Introduction

This PhD dissertation consists of four chapters, which all contain new results. These results have been published in our papers [3], [2], [1] and [5], respectively. Before giving an overview of the contents of the chapters, we make some introductory notes on the subject of our thesis.

In the first three chapters we deal with the numbers of solutions of various important classes of decomposable polynomial equations. Namely, we shall be concerned with diophantine equations of the form

$$F(\mathbf{x}) = b \qquad \text{in} \quad \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}_S^m, \tag{0.1}$$

and, more generally,

$$\begin{aligned} F(\mathbf{x}) = bp_1^{z_1} \dots p_s^{z_s} \quad & \text{in} \quad \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m \\ & \text{and} \ \ z_1, \dots, z_s \in \mathbb{Z}_{\geq 0} \\ & \text{with} \ \ (x_1, \dots, x_m, p_1 \dots p_s) = 1 \text{ if } s > 0. \end{aligned} \tag{0.2}$$

and

$$F(\mathbf{x}) \in b\mathbb{Z}_S^* \qquad \text{in} \quad \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}_S^m, \tag{0.3}$$

where $S = \{p_1, \dots, p_s\}$ is a finite set of $s \geq 0$ primes, $\mathbb{Z}_S$ is the ring of S-integers in $\mathbb{Q}$, $\mathbb{Z}_S^*$ denotes the unit group of $\mathbb{Z}_S$, b is a non-zero rational integer with $(b, p_1 \dots p_s) = 1$ if $s > 0$, and $F(\mathbf{X}) = F(X_1, \dots, X_m) \in \mathbb{Z}_S[X_1, \dots, X_m]$ is a *decomposable polynomial*, that is a polynomial which factorizes into linear factors with algebraic coefficients. Equations (0.1), (0.2) and (0.3) are called *decomposable polynomial equations*, and in particular (0.2) a *decomposable polynomial equation of Mahler type*. Very important is the case when F is homogeneous, in this case (0.1), (0.2) and (0.3) are called *decomposable form equations*. The most important classes of decomposable polynomial equations are Thue equations, Thue-Mahler equations, norm form equations, discriminant form equations, index form equations, resultant form equations, and their inhomogeneous generalizations. They have many significant applications, among others in algebraic number theory. For the extremely rich literature of these equations and their applications, here we refer to the books [7], [19], [39], [40] and the survey papers [12], [16], [27], [28].

As is known, in case of decomposable form equations (0.2) and (0.3) are in fact equivalent, and the number of solutions of (0.2) coincides with the number of so-called "cosets of solutions" of (0.3) (cf. Chapters 2,3). In the inhomogeneous case, we cannot speak about "cosets of solutions", but the number of solutions of (0.2) does not exceed that of (0.3). Hence, in this case it is enough to derive bounds for the number of solutions of (0.3).

The main results of the first chapter provide explicit upper bounds for the number of solutions of general decomposable polynomial equations of the form (0.1) and

(0.3). Some consequences are also presented to certain important special types of decomposable polynomial equations. We note that all the results obtained in the first chapter are valid also in the case of homogeneous equations, but in this special situation slightly better results are known in the literature. Hence in this chapter we shall restrict ourselves mainly to inhomogeneous equations.

The second and third chapters are devoted to norm form, discriminant form and index form equations, which are particularly important from the point of view of applications. Under certain general conditions we give explicit upper bounds for the number of solutions which are much better than the bounds of the first chapter specialized to the homogeneous case, and provide significant improvements of the previous bounds which were earlier obtained on the equations under consideration.

In Chapters 1,2 and 3, we shall use several deep results from the literature to prove our theorems. We remark that some of these results have been established by means of various recent improvements of Schmidt's powerful quantitative subspace theorem; for a survey on Schmidt's theorem and its improvements see [17].

In the fourth chapter, a problem of P. Turán concerning irreducible polynomials is investigated. The problems studied in the first three chapters are clearly of diophantine nature. We note that the study of the irreducibility of polynomials with integer coefficients is also a diophantine question. Indeed, if such a polynomial is not irreducible, then it can be written as a product of two polynomials with integer coefficients. Now regarding the coefficients of the factors as variables, calculating the product of the two factors and comparing the coefficients of the product with the original polynomial we get a system of diophantine equations to be solved.

In what follows, we briefly present the contents of chapters 1,2,3 and 4, respectively.

I

Decomposable form equations have been very extensively studied in the literature. Various ineffective finiteness results have been established which do not provide, however, any algorithm for determining the solutions. Later, several effective theorems have also been obtained which give explicit upper bounds for the solutions. Further, explicit upper bounds were derived for the number of solutions which bounds are independent of the coefficients of F. For such results we refer to the books [19], [39], [40], the survey papers [12], [16], [26], [27], [28] and the references given there.

The first (ineffective) upper bound for the number of solutions of general inhomogeneous decomposable polynomial equations was given by Evertse, Gaál and Győry in [10]. They provided a necessary and sufficient condition for equation (0.1) to have only finitely many solutions for every S and b. Further, under this finiteness condition they derived an upper bound for the numbers of solutions of such equations. This bound was given explicitly in terms of the numbers of solutions of some appropriate unit equations. However, when [10] was written, no explicit upper bound was known for the number of solutions of unit equations. It should be remarked that the results of [10] were established in a more general situation, for the case when the ground ring is an arbitrary finitely generated integral domain over $\mathbb{Z}$.

Combining the upper bound in [10] with a recent bound of Evertse [8] on the number of solutions of unit equations, one can now give easily an explicit upper bound for the number of solutions of equation (0.1). However, the bound so obtained is very large in terms of the degree of F, because it is an exponential function of the degree of the splitting field of F. In the first chapter we follow a completely different approach, and under the finiteness condition of [10] we give much better upper bounds for the the number of solutions of (0.1) which are already polynomial in terms of the degree of F. Further, we show that the finiteness criteria of [10] is not valid in the case of the more general equations of the form (0.3). We also give a sufficient condition for the finiteness of the number of solutions of equation (0.3), and under this condition we derive an explicit upper bound for the number of solutions. Further, in the case $b = 1$ we give a bound for the number of solutions of (0.3) subject only to the condition that this number is finite. Several consequences of these results are also presented for special decomposable polynomial equations. We give among others upper bounds for the numbers of solutions of inhomogeneous Thue equations, discriminant polynomial equations, norm polynomial equations and resultant polynomial equations. To prove our theorems, we utilize some deep results of Győry [23] and Evertse and Győry [14] on the number of families of solutions of decomposable form equations whose proofs depend, among other things, on recent quantitative versions of Schmidt's subspace theorem. The results presented in the first chapter are joint results with my supervisor, Prof. K. Győry, and will be published in our joint paper [3].

II

In the second chapter we deal with norm form equations. A norm form is a decomposable form of the type

$$F(\mathbf{X}) := a_0 N_{K/\mathbb{Q}}(\alpha_1 X_1 + \cdots + \alpha_m X_m),$$

where $\alpha_1, \ldots, \alpha_m$ are linearly independent elements of a number field K of degree n over $\mathbb{Q}$, and a_0 is a non-zero rational number which is chosen so that F has its coefficients in $\mathbb{Z}$.

Let $p_1, \ldots, p_s$ be rational primes with $s \geq 0$, and $b \in \mathbb{Z} \setminus \{0\}$ a rational integer which is relatively prime to $p_1, \ldots, p_s$ if $s > 0$. The equations of the form

$$\begin{aligned} a_0 N_{K/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_m x_m) &= b p_1^{z_1} \ldots p_s^{z_s} \\ &\text{in } \; x_1, \ldots, x_m \in \mathbb{Z}, \\ &\text{and } \; z_1, \ldots, z_s \in \mathbb{Z}_{\geq 0} \\ &\text{with } \; (x_1, \ldots, x_m, p_1 \ldots p_s) = 1 \text{ if } s > 0, \end{aligned} \tag{0.4}$$

are called *norm form equations*, and for $s > 0$, *norm form equations of Mahler type*.

In the case $s = 0$, Schmidt [36] gave a criterion for (0.4) to have only finitely many solutions for any $b \in \mathbb{Z} \setminus \{0\}$. In [35] Schlickewei extended Schmidt's result to the case $s > 0$. These results were proved by using Schmidt's subspace theorem, and thus they are ineffective.

Later, Győry and Papp ($s = 0$) and Győry ($s \geq 0$) derived effective finiteness results for a large class of norm form equations; see [19] and [20] for references. For example, Győry [20] derived explicit bounds for the solutions under the assumption that

(i) $x_m \neq 0$, $\alpha_1 = 1, \alpha_2, \ldots, \alpha_{m-1}$ are $\mathbb{Q}$-linearly independent and α_m has degree at least 3 over $\mathbb{Q}(\alpha_1, \ldots, \alpha_{m-1})$,

or that

(ii) the degree of α_i is at least 3 over $\mathbb{Q}(\alpha_1, \ldots, \alpha_{i-1})$ for $i = 2, \ldots, m$.

