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Norm and almost everywhere convergence and divergence of matrix transform means of Walsh-Fourier series

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Abstract
In this paper, we study the norm and almost everywhere convergence and divergence questions regarding subsequences of the general matrix-based T means $\sigma_n^T(f)$ of the Walsh-Fourier series. We give sufficient conditions that are in a certain sense optimal for convergence for every integrable function f, namely

$$\sum_{k=1}^{n} t_{k,n}^2 = O\left(\frac{1}{n}\right).$$

In comparison with previous results in this topic, we do not use the monotonicity of the defining sequences $(t_{k,n}, 1 \le k \le n, k \in \mathbb{P})$ of T. The T summation is a common generalization of several well-known summation methods, such as Fejér, Cesàro, Weierstrass, Riesz, Picard and Bessel, and Nörlund summation methods.

Keywords Walsh group · Walsh-Paley system · Walsh-Fourier series · Matrix transform means · Norm and almost everywhere convergences · Divergences in norm and almost everywhere

Mathematics Subject Classification 42C10

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1 Definitions and notations

Let $\mathbb{P}$ be the set of positive natural numbers and $\mathbb{N} := \mathbb{P} \cup \{0\}$. Let $\mathbb{Z}_2$ denote the discrete cyclic group of order 2. The group operation is the modulo 2 addition. We endow $\mathbb{Z}_2$ with the discrete topology, in which every subset is open. The normalized Haar measure μ on $\mathbb{Z}_2$ is given in the way that $\mu(\{0\}) = \mu(\{1\}) = 1/2$. That is, the measure of a singleton is $1/2$. $G := \mathop{\times}\limits_{k=0}^{\infty} \mathbb{Z}_2$, G is called the dyadic group. The elements of the dyadic group G are the 0, 1 sequences. That is, $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$ $(k \in \mathbb{N})$ and let $x2^n := (x_n, x_{n+1}, \dots)$.

For representing elements of G in the interval $[0, 1]$, we use the map

$$|x| := \sum_{k=0}^{\infty} x_k 2^{-(k+1)}.$$

Then $|x2^n| = \sum_{k=n}^{\infty} x_k 2^{n-(k+1)}$.

The group operation on G is the coordinate-wise addition (denoted by $+$), the normalized Haar measure μ is the product measure and the topology is the product topology. For another topology on the dyadic group see e.g. [5].

Dyadic intervals are defined in the usual way

$$I_0(x) := G, \quad I_n(x) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}$$

for $x \in G, n \in \mathbb{P}$. Denote $I_n := I_n(0)$ and $J_n := I_n \setminus I_{n+1}$ for any interval, where $n \in \mathbb{N}$. Intervals form a base for the neighbourhoods of G. Denote by $\mathcal{A}_n$ the σ-algebra generated by the intervals $I_n(x)$. That is, $\mathcal{A}_n := \{I_n(x) : x \in G\}$ $(n \in \mathbb{N})$. Throughout, we denote the complement of a set $I \subseteq G$ by $\bar{I} := G \setminus I$.

Let $L_p(G)$ denote the usual Lebesgue spaces on G (with the corresponding norm $\|.\|_p$).

For the sake of brevity in notation, we agree to write L_∞ instead of C and set $\|f\|_\infty := \sup\{|f(x)| : x \in G\}$. It is clear that the space L_∞ is not the same as the space of continuous functions, i.e. C is a proper subspace of L_∞.

But since in the case of continuous functions the supremum norm and the L_∞ norm are the same, for convenience we hope the reader will be able to tolerate this simplification in notation.

Now, we introduce some concepts of Walsh-Fourier analysis. The Rademacher functions are defined as

$$r_n(x) := (-1)^{x_n} \quad (x \in G, n \in \mathbb{N}).$$

The sequence of the Walsh-Paley functions is the product system of the Rademacher functions. Namely, every natural number n has a unique binary representation

$$n = \sum_{k=0}^{\infty} n_k 2^k, \quad n_k \in \{0, 1\} \ (k \in \mathbb{N}),$$

where only a finite number of n_k's are different from zero. We will use the notation $n^{(s)} := \sum_{k=s}^{\infty} n_k 2^k$, where $s \in \mathbb{N}$ and also that $n \oplus m = \sum_{k=0}^{\infty} |n_k - m_k| 2^k$ $(n, m \in \mathbb{N})$. Let the order of $n \in \mathbb{P}$ be denoted by $|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}$, meaning that $2^{|n|} \leq n < 2^{|n|+1}$.

The variation of $n \in \mathbb{N}$ is defined in the following way

$$V(n) := \sum_{k=1}^{\infty} |n_k - n_{k-1}| + n_0.$$

The Walsh-Paley functions are $w_0(x) := 1$ and for $n \in \mathbb{P}$

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{|n|} n_k x_k}.$$

It is known [13] that the Walsh-Paley system $(w_n, n \in \mathbb{N})$ is the character system of $(G, +)$.

The jth Fourier-coefficient, the kth partial sum of the Walsh-Fourier series and the nth Dirichlet kernel is defined as

$$\hat{f}(j) := \int_G f(x) w_j(x) d\mu(x),$$

$$S_k(f) := \sum_{j=0}^{k-1} \hat{f}(j) w_j,$$

$$D_n := \sum_{k=0}^{n-1} w_k.$$

From the definition of the Dirichlet kernel and the Walsh-Paley system, it is clear that

$$|D_n| \leq \sum_{k=0}^{n-1} |w_k| = n. \tag{1}$$

Let $T := \left(t_{k,n}\right)_{k,n=1}^{\infty}$ be a doubly infinite matrix of numbers. It is always assumed that the matrix T is upper triangular. Namely, $t_{k,n} := 0$, if $k > n$. Define the nth matrix transform mean determined by the matrix T as

$$\sigma_n^T(f) := \sum_{k=1}^{n} t_{k,n} S_k(f),$$

where $(t_{k,n}, 1 \leq k \leq n, k \in \mathbb{P})$ be a finite sequence of non-negative numbers for each $n \in \mathbb{P}$.

