$
$$\le \left(W_2^2+\varepsilon\right)\left\|(I_\hbar)_1 - (I_\hbar)_2\right\|_\infty^2$$
$$\le \overline{\kappa}_2\left\|\left((I_\hbar)_1 - (I_\hbar)_2\right)\right\|_{\inf}^2$$

where

$$\overline{\kappa}_2 = W_2^2 + \varepsilon$$

Now

$$\left|J_3\left(t,S_\hbar, I_\hbar, (R_\hbar)_1, S_\upsilon, I_\upsilon, L\right) - J_3\left(t,S_\hbar, I_\hbar, (R_\hbar)_2, S_\upsilon, I_\upsilon, L\right)\right|^2$$
$$= \left|-\left(\sigma^\gamma + \mu_1^\gamma\right)\right|^2\left|(R_\hbar)_1 - (R_\hbar)_2\right|^2$$
$$\le \left[\left(\sigma^\gamma + \mu_1^\gamma\right)^2 + \varepsilon\right]\left|(R_\hbar)_1 - (R_\hbar)_2\right|^2$$
$$\le \left[\left(\sigma^\gamma + \mu_1^\gamma\right)^2 + \varepsilon\right]\sup_{t\in D_{R_\hbar}}\left|(R_\hbar)_1 - (R_\hbar)_2\right|^2$$
$$\le \left[\left(\sigma^\gamma + \mu_1^\gamma\right)^2 + \varepsilon\right]\left\|(R_\hbar)_1 - (R_\hbar)_2\right\|_\infty^2$$
$$\le \overline{\kappa}_3\left\|(R_\hbar)_1 - (R_\hbar)_2\right\|_{\inf}^2$$

where

$$\overline{\kappa}_3 = \left(\sigma^\gamma + \mu_1^\gamma\right)^2 + \varepsilon$$

Now

$$\left|J_4\left(t,S_\hbar, I_\hbar, R_\hbar, (S_\upsilon)_1, I_\upsilon, L\right) - J_4\left(t,S_\hbar, I_\hbar, R_\hbar, (S_\upsilon)_2, I_\upsilon, L\right)\right|^2$$
$$= \left|-\left(\left(\eta^\gamma I_\hbar + \kappa^\gamma L\right) - \upsilon^\gamma\right)\right|^2\left|(S_\upsilon)_1 - (S_\upsilon)_2\right|^2$$
$$\le 2\left[\left|\eta^\gamma I_\hbar\right|^2 + |\kappa^\gamma L|^2 + |\upsilon^\gamma|^2\right]\left|(S_\upsilon)_1 - (S_\upsilon)_2\right|^2$$
$$\le 2\left[(\eta^\gamma)^2\left|I_\hbar\right|^2 + (\kappa^\gamma)^2|L|^2 + (\upsilon^\gamma)^2\right]\sup_{t\in D_{S_\upsilon}}\left|(S_\upsilon)_1 - (S_\upsilon)_2\right|^2$$
$$\le 2\left[(\eta^\gamma)^2\sup_{t\in D_{I_\hbar}}\left|I_\hbar\right|^2 + (\kappa^\gamma)^2\sup_{t\in D_L}|L|^2 + (\upsilon^\gamma)^2\right]\sup_{t\in D_{S_\upsilon}}\left|\left((S_\upsilon)_1 - (S_\upsilon)_2\right)\right|^2$$
$$\le 2\left[(\eta^\gamma)^2\left\|I_\hbar\right\|_\infty^2 + (\kappa^\gamma)^2\|L\|_\infty^2 + (\upsilon^\gamma)^2\right]\left\|(S_\upsilon)_1 - (S_\upsilon)_2\right\|_\infty^2$$
$$\le \overline{\kappa}_4\left\|(S_\upsilon)_1 - (S_\upsilon)_2\right\|_{\inf}^2$$

where

$$\overline{\kappa}_4 = 2\left[(\eta^\gamma)^2\left\|I_\hbar\right\|_\infty^2 + (\kappa^\gamma)^2\|L\|_\infty^2 + (\upsilon^\gamma)^2\right]$$

Now

$$\left|J_5\left(t,S_\hbar, I_\hbar, R_\hbar, S_\upsilon, (I_\upsilon)_1, L\right) - J_5\left(t,S_\hbar, I_\hbar, R_\hbar, S_\upsilon, (I_\upsilon)_2, L\right)\right|^2$$
$$= |-\upsilon^\gamma|^2\left|(I_\upsilon)_1 - (I_\upsilon)_2\right|^2$$
$$\le \left[(\upsilon^\gamma)^2 + \varepsilon\right]\left|(I_\upsilon)_1 - (I_\upsilon)_2\right|^2$$
$$\le \left[(\upsilon^\gamma)^2 + \varepsilon\right]\sup_{t\in D_{I_\upsilon}}\left|(I_\upsilon)_1 - (I_\upsilon)_2\right|^2$$
$$\le \left[(\upsilon^\gamma)^2 + \varepsilon\right]\left\|(I_\upsilon)_1 - (I_\upsilon)_2\right\|_\infty^2$$
$$\le \overline{\kappa}_5\left\|(I_\upsilon)_1 - (I_\upsilon)_2\right\|_{\inf}^2$$

where

$$\overline{\kappa}_5 = \left[(\upsilon^\gamma)^2 + \varepsilon\right]$$

Now

$$\left|J_6\left(t,S_\hbar, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, (L)_1\right) - J_6\left(t,S_\hbar, I_\hbar, R_\hbar, S_\upsilon, I_\upsilon, (L)_2\right)\right|^2$$
$$= |-(\epsilon^\gamma + \tau^\gamma)|^2\left|(L)_1 - (L)_2\right|^2$$
$$\le \left[(\epsilon^\gamma + \tau^\gamma)^2 + \varepsilon\right]\left|(L)_1 - (L)_2\right|^2$$
$$\le \left[(\epsilon^\gamma + \tau^\gamma)^2 + \varepsilon\right]\sup_{t\in D_L}\left|(L)_1 - (L)_2\right|^2$$
$$\le \left[(\epsilon^\gamma + \tau^\gamma)^2 + \varepsilon\right]\left\|(L)_1 - (L)_2\right\|_\infty^2$$
$$\le \overline{\kappa}_6\left\|(L)_1 - (L)_2\right\|_{\inf}^2$$

where

$\overline{\kappa}_6 = (\varepsilon^\gamma + \tau^\gamma)^2 + \varepsilon$

5. Analysis of global stability

In this section, for the purpose of disease eradication, we demonstrate the conditions of global stability analysis with the help of Lyapunov's approach and Lasalle's invariance principle. To identify,

