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Wallach's ratio principle under affine representation

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ABSTRACT

Motivated by Heller (2014) and supplementing the results found there, the main objective of this paper is to study the near-miss to Wallach's ratio principle and the near-miss to illumination invariance, assuming more general psychophysical representations than in the previous works. We employ a model-creation technique founded on functional equations to study the affine and gain-control type representations of these phenomena, respectively.

1. Introduction

One of the most widely accepted principles in relation to the perception of achromatic colors placed in configurations with a central spatial effect is the so-called *Wallach's ratio principle*. Assuming that perceived achromatic colors form a continuum from black to white, this principle states that the sensation produced by the physical intensity of the centrally located color depends on the center–surround ratio and not on the absolute intensity of the centrally located patch.

Seminal experiments described in [1] exemplified the phenomenon of simultaneous contrast, in which the perceived luminance of an object can be altered by changing the luminance of its surroundings. In his experiment, he showed that a disk of constant luminance could be made to appear as any shade between white and black by varying the luminance of a surrounding annulus.

To further validate Wallach's ratio principle, [2] conducted a comprehensive study where they projected an achromatic pattern of five squares, varying from white to black, on a gray background. Three levels of projector intensity were employed to simulate a 12-fold illumination range. At each level, observers were instructed to match the perceived lightness or brightness of each of the five squares by adjusting the luminance of a comparison square displayed in a separate chamber, surrounded by a large white region of constant luminance, brighter than the comparison square itself.

[3] repeated and extended the previous experiments.

Munsell chart Following [4], a series of gray paper samples ranging from white to black are presented on a uniform background, and the observer is asked to select the sample that matches a target surface in the standard display. This method is simple to use, but has been criticized because simultaneous presentation of the entire range of gray shades may artificially influence the results of perceived lightness constancy.

Lightness measure A square of variable luminance is presented on a bright white background.

Brightness measure A square of variable luminance is presented on a totally dark background.

The findings of their study suggest that the ratio principle holds true across a wide range of illumination conditions and is not limited to a narrow range or dependent on absolute intensity. The obtained psychophysical functions are nearly horizontal,

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indicating that perceived lightness is largely invariant with changes in illumination. They also note that it is possible that even the slight positive slope observed in the functions could be eliminated.

[5] aimed to investigate how adult and infant subjects perceive achromatic contrast when the surround luminance is altered between the study and test phases of each trial. Adults were tested using a lightness matching technique where they were asked to adjust the luminance of a test stimulus until it appeared the same shade of gray as a standard stimulus. Adults showed a near-ratio rule with a small deviation towards a luminance match. Infants were tested using a forced-choice novelty preference technique in combination with a cross-familiarization paradigm. Their findings suggests that infants as young as 4 months old may already have an adult-like achromatic contrast system that governs their looking preferences for relatively simple visual displays.

In addition to Wallach's ratio principle, we would like to discuss another principle, the so-called illumination invariance. Eight artists participating in an experiment were given a black and a white disc and were asked to return to their own gallery and paint a gray disc that they thought was 'halfway' between the two (white and black) discs they received. [6] reported that although the lighting conditions were different in each gallery, the gray disks made by the artists were practically indistinguishable.

Using Plateau's observations, [7] set up a purely mathematical model formed by two functional equations. Having solved these, he was able to determine the possible forms of the psychophysical function for the underlying perceptual attribute. Theorem 3.12 of [7] serves as an impressive illustration of how functional equations can be utilized to deduce the form of a specific psychophysical function based on fundamental empirical observations, highlighting the powerful role that mathematics can play in unraveling the intricacies of human perception. This kind of model-creating attitude was further developed by [8,9].

[9] studies the near-miss to Wallach's ratio principle as well as the near-miss to illumination invariance under a subtractive representation, respectively.

The main objective of this paper is to study the near-miss to Wallach's ratio principle and the near-miss to illumination invariance, assuming more general psychophysical representations, using the above-mentioned model-creation method, based on functional equations. Here we will discuss the so-called affine and gain-control type representations, respectively.

1.1. Setup of the model

Let ζ be a fixed positive number and $I =]0, \zeta[\subset \mathbb{R}$ be a nonempty open interval containing 1.[1] The *set of stimuli* is a nonempty and open set $S \subset I \times I$. Further, $(a, s) \in S$ means that *patch a is in a surround s*.

Experiments related to the above-mentioned Lightness measure and Brightness measure require that in these cases we do not work on the entire S set, but on its subsets below. The set of *incremental stimuli* is

$$\Gamma = \{(a, s) \mid (a, s) \in S, a \geq s\}$$

and the set of *decremental stimuli* is

$$D = \{(a, s) \mid (a, s) \in S, a \leq s\}.$$

Assume the functions $P_{l,r}, P_{r,l} : S^2 \to]0, 1[$ to be

(i) strictly increasing on the first and fourth variables
(ii) strictly decreasing in the second and third variables
(iii) continuous.

If $(a, s), (b, t) \in S$, then $P_{l,r}(a, s; b, t)$ is the probability with patch a in surround s presented to the left is considered to be more intense as patch b in surround t presented to the right. The interpretation of the function $P_{r,l}$ is analogous.

Due to the *two alternative forced choice procedure*, we have

$$P_{l,r}(a, s; b, t) + P_{r,l}(b, t; a, s) = 1 \tag{2AFC}$$

for all $(a, s), (b, t) \in S$.

Using this, it is always sufficient to determine only one of the functions $P_{r,l}$ and $P_{l,r}$. Note that if we rearrange the above identity, we get that

$$P_{l,r}(a, s; b, t) = 1 - P_{r,l}(b, t; a, s) \qquad ((a, s), (b, t) \in S).$$

This shows that the values of the function $P_{l,r}$ are completely determined by the values of the function $P_{r,l}$.

Accordingly, the lower indices of the included functions will be omitted and from here on we will work with a function P, defined on S and with a values in $]0, 1[$.

Let $(a, s) \in S$ and assume that we are given a positive λ such that $(\lambda a, \lambda s) \in S$ holds. The pair $(\lambda a, \lambda s)$ represents the stimulus (a, s) after changing illuminance by the factor λ.

In this work, we will deal with the following principles.

[1] Note that the condition $1 \in I$ does not impair generality. Indeed, if $I \neq \emptyset$, then with the introduction of a new variable, the fulfillment of $1 \in I$ can be guaranteed. In practical applications, this means a switch to a new unit of measure.

Wallach's ratio principle

$$P(a,s;b,t)=P(\lambda a,\lambda s;\mu b,\mu t) \tag{WRP}$$

near-miss to Wallach's ratio principle

$$P(a,s;b,t)=P(\lambda^{\rho}a,\lambda s;\mu^{\rho}b,\mu t), \tag{nm-WRP}$$

where $\rho>0$ is given.

illumination invariance

$$P(a,s;b,t)=P(\lambda a,\lambda s;\lambda b,\lambda t) \tag{II}$$

near-miss to illumination invariance

$$P(a,s;b,t)=P(\lambda^{\rho}a,\lambda s;\lambda^{\rho}b,\lambda t) \tag{nm-II}$$

where $\rho>0$ is given.

For all four equation, each equation must hold for all possible values of the variables. That is, Eq. (nm-WRP) holds for all pairs $(a,s),(b,t)\in S$ and $\lambda,\mu>0$ for which $(\lambda^{\rho}a,\lambda s),(\mu^{\rho}b,\mu t)\in S$. Similarly, Eq. (nm-II) holds for all pairs $(a,s),(b,t)\in S$ and $\lambda>0$ for which $(\lambda^{\rho}a,\lambda s),(\lambda^{\rho}b,\lambda t)\in S$.