The nth matrix transform kernel is defined by

$$K_n^T := \sum_{k=1}^{n} t_{k,n} D_k.$$

It is easy to see that

$$\sigma_n^T(f;x) = \int_G f(u) K_n^T(u+x) d\mu(u),$$

where $x, u \in G$.

We say that the sequence $(d_n, n \in \mathbb{P})$ is lacunary, if there exists $1 < \lambda \in \mathbb{R}$, such that inequality $d_{n+1} \geq d_n \lambda$ holds for every $n \in \mathbb{P}$.

2 Introduction

The T summation methods (using matrix transforms means) are common generalizations of several well-known summation methods, such as Fejér, Cesàro, Weierstrass, Riesz, Picard and Bessel, and Nörlund summation methods. For details, see [2], [22], [11] and [23]. For matrix transforms means with respect to trigonometric system we mention the result of Chandra [6, Theorem 3] in the first place: Let f be a continuous and 2π-periodic function and $(t_{k,n} : 1 \leq k \leq n)$ monotone decreasing (in k). Set $b_{k,n} = \sum_{r=1}^{k} t_{r,n}$. Chandra proved the following estimate in the supremum norm:

$$\|\sigma_n^T(f) - f\| = O\left(\omega(\pi/n) + \sum_{k=1}^{n} k^{-1}\omega(\pi/k) b_{k,n}\right) \tag{2}$$

for the trigonometric system. ($\omega(.)$ is the modulus of continuity with respect to the supremum norm.) Leindler [15] improved this result for matrices T satisfying $\sum_{k=m}^{\infty} |t_{k,n} - t_{k+1,n}| \leq C t_{m,n}$. Totik proved [21] that it is not possible to give a better upper estimation in (2). For results concerning the Walsh system, see papers by Blyumin [4], Weisz [22, 23], and Blahota and Nagy [3].

For approximation and norm convergence (in L_1) results for matrix transforms means with respect to the Walsh system see papers of Blyumin [4], Weisz [22, 23], and Blahota and Nagy [3].

Most results mentioned in this section share the property that the defining sequences $(q_k,\ k \in \mathbb{N})$ (Nörlund means, see [7, 16, 18]), $(p_k,\ k \in \mathbb{P})$ (weighted means, see [17]) or $(t_{k,n},\ 1 \leq k \leq n,\ k \in \mathbb{P})$ for all fixed n (matrix transform means, see above) are monotonic. In this paper we do not suppose monotonicity of any kind.

For the time being, there is no known condition on the elements of the matrix T that would be necessary and sufficient for the almost everywhere or norm convergence of the σ_n^T means for an arbitrary integrable function, neither in the trigonometric nor in the Walsh case.

Since the sum of the nonnegative numbers $t_{k,n}$ equals 1, it is obvious that the sum of their squares is not bigger than 1. By the well-known inequality between the arithmetic and quadratic (root mean square) means we have $1/n \le \sum_{k=1}^{n} t_{k,n}^2$. In this paper, we prove that a sufficient condition for the almost everywhere and norm convergence (for some subsequences of) $\sigma_n^T(f) \to f$ for each integrable function f is that this sum of squares of $t_{k,n}$ should be essentially of order $1/n$ (i.e., $O(1/n)$).

In Sections 6 and 7 we prove divergence theorems with respect to subsequences of matrix means. For more divergence and convergence results regarding subsequences of Fourier sums see e.g. [14], [10] and [19].

In 1936, Zalcwasser raised the following problem concerning the trigonometric system: Is there a subsequence $(\delta_j : j \in \mathbb{P})$ of the sequence of the natural numbers and an integrable function f such that the arithmetic means of the partial sums $S_{\delta_j}(f)$ do not converge to f almost everywhere? This question is answered in the affirmative by one of the authors of this paper in [9]. It is quite natural to raise the same question with respect to the Walsh system as well. In the proof of Theorem 5 (among others) we give the construction of an integrable function f, and $t_{k,n} = T'_{k,n}/T'_n$ $(k, n \in \mathbb{N})$, where $T'_{k,n}$ is either 0 or 1 and the number of $T'_{k,n} = 1$ is T'_n. That is, the T-means $\sigma_n^T(f)$ looks like the form $\frac{1}{N}\sum_{j=1}^{N} S_{\delta_j}(f) = \sum_{k=1}^{n} \gamma_k/N S_k(f)$ and does not converge to f almost everywhere, where γ_k is either 0 or 1 and the number of $\gamma_k = 1$ is N. Moreover, we prove almost everywhere divergence. This result does not answer Zalcwasser's question (because in our counterexample the subsequence $\delta_1, \dots, \delta_N$ changes as n changes), but it leads us to conjecture that the answer in the Walsh case is the same as in the trigonometric case (see [9]).

3 Auxiliary results

Lemma 1 *[20] If $n \in \mathbb{N}$, then*

$$\frac{V(n)}{8} \le \|D_n\|_1 \le V(n).$$

Lemma 2 *[2] If $x \in J_v$ and $v, n \in \mathbb{N}$, then we have*

$$D_n(x) = w_{n^{(v)}}(x)\left(\sum_{k=0}^{v-1} n_k 2^k - n_v 2^v\right).$$

Lemma 3 *[20] Let $n \in \mathbb{N},\ x \in G$. Then, for the Walsh-Paley system, we have*

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}$$

Lemma 4 *Let $A, n \in \mathbb{N}$, $x \in G$. Then for Walsh-Paley system*

$$D_{A2^n}(x) = \begin{cases} 2^n D_A(x2^n), & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}$$

Proof Let $r, s \in \mathbb{N}$. Since $(r2^n)_s = 0$ for $0 \le s < n$ and $x_s = (x2^n)_{s-n}$, $(r2^n)_s = r_{s-n}$ for $s \ge n$, then

$$\begin{aligned} w_{r2^n}(x) &= (-1)^{\sum_{s=0}^{\infty} x_s (r2^n)_s} = (-1)^{\sum_{s=n}^{\infty} x_s (r2^n)_s} = (-1)^{\sum_{s=n}^{\infty} (x2^n)_{s-n} r_{s-n}} \\ &= (-1)^{\sum_{s=0}^{\infty} (x2^n)_s r_s} = w_r(x2^n). \end{aligned} \tag{3}$$

