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Characterizations of the equality of two-variable generalized quasiarithmetic means ☆

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ABSTRACT

This paper is motivated by an astonishing result of H. Alzer and S. Ruscheweyh published in 2001, which states that the intersection of the classes two-variable Gini means and Stolarsky means is equal to the class of two-variable power means. The two-variable Gini and Stolarsky means form two-parameter classes of means expressed in terms of power functions. They can naturally be generalized in terms of the so-called Bajraktarević and Cauchy means. Our aim is to show that the intersection of these two classes of functional means, under high-order differentiability assumptions, is equal to the class of two-variable quasiarithmetic means.

1. Introduction

A notion, which subsumes the concept of arithmetic, geometric and harmonic means is the concept of power means. For a real number a, the *two-variable a^{th}-Hölder or a^{th}-power mean* $\mathcal{H}_a : \mathbb{R}_+^2 \to \mathbb{R}$ is defined as

$$\mathcal{H}_a(x,y) := \begin{cases} \left(\frac{x^a + y^a}{2}\right)^{\frac{1}{a}} & \text{if } a \neq 0, \\ \sqrt{xy} & \text{if } a = 0. \end{cases}$$

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Observe that, for $a = 1$, $a = 0$, and $a = -1$, the power mean $\mathcal{H}_a$ equals the arithmetic, geometric and harmonic means, respectively. The theory of power means is well-developed, most of the details of their theory can be found in the monographs [4], [5], [7], [20], and [21].

The class of two-variable power means has been extended in numerous ways in the literature. One early extension was introduced by C. Gini [6] in 1938 who, for two real parameters a, b, defined the mean $\mathcal{G}_{a,b} : \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$\mathcal{G}_{a,b}(x,y) := \begin{cases} \left(\dfrac{x^a + y^a}{x^b + y^b}\right)^{\frac{1}{a-b}} & \text{if } a \neq b, \\ \exp\left(\dfrac{x^a \log(x) + y^a \log(y)}{x^a + y^a}\right) & \text{if } a = b. \end{cases} \tag{1.1}$$

These means are nowadays called *two-variable Gini means*. One can easily see that the two-variable Hölder means form a subclass of two-variable Gini means. Indeed, for $a \in \mathbb{R}$, we have $\mathcal{H}_a = \mathcal{G}_{a,0} = \mathcal{G}_{0,a}$ and $\mathcal{H}_0 = \mathcal{G}_{a,-a}$.

Another extension of the class of two-variable power means was discovered by K. Stolarsky [27] in 1975, who, for two real parameters a, b, defined $\mathcal{S}_{a,b} : \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$\mathcal{S}_{a,b}(x,y) := \begin{cases} \left(\dfrac{b(x^a - y^a)}{a(x^b - y^b)}\right)^{\frac{1}{a-b}} & \text{if } ab(a-b)(x-y) \neq 0, \\ \exp\left(-\dfrac{1}{a} + \dfrac{x^a \log(x) - y^a \log(y)}{x^a - y^a}\right) & \text{if } a = b, ab(x-y) \neq 0, \\ \left(\dfrac{x^a - y^a}{a(\log(x) - \log(y))}\right)^{\frac{1}{a}} & \text{if } b = 0, a(x-y) \neq 0, \\ \left(\dfrac{x^b - y^b}{b(\log(x) - \log(y))}\right)^{\frac{1}{b}} & \text{if } a = 0, b(x-y) \neq 0, \\ \sqrt{xy} & \text{if } a = b = 0, \\ x & \text{if } x = y. \end{cases} \tag{1.2}$$

We call these means *Stolarsky means* today. We note, that these means are sometimes called *extended means* (cf. [8,9]) or difference means [22,23]. One can also see that the two-variable Hölder means form a subclass of Stolarsky means. Indeed, for $a \in \mathbb{R}$, it is easy to see that $\mathcal{H}_a = \mathcal{S}_{2a,a} = \mathcal{S}_{a,2a}$, and $\mathcal{H}_0 = \mathcal{S}_{a,-a}$ hold.

Therefore, the class of two-variable Hölder means is contained in the intersection of the classes of two-variable Gini and Stolarsky means. In 2001, H. Alzer and St. Ruscheweyh [1] established a surprising result which asserts that, instead of inclusion, the equality holds here, i.e., the class of two-variable Hölder means is equal to the intersection of the classes of two-variable Gini and Stolarsky means.

In what follows, we will recall further important classes of two-variable functional means that extend Hölder, Gini and Stolarsky means in a natural way. These classes are the quasiarithmetic, Bajraktarević, and Cauchy means. Motivated by the above-described result of Alzer and Ruscheweyh [1], our aim is to show that, under 8 times differentiability assumptions, the intersection of the classes of two-variable Bajraktarević and Cauchy means equals the class of two-variable quasiarithmetic means.

Throughout this paper let I denote a nonempty open real interval and let

$$\mathcal{CM}(I) := \{f : I \to \mathbb{R} \mid f \text{ is continuous and strictly monotone}\},$$
$$\mathcal{CP}(I) := \{f : I \to \mathbb{R} \mid f \text{ is continuous and positive}\}.$$

For $\varphi \in \mathcal{CM}(I)$, the *two-variable quasiarithmetic mean* generated by φ is the map $\mathcal{A}_\varphi : I^2 \to I$ given as

$$\mathcal{A}_\varphi(x, y) := \varphi^{-1}\left(\frac{\varphi(x) + \varphi(y)}{2}\right).$$

The comprehensive investigation of these means can be found, for instance, in the book [7]. In 1958, M. Bajraktarević [2], [3] created a new generalization of quasiarithmetic means essentially in the following way: For $f, g : I \to \mathbb{R}$ such that $f/g \in \mathcal{CM}(I)$ and $g \in \mathcal{CP}(I)$, define $\mathcal{B}_{f,g} : I^2 \to I$ by

$$\mathcal{B}_{f,g}(x, y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x) + f(y)}{g(x) + g(y)}\right).$$

The mean $\mathcal{B}_{f,g}$ will be called a *two-variable Bajraktarević mean*. It is clear that $\mathcal{B}_{f,g}$ equals $\mathcal{A}_f$ if g is a constant. Consequently, quasiarithmetic means form a subclass of Bajraktarević means. Assuming 6 times continuous differentiability, the equality problem of these means was solved by Losonczi [11,15]. A recent characterization of this equality in terms of eight equivalent conditions has been established in [18, Theorem 15].

Another important generalization of quasiarithmetic means can be obtained as follows: If $f, g : I \to \mathbb{R}$ are continuously differentiable functions with $g' \in \mathcal{CP}(I)$ and $f'/g' \in \mathcal{CM}(I)$, then define $C_{f,g} : I^2 \to I$ by

$$C_{f,g}(x, y) := \begin{cases} \left(\dfrac{f'}{g'}\right)^{-1}\left(\dfrac{f(x) - f(y)}{g(x) - g(y)}\right) & \text{if } x \neq y, \\ x & \text{if } x = y. \end{cases}$$

The mean value property of this mean is a direct consequence of the Cauchy Mean Value Theorem, this is why this mean is called a *Cauchy mean or difference mean* in the literature (cf. [10], [12]). If φ is differentiable with a nonvanishing derivative, then one can easily see that $C_{\varphi^2,\varphi} = \mathcal{A}_\varphi$. Consequently, quasiarithmetic means (with a differentiable generator) form also a subclass of Cauchy means. Assuming 7 times continuous differentiability, the equality problem of these means was solved by Losonczi [12]. A characterization of this equality in terms of eight equivalent conditions has also been established in paper [18, Theorem 16].

In this paper, we recall the following generalization of quasiarithmetic means, which was introduced in [16] and also investigated in [17]. Given two continuous functions $f, g : I \to \mathbb{R}$ with $g \in \mathcal{CP}(I)$, $f/g \in \mathcal{CM}(I)$ and a probability measure μ on the Borel subsets of $[0, 1]$, the two-variable mean $\mathcal{M}_{f,g;\mu} : I^2 \to I$ is defined by

$$\mathcal{M}_{f,g;\mu}(x, y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{\int_{[0,1]} f\big(tx + (1-t)y\big)d\mu(t)}{\int_{[0,1]} g\big(tx + (1-t)y\big)d\mu(t)}\right).$$

Means of the above form, will be called *generalized quasiarithmetic means*.

In what follows, let δ_τ denote the Dirac measure concentrated at the point $\tau \in [0, 1]$. Using this notation, one can see that if $\mu = \frac{\delta_0+\delta_1}{2}$, then $\mathcal{M}_{f,g;\mu} = \mathcal{B}_{f,g}$ provided that $g \in \mathcal{CP}(I)$ and $f/g \in \mathcal{CM}(I)$. On the other hand, if μ is the Lebesgue measure λ restricted to $[0, 1]$, and $g' \in \mathcal{CP}(I)$, $f'/g' \in \mathcal{CM}(I)$, then, using the Fundamental Theorem of Calculus, one can verify the equality $\mathcal{M}_{f',g';\mu} = C_{f,g}$. Therefore, the classes of two-variable Bajraktarević and Cauchy means form subclasses of generalized quasiarithmetic means.

The equality problem of means in various classes of two-variable means has been investigated and solved by now. We refer here to Losonczi's works [11], [12], [13], [14], [15], where the equality of two-variable means is characterized in various settings. A key idea in these papers, under high order differentiability assumptions, is to calculate and then to compare the partial derivatives of the means at diagonal points of the Cartesian product $I \times I$. The mixed equality problem of quasiarithmetic and Lagrangian means was solved by Páles [24].

The equality problem of generalized quasiarithmetic means with the same probability measure μ, i.e., the characterization of those pairs of functions (f, g) and (h, k) such that

$$\mathcal{M}_{f,g;\mu} = \mathcal{M}_{h,k;\mu}$$

holds, was investigated and partially solved in the paper [18]. In the particular cases $\mu = \frac{\delta_0+\delta_1}{2}$ and $\mu = \lambda$ of these results, the equality problem of two-variable Bajraktarević and the equality problem of Cauchy means was solved under 6^{th}-order differentiability assumptions.

The aim of this paper is to study the equality problem of generalized quasiarithmetic means with the possibly different probability measures μ and ν. In other words, we aim to characterize those pairs of functions (f, g) and (h, k) such that

$$\mathcal{M}_{f,g;\mu} = \mathcal{M}_{h,k;\nu}.$$

The investigation of this functional equation will require 8^{th}-order differentiability assumptions. Our approach is to compute the higher-order directional derivatives of the two means at the diagonal points of I^2 and then the equality of these derivatives results an 8^{th}-order system of differential equations for the unknown functions f, g, h, and k. In order to be able to integrate these equations, we will assume that the measures are symmetric with respect to the midpoint of the interval $[0, 1]$. Due to this symmetry, the odd-order directional derivatives will vanish and therefore, the information then is derived only from the equality of the even-order directional derivatives. In total, we have four unknown functions, therefore we need to obtain four differential equations. This explains why it is necessary to perform the differentiation up to the order 8. The final main goal is to solve the system of differential equations so obtained when $\mu = \frac{\delta_0+\delta_1}{2}$ and $\nu = \lambda$. This is exactly the problem of equality of two-variable Bajraktarević means to Cauchy means. As a consequence of this result, it will follow that the intersection of these two classes of means consists of quasiarithmetic means.

The paper is organized as follows. In the next section we introduce the terminology related to measures and their moments. Then we consider a two-parameter subclass of symmetric measures which will include the two basic measures $\mu = \frac{\delta_0+\delta_1}{2}$ and $\mu = \lambda$ that are needed for the description of Bajraktarević and Cauchy means. In the third section, among others, we introduce the functions $\Phi_{f,g}$ and $\Psi_{f,g}$ (in terms of the generalized Wronskians) which will allow us to reduce the degree of the forthcoming system of differential equations from 8 to 6. The recursive formula among the generalized Wronskians is also established here. In Section 4, for any diagonal point of I^2, we introduce a single variable function related to the mean $\mathcal{M}_{f,g;\mu}$ for which we plan to compute the even-order derivatives up to the order 8. The computation involves the standard Leibniz Rule, the Faá di Bruno Formulas for Bell polynomials and the formulas established in Section 3 for the generalized Wronskian. In Section 5, we start analyzing the four differential equations obtained from the comparison of the 2^{nd}, 4^{th}, 6^{th}, and 8^{th}-order directional derivatives. Our main result is presented in Theorem 5.9, where all the pieces of the previous steps are put together and we obtain several equivalent characterizations of the equality of Bajraktarević and Cauchy means. Here, beyond the previously established auxiliary results we use the characterization of the equality of Bajraktarević and Cauchy means to quasiarithmetic means from the papers [26] and [19], respectively. We know that some of our computations cannot be checked by an easy calculation. Thus, one may use computer algebra to verify them, as it was also done by the authors.