5.1. Lyapunov's first derivative

Theorem 4. *If the reproductive number $R_0 > 1$., then the equilibrium points of our model (9) are globally asymptotically stable.*

Proof. We define the Lyapunove function:

$$L\left(S_\hbar^*, I_\hbar^*, R_\hbar^*, S_\upsilon^*, I_\upsilon^*, L^*\right) = \left(S_\hbar - S_\hbar^* - S_\hbar^* \log \frac{S_\hbar^*}{S_\hbar}\right) + \left(I_\hbar - I_\hbar^* - I_\hbar^* \log \frac{I_\hbar^*}{I_\hbar}\right) + \left(R_\hbar - R_\hbar^* - R_\hbar^* \log \frac{R_\hbar^*}{R_\hbar}\right) + \left(S_\upsilon - S_\upsilon^* - S_\upsilon^* \log \frac{S_\upsilon^*}{S_\upsilon}\right) + \left(I_\upsilon^* - I_\upsilon^* - I_\upsilon^* \log \frac{I_\upsilon^*}{I_\upsilon}\right) + \left(L - L^* - L^* \log \frac{L^*}{L}\right) \tag{17}$$

Taking derivative on both sides, we obtain

$$\dot{L} = \left(\frac{S_\hbar - S_\hbar^*}{S_\hbar}\right)\dot{S}_\hbar + \left(\frac{I_\hbar - I_\hbar^*}{I_\hbar}\right)\dot{I}_\hbar + \left(\frac{R_\hbar - R_\hbar^*}{R_\hbar}\right)\dot{R}_\hbar + \left(\frac{S_\upsilon - S_\upsilon^*}{S_\upsilon}\right)\dot{S}_\upsilon + \left(\frac{I_\upsilon - I_\upsilon^*}{I_\upsilon}\right)\dot{I}_\upsilon + \left(\frac{L - L^*}{L}\right)\dot{L}$$

Now, putting the values of $\dot{S}_\hbar, \dot{I}_\hbar, \dot{R}_\hbar, \dot{S}_\upsilon, \dot{I}_\upsilon, \dot{L}$ from equation (17) in the above equation, we have

$$\dot{L} = \left(\frac{S_\hbar - S_\hbar^*}{S_\hbar}\right)\left[\delta^\gamma - \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar + \sigma^\gamma R_\hbar - \mu_1^\gamma S_\hbar\right] + \left(\frac{I_\hbar - I_\hbar^*}{I_\hbar}\right)\left[\left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right) S_\hbar - W_2 I_\hbar\right] + \left(\frac{R_\hbar - R_\hbar^*}{S}\right)\left[\omega^\gamma I_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right) R_\hbar\right] + \left(\frac{S_\upsilon - S_\upsilon^*}{S_\upsilon}\right)\left[\varphi^\gamma - \left(\eta^\gamma I_\hbar + \kappa^\gamma L\right) S_\upsilon - \upsilon^\gamma S_\upsilon\right] + \left(\frac{I_\upsilon - I_\upsilon^*}{I_\upsilon}\right)\left[\left(\eta^\gamma I_\hbar + k^\gamma L\right) S_\upsilon - \upsilon^\gamma I_\upsilon\right] + \left(\frac{L - L^*}{L}\right)\left[\phi^\gamma I_\upsilon - \left(\varepsilon^\gamma + \tau^\gamma\right) L\right]$$

Putting $S_\hbar = S_\hbar - S_\hbar^*, I_\hbar = I_\hbar - I_\hbar^*, R_\hbar = R_\hbar - R_\hbar^*, S_\upsilon = S_\upsilon - S_\upsilon^*, I_\upsilon = I_\upsilon - I_\upsilon^*, L = L - L^*$

$$\dot{L} = \left(\frac{S_\hbar - S_\hbar^*}{S_\hbar}\right)\left[\delta^\gamma - \left[\psi^\gamma (L - L^*) + \varrho^\gamma \left(I_\upsilon - I_\upsilon^*\right)\right]\left(S_\hbar - S_\hbar^*\right) + \sigma^\gamma \left(R_\hbar - R_\hbar^*\right) - \mu_1^\gamma \left(S_\hbar - S_\hbar^*\right)\right] + \left(\frac{I_\hbar - I_\hbar^*}{I_\hbar}\right)\left[\psi^\gamma \left[(L - L^*) + \varrho^\gamma \left(I_\upsilon - I_\upsilon^*\right)\right]\left(S_\hbar - S_\hbar^*\right) - W_2 \left(I_\hbar - I_\hbar^*\right)\right] + \left(\frac{R_\hbar - R_\hbar^*}{R_\hbar}\right)\left[\omega^\gamma \left(I_\hbar - I_\hbar^*\right) - \left(\sigma^\gamma + \mu_1^\gamma\right)\left(R_\hbar - R_\hbar^*\right)\right] + \left(\frac{S_\upsilon - S_\upsilon^*}{S_\upsilon}\right)\left[\varphi^\gamma - \left[\eta^\gamma \left(I_\hbar - I_\hbar^*\right) + \kappa^\gamma (L - L^*) - \upsilon^\gamma\right]\left(S_\upsilon - S_\upsilon^*\right)\right] + \left(\frac{I_\upsilon - I_\upsilon^*}{I_\upsilon}\right)\left[\left[\eta^\gamma \left(I_\hbar - I_\hbar^*\right) + k^\gamma (L - L^*)\right]\left(S_\upsilon - S_\upsilon^*\right) - \upsilon^\gamma \left(I_\upsilon - I_\upsilon^*\right)\right] + \left(\frac{L - L^*}{L}\right)\left[\phi^\gamma \left(I_\upsilon - I_\upsilon^*\right) - (\varepsilon^\gamma + \tau^\gamma)(L - L^*)\right]$$

which can be written as

$$L = \aleph + \Re, \tag{18}$$

where $\aleph$ contains all the positive terms and $\Re$ contains all the negative terms.