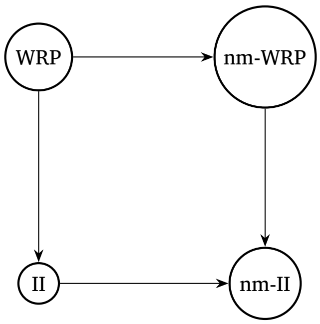
More precisely, the domains of the above equations are

$$\begin{aligned}
D_{WRP}&=\left\{(\lambda,\mu,a,s,b,t)\in\mathbb{R}^6\mid(a,s),(b,t)\in S,\lambda,\mu>0\text{ such that also }(\lambda a,\lambda,s),(\mu b,\mu t)\in S\right\},\\
D_{nm-WRP}&=\left\{(\lambda,\mu,a,s,b,t)\in\mathbb{R}^6\mid(a,s),(b,t)\in S,\lambda,\mu>0\text{ such that also }(\lambda^{\rho}a,\lambda,s),(\mu^{\rho}b,\mu t)\in S\right\},\\
D_{II}&=\left\{(\lambda,a,s,b,t)\in\mathbb{R}^5\mid(a,s),(b,t)\in S,\lambda,\mu>0\text{ such that also }(\lambda a,\lambda,s),(\lambda b,\lambda t)\in S\right\},\\
D_{nm-II}&=\left\{(\lambda,a,s,b,t)\in\mathbb{R}^5\mid(a,s),(b,t)\in S,\lambda,\mu>0\text{ such that also }(\lambda^{\rho}a,\lambda,s),(\lambda^{\rho}b,\lambda t)\in S\right\},
\end{aligned}$$

respectively. As the set $S\subset I\times I$ is assumed to be nonempty and open, the sets $D_{WRP},D_{nm-WRP}\subset\mathbb{R}^6$ and also the sets $D_{II},D_{nm-II}\subset\mathbb{R}^5$ are nonempty and open subsets of $\mathbb{R}^6$ and $\mathbb{R}^5$, respectively. From now on, we suppose that all these sets are connected, too.

Suppose that we are given two stimuli $(a,s),(b,t)\in S$ and two positive factors λ and μ. Eq. (WRP) expresses that the probability with patch a in surround s is considered to be more intense as patch b in surround t remains the same if we change the illuminance of the stimulus by the factor λ and we change the illuminance of the stimulus by the factor μ, leaving illumination constant. At the same time, (II) expresses that the probability with patch a in surround s is considered to be more intense as patch b in surround t remains the same if we change the illuminance of the stimuli (a,s) and (b,t) by the same factor λ.

All four equations express, separately, that the function P in question is a homogeneous function in some sense. Further, the following implications exist between these equations.



Indeed, if a function $P:S^2\to]0,1[$ fulfills (nm-WRP) for all possible $(a,s),(b,t)\in S$ and $\lambda,\mu>0$, then with the substitution $\mu=\lambda$ we obtain that P satisfies (nm-II). An analogous implication holds in connection with the Eqs. (II) and (WRP).

In addition to the above findings, none of the previously mentioned equations can fully determine the function P. Due to this limitation, it is commonly accepted that the function P has a specific psychophysical representation. In this paper we will explore the following representations.

We say that a function $P:S^2\to]0,1[$ admits

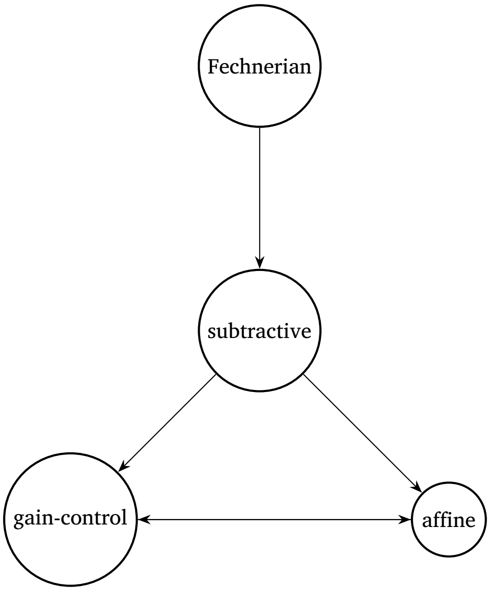
- an *affine representation* if

$$P(a,s;b,t)=F(u(a,s)+h(a,s)\cdot g(b,t))\qquad((a,s),(b,t)\in S)$$

holds with an appropriate function F and with some nonconstant functions u,g, and with a nonconstant and nowhere zero function h.

- a *gain-control type representation* if there exist a function F and nonconstant functions w, v and r such that r is nowhere zero and
$$P(a,s;b,t) = F\left(\frac{w(a,s)-v(b,t)}{r(a,s)}\right) \qquad ((a,s),(b,t) \in S)$$
is fulfilled.
- a *subtractive representation* if there exist a function F and nonconstant functions u and v such that
$$P(a,s;b,t) = F\left(u(a,s)-v(b,t)\right) \qquad ((a,s),(b,t) \in S)$$
holds.
- a *Fechnerian representation* if there exist functions F, u such that
$$P(a,s;b,t) = F\left(u(a,s)-u(b,t)\right) \qquad ((a,s),(b,t) \in S)$$
is satisfied.

The interpretation of having a Fechnerian representation is that the sensory mechanism scales the stimulus intensities (a,s) and (b,t) into the values $u(a,s)$ and $u(b,t)$, respectively, such that the probabilities $P(a,s;b,t)$ are determined solely by the differences between the scaled values $u(a,s)$ and $u(b,t)$. The interpretation of the subtractive representation is similar, with the exception that the stimuli (a,s) and (b,t) may be scaled by different scale functions u and v. The interpretation of $w(a,s)-v(b,t)$ in the gain-control type representation is similar to the above. Here, however, the denominator includes a normalizing factor $r(a,s)$, which somewhat compensates the effect of $v(b,t)$.



Regarding subtractive representations, the interested reader may consult [10], while concerning affine representation we recommend [11–13]. Finally, [14] contains several interesting results about gain-control representations. We note however that the discrimination probabilities in the above-mentioned papers are not four-variable but only two-variable functions. Our definitions have therefore been adapted to the current situation. Accordingly, the scale functions used in this paper have two independent variables.

In the next section, we will use a result of [15] for a general multiplicative Pexider-type equation, which, for the sake of completeness, is formulated below. The interested reader can find additional results regarding the theory of functional equations in the monographs [16,17].

Lemma 1 (*Aczél (Multiplicative Version)*). *Let* $R \subset]0,+\infty[\times]0,+\infty[$ *be a nonempty, open and connected set. Suppose*
$$f(st) = g(s) + h(s)k(t)$$
for all $(s,t) \in R$*, where* f *is philandering and measurable. Then*

(a) either
$$\begin{aligned} f(r) &= \beta_0 \log(r) + \beta_1 + \beta_2\beta_3 \\ g(s) &= \beta_0 \log(s) + \beta_1 \\ h(s) &= \beta_3 \\ k(t) &= \frac{\beta_0}{\beta_3}\log(t) + \beta_2 \end{aligned}$$

(b) or

$$\begin{aligned} f(r) &= \beta_3\beta_4 r^{\beta_0} + \beta_1 \\ g(s) &= \beta_2 s^{\beta_0} + \beta_1 \\ h(s) &= \beta_3 s^{\beta_0} \\ k(t) &= \beta_4 t^{\beta_0} - \frac{\beta_2}{\beta_3} \end{aligned}$$

for all $s \in \{s \mid \text{there exists } t \text{ such that } (s,t) \in R\}$, $t \in \{t \mid \text{there exists } s \text{ such that } (s,t) \in R\}$ *and* $r \in \{st \mid \text{such that } (s,t) \in R\}$*, where* $\beta_i, i = 0,1,2,3,4$ *are constants such that* $\beta_0\beta_3\beta_4 \neq 0$*, but otherwise arbitrary.*

2. Results

As we pointed out in the previous section, the domains of the functional equations studied here are the sets $D_{WRP}, D_{nm-WRP} \subset \mathbb{R}^6$ and $D_{II}, D_{nm-II} \subset \mathbb{R}^6$, respectively. When solving the equations in question, not all 'reasonable' substitutions are allowed for these equations. We will overcome this technical difficulty by first showing that if the function $P : S^2 \to]0,1[$ satisfies some of the above equations on the domain of the equation in question, then P can be extended to a function $\tilde{P} :]0,+\infty[^4 \to \mathbb{R}$ that satisfies the given equation on the entire set $]0,+\infty[^4$.

More precisely, we show that if the function P satisfies one of the Eqs. (WRP), (nm-WRP), (II), and (nm-II), resp., then P can be extended to a function $\tilde{P}:]0,+\infty[^4 \to \mathbb{R}$ that satisfies the given equation for all $(a,s),(b,t) \in]0,+\infty[^2$ and all $\lambda,\mu \in]0,+\infty[$. Accordingly, when solving these equations, we can work with functions defined on the set $]0,+\infty[^4$ instead of the set S. In order to simplify the notations, let $J =]0,+\infty[$ and $Q =]0,+\infty[^2$.