We remark that in (i) and (ii) the conditions concerning the degrees are necessary. Further, it is interesting to note that this finiteness result in the case (i) cannot be deduced even in an ineffective form from the finiteness theorems of Schmidt and Schlickewei. In 1985 Evertse and Győry [11] gave explicit upper bounds for the number of solutions of norm form equations of the form (0.4) under the hypotheses (i) and (ii). Later Győry [23], Evertse [8] and Evertse and Győry [14] derived general upper bounds for arbitrary norm form equations which include the case (ii), but not the case (i). For further results concerning norm form equations, see [39], [27] and the references given there.

In the second chapter we considerably improve the above-mentioned bounds of [11]. Following an idea of Győry, we reduce equation (0.4) to a relative Thue-Mahler equation, and then we use a result of Evertse [9] to estimate the number of solutions of this equation. We also give a further improvement on the number of solutions of (0.4) which is valid for all but at most finitely many possible values of the constant term b of the equation. Our bound obtained under the assumption (ii) is better for almost all b than the general bounds of [23], [8] and [14]. Further, we give an explicit upper bound for the number of exceptional values of b. To prove this result, first we apply Győry's method to reduce equation (0.4) to a system of unit equations and then we use results of Evertse, Győry, Stewart and Tijdeman [15] and Evertse and Győry [14] to derive our upper bound for the number of solutions. The bound for the number of exceptional values is obtained by combining results from [6], [17] and [22]. We note that in [15], [14], [17], and [22] the proofs of the results in question involve, among other things, the Thue-Siegel-Roth-Schmidt method and its recent quantitative versions. The results of Chapter 2 are published in our paper [2].

III

In the third chapter we present our new results concerning the number of solutions of index form equations. An *index form equation of Mahler type* is an equation of the form

$$\begin{aligned} I(x_2,\dots,x_n) = \pm I p_1^{z_1}\dots p_s^{z_s} & \\ \text{in } \; x_2,\dots,x_n \in \mathbb{Z}, & \\ \text{and } \; z_1,\dots,z_s \in \mathbb{Z}_{\geq 0} & \\ \text{with } \; (x_2,\dots,x_n,p_1\dots p_s) = 1 \text{ if } s>0, & \end{aligned} \tag{0.5}$$

where $I(\mathbf{X}) = I(X_2,\dots,X_n)$ is the index form corresponding to a given integral basis of a given number field K, $p_1,\dots,p_s$ are rational primes with $s \geq 0$, and I is a rational integer which is relatively prime to $p_1,\dots,p_s$ if $s>0$.

The first finiteness result concerning index form equations is due to Győry [18]. He considered the most interesting case when $s=0$. He first reduced equation (0.5) to a system of unit equations and using his effective results on these equations he gave explicit upper bounds for the solutions of (0.5). Győry's result was later generalized to Mahler type equations by Trelina [42] and Győry and Papp [29].

In 1985, Evertse and Győry [11] derived an explicit upper bound for the number of solutions of equation (0.5). This bound was a consequence of their more general result on decomposable form equations. Later, Evertse [8] and Evertse and Győry [14] improved their results on decomposable form equations, and thereby also the bounds for the number of solutions of equation (0.5). For further results concerning discriminant and index form equations see the survey paper [28] and the references given there.

For given $n \geq 5$, almost all number fields K of degree n have the property that the Galois group of K (more precisely, that of the normal closure of K over $\mathbb{Q}$) is triply transitive. In the important special case when the Galois group, $\mathcal{G}$, of K is triply transitive, Győry [28] improved considerably the bounds established in [11] and [8] for the number of solutions of (0.5). In the third chapter, under the same assumptions on the Galois group $\mathcal{G}$ we considerably improve the bound of Evertse and Győry [14] for almost all values of I, and we also give a bound for the number of exceptional values of I. To prove this result we use a similar method to that applied in Chapter 2 to norm form equations. We also combine Győry's method on reducing decomposable form equations to unit equations with the above-mentioned results from [15], [14], [6], [17] and [22]. As consequences of our result, we present some results concerning discriminant form equations and the number of elements with given discriminant in a given number field. The results of the third chapter are published in [1].

IV

In the fourth chapter some new results are presented on a problem of P. Turán concerning the "distance" of polynomials to irreducible polynomials. It is well-known that there are infinitely many irreducible polynomials in $\mathbb{Z}[x]$ for each given degree, and that (cf. H.-W. Knobloch [31]) the "majority" of these polynomials are irreducible. This suggests that the "distance" of a polynomial to the "nearest" irreducible polynomial cannot be too large. By the "distance" of two polynomials in $\mathbb{Z}[x]$ we mean the following. The sum of the absolute values of the coefficients of a polynomial $P \in \mathbb{Z}[x]$ is called the length of P, and is denoted by $|P|$. The distance of two polynomials $P, Q \in \mathbb{Z}[x]$ is then defined as $|P - Q|$.

P. Turán asked the following question: Does there exist an absolute constant C_1 such that for every $P(x) \in \mathbb{Z}[x]$ of degree n, there is a polynomial $Q(x) \in \mathbb{Z}[x]$ which is irreducible over $\mathbb{Q}$, and satisfies the conditions that $\deg(Q) \leq n$ and $|P-Q| \leq C_1$?

This problem turned out to be a very hard one. The best known result in this direction is due to Schinzel [34], who gave an affirmative answer to the above question in the case when we omit the condition $\deg(Q) \leq n$. For a related result see Győry [25].

Using computational methods, in [4] we proved with Hajdu that for polynomials of degree ≤ 22 we have $C_1 \leq 4$. This result was included in [30].

Later, we continued our work on Turán's problem, and the fourth chapter contains our new results on the topic. On one hand, we refined our algorithms and we extended our result to polynomials of degree ≤ 24. These results have been published in [5]. On the other hand, quite recently we have obtained some new results concerning extreme polynomials (for a definition see Chapter 4). These results are not published yet.

In the following, we detail chapter by chapter the results obtained in our dissertation.

1 On the number of solutions of general decomposable polynomial equations

The first chapter contains results concerning the number of solutions of general decomposable polynomial equations, and their applications to special decomposable polynomial equations. The results of this chapter are joint results with my supervisor Prof. K. Győry, and will be published in [3].

Let $F(\mathbf{X}) = F(X_1, \ldots, X_m) \in \mathbb{Q}[X_1, \ldots, X_m]$ be a *decomposable polynomial* of degree $n \geq 3$ in $m \geq 2$ variables, that is a polynomial which can be written in the form

$$F(\mathbf{X}) = \prod_{i=1}^{n} l_i(\mathbf{X}),$$

where $l_1(\mathbf{X}), \dots, l_n(\mathbf{X})$ are linear polynomials with coefficients in an algebraic number field G. This factorization is unique up to proportional factors from G^*. Let $S = \{p_1, \dots, p_s\}$ be a finite set of $s \geq 0$ rational primes, and denote by $\mathbb{Z}_S$ the ring of S-integers in $\mathbb{Q}$. Consider the equation

$$F(\mathbf{x}) = b \qquad \text{in} \quad \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}_S^m, \tag{1.1}$$

where b is a given non-zero S-integer. We assume throughout the present chapter that F has its coefficients in $\mathbb{Z}_S$. Then (1.1) is called a *decomposable polynomial equation* over $\mathbb{Z}_S$. If in particular F is a form, (1.1) is a decomposable form equation.

We recall (cf. [13]) that if $\mathcal{F}$ is a finite set of linear forms in $G[X_1, \dots, X_k]$, $k \geq 2$, then a non-zero $\mathbb{Q}$-linear subspace V of the vector space $\mathbb{Q}^k$ is called $\mathcal{F}$-*non-degenerate* if $\mathcal{F}$ contains a subset of at least three linear forms whose restrictions to V are linearly dependent, but pairwise linearly independent. Otherwise V is called $\mathcal{F}$-*degenerate*. Further, V is called $\mathcal{F}$-*admissible* if no form in $\mathcal{F}$ vanishes identically on V.

Denote by $\mathcal{L}$ a maximal subset of pairwise linearly independent polynomials among the linear factors $l_1, \dots, l_n$ of F over G. Put

$$\mathcal{L}^* = \{X_{m+1}\} \cup \left\{ X_{m+1} \cdot l\left(\frac{X_1}{X_{m+1}}, \dots, \frac{X_m}{X_{m+1}}\right) : l \in \mathcal{L} \right\}.$$

Then $\mathcal{L}^*$ consists of linear forms in $X_1, \dots, X_{m+1}$ with coefficients in G.