5.2. Second derivative of Lyapunov

By taking the second order derivative of the Lyapunov function, we obtain the following expression:

$$\ddot{L} = \left(\frac{\dot{S}_\hbar}{S_\hbar}\right)^2 S_\hbar^* + \left(\frac{\dot{I}_\hbar}{I_\hbar}\right)^2 I_\hbar^* + \left(\frac{\dot{R}_\hbar}{R_\hbar}\right)^2 R_\hbar^* + \left(\frac{\dot{S}_\upsilon}{S_\upsilon}\right)^2 S_\upsilon^* + \left(\frac{\dot{I}_\upsilon}{I_\upsilon}\right)^2 I_\upsilon^* + \left(\frac{\dot{L}}{L}\right)^2 L^* + \left(1 - \frac{S_\hbar^*}{S_\hbar}\right)\ddot{S}_\hbar + \left(1 - \frac{I_\hbar^*}{I_\hbar}\right)\ddot{I}_\hbar + \left(1 - \frac{R_\hbar^*}{R_\hbar}\right)\ddot{R}_\hbar + \left(1 - \frac{S_\upsilon^*}{S\upsilon}\right)\ddot{S}_\upsilon + \left(1 - \frac{I_\upsilon^*}{I_\upsilon}\right)\ddot{I}_\upsilon + \left(1 - \frac{L^*}{L}\right)\ddot{L}$$

Here

$$\begin{aligned}
\ddot{S}_\hbar &= -\left[\left(\psi^\gamma \dot{L} + \varrho^\gamma \dot{I}_\upsilon\right) S_\hbar + \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right)\dot{S}_\hbar\right] + \sigma^\gamma \dot{R}_\hbar - \mu_1^\gamma \dot{S}_\hbar \\
\ddot{I}_\hbar &= \left(\psi^\gamma \dot{L} + \varrho^\gamma \dot{I}_\upsilon\right) S_\hbar + \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right)\dot{S}_\hbar - W_2 \dot{I}_\hbar \\
\ddot{R}_\hbar &= \omega^\gamma \dot{I}_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right)\dot{R}_\hbar \\
\ddot{S}_\upsilon &= -\left[\left(\eta^\gamma \dot{I}_\hbar + \kappa^\gamma \dot{L}\right) S_\upsilon + \left(\eta^\gamma I_\hbar + \kappa^\gamma L - \upsilon^\gamma\right)\dot{S}_\upsilon\right] \\
\ddot{I}_\upsilon &= \left(\eta^\gamma \dot{I}_\hbar + k^\gamma \dot{L}\right) S_\upsilon + \left(\eta^\gamma I_\hbar + k^\gamma L\right)\dot{S}_\upsilon - \upsilon^\gamma \dot{I}_\upsilon \\
\ddot{L} &= \phi^\gamma \dot{I}_\upsilon - (\varepsilon^\gamma + \tau^\gamma)\dot{L}
\end{aligned}$$

Then

$$\ddot{L} = \left(\frac{\dot{S}_\hbar}{S_\hbar}\right)^2 S_\hbar^* + \left(\frac{\dot{I}_\hbar}{I_\hbar}\right)^2 I_\hbar^* + \left(\frac{\dot{R}_\hbar}{R_\hbar}\right)^2 R_\hbar^* + \left(\frac{\dot{S}_\upsilon}{S_\upsilon}\right)^2 S_\upsilon^* + \left(\frac{\dot{I}_\upsilon}{I_\upsilon}\right)^2 I_\upsilon^* + \left(\frac{\dot{L}}{L}\right)^2 L^*$$
$$+ \left(1 - \frac{S_\hbar^*}{S_\hbar}\right)\left(-\left[\left(\psi^\gamma \dot{L} + \varrho^\gamma \dot{I}_\upsilon\right) S_\hbar + \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right)\dot{S}_\hbar\right] + \sigma^\gamma \dot{R}_\hbar - \mu_1^\gamma \dot{S}_\hbar\right)$$
$$+ \left(1 - \frac{I_\hbar^*}{I_\hbar}\right)\left(\left(\psi^\gamma \dot{L} + \varrho^\gamma \dot{I}_\upsilon\right) S_\hbar + \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right)\dot{S}_\hbar - W_2 \dot{I}_\hbar\right)$$
$$+ \left(1 - \frac{R_\hbar^*}{R_\hbar}\right)\left[\omega^\gamma \dot{I}_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right)\dot{R}_\hbar\right]$$
$$+ \left(1 - \frac{S_\upsilon^*}{S\upsilon}\right)\left(-\left[\left(\eta^\gamma \dot{I}_\hbar + \kappa^\gamma \dot{L}\right) S_\upsilon + \left(\eta^\gamma I_\hbar + \kappa^\gamma L - \upsilon^\gamma\right)\dot{S}_\upsilon\right]\right)$$
$$+ \left(1 - \frac{I_\upsilon^*}{I_\upsilon}\right)\left(\left(\eta^\gamma \dot{I}_\hbar + k^\gamma \dot{L}\right) S_\upsilon + \left(\eta^\gamma I_\hbar + k^\gamma L\right)\dot{S}_\upsilon - \upsilon^\gamma \dot{I}_\upsilon\right)$$
$$+ \left(1 - \frac{L^*}{L}\right)\left(\phi^\gamma \dot{I}_\upsilon - (\varepsilon^\gamma + \tau^\gamma)\dot{L}\right) \tag{19}$$

Now, let us consider that

$$\dot{\Lambda}\left(S_\hbar, I_\hbar . R_\hbar, S_\upsilon, I_\upsilon, L\right) = \left(\frac{\dot{S}_\hbar}{S_\hbar}\right)^2 S_\hbar^* + \left(\frac{\dot{I}_\hbar}{I_\hbar}\right)^2 I_\hbar^* + \left(\frac{\dot{R}_\hbar}{R_\hbar}\right)^2 R_\hbar^* + \left(\frac{\dot{S}_\upsilon}{S_\upsilon}\right)^2 S_\upsilon^* + \left(\frac{\dot{I}_\upsilon}{I_\upsilon}\right)^2 I_\upsilon^* + \left(\frac{\dot{L}}{L}\right)^2 L^*$$