Lemma 2 (*Extension for* (nm-WRP)). *Suppose that the function* $P : S^2 \to \mathbb{R}$ *satisfies the near-miss to Wallach' ratio principle* (nm-WRP) *with a fixed parameter* ρ*. Then there exists a function* $\tilde{P} : Q^2 \to \mathbb{R}$ *that satisfies* (nm-WRP) *on* Q^2 *such that* $\tilde{P}|_{S^2} = P$ *holds. If* $\rho = 1$*, we obtain the corresponding result for Wallach's ratio principle* (WRP).

Proof. Consider the set

$$C(S) = \{(a,s) \in Q \mid (\lambda^\rho a, \lambda s) \in S \text{ for some } \lambda \in J\}.$$

If $(a,s),(b,t) \in C(S)$ and $\lambda,\mu \in J$ are such that $(\lambda^\rho a, \lambda s) \in S$ and $(\mu^\rho b, \mu t) \in S$, then let

$$\tilde{P}(a,s;b,t) = P(\lambda^\rho a, \lambda s; \mu^\rho b, \mu t),$$

otherwise let $\tilde{P}(a,s;b,t) = 0$. Then the function $\tilde{P} : Q^2 \to \mathbb{R}$ is well-defined, $\tilde{P}|_{S^2} = P$ and $\tilde{P}$ fulfills (nm-WRP) for all $(a,s),(b,t) \in Q$ and for all $\lambda,\mu \in J$. □

Lemma 3 (*Extension for* (nm-II)). *Suppose that the function* $P : S^2 \to \mathbb{R}$ *satisfies the near-miss to illumination invariance principle* (nm-II) *with a fixed parameter* ρ*. Then there exists a function* $\tilde{P} : Q^2 \to \mathbb{R}$ *that satisfies* (nm-II) *on* Q^2 *such that* $\tilde{P}|_{S^2} = P$ *holds. If* $\rho = 1$*, we obtain the corresponding result for illumination invariance principle* (II).

Proof. Let us consider the set

$$C(S) = \left\{(a,s,b,t) \in]0,+\infty[^4 \mid (\lambda^\rho a, \lambda s) \in S \text{ and } (\lambda^\rho b, \lambda t) \in S \text{ for some } \lambda \in]0,+\infty[\right\}.$$

If $(a,s,b,t) \in C(S)$ and $\lambda \in J$ are such that $(\lambda^\rho a, \lambda s) \in S$ and $(\lambda^\rho b, \lambda t) \in S$, then let

$$\tilde{P}(a,s;b,t) = P(\lambda^\rho a, \lambda s; \lambda^\rho b, \lambda t),$$

otherwise let $\tilde{P}(a,s;b,t) = 0$. Then the function $\tilde{P} : Q^2 \to \mathbb{R}$ is well-defined, $\tilde{P}|_{S^2} = P$ and $\tilde{P}$ fulfills (nm-II) for all $(a,s),(b,t) \in Q$ and for all $\lambda,\mu \in J$. □

In order to make the following proofs simpler, we first show that the discrimination probabilities have an affine representation exactly when they have a gain-control type representation.

Remark 1. Let $P : S^2 \to]0,1[$ be a function. Assume that the function P has an affine representation with the functions F,u,h and g. Define the functions U,V and σ by

$$\begin{aligned} U(a,s) &= \frac{u(a,s)}{h(a,s)} \\ V(b,t) &= -g(b,t) \qquad ((a,s),(b,t) \in S). \\ \sigma(a,s) &= \frac{1}{h(a,s)} \end{aligned}$$

Then

$$P(a,s;b,t) = F(u(a,s)+h(a,s)\cdot g(b,t)) = F\left(h(a,s)\cdot\left(\frac{u(a,s)}{h(a,s)}-(-g(b,t))\right)\right)$$
$$= F\left(\frac{\frac{u(a,s)}{h(a,s)}-(-g(b,t))}{\frac{1}{h(a,s)}}\right) = F\left(\frac{U(a,s)-V(a,s)}{\sigma(a,s)}\right)$$

for all $(a,s),(b,t)\in S$. So P admits a gain-control representation. Conversely, suppose that P admits a gain-control representation with the functions F, U, V and σ and consider the functions

$$\begin{aligned} u(a,s) &= \frac{U(a,s)}{\sigma(a,s)} \\ g(b,t) &= -V(b,t) \qquad ((a,s),(b,t)\in S)\,. \\ h(a,s) &= \frac{1}{\sigma(a,s)} \end{aligned}$$

Then

$$P(a,s;b,t) = F\left(\frac{U(a,s)-V(a,s)}{\sigma(a,s)}\right) = F\left(\frac{U(a,s)}{\sigma(a,s)}+\frac{1}{\sigma(a,s)}\cdot(-V(b,t))\right) = F(u(a,s)+h(a,s)\cdot g(b,t))$$

for all $(a,s),(b,t)\in S$. So P admits an affine representation.

Lemma 4 (*Extension of Affine Representations*). *Let $P: Q^2\to\mathbb{R}$ be a function for which*

$$P(a,s;b,t) = F(u(a,s)+h(a,s)\cdot g(b,t)) \qquad ((a,s),(b,t)\in S)$$

holds with an appropriate strictly monotonic function F and with some strictly monotonic functions $u,g: S\to\mathbb{R}$, and with a nowhere zero function $h: S\to\mathbb{R}$. Then there exist functions $\tilde{u},\tilde{g},\tilde{h}: Q\to\mathbb{R}$ such that

$$\tilde{u}|_S = u \qquad \tilde{g}|_S = g \qquad \text{and} \qquad \tilde{h}|_S = h$$

and also

$$P(a,s;b,t) = F(\tilde{u}(a,s)+\tilde{h}(a,s)\cdot\tilde{g}(b,t))$$

hold for all $(a,s),(b,t)\in Q$.

Proof. Let $(a^*,s^*)\in S$ be arbitrarily fixed. Since the function F is strictly monotonic, we get from the above representation that

$$g(b,t) = \frac{F^{-1}\circ P(a^*,s^*;b,t)-u(a^*,s^*)}{h(a^*,s^*)} \qquad ((b,t)\in S).$$

As the function P is defined on the set Q^2, this shows that the function g can be extended to Q via this formula. Since the function $g: Q\to\mathbb{R}$ is strictly monotonic, there exist $(b_1,t_1),(b_2,t_2)\in S$ for which $g(b_1,t_1)\neq g(b_2,t_2)$. Using these points, we obtain that

$$\begin{aligned} F^{-1}\circ P(a,s;b_1,t_1) &= u(a,s)+h(a,s)\cdot g(b_1,t_1) \\ F^{-1}\circ P(a,s;b_2,t_2) &= u(a,s)+h(a,s)\cdot g(b_2,t_2) \qquad ((a,s)\in S)\,. \end{aligned}$$

From these equations, the function h can be expressed,

$$h(a,s) = \frac{F^{-1}\circ P(a,s;b_1,t_1)-F^{-1}\circ P(a,s;b_2,t_2)}{g(b_1,t_1)-g(b_2,t_2)} \qquad ((a,s)\in S)\,.$$

As the right hand side of this formula is defined on Q, we get that the function h admits an extension from S to Q. These however imply that the function u can also be extended, as we have

$$u(a,s) = F^{-1}\circ P(a,s;b^*,t^*)-h(a,s)\cdot g(b^*,t^*) \qquad ((a,s)\in S)\,,$$

where $(b^*,t^*)\in S$ is arbitrarily but fixed. □

2.1. Near-miss to Wallach's ratio principle

Theorem 1. *Suppose that the function $P: Q^2\to\mathbb{R}$ fulfills the near-miss to Wallach's ratio principle* (nm-WRP) *with a fixed parameter ρ and admits an affine representation, i.e., we have*

$$P(a,s;b,t) = F(u(a,s)+h(a,s)\cdot g(b,t)) \qquad ((a,s),(b,t)\in Q)$$

holds with an appropriate function F and with some functions $u,h,g: Q\to\mathbb{R}$. Then there exists one-variable functions $\tilde{u},\tilde{h}$ and $\tilde{g}: J\to\mathbb{R}$ such that

$$P(a,s;b,t) = F\left(\tilde{u}\left(\frac{a}{s^\rho}\right)+\tilde{h}\left(\frac{a}{s^\rho}\right)\cdot\tilde{g}\left(\frac{b}{t^\rho}\right)\right)$$

for all $(a,s),(b,t)\in Q$. Conversely, any function P defined via this formula, satisfies (nm-WRP) *for all $(a,s),(b,t)\in Q$ and for all $\lambda,\mu\in J$.*

Especially, if instead of the near-miss to Wallach's ratio principle (nm-WRP), the function $P : Q^2 \to \mathbb{R}$ fulfills the Wallach's ratio principle (WRP), then we have

$$P(a,s;b,t) = F\left(\tilde{u}\left(\frac{a}{s}\right) + \tilde{h}\left(\frac{a}{s}\right)\cdot \tilde{g}\left(\frac{b}{t}\right)\right)$$

for all $(a,s),(b,t) \in Q$. Conversely, any function P defined via this formula, satisfies (WRP) for all $(a,s),(b,t) \in Q$ and for all $\lambda,\mu \in J$.