Using equation (3) we obtain

$$\begin{aligned} D_{A2^n}(x) &= \sum_{s=0}^{A2^n-1} w_s(x) = \sum_{j=0}^{2^n-1} \sum_{r=0}^{A-1} w_{r2^n+j}(x) = \sum_{j=0}^{2^n-1} w_j(x) \sum_{r=0}^{A-1} w_{r2^n}(x) \\ &= D_{2^n}(x) \sum_{r=0}^{A-1} w_r(x2^n) = D_{2^n}(x) D_A(x2^n). \end{aligned}$$

From this equality, the statement of Lemma 4 is derived using Lemma 3. □

Remark 1 We denote by c a positive absolute constant, which may vary from line to line.

4 Norm convergence

Theorem 1 *Suppose that $f \in L_1(G)$, and let $(t_{k,a_n}, 1 \le k \le a_n,\ k \in \mathbb{P})$ be a finite sequence of non-negative numbers, where $(a_n, n \in \mathbb{P})$ be a strictly increasing sequence of positive integers such that*

$$\sum_{k=1}^{a_n} t_{k,a_n} = 1$$

and

$$\sum_{k=1}^{a_n} t_{k,a_n}^2 = O\left(\frac{1}{a_n}\right) \tag{4}$$

are satisfied for every $n \in \mathbb{P}$.

Then $\sigma_{a_n}^T(f) \to f$ in the L_1-norm.

Theorem 2 *Let $(a_n, n \in \mathbb{P})$ be a strictly increasing sequence of positive integers and let $(t_{k,a_n}, 1 \le k \le a_n,\ k \in \mathbb{P})$ be a finite sequence of non-negative numbers satisfying*

$$\sum_{k=1}^{a_n} t_{k,a_n}^2 = O\left(\frac{1}{a_n}\right)$$

for every $n \in \mathbb{P}$. Then

$$\left\| K_{a_n}^T \right\|_1 \le c.$$

Proof Using inequality (1), we have

$$\begin{aligned}
\left\| K_{a_n}^T \right\|_1 &= \int_G \left| \sum_{k=1}^{a_n} t_{k,a_n} D_k \right| d\mu \\
&\le \int_{I_{|a_n|}} \sum_{k=1}^{a_n} t_{k,a_n} |D_k| d\mu + \int_{\bar{I}_{|a_n|}} \left| \sum_{k=1}^{a_n} t_{k,a_n} D_k \right| d\mu \\
&\le \mu\left(I_{|a_n|}\right) a_n + \int_{\bar{I}_{|a_n|}} \left| \sum_{s=0}^{|a_n|} \sum_{k=a_n^{(s+1)}+1}^{a_n^{(s)}} t_{k,a_n} D_k \right| d\mu \\
&< 2 + \sum_{s=0}^{|a_n|} \int_{\bar{I}_{|a_n|}} \left| \sum_{k=a_n^{(s+1)}+1}^{a_n^{(s)}} t_{k,a_n} D_k \right| d\mu \\
&\le 2 + \sum_{s=0}^{|a_n|} \sum_{v=0}^{|a_n|-1} \int_{J_v} \left| \sum_{k=a_n^{(s+1)}+1}^{a_n^{(s)}} t_{k,a_n} D_k \right| d\mu \\
&\le 2 + \sum_{v=0}^{|a_n|-1} \sum_{s=0}^{v} \int_{J_v} \left| \sum_{k=a_n^{(s+1)}+1}^{a_n^{(s)}} t_{k,a_n} D_k \right| d\mu \\
&\quad + \sum_{v=0}^{|a_n|-1} \sum_{s=v+1}^{|a_n|} \int_{J_v} \left| \sum_{k=a_n^{(s+1)}+1}^{a_n^{(s)}} t_{k,a_n} D_k \right| d\mu =: 2 + K_1 + K_2.
\end{aligned}$$

With the help of Lemma 2 and Cauchy–Bunyakovsky–Schwarz inequality, we get

$$\begin{aligned}
K_1 &\le \sum_{v=0}^{|a_n|-1} \sum_{s=0}^{v} \int_{J_v} \sum_{k=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} \left| D_{a_n^{(s+1)}+k} \right| d\mu \\
&\le \sum_{v=0}^{|a_n|-1} \sum_{s=0}^{v} \frac{1}{2^v} \sum_{k=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} 2^v
\end{aligned}$$

$$\leq \sum_{v=0}^{|a_n|-1} \sum_{s=0}^{v} \left(\sum_{k=1}^{2^s-1} 1 \sum_{k=1}^{2^s-1} t^2_{a_n^{(s+1)}+k,a_n} \right)^{1/2}$$

$$\leq c \sum_{v=0}^{|a_n|-1} \sum_{s=0}^{v} \frac{2^{s/2}}{a_n^{1/2}} \leq \frac{c}{a_n^{1/2}} \sum_{v=0}^{|a_n|-1} 2^{v/2} \leq c.$$

