

Double-Sources Queuing-Inventory Systems with Finite Waiting Room and Destructible Stocks

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Abstract: Models of double-source queuing-inventory systems are studied in the presence of a finite buffer for waiting in the queue of consumer customers, where instant destruction of inventory is possible. It is assumed that the lead times of orders, as well as the cost of delivery from various sources, differ from each other. Replenishment of stocks from various sources is carried out according to the following scheme: if the inventory level drops to the reorder point s , then a regular order for the supply of inventory to a slow source is generated; if the inventory level falls below a certain threshold value r , where $r < s$, then the system instantly cancels the regular order and generates an emergency order to the fast source. Models of systems that use (s, S) or (s, Q) replenishment policies are studied. Exact and approximate methods for finding the performance measures of the models under study are proposed. The problems of minimizing the total cost are solved by choosing the appropriate values of the parameters s and r when using different replenishment policies. Numerical examples demonstrated the high accuracy of an approximate method as well as compared performance measures of the system under various replenishment policies.

Keywords: queuing-inventory system; double sources; finite buffer

MSC: 60K25; 90B22; 68M20

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1. Introduction

For the first time, the heading “Queuing-Inventory System” (QIS) was introduced in the papers [1,2]. These are the systems in which in order to service incoming consumer customers (c -customers), along with idle server, some resources are required. These resources are called inventories.

The QIS theory has been intensively developed in the past three decades. The first publications in this direction are [3,4]. In [4] an $M/G/1$ type QIS with exponentially distributed lead time and under light traffic is considered. They obtained product form solution for steady-state analysis. In [3] a Markov Decision Process (MDP) approach to optimize the finite QIS is used where order size is a controllable parameter. They developed an exact and approximate (for large scale QIS) methods to solution of the constructed MDP.

The current state of QIS theory and its applications are described in detail in a recent review paper [5]. Therefore, we indicate here only related works published after this review work, as well as those works that were not included in its bibliography.

This work is related to (i) QIS with destructible stocks, (ii) multi-parametric replenishment policies (RP) in double sources QIS and (iii) multiple type c -customers. In what follows, we shortly review the relevant literature.

Let us note that here we distinguish between the QISs with destructible and perishable stocks. In turn, in a class of perishable QISs two sub-classes of systems are differentiated: (1) QIS with individual life time (ILF) and (2) QIS with common life time (CLT). In QIS with ILT each item can perish independently of the others, while in QIS with CLT all items perish together, e.g., foods with the same expiry date, medicines manufactured with the same expiry date and so on. In considering perishable QIS with ILT, we draw from a vast stream of literature, see, e.g., [6–12]. Models of QIS with CLT has been little studied. For the first time, such a model was examined in [13], next in [14] and recently, in [15]. Some review of works in this direction can be found in the last paper. Note that QIS with CLT can be treated as QIS with catastrophes in inventory part (but not in a service facility).

Almost all studies on perishable QIS of both types are devoted to models in which items can perish within a certain time interval. However, in practice, there are QIS in which items can be destroyed instantly, e.g., (1) a shop selling breakable items in which destruction of the items may be the result of a worker's careless actions; (2) a company selling electrical/electronic goods in which destruction of the item can occur even during the sale as a result of a power surge in the electrical network. Models of QIS with destructible inventory are similar to models of QIS with ILT, but their main difference is as follows: in QIS with ILT a perish rate of inventory is a linear function of inventory level (i.e., if inventory level is m , then perish rate is $m\gamma$, where γ^{-1} is a life time of individual item), but in QIS with destructible inventory, destruction rate is independent on inventory level, i.e., it is a constant quantity. In other words, QIS with destructible inventory can be treated as QIS with ILT in which perish rate is a constant quantity (as in models of retrial queues with linear and constant retrial rates from orbit). Models of QIS with destructible inventory have hardly been studied, see [16]. In this paper, we consider models of QIS with destructible inventory.

In the vast majority of works, models of QIS with a single source are studied. One problem of optimal choosing of the source from a set of sources with different supply rates and costs was solved in [17], where the optimality criterion was the total cost. In this work, it was assumed that in the entire period of the system operation, an order is made to only one source. However, the models of QIS with multiple sources in which an order can be sent to different sources depending on the current inventory level represents important scientific and practical interest. In other words, in such QISs the distribution of the order between a fast and an expensive supplier and a slow but inexpensive supplier are very important problems. Solutions to these problems require determining appropriate replenishment policies (RPs).

Recently, only a few papers have considered models of double-source QISs. Original RP in double-source QIS was recently proposed in [18,19]. Proposed in these papers, RP is defined as follows: whenever the inventory level falls to r ($>S/2$), an order with the temporary (regular) supplier for Q_1 items is placed, $Q_1 = S - r$, where S is maximal capacity of store: whenever the inventory level reaches $s < Q_1 < r$, s , an order of $Q_2 = S - s$ ($> s + 1$) items to the regular supplier is placed and immediately cancels the order for Q_1 items from the temporary supplier (to be fair, in [18] do not explicitly state the cancellation procedure but in [19] this fact is shown clearly). In both papers, the QIS models with infinite queue for c -customers are considered and Neuts' matrix-analytic method (MAM) [20] is used.

Other kinds of RP in double-sources QISs were proposed in [21,22]. The main feature of the proposed policies is that they are based on classical RPs, i.e., (s, S) and (s, Q) policies are used for construction of the new RPs. In other words, in both RPs no order is placed until the inventory level is above the reorder point s , $s < S/2$.

In [21] two models of double-source QIS with instantly damaging inventory were proposed: one model uses the (s, S) -policy, and the other model uses the (s, Q) -policy during the entire period of the system's operation (for definiteness, we note that (s, S) -policy (sometimes it is called "Up to S " policy) is a RP in which the order volume is as much as it is able to bring the level back to S at the replenishment epoch and in (s, Q) -policy the

order volume is fixed and is equal to $Q = S - s$). In both RPs, choice of the source is made depending on the current inventory level as follows: a regular order to SS is placed when the inventory level drops to the value s , and an emergency order to FS is placed when the inventory level drops to some critical value r , where $0 \leq r < s$. Similar but hybrid RP in double-source QIS was proposed in [22]. In hybrid RP, it is assumed that a regular order to the Slow Source (SS) is placed in accordance with (s, Q) -policy while an emergency order to the Fast Source (FS) is placed in accordance with (s, S) -policy. In all cases, at the time of placing an emergency order, the regular order is immediately canceled.