Proof. Let $\rho > 0$ be arbitrarily fixed and assume that the function $P : Q^2 \to \mathbb{R}$ fulfills the near-miss to Wallach's ratio principle (nm-WRP) for all $(a,s),(b,t) \in Q$ and for all $\lambda,\mu > 0$. Let us substitute $\frac{1}{s}$ and $\frac{1}{t}$ in place of λ and μ, respectively in (nm-WRP) to deduce that

$$P(a,s;b,t) = P\left(\frac{a}{s^\rho},1;\frac{b}{t^\rho},1\right)$$

for all $(a,s),(b,t) \in Q$. At the same time, the function P admits an affine representation. Combining these, we get that

$$P(a,s;b,t) = P\left(\frac{a}{s^\rho},1;\frac{b}{t^\rho},1\right) = F\left(u\left(\frac{a}{s^\rho},1\right) + h\left(\frac{a}{s^\rho},1\right)\cdot g\left(\frac{b}{t^\rho},1\right)\right) \qquad ((a,s),(b,t) \in Q)\,.$$

This already gives the statement of the theorem, if we consider the functions $\tilde{u}, \tilde{h}$ and $\tilde{g}$ defined by

$$\tilde{u}(a) = u(a,1) \qquad \tilde{h}(a) = h(a,1) \qquad \text{and} \qquad \tilde{h}(a) = h(a,1) \qquad (a \in I)\,.$$

The second part of the statement of the theorem follows immediately from the first by choosing $\rho = 1$. Proving the converse statement in both cases is a simple calculation. □

With the help of Remark 1 and Theorem 1 we conclude the following statement.

Proposition 1. *Suppose that the function $P : Q^2 \to \mathbb{R}$ fulfills the near-miss to Wallach's ratio principle (nm-WRP) with a fixed parameter ρ and admits a gain-control representation, i.e., we have*

$$P(a,s;b,t) = F\left(\frac{u(a,s) - v(b,t)}{\sigma(a,s)}\right) \qquad ((a,s),(b,t) \in Q)$$

holds with an appropriate function F and with some functions u, v, and σ where the function σ is assumed to be nowhere zero. Then there exists one-variable functions $\tilde{u}, \tilde{v}$ and $\tilde{\sigma}$ defined on the set J such that

$$P(a,s;b,t) = F\left(\frac{\tilde{u}\left(\frac{a}{s^\rho}\right) - \tilde{v}\left(\frac{b}{t^\rho}\right)}{\tilde{\sigma}\left(\frac{a}{s^\rho}\right)}\right)$$

for all $(a,s),(b,t) \in Q$. Conversely, any function P defined via this formula, satisfies (nm-WRP) for all $(a,s),(b,t) \in Q$ and for all $\lambda,\mu \in J$.

Especially, if instead of the near-miss to Wallach's ratio principle (nm-WRP), the function $P : Q^2 \to \mathbb{R}$ fulfills the Wallach's ratio principle (WRP), then we have

$$P(a,s;b,t) = F\left(\frac{\tilde{u}\left(\frac{a}{s}\right) - \tilde{v}\left(\frac{b}{t}\right)}{\tilde{\sigma}\left(\frac{a}{s}\right)}\right)$$

for all $(a,s),(b,t) \in Q$. And also conversely, any function P defined via this formula, satisfies (nm-WRP) for all $(a,s),(b,t) \in Q$ and for all $\lambda,\mu \in J$.

Corollary 1. *The function $P : Q^2 \to \mathbb{R}$ fulfills the near-miss to Wallach's ratio principle (nm-WRP) with a fixed parameter ρ and admits a special gain-control representation, i.e., we have*

$$P(a,s;b,t) = F\left(\frac{u(a,s) - u(b,t)}{\sigma(a,s)}\right) \qquad ((a,s),(b,t) \in Q)$$

with an appropriate function F and with some functions u and σ, where the function σ is assumed to be nowhere zero, if and only if there exists one-variable functions $\tilde{u}, \tilde{\sigma} : J \to \mathbb{R}$ such that

$$P(a,s;b,t) = F\left(\frac{\tilde{u}\left(\frac{a}{s^\rho}\right) - \tilde{u}\left(\frac{b}{t^\rho}\right)}{\tilde{\sigma}\left(\frac{a}{s^\rho}\right)}\right)$$

for all $(a,s),(b,t) \in Q$.

Especially, the function $P : Q^2 \to \mathbb{R}$ fulfills the Wallach's ratio principle (WRP), if and only if we have

$$P(a,s;b,t) = F\left(\frac{\tilde{u}\left(\frac{a}{s}\right) - \tilde{u}\left(\frac{b}{t}\right)}{\tilde{\sigma}\left(\frac{a}{s}\right)}\right)$$

for all $(a,s),(b,t) \in Q$.

2.2. Near-miss to illumination invariance

Theorem 2. *Let be $P : Q^2 \to \mathbb{R}$ be a function that admits an affine representation, i.e., suppose that we have*

$$P(a, s; b, t) = F(u(a, s) + h(a, s) \cdot g(b, t)) \qquad ((a, s), (b, t) \in Q)$$

with an appropriate strictly monotonic function F and with some nonconstant function $g : Q \to \mathbb{R}$, with a continuous and nonconstant function $u : Q \to \mathbb{R}$, and with a nowhere zero, continuous function $h : Q \to \mathbb{R}$. Then P satisfies the near-miss to illumination invariance principle (nm-II) with a fixed parameter ρ if and only if one of the alternatives

(a)

$$\begin{aligned} u(a,s) &= \tilde{u}\left(\frac{a}{s^\rho}\right) - \alpha\tilde{h}\left(\frac{a}{s^\rho}\right)\cdot\log(s) \\ h(a,s) &= \tilde{h}\left(\frac{a}{s^\rho}\right) \\ g(b,t) &= \tilde{g}\left(\frac{b}{t^\rho}\right) + \alpha\log(t), \end{aligned}$$

(b)

$$\begin{aligned} u(a,s) &= \tilde{u}\left(\frac{a}{s^\rho}\right) + \alpha\left(1 - s^{-\theta}\right)\cdot\tilde{h}\left(\frac{a}{s^\rho}\right) \\ h(a,s) &= s^{-\theta}\tilde{h}\left(\frac{a}{s^\rho}\right) \\ g(b,t) &= t^\theta\cdot\tilde{g}\left(\frac{b}{t^\rho}\right) + \alpha \end{aligned}$$

hold for all $(a, s), (b, t) \in Q$ with appropriate one-variable functions $\tilde{u}, \tilde{h}, \tilde{g} : J \to \mathbb{R}$, and with some real constants α and nonzero θ.