By Cauchy–Bunyakovsky–Schwarz inequality for integrals, using the notation 1_B for the characteristic function of the set $B \subseteq G$, we have

$$K_2 = \sum_{v=0}^{|a_n|-1} \sum_{s=v+1}^{|a_n|} \int_G 1_{J_v} \left| \sum_{k=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} D_{a_n^{(s+1)}+k} \right| d\mu$$

$$\leq \sum_{v=0}^{|a_n|-1} \sum_{s=v+1}^{|a_n|} \mu(J_v)^{1/2} \left(\int_{J_v} \left(\sum_{k=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} D_{a_n^{(s+1)}+k} \right)^2 d\mu \right)^{1/2}$$

$$\leq \sum_{v=0}^{|a_n|-1} \sum_{s=v+1}^{|a_n|} 2^{-v/2} \left(\int_{J_v} \left(\sum_{k=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} D_{a_n^{(s+1)}+k} \right)^2 d\mu \right)^{1/2}$$

holds. However, from Lemma 2, we have

$$\int_{J_v} \left(\sum_{k=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} D_{a_n^{(s+1)}+k} \right)^2 d\mu$$

$$= \int_{J_v} \sum_{k,l=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} t_{a_n^{(s+1)}+l,a_n} w_{a_n^{(s+1)}+k^{(v)}}(z) w_{a_n^{(s+1)}+l^{(v)}}(z)$$

$$\times \left(\sum_{j=0}^{v-1} k_j 2^j - k_v 2^v \right) \left(\sum_{j=0}^{v-1} l_j 2^j - l_v 2^v \right) d\mu(z)$$

$$\leq 2^{2v} \sum_{k,l=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} t_{a_n^{(s+1)}+l,a_n} \left| \int_{J_v} w_{k^{(v)}}(z) w_{l^{(v)}}(z) d\mu(z) \right| =: (*).$$

The integral $\int\limits_{J_v} w_{k^{(v)}}(z) w_{l^{(v)}}(z) d\mu(z)$ can be different from zero only in the case when

$$k^{(v)} \oplus l^{(v)} = 0,$$

so $k_v = l_v, k_{v+1} = l_{v+1}, \ldots$. Then we obtain the following upper estimate (using the inequality $\alpha\beta \le \alpha^2/2 + \beta^2/2$):

$$\begin{aligned}
(*) \le & 2^v \sum_{k=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} \sum_{\{l:l_j=k_j, j\ge v\}} t_{a_n^{(s+1)}+l,a_n} \\
= & 2^v \sum_{k=1}^{2^s-1} t_{a_n^{(s+1)}+k,a_n} \sum_{l=1}^{2^v} t_{a_n^{(s+1)}+k^{(v)}+l,a_n} \\
\le & 2^{v-1} \sum_{k=1}^{2^s-1} \sum_{l=1}^{2^v} \left(t^2_{a_n^{(s+1)}+k,a_n} + t^2_{a_n^{(s+1)}+k^{(v)}+l,a_n} \right) \\
= & 2^{2v-1} \sum_{k=1}^{2^s-1} t^2_{a_n^{(s+1)}+k,a_n} + 2^v \sum_{l=1}^{2^{v-1}} \sum_{k=1}^{2^s-1} t^2_{a_n^{(s+1)}+k^{(v)}+l,a_n} \\
\le & 2^{2v-1} \sum_{k=1}^{a_n} t^2_{k,a_n} + 2^{2v-1} \sum_{k=1}^{a_n} t^2_{k,a_n} \\
\le & c 2^{2v}/a_n.
\end{aligned}$$

From this

$$\begin{aligned}
K_2 \le & \sum_{v=0}^{|a_n|-1} \sum_{s=v+1}^{|a_n|} 2^{-v/2} 2^v c / a_n^{1/2} \\
= & \frac{c}{a_n^{1/2}} \sum_{v=0}^{|a_n|-1} (|a_n| - v) 2^{v/2} \le c.
\end{aligned}$$

□

Proof of Theorem 1 The proof is trivial from the Banach principle with respect to convergence of sequence of uniformly bounded operators and Theorem 2. □

5 Divergence in norm

Theorem 3 *Let $(\lambda_j, j \in \mathbb{P})$ be an arbitrary sequence of positive numbers tending to infinity. Then there exist a sequence $(m_j, j \in \mathbb{P})$ of positive integers tending monotonically to infinity, a function $f \in L_1(G)$, and a finite sequence of non-negative numbers $(t_{k,m_j}, 1 \le k \le m_j, \ k, j \in \mathbb{P})$ such that*

$$\sum_{k=1}^{m_j} t_{k,m_j} = 1 \tag{5}$$

holds for every $j \in \mathbb{P}$*,*

$$\sum_{k=1}^{m_j} t_{k,m_j}^2 = O\left(\frac{\lambda_{m_j}}{m_j}\right) \tag{6}$$

and

$$\sigma_{m_j}^T(f) \to \infty$$

in L_1*-norm, as* $j \to \infty$*.*

Proof Let $(A_s, s \in \mathbb{P})$ and $(L_s, s \in \mathbb{P})$ be sequences of natural numbers such that $A_s \geq L_s \nearrow \infty$, and suppose that the sequence $A_s - L_s$ is increasing; their exact definitions will be specified later.

Let $A_s, L_s \in \mathbb{P}$ be sequences of naturals such that $A_s \geq L_s \nearrow \infty$ and $A_s - L_s$ is increasing; their exact definitions will be specified later. For any $n \in \mathbb{P}$ we set

$$n_{(A_s,L_s)} := n_{A_s} 2^{A_s} + \cdots + n_{A_s - L_s} 2^{A_s - L_s},$$

where n_k are the binary digits of n. We shall always consider integers n for which $n_{A_s} = 1$.

It is clear, that

$$2^{A_s} \leq n_{(A_s,L_s)} \leq 2^{A_s+1}. \tag{7}$$

Let $k \in \mathbb{P}$ and

$$t_k' := \begin{cases} 1, & \text{if } k \in \cup_{s \in \mathbb{P}} \left[n_{(A_s,L_s)}, n_{(A_s,L_s)} + 2^{A_s - L_s}\right), \\ 0, & \text{otherwise.} \end{cases}$$

Let us define (only in this section) $T_n' := \sum_{k=1}^n t_k'$ and $t_{k,n} := t_k'/T_n'$, where $k, n \in \mathbb{P}$, $k \leq n$ and $t_{k,n} = 0$ for $k > n$. Then condition (5) is satisfied. From the definitions, we obtain that

$$T'_{2^{A_j+1}-1} = \sum_{s=1}^{j} \sum_{k=n_{(A_s,L_s)}}^{n_{(A_s,L_s)}+2^{A_s-L_s}-1} 1 = \sum_{s=1}^{j} 2^{A_s - L_s}.$$

This implies that for every $j \in \mathbb{P}$,

$$n_{(A_j,L_j)} \leq T'_{2^{A_j+1}-1} < n_{(A_{j+1},L_{j+1})}$$

and

$$2^{A_j - L_j} \leq T'_{2^{A_j+1}-1} < 2^{A_j - L_j + 1} \tag{8}$$

are true.