After the publications of papers [21,22], the authors became aware of works [23,24]. It appears that the first paper dealing with QIS with multiple sources was [23]. In that paper, the service facility of examined QIS is a $M/M/1/\infty$ queue, and the selection of source among two sources with different lead times is determined according to Bernoulli trials. Arriving customers are lost when inventory is out of stock (lost sales scheme) and fixed size order RP is used. The authors obtained explicit formulas for calculating the steady-state distribution of constructed 3D Markov chain. Note that the implementation of randomized source selection schemes in practice encounters certain difficulties, and sometimes becomes impossible. The following single-server Markovian QIS model with lost sales scheme and a deterministic rule for source choosing is explored in [22]: a fixed quantity Q_i is ordered to supplier i when on-hand inventory level reaches r_i where $\max_i(r_i) \leq \min_i(Q_i)$. It is assumed that when a supplier is about to deliver an order, the existing outstanding order to the other supplier, if any, is automatically cancelled. By using the results of [25] authors prove that the stationary distribution of queue length is independent of the inventory level, and is identical to the stationary distribution of the queue length in a classical $M/M/1$ -distribution of, i.e., the joint stationary distribution of queue length and inventory level has multiplicative form. Note that in order to prove this fact, the key assumption is the use of the lost sales scheme. It is important to note that in the works [21,22], as well as in this paper, it is assumed that an arriving c -customer can queue up even if the warehouse is empty, i.e., no lost sale scheme is used.

Classification of c -customers in QIS can be done by using their various property. So, models of QIS with high and low priorities of c -customers and with different RPs were considered in [26–34].

In the indicated above papers the following admission scheme for different types of c -customers is applied: if inventory level is exceeding some critical level then c -customers of both types are serviced, but if it is lower than the critical one the service is given only to high priority c -customers. Another admission scheme was proposed in [35]: high-priority c -customers are accepted if there is at least one free space in the buffer, and low priority c -customers are only accepted when the total queue length is less than the given threshold value. Models of QISs in which c -customers are required various size of inventory was examined in [36].

Note that the classification of c -customers can be carried out not at the instant of their arrival, but after the completion of the service process. In other words, in some QIS part of c -customers may refuse to purchase the item after being served due to some reasons, i.e., they require only service and do not require items. Such a multi-class QIS model was first studied in [37] and since then a series of papers have taken this phenomenon into account; see, e.g., the recent paper [38] and its reference list. In this paper, we also take into account this phenomenon.

We refer the readers to [14,39–44] for a discussion of more complex QIS models with Markovian Arrival Process (MAP) and phase-type distribution of service and/or lead times.

In all available papers devoted to double-source QIS, the models of systems with infinite queue for c -customers are considered and lost sale scheme is used. In this paper, we propose novel RPs in double-source QIS with finite room for c -customers and destructible stocks items to improve its performance measures by switching the order from

slow source to fast one depending on the current level of inventory. Moreover, here unlike the known papers, no lost sale scheme for c -customers is used.

The main contributions of this paper can be summarized as follows:

- A novel double-source QIS with finite room for waiting an impatient c -customers and destructible stocks under no lost sale scheme is formulated as a two-dimensional Markov process;
- The novel replenishment policies in double-source QIS based on classical (s, S) and (s, Q) policies are proposed. An additional level-dependent parameter in these policies is defined that determine switching of order from slow to fast source;
- An approximate method to derive closed-form expressions for the system's steady-state probabilities and for its performance measures is developed;
- By numerical experiments it is shown that under certain realistic assumptions related to the ratio of some system's load parameters, the developed approximate method has high accuracy;
- Assuming linear costs for each waiting and losing customer and each stored item, results of a cost analysis are provided, demonstrating how the optimal values of the replenishment policies is affected by the load parameters of the system.

The paper has the following structure. In Section 2 we describe the models by spelling out the assumptions of the underlying variables and proposed replenishment policies. Exact and approximate approaches to steady-state analysis of the models including the list of main performance measures are given in Section 3. Results of illustrative numerical examples along with optimization problems are demonstrated in Section 4. Concluding remarks are given in Section 5.

2. The Models

Below we accept the following assumptions:

- Single-server QIS contains the store of maximal capacity $S, S < \infty$, and c -customers forms the Poisson flow with rate λ ;
- An incoming c -customer is accepted for servicing immediately if the server is idle at that moment and the inventory level is positive; if the inventory level is positive and the server is busy, then this c -customer is placed in a queue of finite length. The maximum buffer size for waiting of c -customers is $N - 1, N < \infty$; if upon arrival of the c -customer buffer is full, then this c -customer is lost with probability (w.p.) one;
- An incoming c -customer can join the queue even when the inventory level is zero, i.e., if upon arrival of the c -customer an inventory level is zero, but there is free space in the buffer, then it either joins the queue w.p. φ_1 , or leaves the system w.p. φ_2 , where $\varphi_1 + \varphi_2 = 1$;
- After the service is completed, each c -customer either does not receive items w.p. σ_1 , or receives items w.p. σ_2 , where $\sigma_1 + \sigma_2 = 1$. In both cases, the service time of c -customers has exponential distribution function (d.f.) with different mean values, i.e., if the c -customer does not receive the items, then the average time of its service is equal to μ_1^{-1} ; otherwise, this time is equal to μ_2^{-1} ;
- The c -customer at the head of the queue becomes impatient if the inventory level drops to zero, i.e., in such cases, the c -customer at the head of the queue waits for a random time, which has an exponential d.f. with an average τ^{-1} , and after this time it leaves the system without items;
- Along with c -customers, the system receives a Poisson flow of destructive customers (d -customers) with rate κ , i.e., upon arrival of d -customers, the inventory level is instantly reduced by one. It is assumed that the d -customer may even destroy the item that is in the release status to the c -customer; if the inventory level is equal to zero, then the arrived d -customer does not affect the operation of the system;
- Replenishment can be done either from the SS or from the FS. The lead time from each source has an exponential d.f., i.e., if an order is made to SS, then the average

waiting time for the supply of stock is v_1^{-1} , and if an order is made to FS, then the average waiting time for the supply of stock is v_2^{-1} , where $v_2 > v_1$;

- The system can use either (s, S) or (s, Q) replenishment policies. In both policies, it is assumed that if the stock level drops to the order point $s, 0 < s < (S/2)$, then an order is made to SS, and when the stock level drops to the threshold value $r, 0 \leq r < s$, then the order is instantly canceled from SS and the order is sent to FS. The time required for a cancellation issue is negligible, i.e., this time is zero. It is also assumed that canceling an order from an SS results in certain penalties.