2. Auxiliary results on measures

We say that a Borel probability measure μ on $[0,1]$ is symmetric if $\mu(A) = \mu(1-A)$ for all Borel sets $A \subseteq [0,1]$. The collection of all such measures on $[0,1]$ will be denoted by $\mathcal{SM}([0,1])$. For $\mu \in \mathcal{SM}([0,1])$, we introduce its k^{th} *centralized moment* as follows

$$\mu_k := \int_0^1 (t - \tfrac{1}{2})^k d\mu(t) \qquad (k \in \mathbb{N} \cup \{0\}).$$

Obviously, $\mu_0 = 1$, and the even order centralized moments are nonnegative. On the other hand, due to the symmetry of the measure, the odd-order ones are equal to zero.

In what follows we define a two-parameter class of $\mathcal{SM}([0,1])$ which will be instrumental for our investigations. For two given positive parameters ℓ and p, we define a probability measure $\pi := \pi(\ell, p)$ via the following equalities for its centralized moments

$$\pi_0 := 1, \quad \pi_{2n-1} := 0, \quad \pi_{2n} := \frac{(2n)!}{n!} \ell^n p^{\langle n \rangle} \qquad (n \in \mathbb{N}),$$

where, the modified power $p^{\langle n \rangle}$ is defined by

$$p^{\langle n \rangle} := \prod_{i=0}^{n-1} \frac{p}{1+ip} \qquad (n \in \mathbb{N}).$$

Clearly, $p^{\langle 1 \rangle} = p^1 = p$ and the recursive formula $p^{\langle n+1 \rangle} = \frac{p}{1+np} \cdot p^{\langle n \rangle}$ holds.

By a classical result related to the Hausdorff moment problem, the measure $\pi(\ell, p)$ is uniquely determined, however, it may not exist for every $\ell, p > 0$. It also follows that $\pi(\ell, p)$ has to be a symmetric measure with respect to the point $\frac{1}{2}$. The set of those parameters (ℓ, p) for which the probability measure $\pi(\ell, p)$ exists will be denoted by Π.

Lemma 2.1. *For the parameter set* Π*, we have the following inclusion*

$$\Pi \subseteq \,]0, \tfrac{1}{16}] \times \,]0, 2].$$

Proof. Let $(\ell, p) \in \Pi$. Then, due to the estimate $|t - \frac{1}{2}| \le \frac{1}{2}$, we get

$$\frac{(2n)!}{n!} \ell^n p^{\langle n \rangle} = \pi_{2n}(\ell, p) = \int_{[0,1]} (t - \tfrac{1}{2})^{2n} d\pi(\ell, p)(t) \le \int_{[0,1]} \frac{1}{2^{2n}} d\pi(\ell, p)(t) = \frac{1}{4^n}$$

for all $n \in \mathbb{N}$. This inequality yields

$$4\ell \le \sqrt[n]{\frac{n!}{(2n)! p^{\langle n \rangle}}}. \tag{2.1}$$

In order to compute the limit of the right hand side, we shall use the multiplicative version of the Cesaro–Stolz theorem. Denote by a_n the expression under the nth root on the right hand side of (2.1). Then, according to this classical result, we get

$$4\ell \le \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)(1+np)}{(2n+2)(2n+1)p} = \frac{1}{4},$$

which implies the inequality $\ell \le \frac{1}{16}$.

To prove the inequality $p \le 2$, we apply the Cauchy–Bunyakovski–Schwartz inequality:

$$\begin{aligned}\big(\pi_2(\ell,p)\big)^2 &= \Big(\int\limits_{[0,1]} (t-\tfrac{1}{2})^2 \cdot 1\, d\pi(\ell,p)(t)\Big)^2 \\ &\le \Big(\int\limits_{[0,1]} (t-\tfrac{1}{2})^4 d\pi(\ell,p)(t)\Big)\Big(\int\limits_{[0,1]} 1\, d\pi(\ell,p)(t)\Big) = \pi_4(\ell,p),\end{aligned}$$

which reduces to

$$(2\ell p)^2 \le \frac{12(\ell p)^2}{1+p}.$$

This simplifies to the inequality $p \le 2$. $\square$

Lemma 2.2. *Let $\tau \in [0, \frac{1}{2}[$. Then*

$$\tfrac{1}{2}(\delta_\tau + \delta_{1-\tau}) = \pi\big((\tfrac{1-2\tau}{4})^2, 2\big) \qquad \textit{and} \qquad \tfrac{1}{1-2\tau}\lambda|_{[\tau,1-\tau]} = \pi\big((\tfrac{1-2\tau}{4})^2, \tfrac{2}{3}\big),$$

where λ denotes the standard Lebesgue measure restricted to $[0,1]$.

Proof. Denote $\mu := \frac{1}{2}(\delta_\tau + \delta_{1-\tau})$. Then, $\mu_0 = 1$, $\hat{\mu}_1 = \frac{1}{2}$ and, for $n \in \mathbb{N}$, we have

$$\mu_n = \int\limits_{[0,1]} (t-\tfrac{1}{2})^n d\mu(t) = \frac{(\tau-\frac{1}{2})^n + (1-\tau-\frac{1}{2})^n}{2} = \begin{cases} (\frac{1}{2}-\tau)^n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, if $p = 2$, $\ell = (\frac{1-2\tau}{4})^2$, then, for all $n \in \mathbb{N}$, $\pi_{2n-1}((\frac{1-2\tau}{4})^2, 2) = 0 = \mu_{2n-1}$ and

$$\pi_{2n}((\tfrac{1-2\tau}{4})^2, 2) = \frac{(2n)!}{n!}\ell^n p^{\langle n\rangle} = \frac{(2n)! \cdot (2\ell)^n}{n! \cdot 1 \cdot 3 \cdots (2n-1)} = (4\ell)^n = (\tfrac{1}{2}-\tau)^{2n} = \mu_{2n}.$$

Therefore, all the moments of μ and $\pi((\frac{1-2\tau}{4})^2, 2)$ are the same, which proves that $\mu = \pi((\frac{1-2\tau}{4})^2, 2)$.

For the second assertion, denote $\nu := \frac{1}{1-2\tau}\lambda|_{[\tau,1-\tau]}$. Then, $\mu_0 = 1$, $\hat{\mu}_1 = \frac{1}{2}$ and, for $n \in \mathbb{N}$, we have

$$\begin{aligned}\nu_n &= \int\limits_{[0,1]} (t-\tfrac{1}{2})^n d\nu(t) = \frac{1}{1-2\tau}\int\limits_{[\tau,1-\tau]} (t-\tfrac{1}{2})^n dt = \frac{1}{1-2\tau}\left[\frac{(t-\frac{1}{2})^{n+1}}{n+1}\right]_{t=\tau}^{t=1-\tau} \\ &= \frac{(1-\tau-\frac{1}{2})^{n+1} - (\tau-\frac{1}{2})^{n+1}}{(1-2\tau)(n+1)} = \begin{cases} \frac{(\frac{1}{2}-\tau)^n}{n+1} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}\end{aligned}$$

On the other hand, if $p = \frac{2}{3}$, $\ell = (\frac{1-2\tau}{4})^2$, then, for all $n \in \mathbb{N}$,

$$\pi_{2n}((\tfrac{1-2\tau}{4})^2, \tfrac{2}{3}) = \frac{(2n)!}{n!}\ell^n p^{\langle n\rangle} = \frac{(2n)! \cdot (2\ell)^n}{n! \cdot 3 \cdot 5 \cdots (2n+1)} = \frac{(4\ell)^n}{2n+1} = \frac{(\frac{1}{2}-\tau)^{2n}}{2n+1} = \nu_{2n}.$$

Therefore, all the moments of ν and $\pi((\frac{1-2\tau}{4})^2, \frac{2}{3})$ are the same, which proves that $\nu = \pi((\frac{1-2\tau}{4})^2, \frac{2}{3})$. $\square$

3. Auxiliary results on generalized quasiarithmetic means

We introduce the following notations:

$$\begin{aligned}\mathcal{C}_0(I) &:= \{(f,g)\,|\,f,g : I \to \mathbb{R},\ g \in \mathcal{CP}(I),\ f/g \in \mathcal{CM}(I)\},\\ \mathcal{C}_n(I) &:= \{(f,g)\,|\,(f,g) \in \mathcal{C}_0(I),\ f,g \text{ are } n\text{-times continuously}\\ &\qquad \text{differentiable such that } f'g - fg' \text{ does not vanish anywhere}\} \qquad (n \in \mathbb{N}).\end{aligned}$$

Whenever $n \geq 2$ and $(f,g) \in \mathcal{C}_n(I)$ and $i,j \in \{0,\dots,n\}$, then we define

$$W^{i,j}_{f,g} := \begin{vmatrix} f^{(i)} & f^{(j)} \\ g^{(i)} & g^{(j)} \end{vmatrix}, \qquad \Phi_{f,g} := \frac{W^{2,0}_{f,g}}{W^{1,0}_{f,g}} \qquad \text{and} \qquad \Psi_{f,g} := -\frac{W^{2,1}_{f,g}}{W^{1,0}_{f,g}}.$$

We can now restate [18, Lemma 1]:

Lemma 3.1. *Let $n \geq 2$ and $(f,g) \in \mathcal{C}_n(I)$ and define $(\varphi_0,\dots,\varphi_n)$ and $(\psi_0,\dots,\psi_n)$ by*

$$\begin{aligned} &\varphi_0 := 0, \qquad \varphi_{i+1} := \varphi_i' + \varphi_i\Phi_{f,g} + \psi_i \qquad (i \in \{0,\dots,n-1\}),\\ &\psi_0 := 1, \qquad \psi_{i+1} := \varphi_i\Psi_{f,g} + \psi_i' \qquad (i \in \{0,\dots,n-1\}).\end{aligned} \tag{3.1}$$

Then

$$W^{i,j}_{f,g} = \begin{vmatrix} \varphi_i & \varphi_j \\ \psi_i & \psi_j \end{vmatrix} \cdot W^{1,0}_{f,g} \qquad (i,j \in \{0,\dots,n\}). \tag{3.2}$$

In particular,

$$W^{i,0}_{f,g} = \varphi_i W^{1,0}_{f,g}, \qquad W^{i,1}_{f,g} = -\psi_i W^{1,0}_{f,g}, \qquad W^{i,2}_{f,g} = (\varphi_i\Psi_{f,g} - \psi_i\Phi_{f,g})W^{1,0}_{f,g} \qquad (i \in \{0,\dots,n\}). \tag{3.3}$$

For small i, we have:

$$\begin{aligned} &\varphi_1 = 1, \qquad \varphi_2 = \Phi_{f,g}, \qquad \varphi_3 = \Phi'_{f,g} + \Phi^2_{f,g} + \Psi_{f,g},\\ &\psi_1 = 0, \qquad \psi_2 = \Psi_{f,g}, \qquad \psi_3 = \Phi_{f,g}\Psi_{f,g} + \Psi'_{f,g}.\end{aligned}$$

The subsequent elements can easily be computed by (3.1). We shall need the following consequence of (3.1).

Lemma 3.2. *Under the same assumptions as in Lemma 3.1, for the sequences (φ_i) and (ψ_i), we have*

$$\begin{aligned} &\varphi_{i+2} = \varphi_i'' + 2\varphi_i'\Phi_{f,g} + \varphi_i\varphi_3 + 2\psi_i' + \psi_i\Phi_{f,g} \qquad (i \in \{0,\dots,n-2\}),\\ &\psi_{i+2} = 2\varphi_i'\Psi_{f,g} + \varphi_i\psi_3 + \psi_i'' + \psi_i\Psi_{f,g} \qquad (i \in \{0,\dots,n-2\}).\end{aligned} \tag{3.4}$$

Proof. Applying the recursion (3.1) twice, for $i \in \{0,\dots,n-2\}$, we get

$$\begin{aligned} \varphi_{i+2} &= \varphi_{i+1}' + \varphi_{i+1}\Phi_{f,g} + \psi_{i+1}\\ &= (\varphi_i' + \varphi_i\Phi_{f,g} + \psi_i)' + (\varphi_i' + \varphi_i\Phi_{f,g} + \psi_i)\Phi_{f,g} + \varphi_i\Psi_{f,g} + \psi_i',\\ &= \varphi_i'' + 2\varphi_i'\Phi_{f,g} + \varphi_i\varphi_3 + 2\psi_i' + \psi_i\Phi_{f,g},\\ \psi_{i+2} &= \varphi_{i+1}\Psi_{f,g} + \psi_{i+1}'\\ &= (\varphi_i' + \varphi_i\Phi_{f,g} + \psi_i)\Psi_{f,g} + (\varphi_i\Psi_{f,g} + \psi_i')'\\ &= 2\varphi_i'\Psi_{f,g} + \varphi_i\psi_3 + \psi_i'' + \psi_i\Psi_{f,g},\end{aligned}$$

which completes the proof of (3.4). □

The next statement is instrumental for the computation of higher-order derivatives of means.

Lemma 3.3. ([16, Lemma 1], [26, Lemma 1.1]) *Let $(f,g) \in \mathcal{C}_0(I)$ and $\mu \in \mathcal{SM}([0,1])$. Then, for all $x, y \in I$, the unique solution z of the equation*

$$\int\limits_{[0,1]} \begin{vmatrix} f\big(tx+(1-t)y\big) & f(z) \\ g\big(tx+(1-t)y\big) & g(z) \end{vmatrix} d\mu(t) = 0 \tag{3.5}$$

equals $\mathcal{M}_{f,g;\mu}(x,y)$.