On the other hand, we have

$$\begin{aligned}\sigma^T_{2^{A_j+1}-1}(f) &= \frac{1}{T'_{2^{A_j+1}-1}} \sum_{k=1}^{2^{A_j+1}-1} t'_k S_k(f) \\ &= \frac{1}{T'_{2^{A_j+1}-1}} \sum_{k=n_{(A_j,L_j)}}^{n_{(A_j,L_j)}+2^{A_j-L_j}-1} S_k(f) + \frac{1}{T'_{2^{A_j+1}-1}} \sum_{s=1}^{j-1} \sum_{k=n_{(A_s,L_s)}}^{n_{(A_s,L_s)}+2^{A_s-L_s}-1} S_k(f) \\ &=: \sum_1 + \sum_2 .\end{aligned}$$

Since $\|S_k f\|_\infty \le k\|f\|_1$, from (8) we get

$$\begin{aligned}\left|\sum_2\right| &\le \frac{1}{2^{A_j-L_j}} \sum_{s=1}^{j-1} \sum_{k=n_{(A_s,L_s)}}^{n_{(A_s,L_s)}+2^{A_s-L_s}-1} k\|f\|_1 \\ &\le c\|f\|_1 \frac{1}{2^{A_j-L_j}} \sum_{s=1}^{j-1} \left(2^{A_s}\right)^2 \\ &\le c\|f\|_1 \frac{2^{2A_{j-1}}}{2^{A_j-L_j}}.\end{aligned}$$

Thus, the inequality $\left|\sum_2\right| \le c\|f\|_1$ holds if

$$2A_{j-1} \le A_j - L_j. \tag{9}$$

Let $m_j := 2^{A_j+1} - 1$. Since $t'_k \in \{0, 1\}$, it is easy to see that (using (8))

$$\sum_{k=1}^{m_j} t^2_{k,m_j} = \sum_{k=1}^{m_j} \left(\frac{t'_k}{T'_{m_j}}\right)^2 = \frac{1}{T'_{m_j}} \left(\frac{1}{T'_{m_j}} \sum_{k=1}^{m_j} t'_k\right) = \frac{1}{T'_{m_j}} \le 2^{L_j-A_j}.$$

Thus, to satisfy (6), it suffices to assume that

$$L_j \le \log_2(\lambda_{2^{A_j+1}-1}). \tag{10}$$

Indeed, then

$$2^{L_j-A_j} \le \frac{\lambda_{2^{A_j+1}-1}}{2^{A_j}} \le c\frac{\lambda_{m_j}}{m_j}.$$

Let $k := n_{(A_j,L_j)} + u$, where $u < 2^{A_j-L_j}$. Then

$$S_k(f) = S_{n_{(A_j,L_j)}}(f) + w_{n_{(A_j,L_j)}} S_u\left(f w_{n_{(A_j,L_j)}}\right).$$

Using this equality, we obtain

$$\sum_{1} = \frac{2^{A_j - L_j}}{T'_{2^{A_j+1}-1}} S_{n_{(A_j,L_j)}}(f) + \frac{w_{n_{(A_j,L_j)}}}{T'_{2^{A_j+1}-1}} \sum_{u=1}^{2^{A_j-L_j}-1} S_u\left(f w_{n_{(A_j,L_j)}}\right)$$
$$=: \sum_{1,1} + \sum_{1,2}.$$

From the property of the Fejér mean and inequality (8), we obtain

$$\left\| \sum_{1,2} \right\|_1 \leq c \left\| \sigma_{2^{A_j-L_j}-1}\left(f w_{n_{(A_j,L_j)}}\right) \right\|_1 \leq c\|f\|_1.$$

On the other hand, from (8) again

$$\left\| \sum_{1,1} \right\|_1 > \frac{1}{2} \left\| S_{n_{(A_j,L_j)}}(f) \right\|_1.$$

Summarizing these facts, assuming condition (9) it follows

$$\left\| \sigma^T_{2^{A_j+1}-1}(f) \right\|_1 > \frac{1}{2} \left\| S_{n_{(A_j,L_j)}}(f) \right\|_1 - c\|f\|_1. \tag{11}$$

Now, we define the function f as follows:

$$f := \sum_{v=1}^{\infty} \mu_v \left(D_{2^{A_v+1}} - D_{2^{A_v}}\right),$$

where $\sum_{v=1}^{\infty} \mu_v < \infty$ and $\mu_v > 0$.

It is known that

$$S_a(D_b) = D_{\min\{a,b\}}$$

holds for all $a, b \in \mathbb{N}$, so, using inequalities (7) we have

$$S_{n_{(A_j,L_j)}}\left(D_{2^{A_v+1}} - D_{2^{A_v}}\right) = \begin{cases} 0, & \text{if } v \geq j+1, \\ D_{n_{(A_j,L_j)}} - D_{2^{A_j}}, & \text{if } v = j, \\ D_{2^{A_v+1}} - D_{2^{A_v}}, & \text{if } v \leq j-1. \end{cases}$$

Hence

$$S_{n_{(A_j,L_j)}}(f) = \mu_j \left(D_{n_{(A_j,L_j)}} - D_{2^{A_j}}\right) + \sum_{v=1}^{j-1} \mu_v \left(D_{2^{A_v+1}} - D_{2^{A_v}}\right).$$

Now let us choose coordinates of $n_{(A_j,L_j)}$ (from A_j to $A_j - L_j$) alternately as 1 and 0. Then, by Lemma 1, we obtain

$$\left\| S_{n_{(A_j,L_j)}}(f) \right\|_1 > \mu_j \left\| D_{n_{(A_j,L_j)}} \right\|_1 - \mu_j - 2\sum_{v=1}^{j-1} \mu_v \geq c\mu_j L_j.$$

Therefore, combining this with inequality (11), we obtain

$$\left\| \sigma^T_{2^{A_j+1}-1}(f) \right\|_1 > c\mu_j L_j - c\|f\|_1. \tag{12}$$

Let us choose $\mu_j := 1/j^2$. Then, we have $\sum_{v=1}^{\infty} \mu_v < \infty$ and consequently $f \in L_1(G)$.