The problem is to find the joint distribution of the number of c -customers in the system and the inventory level of the system and to determine the main performance measures: average inventory level, average supply from each source, average reorder rate to each source, average destruction rate of inventory due to the receipt of d -customers, average length of queue of c -customers and loss probability of c -customers. In addition, the problems of optimizing the system by choosing the optimal values of the reorder point s and the threshold value r are of interest.

3. The Steady State Analysis of the Models

First consider the model QIS under (s, S) policy. Mathematical model of this QIS is two dimensional Markov chain (2D MC) with states (n, m) , where n is the total number of c -customers in the system (in buffer or being serviced), $n = 0, 1, \dots, N$, and m is on-hand inventory level, $m = 0, 1, \dots, S$. State space of this 2D MC is given as $E = \{0, 1, \dots, N\} \times \{0, 1, \dots, S\}$.

Transition rates from state $(n_1, m_1) \in E$ to state $(n_2, m_2) \in E$ are denoted by $q((n_1, m_1); (n_2, m_2))$. So, the investigated 2D MC has an infinitesimal generator $Q_1 = (q_1(n_1, m_1); (n_2, m_2)): (n_1, m_1), (n_2, m_2) \in E$ with the following transition rates for $(n_1, m_1) \in E$:

$$q_1((n_1, m_1); (n_1 + 1, m_1)) = \lambda \varphi_1 I(n_1 < N) \cdot I(m_1 = 0); \quad (1)$$

$$q_1((n_1, m_1); (n_1 + 1, m_1)) = \lambda I(n_1 < N) \cdot I(m_1 > 0); \quad (2)$$

$$q_1((n_1, m_1); (n_1 - 1, m_1)) = \mu_1 \sigma_1 \cdot I(m_1 > 0) \quad (3)$$

$$q_1((n_1, m_1); (n_1 - 1, m_1 - 1)) = \mu_2 \sigma_2 \cdot I(m_1 > 0) \quad (4)$$

$$q_1((n_1, m_1); (n_1, m_1 - 1)) = \kappa \cdot I(m_1 > 0) \quad (5)$$

$$q_1((n_1, m_1); (n_1 - 1, m_1)) = \tau \cdot I(n_1 > 0) \cdot I(m_1 = 0) \quad (6)$$

$$q_1((n_1, m_1); (n_1, S)) = v_1 \cdot I(r < m_1 \leq s) \quad (7)$$

$$q_1((n_1, m_1); (n_1, S)) = v_2 \cdot I(0 \leq m_1 \leq r). \quad (8)$$

Hereinafter, $I(A)$ is the indicator function of the event A , which is 1 if A is true and 0 otherwise.

Based on relations (1)–(8) we conclude that the constructed 2D MC is a reducible chain, i.e., for any positive values of the initial parameters, stationary mode is existing. Let $p(n, m)$ represent the probability of the state $(n, m) \in E$. The system of global balance equations (SGBE) is expressed as follows:

$$p(n_1, m_1) \sum_{(n_2, m_2) \in E_{(n_1, m_1)}^+} q((n_1, m_1), (n_2, m_2)) = \sum_{(n_2, m_2) \in E_{(n_1, m_1)}^-} q((n_2, m_2), (n_1, m_1)) p(n_2, m_2), (n_1, m_1) \in E \quad (9)$$

In Equation (9) the following notations are used: $E_{(n_1, m_1)}^+$ is the subset of states of E , that can be reached from state (n_1, m_1) in one step; $E_{(n_1, m_1)}^-$ is the subset of states of E , from which it is possible to go to the state (n_1, m_1) in one step.

Normalizing condition should be added to SGBE (9):

$$\sum_{(n, m) \in E} p(n, m) = 1. \quad (10)$$

Note that SGBE (9), (10) represent the system of linear algebraic equations with dimension $(N + 1)(S + 1)$. For moderate values of the parameters N and S , constructed SGBE (9), (10) can be solved by using existing software. However, certain computational difficulties arise in cases where the dimension of the SGBE is large. To overcome these difficulties, an approximate method for calculating the steady-state probabilities is proposed below.

The method developed below can be correctly applied to models of QIS in which the following assumption is fulfilled: the rate of c -customers exceeds both the rate of d -customers and the replenishment rate, $\lambda > \max(\kappa, \nu)$. Note that this assumption corresponds to the working mode of real QISs. In addition, as a rule, in real QISs, the average service time of c -customers that receive inventory is much longer than the average service time of c -customers that receive only service, i.e., $\mu_1 > \mu_2$.

Under these assumptions, consider the following splitting of the initial state space E of the studied 2D MC:

$$E = \bigcup_{m=0}^S E_m, E_{m_1} \cap E_{m_2} = \emptyset, m_1 \neq m_2, \quad (11)$$

where $E_m = \{(n, m) \in E : n = 0, 1, \dots, N\}, m = 0, 1, \dots, S$.

Note 1. From relations (1)–(8) we conclude that when indicated above assumptions are fulfilled, the transitions intensities between the states within classes $E_m, m = 0, 1, \dots, S$, is greater than the transitions intensities between the states from different classes. Therefore, the application of the space merging method for the examined 2D MC is correct, see Appendix of the book [45].

Next, each subset of states E_m is combined into one merged state $\langle m \rangle$, and the following merging function is determined in the initial state space $E: U(n, m) = \langle m \rangle, (n, m) \in E$. The merged states form the set $\hat{E} = \{\langle m \rangle : m = 0, 1, \dots, S\}$.