Proof. Suppose that the function $P : Q^2 \to \mathbb{R}$ fulfills the near-miss to illumination invariance principle (nm-II) with a fixed parameter ρ and admits an affine representation. Combining these two assumptions we arrive to

$$F(u(\lambda^\rho a, \lambda s) + h(\lambda^\rho a, \lambda s)\cdot g(\lambda^\rho b, \lambda t)) = F(u(a, s) + h(a, s)\cdot g(b, t)) \qquad ((a, s), (b, t) \in Q),$$

or since F is strictly monotonic,

$$u(\lambda^\rho a, \lambda s) + h(\lambda^\rho a, \lambda s)\cdot g(\lambda^\rho b, \lambda t) = u(a, s) + h(a, s)\cdot g(b, t) \qquad ((a, s), (b, t) \in Q). \tag{1}$$

Let now $(a^*, s^*) \in Q$ be arbitrarily fixed. This latter identity with (a^*, s^*) becomes

$$u(\lambda^\rho a^*, \lambda s^*) + h(\lambda^\rho a^*, \lambda s^*)\cdot g(\lambda^\rho b, \lambda t) = u(a^*, s^*) + h(a^*, s^*)\cdot g(b, t) \qquad ((b, t) \in Q),$$

or after some rearrangement,

$$g(\lambda^\rho b, \lambda t) = \frac{h(a^*, s^*)}{h(\lambda^\rho a^*, \lambda s^*)}\cdot g(b, t) + \frac{u(a^*, s^*) - u(\lambda^\rho a^*, \lambda s^*)}{h(\lambda^\rho a^*, \lambda s^*)} \qquad ((b, t) \in Q)$$

follows, where we also use that the function h is nowhere zero. In other words, we have

$$g(\lambda^\rho b, \lambda t) = H(\lambda)\cdot g(b, t) + U(\lambda) \tag{2}$$

for all $(b, t) \in Q$ and $\lambda \in J$, where the functions H and U are defined on J by

$$H(\lambda) = \frac{h(a^*, s^*)}{h(\lambda^\rho a^*, \lambda s^*)} \quad \text{and} \quad U(\lambda) = \frac{u(a^*, s^*) - u(\lambda^\rho a^*, \lambda s^*)}{h(\lambda^\rho a^*, \lambda s^*)} \qquad (\lambda \in J).$$

If we substitute $\lambda\mu$ in place of λ in Eq. (2), we obtain that

$$g(\lambda^\rho \mu^\rho b, \lambda\mu t) = H(\lambda\mu)\cdot g(b, t) + U(\lambda\mu)$$

holds. At the same time, by a successive application of Eq. (2),

$$\begin{aligned} g(\lambda^\rho\mu^\rho b, \lambda\mu t) &= g(\lambda^\rho(\mu^\rho b), \lambda(\mu t)) = H(\lambda)\cdot g(\mu^\rho b, \mu t) + U(\lambda) \\ &= H(\lambda)\cdot[H(\mu)\cdot g(b, t) + U(\mu)] + U(\lambda) \\ &= H(\lambda)H(\mu)\cdot g(b, t) + H(\lambda)U(\mu) + U(\lambda) \end{aligned}$$

follows. Thus

$$[H(\lambda\mu) - H(\lambda)H(\mu)]\cdot g(b, t) = -U(\lambda\mu) + H(\lambda)U(\mu) + U(\lambda).$$

Since the function g is nonconstant, this is possible only if

$$\begin{aligned} H(\lambda\mu) &= H(\lambda)H(\mu) \\ U(\lambda\mu) &= H(\lambda)U(\mu) + U(\lambda) \qquad (\lambda, \mu \in J) \end{aligned}$$

As the function $u : Q \to \mathbb{R}$ was assumed to be nonconstant and continuous and the function $h : Q \to \mathbb{R}$ is supposed to be nowhere zero and continuous, the functions $U : J \to \mathbb{R}$ and $H : J \to \mathbb{R}$ are continuous and U is philandering. Notice that the first equation says that H is multiplicative, while in case of the second equation Lemma 1 can be applied. Thus we have the following alternative

(a) either H is constant on J and then necessarily this constant is 1, yielding that U is logarithmic, so (2) takes the form

$$g(\lambda^{\rho}b, \lambda t) = g(b,t) + \alpha \log(\lambda) \qquad ((b,t) \in Q, \lambda \in J)$$

with an appropriate real number α. From this, with $\lambda = \frac{1}{t}$,

$$g\left(\frac{b}{t^{\rho}}, 1\right) = g(b,t) - \alpha \log(t) \qquad ((b,t) \in Q).$$

follows. Thus

$$g(b,t) = \tilde{g}\left(\frac{b}{t^{\rho}}\right) + \alpha \log(t)$$

holds with an appropriate real number α and with a single-variable function $\tilde{g}$.

(b) or the function H is a nonconstant multiplicative function, i.e. we have $H(\lambda) = \lambda^{\beta_0}$ $(\lambda \in J)$ with some nonzero real number β_0 and (2) reduces to

$$g(\lambda^{\rho}b, \lambda t) = \lambda^{\beta_0} \cdot g(b,t) + \beta_2 \lambda^{\beta_0} = \lambda^{\beta_0} \cdot \left(g(b,t) + \beta_2\right),$$

where we also used that $U(1) = 0$. From this, with $\lambda = \frac{1}{t}$,

$$g\left(\frac{b}{t^{\rho}}, 1\right) = t^{-\beta_0} \cdot \left(g(b,t) + \beta_2\right)$$

follows. Therefore, after some rearrangement we deduce finally that

$$g(b,t) = t^{\theta} \cdot \tilde{g}\left(\frac{b}{t^{\rho}}\right) + \alpha$$

holds with an appropriate α and nonzero θ and with a single-variable function $\tilde{g}$.

We will now determine the functions u and h. If alternative (a) holds above, then Eq. (1) takes the form

$$u(\lambda^{\rho}a, \lambda s) + h(\lambda^{\rho}a, \lambda s) \cdot (g(b,t) + \alpha \log(\lambda)) = u(a,s) + h(a,s) \cdot g(b,t)$$

for all $(a,s), (b,t) \in Q$ and $\lambda \in J$. Substituting $\frac{1}{s}$ in place of λ in the above identity, we obtain that

$$u(a,s) + h(a,s) \cdot g(b,t) = u\left(\frac{a}{s^{\rho}}, 1\right) + h\left(\frac{a}{s^{\rho}}, 1\right) \cdot (g(b,t) - \alpha \log(s))$$

holds for all $(a,s), (b,t) \in Q$. Since the function g is nonconstant, there exist $(b^*, t^*), (\bar{b}, \bar{t}) \in Q$ such that $g(b^*, t^*) \neq g(\bar{b}, \bar{t})$. Using these pairs, the system of equations

$$u(a,s) + h(a,s) \cdot g(b^*, t^*) = u\left(\frac{a}{s^{\rho}}, 1\right) + h\left(\frac{a}{s^{\rho}}, 1\right) \cdot (g(b^*, t^*) - \alpha \log(s))$$

$$u(a,s) + h(a,s) \cdot g(\bar{b}, \bar{t}) = u\left(\frac{a}{s^{\rho}}, 1\right) + h\left(\frac{a}{s^{\rho}}, 1\right) \cdot (g(\bar{b}, \bar{t}) - \alpha \log(s))$$

follows for all $(a,s) \in Q$. Subtracting the first equation from the second one side by side, we get that

$$h(a,s) = h\left(\frac{a}{s^{\rho}}, 1\right) \qquad ((a,s) \in Q).$$

Furthermore, inserting this form of the function h into the equation above, we obtain that

$$u(a,s) = u\left(\frac{a}{s^{\rho}}, 1\right) - \alpha h\left(\frac{a}{s^{\rho}}, 1\right) \log(s)$$

for all $(a,s) \in Q$. Consider the single-variable function $\tilde{u}$ and $\tilde{h}$ defined by

$$\tilde{u}(a) = u(a,1) \qquad \text{and} \qquad \tilde{h}(a) = h(a,1) \qquad (s \in J)$$

to deduce that

$$u(a,s) = \tilde{u}\left(\frac{a}{s^{\rho}}\right) - \alpha \tilde{h}\left(\frac{a}{s^{\rho}}\right) \cdot \log(s) \quad \text{and} \quad h(a,s) = \tilde{h}\left(\frac{a}{s^{\rho}}\right)$$

holds for all $(a,s) \in Q$. We now turn to the discussion of alternative (b). In this case

$$g(\lambda^{\rho}b, \lambda t) = \lambda^{\theta} g(b,t) + \alpha(1 - \lambda^{\theta})$$

holds for all $(b,t) \in Q$ and $\lambda \in J$. Hence Eq. (1) becomes

$$u(\lambda^{\rho}a, \lambda s) + h(\lambda^{\rho}a, \lambda s) \cdot \left(\lambda^{\theta} g(b,t) + \alpha(1 - \lambda^{\theta})\right) = u(a,s) + h(a,s) \cdot g(b,t)$$

for all $(a,s), (b,t) \in Q$ and $\lambda \in J$. Substituting $\frac{1}{s}$ in place of λ in the above identity, we obtain that