By choosing sufficiently large A_j so that $L_j \leq \log_2(\lambda_{2^{A_j+1}-1})$ we see that assumption (6) is satisfied and from (12)

$$\left\| \sigma^T_{m_j}(f) \right\|_1 \to \infty$$

follows, as $j \to \infty$. □

6 Almost everywhere convergence

Theorem 4 *Let $f \in L_1(G)$. Let $(d_n, n \in \mathbb{P})$ be a lacunary sequence. For every $n \in \mathbb{P}$ let $(t_{k,d_n}, 1 \leq k \leq d_n)$ be a finite sequence of non-negative numbers with properties*

$$\sum_{k=1}^{d_n} t_{k,d_n} = 1 \tag{13}$$

holds for every $n \in \mathbb{P}$, and

$$\sum_{k=1}^{d_n} t^2_{k,d_n} = O(1/d_n). \tag{14}$$

Then we have

$$\sigma^T_{d_n}(f) \to f$$

almost everywhere as $n \to \infty$.

Remark 2 We will prove Theorem 4 under the assumption that $\lambda \geq 2$.

If $1 < \lambda \leq 2$, then it is easy to prove that the sequence (d_n) can be decomposed into finitely many subsequences whose "associated lacunarity index" (λ) is not less than 2. Since for these finitely many subsequences we have almost everywhere convergence

$\sigma^T_{d_{n_j}}(f) \to f$, the original sequence $\sigma_{d_{n_j}}(f)$ converges to f almost everywhere as well. That is, Theorem 4 is fulfilled for each $\lambda > 1$. In the following, we can assume that the sequence (d_n) satisfies condition (14) and $d_{n+1} \geq 2d_n$.

We would especially like to emphasize that in Theorem 4 the sequence $(t_{k,d_n}, 1 \leq k \leq d_n)$ is not assumed to be monotonic. We only assume that the sum of these non-negative numbers is 1 and the sum of their squares is $O(1/d_n)$.

The proof of Theorem 4 is based on some lemmas below. To formulate the first lemma, we introduce a new operator and kernel function. Let $h_n(f) := f * H_n$, the definition of the kernel functions H_n is given as

$$H_n := \left| K^T_{d_n} \right|.$$

In Lemma 7 we prove the operator $h_*(f) := \sup_n |h_n(f)|$ is quasi-local. This means that for every $f \in L_1(G)$ and interval $Q \subseteq G$ for which $\int_Q f d\mu = 0$ and $\operatorname{supp} f \subseteq Q$, inequality $\int_{G\setminus Q} h_*(f) d\mu < c\|f\|_1$ holds (see e.g. [20]). Before proving this, we first state Lemma 5. Set $b, m, n, p \in \mathbb{N}$. Let

$$M_{m,n} := \left| \sum_{l=2^{n-1}+1}^{2^n} t_{l,d_m} D_l \right|,$$
$$H_{m,b} := \left| \sum_{l=2^b+1}^{d_m} t_{l,d_m} D_l \right|.$$

Lemma 5 *Let $m, n, v \in \mathbb{N}$. Let $(t_{k,d_m}, 1 \leq k \leq d_m)$ be a finite sequence of non-negative numbers such that conditions (13) and (14) are fulfilled. Let $v < n-1 < |d_m|$. Then we have*

$$\int_{J_v} M_{m,n} d\mu \leq c\frac{2^{v/2}}{d_m^{1/2}} \quad \text{and} \quad \int_{J_v} H_{m,|d_m|} d\mu \leq c\frac{2^{v/2}}{d_m^{1/2}}.$$

Proof Since

$$\sum_{l=2^{n-1}+1}^{2^n} t_{l,d_m} D_l = \sum_{l=1}^{2^{n-1}} t_{2^{n-1}+l,d_m} D_{2^{n-1}+l},$$
$$\sum_{l=2^{|d_m|}+1}^{d_m} t_{l,d_m} D_l = \sum_{l=1}^{d_m-2^{|d_m|}} t_{2^{|d_m|}+l,d_m} D_{2^{|d_m|}+l}$$

and in case of $z \in J_v$ by Lemma 2, we obtain

$$D_{2^{n-1}+l}(z) = w_{2^{n-1}+l^{(v)}}(z)\left(\sum_{j=0}^{v-1} l_j 2^j - l_v 2^v\right),$$
$$D_{2^{|d_m|}+l}(z) = w_{2^{|d_m|}+l^{(v)}}(z)\left(\sum_{j=0}^{v-1} l_j 2^j - l_v 2^v\right).$$

With the help of the Cauchy–Bunyakovsky–Schwarz inequality, we have

$$\int_{J_v} M_{m,n} d\mu \leq \left(\frac{1}{2^v}\int_{J_v}\left(\sum_{l=1}^{2^{n-1}} t_{2^{n-1}+l,d_m} D_{2^{n-1}+l}\right)^2 d\mu\right)^{1/2}, \tag{15}$$

$$\int_{J_v} H_{m,|d_m|} d\mu \leq \left(\frac{1}{2^v}\int_{J_v}\left(\sum_{l=1}^{d_m-2^{|d_m|}} t_{2^{|d_m|}+l,d_m} D_{2^{|d_m|}+l}\right)^2 d\mu\right)^{1/2}. \tag{16}$$