Below we show that when using the (s, S) -policy, the approximate values of the steady-state probabilities are calculated as

$$p(n, m) \approx \rho_m(n) \pi_1(\langle m \rangle), \quad (12)$$

where

$$\rho_m(n) = \begin{cases} \theta^n \frac{1 - \theta}{1 - \theta^{N+1}}, & m > 0, n = 0, 1, \dots, N, \\ \psi^n \frac{1 - \psi}{1 - \psi^{N+1}}, & m = 0, n = 0, 1, \dots, N, \end{cases} \quad (13)$$

$$\theta = \lambda / \mu_1 \sigma_1, \psi = \lambda \varphi_1 / \tau;$$

$$\pi_1(\langle m \rangle) = \begin{cases} \alpha_m \pi_1(\langle 0 \rangle), & 1 \leq m \leq r, \\ \beta_m \pi_1(\langle 0 \rangle), & r + 1 \leq m \leq s, \\ \beta_s \pi_1(\langle 0 \rangle), & s + 1 \leq m \leq S, \end{cases}$$

$$\pi_1(\langle 0 \rangle) = \left(1 + \sum_{m=1}^r \alpha_m + \sum_{m=r+1}^s \beta_m + (S - s) \beta_s \right)^{-1};$$

$$\alpha_m = a_2(1 + a_2)^{m-1}, \beta_m = a_2(1 + a_2)^r(1 + a_1)^{m-r-1}; a_k = \frac{v_k}{\mu_2\sigma_2(1-\rho(0))+\kappa}, k = 1, 2; \quad (14)$$

Note 2. Since the values of state probabilities $\rho_m(n)$ for cases $m > 0$ do not depend on parameter m (see first row of the formula (13)), in the rest of the paper in their notations subscript m for cases $m > 0$ is omitted for simplification. Additionally, note that in cases $\theta = 1$ and/or $\psi = 1$, all state probabilities $\rho_m(n) = 1/(N + 1)$ for any $n, n = 0, 1, \dots, N$ and $m, m = 0, 1, \dots, S$.

Indeed, when using splitting (11) from relations (1)–(8), we conclude that the stationary distribution of all split models with state spaces $E_m, m = 1, \dots, S$, coincide with the stationary distribution of a single-server Markov queueing system M/M/1/N with a load $\theta = \lambda/\mu_1\sigma_1$. Stationary probability of state $(n, m) \in E_m$ within these split models are denoted by $\rho_m(n)$. These quantities do not depend on $m = 1, \dots, S$, and they are calculated as

$$\rho(n) = \theta^n \frac{1 - \theta}{1 - \theta^{N+1}}, n = 0, 1, \dots, N. \quad (15)$$

In addition, from relations (1)–(8) we conclude that the stationary distribution of a split model with state space E_0 coincide with the stationary distribution of a single-server Markov queueing system M/M/1/N with a load $\psi = \lambda\varphi_1/\tau$. In other words, stationary probabilities of states $(n, 0) \in E_0$ within this split model are calculated as

$$\rho_0(n) = \psi^n \frac{1 - \psi}{1 - \psi^{N+1}}, n = 0, 1, \dots, N. \quad (16)$$

Combining Formulas (15) and (16), we obtain Formula (13).

Merged model represents 1D MC with state space $\hat{E}$. Transition rate from merged state $\langle m_1 \rangle \in \hat{E}$ to other merged state $\langle m_2 \rangle \in \hat{E}$ is denoted by $q(\langle m_1 \rangle, \langle m_2 \rangle)$. These quantities are calculated as

$$q(\langle m_1 \rangle, \langle m_2 \rangle) = \sum_{(n_2, m_2) \in E_{m_2}} q((n_1, m_1), (n_2, m_2)) \rho_{m_1}(n_1). \quad (17)$$

By taking into account relations (1)–(8), (15)–(17), after some algebra we obtain:

$$q(\langle m_1 \rangle, \langle m_2 \rangle) = \begin{cases} v_2, & 0 \leq m_1 \leq r, m_2 = S, \\ v_1, & r + 1 \leq m_1 \leq s, m_2 = S, \\ \mu_2\sigma_2(1 - \rho(0)) + \kappa, & m_1 > 0, m_2 = m_1 - 1. \end{cases} \quad (18)$$

From (18) after some algebra we obtain that the stationary probabilities of merged states, $\pi_1(\langle m \rangle), \langle m \rangle \in \hat{E}$, are calculated by (14).

Finally, according to the space merging algorithm, we obtain that the steady-state probabilities of the initial 2D MC are calculated from Formula (12).

Now consider the model under (s, Q) -policy. For this model state space is also defined by the set E . Here, the infinitesimal generator

$$Q_2 = (q_2(n_1, m_1); (n_2, m_2)): (n_1, m_1), (n_2, m_2) \in E$$

of the appropriate 2D MC are determined via (1)–(6), but here Formulas (7) and (8) are replaced by the following formulas:

$$q_1((n_1, m_1); (n_1, m_1 + S - s)) = v_1 \cdot I(r < m_1 \leq s)$$

$$q_1((n_1, m_1); (n_1, m_1 + S - s)) = v_2 \cdot I(0 \leq m_1 \leq r)$$

Similarly, we can prove that when using the (s, Q) -policy, the approximate values of the steady-state probabilities are calculated as

$$p(n, m) \approx \rho_m(n)\pi_2(\langle m \rangle), \quad (19)$$

where

$$\pi_2(< m >) = \begin{cases} \alpha_m \pi_2(< 0 >), & 1 \leq m \leq r, \\ \beta_m \pi_2(< 0 >), & r + 1 \leq m \leq s, \\ \beta_s \pi_2(< 0 >), & s + 1 \leq m \leq S - s, \\ \eta_m \pi_2(< 0 >), & S - s + 1 \leq m \leq S, \end{cases} \quad (20)$$

$$\pi_2(< 0 >) = \left(1 + \sum_{m=1}^r \alpha_m + \sum_{m=r+1}^s \beta_m + (S - 2s) \beta_s + \sum_{m=S-s+1}^S \eta_m \right)^{-1},$$

$$\eta_m = \frac{1}{\mu_2 \sigma_2 (1 - \rho(0)) + \kappa} \sum_{i=m-S+s}^s \chi_i x_i, \quad S - s + 1 \leq m \leq S;$$

$$\chi_i = \begin{cases} \alpha_i, & 1 \leq i \leq r, \\ \beta_i, & r + 1 \leq i \leq s; \end{cases} \quad x_i = \begin{cases} v_2, & 1 \leq i \leq r, \\ v_1, & r + 1 \leq i \leq s. \end{cases}$$

4. Performance Measures

Desired performance measures are calculated via steady-state probabilities as follows.