It was shown in [10], Theorem 1, that (1.1) has only finitely many solutions for every S and b if and only if the following condition holds: (i) the linear forms in $\mathcal{L}^*$ have rank $m+1$ over G, and every $\mathcal{L}^*$-admissible linear subspace of $\mathbb{Q}^{m+1}$ of dimension ≥ 3 is $\mathcal{L}^*$-non-degenerate. Further, under the assumption (i), a non-explicit bound was derived (cf. [10], Theorem 2) for the number of solutions of (1.1) which does not depend on the coefficients of F. This bound was given explicitly in terms of the numbers of solutions of some unit equations. However, when the paper [10] was written, no explicit upper bound was available on the number of solutions of unit equations. On combining the bound of [10] with an explicit upper bound of Evertse [8] on the number of solutions of unit equations, one can easily show that under the assumption (i), our equation (1.1) has at most

$$n\left(2^{18}m\right)^{g(m+2)^4(s+\omega_S(b)+1)/2} \tag{1.2}$$

solutions. Here $\omega_S(b)$ denotes the number of those distinct primes p, not contained in S, for which $p|b$ in $\mathbb{Z}_S$, and g denotes the degree of the field G over $\mathbb{Q}$. If G is chosen to be the splitting field of F over $\mathbb{Q}$ then $g \leq n!$, and this bound for g cannot be diminished in general.

In this chapter we give a much better explicit upper bound for the number of solutions of (1.1) which is already polynomial in terms of n. Further, we extend our result to the more general equation

$$F(\mathbf{x}) \in b\mathbb{Z}_S^* \qquad \text{in } \mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{Z}_S^m, \tag{1.3}$$

where $\mathbb{Z}_S^*$ denotes the group of S-units in $\mathbb{Z}_S$. We point out that under the assumption (i) this equation may have infinitely many solutions. Then we show that (i) together with the condition that (i') for at least one polynomial $l^* \in \mathcal{L}^*$ we have $l^*(\mathbf{a}) \neq 0$ for any $\mathbf{0} \neq \mathbf{a} \in \mathbb{Q}^{m+1}$, imply already the finiteness of the number of solutions. Further, under these assumptions we derive explicit upper bounds for the number of solutions of (1.3). Moreover, for $b = 1$, we give a similar upper bound, provided only that the number of solutions of (1.3) is finite.

The significant improvement in our bounds is due to a new approach which is different from that of [10]. As a generalization of Schmidt's famous results [37] on norm form equations, Győry [23] proved that the set of solutions of both (1.1) and (1.3) is the union of finitely many so-called families of solutions. Further, he gave an explicit upper bound for the number of these families. This bound was later improved by Evertse and Győry [14]. In the proofs of our main results we first reduce equations (1.1) and (1.3) to homogeneous decomposable polynomial equations in $m + 1$ unknowns. Then we apply the above-mentioned results of [23] and [14] to derive bounds for the numbers of solutions of (1.1) and (1.3).

We give several consequences for inhomogeneous Thue equations, discriminant polynomial equations, norm polynomial equations and resultant polynomial equations. In particular, our results are valid also in the homogeneous case. However, in this case slightly better estimates are known in the literature, hence we shall not deal here with applications of our results to decomposable form equations.

To state **our results** we need some further notation. For a prime p not contained in S, denote by $\mathrm{ord}_p(b)$ the greatest integer a such that $p^a | b$ in $\mathbb{Z}_S$. Put

$$\psi_S(b, n, m) = \binom{n+1}{m}^{\omega_S(b)} \prod_{\substack{p \text{ prime} \\ p \notin S}} \binom{\mathrm{ord}_p(b) + m}{m}$$

where the product is taken over all primes p not contained in S, and let

$$\delta(m) = \frac{2}{3}(m+1)(m+2)(2m+3) - 4.$$

We note that $\delta(m) \leq 2(m+1)^3$.

Theorem 1.1. *Suppose that*

(i) *the linear forms in $\mathcal{L}^*$ have rank $m + 1$ over G, and every $\mathcal{L}^*$-admissible linear subspace of $\mathbb{Q}^{m+1}$ of dimension ≥ 3 is $\mathcal{L}^*$-non-degenerate.*

Then the number of solutions of equation (1.1) *does not exceed the bounds*

$$n\left(2^{17}n\right)^{\delta(m)(s+1)} \cdot \psi_S(b,n,m) \tag{1.4}$$

and

$$n\left(2^{17}n\right)^{\delta(m)(s+\omega_S(b)+1)}. \tag{1.5}$$

As is easily seen, the bound (1.5) is in general much better than (1.2). In the special case when in (1.1) F is homogeneous, the assumption (i) is equivalent (cf. [10], Corollary 1) to the fact that every $\mathcal{L}$-admissible linear subspace of $\mathbb{Q}^m$ of dimension ≥ 2 is $\mathcal{L}$-non-degenerate. In this case slightly better bounds are given in [8] and [14] for the number of solutions of (1.1).

Our example given below shows that in contrast with the case of decomposable form equations, Theorem 1.1 cannot be generalized for decomposable polynomial equations of the form (1.3). However, if we replace condition (i) of Theorem 1.1 with a stronger assumption, we are able to derive a finiteness theorem and explicit bounds for the number of solutions of (1.3).

Theorem 1.2. *Suppose that the condition* (i) *holds, and that*

(i') *for at least one polynomial $l^* \in \mathcal{L}^*$ we have $l^*(\boldsymbol{a}) \neq 0$ for any $\mathbf{0} \neq \mathbf{a} \in \mathbb{Q}^{m+1}$.*

Then the number of solutions of equation (1.3) *does not exceed the bound*

$$\left(2^{17}n\right)^{\delta(m)(s+\omega_S(b)+1)}. \tag{1.6}$$

As a consequence of Theorem 1.2 we give now another finiteness condition for the number of solutions of (1.3) which is sometimes easier to check. Let $\mathcal{L}_0^* = \mathcal{L}^* \backslash \{X_{m+1}\}$.

Corollary 1.1. *Suppose that*

(ii') *$\mathcal{L}_0^*$ has rank $m+1$ over G, and that $l^*(\mathbf{a}) \neq 0$ for each $l^* \in \mathcal{L}_0^*$ and for any $\mathbf{0} \neq \mathbf{a} \in \mathbb{Q}^{m+1}$.*

Then the number of solutions of (1.3) *does not exceed the bound* (1.6).

We note that in the inhomogeneous case our Theorem 1.2 and Corollary 1.1 provide bounds also for the number of solutions of the corresponding Mahler type equation of the form (0.2).

The following example shows that condition (i') in Theorem 1.2 is also necessary.

Example: Put $S = \{5, 13\}$ and consider the polynomial $F(X_1, X_2) = (4X_1 + 6X_2 - 5)(X_2 + 4)(X_2 + 12) \in \mathbb{Z}_S[X_1, X_2]$. This polynomial satisfies the condition (i) of Theorem 1.2, but there is no linear factor l of F for which the corresponding linear form $l^*(X_1, X_2, X_3)$ has the property that $l^*(x_1, x_2, x_3) \neq 0$ for $(0,0,0) \neq (x_1, x_2, x_3) \in \mathbb{Q}^3$. It is easy to see that the equation

$$F(x_1, x_2) \in \mathbb{Z}_S^* \ \text{ in } \ x_1, x_2 \in \mathbb{Z}_S$$

has infinitely many solutions of the form $(x_1, 1)$.

Theorem 1.3. *Assume that* $\operatorname{rank}\mathcal{L}^* = m+1$ *and* $b=1$. *If the number of solutions of* (1.3) *is finite, then this number does not exceed the bound*

$$(2^{17}n)^{\delta(m)(s+1)}. \tag{1.7}$$

Now we formulate some **consequences** and **applications** of our Theorems 1.1 to 1.3.

First let $F_0(\mathbf{X}) = F_0(X_1,\dots,X_m) \in \mathbb{Z}_S[X_1,\dots,X_m]$ be a *decomposable form* in $m \geq 2$ variables and assume that

$$F_0(\mathbf{X}) = \prod_{i=1}^{n} h_i(\mathbf{X})$$

with linear forms $h_i(\mathbf{X}) \in G[X_1,\dots,X_m]$ for $i=1,\dots,n$. Let $\mathcal{L}_0$ denote a maximal subset of pairwise non-proportional linear forms in $\{h_1,\dots,h_n\}$ over G. Let Λ be the set of all n-tuples $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \in G^n$ for which the decomposable polynomial

$$F_{\boldsymbol{\lambda}}(\mathbf{X}) = \prod_{i=1}^{n} (h_i(\mathbf{X}) + \lambda_i)$$

has its coefficients in $\mathbb{Z}_S$. Clearly, $F_{\mathbf{0}}(\mathbf{X}) = F_0(\mathbf{X})$, hence $\mathbf{0} \in \Lambda$. Our Theorem 1.1 implies the following.

Corollary 1.2. *Suppose that*

(ii) *every subspace of* $\mathbb{Q}^m$ *of dimension* ≥ 2 *is* $\mathcal{L}_0$*-non-degenerate.*

Then for any $b \in \mathbb{Z}_S \setminus \{0\}$ *and for every fixed* $\boldsymbol{\lambda} \in \Lambda$, *the number of solutions of the equation*

$$F_{\boldsymbol{\lambda}}(\mathbf{x}) = b \quad \text{in} \quad \mathbf{x} \in \mathbb{Z}_S^m \tag{1.8}$$

does not exceed the bounds occurring in (1.4) *and* (1.5).