$$u\left(\frac{a}{s^{\rho}}, 1\right) + h\left(\frac{a}{s^{\rho}}, 1\right) \cdot \left(s^{-\theta} g(b,t) + \alpha(1 - s^{-\theta})\right) = u(a,s) + h(a,s) \cdot g(b,t)$$

holds for all $(a,s), (b,t) \in Q$. Since the function g is nonconstant, there exist $(b^*, t^*), (\bar{b}, \bar{t}) \in Q$ such that $g(b^*, t^*) \neq g(\bar{b}, \bar{t})$. Using these pairs, the system of equations

$$u(a,s)+h(a,s)\cdot g(b^*,t^*)=u\left(\frac{a}{s^{\rho}},1\right)+h\left(\frac{a}{s^{\rho}},1\right)\cdot\left(s^{-\theta}g(b^*,t^*)+\alpha(1-s^{-\theta})\right)$$
$$u(a,s)+h(a,s)\cdot g(\bar{b},\bar{t})=u\left(\frac{a}{s^{\rho}},1\right)+h\left(\frac{a}{s^{\rho}},1\right)\cdot\left(s^{-\theta}g(\bar{b},\bar{t})+\alpha(1-s^{-\theta})\right)$$

follows for all $(a,s)\in Q$. Subtracting the first equation from the second one side by side, we get that

$$h(a,s)=s^{-\theta}h\left(\frac{a}{s^{\rho}},1\right) \qquad ((a,s)\in Q).$$

Furthermore, inserting this form of the function h into the above identity, we get that

$$u(a,s)=u\left(\frac{a}{s^{\rho}},1\right)+\alpha\left(1-s^{\theta}\right)\cdot h\left(\frac{a}{s^{\rho}},1\right)$$

for all $(a,s)\in Q$. Introducing the single-variable functions $\tilde{u}$ and $\tilde{h}$ as

$$\tilde{u}(a)=u(a,1) \qquad \text{and} \qquad \tilde{h}(a)=h(a,1) \qquad (s\in J),$$

we conclude that

$$u(a,s)=\tilde{u}\left(\frac{a}{s^{\rho}}\right)+\alpha\left(1-s^{-\theta}\right)\cdot\tilde{h}\left(\frac{a}{s^{\rho}}\right) \qquad \text{and} \qquad h(a,s)=s^{-\theta}\tilde{h}\left(\frac{a}{s^{\rho}}\right)$$

□

Remark 2. The assumptions that the function $u:Q\to\mathbb{R}$ is nonconstant and continuous and the function $h:Q\to\mathbb{R}$ is nowhere zero and continuous were used in the above proof only when we determined the solutions of the system of functional equations

$$H(\lambda\mu)=H(\lambda)H(\mu)$$
$$U(\lambda\mu)=H(\lambda)U(\mu)+U(\lambda) \qquad (\lambda,\mu\in J)$$

These assumptions guaranteed that the functions $U:J\to\mathbb{R}$ and $H:J\to\mathbb{R}$ are continuous and U is philandering. However, for Lemma 1 to be applicable, it is sufficient to guarantee that the function U is philandering and measurable. For this, instead of continuity, it is sufficient to assume, for example, the milder measurability of the function u. Additionally, we note that the above system of functional equations can be solved without assuming any regularity at all. Indeed, the first equation states that H is a multiplicative function on J. If H is the identically 1 multiplicative function, the second equation says that the function U is logarithmic on J. The other possibility is that H is a multiplicative function which is not the identically 1 function. In this case however

$$U(\lambda)=c\left(H(\lambda)-1\right)$$

holds for all $\lambda\in J$ with some real constant c. Finally, we note that, at least using the present proof, the condition that F is strictly monotonic and g is nonconstant, cannot be omitted.

Remark 3. By choosing $\rho=1$, it immediately follows that those functions P that satisfy the identity (II) and have an affine representation can be written using the functions $u,h,g:Q\to\mathbb{R}$ below.

(a) Either

$$\begin{aligned} u(a,s) &= \tilde{u}\left(\tfrac{a}{s}\right)-\alpha\tilde{h}\left(\tfrac{a}{s}\right)\cdot\log(s)\\ h(a,s) &= \tilde{h}\left(\tfrac{a}{s}\right)\\ g(b,t) &= \tilde{g}\left(\tfrac{b}{t}\right)+\alpha\log(t), \end{aligned}$$

(b) or

$$\begin{aligned} u(a,s) &= \tilde{u}\left(\tfrac{a}{s}\right)+\alpha\left(1-s^{-\theta}\right)\cdot\tilde{h}\left(\tfrac{a}{s}\right)\\ h(a,s) &= s^{-\theta}\tilde{h}\left(\tfrac{a}{s}\right)\\ g(b,t) &= t^{\theta}\cdot\tilde{g}\left(\tfrac{b}{t}\right)+\alpha \end{aligned}$$

for all $(a,s),(b,t)\in Q$ with appropriate one-variable functions $\tilde{u},\tilde{h},\tilde{g}:J\to\mathbb{R}$, and with some real constants α and nonzero θ.

From Theorem 2 we immediately get the following.

Corollary 2. *If the function $P:Q^2\to\mathbb{R}$ fulfills the near-miss to illumination invariance (nm-II) with a fixed parameter ρ and admits an affine representation, then*

(a) either

$$P(a,s;b,t)=F\left(\tilde{u}\left(\frac{a}{s^{\rho}}\right)+\tilde{h}\left(\frac{a}{s^{\rho}}\right)\tilde{g}\left(\frac{b}{t^{\rho}}\right)+\alpha\tilde{h}\left(\frac{a}{s^{\rho}}\right)(\log(t)-\log(s))\right)$$

(b) or

$$P(a,s;b,t) = F\left(\tilde{u}\left(\frac{a}{s^{\rho}}\right) + \alpha s^{-\theta} t^{\theta} \tilde{h}\left(\frac{a}{s^{\rho}}\right)\tilde{g}\left(\frac{b}{t^{\rho}}\right)\right)$$

holds.

Below we determine those functions P that admit a gain-control type representation and also fulfill (nm-II).

Corollary 3. *A function $P : Q^2 \to \mathbb{R}$ admitting a gain-control type representation, i.e.,*

$$P(a,s;b,t) = F\left(\frac{u(a,s) - v(b,t)}{\sigma(a,s)}\right) \qquad ((a,s),(b,t) \in Q)$$

with some strictly monotonic function F, with some nonconstant, continuous functions u, v, and with a nowhere zero, continuous function σ, fulfills the near-miss to illumination invariance principle (nm-II) with a fixed parameter ρ if and only if

(a) either

$$\begin{aligned} u(a,s) &= \bar{u}\left(\tfrac{a}{s^{\rho}}\right) - \alpha \log(s) \\ \sigma(a,s) &= \bar{h}\left(\tfrac{a}{s^{\rho}}\right) \\ v(b,t) &= \bar{g}\left(\tfrac{b}{t^{\rho}}\right) + \alpha \log(t), \end{aligned}$$

(b) or

$$\begin{aligned} u(a,s) &= s^{\theta}\bar{u}\left(\tfrac{a}{s^{\rho}}\right) + \alpha \\ \sigma(a,s) &= s^{\theta}\bar{h}\left(\tfrac{a}{s^{\rho}}\right) \\ v(b,t) &= t^{\theta} \cdot \bar{g}\left(\tfrac{b}{t^{\rho}}\right) + \alpha \end{aligned}$$

hold for all $(a,s),(b,t) \in Q$ with appropriate one-variable functions $\bar{u}, \bar{h}, \bar{g} : J \to \mathbb{R}$, and with some real constants α and nonzero θ.

Proof. Suppose that the function $P : Q^2 \to \mathbb{R}$ fulfills the near-miss to illumination invariance principle (nm-II) with a fixed parameter ρ and admits a gain-control type representation with some strictly monotonic function F, with some nonconstant, continuous functions u, v, and with a nowhere zero, continuous function σ. Then

$$P(a,s;b,t) = F\left(\frac{u(a,s) - v(b,t)}{\sigma(a,s)}\right) = F\left(U(a,s) + H(a,s)\cdot G(b,t)\right) \qquad ((a,s),(b,t) \in Q)$$

holds, where the functions U, H and G are defined by

$$U(a,s) = \frac{u(a,s)}{\sigma(a,s)} \qquad H(a,s) = \frac{1}{\sigma(a,s)} \qquad \text{and} \qquad G(b,t) = -v(b,t).$$

Applying Theorem 2, we obtain the form of the functions U, H and G. Thus

$$\begin{aligned} u(a,s) &= \frac{U(a,s)}{H(a,s)} \\ \sigma(a,s) &= \frac{1}{H(a,s)} \qquad ((a,s),(b,t) \in Q) \\ v(b,t) &= -G(b,t), \end{aligned}$$

from which we finally get the possible forms of the functions u, v and σ. □

Using this corollary, we can describe those functions $P : Q \to \mathbb{R}$ that have subtractive representation and satisfy Eq. (nm-II).