We say that $(f,g),(h,k) \in \mathcal{C}_0(I)$ *are equivalent pairs*, denoted by $(f,g) \sim (h,k)$ if h and k are linear combinations of f and g. As we have seen it in [26, Theorem 2.1], $(f,g) \sim (h,k)$ holds if and only if $\Phi_{f,g} = \Phi_{h,k}$ and $\Psi_{f,g} = \Psi_{h,k}$. It also easily follows from Lemma 3.3 that equivalent pairs of generating functions determine identical means, i.e., $\mathcal{M}_{f,g;\mu} = \mathcal{M}_{h,k;\mu}$ holds whenever $(f,g) \sim (h,k)$.

4. Higher-order directional derivatives of generalized quasiarithmetic means

In view of [18, Lemma 5], we have that $\mathcal{M}_{f,g;\mu}$ is n-times continuously differentiable on $I \times I$ provided that $(f,g) \in \mathcal{C}_n(I)$ and $\mu \in \mathcal{SM}([0,1])$. For $(f,g) \in \mathcal{C}_0(I)$ and $x \in I$, we construct $m_x : 2(I-x) \cap 2(x-I) \to \mathbb{R}$ by

$$m_x(u) := m_{x;f,g;\mu}(u) := \mathcal{M}_{f,g;\mu}\big(x + \tfrac{1}{2}u, x - \tfrac{1}{2}u\big). \tag{4.1}$$

The symmetry of μ yields that m_x is even, that is, $m_x(u) = m_x(-u)$ holds for $u \in U_x := 2(I-x) \cap 2(x-I)$.

In the subsequent results, the derivatives of any real function h up to the order 8 will be denoted by $h^{(i)}$ for all $i \in \{0,\dots,8\}$. Alternatively, $h^{(0)}, h^{(1)}, h^{(2)}$ and $h^{(3)}$, will be denoted by h, h', h'' and h''', respectively.

Proposition 4.1. *Let $n \in \mathbb{N}$, $(f,g) \in \mathcal{C}_n(I)$ and $\mu \in \mathcal{SM}([0,1])$. Then, for all fixed $x \in I$, the function m_x defined by (4.1) is n-times differentiable at $u = 0$. Furthermore, $m_x(0) = x$ and, in the cases $n \in \{1,3,5,7\}$, the equality $m_x^{(n)}(0) = 0$ holds and, in the cases $n \in \{2,4,6,8\}$, we have*

$$\begin{aligned}
m_x''(0) &= \mu_2\varphi_2(x),\\
m_x^{(4)}(0) &= \mu_4\varphi_4(x) - 3\mu_2^2(\varphi_2^3 + 2\psi_2\varphi_2)(x),\\
m_x^{(6)}(0) &= \mu_6\varphi_6(x) - 15\mu_4\mu_2(\varphi_4(\varphi_2^2+\psi_2) + \varphi_2\psi_4)(x) - 15\mu_2^3\varphi_2(\varphi_3\varphi_2^2 - 3(\varphi_2^2+\psi_2)(\varphi_2^2+2\psi_2))(x),\\
m_x^{(8)}(0) &= \mu_8\varphi_8(x) - 28\mu_6\mu_2(\varphi_6(\varphi_2^2+\psi_2) + \varphi_2\psi_6)(x) - 35\mu_4^2(\varphi_4^2\varphi_2 + 2\varphi_4\psi_4)(x)\\
&\quad + 210\mu_4\mu_2^2(\varphi_4(3\varphi_2^4 + \varphi_2^2(7\psi_2 - \varphi_3) + 2\psi_2^2) + 2\varphi_2\psi_4(\varphi_2^2 + 2\psi_2))(x)\\
&\quad - 105\mu_2^4(\varphi_4\varphi_2^4 + 15\varphi_2^7 + 2\varphi_2^3(5\varphi_2^2 + 6\psi_2)(6\psi_2 - \varphi_3) + 4\varphi_2(6\psi_2^3 - \varphi_2^3\psi_3))(x),
\end{aligned}$$

where $\varphi_i = \varphi_{i;f,g}$ $(i \in \{2,3,4,6,8\})$ and $\psi_i = \psi_{i;f,g}$ $(i \in \{2,3,4,6\})$ are given by (3.1).

Proof. Let $x \in I$ be fixed. The equality $m_x(0) = x$ is a direct consequence of (4.1). The n-times differentiability of m_x at $u = 0$ follows from [18, Lemma 5]. By the symmetry of the mean $\mathcal{M}_{f,g;\mu}$, $m_x : U_x \to \mathbb{R}$ is even. This implies that all odd-order derivatives of m_x vanish at the point $u = 0$.

For brevity, let F denote the vector valued map $\binom{f}{g} : I \to \mathbb{R}^2$. Then, for $u \in U_x$, (3.5) can be rewritten as

$$\int\limits_{[0,1]} \Big| F(x+(t-\tfrac{1}{2})u) \quad (F\circ m_x)(u) \Big| \, d\mu(t) = 0.$$

Performing an n^{th}-order differentiation with respect to u the standard Leibniz Rule implies

$$\sum_{i=0}^{n} \binom{n}{i} \int\limits_{[0,1]} \Big| F^{(i)}(x+(t-\tfrac{1}{2})u) \quad (F\circ m_x)^{(n-i)}(u) \Big| (t-\tfrac{1}{2})^i d\mu(t) = 0.$$

Now putting $u=0$, we get

$$\sum_{i=0}^{n} \binom{n}{i} \mu_i \Big| F^{(i)}(x) \quad (F\circ m_x)^{(n-i)}(0) \Big| = 0.$$

Using that the odd-order centralized moments of μ are equal to zero, this equality finally simplifies to

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} \mu_{2i} \Big| F^{(2i)}(x) \quad (F\circ m_x)^{(n-2i)}(0) \Big| = 0. \tag{4.2}$$

The evenness of m_x implies that $F\circ m_x$ is also even on U_x and hence all of its (existing) odd-order derivatives vanish at $u=0$. This shows that (4.2) is nontrivial only when n is even. To elaborate the condition (4.2) in the cases $n\in\{2,4,6,8\}$, we shall use Faá di Bruno's formula to compute $(F\circ m_x)^{(N)}$ as follows

$$(F\circ m_x)^{(N)}(0) := \sum_{k=1}^{N} (F^{(k)}\circ m_x)\cdot \mathcal{B}_{N,k}\Big(m_x'(0), m_x''(0), \dots, m_x^{(N-k+1)}(0)\Big). \tag{4.3}$$

Here $\mathcal{B}_{N,k}:\mathbb{R}^{N-k+1}\to\mathbb{R}$ is the incomplete Bell polynomial, which is defined by the recursive formula

$$\mathcal{B}_{N,k}(x_1,\dots,x_{N-k+1}) := \sum_{j=1}^{N-k+1} \binom{N-1}{j-1} x_j \mathcal{B}_{N-j,k-1}(x_1,\dots,x_{N-j-k+2}),$$

such that $\mathcal{B}_{0,0}=1$ and $\mathcal{B}_{N,0}=0=\mathcal{B}_{0,k}$ for all $k,N\in\mathbb{N}$. When applying the formula (4.3), we have that the odd-order derivatives of m_x are zero at $u=0$, therefore, we need to compute the Bell polynomials only for such arguments where the odd coordinates are equal to zero. Now, an easy calculation shows that, for all $N\in\{1,\dots,8\}$, $k\in\{1,\dots,N\}$ and argument $\boldsymbol{x}_i := (x_1,x_2,\dots,x_i)$ where $x_{2j-1}=0$ for all $j\in\{1,\dots,\lfloor\frac{i+1}{2}\rfloor\}$,

$$\mathcal{B}_{N,k}(\boldsymbol{x}_{N-k+1}) = 0 \qquad \text{if either } N \text{ is odd } \& \ k\in\{1,\dots,N\} \quad \text{or } N \text{ is even } \& \ k\in\{\tfrac{N+2}{2},\dots,N\},$$

and, for the remaining cases, we have

$$\begin{aligned}
&\mathcal{B}_{2,1}(\boldsymbol{x}_2) = x_2, \\
&\mathcal{B}_{4,1}(\boldsymbol{x}_4) = x_4, && \mathcal{B}_{4,2}(\boldsymbol{x}_3) = 3x_2^2, \\
&\mathcal{B}_{6,1}(\boldsymbol{x}_6) = x_6, && \mathcal{B}_{6,2}(\boldsymbol{x}_5) = 15x_2x_4, && \mathcal{B}_{6,3}(\boldsymbol{x}_4) = 15x_2^3, \\
&\mathcal{B}_{8,1}(\boldsymbol{x}_8) = x_8, && \mathcal{B}_{8,2}(\boldsymbol{x}_7) = 28x_2x_6+35x_4^2, && \mathcal{B}_{8,3}(\boldsymbol{x}_6) = 210x_2^2x_4, && \mathcal{B}_{8,4}(\boldsymbol{x}_5) = 105x_2^4.
\end{aligned}$$

These identities, together with formula (4.3), imply that

$$\begin{aligned}
(F\circ m_x)(0) &= F(x),\\
(F\circ m_x)''(0) &= F'(x)m_x''(0),\\
(F\circ m_x)^{(4)}(0) &= F'(x)m_x^{(4)}(0)+3F''(x)(m_x''(0))^2,\\
(F\circ m_x)^{(6)}(0) &= F'(x)m_x^{(6)}(0)+15F''(x)m_x''(0)m_x^{(4)}(0)+15F'''(x)(m_x''(0))^3,\\
(F\circ m_x)^{(8)}(0) &= F'(x)m_x^{(8)}(0)+F''(x)(28m_x''(0)m_x^{(6)}(0)+35(m_x^{(4)}(0))^2)\\
&\quad +210F'''(x)(m_x''(0))^2m_x^{(4)}(0)+105F^{(4)}(x)(m_x''(0))^4.
\end{aligned}\tag{4.4}$$

From now on, we shall use the notations φ_i and ψ_i which were introduced in Lemma 3.1. If $n=2$, then equation (4.2) yields

$$\begin{vmatrix} F(x) & (F\circ m_x)''(0)\end{vmatrix}+\mu_2\begin{vmatrix} F''(x) & (F\circ m_x)(0)\end{vmatrix}=0,$$

which, by the first formula in (4.4), simplifies to

$$-W_{f,g}^{1,0}(x)m_x''(0)+\mu_2W_{f,g}^{2,0}(x)=0.$$

This shows $m_x''(0)=\mu_2\varphi_2(x)$.

If $n=4$, then (4.2) states that

$$\begin{vmatrix} F(x) & (F\circ m_x)^{(4)}(0)\end{vmatrix}+6\mu_2\begin{vmatrix} F''(x) & (F\circ m_x)''(0)\end{vmatrix}+\mu_4\begin{vmatrix} F^{(4)}(x) & (F\circ m_x)(0)\end{vmatrix}=0,$$

which, by (4.4), gives

$$-W_{f,g}^{1,0}(x)m_x^{(4)}(0)-3W_{f,g}^{2,0}(x)(m_x''(0))^2+6\mu_2W_{f,g}^{2,1}(x)m_x''(0)+\mu_4W_{f,g}^{4,0}(x)=0.$$

Thus, applying (3.3) and the expression obtained for $m_x''(0)$, we get

$$\begin{aligned}
m_x^{(4)}(0) &= -3\varphi_2(x)(m_x''(0))^2-6\mu_2\psi_2(x)m_x''(0)+\mu_4\varphi_4(x)\\
&= -3\mu_2^2\varphi_2^3(x)-6\mu_2^2\psi_2(x)\varphi_2(x)+\mu_4\varphi_4(x).
\end{aligned}$$

If $n=6$, then (4.2) results

$$\begin{aligned}
&\begin{vmatrix} F(x) & (F\circ m_x)^{(6)}(0)\end{vmatrix}+15\mu_2\begin{vmatrix} F''(x) & (F\circ m_x)^{(4)}(0)\end{vmatrix}\\
&\qquad +15\mu_4\begin{vmatrix} F^{(4)}(x) & (F\circ m_x)''(0)\end{vmatrix}+\mu_6\begin{vmatrix} F^{(6)}(x) & (F\circ m_x)(0)\end{vmatrix}=0,
\end{aligned}$$

which, by (4.4), amounts to

$$\begin{aligned}
&-W_{f,g}^{1,0}(x)m_x^{(6)}(0)-15W_{f,g}^{2,0}(x)m_x''(0)m_x^{(4)}(0)-15W_{f,g}^{3,0}(x)(m_x''(0))^3\\
&\qquad +15\mu_2W_{f,g}^{2,1}(x)m_x^{(4)}(0)+15\mu_4W_{f,g}^{4,1}(x)m_x''(0)+\mu_6W_{f,g}^{6,0}(x)=0.
\end{aligned}$$

From here, using (3.3), we arrive at

$$\begin{aligned}
m_x^{(6)}(0) &= -15\varphi_2(x)m_x''(0)m_x^{(4)}(0)-15\varphi_3(x)(m_x''(0))^3\\
&\quad -15\mu_2\psi_2(x)m_x^{(4)}(0)-15\mu_4\psi_4(x)m_x''(0)+\mu_6\varphi_6(x).
\end{aligned}$$

By substituting the expressions obtained for $m_x''(0)$ and $m_x^{(4)}(0)$, this equality reduces to the assertion for $m_x^{(6)}(0)$.