Then equation (19) becomes

$$\ddot{L} = \dot{\Lambda}\left(S_\hbar, I_\hbar . R_\hbar, S_\upsilon, I_\upsilon, L\right)$$
$$+ \left(1 - \frac{S_\hbar^*}{S_\hbar}\right)\left(-\left[\left(\psi^\gamma \dot{L} + \varrho^\gamma \dot{I}_\upsilon\right) S_\hbar + \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right)\dot{S}_\hbar\right] + \sigma^\gamma \dot{R}_\hbar - \mu_1^\gamma \dot{S}_\hbar\right)$$
$$+ \left(1 - \frac{I_\hbar^*}{I_\hbar}\right)\left(\left(\psi^\gamma \dot{L} + \varrho^\gamma \dot{I}_\upsilon\right) S_\hbar + \left(\psi^\gamma L + \varrho^\gamma I_\upsilon\right)\dot{S}_\hbar - W_2 \dot{I}_\hbar\right)$$
$$+ \left(1 - \frac{R_\hbar^*}{R_\hbar}\right)\left[\omega^\gamma \dot{I}_\hbar - \left(\sigma^\gamma + \mu_1^\gamma\right)\dot{R}_\hbar\right]$$
$$+ \left(1 - \frac{S_\upsilon^*}{S\upsilon}\right)\left(-\left[\left(\eta^\gamma \dot{I}_\hbar + \kappa^\gamma \dot{L}\right) S_\upsilon + \left(\eta^\gamma I_\hbar + \kappa^\gamma L - \upsilon^\gamma\right)\dot{S}_\upsilon\right]\right)$$
$$+ \left(1 - \frac{I_\upsilon^*}{I_\upsilon}\right)\left(\left(\eta^\gamma \dot{I}_\hbar + k^\gamma \dot{L}\right) S_\upsilon + \left(\eta^\gamma I_\hbar + k^\gamma L\right)\dot{S}_\upsilon - \upsilon^\gamma \dot{I}_\upsilon\right)$$
$$+ \left(1 - \frac{L^*}{L}\right)\left(\phi^\gamma \dot{I}_\upsilon - (\varepsilon^\gamma + \tau^\gamma)\dot{L}\right)$$

After replacing the values of first order derivatives, we obtain

$$
\begin{aligned}
\ddot{L} &= \dot{\Lambda}\left(S_\hbar, I_\hbar . R_\hbar, S_\upsilon, I_\upsilon, L\right) + \\
&\left(1-\tfrac{S_\hbar^*}{S_\hbar}\right)\begin{pmatrix} -\begin{bmatrix} \left(\psi^\gamma\left(\phi^\gamma I_\upsilon-(\varepsilon^\gamma+\tau^\gamma)L\right)+\varrho^\gamma\left(\left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar-W_2 I_\hbar\right)\right)S_\hbar+ \\ \left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)\left(\delta^\gamma-\left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar+\sigma^\gamma R_\hbar-\mu_1^\gamma S_\hbar\right)\end{bmatrix} \\ +\sigma^\gamma\left(\omega^\gamma I_\hbar-\left(\sigma^\gamma+\mu_1^\gamma\right)R_\hbar\right)-\mu_1^\gamma\left(\delta^\gamma-\left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar+\sigma^\gamma R_\hbar-\mu_1^\gamma S_\hbar\right)\end{pmatrix} \\
&+\left(1-\tfrac{I_\hbar^*}{I_\hbar}\right)\begin{pmatrix} \left(\psi^\gamma\left(\phi^\gamma I_\upsilon-(\varepsilon^\gamma+\tau^\gamma)L\right)+\varrho^\gamma\left(\left(\eta^\gamma I_\hbar+k^\gamma L\right)S_\upsilon-\upsilon^\gamma I_\upsilon\right)\right)S_\hbar+ \\ \left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)\left(\delta^\gamma-\left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar+\sigma^\gamma R_\hbar-\mu_1^\gamma S_\hbar\right) \\ -W_2\left(\left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar-W_2 I_\hbar\right)\end{pmatrix} \\
&+\left(1-\tfrac{R_\hbar^*}{R_\hbar}\right)\begin{bmatrix} \omega^\gamma\left(\left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar-W_2 I_\hbar\right)- \\ \left(\sigma^\gamma+\mu_1^\gamma\right)\left(\omega^\gamma I_\hbar-\left(\sigma^\gamma+\mu_1^\gamma\right)R_\hbar\right)\left(\omega^\gamma I_\hbar-\left(\sigma^\gamma+\mu_1^\gamma\right)R_\hbar\right)\end{bmatrix} \\
&+\left(1-\tfrac{S_\upsilon^*}{S\upsilon}\right)\left(-\begin{bmatrix} \left(\eta^\gamma\left(\left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar-W_2 I_\hbar\right)+\kappa^\gamma\left(\phi^\gamma I_\upsilon-(\varepsilon^\gamma+\tau^\gamma)L\right)\right)S_\upsilon+ \\ \left(\eta^\gamma I_\hbar+\kappa^\gamma L-\upsilon^\gamma\right)\left(\varphi^\gamma-\left(\eta^\gamma I_\hbar+\kappa^\gamma L\right)S_\upsilon-\upsilon^\gamma S_\upsilon\right)\end{bmatrix}\right) \\
&+\left(1-\tfrac{I_\upsilon^*}{I_\upsilon}\right)\begin{pmatrix} \begin{pmatrix}\eta^\gamma\left(\left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar-W_2 I_\hbar\right)+ \\ k^\gamma\left(\phi^\gamma I_\upsilon-(\varepsilon^\gamma+\tau^\gamma)L\right)\end{pmatrix}S_\upsilon+ \\ \left(\eta^\gamma I_\hbar+k^\gamma L\right)\left(\varphi^\gamma-\left(\eta^\gamma I_\hbar+\kappa^\gamma L\right)S_\upsilon-\upsilon^\gamma S_\upsilon\right)-\upsilon^\gamma\left(\left(\eta^\gamma I_\hbar+k^\gamma L\right)S_\upsilon-\upsilon^\gamma I_\upsilon\right)\end{pmatrix} \\
&+\left(1-\frac{L^*}{L}\right)\left(\phi^\gamma\left(\left(\eta^\gamma I_\hbar+k^\gamma L\right)S_\upsilon-\upsilon^\gamma I_\upsilon\right)-(\varepsilon^\gamma+\tau^\gamma)\left(\phi^\gamma I_\upsilon-(\varepsilon^\gamma+\tau^\gamma)L\right)\right)
\end{aligned}
$$

6. Computational analysis with fractal fractional operator

In this section, Leptospirosis model has been applied to the new differential and integral operators. So the operators with the new Mittag-Leffler kernel will be used in the place of the integer-order differential operator.