We now analyse the integrals over the sets J_v. They are,

$$\begin{aligned}
&\int_{J_v} \sum_{k,l=1}^{2^{n-1}} t_{2^{n-1}+k,d_m} t_{2^{n-1}+l,d_m} w_{2^{n-1}+k^{(v)}}(z) w_{2^{n-1}+l^{(v)}}(z) \\
&\qquad \times \left(\sum_{j=0}^{v-1} k_j 2^j - k_v 2^v\right)\left(\sum_{j=0}^{v-1} l_j 2^j - l_v 2^v\right) d\mu(z) \\
&\quad = \sum_{k,l=1}^{2^{n-1}} t_{2^{n-1}+k,d_m} t_{2^{n-1}+l,d_m}\left(\sum_{j=0}^{v-1} k_j 2^j - k_v 2^v\right)\left(\sum_{j=0}^{v-1} l_j 2^j - l_v 2^v\right) \\
&\qquad \times \int_{J_v} w_{2^{n-1}+k^{(v)}}(z) w_{2^{n-1}+l^{(v)}}(z) d\mu(z)
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\int_{J_v} \sum_{k,l=1}^{d_m-2^{|d_m|}} t_{2^{|d_m|}+k,d_m} t_{2^{|d_m|}+l,d_m} w_{2^{|d_m|}+k^{(v)}}(z) w_{2^{|d_m|}+l^{(v)}}(z) \\
&\qquad \times \left(\sum_{j=0}^{v-1} k_j 2^j - k_v 2^v\right)\left(\sum_{j=0}^{v-1} l_j 2^j - l_v 2^v\right) d\mu(z)
\end{aligned}$$

$$= \sum_{k,l=1}^{d_m-2^{|d_m|}} t_{2^{|d_m|}+k,d_m} t_{2^{|d_m|}+l,d_m} \left(\sum_{j=0}^{v-1} k_j 2^j - k_v 2^v\right)\left(\sum_{j=0}^{v-1} l_j 2^j - l_v 2^v\right) \times \int_{J_v} w_{2^{|d_m|}+k^{(v)}}(z) w_{2^{|d_m|}+l^{(v)}}(z) d\mu(z) \tag{18}$$

respectively.

The integrals $\int_{J_v} w_{2^{n-1}+k^{(v)}}(z) w_{2^{n-1}+l^{(v)}}(z) d\mu(z)$ and $\int_{J_v} w_{2^{|d_m|}+k^{(v)}}(z) w_{2^{|d_m|}+l^{(v)}}(z) d\mu(z)$ can be nonzero only when

$$k^{(v)} \oplus l^{(v)} = 0,$$

so $k_v = l_v, k_{v+1} = l_{v+1}, \ldots$ in expressions (17) and (18), too. In these cases, the values of the integrals are 2^{-v}.

Thus, for integrals $\int_{J_v} (\cdot)^2$, we obtain the following upper estimates respectively:

$$\frac{c}{2^v} \sum_{k=1}^{2^{n-1}} t_{2^{n-1}+k,d_m} \sum_{\{l: l_j = k_j, j \geq v\}} t_{2^{n-1}+l,d_m} 2^{2v},$$

$$\frac{c}{2^v} \sum_{k=1}^{d_m-2^{|d_m|}} t_{2^{|d_m|}+k,d_m} \sum_{\{l: l_j = k_j, j \geq v\}} t_{2^{|d_m|}+l,d_m} 2^{2v}.$$

The obvious inequality $\alpha\beta \leq \alpha^2/2 + \beta^2/2$ and (14) give

$$\begin{aligned}
&\sum_{k=1}^{2^{n-1}} t_{2^{n-1}+k,d_m} \sum_{\{l: l_j = k_j, j \geq v\}} t_{2^{n-1}+l,d_m} \\
&= \sum_{k=1}^{2^{n-1}} t_{2^{n-1}+k,d_m} \sum_{l=1}^{2^v} t_{2^{n-1}+k^{(v)}+l,d_m} \\
&\leq c \sum_{k=1}^{2^{n-1}} \sum_{l=1}^{2^v} \left(t^2_{2^{n-1}+k,d_m} + t^2_{2^{n-1}+k^{(v)}+l,d_m}\right) \\
&\leq c 2^v \sum_{k=1}^{2^{n-1}} t^2_{2^{n-1}+k,d_m} + c \sum_{k_0,\ldots,k_{v-1} \in \{0,1\}} \sum_{k_v,\ldots,k_{n-1} \in \{0,1\}} \sum_{l=1}^{2^v} t^2_{2^{n-1}+k^{(v)}+l,d_m} \\
&\leq c 2^v \sum_{k=1}^{2^{n-1}} t^2_{2^{n-1}+k,d_m} \leq c 2^v \sum_{k=1}^{d_m} t^2_{k,d_m} \leq \frac{c 2^v}{d_m}
\end{aligned}$$

and similarly

$$\sum_{k=1}^{d_m-2^{|d_m|}} t_{2^{|d_m|}+k,d_m} \sum_{\{l:l_j=k_j,j\geq v\}} t_{2^{|d_m|}+l,d_m}$$
$$= \sum_{k=1}^{d_m-2^{|d_m|}} t_{2^{|d_m|}+k,d_m} \sum_{l=1}^{2^v} t_{2^{|d_m|}+k^{(v)}+l,d_m}$$
$$\leq c \sum_{k=1}^{d_m-2^{|d_m|}} \sum_{l=1}^{2^v} \left(t^2_{2^{|d_m|}+k,d_m} + t^2_{2^{|d_m|}+k^{(v)}+l,d_m}\right)$$
$$\leq c2^v \sum_{k=1}^{d_m-2^{|d_m|}} t^2_{2^{|d_m|}+k,d_m} + c \sum_{k_0,\dots,k_{v-1}\in\{0,1\}} \sum_{k_v,\dots,k_{n-1}\in\{0,1\}} \sum_{l=1}^{2^v} t^2_{2^{|d_m|}+k^{(v)}+l,d_m}$$
$$\leq c2^v \sum_{k=1}^{d_m-2^{|d_m|}} t^2_{2^{|d_m|}+k,d_m} \leq c2^v \sum_{k=1}^{d_m} t^2_{k,d_m} \leq \frac{c2^v}{d_m},$$