Finally, if $n = 8$, then (4.2) implies

$$\begin{vmatrix} F(x) & (F\circ m_x)^{(8)}(0) \end{vmatrix} + 28\mu_2 \begin{vmatrix} F''(x) & (F\circ m_x)^{(6)}(0) \end{vmatrix} + 70\mu_4 \begin{vmatrix} F^{(4)}(x) & (F\circ m_x)^{(4)}(0) \end{vmatrix} + 28\mu_6 \begin{vmatrix} F^{(6)}(x) & (F\circ m_x)''(0) \end{vmatrix} + \mu_8 \begin{vmatrix} F^{(8)}(x) & (F\circ m_x)(0) \end{vmatrix} = 0.$$

Now, applying the identities (4.4), this gives

$$\begin{aligned} &-W^{1,0}_{f,g}(x)m_x^{(8)}(0) - W^{2,0}_{f,g}(x)(28m''_x(0)m_x^{(6)}(0) + 35(m_x^{(4)}(0))^2) - 210W^{3,0}_{f,g}(x)(m''_x(0))^2 m_x^{(4)}(0) \\ &- 105W^{4,0}_{f,g}(x)(m''_x(0))^4 + 28\mu_2 W^{2,1}_{f,g}(x)m_x^{(6)}(0) - 420\mu_2 W^{3,2}_{f,g}(x)(m''_x(0))^3 \\ &+ 70\mu_4 W^{4,1}_{f,g}(x)m_x^{(4)}(0) + 210\mu_4 W^{4,2}_{f,g}(x)(m''_x(0))^2 + 28\mu_6 W^{6,1}_{f,g}(x)m''_x(0) + \mu_8 W^{8,0}_{f,g}(x) = 0. \end{aligned}$$

From here, using (3.2) and (4.4), we get

$$\begin{aligned} m_x^{(8)}(0) = &-\varphi_2(x)(28m''_x(0)m_x^{(6)}(0) + 35(m_x^{(4)}(0))^2) - 210\varphi_3(x)(m''_x(0))^2 m_x^{(4)}(0) \\ &- 105\varphi_4(x)(m''_x(0))^4 - 28\mu_2\psi_2(x)m_x^{(6)}(0) - 420\mu_2(\varphi_3\psi_2 - \psi_3\varphi_2)(x)(m''_x(0))^3 \\ &- 70\mu_4\psi_4(x)m_x^{(4)}(0) + 210\mu_4(\varphi_4\psi_2 - \psi_4\varphi_2)(x)(m''_x(0))^2 - 28\mu_6\psi_6(x)m''_x(0) + \mu_8\varphi_8(x). \end{aligned}$$

This, together with the formulae for $m''_x(0), m_x^{(4)}(0)$, and $m_x^{(6)}(0)$, implies the required result. □

5. Necessary and sufficient conditions for the equality of generalized quasiarithmetic means

If $(f, g), (h, k) \in \mathcal{C}_0(I)$ and $\mu, \nu \in \mathcal{SM}([0, 1])$, then $\mathcal{M}_{f,g;\mu}$ and $\mathcal{M}_{h,k;\nu}$ are said to be *equal near the diagonal* $\Delta(I) := \{(x, x) \mid x \in I\}$ if, for some open set $U \subseteq I^2$ containing a dense subset D of $\Delta(I)$, they are equal at each pair belonging to U.

Lemma 5.1. ([18, Lemma 7]) *Let $\mu, \nu \in \mathcal{SM}([0, 1])$, $n \in \mathbb{N}$ and $(f, g), (h, k) \in \mathcal{C}_{2n}(I)$ and assume that $\mathcal{M}_{f,g;\mu}$ equals $\mathcal{M}_{h,k;\nu}$ near $\Delta(I)$. Then, for all $i \in \{1, \dots, n\}$ and $x \in I$,*

$$m^{(2i)}_{x;f,g;\mu}(0) = m^{(2i)}_{x;h,k;\nu}(0). \tag{5.1}$$

In the next statement, we consider first the particular case when $\mu = \nu$ and $\Psi_{f,g} = \Psi_{h,k}$, and then we characterize the equality of the means $\mathcal{M}_{f,g;\mu}$ and $\mathcal{M}_{h,k;\nu}$.

Theorem 5.2. *Let $\mu \in \mathcal{SM}([0, 1])$ with $\mu_2 > 0$ and $(f, g), (h, k) \in \mathcal{C}_2(I)$ such that $\Psi_{f,g} = \Psi_{h,k}$. Then the following assertions are equivalent:*

(i) $\mathcal{M}_{f,g;\mu}$ *and* $\mathcal{M}_{h,k;\mu}$ *are equal on* I^2*.*
(ii) $\mathcal{M}_{f,g;\mu}$ *and* $\mathcal{M}_{h,k;\mu}$ *are equal near* $\Delta(I)$*.*
(iii) $m''_{x;f,g;\mu}(0) = m''_{x;h,k;\mu}(0)$ *holds for all* $x \in I$*.*
(iv) $\Phi_{f,g} = \Phi_{h,k}$ *holds on* I*.*
(v) $(f, g) \sim (h, k)$ *holds.*

Proof. The assertion (i)⇒(ii) is clear. The assertion (ii)⇒(iii) follows from Lemma 5.1. If (iii) holds, then, by Proposition 4.1, we get $\mu_2\Phi_{f,g} = \mu_2\Phi_{h,k}$, which reduces to the equality $\Phi_{f,g} = \Phi_{h,k}$ proving (iv). Finally, [26, Theorem 2.1] implies that $(f, g) \sim (h, h)$ which yields both of the implications (iv)⇒(v) and (v)⇒(i). □

Due to the symmetry of the measure μ, the third-order necessary condition $m'''_{x;f,g;\mu}(0) = m'''_{x;h,k;\mu}(0)$ does not imply the equality $\Psi_{f,g} = \Psi_{h,k}$ (as it is obtained in [18]). Therefore, in the sequel, we want to consider problems where either the measures μ and ν are different or the equality $\Psi_{f,g} = \Psi_{h,k}$ cannot be established. In view of Proposition 4.1 and Lemma 3.1, without any further assumption on the measures μ and ν, we can easily see that the ith equation in (5.1) is a nonlinear differential equation of order $2(i-1)$ for the unknown functions $\Phi_{f,g}$, $\Psi_{f,g}$, $\Phi_{h,k}$, and $\Psi_{h,k}$. Therefore, for the solution of the equality problem of the means, it seems to be enough to take these equations for $i \in \{1, 2, 3, 4\}$ and solve the system of differential equations so obtained. However, the integration of this system in its full generality seems to be hopeless. Therefore, we take μ and ν from the two-parameter family of measures $\{\pi(\ell, p) : (\ell, p) \in \Pi\}$ introduced in Section 2, for which we will be able to derive a solution method.

In the subsequent lemmas, we will take the following additional hypothesis:

($\mathcal{H}$) $\mu = \pi(\ell, p)$ and $\nu = \pi(\ell, q)$ for some $(\ell, p), (\ell, q) \in \Pi$.

Lemma 5.3. *Assume ($\mathcal{H}$), let $n \geq 2$ and $(f, g), (h, k) \in \mathcal{C}_n(I)$ and define $V := \big|W^{1,0}_{f,g}\big|^{-p}$. Then V is $(n-1)$-times continuously differentiable. If, for all $x \in I$,*

$$m''_{x;f,g;\mu}(0) = m''_{x;h,k;\nu}(0) \tag{5.2}$$

holds, then

$$q\Phi_{h,k} = p\Phi_{f,g} = -\frac{V'}{V} =: \Phi. \tag{5.3}$$

Proof. If $(f, g), (h, k) \in \mathcal{C}_n(I)$, then $W^{1,0}_{f,g}$ is a nowhere zero function which is $(n-1)$-times continuously differentiable, hence so is V.

Due to the formula for the second-order derivative in Proposition 4.1 and Lemma 3.1, we have that

$$m''_{x;f,g;\mu}(0) = \mu_2\varphi_{2;f,g}(x) = \mu_2\Phi_{f,g} = 2p\ell\Phi_{f,g}, \qquad m''_{x;h,k;\nu}(0) = \nu_2\varphi_{2;h,k}(x) = \nu_2\Phi_{h,k} = 2q\ell\Phi_{h,k}.$$

Thus, the equality (5.2) yields the first equality in (5.3). From the definition of V, we get that

$$V' = (-p)\big|W^{1,0}_{f,g}\big|^{-p}\frac{W^{2,0}_{f,g}}{W^{1,0}_{f,g}} = (-V)p\Phi_{f,g} = (-V)\Phi.$$

This proves the second equality in (5.3). □

As an easy consequence of the equality $V' = -V\Phi$, we get the following statement.

Lemma 5.4. *Under the notations of the previous lemma and appropriate differentiability assumptions, we have*

$$\begin{aligned}
V' &= -V\Phi, \\
V'' &= V(\Phi^2 - \Phi'), \\
V''' &= V(-\Phi^3 + 3\Phi'\Phi - \Phi''), \\
V^{(4)} &= V(\Phi^4 - 6\Phi'\Phi^2 + 4\Phi''\Phi + 3\Phi'^2 - \Phi'''), \\
V^{(5)} &= V(-\Phi^5 + 10\Phi'\Phi^3 - 15\Phi'^2\Phi - 10\Phi''\Phi^2 + 10\Phi''\Phi' + 5\Phi'''\Phi - \Phi^{(4)}).
\end{aligned}$$

Proof. Using the equality $V' = -V\Phi$, we recursively get

$$\begin{aligned}
V'' &= (-V\Phi)' = -V'\Phi - V\Phi' = V(\Phi^2 - \Phi'),\\
V''' &= (V(\Phi^2 - \Phi'))' = V'(\Phi^2 - \Phi') + V(\Phi^2 - \Phi')' = V(-\Phi^3 + 3\Phi'\Phi - \Phi''),\\
V^{(4)} &= (V(-\Phi^3 + 3\Phi'\Phi - \Phi''))' = V'(-\Phi^3 + 3\Phi'\Phi - \Phi'') + V(-\Phi^3 + 3\Phi'\Phi - \Phi'')'\\
&= V(\Phi^4 - 6\Phi'\Phi^2 + 4\Phi''\Phi + 3\Phi'^2 - \Phi'''),\\
V^{(5)} &= (V(\Phi^4 - 6\Phi'\Phi^2 + 4\Phi''\Phi + 3\Phi'^2 - \Phi'''))'\\
&= V'(\Phi^4 - 6\Phi'\Phi^2 + 4\Phi''\Phi + 3\Phi'^2 - \Phi''') + V(\Phi^4 - 6\Phi'\Phi^2 + 4\Phi''\Phi + 3\Phi'^2 - \Phi''')'\\
&= V(-\Phi^5 + 10\Phi'\Phi^3 - 15\Phi'^2\Phi - 10\Phi''\Phi^2 + 10\Phi''\Phi' + 5\Phi'''\Phi - \Phi^{(4)}). \quad \square
\end{aligned}$$

Lemma 5.5. *Assume $(\mathcal{H})$, let $n \geq 4$ and $(f,g),(h,k) \in \mathcal{C}_n(I)$ and define $V := \big|W^{1,0}_{f,g}\big|^{-p}$. If, for all $x \in I$ and $i \in \{1,2\}$,*

$$m^{(2i)}_{x;f,g;\mu}(0) = m^{(2i)}_{x;h,k;\nu}(0) \tag{5.4}$$

is satisfied, then (5.3) holds and there exist a constant $c \in \mathbb{R}$ and an $(n-3)$-times continuously differentiable function $B : I \to \mathbb{R}$ such that

$$\Psi_{f,g} = \frac{B+c}{6p^{\langle 2\rangle}V} - \frac{(p-2)(\Phi' - \Phi^2)}{6p^2} \qquad \text{and} \qquad \Psi_{h,k} = \frac{B-c}{6q^{\langle 2\rangle}V} - \frac{(q-2)(\Phi' - \Phi^2)}{6q^2}. \tag{5.5}$$

Proof. It follows from Lemma 5.3 that V is $(n-1)$-times, i.e., at least 3-times continuously differentiable and the case $i = 1$ in (5.4) that (5.3) holds on I. Using the recursion (3.1), we obtain

$$\begin{aligned}
\varphi_{2;f,g} &= \Phi_{f,g} = \frac{\Phi}{p}, \qquad \psi_{2;f,g} = \Psi_{f,g},\\
\varphi_{3;f,g} &= \varphi'_{2;f,g} + \varphi_{2;f,g}\Phi_{f,g} + \psi_{2;f,g} = \frac{\Phi'}{p} + \frac{\Phi^2}{p^2} + \Psi_{f,g},\\
\psi_{3;f,g} &= \varphi_{2;f,g}\Psi_{f,g} + \psi'_{2;f,g} = -\frac{V'}{pV}\Psi_{f,g} + \Psi'_{f,g},\\
\varphi_{4;f,g} &= \varphi'_{3;f,g} + \varphi_{3;f,g}\Phi_{f,g} + \psi_{3;f,g} = \frac{\Phi''}{p} + \frac{3\Phi'\Phi}{p^2} + \frac{\Phi^3}{p^3} - \frac{2V'}{pV}\Psi_{f,g} + 2\Psi'_{f,g}.
\end{aligned} \tag{5.6}$$