Corollary 4. *A function $P : Q^2 \to \mathbb{R}$ admitting a subtractive representation, i.e.,*

$$P(a,s;b,t) = F(u(a,s) - v(b,t)) \qquad ((a,s),(b,t) \in Q)$$

with some strictly monotonic F and with some nonconstant and continuous functions $u, v : Q \to \mathbb{R}$, fulfills the near-miss to illumination invariance principle (nm-II) with a fixed parameter ρ if and only if

$$\begin{aligned} u(a,s) &= \bar{u}\left(\tfrac{a}{s^{\rho}}\right) - \alpha \log(s) \\ v(b,t) &= \bar{g}\left(\tfrac{b}{t^{\rho}}\right) + \alpha \log(t), \end{aligned}$$

for all $(a,s),(b,t) \in Q$ with appropriate one-variable functions $\bar{u}, \bar{g} : J \to \mathbb{R}$, and with some real constant α.

Proof. If for the function $P : Q^2 \to \mathbb{R}$ there exists a strictly monotonic function F and there are nonconstant and continuous functions $u, v : Q \to \mathbb{R}$ such that we have

$$P(a, s; b, t) = F(u(a, s) - v(b, t)) \qquad ((a, s), (b, t) \in Q),$$

then the function P has a gain-control representation, too, if we consider $\sigma \equiv 1$. Thus in view of Corollary 3,

(a) either

$$\begin{aligned} u(a, s) &= \bar{u}\left(\tfrac{a}{s^\rho}\right) - \alpha \log(s) \\ \sigma(a, s) &= \bar{h}\left(\tfrac{a}{s^\rho}\right) \\ v(b, t) &= \bar{g}\left(\tfrac{b}{t^\rho}\right) + \alpha \log(t), \end{aligned}$$

(b) or

$$\begin{aligned} u(a, s) &= s^\theta \bar{u}\left(\tfrac{a}{s^\rho}\right) + \alpha \\ \sigma(a, s) &= s^\theta \bar{h}\left(\tfrac{a}{s^\rho}\right) \\ v(b, t) &= t^\theta \cdot \bar{g}\left(\tfrac{b}{t^\rho}\right) + \alpha \end{aligned}$$

hold for all $(a, s), (b, t) \in Q$ with appropriate one-variable functions $\bar{u}, \bar{h}, \bar{g} : J \to \mathbb{R}$, and with some real constants α and nonzero θ.

At the same time, in this case $\sigma \equiv 1$ should hold. Thus, in case of alternative (a) we have

$$\bar{h}\left(\frac{a}{s^\rho}\right) = \sigma(a, s) = 1 \qquad (a, s \in J)$$

which is possible only if the function h is the identically one function. Alternative (b) leads to

$$s^\theta \bar{h}\left(\frac{a}{s^\rho}\right) = \sigma(a, s) = 1 \qquad (a, s \in J),$$

which is impossible in this case. Therefore, there exist functions $\bar{u}, \bar{g} : J \to \mathbb{R}$ and a real constant α such that

$$\begin{aligned} u(a, s) &= \bar{u}\left(\tfrac{a}{s^\rho}\right) - \alpha \log(s) \\ v(b, t) &= \bar{g}\left(\tfrac{b}{t^\rho}\right) + \alpha \log(t), \end{aligned}$$

for all $(a, s), (b, t) \in Q$. $\square$

In the next statement we will consider special gain-control type representations.

Corollary 5. *Suppose that the function $P : Q^2 \to \mathbb{R}$ admits a special gain-control type representation of the form*

$$P(a, s; b, t) = F\left(\frac{u(a, s) - u(b, t)}{\sigma(a, s)}\right) \qquad ((a, s), (b, t) \in Q)$$

with some strictly monotonic function F, nonconstant and continuous function $u : Q \to \mathbb{R}$ and with a nowhere zero, continuous function $\sigma : Q \to \mathbb{R}$. Then P fulfills the near-miss to illumination invariance principle (nm-II) with a fixed parameter ρ if and only if we have the following alternatives.

(a) Either

$$\begin{aligned} u(a, s) &= \bar{u}\left(\tfrac{a}{s^\rho}\right) - \alpha \log(s) \\ \sigma(a, s) &= \bar{h}\left(\tfrac{a}{s^\rho}\right) \end{aligned}$$

(b) or

$$\begin{aligned} u(a, s) &= s^\theta \bar{u}\left(\tfrac{a}{s^\rho}\right) - \alpha \\ \sigma(a, s) &= s^\theta \bar{h}\left(\tfrac{a}{s^\rho}\right) \end{aligned}$$

hold for all $(a, s) \in Q$ with appropriate one-variable functions $\bar{u}$ and $\bar{h}$, and with some real constant α and nonzero θ.

Proof. The beginning of the proof of this corollary is similar as that of Corollary 3. The only exception is that the functions U, H and G are defined by

$$U(a, s) = \frac{u(a, s)}{\sigma(a, s)} \qquad H(a, s) = \frac{1}{\sigma(a, s)} \qquad \text{and} \qquad G(b, t) = -u(b, t).$$

Thus, after the application of Theorem 2, we obtain that

(a)

$$\begin{aligned} U(a, s) &= \tilde{u}\left(\tfrac{a}{s^\rho}\right) - \alpha \tilde{h}\left(\tfrac{a}{s^\rho}\right) \cdot \log(s) \\ H(a, s) &= \tilde{h}\left(\tfrac{a}{s^\rho}\right) \\ G(b, t) &= \tilde{g}\left(\tfrac{b}{t^\rho}\right) + \alpha \log(t), \end{aligned}$$

(b)

$$\begin{aligned} U(a,s) &= \tilde{u}\left(\frac{a}{s^{\rho}}\right) + \alpha\left(1 - s^{-\theta}\right)\cdot\tilde{h}\left(\frac{a}{s^{\rho}}\right) \\ H(a,s) &= s^{-\theta}\tilde{h}\left(\frac{a}{s^{\rho}}\right) \\ G(b,t) &= t^{\theta}\cdot\tilde{g}\left(\frac{b}{t^{\rho}}\right) + \alpha \end{aligned}$$

hold for all $(a,s),(b,t)\in Q$ with appropriate one-variable functions $\tilde{u},\tilde{h},\tilde{g}\colon J\to\mathbb{R}$, and with some real constants α and nonzero θ.

In case of alternative (a) this leads to

$$u(a,s) = \frac{U(a,s)}{H(a,s)} = \frac{\tilde{u}\left(\frac{a}{s^{\rho}}\right) - \alpha\tilde{h}\left(\frac{a}{s^{\rho}}\right)\cdot\log(s)}{\tilde{h}\left(\frac{a}{s^{\rho}}\right)} = \frac{\tilde{u}\left(\frac{a}{s^{\rho}}\right)}{\tilde{h}\left(\frac{a}{s^{\rho}}\right)} - \alpha\log(s)$$

and

$$\sigma(a,s) = \frac{1}{H(a,s)} = \frac{1}{\tilde{h}\left(\frac{a}{s^{\rho}}\right)}$$

for all $(a,s)\in Q$. If we introduce the functions $\bar{u},\bar{h}\colon J\to\mathbb{R}$ through

$$\bar{u}(a) = \frac{\tilde{u}(a)}{\tilde{h}(a)} \qquad \text{and} \qquad \bar{h}(a) = \frac{1}{\tilde{h}(a)} \qquad (a\in J),$$

we get alternative (a) of our corollary. In case of alternative (b), we obtain that

$$u(a,s) = \frac{U(a,s)}{H(a,s)} = \frac{\tilde{u}\left(\frac{a}{s^{\rho}}\right) + \alpha\left(1-s^{-\theta}\right)\cdot\tilde{h}\left(\frac{a}{s^{\rho}}\right)}{s^{-\theta}\tilde{h}\left(\frac{a}{s^{\rho}}\right)} = s^{\theta}\cdot\left(\frac{\tilde{u}\left(\frac{a}{s^{\rho}}\right)}{\tilde{h}\left(\frac{a}{s^{\rho}}\right)} + \alpha\right) - \alpha$$

and

$$\sigma(a,s) = \frac{1}{H(a,s)} = \frac{s^{\theta}}{\tilde{h}\left(\frac{a}{s^{\rho}}\right)}$$

for all $(a,s)\in Q$. If we introduce the functions $\bar{u},\bar{h}\colon Q\to\mathbb{R}$ through

$$\bar{u}(a) = \frac{\tilde{u}(a)}{\tilde{h}(a)} + \alpha \qquad \text{and} \qquad \bar{h}(a) = \frac{1}{\tilde{h}(a)} \qquad (a\in J),$$

we get alternative (b) in our corollary. □

In this paper, we have investigated the (near-miss to) Wallach's ratio principle and (near-miss to) illumination invariance principle, resp. under various psychophysical representations. In the following table, we have summarized our findings from this section.