In what follows, we apply Corollary 1.2 to inhomogeneous Thue equations, discriminant polynomial equations and norm polynomial equations. In case of equations considered over $\mathbb{Z}_S$, our Corollary 1.2 and Corollaries 1.3, 1.5 and 1.7 below give improved, explicit versions of Corollaries 2 to 5 of [10] in which non-explicit bounds were given for the numbers of solutions of the corresponding equations.

Let $F(X_1,X_2) \in \mathbb{Z}_S[X_1,X_2]$ be a decomposable polynomial of degree n. Assume that

$$F(X_1,X_2) = \prod_{i=1}^{n} (h_i(X_1,X_2) + \lambda_i) \tag{1.9}$$

where $h_i(X_1,X_2)$ is a linear form with coefficients in G and $\lambda_i \in G$ for $i=1,\dots,n$. Let b be a non-zero S-integer.

Corollary 1.3. *Suppose that*

(iii) *there are at least three pairwise linearly independent forms among* $h_1(X_1, X_2), \ldots, h_n(X_1, X_2)$.

Then the number of solutions of the inhomogeneous Thue equation

$$F(x_1, x_2) = b \quad \text{in } x_1, x_2 \in \mathbb{Z}_S \tag{1.10}$$

does not exceed the bounds

$$n\left(2^{17} n\right)^{52(s+1)} \cdot \psi_S(b, n, 2)$$

and

$$n\left(2^{17} n\right)^{52(s+\omega_S(b)+1)}.$$

From Theorem 1.2 we shall deduce the following.

Corollary 1.4. *Suppose that*

(iii') *there are at least three linearly independent polynomials among* $h_1(X_1, X_2) + \lambda_1, \ldots, h_n(X_1, X_2) + \lambda_n$, *and that for some* i, $h_i(x_1, x_2) \notin \{0, -\lambda_i\}$ *for any* $(0,0) \neq (x_1, x_2) \in \mathbb{Q}^2$.

Then the number of solutions of the equation

$$F(x_1, x_2) \in b\mathbb{Z}_S^* \quad \text{in} \quad x_1, x_2 \in \mathbb{Z}_S \tag{1.11}$$

does not exceed the bound

$$\left(2^{17} n\right)^{52(s+\omega_S(b)+1)}.$$

The bounds in Corollaries 1.3 and 1.4 are valid also in the case of Thue and Thue-Mahler equations, when $\lambda_i = 0$ for $i = 1, \ldots, n$. However, when in (1.10) $F(X_1, X_2)$ is an irreducible binary form, Evertse [9] obtained a much better bound for the numbers of solutions of (1.10) and (1.11).

Let M be a number field of degree $n \geq 3$, $\alpha_0 = 1, \alpha_1, \ldots, \alpha_m$ linearly independent elements of M over $\mathbb{Q}$ such that $M = \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$, and λ an arbitrary element of M. Let $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_n$ be the $\mathbb{Q}$-isomorphisms of M into $\mathbb{C}$. For any $\alpha \in M$, let $\alpha^{(i)} = \sigma_i(\alpha)$. Put

$$L^{(i)}(\mathbf{X}) := X_0 + \alpha_1^{(i)} X_1 + \cdots + \alpha_m^{(i)} X_m + \lambda^{(i)}$$

with the convention that $L(\mathbf{X}) = L^{(1)}(\mathbf{X})$. Then the decomposable polynomial

$$D_{M/\mathbb{Q}}(L(\mathbf{X})) := \prod_{1 \leq i < j \leq n} (L^{(i)}(\mathbf{X}) - L^{(j)}(\mathbf{X}))^2$$

is called a *discriminant polynomial.* We consider the *discriminant polynomial equation*

$$a_0 D_{M/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_m x_m + \lambda) = b \quad \text{in } x_1, \ldots, x_m \in \mathbb{Z}_S, \tag{1.12}$$

where $a_0 \in \mathbb{Q}^*$, $b \in \mathbb{Z}_S \setminus \{0\}$. Further, suppose that a_0 is chosen such that the discriminant polynomial $a_0 D_{M/\mathbb{Q}}(\alpha_1 X_1 + \cdots + \alpha_m X_m + \lambda)$ has its coefficients in $\mathbb{Z}_S$.

Corollary 1.5. *Under the above conditions, the number of solutions of equation* (1.12) *does not exceed the bounds* (1.4) *and* (1.5) *with n replaced by $n(n-1)$.*

In the important special case $\lambda = 0$, somewhat better bounds follow for the number of solutions of (1.12) from the results of [8] and [14] concerning decomposable form equations. Under some conditions on the normal closure of $M/\mathbb{Q}$, even better bounds can be found in [11], [1] and [28] for $\lambda = 0$.

With the above assumptions, consider the following generalization

$$a_0 D_{M/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_m x_m + \lambda) \in b\mathbb{Z}_S^* \quad \text{in } x_1, \ldots, x_m \in \mathbb{Z}_S \tag{1.13}$$

of equation (1.12). From Corollary 1.1 we shall deduce the next

Corollary 1.6. *Suppose that the number field M has no proper subfield, and that $1, \alpha_1, \ldots, \alpha_m, \lambda$ are linearly independent over $\mathbb{Q}$. Then the number of solutions of the equation* (1.13) *does not exceed the bound* (1.6) *with n replaced by $n(n-1)$.*

In the case $\lambda = 0$, the results of [8] and [14] concerning decomposable form equations provide somewhat better bounds for the number of solutions of (1.13).

Let again M be a number field of degree $n \geq 3$, $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m$ linearly independent elements of M over $\mathbb{Q}$ such that $M = \mathbb{Q}(\alpha_2, \ldots, \alpha_m)$, and $\lambda \in M$. As above, put

$$L^{(i)}(\mathbf{X}) := \alpha_1^{(i)} X_1 + \cdots + \alpha_m^{(i)} X_m + \lambda^{(i)}.$$

Then the polynomial

$$N_{M/\mathbb{Q}}(\alpha_1 X_1 + \cdots + \alpha_m X_m + \lambda) = \prod_{i=1}^{n} L^{(i)}(\mathbf{X})$$

is called a *norm polynomial*. Consider the *norm polynomial equation*

$$a_0 N_{M/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_m x_m + \lambda) = b \quad \text{in } x_1, \ldots, x_m \in \mathbb{Z}_S, \tag{1.14}$$

where $b \in \mathbb{Z}_S \backslash \{0\}$, and $a_0 \in \mathbb{Q}^*$ is chosen so that the norm polynomial $a_0 N_{M/\mathbb{Q}}(\alpha_1 X_1 + \cdots + \alpha_m X_m + \lambda)$ has its coefficients in $\mathbb{Z}_S$.

Denote by $\mathcal{M}$ the $\mathbb{Z}$-module generated by $\alpha_1, \ldots, \alpha_m$ in M. $\mathcal{M}$ is called *non-degenerate* if the $\mathbb{Q}$-vector space generated by $\mathcal{M}$ does not contain any subspace of the form $\mu M'$ where $\mu \in M^*$ and M' is a subfield of M such that $\mathbb{Q} \subsetneq M' \subseteq M$. Otherwise $\mathcal{M}$ is called *degenerate*.

Corollary 1.7. *Suppose that*

(iv) *$\mathcal{M}$ is non-degenerate.*

Then the number of solutions of the equation (1.14) *does not exceed the bounds occurring in* (1.4) *and* (1.5).

We note that for $\lambda = 0$, slightly better bounds are given in [8] and [14] for the number of solutions of (1.14).

As a generalization of (1.14), consider now the equation

$$a_0 N_{M/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_m x_m + \lambda) \in b\mathbb{Z}_S^* \quad \text{in} \quad x_1, \dots, x_m \in \mathbb{Z}_S. \tag{1.15}$$

It is an immediate consequence of Corollary 1.1 that if $\alpha_1, \dots, \alpha_m$ and λ are linearly independent over $\mathbb{Q}$ then the number of solutions of equation (1.15) does not exceed the bound (1.6). However, the number of solutions of (1.15) can be finite also in the case when $\alpha_1, \dots, \alpha_m$ and λ are linearly dependent. The following corollary follows immediately from our Theorem 1.3.

Corollary 1.8. *Suppose that for $b = 1$ equation* (1.15) *has only finitely many solutions. Then the number of these solutions does not exceed the bound* (1.7).