These formulae, together with $\mu = \pi(\ell, p)$ and the formula for the fourth-order derivative by Proposition 4.1, give

$$\begin{aligned}
m^{(4)}_{x;f,g;\mu}(0) &= \mu_4\varphi_{4;f,g}(x) - 3\mu_2^2(\varphi^3_{2;f,g} + 2\psi_{2;f,g}\varphi_{2;f,g})(x)\\
&= 12\ell^2\bigg(-\frac{p^{\langle 2\rangle}V'''}{pV} + \frac{3V''V'}{V^2} - \frac{2V'^3}{V^3} + \frac{2p^{\langle 2\rangle}(V\Psi_{f,g})'}{V}\bigg)(x)\\
&= 12\ell^2\bigg(\frac{p\Phi''}{1+p} + \frac{3\Phi'\Phi}{1+p} - \frac{\Phi^3}{1+p} + \frac{2p^2}{1+p}\frac{(V\Psi_{f,g})'}{V}\bigg)(x).
\end{aligned}$$

Similarly, using that $\nu = \pi(\ell, q)$, we get

$$m^{(4)}_{x;h,k;\nu}(0) = 12\ell^2\bigg(\frac{q\Phi''}{1+q} + \frac{3\Phi'\Phi}{1+q} - \frac{\Phi^3}{1+q} + \frac{2q^2}{1+q}\frac{(V\Psi_{h,k})'}{V}\bigg)(x).$$

Therefore, the case $i = 2$ in (5.4) reduces to

$$2p^{\langle 2\rangle}(V\Psi_{f,g})' + \frac{p^{\langle 2\rangle}}{p}V(\Phi'' - 3\Phi'\Phi + \Phi^3) = 2q^{\langle 2\rangle}(V\Psi_{h,k})' + \frac{q^{\langle 2\rangle}}{q}V(\Phi'' - 3\Phi'\Phi + \Phi^3),$$

or equivalently, using that $V' = -V\Phi$,

$$2p^{\langle 2\rangle}(V\Psi_{f,g})' + \frac{p^{\langle 2\rangle}}{p}(V(\Phi' - \Phi^2))' = 2q^{\langle 2\rangle}(V\Psi_{h,k})' + \frac{q^{\langle 2\rangle}}{q}(V(\Phi' - \Phi^2))'.$$

Hence, after integration, for some real constant c,

$$2p^{\langle 2\rangle}V\Psi_{f,g} + \frac{p^{\langle 2\rangle}}{p}V(\Phi' - \Phi^2) - \frac{c}{3} = 2q^{\langle 2\rangle}V\Psi_{h,k} + \frac{q^{\langle 2\rangle}}{q}V(\Phi' - \Phi^2) + \frac{c}{3}.$$

Denote by C the function standing on the left hand side of this equality. The regularity assumptions imply that C is $(n-3)$-times continuously differentiable. Now defining B as $B := 3C - 2V(\Phi' - \Phi^2)$, we can see that B is also $(n-3)$-times continuously differentiable and the above equality yields (5.5). □

Lemma 5.6. *Assume $(\mathcal{H})$, let $n \geq 6$ and $(f, g), (h, k) \in \mathcal{C}_n(I)$ and define $V := \left|W^{1,0}_{f,g}\right|^{-p}$. If, for all $x \in I$ and $i \in \{1, 2, 3\}$, the equality (5.4) is satisfied, then the equality (5.3) holds and the identities (5.5) are valid for some real constant c and an $(n-3)$-times continuously differentiable function $B : I \to \mathbb{R}$. In addition, we get*

$$(p-q)\left(\frac{5pq - p - q - 4}{(1+p)(1+q)}V^{(5)}V + (2B''V + B'V')' + B'B\right) = c(4pq + 3p + 3q + 2)B'. \tag{5.7}$$

In particular, if $p = q$, then cB is a constant and if $p \neq q$ and either $5pq = p + q + 4$ or V is an at most 4-degree polynomial, then there exist real constants d, e such that

$$(B')^2V = -\frac{B^3}{6} + \frac{c(4pq + 3p + 3q + 2)}{2(p-q)}B^2 + dB + e. \tag{5.8}$$

Proof. The equalities (5.3) and (5.5) are consequences of Lemma 5.3 and Lemma 5.5, respectively. Using the recursive formulas (3.1), (3.4), the formulas from (5.6), and the first identity in (5.5), we get

$$\begin{aligned}
\psi_{4;f,g} &= \varphi_{3;f,g}\Psi_{f,g} + \psi'_{3;f,g} \\
&= \left(\frac{2\Phi'}{p} + \frac{\Phi^2}{p^2}\right)\Psi_{f,g} + \Psi^2_{f,g} + \frac{\Phi}{p}\Psi'_{f,g} + \Psi''_{f,g}, \\
\varphi_{6;f,g} &= \varphi''_{4;f,g} + 2\varphi'_{4;f,g}\Phi_{f,g} + \varphi_{4;f,g}\varphi_{3;f,g} + 2\psi'_{4;f,g} + \psi_{4;f,g}\Phi_{f,g} \\
&= \frac{\Phi^{(4)}}{p} + \frac{5\Phi'''\Phi}{p^2} + \frac{10\Phi''\Phi'}{p^2} + \frac{10\Phi''\Phi^2}{p^3} + \frac{15\Phi'^2\Phi}{p^3} + \frac{10\Phi'\Phi^3}{p^4} + \frac{\Phi^5}{p^5} + \frac{3\Phi}{p}\Psi^2_{f,g} \\
&\quad + \left(\frac{7\Phi''}{p} + \frac{15\Phi'\Phi}{p^2} + \frac{4\Phi^3}{p^3}\right)\Psi_{f,g} + \left(\frac{12\Phi'}{p} + \frac{9\Phi^2}{p^2}\right)\Psi'_{f,g} + 6\Psi'_{f,g}\Psi_{f,g} + \frac{9\Phi}{p}\Psi''_{f,g} + 4\Psi'''_{f,g}.
\end{aligned} \tag{5.9}$$

Combining these identities with $\mu = \pi(\ell, p)$ and the formula for the sixth-order derivative by Proposition 4.1 yields

$$\begin{aligned}
m^{(6)}_{x;f,g;\mu}(0) &= \mu_6\varphi_{6;f,g}(x) - 15\mu_4\mu_2(\varphi_{4;f,g}(\varphi^2_{2;f,g} + \psi_{2;f,g}) + \varphi_{2;f,g}\psi_{4;f,g})(x) \\
&\quad - 15\mu_2^3\varphi_{2;f,g}(\varphi_{3;f,g}\varphi^2_{2;f,g} - 3(\varphi^2_{2;f,g} + \psi_{2;f,g})(\varphi^2_{2;f,g} + 2\psi_{2;f,g}))(x) \\
&= 120\ell^3 p^{\langle 3\rangle}\Big(\frac{\Phi^{(4)}}{p} + \frac{5\Phi'''\Phi}{p^2} + \frac{10\Phi''\Phi'}{p^2} + \frac{(7-6p)\Phi''\Phi^2}{p^3} + \frac{15\Phi'^2\Phi}{p^3} - \frac{(2p+21)\Phi'\Phi^3}{p^3} + \frac{4\Phi^5}{p^3} \\
&\quad + 12p\Phi\Psi^2_{f,g} + \frac{2\Psi_{f,g}}{p}((2-3p)\Phi'' - 15\Phi'\Phi + 8\Phi^3) + \frac{6\Psi'_{f,g}}{p}(2\Phi' - 3\Phi^2) - 12p\Psi'_{f,g}\Psi_{f,g} \\
&\quad + \frac{6(1-p)\Psi''_{f,g}}{p}\Phi + 4\Psi'''_{f,g}\Big)(x).
\end{aligned} \tag{5.10}$$

Using the first identity in (5.5) and $V' = -V\Phi$, we recursively get

$$\begin{aligned}
\Psi'_{f,g} &= -\frac{p-2}{6p^2}(\Phi'' - 2\Phi'\Phi) + \frac{B' + (B+c)\Phi}{6p^{\langle 2\rangle}V}, \\
\Psi''_{f,g} &= -\frac{p-2}{6p^2}(\Phi''' - 2\Phi''\Phi - 2\Phi'^2) + \frac{B'' + 2B'\Phi + (B+c)(\Phi' + \Phi^2)}{6p^{\langle 2\rangle}V}, \\
\Psi'''_{f,g} &= -\frac{p-2}{6p^2}(\Phi^{(4)} - 2\Phi'''\Phi - 6\Phi''\Phi') + \frac{B''' + 3B''\Phi + 3B'(\Phi' + \Phi^2) + (B+c)(\Phi'' + 3\Phi'\Phi + \Phi^3)}{6p^{\langle 2\rangle}V}.
\end{aligned} \tag{5.11}$$

Now, substituting these identities into (5.10), we arrive at

$$\begin{aligned}
m^{(6)}_{x;f,g;\mu}(0) &= \frac{40\ell^3}{(1+p)(1+2p)}\Big(p(p+4)(\Phi^{(4)} + \Phi^5) + (7p^2 - 2p + 6)(\Phi^{(3)}\Phi + 2\Phi''\Phi') \\
&\quad - (8p^2 - 13p + 9)\Phi''\Phi^2 - 3(p^2 - 11p + 3)\Phi'^2\Phi - (4p^2 + 31p - 3)\Phi'\Phi^3 \\
&\quad + \frac{(p+1)}{V}\Big(2pB''' + (7p+4)B'\Phi'\Big) + \frac{(p+1)^2}{V}\Big(3B''\Phi - B'\Phi^2 - \frac{B'(B+c)}{V}\Big)\Big)(x).
\end{aligned}$$

Similar argument applies to the case $\nu = \pi(\ell, q)$, we have

$$\begin{aligned}
m^{(6)}_{x;h,k;\nu}(0) &= \frac{40\ell^3}{(1+q)(1+2q)}\Big(q(q+4)(\Phi^{(4)} + \Phi^5) + (7q^2 - 2q + 6)(\Phi^{(3)}\Phi + 2\Phi''\Phi') \\
&\quad - (8q^2 - 13q + 9)\Phi''\Phi^2 - 3(q^2 - 11q + 3)\Phi'^2\Phi - (4q^2 + 31q - 3)\Phi'\Phi^3 \\
&\quad + \frac{(q+1)}{V}\Big(2qB''' + (7q+4)B'\Phi'\Big) + \frac{(q+1)^2}{V}\Big(3B''\Phi - B'\Phi^2 - \frac{B'(B-c)}{V}\Big)\Big)(x).
\end{aligned}$$

Now the case $i = 3$ in (5.4) simplifies to

$$\begin{aligned}
(p-q)\Big(\frac{(5pq - p - q - 4)}{(p+1)(q+1)}V^2\Big(-(\Phi^{(4)} + \Phi^5) + 5(\Phi^{(3)}\Phi + 2\Phi''\Phi') - 10\Phi''\Phi^2 - 15\Phi'^2\Phi + 10\Phi'\Phi^3\Big) \\
+ 2B''' - 3B''\Phi + B'(\Phi^2 - \Phi') + B'B\Big) = c(4pq + 3p + 3q + 2)B'.
\end{aligned}$$

Using the first, second and last identities from Lemma 5.4, we can easily see that this equality is exactly equivalent to (5.7).