$$
\begin{aligned}
{}_0^{FFM}D_t^{\gamma,\beta} S_\hbar(t) &= \delta^\gamma-\left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar+\sigma^\gamma R_\hbar-\mu_1^\gamma S_\hbar \\
{}_0^{FFM}D_t^{\gamma,\beta} I_\hbar(t) &= \left(\psi^\gamma L+\varrho^\gamma I_\upsilon\right)S_\hbar-W_2 I_\hbar, \\
{}_0^{FFM}D_t^{\gamma,\beta} R_\hbar(t) &= \omega^\gamma I_\hbar-\left(\sigma^\gamma+\mu_1^\gamma\right)R_\hbar, \\
{}_0^{FFM}D_t^{\gamma,\beta} S_\upsilon(t) &= \varphi^\gamma-\left(\eta^\gamma I_\hbar+\kappa^\gamma L\right)S_\upsilon-\upsilon^\gamma S_\upsilon, \\
{}_0^{FFM}D_t^{\gamma,\beta} I_\upsilon(t) &= \left(\eta^\gamma I_\hbar+k^\gamma L\right)S_\upsilon-\upsilon^\gamma I_\upsilon, \\
{}_0^{FFM}D_t^{\gamma,\beta} L(t) &= \phi^\gamma I_\upsilon-(\varepsilon^\gamma+\tau^\gamma)L,
\end{aligned}
$$

with initial states given by $S_\hbar(0)\ge 0, I_\hbar(0)\ge 0, R_\hbar(0)\ge 0, I_\upsilon(0)\ge 0, S_\upsilon(0)\ge 0, L(0)\ge 0$.

So we have

$$
\begin{aligned}
{}_0^{FFM}D_t^{\gamma,\beta} S_\hbar(t) &= \left(S_\hbar\right)_1(t,\xi), \\
{}_0^{FFM}D_t^{\gamma,\beta} I_\hbar(t) &= \left(I_\hbar\right)_1(t,\xi), \\
{}_0^{FFM}D_t^{\gamma,\beta} R_\hbar(t) &= \left(R_\hbar\right)_1(t,\xi), \\
{}_0^{FFM}D_t^{\gamma,\beta} S_\upsilon(t) &= \left(S_\upsilon\right)_1(t,\xi), \\
{}_0^{FFM}D_t^{\gamma,\beta} I_\upsilon(t) &= \left(I_\upsilon\right)_1(t,\xi), \\
{}_0^{FFM}D_t^{\gamma,\beta} L(t) &= (L)_1(t,\xi).
\end{aligned}
$$

By using definition (8), we obtain,

$$
\begin{aligned}
&S_\hbar\left(t_{\delta+1}\right)=\left(S_\hbar\right)_0 \\
&+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}\left(S_\hbar\right)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right) \\
&+h\sum_{\mu=2}^{\infty}\int_{t_\mu}^{t_{\mu+1}}\left(S_\hbar\right)_1(\tau,\xi)\tau^{1-\beta}\left(t_{\delta+1}-\tau\right)^{\alpha-1}d\tau \\
&I_\hbar\left(t_{\delta+1}\right)=\left(I_\hbar\right)_0 \\
&+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}\left(I_\hbar\right)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right) \\
&+h\sum_{\mu=2}^{\infty}\int_{t_\mu}^{t_{\mu+1}}\left(I_\hbar\right)_1(\tau,\xi)\tau^{1-\beta}\left(t_{\delta+1}-\tau\right)^{\alpha-1}d\tau \\
&R_\hbar\left(t_{\delta+1}\right)=\left(R_\hbar\right)_0 \\
&+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}\left(R_\hbar\right)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right) \\
&+h\sum_{\mu=2}^{\infty}\int_{t_\mu}^{t_{\mu+1}}\left(R_\hbar\right)_1(\tau,\xi)\tau^{1-\beta}\left(t_{\delta+1}-\tau\right)^{\alpha-1}d\tau \\
&S_\upsilon\left(t_{\delta+1}\right)=\left(S_\upsilon\right)_0 \\
&+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}\left(S_\upsilon\right)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right) \\
&+h\sum_{\mu=2}^{\infty}\int_{t_\mu}^{t_{\mu+1}}\left(S_\upsilon\right)_0(\tau,\xi)\tau^{1-\beta}\left(t_{\delta+1}-\tau\right)^{\alpha-1}d\tau \\
&I_\upsilon\left(t_{\delta+1}\right)=\left(I_\upsilon\right)_0 \\
&+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}\left(I_\upsilon\right)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right) \\
&+h\sum_{\mu=2}^{\infty}\int_{t_\mu}^{t_{\mu+1}}\left(I_\upsilon\right)_1(\tau,\xi)\tau^{1-\beta}\left(t_{\delta+1}-\tau\right)^{\alpha-1}d\tau \\
&L(t)=(L)_0 \\
&+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}(L)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right) \\
&+h\sum_{\mu=2}^{\infty}\int_{t_\mu}^{t_{\mu+1}}(L)_1(\tau,\xi)\tau^{1-\beta}\left(t_{\delta+1}-\tau\right)^{\alpha-1}d\tau
\end{aligned}
$$

where $\xi = S_\hbar, I_\hbar, R_\hbar, I_\upsilon, S_\upsilon, L$ and $h=\frac{\gamma}{AB(\gamma)\Gamma(\gamma)}$, we get