Average inventory level (S_{av}) is

$$S_{av} = \sum_{m=1}^S m \sum_{n=0}^N p(n, m). \quad (21)$$

Average supply from source- i when using (s, S) -policy ($V_{av}(i)$) are

$$V_{av}(1) = \sum_{m=r+1}^S (S - m) \sum_{n=0}^N p(n, m); \quad V_{av}(2) = \sum_{m=0}^r (S - m) \sum_{n=0}^N p(n, m); \quad (22)$$

Average supply from source- i when using (s, Q) -policy are

$$V_{av}(1) = (S - s) \sum_{m=r+1}^S \sum_{n=0}^N p(n, m); \quad V_{av}(2) = (S - s) \sum_{m=0}^r \sum_{n=0}^N p(n, m); \quad (23)$$

Average number of c -customers in system (L_{av}) is

$$L_{av} = \sum_{n=1}^N n \sum_{m=0}^S p(n, m); \quad (24)$$

Average destruction rate of the stock (DRS) is

$$DRS = \kappa \left(1 - \sum_{n=0}^N p(n, 0) \right); \quad (25)$$

Average reorder rate of regular supply (RR_1) is

$$RR_1 = \kappa p(0, s + 1) + (\mu_2 \sigma_2 + \kappa) \sum_{n=1}^N p(n, s + 1); \quad (26)$$

Average reorder rate of emergency supply (RR_2) is

$$RR_2 = \kappa p(0, r + 1) + (\mu_2 \sigma_2 + \kappa) \sum_{n=1}^N p(n, r + 1). \quad (27)$$

Note 3. The average rate of emergency orders is equal to the average rate of cancellation (RC) of regular orders.

Loss probability of c -customers (PL) is

$$PL = \varphi_2 \sum_{n=0}^N p(n, 0) + \sum_{m=0}^S p(N, m) + \frac{\tau}{\tau + \lambda\varphi_1 + \nu_2} \sum_{n=1}^{N-1} p(n, 0) + \frac{\tau}{\tau + \nu_2} p(N, 0). \quad (28)$$

In (28), the first and second terms of the sum estimate the loss probabilities of c -customers when they enter the system, provided that at this moment the inventory level is zero and there is no free space in the buffer, respectively, the third and fourth terms determine the loss probabilities c -customers due to their impatience in the queue, provided that before the start of their service, the inventory level drops to zero.

Taking into account relations (12)–(14), after standard transformations, it can be shown that when using the (s, S) -policy, the approximate values of the performance measures are calculated as

$$S_{av} \approx \sum_{m=1}^S m\pi_1(< m >) ; \quad (29)$$

$$V_{av}(1) \approx \sum_{m=r+1}^S (S - m) \pi_1(< m >) ; V_{av}(2) \approx \sum_{m=0}^r (S - m) \pi_1(< m >) ; \quad (30)$$

$$L_{av} \approx \sum_{n=1}^N n \left((1 - \pi_1(< 0 >)) \rho(n) + \pi_1(< 0 >) \rho_0(n) \right) ; \quad (31)$$

$$DRS \approx \kappa(1 - \pi_1(< 0 >)) ; \quad (32)$$

$$RR_1 \approx \pi_1(< s + 1 >) \left(\kappa\rho(0) + (\mu_2\sigma_2 + \kappa)(1 - \rho(0)) \right) ; \quad (33)$$

$$RR_2 \approx \pi_1(< r + 1 >) \left(\kappa\rho(0) + (\mu_2\sigma_2 + \kappa)(1 - \rho(0)) \right) ; \quad (34)$$

$$PL \approx \varphi_2\pi(0)(1 - \rho_0(N)) + \frac{\tau}{\tau + \lambda\varphi_1 + \nu_2} \pi(0)(1 - \rho_0(0) - \rho_0(N)) + \frac{\tau}{\tau + \nu_2} \rho_0(N)\pi(0)(1 - \pi(0))\rho(N) + \pi(0)\rho_0(N). \quad (35)$$

In a similar way, taking into account relations (19) and (20), we can show that when using the (s, Q) -policy, the approximate values of the performance measures are calculated via (29), (31)–(35), where the quantities $\pi_1(m)$ are replaced by the $\pi_2(m)$, and an average supply from various sources are calculated as

$$V_{av}(1) = (S - s) \sum_{m=r+1}^S \pi_2(< m >) ; \quad V_{av}(2) = (S - s) \sum_{m=0}^r \pi_2(< m >) ; \quad (36)$$

Below, we consider the results of numerical experiments carried out using the obtained formulas.

5. Numerical Results

This section is threefold. Firstly, we study the accuracy of the developed approximate formulas to calculate the steady-state probabilities, secondly, we investigate the behavior of performance measures versus initial parameters of the model and thirdly, we consider the optimization problem.

Consider the first goal of the numerical experiments. Analytical investigation of the accuracy of the developed approximate formulas is impossible. Therefore, we consider solution of the problem via numerical experiments. Exact values of the steady-state probabilities are calculated via SGBE (9), (10). Some results of numerical experiments are

demonstrated in Table 1. From this table, we conclude that under both RPs the accuracy of the developed formulas to calculation of the steady-state probabilities are sufficiently high for practice. Results of numerical experiments show that the accuracy of the developed approximate formulas systematically increases with the increases in the dimension of the system, i.e., when N and S take on large values. The last fact is very important, since the approximate approach was developed specifically for large scale models.