It is easily seen that the case $p = q$ implies that cB is a constant. On the other hand, if either $5pq = p+q+4$ or V is an at most 4-degree polynomial, then identity (5.7) reduces to

$$(2B''V + B'V')' + B'B = \frac{c(4pq + 3p + 3q + 2)B'}{(p-q)}.$$

Thus, after integration for some real constant d, we get the following first-order inhomogeneous linear differential equation for the function V:

$$2B''V + B'V' + \frac{B^2}{2} = \frac{c(4pq+3p+3q+2)B}{(p-q)} + d. \tag{5.12}$$

Multiplying this equality by B' side by side, the equation so obtained is again integrable, therefore there exists a constant e such that formula (5.8) holds. □

Lemma 5.7. *Under the same assumptions of the previous lemma provided that the function $B : I \to \mathbb{R}$ is 5-times continuously differentiable and if $p \neq q$ and either $5pq = p + q + 4$ or V is an at most 4-degree polynomial. Then we have*

$$\begin{aligned}
B''V &= \frac{B'V\Phi}{2} + R\circ B,\\
B'''V &= \frac{B'V}{4}(2\Phi' + \Phi^2) + \frac{3\Phi}{2}(R\circ B) + B'(R'\circ B),\\
B^{(4)}V &= \frac{B'V}{8}\big(4\Phi'' + 6\Phi'\Phi + \Phi^3\big) + \Big(\frac{7\Phi^2}{4} + 2\Phi'\Big)(R\circ B) + 3B'\Phi(R'\circ B) + \frac{1}{V}((R'R)\circ B)\\
&\quad + B'^2(R''\circ B),\\
B^{(5)}V &= \frac{B'V}{16}\big(8\Phi''' + 16\Phi''\Phi + 12\Phi'^2 + 12\Phi'\Phi^2 + \Phi^4\big) + \frac{1}{8}(20\Phi'' + 50\Phi'\Phi + 15\Phi^3)(R\circ B)\\
&\quad + \frac{B'}{4}\big(25\Phi^2 + 20\Phi'\big)(R'\circ B) + \frac{5\Phi}{V}((R'R)\circ B) + \frac{B'}{V}(R'^2\circ B)\\
&\quad + \frac{3B'}{V}(R''R\circ B) + 5B'^2\Phi(R''\circ B),
\end{aligned}$$

where R of at most second degree polynomial which takes the form

$$R(u) := -\frac{u^2}{4} + \frac{c(4pq+3p+3q+2)u}{2(p-q)} + \frac{d}{2}.$$

Proof. Using Lemma 5.6, we get that identity (5.12) is valid. Thus, using $V' = -V\Phi$, we obtain

$$\begin{aligned}
B''V &= \frac{B'V\Phi}{2} - \frac{B^2}{4} - \frac{c(4pq+3p+3q+2)B}{2(p-q)} + \frac{d}{2} = \frac{B'V\Phi}{2} + R\circ B,\\
B'''V &= -B''V' + \frac{B''V\Phi}{2} + \frac{B'V'\Phi}{2} + \frac{B'V\Phi'}{2} + (R'\circ B)B'\\
&= \frac{B'V}{4}(2\Phi' + \Phi^2) + \frac{3\Phi}{2}(R\circ B) + B'(R'\circ B),\\
B^{(4)}V &= -B'''V' + \frac{B''V}{4}(2\Phi' + \Phi^2) + \frac{B'V'}{4}(2\Phi' + \Phi^2) + \frac{B'V}{4}(2\Phi'' + 2\Phi\Phi')\\
&\quad + \frac{3\Phi'}{2}(R\circ B) + \frac{3\Phi}{2}(R'\circ B)B' + (R''\circ B)B'^2 + (R'\circ B)B''\\
&= \Big(\frac{B'V}{4}(2\Phi' + \Phi^2) + \frac{3\Phi}{2}(R\circ B) + (R'\circ B)B'\Big)\Phi\\
&\quad + \Big(\frac{B'V\Phi}{8} + \frac{R\circ B}{4}\Big)(2\Phi' + \Phi^2) - \frac{B'V\Phi}{4}(2\Phi' + \Phi^2) + \frac{B'V}{4}(2\Phi'' + 2\Phi\Phi')\\
&\quad + \frac{3\Phi'}{2}(R\circ B) + \frac{3\Phi}{2}(R'\circ B)B' + (R''\circ B)B'^2 + \frac{R'\circ B}{V}\Big(\frac{B'V\Phi}{2} + R\circ B\Big)
\end{aligned}$$

$$= \frac{B'V}{8}\big(4\Phi'' + 6\Phi'\Phi + \Phi^3\big) + \Big(\frac{7\Phi^2}{4} + 2\Phi'\Big)(R\circ B) + 3B'\Phi(R'\circ B) + \frac{1}{V}((R'R)\circ B)$$
$$+ B'^2(R''\circ B),$$

and

$$\begin{aligned}
B^{(5)}V &= -B^{(4)}V' + \frac{B''V + B'V'}{8}\big(4\Phi'' + 6\Phi'\Phi + \Phi^3\big) + \frac{B'V}{8}\big(4\Phi''' + 6\Phi''\Phi + 6\Phi'^2 + 3\Phi'\Phi^2\big)\\
&\quad + \Big(\frac{7\Phi'\Phi}{2} + 2\Phi''\Big)(R\circ B) + \Big(\frac{7\Phi^2}{4} + 2\Phi'\Big)(R'\circ B)B' + 3(B''\Phi + B'\Phi')(R'\circ B)\\
&\quad + 3B'\Phi(R''\circ B)B' - \frac{V'}{V^2}((R'R)\circ B) + \frac{1}{V}((R''R + R'^2)\circ B)B' + 2B''B'(R''\circ B)\\
&= \Big(\frac{B'V}{8}\big(4\Phi'' + 6\Phi'\Phi + \Phi^3\big) + \Big(\frac{7\Phi^2}{4} + 2\Phi'\Big)(R\circ B) + 3B'\Phi(R'\circ B) + \frac{1}{V}((R'R)\circ B)\\
&\quad + B'^2(R''\circ B)\Big)\Phi + \frac{2R\circ B - B'V\Phi}{16}\big(4\Phi'' + 6\Phi'\Phi + \Phi^3\big) + \frac{B'V}{8}\big(4\Phi''' + 6\Phi''\Phi + 6\Phi'^2 + 3\Phi'\Phi^2\big)\\
&\quad + \Big(\frac{7\Phi'\Phi}{2} + 2\Phi''\Big)(R\circ B) + \Big(\frac{7\Phi^2}{4} + 2\Phi'\Big)(R'\circ B)B' + 3\Big(\frac{\Phi}{V}\Big(\frac{B'V\Phi}{2} + R\circ B\Big) + B'\Phi'\Big)(R'\circ B)\\
&\quad + 3B'\Phi(R''\circ B)B' + \frac{\Phi}{V}((R'R)\circ B) + \frac{B'}{V}((R''R + R'^2)\circ B) + \frac{2B'}{V}\Big(\frac{B'V\Phi}{2} + R\circ B\Big)(R''\circ B)\\
&= \frac{B'V}{16}\big(8\Phi''' + 16\Phi''\Phi + 12\Phi'^2 + 12\Phi'\Phi^2 + \Phi^4\big) + \frac{1}{8}(20\Phi'' + 50\Phi'\Phi + 15\Phi^3)(R\circ B)\\
&\quad + \frac{B'}{4}\big(25\Phi^2 + 20\Phi'\big)(R'\circ B) + \frac{5\Phi}{V}((R'R)\circ B) + \frac{B'}{V}(R'^2\circ B) + \frac{3B'}{V}(R''R\circ B)\\
&\quad + 5B'^2\Phi(R''\circ B). \qquad \square
\end{aligned}$$

Lemma 5.8. *Assume* ($\mathcal{H}$) *with* $(p,q) = (2,\frac{2}{3})$, *let* $(f,g),(h,k) \in \mathcal{C}_8(I)$. *If, for all* $x \in I$ *and* $i \in \{1,2,3,4\}$, *the equality* (5.4) *is satisfied, then* $\Phi_{h,k} = 3\Phi_{f,g}$ *holds and there exist real constants* a, b *such that*

$$\Psi_{f,g} = a\Big(W_{f,g}^{1,0}\Big)^2 \qquad \text{and} \qquad \Psi_{h,k} = b\Big(W_{h,k}^{1,0}\Big)^{\frac{2}{3}} + \frac{1}{3}\Phi'_{h,k} - \frac{2}{9}\Phi_{h,k}^2. \tag{5.13}$$

Proof. The equality $\Phi_{h,k} = 3\Phi_{f,g}$ follows from Lemma 5.3 and we also have (5.3) with $V := \big(W_{f,g}^{1,0}\big)^{-2}$. Then V is seven times differentiable and hence Φ is six times differentiable. The validity of (5.4) for $i = 2$ and Lemma 5.5 imply the existence of a constant c and a five times differentiable function $B : I \to \mathbb{R}$ such that the equalities in (5.5) hold. The parameters $p = 2$ and $q = \frac{2}{3}$ also satisfy the condition $5pq = p + q + 4$, hence validity of (5.4) for $i = 3$ yields that we have the conclusions of Lemma 5.7 with

$$R(u) = -\frac{u^2}{4} + \frac{23cu}{4} + \frac{d}{2}. \tag{5.14}$$

Using the two-step recursion (3.4) for $i \in \{4,6\}$, we arrive at

$$\begin{aligned}
\psi_{6;f,g} &= 2\varphi'_{4;f,g}\Psi_{f,g} + \varphi_{4;f,g}\psi_{3;f,g} + \psi''_{4;f,g} + \psi_{4;f,g}\Psi_{f,g}\\
&= \Psi_{f,g}^{(4)} + \Psi'''_{f,g}\frac{\Phi}{p} + 7\Psi''_{f,g}\Psi_{f,g} + \Psi''_{f,g}\Big(\frac{\Phi^2}{p^2} + \frac{4\Phi'}{p}\Big) + \Psi'_{f,g}\Big(\frac{6\Phi''}{p} + \frac{7\Phi'\Phi}{p^2} + \frac{\Phi^3}{p^3}\Big) + 4\Psi'^2_{f,g}\\
&\quad + \Psi'_{f,g}\Psi_{f,g}\frac{9\Phi}{p} + \Psi_{f,g}\Big(\frac{4\Phi'''}{p} + \frac{9\Phi''\Phi}{p^2} + \frac{8\Phi'^2}{p^2} + \frac{9\Phi'\Phi^2}{p^3} + \frac{\Phi^4}{p^4}\Big) + \Psi_{f,g}^2\Big(\frac{6\Phi'}{p} + \frac{3\Phi^2}{p^2}\Big) + \Psi_{f,g}^3,
\end{aligned}$$

and

$$\begin{aligned}
\varphi_{8;f,g} &= \varphi''_{6;f,g} + 2\varphi'_{6;f,g}\Phi_{f,g} + \varphi_{6;f,g}\varphi_{3;f,g} + 2\psi'_{6;f,g} + \psi_{6;f,g}\Phi_{f,g} \\
&= \frac{\Phi^{(6)}}{p} + \frac{7\Phi^{(5)}\Phi}{p^2} + \Phi^{(4)}\Big(\frac{12\Phi'}{p^2} + \frac{21\Phi^2}{p^3}\Big) + 35\Phi'''\Big(\frac{\Phi''}{p^2} + \frac{3\Phi'\Phi}{p^3} + \frac{\Phi^3}{p^4}\Big) + \frac{70\Phi''^2\Phi}{p^3} \\
&\quad + 35\Phi''\Big(\frac{3\Phi'^2}{p^3} + \frac{6\Phi'\Phi^2}{p^4} + \frac{\Phi^4}{p^5}\Big) + \frac{105\Phi'^3\Phi}{p^4} + \frac{105\Phi'^2\Phi^3}{p^5} + \frac{21\Phi'\Phi^5}{p^6} + \frac{\Phi^7}{p^7} + \frac{4\Phi\Psi^3_{f,g}}{p} \\
&\quad + \Big(\frac{22\Phi''}{p} + \frac{42\Phi'\Phi}{p^2} + \frac{10\Phi^3}{p^3}\Big)\Psi^2_{f,g} + \Big(\frac{16\Phi^{(4)}}{p} + \frac{56\Phi'''\Phi}{p^2} + \frac{86\Phi''\Phi^2}{p^3} + \frac{112\Phi''\Phi'}{p^2} + \frac{128\Phi'^2\Phi}{p^3} \\
&\quad + \frac{70\Phi'\Phi^3}{p^4} + \frac{6\Phi^5}{p^5}\Big)\Psi_{f,g} + \Big(\frac{46\Phi'''}{p} + \frac{124\Phi''\Phi}{p^2} + \frac{90\Phi'^2}{p^2} + \frac{142\Phi'\Phi^2}{p^3} + \frac{20\Phi^4}{p^4}\Big)\Psi'_{f,g} \\
&\quad + \frac{40\Phi}{p}\Psi'^2_{f,g} + \Big(\frac{72\Phi'}{p} + \frac{48\Phi^2}{p^2}\Big)\Psi'_{f,g}\Psi_{f,g} + \Big(\frac{60\Phi''}{p} + \frac{124\Phi'\Phi}{p^2} + \frac{34\Phi^3}{p^3}\Big)\Psi''_{f,g} + \frac{52\Phi}{p}\Psi''_{f,g}\Psi_{f,g} \\
&\quad + \Big(\frac{44\Phi'}{p} + \frac{34\Phi^2}{p^2}\Big)\Psi'''_{f,g} + \frac{20\Phi}{p}\Psi^{(4)}_{f,g} + 12\Psi'_{f,g}\Psi^2_{f,g} + 48\Psi''_{f,g}\Psi'_{f,g} + 24\Psi'''_{f,g}\Psi_{f,g} + 6\Psi^{(5)}_{f,g}.
\end{aligned}$$