Finally, we apply Corollary 1.2 to resultant equations. Let $P \in \mathbb{Z}_S[X]$ be a polynomial of degree $n \geq 3$ with leading coefficient a_0 and without multiple zeros. Let $\alpha_1, \dots, \alpha_n$ denote the zeros of P, and put $G = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$. Fix a positive integer m, and let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in G^n$ such that the decomposable polynomial

$$a_0^m \prod_{i=1}^{n} \left(\alpha_i^m X_0 + \alpha_i^{m-1} X_1 + \cdots + X_m + \lambda_i \right) \tag{1.16}$$

has its coefficients in $\mathbb{Z}_S$. This is the case e.g. for $\boldsymbol{\lambda} = \mathbf{0}$ when the corresponding decomposable form in (1.16) is just the resultant $\mathrm{Res}(P, Q)$ of the polynomials $P(X)$ and $Q(X) = X_0 X^m + X_1 X^{m-1} + \cdots + X_m$. Hence in the case $\boldsymbol{\lambda} = \mathbf{0}$ we call (1.16) a *resultant form*, and in general a *resultant polynomial.* We denote this polynomial by $\mathrm{Res}_{\boldsymbol{\lambda}}(P, Q)$ with the above Q. Consider the *resultant polynomial equation*

$$\mathrm{Res}_{\boldsymbol{\lambda}}(P, Q) = b \quad \text{in} \quad Q \in \mathbb{Z}_S[X], \tag{1.17}$$

where b is a given non-zero element of $\mathbb{Z}_S$.

It follows from Theorem 5 of [25] that for $\boldsymbol{\lambda} = \mathbf{0}$, $b \in \mathbb{Z}_S^*$ and $m < n/2$, the number of solutions of (1.17) in $Q \in \mathbb{Z}_S[X]$ with degree m is at most

$$2 \left(2^{34} n^2 \right)^{(m+1)^3 (s+1)}. \tag{1.18}$$

Further, it was pointed out in [25] that the assumption $m < n/2$ cannot be replaced by $m \leq n/2$ in general.

From our Corollary 1.2 we shall deduce the following.

Corollary 1.9. *Let m be a positive integer with $m < n/2$, and suppose that $P(X)$ has no non-constant divisor of degree $< m$ in $\mathbb{Q}[X]$. Then the number of solutions $Q(X)$ of* (1.17) *with degree m does not exceed the bounds* (1.4) *and* (1.5) *with m replaced by $m + 1$.*

For $\boldsymbol{\lambda} = \mathbf{0}$ and $b \in \mathbb{Z}_S^*$, our bound (1.5) provided by Corollary 1.9 can be compared with (1.18). It is likely that Corollary 1.9 remains still valid without the assumption that P has no non-constant factor with rational coefficients and with degree less than m.

We note that similar results can be deduced from Theorem 1.2 and Corollary 1.1 for the number of those polynomials $Q \in \mathbb{Z}_S[X]$ for which $\mathrm{Res}_{\boldsymbol{\lambda}}(P, Q) \in b\mathbb{Z}_S^*$.

2 On the number of solutions of norm form equations

In the second chapter some results concerning the number of solutions of norm form equations are presented. The contents of this chapter are published in [2].

Let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_m$ be linearly independent algebraic numbers over $\mathbb{Q}$ and put $K := \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$. Denote by n the degree $[K : \mathbb{Q}]$ of the field K over the rationals and let $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_n$ be the $\mathbb{Q}$-isomorphisms of K into $\mathbb{C}$. For any $\alpha \in K$, put $\alpha^{(i)} = \sigma_i(\alpha)$. Consider the linear forms $l^{(i)}(\mathbf{X}) = \alpha_1^{(i)} X_1 + \cdots + \alpha_m^{(i)} X_m$ for $i = 1, \ldots, n$. We can choose a non-zero rational integer a_0 such that the *norm form*

$$F(\mathbf{X}) := a_0 N_{K/\mathbb{Q}}(\alpha_1 X_1 + \cdots + \alpha_m X_m) = a_0 \prod_{i=1}^{n} l^{(i)}(\mathbf{X})$$

has its coefficients in $\mathbb{Z}$. Let $S = \{p_1, \ldots, p_s\}$ be a set of $s \geq 0$ rational primes, and $b \in \mathbb{Z} \setminus \{0\}$ a rational integer which is relatively prime to $p_1, \ldots, p_s$ if $s > 0$. The equation

$$\begin{aligned} a_0 N_{K/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_m x_m) &= b p_1^{z_1} \ldots p_s^{z_s} \\ &\text{in } \; x_1, \ldots, x_m \in \mathbb{Z}, \\ &\text{and } \; z_1, \ldots, z_s \in \mathbb{Z}_{\geq 0} \\ &\text{with } \; (x_1, \ldots, x_m, p_1 \ldots p_s) = 1 \text{ if } s > 0 \end{aligned} \tag{2.1}$$

is called a *norm form equation*, more precisely, for $s > 0$, a *norm form equation of Mahler type*. We note that the well-known Thue equations and Thue-Mahler equations are special cases of equation (2.1).

Throughout this chapter we identify solutions $\mathbf{x} = (x_1, \ldots, x_m), z_1, \ldots, z_s$ and $\mathbf{x}' = (x_1', \ldots, x_m'), z_1', \ldots, z_s'$ of equation (2.2) if $\mathbf{x}' = \pm\mathbf{x}$.

Denote by $\mathcal{M} = \{\alpha_1, \ldots, \alpha_m\}$ the $\mathbb{Z}$-module generated by $\alpha_1, \ldots, \alpha_m$ in K. In the case when $\mathcal{M}$ is degenerate there is a $b \in \mathbb{Z} \setminus \{0\}$ for which (2.1) has infinitely many solutions for $s = 0$. Using his subspace theorem, Schmidt [36] in the case $s = 0$ showed that if $\mathcal{M}$ is non-degenerate then (2.1) has only finitely many solutions for every $b \in \mathbb{Z} \setminus \{0\}$. This result was generalized to the case $s > 0$ by Schlickewei [35]. Later Schmidt [37] proved that for $s = 0$, the set of solutions of (2.1) is the union of finitely many so-called "families of solutions". This theorem was generalized to equation (2.1) by Schlickewei [35], to the so-called finitely generated case by Laurent [32], and to arbitrary decomposable form equations by Győry [23].

The results mentioned above are all ineffective. For a large class of norm form equations, effective finiteness results were established by Győry and Papp and Győry; for references see [19] and [20]. In [20] Győry proved in an effective form that if α_m has degree at least 3 over the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_{m-1})$ then equation (2.1) has only finitely many solutions with $x_m \neq 0$. In this theorem $x_m \neq 0$ and the condition concerning the degree of α_m are necessary. Further, in this finiteness assertion it can happen that $\mathcal{M}$ is degenerate. Hence this result of Győry does not follow from the theorems of Schmidt and Schlickewei. Evertse and Győry [11] proved in 1985 that under the assumption of Győry's theorem the number of solutions of (2.1) with $x_m \neq 0$ does not exceed

$$(4 \cdot 7^{g(2s+2\omega(b)+3)})^{m-1}, \tag{2.2}$$

where g denotes the degree of the normal closure of K over $\mathbb{Q}$, and $\omega(b)$ denotes the number of distinct prime factors of b. As a consequence, they obtained a similar bound for the number of all solutions, subject to the condition that the degree of α_i is at least 3 over $\mathbb{Q}(\alpha_1, \ldots, \alpha_{i-1})$ for $i = 2, \ldots, m$.

For non-degenerate $\mathcal{M}$ and $s = 0$, Schmidt ([38], see also [39]) derived an explicit upper bound for the number of solutions of (2.1), which depends only on n, m and the total number $\Omega(b)$ of prime factors of b. In the proof Schmidt used his quantitative subspace theorem. As a consequence of a more general theorem concerning decomposable form equations, Győry [23] derived an explicit upper bound for the number of families of solutions of (2.1). In particular, this gave an explicit upper bound for the number of solutions, which depends only on n, m, s and $\Omega(b)$, provided that the number of solutions is finite. Under this assumption, Evertse [8] derived later the bound $(2^{33}n^2)^{m^3(s+1)}$ for the number of solutions when $b = 1$. Under the same assumption, Evertse and Győry [14] have recently established the bound

$$(2^{33}n^2)^{e(m)(s+1)} \cdot \psi_m(b) \tag{2.3}$$

for the number of solutions, where $e(m) = \frac{1}{3}m(m+1)(2m+1) - 2$ and

$$\psi_m(b) = \binom{n}{m-1}^{\omega(b)} \prod_{\substack{p \mid b \\ p \text{ prime}}} \binom{\mathrm{ord}_p(b) + m - 1}{m-1}.$$

Further, they gave the upper bound

$$(2^{33}n^2)^{e(m)(s+\omega(b)+1)} \tag{2.4}$$

in the non-degenerate case.

Our Corollaries 1.7 and 1.8 in Chapter 1 provide bounds also for the number of solutions of norm *form* equations, but these bounds are slightly weaker than the above-mentioned bounds of Evertse and Győry [14].

In this chapter we give **significant improvements** of the earlier bounds under the assumptions considered in Győry [20] and Evertse and Győry [11].

We keep the above notation. Further, denote by $\mathbb{Z}_S$ the ring of S-integers and by Z_S^* the group of S-units in $\mathbb{Q}$.