Table 1. Dependence of the absolute error of the steady-state probabilities on parameter s ; $\lambda = 8$, $\kappa = 2$, $\mu_1 = 45$, $\mu_2 = 15$, $\tau = 1.5$, $v_1 = 2$, $v_2 = 8$, $r = 5$, $S = 22$, $N = 100$.

s	Max of Error	
	(s,S)-Policy	(s,Q)-Policy
6	1.06×10^{-2}	1.31×10^{-2}
7	1.15×10^{-2}	1.66×10^{-2}
8	1.26×10^{-2}	1.98×10^{-2}
9	1.38×10^{-2}	2.26×10^{-2}
10	1.37×10^{-2}	2.54×10^{-2}

Now, consider the second goal of the numerical experiments. The proposed approach makes it possible to carry out numerical experiments to study the behavior of performance measures (21)–(28) versus any initial system's parameter. Below, in Tables 2–10, we present the results of appropriate calculations where in each column the top row corresponds to the (s, S) -policy, and the lower one, to the (s, Q) -policy. Due to the limited size of the paper, conclusions regarding the behavior of performance measures are left to the reader.

In all numerical experiments (in Tables 2–9), it is assumed that $\sigma_1 = 0.4$, $\varphi_1 = 0.6$ and $S = 30$, $N = 100$. The values of other parameters are indicated in the title of the tables.

Table 2. Dependence of the performance measures on parameter λ ; $\mu_1 = 45$, $\mu_2 = 15$, $\kappa = 2$, $\tau = 1.5$, $v_1 = 2$, $v_2 = 8$, $r = 5$, $s = 10$.

λ	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	DRS	RR_1	RR_2	PL
8	10,968	0.262	18,770	0.815	2.000	0.258	8.1520×10^{-2}	8.1520×10^{-2}
	3.291	0.079	5.633	0.804	2.000	0.077	2.4463×10^{-2}	4.2171×10^{-5}
9	10,993	0.306	18,701	1.021	2.000	0.278	9.4923×10^{-2}	9.4923×10^{-2}
	3.399	0.095	5.783	1.007	2.000	0.086	2.9353×10^{-2}	6.4781×10^{-5}
10	11,013	0.352	18,635	1.280	1.999	0.297	1.0882×10^{-1}	1.0882×10^{-1}
	3.501	0.112	5.923	1.259	2.000	0.095	3.4592×10^{-2}	9.4994×10^{-5}
11	11,030	0.399	18,572	1.611	1.999	0.317	1.2315×10^{-1}	1.2315×10^{-1}
	3.596	0.130	6.056	1.584	2.000	0.103	4.0156×10^{-2}	1.3405×10^{-4}
12	11,042	0.447	18,510	2.053	1.999	0.337	1.3785×10^{-1}	1.3785×10^{-1}
	3.687	0.149	6.181	2.018	2.000	0.112	4.6028×10^{-2}	1.8316×10^{-4}

Table 3. Dependence of the performance measures on parameter κ ; $\lambda = 8$, $\mu_1 = 45$, $\mu_2 = 15$, $\tau = 1.5$, $v_1 = 2$, $v_2 = 8$, $r = 5$, $s = 10$.

κ	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	DRS	RR_1	RR_2	PL
2	10,968	0.262	18,770	0.815	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.633	0.804	2.000	0.077	2.4463×10^{-2}	4.2171×10^{-5}
3	11,013	0.352	18,635	0.830	2.999	0.297	1.0882×10^{-1}	2.8717×10^{-4}
	3.501	0.112	5.923	0.809	3.000	0.095	3.4592×10^{-2}	9.1282×10^{-5}
4	11,042	0.447	18,510	0.853	3.998	0.337	1.3785×10^{-1}	5.1361×10^{-4}
	3.687	0.149	6.181	0.818	3.999	0.112	4.6028×10^{-2}	1.7150×10^{-4}
5	11,058	0.548	18,394	0.886	4.996	0.375	1.6813×10^{-1}	8.3369×10^{-4}

	3.856	0.191	6.413	0.830	4.998	0.131	5.8624×10^{-2}	2.9069×10^{-4}
6	11,064	0.652	18,283	0.930	5.992	0.413	1.9935×10^{-1}	1.2579×10^{-3}
	4.011	0.237	6.628	0.847	5.997	0.150	7.2263×10^{-2}	4.5597×10^{-4}

Table 4. Dependence of the performance measures on parameter μ_1 ; $\lambda = 8$, $\kappa = 2$, $\mu_2 = 15$, $\tau = 1.5$, $v_1 = 2$, $v_2 = 8$, $r = 5$, $s = 10$.

μ_1	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	DRS	RR_1	RR_2	PL
45	10,968	0.262	18,770	0.815	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.633	0.804	2.000	0.077	2.4463×10^{-2}	4.2171×10^{-5}
50	10,944	0.229	18,827	0.677	2.000	0.242	7.1212×10^{-2}	1.0050×10^{-4}
	3.199	0.067	5.504	0.670	2.000	0.071	2.0819×10^{-2}	2.9382×10^{-5}
55	10,922	0.202	18,876	0.579	2.000	0.228	6.3090×10^{-2}	7.4425×10^{-5}
	3.120	0.058	5.392	0.574	2.000	0.065	1.8023×10^{-2}	2.1261×10^{-5}
60	10,901	0.181	18,918	0.506	2.000	0.217	5.6559×10^{-2}	5.6757×10^{-5}
	3.051	0.051	5.294	0.502	2.000	0.061	1.5828×10^{-2}	1.5883×10^{-5}
65	10,882	0.164	18,954	0.449	2.000	0.208	5.1217×10^{-2}	4.4383×10^{-5}
	2.989	0.045	5.207	0.446	2.000	0.057	1.4070×10^{-2}	1.2192×10^{-5}

Table 5. Dependence of the performance measures on parameter μ_2 ; $\lambda = 8$, $\kappa = 2$, $\mu_1 = 45$, $\tau = 1.5$, $v_1 = 2$, $v_2 = 8$, $r = 5$, $s = 10$.