These identities, together with the formulas from (5.6) and (5.9), $\mu = \pi(\ell, 2)$, and the formula for the eighth-order derivative by Proposition 4.1, imply that

$$\begin{aligned}
m^{(8)}_{x;f,g;\mu}(0) &= 256\ell^4\Big(\varphi_{8;f,g}(x) - 28(\varphi_{6;f,g}(\varphi^2_{2;f,g} + \psi_{2;f,g}) + \varphi_{2;f,g}\psi_{6;f,g})(x) - 35(\varphi^2_{4;f,g}\varphi_{2;f,g} \\
&\quad + 2\varphi_{4;f,g}\psi_{4;f,g})(x) + 210(\varphi_{4;f,g}(3\varphi^4_{2;f,g} + \varphi^2_{2;f,g}(7\psi_{2;f,g} - \varphi_{3;f,g}) + 2\psi^2_{2;f,g}) \\
&\quad + 2\varphi_{2;f,g}\psi_{4;f,g}(\varphi^2_{2;f,g} + 2\psi_{2;f,g}))(x) - 105(\varphi_{4;f,g}\varphi^4_{2;f,g} + 15\varphi^7_{2;f,g} \\
&\quad + 2\varphi^3_{2;f,g}(5\varphi^2_{2;f,g} + 6\psi_{2;f,g})(6\psi_{2;f,g} - \varphi_{3;f,g}) + 4\varphi_{2;f,g}(6\psi^3_{2;f,g} - \varphi^3_{2;f,g}\psi_{3;f,g}))(x)\Big) \\
&= 16\ell^4\Big(8\Phi^{(6)} + 28\Phi^{(5)}\Phi + 14\Phi^{(4)}(6\Phi' - \Phi^2) + 35\Phi'''(4\Phi'' + 6\Phi'\Phi - 3\Phi^3) + 70\Phi''^2\Phi \\
&\quad + 70\Phi''\Phi'(3\Phi' - 7\Phi^2) + 105\Phi'^3\Phi - 630\Phi'^2\Phi^3 + 329\Phi'\Phi^5 - 34\Phi^7 - 8704\Phi\Psi^3_{f,g} \\
&\quad + 64(22\Phi'' + 119\Phi'\Phi - 85\Phi^3)\Psi^2_{f,g} - 16(6\Phi^{(4)} + 49\Phi'''\Phi + 77\Phi''\Phi' - 51\Phi''\Phi^2 + 117\Phi'^2\Phi \\
&\quad - 238\Phi'\Phi^3 + 51\Phi^5)\Psi_{f,g} + 8(46\Phi''' - 127\Phi''\Phi + 45\Phi'^2 + 360\Phi'\Phi^2 + 125\Phi^4)\Psi'_{f,g} \\
&\quad - 4352(\Phi' - 2\Phi^2)\Psi'_{f,g}\Psi_{f,g} - 2816\Phi\Psi'^2_{f,g} + 2432\Phi\Psi''_{f,g}\Psi_{f,g} - 8(10\Phi'' + 99\Phi'\Phi - 26\Phi^3)\Psi''_{f,g} \\
&\quad + 8(44\Phi' - 53\Phi^2)\Psi'''_{f,g} - 64\Phi\Psi^{(4)}_{f,g} + 32((272\Psi^2_{f,g} - 46\Psi''_{f,g})\Psi'_{f,g} - 44\Psi'''_{f,g}\Psi_{f,g} + 3\Psi^{(5)}_{f,g})\Big)(x).
\end{aligned} \tag{5.15}$$

Using the identities (5.11) and $V' = -V\Phi$, we get

$$\begin{aligned}
\Psi^{(4)}_{f,g} &= -\frac{p-2}{6p^2}(\Phi^{(5)} - 2\Phi^{(4)}\Phi - 8\Phi'''\Phi' - 6\Phi''^2) + \frac{1}{6p^{\langle 2\rangle}V}\Big(B^{(4)} + 4B'''\Phi + 6B''(\Phi' + \Phi^2) \\
&\quad + 3B'(\Phi'' + 3\Phi'\Phi + \Phi^3) + (B + c)(\Phi''' + 4\Phi''\Phi + 6\Phi'\Phi^2 + 3\Phi'^2 + \Phi^4)\Big), \\
\Psi^{(5)}_{f,g} &= -\frac{p-2}{6p^2}(\Phi^{(6)} - 2\Phi^{(5)}\Phi - 10\Phi^{(4)}\Phi' - 20\Phi'''\Phi'') + \frac{1}{6p^{\langle 2\rangle}V}\Big(B^{(5)} + 4B^{(4)}\Phi + 10B'''(\Phi' + \Phi^2) \\
&\quad + 9B''(\Phi'' + 3\Phi'\Phi + \Phi^3) + B'(4\Phi''' + 16\Phi''\Phi + 12\Phi'^2 + 21\Phi'\Phi^2 + 4\Phi^4) \\
&\quad + (B + c)(\Phi^{(4)} + 5\Phi'''\Phi + 10\Phi''\Phi' + 10\Phi''\Phi^2 + 15\Phi'^2\Phi + 10\Phi'\Phi^3 + \Phi^5)\Big).
\end{aligned}$$

Substituting these identities and (5.11) into (5.15), we have

$$m^{(8)}_{x;f,g;\mu}(0) = 16\ell^4\Bigg(8\Phi^{(6)} + 28\Phi^{(5)}\Phi + 14\Phi^{(4)}(6\Phi' - \Phi^2) + 35\Phi'''(4\Phi'' + 6\Phi'\Phi - 3\Phi^3) + 70\Phi''^2\Phi$$
$$+ 70\Phi''\Phi'(3\Phi' - 7\Phi^2) + 105\Phi'^3\Phi - 630\Phi'^2\Phi^3 + 329\Phi'\Phi^5 - 34\Phi^7 + \frac{B'''(164\Phi' + 35\Phi^2)}{V}$$
$$+ \frac{1}{V}\Big(B''(110\Phi'' + 345\Phi'\Phi - 61\Phi^3) + B'(106\Phi''' + 61\Phi''\Phi + 357\Phi'^2 - 321\Phi'\Phi^2 + 46\Phi^4)\Big)$$
$$- \frac{B+c}{V^2}\Big(22B''' + 51B''\Phi + B'(157\Phi' + 11\Phi^2) - \frac{17B'(B+c)}{V}\Big) + \frac{\Phi}{V}\Big(52B^{(4)} - \frac{90B'^2}{V}\Big)$$
$$+ \frac{1}{V}\Big(12B^{(5)} - \frac{23B''B'}{V}\Big)\Bigg)(x).$$

Similarly, using that $\nu = \pi(\ell, \frac{2}{3})$, we arrive at

$$m^{(8)}_{x;h,k;\nu}(0) = 16\ell^4\Bigg(8\Phi^{(6)} + 28\Phi^{(5)}\Phi + 14\Phi^{(4)}(6\Phi' - \Phi^2) + 35\Phi'''(4\Phi'' + 6\Phi'\Phi - 3\Phi^3) + 70\Phi''^2\Phi$$
$$+ 70\Phi''\Phi'(3\Phi' - 7\Phi^2) + 105\Phi'^3\Phi - 630\Phi'^2\Phi^3 + 329\Phi'\Phi^5 - 34\Phi^7 + \frac{B'''(205\Phi' + 380\Phi^2)}{3V}$$
$$+ \frac{1}{3V}\Big(5B''(74\Phi'' + 267\Phi'\Phi - 55\Phi^3) + B'(326\Phi''' + 187\Phi''\Phi + 1139\Phi'^2$$
$$- 1111\Phi'\Phi^2 + 162\Phi^4)\Big) - \frac{B-c}{3V^2}\Big(50B''' + 225B''\Phi + B'(575\Phi' + 25\Phi^2) - \frac{75B'(B-c)}{V}\Big)$$
$$+ \frac{70\Phi}{3V}\Big(2B^{(4)} - \frac{5B'^2}{V}\Big) + \frac{5}{3V}\Big(4B^{(5)} - \frac{B''B'}{V}\Big)\Bigg)(x).$$

Therefore, the case $i = 4$ in (5.4) reduces to

$$16B^{(5)}V + 16B^{(4)}V\Phi - 4B'''(4B + 29c) - 64B''B' - \frac{12B'(2B^2 - 21Bc + 2c^2)}{V} + 80B'^2\Phi$$
$$+ B'\Big(B(104\Phi' - 8\Phi^2) - c(1046\Phi' + 58\Phi^2)\Big) + B''(72B - 378c)\Phi + B'''\Big(112\Phi' - 100\Phi^2\Big)V$$
$$- B''\Big(40\Phi'' + 300\Phi'\Phi - 92\Phi^3\Big)V - B'\Big(8\Phi''' + 4\Phi''\Phi + 68\Phi'^2 - 148\Phi'\Phi^2 + 24\Phi^4\Big)V = 0.$$

Using Lemma 5.7, this identity simplifies to

$$\begin{aligned}&\Big(-16B' + 6\Phi(2B - 23c)\Big)(R \circ B) + B'\Big(12V(4\Phi' + \Phi^2) - (4B + 29c)\Big)(R' \circ B)\\ &+ 24\Phi((R'R) \circ B) + 4B'(R'^2 \circ B) + 12B'(R''R \circ B) + 12B'^2V\Phi\Big(2(R'' \circ B) + 1\Big)\\ &- 3B'(2B^2 - 21Bc + 2c^2) + 3B'V(2B - 23c)(4\Phi' + \Phi^2) = 0.\end{aligned} \tag{5.16}$$

Now, by (5.14), we get

$$R \circ B = -\frac{B^2}{4} + \frac{23cB}{4} + \frac{d}{2}, \qquad R' \circ B = -\frac{B}{2} + \frac{23c}{4} \qquad \text{and} \qquad R'' \circ B = -\frac{1}{2}.$$

Using these identities, then (5.16) reduces to

$$B'\big(5B^2 - 190cB - 81c^2 - 22d\big) = 0. \tag{5.17}$$

We show that this equality implies that B is constant on I. To the contrary, assume that B is nonconstant on I. Then B' is not identically zero, and hence $B'(x_0) \neq 0$ for some $x_0 \in I$. By the continuity of B', there exists an open subinterval J of I containing x_0 such that $B'(x) \neq 0$ for all $x \in J$. Then, for all $x \in J$, the equality (5.17) yields that

$$5B^2(x) - 190cB(x) - 81c^2 - 22d = 0.$$

That is, for all $x \in J$, the value $B(x)$ is equal to one of the roots of the second degree polynomial $5u^2 - 190cu - 81c^2 - 22d$. By the continuity of B, now it follows that B is constant on J, which yields $B'(x_0) = 0$, contradicting the choice of x_0. Therefore, B has to be a constant function. Then denoting $\frac{B+c}{6p^{\langle 2\rangle}}$ by a and $\frac{B-c}{6q^{\langle 2\rangle}}$ by b, the equalities in (5.5) simplify to (5.13). □

Now we are ready to present and prove our main result. We shall need a further notation. For $a \in \mathbb{R}$, define $S_a, C_a : \mathbb{R} \to \mathbb{R}$ as follows

$$S_a(x) := \begin{cases} \sin(\sqrt{-a}x) & \text{if } a < 0, \\ x & \text{if } a = 0, \\ \sinh(\sqrt{a}x) & \text{if } a > 0, \end{cases} \qquad \text{and} \qquad C_a(x) := \begin{cases} \cos(\sqrt{-a}x) & \text{if } a < 0, \\ 1 & \text{if } a = 0, \\ \cosh(\sqrt{a}x) & \text{if } a > 0. \end{cases}$$

Theorem 5.9. *Assume* $(\mathcal{H})$ *with* $(p,q) = (2, \frac{2}{3})$ *and let* $(f,g),(h,k) \in \mathcal{C}_8(I)$. *Then the following assertions are equivalent:*

(i) $\mathcal{M}_{f,g;\mu} = \mathcal{M}_{h,k;\nu}$.
(ii) $\mathcal{M}_{f,g;\mu}$ *and* $\mathcal{M}_{h,k;\nu}$ *are equal near* $\Delta(I)$.
(iii) $m^{(2i)}_{x;f,g;\mu}(0) = m^{(2i)}_{x;h,k;\nu}(0)$ *hold for all* $x \in I$ *and* $i \in \{1,2,3,4\}$.
(iv) $\Phi_{h,k} = 3\Phi_{f,g}$ *and, for some* $a, b \in \mathbb{R}$, *the equalities in* (5.13) *hold.*
(v) *For some* $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$,

$$\alpha f^2 + \beta fg + \gamma g^2 = 1 \qquad \text{and} \qquad \delta h^2 + \varepsilon hk + \zeta k^2 = \Big(W^{1,0}_{h,k}\Big)^{\frac{2}{3}} \tag{5.18}$$

and $W^{1,0}_{h,k} = \eta\Big(W^{1,0}_{f,g}\Big)^3$.
(vi) *For some polynomials* P *and* Q *of at most second degree which are positive on the range of* f/g *and* h/k, *respectively, and for some* $\eta, \rho \in \mathbb{R}$,

$$g = \frac{1}{\sqrt{P}} \circ \frac{f}{g}, \quad k = \left|\left(\frac{h}{k}\right)'\right|\left(\frac{1}{\sqrt{Q^3}} \circ \frac{h}{k}\right), \quad \text{and} \quad \left(\int \frac{1}{Q}\right) \circ \frac{h}{k} = \eta^{\frac{1}{3}}\left(\int \frac{1}{P}\right) \circ \frac{f}{g} + \rho. \tag{5.19}$$

(vii) $\mathcal{M}_{f,g;\mu} = \mathcal{A}_\varphi = \mathcal{M}_{h,k;\nu}$ *holds with* $\varphi := \int W^{1,0}_{f,g}$.
(viii) $\mathcal{M}_{f,g;\mu} = \mathcal{A}_\varphi = \mathcal{M}_{h,k;\nu}$ *holds for some* $\varphi \in \mathcal{CM}(I)$.
(ix) *For some differentiable function* $\varphi \in \mathcal{CM}(I)$ *and* $a, b \in \mathbb{R}$,

$$(f,g) \sim (S_a \circ \varphi, C_a \circ \varphi) \qquad \text{and} \qquad (h,k) \sim (\varphi' \cdot S_b \circ \varphi, \varphi' \cdot C_b \circ \varphi).$$

Proof. The assertion (i)⇒(ii) is trivial. The assertions (ii)⇒(iii) and (iii)⇒(iv) are consequences of Lemma 5.1 and Lemma 5.8, respectively.