$$
\begin{aligned}
&S_\hbar^{\delta+1} \\
&=\left(S_\hbar\right)_0+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}\left(S_\hbar\right)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),\right. \\
&\quad\left.S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right)+ \\
&\frac{\beta(\Delta t)^\gamma}{AB(\gamma)\Gamma(\gamma+2)} \\
&\times\sum_{\mu=2}^{\delta}\left[t_\mu^{\beta-1}\left(S_\hbar\right)_1\left(t,S_\hbar\left(t_\mu\right),I_\hbar\left(t_\mu\right),R_\hbar\left(t_\mu\right),\right.\right. \\
&\quad\left.S_\upsilon\left(t_\mu\right),I_\upsilon\left(t_\mu\right),L\left(t_\mu\right)\right) \\
&\left((\delta+1-\mu)^\gamma(\delta-\mu+2+\gamma)-(\delta-\mu)^\gamma(\delta-\mu+2+2\gamma)\right)- \\
&-t_{\mu-1}^{\beta-1}\left(S_\hbar\right)_1\left(t,S_\hbar\left(t_{\mu-1}\right),I_\hbar\left(t_{\mu-1}\right),R_\hbar\left(t_{\mu-1}\right),\right. \\
&\quad\left.S_\upsilon\left(t_{\mu-1}\right),I_\upsilon\left(t_{\mu-1}\right),L\left(t_{\mu-1}\right)\right) \\
&\left.\left((\delta-\mu+1)^{\gamma+1}-(\delta-\mu)^\gamma(\delta-\mu+1+\gamma)\right)\right] \\
&I_\hbar^{\delta+1}=\left(I_\hbar\right)_0 \\
&+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}\left(I_\hbar\right)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),\right. \\
&\quad\left.S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right) \\
&+\frac{\beta(\Delta t)^\gamma}{AB(\gamma)\Gamma(\gamma+2)} \\
&\times\sum_{\mu=2}^{\delta}\left[t_\mu^{\beta-1}\left(I_\hbar\right)_1\left(t,S_\hbar\left(t_\mu\right),I_\hbar\left(t_\mu\right),R_\hbar\left(t_\mu\right),\right.\right. \\
&\quad\left.S_\upsilon\left(t_\mu\right),I_\upsilon\left(t_\mu\right),L\left(t_\mu\right)\right) \\
&\left((\delta+1-\mu)^\gamma(\delta-\mu+2+\gamma)-(\delta-\mu)^\gamma(\delta-\mu+2+2\gamma)\right)- \\
&-t_{\mu-1}^{\beta-1}\left(I_\hbar\right)_1\left(t,S_\hbar\left(t_{\mu-1}\right),I_\hbar\left(t_{\mu-1}\right),R_\hbar\left(t_{\mu-1}\right),\right. \\
&\quad\left.S_\upsilon\left(t_{\mu-1}\right),I_\upsilon\left(t_{\mu-1}\right),L\left(t_{\mu-1}\right)\right) \\
&\left.\left((\delta-\mu+1)^{\gamma+1}-(\delta-\mu)^\gamma(\delta-\mu+1+\gamma)\right)\right] \\
&R_\hbar^{\delta+1}=\left(R_\hbar\right)_0 \\
&+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}\left(R_\hbar\right)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right)+ \\
&\frac{\beta(\Delta t)^\gamma}{AB(\gamma)\Gamma(\gamma+2)} \\
&\times\sum_{\mu=2}^{\delta}\left[t_\mu^{\beta-1}\left(R_\hbar\right)_1\left(t,S_\hbar\left(t_\mu\right),I_\hbar\left(t_\mu\right),R_\hbar\left(t_\mu\right),S_\upsilon\left(t_\mu\right),I_\upsilon\left(t_\mu\right),L\left(t_\mu\right)\right)\right. \\
&\left((\delta+1-\mu)^\gamma(\delta-\mu+2+\gamma)-(\delta-\mu)^\gamma(\delta-\mu+2+2\gamma)\right)- \\
&-t_{\mu-1}^{\beta-1}\left(R_\hbar\right)_1\left(t,S_\hbar\left(t_{\mu-1}\right),I_\hbar\left(t_{\mu-1}\right),R_\hbar\left(t_{\mu-1}\right),\right. \\
&\quad\left.S_\upsilon\left(t_{\mu-1}\right),I_\upsilon\left(t_{\mu-1}\right),L\left(t_{\mu-1}\right)\right) \\
&\left.\left((\delta-\mu+1)^{\gamma+1}-(\delta-\mu)^\gamma(\delta-\mu+1+\gamma)\right)\right] \\
&S_\upsilon^{\delta+1}=\left(S_\upsilon\right)_0 \\
&+\frac{1-\gamma}{AB(\gamma)}t_\delta^{1-\beta}\left(S_\upsilon\right)_1\left(t,S_\hbar\left(t_\delta\right),I_\hbar\left(t_\delta\right),R_\hbar\left(t_\delta\right),\right. \\
&\quad\left.S_\upsilon\left(t_\delta\right),I_\upsilon\left(t_\delta\right),L\left(t_\delta\right)\right)+
\end{aligned}
$$

$$\frac{\beta (\Delta t)^{\gamma}}{AB(\gamma)\Gamma(\gamma+2)}$$
$$\times \sum_{\mu=2}^{\delta} \left[t_{\mu}^{\beta-1} \left(S_h\right)_1 \left(t, S_h\left(t_\mu\right), I_h\left(t_\mu\right), R_h\left(t_\mu\right), S_v\left(t_\mu\right), I_v\left(t_\mu\right), L\left(t_\mu\right)\right)\right.$$
$$\left((\delta+1-\mu)^{\gamma}(\delta-\mu+2+\gamma)-(\delta-\mu)^{\gamma}(\delta-\mu+2+2\gamma)\right)-$$
$$- t_{\mu-1}^{\beta-1} \left(S_v\right)_1 \left(t, S_h\left(t_{\mu-1}\right), I_h\left(t_{\mu-1}\right), R_h\left(t_{\mu-1}\right),\right.$$
$$\left. S_v\left(t_{\mu-1}\right), I_v\left(t_{\mu-1}\right), L\left(t_{\mu-1}\right)\right)$$
$$\left.\left((\delta-\mu+1)^{\gamma+1}-(\delta-\mu)^{\gamma}(\delta-\mu+1+\gamma)\right)\right]$$
$$I_v^{\delta+1} = \left(I_v\right)_0$$
$$+ \frac{1-\gamma}{AB(\gamma)} t_{\delta}^{1-\beta} \left(I_v\right)_1 \left(t, S_h\left(t_\delta\right), I_h\left(t_\delta\right), R_h\left(t_\delta\right), S_v\left(t_\delta\right), I_v\left(t_\delta\right), L\left(t_\delta\right)\right) +$$
$$\frac{\beta (\Delta t)^{\gamma}}{AB(\gamma)\Gamma(\gamma+2)}$$
$$\times \sum_{\mu=2}^{\delta} \left[t_{\mu}^{\beta-1} \left(I_v\right)_1 \left(t, S_h\left(t_\mu\right), I_h\left(t_\mu\right), R_h\left(t_\mu\right), S_v\left(t_\mu\right), I_v\left(t_\mu\right), L\left(t_\mu\right)\right)\right.$$
$$\left((\delta+1-\mu)^{\gamma}(\delta-\mu+2+\gamma)-(\delta-\mu)^{\gamma}(\delta-\mu+2+2\gamma)\right)$$
$$- t_{\mu-1}^{\beta-1} \left(I_v\right)_1 \left(t, S_h\left(t_{\mu-1}\right), I_h\left(t_{\mu-1}\right), R_h\left(t_{\mu-1}\right),\right.$$
$$\left. S_v\left(t_{\mu-1}\right), I_v\left(t_{\mu-1}\right), L\left(t_{\mu-1}\right)\right)$$
$$\left.\left((\delta-\mu+1)^{\gamma+1}-(\delta-\mu)^{\gamma}(\delta-\mu+1+\gamma)\right)\right]$$
$$L^{\delta+1} = L_0$$
$$+ \frac{1-\gamma}{AB(\gamma)} t_{\delta}^{1-\beta} L_1 \left(t, S_h\left(t_\delta\right), I_h\left(t_\delta\right), R_h\left(t_\delta\right), S_v\left(t_\delta\right), I_v\left(t_\delta\right), L\left(t_\delta\right)\right) +$$
$$\frac{\beta (\Delta t)^{\gamma}}{AB(\gamma)\Gamma(\gamma+2)}$$
$$\times \sum_{\mu=2}^{\delta} \left[t_{\mu}^{\beta-1} L_1 \left(t, S_h\left(t_\mu\right), I_h\left(t_\mu\right), R_h\left(t_\mu\right),\right.\right.$$
$$\left. S_v\left(t_\mu\right), I_v\left(t_\mu\right), L\left(t_\mu\right)\right)$$
$$\left((\delta+1-\mu)^{\gamma}(\delta-\mu+2+\gamma)-(\delta-\mu)^{\gamma}(\delta-\mu+2+2\gamma)\right)$$
$$- t_{\mu-1}^{\beta-1} L_1 \left(t, S_h\left(t_{\mu-1}\right)\left(I_h\right)\left(t_{\mu-1}\right), R_h\left(t_{\mu-1}\right),\right.$$
$$\left. S_v\left(t_{\mu-1}\right), I_v\left(t_{\mu-1}\right), L\left(t_{\mu-1}\right)\right)$$
$$\left.\left((\delta-\mu+1)^{\gamma+1}-(\delta-\mu)^{\gamma}(\delta-\mu+1+\gamma)\right)\right]$$