μ_2	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	DRS	RR_1	RR_2	PL
15	10,968	0.262	18,770	0.815	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.633	0.804	2.000	0.077	2.4463×10^{-2}	4.2171×10^{-5}
20	11,025	0.383	18,592	0.837	1.999	0.310	1.1833×10^{-1}	3.5303×10^{-4}
	3.565	0.124	6.012	0.812	2.000	0.100	3.8266×10^{-2}	1.1416×10^{-4}
25	11,054	0.514	18,432	0.874	1.998	0.362	1.5792×10^{-1}	7.1588×10^{-4}
	3.801	0.177	6.338	0.825	1.999	0.125	5.4305×10^{-2}	2.4617×10^{-4}
30	11,064	0.652	18,283	0.930	1.997	0.413	1.9935×10^{-1}	1.2579×10^{-3}
	4.011	0.237	6.628	0.847	1.999	0.150	7.2263×10^{-2}	4.5597×10^{-4}
35	11,060	0.796	18,144	1.007	1.996	0.464	2.4198×10^{-1}	1.9972×10^{-3}
	4.201	0.302	6.891	0.879	1.998	0.176	9.1906×10^{-2}	7.5854×10^{-4}

Table 6. Dependence of the performance measures on parameter τ ; $\lambda = 8$, $\kappa = 2$, $\mu_1 = 45$, $\mu_2 = 15$, $v_1 = 2$, $v_2 = 8$, $r = 5$, $s = 10$.

τ	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	DRS	RR_1	RR_2	PL
1.5	10,968	0.262	18,770	0.815	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.633	0.804	2.000	0.077	2.4463×10^{-2}	4.2171×10^{-5}
3	10,968	0.262	18,770	0.814	2.000	0.258	8.1520×10^{-2}	1.2463×10^{-4}
	3.291	0.079	5.633	0.804	2.000	0.077	2.4463×10^{-2}	4.1657×10^{-5}
4.5	10,968	0.262	18,770	0.812	2.000	0.258	8.1520×10^{-2}	1.0370×10^{-4}
	3.291	0.079	5.633	0.804	2.000	0.077	2.4463×10^{-2}	4.1100×10^{-5}
6	10,968	0.262	18,770	0.800	2.000	0.258	8.1520×10^{-2}	9.6548×10^{-5}
	y	0.079	5.633	0.804	2.000	0.077	2.4463×10^{-2}	4.0504×10^{-5}
7.5	10,968	0.262	18,770	0.800	2.000	0.258	8.1520×10^{-2}	9.3768×10^{-5}
	3.291	0.079	5.633	0.804	2.000	0.077	2.4463×10^{-2}	3.9870×10^{-5}

Table 7. Dependence of the performance measures on parameter v_1 ; $\lambda = 8$, $\kappa = 2$, $\mu_1 = 45$, $\mu_2 = 15$, $\tau = 1.5$, $v_2 = 8$, $r = 5$, $s = 10$.

v_1	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	DRS	RR_1	RR_2	PL
2	10,968	0.262	18,770	0.815	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	0.079	0.804	2.000	0.077	2.4463×10^{-2}	4.2171×10^{-5}
3	10,782	0.168	19,050	0.809	2.000	0.264	5.2089×10^{-2}	8.9793×10^{-5}
	2.774	0.043	0.043	0.802	2.000	0.068	1.3401×10^{-2}	2.3102×10^{-5}
4	10,650	0.112	19,238	0.806	2.000	0.268	3.4710×10^{-2}	5.9835×10^{-5}
	2.419	0.025	0.025	0.801	2.000	0.061	7.8834×10^{-3}	1.3590×10^{-5}
5	10,553	0.077	19,370	0.804	2.000	0.271	2.3967×10^{-2}	4.1315×10^{-5}
	2.156	0.016	0.016	0.801	2.000	0.055	4.8959×10^{-3}	8.4397×10^{-6}
6	10,479	0.055	19,466	0.803	2.000	0.273	1.7058×10^{-2}	2.9405×10^{-5}
	1.950	0.010	0.010	0.801	2.000	0.051	3.1749×10^{-3}	5.4731×10^{-6}

Table 8. Dependence of the performance measures on parameter v_2 ; $\lambda = 8$, $\kappa = 2$, $\mu_1 = 45$, $\mu_2 = 15$, $\tau = 1.5$, $v_1 = 2$, $r = 5$, $s = 10$.

v_2	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	DRS	RR_1	RR_2	PL
8	10,968	0.262	18,770	0.815	2.000	2.5764×10^{-1}	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.633	0.804	2.000	7.7317×10^{-2}	2.4463×10^{-2}	4.2171×10^{-5}
9	10,980	0.233	18,787	0.809	2.000	2.5793×10^{-1}	8.1612×10^{-2}	8.7411×10^{-5}
	3.285	0.070	5.620	0.803	2.000	7.7163×10^{-2}	2.4415×10^{-2}	2.6150×10^{-5}
10	10,990	0.209	18,801	0.806	2.000	2.5817×10^{-1}	8.1686×10^{-2}	5.6393×10^{-5}
	3.280	0.062	5.610	0.802	2.000	7.7041×10^{-2}	2.4376×10^{-2}	1.6828×10^{-5}
11	10,998	0.190	18,812	0.804	2.000	2.5836×10^{-1}	8.1747×10^{-2}	3.7529×10^{-5}
	3.275	0.057	5.602	0.801	2.000	7.6942×10^{-2}	2.4345×10^{-2}	1.1176×10^{-5}
12	11,005	0.174	18,821	0.803	2.000	2.5852×10^{-1}	8.1798×10^{-2}	2.5653×10^{-5}
	3.272	0.052	5.596	0.801	2.000	7.6860×10^{-2}	2.4319×10^{-2}	7.6268×10^{-6}

Table 9. Dependence of the performance measures on parameter s ; $\lambda = 8$, $\kappa = 2$, $\mu_1 = 45$, $\mu_2 = 15$, $\tau = 1.5$, $v_1 = 2$, $v_2 = 8$, $r = 5$.

s	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	DRS	RR_1	RR_2	PL
6	7.662	0.750	13,588	0.860	1.999	0.338	0.338	5.8271×10^{-4}
	7.760	0.739	13,502	0.859	1.999	0.333	0.333	5.7427×10^{-4}
7	7.625	0.576	13,799	0.846	1.999	0.347	0.260	4.4808×10^{-4}
	7.887	0.575	13,538	0.846	1.999	0.346	0.259	4.4733×10^{-4}
8	7.485	0.447	14,068	0.836	1.999	0.359	0.202	3.4767×10^{-4}
	7.868	0.449	13,683	0.836	1.999	0.360	0.202	3.4895×10^{-4}
9	7.270	0.350	14,380	0.828	1.999	0.374	0.158	2.7185×10^{-4}
	7.753	0.351	13,896	0.828	1.999	0.375	0.158	2.7264×10^{-4}
10	7.000	0.275	14,725	0.822	2.000	0.392	0.124	2.1406×10^{-4}
	7.582	0.274	14,144	0.822	2.000	0.391	0.124	2.1338×10^{-4}

Table 10. Dependence of the performance measures on parameter N ; $\lambda = 8$, $\kappa = 2$, $\mu_1 = 45$, $\mu_2 = 15$, $\tau = 1.5$, $v_1 = 2$, $v_2 = 8$, $r = 5$, $s = 10$.