Theorem 2.1. *Under the above assumptions, let* $L = \mathbb{Q}(\alpha_1, \ldots, \alpha_{m-1})$*,* $d = [K : L]$ *and* $f = [L : \mathbb{Q}]$*. Suppose that* $d \geq 3$*. Then the number of solutions of equation* (2.1) *with* $x_m \neq 0$ *is at most*

$$(5 \cdot 10^6 d)^{(s+\omega(b)+1)f}. \tag{2.5}$$

The bound (2.5) is a significant improvement of the bound (2.2), since it does not depend on the degree of the normal closure of K. It is likely that the factor f is unnecessary in the exponent of our bound. In our case $f \geq m - 1$ holds. When $e(m) > f$, (2.5) is better than (2.4). However, in the non-degenerate case (2.4) is valid for all solutions, including the case $x_m = 0$.

We note that under the conditions of Theorem 2.1, equation (2.1) can have infinitely many solutions with $x_m = 0$ if the $\mathbb{Z}$-module $\{\alpha_1, \ldots, \alpha_{m-1}\}$ is degenerate. Hence the above-mentioned results of [38], [23], [8] and [14] cannot be applied to our situation.

Following a standard argument from [20] or [11], our Theorem 2.1 could be applied to the situation when α_i has degree $d \geq 3$ over $\mathbb{Q}(\alpha_1, \ldots, \alpha_{i-1})$ for $i = 2, \ldots, m$. Then, with the notation $f = [\mathbb{Q}(\alpha_1, \ldots, \alpha_{m-1}) : \mathbb{Q}]$, Theorem 2.1 would give essentially the same bound as in (2.5) for the number of solutions of (2.1). But in this case we have $f \geq 3^{m-1}$, hence (2.4) gives in general a better bound for the number of solutions.

If $s > 0$ and the equation

$$a_0 N_{K/\mathbb{Q}}(\alpha_1 x_1 + \cdots + \alpha_m x_m) \in b\mathbb{Z}_S^* \text{ in } x_1, \ldots, x_m \in \mathbb{Z}_S \tag{2.6}$$

has a solution, then it has infinitely many solutions. Indeed, if $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{Z}_S^m$ is a solution of equation (2.6) then each element of the set

$$\mathbf{x}\mathbb{Z}_S^* := \{(\varepsilon x_1, \ldots, \varepsilon x_m) : \varepsilon \in \mathbb{Z}_S^*\}$$

is also a solution of equation (2.6). Such a set is called a $\mathbb{Z}_S^*$-*coset of solutions* of the equation (2.6). Further, for any solution $\mathbf{x}, z_1, \ldots, z_s$ of equation (2.1) $\mathbf{x}\mathbb{Z}_S^*$ is a coset of solutions of equation (2.6), and conversely, in each coset of solutions of equation (2.6) there exists exactly one solution of equation (2.1). Thus Theorem 2.1 implies the following corollary.

Corollary 2.1. *Under the assumptions of Theorem 2.1 the number of* $\mathbb{Z}_S^*$*-cosets of solutions of the equation* (2.6) *does not exceed the bound in* (2.5).

Let $\psi_m(b)$ denote the expression defined above.

Theorem 2.2. *Let the assumptions be the same as in Theorem 2.1, and let $n = [K : \mathbb{Q}]$. Then apart from finitely many values of b, equation* (2.1) *has at most*

$$2\psi_m(b) \tag{2.7}$$

solutions with $x_m \neq 0$. Further, the number of exceptional b's is at most

$$e^{3 \cdot 30^{20}(n(s+1)+1)}. \tag{2.8}$$

When b is not exceptional and $\mathrm{ord}_p(b)$ is small for all prime divisors p of b, (2.7) gives a considerable improvement of (2.2) As a consequence of Theorem 2.2, we get the following:

Corollary 2.2. *Suppose that in* (2.1) *α_i has degree at least 3 over the field $\mathbb{Q}(\alpha_1, \dots, \alpha_{i-1})$ for $i = 2, \dots, m$. Then apart from finitely many values of b, equation* (2.1) *has at most*

$$2m \binom{n}{r}^{\omega(b)} \prod_{\substack{p|b \\ p \ prime}} \binom{\mathrm{ord}_p(b) + m - 1}{m - 1} \tag{2.9}$$

solutions. Here $r = [n/2]$ is the integer part of $n/2$. Further, the number of the exceptional b's is at most

$$me^{3 \cdot 30^{20}(n(s+1)+1)}.$$

In the case when b is not exceptional and $\mathrm{ord}_p(b)$ is small for each prime $p|b$, our Corollary 2.2 considerably improves Theorem 8 of Evertse and Győry [11] over $\mathbb{Q}$. Furthermore, under the assumptions of the Corollary 2.2 our bound (2.9) is much better then the general bound in (2.3).

Applying the results of our Theorem 2.2 to the case of equation (2.6) we get the following corollary.

Corollary 2.3. *Let the assumptions be the same as in Theorem 2.2. Apart from finitely many values of b, the number of $\mathbb{Z}_S^*$-cosets of solutions of equation* (2.6) *does not exceed the bound in* (2.7)*, and the number of exceptional values of b is bounded from above by the expression in* (2.8).

We note that Corollary 2.2 can also be applied similarly to equation (2.6).

Our Theorems 2.1 and 2.2 are valid for $m = 2$, i.e. for the Thue-Mahler equation as well. However, in this special case much better bounds have been established by Evertse [9] and Stewart [41], respectively.

3 On the number of solutions of index form and discriminant form equations

In the third chapter we present our results concerning the number of solutions of index form and discriminant form equations. These results are published in [1].

Let K be an algebraic number field of degree $n \geq 3$ with discriminant D_K and ring of integers $\mathcal{O}_K$. Let $\sigma_1 = \mathrm{id}, \sigma_2, \ldots, \sigma_n$ denote the $\mathbb{Q}$-isomorphisms of K in $\mathbb{C}$. For any $\alpha \in K$, put $\alpha^{(i)} = \sigma_i(\alpha)$. Consider an integral basis $\{1, \alpha_2, \ldots, \alpha_n\}$ in $\mathcal{O}_K$, and the linear forms $l^{(i)}(\mathbf{X}) = X_1 + \alpha_2^{(i)} X_2 + \cdots + \alpha_n^{(i)} X_n$ for $i = 1, \ldots, n$, with the convention that $l^{(1)}(\mathbf{X}) = l(\mathbf{X})$. Putting $l_{ij}(\mathbf{X}) = l^{(i)}(\mathbf{X}) - l^{(j)}(\mathbf{X})$,

$$D_{K/\mathbb{Q}}(l(\mathbf{X})) := \prod_{1 \leq i < j \leq n} l_{ij}^2(\mathbf{X}) \tag{3.1}$$

is a decomposable form with coefficients in $\mathbb{Z}$, which is called the *discriminant form* corresponding to the basis $\{1, \alpha_2, \ldots, \alpha_n\}$. It can be written in the form

$$D_{K/\mathbb{Q}}(l(\mathbf{X})) = (I(\mathbf{X}))^2 D_K, \tag{3.2}$$

where $I(\mathbf{X}) = I(X_2, \ldots, X_n)$ is a decomposable form of degree $n(n-1)/2$ with coefficients in $\mathbb{Z}$. If α is a primitive integral element of K and $\alpha = x_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$ with $x_1, \ldots, x_n \in \mathbb{Z}$, then $|I(x_2, \ldots, x_n)|$ is precisely the index $I(\alpha)$ of α, i.e. the index of the subgroup $\mathbb{Z}^+[\alpha]$ in the additive group $\mathcal{O}_K^+$ of $\mathcal{O}_K$. Hence $I(\mathbf{X})$ is called the *index form* of the basis $\{1, \alpha_2, \ldots, \alpha_n\}$.

Let I denote a positive rational integer, and $S = \{p_1, \ldots, p_s\}$ a finite set of $s \geq 0$ distinct rational primes. Consider the *index form equation*

$$\begin{aligned} I(x_2, \ldots, x_n) = \pm I p_1^{z_1} \ldots p_s^{z_s} & \\ & \text{in } \ x_2, \ldots, x_n \in \mathbb{Z}, \\ & \text{and } \ z_1, \ldots, z_s \in \mathbb{Z}_{\geq 0} \\ & \text{with } \ (x_2, \ldots, x_n, p_1 \ldots p_s) = 1 \text{ if } s > 0. \end{aligned} \tag{3.3}$$

We may and shall assume that I is relatively prime to $p_1, \ldots, p_s$. For $s = 0$, the assumption $(x_2, \ldots, x_n, p_1 \ldots p_s) = 1$ is omitted. We identify the solutions $\mathbf{x} = (x_2, \ldots, x_n), z_1, \ldots, z_s$ and $\mathbf{x}' = (x_2', \ldots, x_n'), z_1', \ldots, z_s'$ of (3.3) if $\mathbf{x}' = \pm\mathbf{x}$.

In the most interesting case when $s = 0$, Győry [18] proved that (3.3) has only finitely many solutions, and gave an effective upper bound for the solutions. Later, this theorem was extended to the case $s > 0$ by Trelina [42] and Győry and Papp [29]. For surveys presenting further generalizations, we refer to [19], [21], [17].