7. Numerical results and discussions

The effectiveness of the obtained theoretical outcomes are established by using advanced technique for the numerical treatment to see its real impact using fractional operator and analysis of the spread of disease. The mathematical analysis of Leptospirosis with environmental effects are analyzed through simulation. Intrusting findings are achieved by implementing the non-integer parametric choices of the Leptospirosis with environmental effects. In Figs. 2–7, solution for $S_h(t), I_h(t), R_h(t), S_v(t), I_v(t)$ and $L(t)$ comes according to desired value by decreasing the fractional values. The effectiveness of the obtained theoretical outcomes are established by the following examples. Matlab coding is employed to find the numerical simulation for fractional order Leptospirosis model. The initial values for all cases of the given system are $S_h(t) = 100, I_h(t) = 40, R_h(t) = 30, S_v(t) = 50, I_v(t) = 10$ and $L(t) = 4$. Fig. 2 and 5 represents the dynamics of susceptible humans S_h and animal respectively in which the susceptible people are decreasing and transferring to the infected individual. Fig. 3 and 4 show the dynamics of infected and recovered individuals respectively in humans at different fractional-orders in which the infected individual rises but after certain time start decline approach to its desired values, also recovered individuals of human population decline gradually but rises by decreasing fractional values. In Fig. 6 and 7, the dynamics of animal infectious and number of infected units in environment have been shown at different fractional orders in which the both rises gradually and approach to desired values. It is observed that infected human declined after certain time while animal infection rises gradually due to in-contact infection between human and animals. But It can be easily observed from Fig. 7 that the number of infection units consist in the environment start reducing by decreasing the fractional values. It is deduced from the outcomes that Fractional order provide us the continuous monitoring for the spread of disease while integer order can not, continuous monitoring help us for future prediction more accurately as well as help us to provide its complete analysis and its actual behavior. It is observed that susceptible human and susceptible vector population decreases then suddenly approach to stable position after almost 8 days which can be seen in Fig. 2 and 5 respectively. Similarly the infected human and infected vectors population rises strictly due to infection then after almost 10 days both approaches to stable position due rise in recovered individuals which can be seen in Fig. 3 and 6 respectively. Similar behavior can be seen that the recovered human individual rises almost after 10 days by decreasing fractional values due decrease in leptospirosis virus which can be seen in Fig. 4 and 7 respectively. It is also observed that results are more appropriate according to our desired values by reducing the fractional values and can be seen in all Figs. 2–7. Comparisons are made between integer order derivative and fractional order derivative. It is observed that the fractal-fractional method provide reliable findings for all compartment according to steady state at non-integer order derivatives as compare to classical derivative.

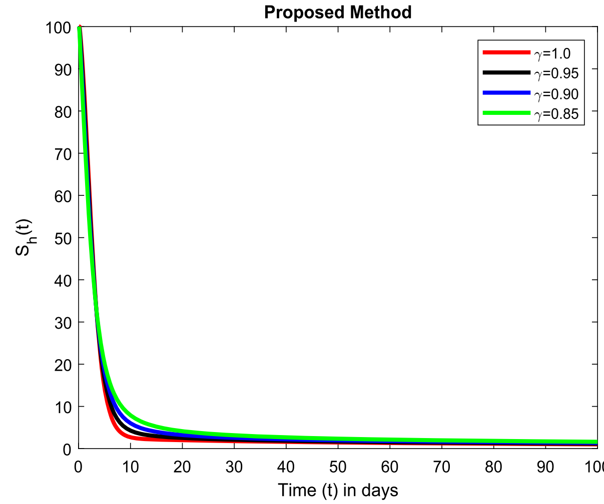

Fig. 2. $S_h(t)$ using fractal fractional operator with different fractional values.

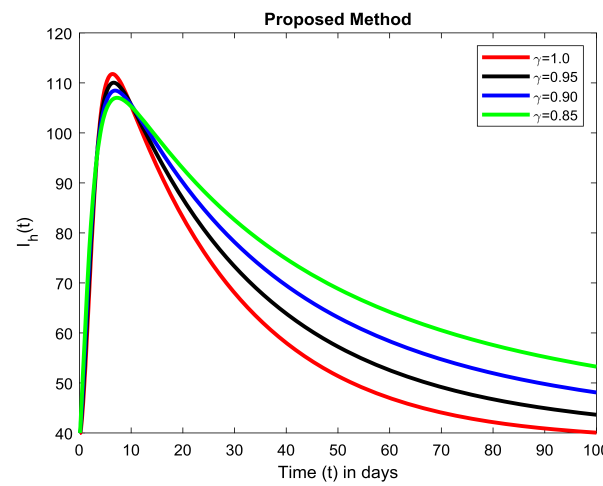

Fig. 3. $I_h(t)$ using fractal fractional operator with different fractional values.