N	$V_{av}(1)$	$V_{av}(2)$	S_{av}	L_{av}	DRS	RR_1	RR_2	PL
50	10,968	0.262	18,770	0.807	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.632	0.799	2.000	2.000	2.4457×10^{-2}	2.0923×10^{-4}
70	10,968	0.262	18,770	0.810	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.633	0.802	2.000	2.000	2.4463×10^{-2}	4.2171×10^{-5}

90	10,968	0.262	18,770	0.813	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.633	0.803	2.000	2.000	2.4463×10^{-2}	4.2171×10^{-5}
110	10,968	0.262	18,770	0.816	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.633	0.804	2.000	2.000	2.4463×10^{-2}	4.2171×10^{-5}
130	10,968	0.262	18,770	0.819	2.000	0.258	8.1520×10^{-2}	1.4053×10^{-4}
	3.291	0.079	5.633	0.806	2.000	2.000	2.4463×10^{-2}	4.2171×10^{-5}

Finally, consider the third aim of the numerical experiments, i.e., minimization of the expected total cost (TC), which is calculated as follows:

$$TC(s, r) = \sum_{i=1}^2 (K_i + c_r(i)V_{av}(i))RR_i + c_cRR_2 + c_hS_{av} + c_dDRS + c_l\lambda PL + c_wL_{av}, \quad (37)$$

where K_i are the fixed costs of one order from Source- i ; $c_r(i)$ are the procurement costs per unit inventory from Source- i , $i = 1, 2$; c_c is the cost due to order cancellation from Source-1; c_h is the holding cost of an inventory unit per unit time; c_d is the cost due to damage per unit inventory; c_l is the cost of losing a c -customer; c_w is the waiting cost of a c -customer per unit time.

Let us assume that controllable are only parameters s and r . Then, the optimization problem is formulated as follows: it is required to find such pairs of optimal values (s^*, r^*) in order to minimize the functional (37). In other words, we need to solve the following task:

$$(s^*, r^*) = \arg \min_{(s, r) \in X} TC(s, r). \quad (38)$$

Some results of solving this problem for a model when using the (s, S) and (s, Q) -policies are shown in Table 11. The optimal solution of the problem for both policies is $(s^*, r^*) = (6, 1)$.

Table 11. Results of the task (38); $\lambda = 8$, $\kappa = 2$, $\mu_1 = 45$, $\mu_2 = 15$, $\tau = 1.5$, $v_1 = 2$, $v_2 = 8$, $S = 22$, $K_1 = 100$, $K_2 = 200$, $c_r(1) = 50$, $c_r(2) = 100$, $c_c = 50$, $c_h = 35$, $c_d = 75$, $c_l = 200$, $c_w = 50$. The minimal value of TC is indicated in bold.

		Total Cost								
$s \backslash r$		10	9	8	7	6	5	4	3	2
1		896.90	885.21	875.95	869.80	867.73	871.11	881.95	903.24	939.62
		870.62	850.90	835.46	824.94	820.45	823.69	837.22	864.91	912.74
2		896.22	884.54	875.36	869.44	867.88	872.30	885.20	910.49	
		872.35	853.23	838.68	829.49	827.00	833.34	851.80	<u>887.57</u>	
3		899.59	889.07	881.50	877.89	879.71	889.26	910.23		
		879.47	862.48	850.85	845.75	849.10	863.98	<u>895.31</u>		
4		906.67	898.44	894.07	895.01	903.51	923.23			
		892.02	878.78	872.33	874.49	888.29	<u>918.65</u>			
5		917.85	913.31	914.16	922.68	942.6				
		910.98	903.58	905.28	919.10	<u>950.08</u>				
6		934.27	935.39	944.45	965.31					
		938.43	939.84	954.14	<u>986.50</u>					
7		958.00	967.89	990.1						
		977.88	992.76	<u>1026.87</u>						
8		992.67	1016.55							
		1035.16	<u>1071.21</u>							
9		1044.60								
		1120.16								

6. Conclusions

Models of double-source QIS are studied, where the delivery rates from various sources differ from each other. They use two replenishment policies, i.e., with a fixed supply volume and with a variable supply volume. In each policy, along with the order point, a threshold parameter is introduced that determines the moment when an order is canceled from a slow source and at the same time a new order is generated to a fast source. Models with a finite buffer for c -customers are studied in which c -customers are queued even if inventory level is zero and there is free space in the waiting room. However, c -customers in the queue can be impatient when the inventory level drops to zero before their service is started. After the service is completed, part of the c -customers, according to the Bernoulli trials, either leaves the system without receiving items, or receives items. Along with c -customers, d -customers also enter the system, the appearance of which leads to an instant decrease in the system's inventory level by one.

Exact and approximate methods for calculating the performance measures of the considered QIS are proposed. The exact method is based on the balance equations of the constructed 2D MC, while the approximate method uses the space merging approach for the 2D MC. When using the approximate method, explicit formulas for calculating both the steady-state probabilities and the performance measures are obtained. The conducted numerical experiments demonstrate the high accuracy of the developed approximate formulas. By the numerical experiments, the behavior of the performance measures versus initial parameters of the models was also studied. In addition, the problems of finding the optimal values of the parameters of the used RPs to minimize the total cost were solved.

Suggestions for future directions of research are:

- Development and investigation of replenishment policies in double source QIS in which switching the order from slow source to fast depends both on inventory level and number of c -customers in the system;
- Investigation of similar QIS models with more complex d.f. for c - and d -customers, service time of c -customers, lead times etc.;
- Investigation of similar QIS models with common life times of the items.

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