The first explicit upper bound for the number of solutions of (3.3) was derived by Evertse and Győry [11]. They showed as a consequence of a more general result that (3.3) has at most

$$(4 \cdot 7^{g(2s+2\omega(I)+3)})^{n-2} \tag{3.4}$$

solutions. Here $\omega(I)$ denotes the number of distinct prime factors of I, and g is the degree of the normal closure of K over $\mathbb{Q}$. Hence $n \le g \le n!$.

It follows from a result of Evertse [8] on decomposable form equations that the number of solutions of (3.3) does not exceed

$$(2^{33}r^2)^{(n-1)^3(s+\omega(I)+1)}, \tag{3.5}$$

where $r = n(n-1)/2$. When n is large and g is large with respect to n, this bound is better than (3.4).

We recall some notation from Chapters 1 and 2. Put

$$\psi(I) = \binom{r}{n-2}^{\omega(I)} \prod_{\substack{p|I \\ p \text{ prime}}} \binom{\mathrm{ord}_p(I)+n-2}{n-2}$$

with $r = n(n-1)/2$, where the product is taken over all distinct prime factors of I and $\mathrm{ord}_p(I)$ denotes the greatest rational integer a for which p^a divides I in $\mathbb{Z}$. As a consequence of a more general theorem concerning decomposable form equations, Evertse and Győry [14] derived in 1997 the upper bound

$$(2^{33}r^2)^{e(n)(s+1)}\psi(I) \tag{3.6}$$

for the number of solutions of (3.3). Here $e(n) = \frac{1}{3}(n-1)n(2n-1)-2$. The bound (3.6) is better than (3.5) when all the exponents $\mathrm{ord}_p(I)$ are small.

Our Corollaries 1.5 and 1.6 in Chapter 1 provide bounds also for the number of solutions of index *form* and discriminant *form* equations, but these bounds are slightly worse than the above-mentioned bounds of Evertse and Győry [14].

An important special case is when the Galois group, $\mathcal{G}$, of the normal closure of K over $\mathbb{Q}$ is triply transitive. In other words, for any ordered subsets $\{i_1, i_2, i_3\}$ and $\{i'_1, i'_2, i'_3\}$ of $\{1, \dots, n\}$ there is a $\sigma \in \mathcal{G}$ such that if $\alpha \in K$ then $\sigma(\alpha^{(i_k)}) = \alpha^{(i'_k)}$ for $k = 1, 2, 3$. For example, $\mathcal{G}$ is triply transitive if $n \ge 5$ and $\mathcal{G} = S_n$ or A_n. It is important to note that for given n, the Galois group $\mathcal{G}$ is triply transitive for almost all number fields of degree n. Under this assumption concerning $\mathcal{G}$, Győry [28] has recently showed that for $s = 0$ *, equation (3.3) has at most

$$2^{4n(n-1)(\omega(I)+1)+8}$$

solutions. This is a considerable improvement of (3.4) and (3.5) for $s = 0$. In this chapter we **considerably improve** (3.6) under the same assumption concerning $\mathcal{G}$.

Theorem 3.1. *Suppose that the Galois group $\mathcal{G}$ is triply transitive. Then apart from finitely many values of I, equation* (3.3) *has at most*

$$2\psi(I)$$

*For simplicity, this result was proved in [28] for $s = 0$ only, but the same arguments work for $s > 0$ as well and give the same upper bound with $\omega(I) + s$ instead of $\omega(I)$.

solutions. Further, the number of the exceptional I's is at most

$$e^{30^{20}n^2(s+1)}.$$

This means that under the assumption of Theorem 3.1 and apart from finitely many values of I, the factor $(2^{33}r^2)^{e(n)(s+1)}$ in (3.6) can be replaced by 2.

Denote by $\mathbb{Z}_S$ the ring of S-integers, and by $\mathbb{Z}_S^*$ the group of S-units in $\mathbb{Z}$, respectively. The equation

$$I(x_2, \dots, x_n) \in I\mathbb{Z}_S^* \ \text{ in } \ x_2, \dots, x_n \in \mathbb{Z}_S \tag{3.7}$$

clearly has infinitely many solutions, provided that it has at least one solution. Indeed, if $\mathbf{x} = (x_2, \dots, x_n) \in \mathbb{Z}_S^{n-1}$ is a solution of equation (3.7) then each element of the set

$$\mathbf{x}\mathbb{Z}_S^* := \{(\varepsilon x_2, \dots, \varepsilon x_n) : \varepsilon \in \mathbb{Z}_S^*\}$$

is also a solution of equation (3.7). Such a set is called a $\mathbb{Z}_S^*$-coset of solutions of (3.7). Further, for any solution $\mathbf{x}, z_1, \dots, z_s$ of equation (3.3) $\mathbf{x}\mathbb{Z}_S^*$ is a $\mathbb{Z}_S^*$-coset of solutions of equation (3.7), and conversely, in each coset of solutions of equation (3.7) there exists exactly one solution of equation (3.3). Thus Theorem 3.1 implies the following corollary.

Corollary 3.1. *Let the assumptions be the same as in Theorem 3.1. For all but finitely many values of I the number of $\mathbb{Z}_S^*$-cosets of solutions of the equation* (3.7) *is at most $2\psi(I)$. Further, the number of the exceptional values of I is at most $e^{30^{20}n^2(s+1)}$.*

In view of (3.2) the equation (3.7) is equivalent to the discriminant form equation

$$D_{K/\mathbb{Q}}(l(\mathbf{x})) \in D\mathbb{Z}_S^* \ \text{ in } \ x_2, \dots, x_m \in \mathbb{Z}_S, \tag{3.8}$$

provided that e.g. $D = I^2 D_K$. We may assume that D is an integer with the property $(D, p_1 \dots p_s) = 1$ if $s > 0$. Denote by D_K^* the S-free part of the discriminant D_K of the field K. Then (3.8) can have a solution only if $I := \sqrt{|D/D_K^*|}$ is a rational integer. From Theorem 3.1 we get the following corollary.

Corollary 3.2. *Under the assumptions of Theorem 3.1, for all but finitely many values of D, the number of cosets of solutions of the discriminant form equation* (3.8) *is at most $2\psi(I)$ with $I := \sqrt{|D/D_K^*|} \in \mathbb{Z}$. Further, the number of exceptional values of D is at most $e^{30^{20}n^2(s+1)}$.*

Applying Theorem 3.1 to equation (3.3) in the case $s = 0$, we obtain as a special case the following result for the equation

$$I(x_2, \dots, x_n) = \pm I \ \text{ in } \ x_2, \dots, x_n \in \mathbb{Z}. \tag{3.9}$$

Theorem 3.2. *Suppose again that the Galois group $\mathcal{G}$ is triply transitive. Then apart from at most $e^{30^{20}n^2}$ values of I, equation* (3.9) *has at most $2\psi(I)$ solutions.*

Theorem 3.2 implies e.g. results concerning the numbers of elements of given discriminant. If in K there exists at least one element with discriminant D then $I := \sqrt{|D/D_K|}$ is a rational integer. Further, if the element $\alpha \in \mathcal{O}_K$ takes the form

$$\alpha = x_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$$

in the integral basis $1, \alpha_2, \ldots, \alpha_n$ with some $x_1, \ldots, x_n \in \mathbb{Z}$ then $D_{K/\mathbb{Q}}(x_1 + x_2\alpha_2 + \cdots + x_n\alpha_n)$ is just the discriminant of α. Two elements $\alpha, \beta \in \mathcal{O}_K$ are called *equivalent* if $\alpha - \beta$ is a rational integer. Clearly, equivalent elements have the same discriminant. It follows from the above-mentioned result of Evertse and Győry [14] on the number of solutions of (3.3) that for given $0 \neq D \in \mathbb{Z}$ for which $I = \sqrt{|D/D_K|} \in \mathbb{Z}$, the number of pairwise inequivalent elements in $\mathcal{O}_K$ having discriminant D does not exceed the bound (3.6) with the choice $s = 0$. Our Theorem 3.2 implies that this bound can be considerably improved for almost all D, provided that the Galois group of K is triply transitive.

Corollary 3.3. *Suppose that the Galois group $\mathcal{G}$ is triply transitive. Then for all but finitely many integer values of D, the number of pairwise inequivalent elements of $\mathcal{O}_K$ having discriminant D is at most $4\psi(I)$ with $I = \sqrt{|D/D_K|} \in \mathbb{Z}$, and the number of exceptional values of D is at most $e^{30^{20}n^2}$.*

4 On a problem of P. Turán concerning irreducible polynomials

In the fourth chapter some new results are presented on a problem of P. Turán concerning irreducible polynomials. Most of these results are published in [5], but also some unpublished results are included.