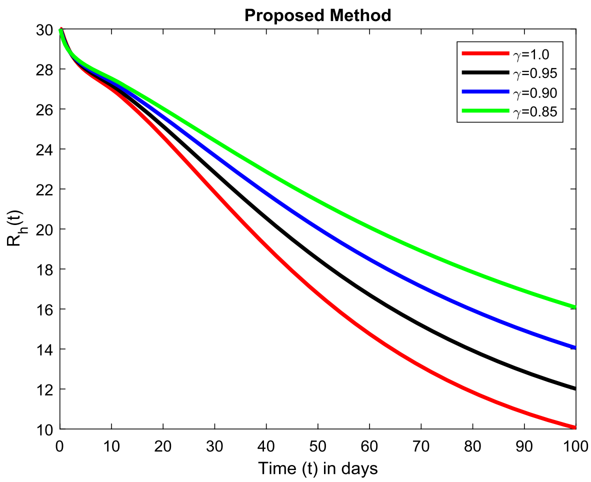

Fig. 4. $R_h(t)$ using fractal fractional operator with different fractional values.

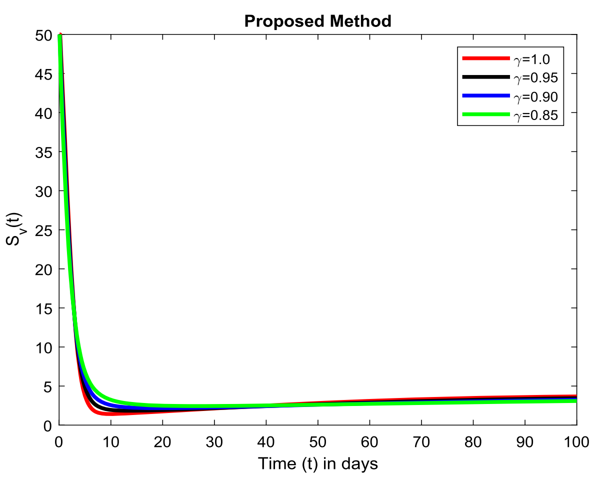

Fig. 5. $S_v(t)$ using fractal fractional operator with different fractional values.

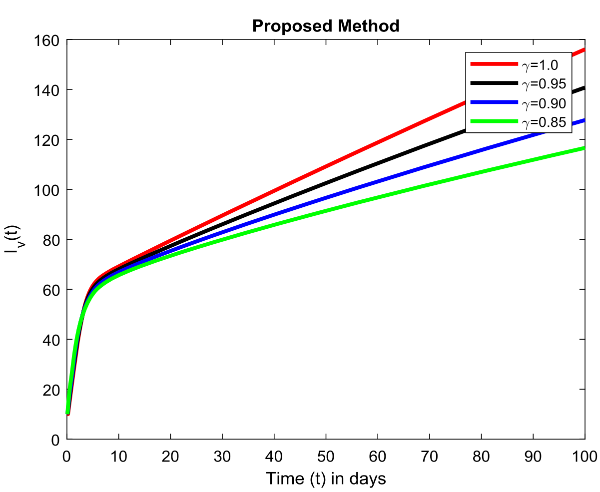

Fig. 6. $I_v(t)$ using fractal fractional operator with different fractional values.

8. Conclusion

In this article, the advanced technique developed by the Atangana Baleanu with Mittag-Leffler kernel known as fractal fractional operator is employed to find reliable solutions and MATLAB code is used on developed solution for simulation. We have presented some advises to

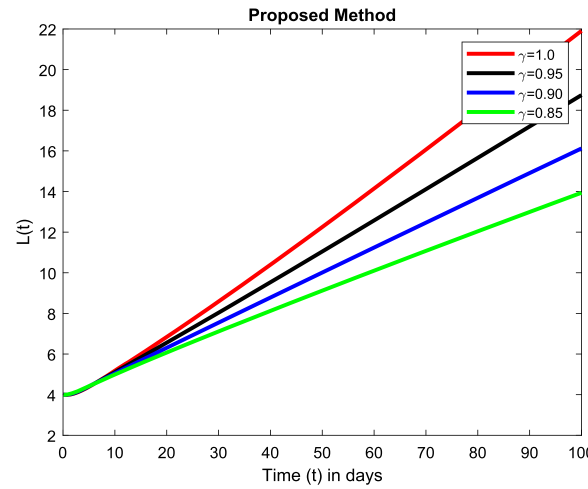

Fig. 7. $L(t)$ using fractal fractional operator with different fractional values.

control this virus so that our community can overcome this pandemic. The dangerous Leptospirosis with environmental effects is analyzed to see its real effects in the community. In this regard, qualitative and quantitative analysis has been made and derived its disease free as well as endemic equilibria including flip bifurcation analysis for continuous dynamical system. It is verified that no flip bifurcation exist in this dynamics Also verification has been made to achieve bounded and unique solution for fractional order Leptospirosis model with environmental effects. Local as well as global stability has been verified to capture its realistic bounded findings. Solutions are derived with the help of fractal fractional operator which provide us the continuous monitoring of spread of the Leptospirosis disease with environmental effects in the society. Effect of global derivative is examine to see how rate of change of Leptospirosis with environmental effects changes. Numerical simulation has been made to check the actual behavior of Leptospirosis with environmental effect in the community using different fractional values as well as provide future predictions on the basis of the our justified findings which will be helpful for developing control strategies to overcome the environmental risk factor of Leptospirosis in society. Leptospirosis developed model with environmental effect must have the exposed human population which is missing, because its symptoms usually start within two to fourteen days. In future, we may study this disease to identify asymptomatic individuals for treatment before its chronic stage because the exposed individuals can be record early as compared to infected individuals. Also its can be analyzed completely after adding exposed population.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The authors are thankful to the Deanship of Scientific Research at Najran University for funding this work under the Research Priorities and Najran Research funding program, grant code (NU/NRP/SERC/12/3).

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