	Affine	Gain-control	Spec. gain-control	Subtractive
WRP	Theorem 1 with $\rho = 1$	Proposition 1 with $\rho = 1$	Corollary 1 with $\rho = 1$	[9]
nm-WRP	Theorem 1	Proposition 1	Corollary 1	[9]
II	Remark 3	Corollary 3 with $\rho = 1$	Corollary 5 with $\rho = 1$	Corollary 4 with $\rho = 1$
nm-II	Theorem 2	Corollary 3	Corollary 5	Corollary 4

3. Conclusions and perspectives

In the previous section, we investigated (the near-miss to) Wallach's ratio principle and also (the near-miss to) illumination invariance, respectively with a focus on the more general affine representation, building on the work of [9] that used a subtractive representation. The affine representation offers a broader range of possibilities for the shapes of the discrimination probabilities, allowing us to account for cases where (the near-miss to) Wallach' ratio principle is only conditionally fulfilled. This approach provides greater flexibility in modeling the perception of luminance or color differences under different conditions.

While the ratio principle was found to be important for chromatic stimuli, [18,19] also revealed small asymmetries in each opponent channel's perceived lightness with respect to their opponent hues. This suggests that the ratio principle might not be entirely symmetrical and could have some exceptions in certain chromatic perception conditions.

Our hypothesis is that, while the ratio principle is somewhat important for the perception of chromatic colors, color constancy is more likely to play a role here. This conjecture is supported by [20]. Their study revealed a number of important insights into the mechanisms of color constancy. By manipulating the stimuli in their experiment and measuring color constancy across different levels of stimulus conditions, they discovered that the classic hypotheses alone cannot fully explain the mechanisms of color constancy. Their findings suggest that more complex visual mechanisms may be involved, as the degree of constancy varied from as low as 11% to as high as 83% under different conditions.

Experiments of [21] on chromatic illumination change discrimination found the following:

- Chromatic discrimination thresholds vary depending on the reference illumination.
- For a neutral reference, there is a trend for higher thresholds for bluer illumination changes, consistent with previous findings of a "blue bias".
- For chromatic reference illuminations (blue, yellow, red, and green), changes towards the neutral illumination are less well discriminated, suggesting a "neutral bias".

The wavelength of visible light spans the range from approximately 380 to 740 nanometers. Within this range, even minute variations in wavelength can lead to discernible differences in the perceived hue. The just-noticeable difference in wavelength can range from as little as 1 nanometer in the blue–green and yellow regions of the spectrum to upwards of 10 nanometers in the longer red and shorter blue wavelengths. In the previous works and also in this work, we identified colors solely by their wavelength without considering the perceived hue, value, or chroma, which are crucial aspects in human color perception. To address this limitation, we propose to employ the Munsell system (hue, value, chroma) for representing colors in our models. This would require us to work on a set of the form $I_h \times I_v \times I_c$, rather than the interval I, allowing for a more comprehensive and accurate representation of color perception. In addition, we expect that, adapted to the given situation and phenomenon, we will have to solve other function equations.

We also note that [9] suggests that it can be worth switching from discrimination probabilities to sensitivity functions. If $(a, s), (b, t) \in Q$ and

$$P(a, s; b, t) = p,$$

then the sensitivity function $\xi_{b,t;p}$ is defined as

$$\xi_{b,t;p}(s) = a.$$

Using this function, (WRP) becomes

$$\xi_{\mu^\rho b,\mu t;p}(\lambda s) = \lambda^\rho \xi_{b,t;p}(s)$$

for all possible $(a, s), (b, t) \in Q$ such that $(\lambda^\rho a, \lambda s), (\mu^\rho b, \mu t) \in Q$. More information on how to switch from discrimination probabilities to sensitivity functions and why this switch is useful can be found in [22,23]. This approach would introduce the possibility of ρ being not merely a constant but a variable that can depend on p, thereby allowing for greater flexibility in the resulting model. We intend to further explore these directions in our future research, uncovering new insights and refining our understanding of lightness constancy and color perception.

Declaration of competing interest

The author declares that she has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this article.

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Data availability

No data was used for the research described in the article.

References

[1] Wallach H. Brightness constancy and the nature of achromatic colors. J Exp Psychol 1948;38(3):310.
[2] Jameson D, Hurvich LM. Complexities of perceived brightness: Apparent brightness may increase, decrease, or remain constant as illumination is increased. Science 1961;133(3447):174–9.
[3] Jacobsen A, Gilchrist A. The ratio principle holds over a million-to-one range of illumination. Percept Psychophys 1988;43:1–6.
[4] Burzlaff W. Methodologische Beiträge zum Problem der Farbenkonstanz. In: Zeitschrift für Psychologie und Physiologie der Sinnesorgane. Abt. 1. Zeitschrift für Psychologie. 1931.
[5] Chien SH-L, Palmer J, Teller DY. Achromatic contrast effects in infants: Adults and 4-month-old infants show similar deviations from Wallach's ratio rule. Vis Res 2005;45(22):2854–61.
[6] Plateau JAF. Sur la mesure des sensations physiques, et sur la loi qui lie l'intensité de ces sensations à l'intensité de la cause excitante. 1872.
[7] Falmagne J-C. Elements of psychophysical theory. no. 6. USA: Oxford University Press; 2002.

[8] Heller J. Illumination-invariance of Plateau's midgray. J Math Psych 2006;50(3):263–70.
[9] Heller J. On the perception of achromatic colors: Wallach's ratio principle revisited. In: TPM: testing, psychometrics, methodology in applied psychology. 2014.
[10] Falmagne J-C. Psychometric functions theory. J Math Psych 1982;25(1):1–50.
[11] Levine MV. Transformations that render curves parallel. J Math Psych 1970;7(3):410–43.
[12] Levine MV. Transforming curves into curves with the same shape. J Math Psych 1972;9(1):1–16.
[13] Hsu Y-F, Iverson GJ, Doble CW. Affine representations in psychophysics and the near-miss to Weber's law. J Math Psych 2010;54(2):256–65.
[14] Falmagne J-C, Lundberg A. Compatibility of gain control and power law representations – a János Aczél connection. Aequationes Math 2000;60(3):201–10.
[15] Aczél J. Extension of a generalized Pexider equation. Proc Amer Math Soc 2005;133(11):3227–33.
[16] Aczél J. Lectures on functional equations and their applications. Academic Press; 1966.
[17] Aczél J. A short course on functional equations: based upon recent applications to the social and behavioral sciences, vol. 3. Springer Science & Business Media; 2012.
[18] Bäuml K-H. A ratio principle for a red/green and a yellow/blue channel? Percept Psychophys 1993;53(3):338–44.
[19] Bäuml K-H. Color appearance: effects of illuminant changes under different surface collections. J Opt Soc Amer A 1994;11(2):531–42.
[20] Kraft JM, Brainard DH. Mechanisms of color constancy under nearly natural viewing. Proc Natl Acad Sci 1999;96(1):307–12.
[21] Aston S, Radonjić A, Brainard DH, Hurlbert AC. Illumination discrimination for chromatically biased illuminations: Implications for color constancy. J Vis 2019;19(3):15.
[22] Augustin T. The parameters in the near-miss-to-Weber's law. J Math Psych 2008;52(1):37–47.
[23] Augustin T, Roscher T. Empirical evaluation of the near-miss-to-Weber's law: a visual discrimination experiment. Psychol Sci Q 2008;50(4):469.