Many important and interesting problems of mathematics are related to the distribution of irreducible elements in various special structures. It is well-known that the number of primes in $\mathbb{N}$ is infinite. However, the set of prime numbers is of density zero and the gap between two consecutive primes can be arbitrarily large. In $\mathbb{Z}[x]$ there are infinitely many irreducible polynomials. Nevertheless, it seems that there are only few common properties of the distribution of irreducible elements in $\mathbb{Z}$ and in $\mathbb{Z}[x]$. Indeed, if we denote by $P(N)$ resp. $R(N)$ the number of polynomials resp. irreducible polynomials in $\mathbb{Z}[x]$ of given degree and of height at most N, then we have (cf. [31])

$$\frac{R(N)}{P(N)} \to 1 \quad \text{as} \quad N \to \infty.$$

In other words 'almost all' polynomials in $\mathbb{Z}[x]$ are irreducible.

The above result suggests that the 'gap' between 'neighbouring' irreducible polynomials in $\mathbb{Z}[x]$ cannot be too large. These facts led P. Turán in 1962 to propose the following interesting problem. To formulate his problem, we need the concept of the length $|P|$ of a polynomial $P(x) = a_nx^n + ... + a_1x + a_0 \in \mathbb{Z}[x]$ which is defined by the expression

$$|P| = \sum_{k=0}^{n} |a_k| \ .$$

By the distance of $P, Q \in \mathbb{Z}[x]$ we mean $|P - Q|$. It follows easily from Eisenstein's theorem that for given $P \in \mathbb{Z}[x]$ of degree n there is an irreducible polynomial $Q \in \mathbb{Z}[x]$ of degree n such that $|P - Q| \leq n + 2$. P. Turán asked the following (cf.[33]):

Does there exist an absolute constant C_1 such that for every $P(x) \in \mathbb{Z}[x]$ of degree n, there is a polynomial $Q(x) \in \mathbb{Z}[x]$ irreducible over $\mathbb{Q}$, satisfying $\deg(Q) \leq n$ and $|P - Q| \leq C_1$?

This problem is very difficult. It turns to be somewhat easier if one removes the condition $\deg(Q) \leq n$. In 1970, A. Schinzel [34] proved the following deep theorem:

Theorem A. (A. Schinzel [34]) *For any nonzero integers a, b and any polynomial P with integral coefficients, such that $P(0) \neq 0$ and $P(1) \neq -a - b$, there exist infinitely many irreducible polynomials $ax^n + bx^m + P(x)$ with $n > m > deg(P)$. One of them satisfies*

$$n < \exp\{(5 \deg(P) + 2 \log |ab| + 7)(||P|| + a^2 + b^2)\},$$

where $||P||$ denotes the sum of the squares of the coefficients of P.

As a consequence of this theorem Schinzel showed that for every $P \in \mathbb{Z}[x]$ of degree n there are infinitely many irreducible $Q \in \mathbb{Z}[x]$ such that

$$|P - Q| \leq \begin{cases} 2 & \text{if } P(0) \neq 0, \\ 3 & \text{otherwise.} \end{cases}$$

Further, one of these irreducible polynomials Q satisfies

$$\deg(Q) \leq e^{(5n+7)(|P|^2+3)}.$$

This result gives a partial answer to Turán's question.

For the sake of completeness, now we present another, similar problem, which was proposed in 1984 by M. Szegedy (cf. [25]). He asked the following:

Does there exist a constant C_2 depending only on n such that for any $P \in \mathbb{Z}[x]$ of degree n, $P(x) + b$ is irreducible over $\mathbb{Q}$ for some $b \in \mathbb{Z}$ with $|b| \leq C_2$?

This seems also to be a very hard question. In 1994, Győry [25] succeeded to give an affirmative answer to Szegedy's problem in case of monic polynomials. This is a consequence of his following theorem.

Theorem B. (K. Győry [25]) *Let $P \in \mathbb{Z}[x]$ be a polynomial of degree n with leading coefficient a_0. There exist an effectively computable constant C_3 depending only on n and $\omega(a_0)$, and $b \in \mathbb{Z}$ with $|b| \leq C_3$ for which $P(x) + b$ is irreducible over $\mathbb{Q}$. (Here $\omega(a_0)$ denotes the number of distinct prime divisors of a_0.)*

We remark that in [25] C_3 is given explicitly.

In our paper [4] we gave upper bounds for the Turán constant C_1 for monic polynomials P of degree not greater than 22. In fact we could prove that for such polynomials $C_1 = 4$ can be chosen. Slightly improving our algorithms, in [5] we extended our result to polynomials of degree at most 24.

For a positive integer n, denote by c_n the smallest integer with the property that for any monic polynomial $P \in \mathbb{Z}[x]$ of degree n one can choose an irreducible monic polynomial $Q \in \mathbb{Z}[x]$ of degree n, such that $|P - Q| \leq c_n$. One can verify easily that for every positive n, c_n exists. Using this notation, our result in [4] says that

$$c_n \leq 4 \quad \text{for every positive integer } n \leq 22.$$

We prove the following **extension**.

Theorem 4.1. *For every positive integer $n \leq 24$ and for every monic polynomial $P \in \mathbb{Z}[x]$ of degree n there exists an irreducible monic polynomial $Q \in \mathbb{Z}[x]$ of degree n such that*

$$|P - Q| \leq 4.$$

For smaller degrees, our computations imply a slightly better result. In fact we could prove that $c_1 = 0$, $c_2 = 1$, $c_n = 2$ for $3 \leq n \leq 6$, $c_n \leq 3$ for $7 \leq n \leq 12$, and $c_n \leq 4$ for $13 \leq n \leq 24$ (see also Table 1). Summarizing these assertions, we can state that for any positive integer $n \leq 24$, we have $c_n \leq 4$.

To make clear the main idea behind the proof we need some further notation. Denote by $\mathbb{Z}_p[x]$ the residue class ring $\mathbb{Z}[x] \pmod p$. For any polynomial $R \in \mathbb{Z}[x]$ denote by R_p the image of R under the natural homomorphism from $\mathbb{Z}[x]$ to $\mathbb{Z}_p[x]$. Each polynomial $P \in \mathbb{Z}[x]$ has a representative which has its coefficients in the interval $]-\frac{p}{2}, \frac{p}{2}]$. The sum of the absolute values of the coefficients of the representative of this type of a polynomial $P \in \mathbb{Z}_p[x]$ is called the p-length of P and it is denoted by $|P|_p$. By the (mod p) distance of two polynomials $P, Q \in \mathbb{Z}[x]$ we mean the p-length of the polynomial $(P - Q)_p$. Similarly, by the (mod p) distance of two polynomials $P, Q \in \mathbb{Z}_p[x]$ we mean the p-length of the polynomial $P - Q$.

For every positive integer n let us denote by $c_n(p)$ the smallest integer with the property that for each monic polynomial $P \in \mathbb{Z}_p[x]$ of degree n there exists an irreducible monic polynomial $Q \in \mathbb{Z}_p[x]$ of degree n for which $|P - Q|_p \leq c_n(p)$.

The main idea of the proof is based on the fact that $c_n \geq c_n(p)$ for any prime number p. Indeed, by the definition of $c_n(p)$, for an arbitrary but fixed polynomial $P \in \mathbb{Z}[x]$ there exists an irreducible monic polynomial $R \in \mathbb{Z}_p[x]$ such that $|P_p - R|_p \leq c_n(p)$. Now it is easy to prove that there exists a monic polynomial $Q \in \mathbb{Z}[x]$ such that $|P - Q| = |P_p - R|_p \leq c_n(p)$ and $Q_p = R$. Finally, this last equality, together with the assumption that R is irreducible in $\mathbb{Z}_p[x]$ shows that Q is irreducible in $Z[x]$.

Thus it is enough to give upper bounds for $c_n(p)$ for each $1 \leq n \leq 24$ with at least one prime p. But then, for fixed n, the polynomials in $\mathbb{Z}_p[x]$ of degree at most n are finite in number. This makes it possible to use computational methods in order to obtain bounds for $c_n(p)$. In fact we have investigated the case $p = 2$ in the range $1 \le n \le 24$ and the case $p = 3$ in the range $1 \le n \le 12$. The case $1 \le n \le 22$ was included in our paper [4]. Our algorithms are exponential in n, and thus we were not able to use them for large values of n. However, later we have slightly refined our algorithms, slightly decreasing the time of computation and so we managed to investigate the case $p = 2$ in the range $1 \le n \le 24$. We summarized our results in a table (see Table 1 in the dissertation) which contains the values of $c_n(2)$, $c_n(3)$, and bounds for c_n in the range $1 \le n \le 24$.

A polynomial $P \in \mathbb{Z}_2[x]$ is called an extreme polynomial, if every polynomial $Q \in \mathbb{Z}_2[x]$ with $\deg(Q) = \deg(P)$ and $|P - Q| \le c_n(2) - 1$ is reducible. In Chapter 4 we give all extreme polynomials of minimal 2-length of degree $2 \le n \le 24$ (see Table 2 in the dissertation). The results concerning extreme polynomials of minimal p-length are also joint results with L. Hajdu, but they have not yet been published.

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