Suppose that (iv) is valid for some $a, b \in \mathbb{R}$. Then, the integration of the identity $\Phi_{h,k} = 3\Phi_{f,g}$ implies

$$W^{1,0}_{h,k} = \eta\Big(W^{1,0}_{f,g}\Big)^3 \tag{5.20}$$

for some real constant η. Now the first identity in (5.13) and implication (iv)⇒(ii) of [25, Theorem 10] yield that there exist real constants α, β, γ such that the first equality in (5.18) holds. An easy computation, using the second identity in (5.13), implies that the expression

$$\frac{3W_{h,k}^{3,0} + 12W_{h,k}^{2,1}}{\left(W_{h,k}^{1,0}\right)^{\frac{5}{3}}} - 5\frac{\left(W_{h,k}^{2,0}\right)^2}{\left(W_{h,k}^{1,0}\right)^{\frac{8}{3}}}$$

is constant. Applying implication (vii)⇒(iv) of [19, Theorem 7], then there exist real constants $\delta, \varepsilon, \zeta$ such that the second equality in (5.18) is valid. Thus, we have proved that assertion (v) holds.

To prove (v)⇒(vi), assume that (v) is valid for some real constants $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$ and define

$$P(u) := \alpha u^2 + \beta u + \gamma \qquad \text{and} \qquad Q(u) := \delta u^2 + \varepsilon u + \zeta \qquad (u \in \mathbb{R}).$$

It is easily seen that the two identities in (5.18) are equivalent to the following two identities

$$P \circ \frac{f}{g} = \frac{1}{g^2} \qquad \text{and} \qquad Q \circ \frac{h}{k} = \frac{\left(W_{h,k}^{1,0}\right)^{\frac{2}{3}}}{k^2}, \tag{5.21}$$

respectively. Thus, P and Q are positive real polynomials on the range of the ratio functions f/g and h/k, respectively. The first equality in (5.19) is a result of the first identity in (5.21). The second equality in (5.19) is a consequence of $W_{h,k}^{1,0} = k^2(h/k)'$ and the second identity in (5.21). Furthermore, using (5.21), we have that

$$\left(\left(\int \frac{1}{P}\right) \circ \frac{f}{g}\right)' = \left(\frac{1}{P} \circ \frac{f}{g}\right) \cdot \left(\frac{f}{g}\right)' = g^2 \cdot \frac{W_{f,g}^{1,0}}{g^2} = W_{f,g}^{1,0}.$$

This identity, together with (5.21) and (5.20), implies

$$\left(\left(\int \frac{1}{Q}\right) \circ \frac{h}{k}\right)' = \left(\frac{1}{Q} \circ \frac{h}{k}\right) \cdot \left(\frac{h}{k}\right)' = \frac{k^2}{\left(W_{h,k}^{1,0}\right)^{\frac{2}{3}}} \cdot \frac{W_{h,k}^{1,0}}{k^2} = \left(W_{h,k}^{1,0}\right)^{\frac{1}{3}} = \eta^{\frac{1}{3}} W_{f,g}^{1,0}$$
$$= \eta^{\frac{1}{3}}\left(\left(\int \frac{1}{P}\right) \circ \frac{f}{g}\right)'.$$

Integrating both sides, then we arrive at the last equality in (5.19) for some real constant ρ. Therefore, assertion (vi) is valid. The implication (vi)⇒(v) is obvious by reversing all the implications in the previous calculation. Thus (vi) and (v) are shown to be equivalent.

To prove that (v) implies (vii). Assume that (v) holds for some real constants $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$. Then implication (ii)⇒(iii) of [25, Theorem 10] and implication (iv)⇒(vi) of [19, Theorem 7] imply that

$$\mathcal{M}_{f,g;\mu} = \mathcal{A}_\varphi \qquad \text{and} \qquad \mathcal{M}_{h,k;\nu} = \mathcal{A}_\psi,$$

respectively, hold on I^2 with $\varphi = \int W_{f,g}^{1,0}$ and $\psi = \int \left(W_{h,k}^{1,0}\right)^{\frac{1}{3}}$. The identity (5.20) gives $\psi = \eta^{\frac{1}{3}}\varphi$, from which we get that $\mathcal{A}_\varphi = \mathcal{A}_\psi$ is valid on I^2. Consequently, (vii) holds.

The implications (vii)⇒(viii) and (viii)⇒(i) are straightforward. Therefore, all the assertions from (i) to (viii) are equivalent. Finally, the equivalence of (viii) and (ix) is a direct consequence of [25, Corollary 9] and [19], respectively. □

For the proof of the equivalence of the nine conditions of the theorem, we essentially needed the eight times differentiability of the generating functions f, g, h, k. On the other hand, one can observe that some particular implications can be verified under weaker regularity assumptions. The assertion (i) trivially follows from (viii) in the case $(f, g), (h, h) \in \mathcal{C}_0(I)$. It is an open problem whether the reversed implication is also true with this natural regularity assumption.

Finally, we can answer the question formulated in the introduction about two-variable means that are equal to a Bajraktarević mean and to a Cauchy mean at the same time.

Corollary 5.10. *Assume* $(\mathcal{H})$ *with* $(p, q) = (2, \frac{2}{3})$. *Then the intersection of the classes of means*

$$\{\mathcal{M}_{f,g;\mu} \mid (f, g) \in \mathcal{C}_8(I)\} \cap \{\mathcal{M}_{h,k;\nu} \mid (h, k) \in \mathcal{C}_8(I)\} \tag{5.22}$$

consists of the symmetric two-variable quasiarithmetic means. In other words, if a two-variable mean is simultaneously is a Bajraktarević mean and a Cauchy mean with eight times differentiable generators, then it has to be a quasiarithmetic mean.

Proof. Assume that a two-variable mean $M : I^2 \to I$ belongs to both of the classes of means. Then there exist $(f, g) \in \mathcal{C}_8(I)$ and $(h, k) \in \mathcal{C}_8(I)$ such that $M = \mathcal{M}_{f,g;\mu}$ and $M = \mathcal{M}_{h,k;\nu}$. Hence, assertion (i) of Theorem 5.9 holds. Then, by this theorem assertion (viii) is also valid, hence M has to be quasiarithmetic. □

As an application of this corollary, we can deduce the result of Alzer and Ruscheweyh.

Corollary 5.11. *Assume that a two-variable mean* $M : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is simultaneously equal to a Gini mean and to a Stolarsky mean. Then* M *has to be a power mean.*

Proof. If M equals to a Gini mean, then there exist real constants a, b such that $M = \mathcal{G}_{a,b}$, where $\mathcal{G}_{a,b}$ was defined by (1.1). This shows that M is homogeneous and it also equals a Bajraktarević mean with generators f and g defined by $f(x) := x^a$ and $g(x) := x^b$ if $a \neq b$ and $f(x) := x^a \log(x)$ if $a = b$.

If M also equals to a Stolarsky mean, then there exist real constants c, d such that $M = \mathcal{S}_{c,d}$, where $\mathcal{S}_{c,d}$ was defined by (1.2). Then M equals a Cauchy mean with obviously chosen generators. Thus, M belongs to the intersection (5.22) and hence it has to be a quasiarithmetic mean. Being also homogeneous, it follows that it has to be a power mean. □

References

[1] H. Alzer, St. Ruscheweyh, On the intersection of two-parameter mean value families, Proc. Am. Math. Soc. 129 (9) (2001) 2655–2662 (electronic), MR 2002c:26024.
[2] M. Bajraktarević, Sur une équation fonctionnelle aux valeurs moyennes, Glas. Mat.-Fiz. Astron. Druš. Mat. Fiz. Hrvat. Ser. II 13 (1958) 243–248, MR 23 #A442.
[3] M. Bajraktarević, Sur une généralisation des moyennes quasilinéaires, Publ. Inst. Math. (Beograd) (N. S.) 3 (17) (1963) 69–76, MR 30 #5077.
[4] P.S. Bullen, Handbook of Means and Their Inequalities, Mathematics and Its Applications, vol. 560, Kluwer Academic Publishers Group, Dordrecht, 2003, MR 2005a:26001.
[5] P.S. Bullen, D.S. Mitrinović, P.M. Vasić, Means and Their Inequalities, Mathematics and Its Applications (East European Series), vol. 31, D. Reidel Publishing Co., Dordrecht, 1988, translated and revised from the Serbo-Croatian, MR 89d:26003.
[6] C. Gini, Di una formula compressiva delle medie, Metron 13 (1938) 3–22.
[7] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, first edition, Cambridge University Press, Cambridge, 1934, second edition, 1952, MR 13,727e.
[8] E. Leach, M. Sholander, Extended mean values, Am. Math. Mon. 85 (2) (1978) 84–90.
[9] E. Leach, M. Sholander, Extended mean values II, J. Math. Anal. Appl. 92 (1983) 207–223.
[10] E. Leach, M. Sholander, Multivariable extended mean values, J. Math. Anal. Appl. 104 (1984) 390–407.
[11] L. Losonczi, Equality of two variable weighted means: reduction to differential equations, Aequ. Math. 58 (3) (1999) 223–241, MR 2001c:39046.
[12] L. Losonczi, Equality of Cauchy mean values, Publ. Math. (Debr.) 57 (2000) 217–230, MR 2001g:39051.

[13] L. Losonczi, Homogeneous Cauchy mean values, in: Z. Daróczy, Zs. Páles (Eds.), Functional Equations — Results and Advances, in: Advances in Mathematics, vol. 3, Kluwer Acad. Publ., Dordrecht, 2002, pp. 209–218, MR 2003f:39073.
[14] L. Losonczi, Equality of two variable Cauchy mean values, Aequ. Math. 65 (1–2) (2003) 61–81, MR 2004m:39039.
[15] L. Losonczi, Equality of two variable means revisited, Aequ. Math. 71 (3) (2006) 228–245, MR 2007e:39035.
[16] L. Losonczi, Zs. Páles, Comparison of means generated by two functions and a measure, J. Math. Anal. Appl. 345 (1) (2008) 135–146, MR 2009d:26041.
[17] L. Losonczi, Zs. Páles, Equality of two-variable functional means generated by different measures, Aequ. Math. 81 (1–2) (2011) 31–53, MR 2773088 (2012b:39044).
[18] L. Losonczi, Zs. Páles, A. Zakaria, On the equality of two-variable general functional means, Aequ. Math. 95 (2021).
[19] R.L. Lovas, Zs. Páles, A. Zakaria, Characterization of the equality of Cauchy means to quasiarithmetic means, J. Math. Anal. Appl. 484 (1) (2020) 123700, MR 4042621.
[20] D.S. Mitrinović, Analytic Inequalities, Springer-Verlag, Berlin–Heidelberg–New York, 1970.
[21] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Classical and New Inequalities in Analysis, Mathematics and Its Applications (East European Series), vol. 61, Kluwer Academic Publishers Group, Dordrecht, 1993, MR 94c:00004.
[22] Zs. Páles, Inequalities for differences of powers, J. Math. Anal. Appl. 131 (1) (1988) 271–281, MR 89f:26023.
[23] Zs. Páles, Comparison of two variable homogeneous means, in: W. Walter (Ed.), General Inequalities, 6, Oberwolfach, 1990, in: International Series of Numerical Mathematics, Birkhäuser, Basel, 1992, pp. 59–70, MR 94b:26016.
[24] Zs. Páles, On the equality of quasi-arithmetic and Lagrangian means, J. Math. Anal. Appl. 382 (1) (2011) 86–96, MR 2805497 (2012d:26034).
[25] Zs. Páles, A. Zakaria, Equality and homogeneity of generalized integral means, Acta Math. Hung. 160 (2) (2020) 412–443, MR 4075411.
[26] Zs. Páles, A. Zakaria, On the equality of Bajraktarević means to quasi-arithmetic means, Results Math. 75 (1) (2020) 19, MR 4045493.
[27] K.B. Stolarsky, Generalizations of the logarithmic mean, Math. Mag. 48 (1975) 87–92, MR 50 #10186.