That is, using (15) and (16) we get

$$\int_{J_v} M_{m,n} \leq c2^{-v/2} \left(\frac{2^{2v}}{d_m}\right)^{1/2} = c\frac{2^{v/2}}{d_m^{1/2}}$$

and

$$\int_{J_v} H_{m,|d_m|} \leq c2^{-v/2} \left(\frac{2^{2v}}{d_m}\right)^{1/2} = c\frac{2^{v/2}}{d_m^{1/2}}.$$

□

Lemma 6 *For every $a, n \in \mathbb{N}$ let $(t_{k,d_n}, 1 \leq k \leq d_n)$ be a finite sequence of non-negative numbers such that conditions (13) and (14) are fulfilled. Then, we have*

$$\int_{\overline{I}_a} \sup_{\{n:d_n>2^a\}} H_{n,a} d\mu \leq c.$$

Proof Since

$$H_{n,a} \leq \left|\sum_{l=2^a+1}^{2^{|d_n|}} t_{l,d_n} D_l\right| + \left|\sum_{l=2^{|d_n|}+1}^{d_n} t_{l,d_n} D_l\right|$$
$$\leq \sum_{m=a+1}^{|d_n|} \left|\sum_{l=2^{m-1}+1}^{2^m} t_{l,d_n} D_l\right| + \left|\sum_{l=2^{|d_n|}+1}^{d_n} t_{l,d_n} D_l\right|,$$

so

$$\sup_{\{n:d_n>2^a\}} H_{n,a} \leq \sum_{\{n:d_n>2^a\}} \sum_{m=a+1}^{|d_n|} \left| \sum_{l=2^{m-1}+1}^{2^m} t_{l,d_n} D_l \right| + \sum_{\{n:d_n>2^a\}} \left| \sum_{l=2^{|d_n|}+1}^{d_n} t_{l,d_n} D_l \right|$$
$$=: B_1 + B_2.$$

Set $n_0 := \min\{n : d_n > 2^a\}$. Then from Lemma 5 we obtain

$$\begin{aligned}
\int_{\overline{I}_a} B_1 d\mu &\leq c \sum_{v=0}^{a-1} \sum_{n=n_0}^{\infty} \sum_{m=a+1}^{|d_n|} \int_{J_v} M_{n,m} d\mu \\
&\leq c \sum_{v=0}^{a-1} \sum_{n=n_0}^{\infty} \sum_{m=a+1}^{|d_n|} \frac{2^{v/2}}{d_n^{1/2}} \\
&\leq c \sum_{n=n_0}^{\infty} (|d_n| - a) \frac{2^{a/2}}{d_n^{1/2}} \\
&\leq c \sum_{n=n_0}^{\infty} (|d_n| - a) \frac{2^{a/2}}{2^{|d_n|/2}} \leq c
\end{aligned}$$

and

$$\begin{aligned}
\int_{\overline{I}_a} B_2 d\mu &\leq c \sum_{v=0}^{a-1} \sum_{n=n_0}^{\infty} \int_{J_v} H_{n,|d_n|} d\mu \\
&\leq c \sum_{v=0}^{a-1} \sum_{n=n_0}^{\infty} \frac{2^{v/2}}{d_n^{1/2}} \\
&\leq c \sum_{n=n_0}^{\infty} \frac{2^{a/2}}{d_n^{1/2}} \leq c.
\end{aligned}$$

This completes the proof of Lemma 6. □

Lemma 7 *Suppose that the finite sequence $(t_{k,d_n}, 1 \leq k \leq d_n)$ satisfies conditions (13) and (14). Then the operator h_* is quasi-local.*

Proof Let $Q \subseteq G$ be a dyadic interval and $f \in L_1$ such that $\operatorname{supp} f \subseteq Q$ and $\int_G f = 0$. Due to shift invariance, we may assume without loss of generality that the interval Q is of the form: $Q = I_a$ for some natural number a. In this proof, we also define $n_0 := \min\{n : d_n > 2^a\}$. Then

$$\begin{aligned}\int_{G\setminus Q} h_*(f)d\mu \le & \int_{\overline{I_a}} \sup_{n\ge n_0}\left|\int_{I_a} f(z)H_{n,a}(z+u)d\mu(z)\right|d\mu(u)\\ &+\int_{\overline{I_a}} \sup_{n\ge n_0}\left|\int_{I_a} f(z)\left|\sum_{l=1}^{2^a} t_{l,d_n}D_l(z+u)\right|d\mu(z)\right|d\mu(u)\\ &+\int_{\overline{I_a}} \sup_{n< n_0}\left|\int_{I_a} f(z)H_n(z+u)d\mu(z)\right|d\mu(u)\\ =:& B_1+B_2+B_3.\end{aligned}$$

Since for $l \le 2^a$ the function D_l is $\mathcal{A}_a$ measurable, then we have

$$B_2=\int_{\overline{I_a}} \sup_{n\ge n_0}\left|\sum_{l=1}^{2^a} t_{l,d_n}D_l(u)\right|\left|\int_{I_a} f(z)d\mu(z)\right|d\mu(u)=0$$

by the usual technique based on the definition of quasi-locality, since $\int_{I_a} fd\mu = 0$. Similarly, we also have

$$B_3 = 0.$$

That is,

$$\int_{G\setminus Q} h_*(f)d\mu \le \int_{\overline{I_a}} \sup_{n\ge n_0}\left|\int_{I_a} f(z)H_{n,a}(z+u)d\mu(z)\right|d\mu(u).$$

Finally, applying Lemma 6, we obtain

$$\begin{aligned}&\int_{\overline{I_a}} \sup_{n\ge n_0}\left|\int_{I_a} f(z)H_{n,a}(z+u)d\mu(z)\right|d\mu(u)\\ &\le \int_{I_a}|f(z)|\int_{\overline{I_a}} \sup_{n\ge n_0} H_{n,a}(u)d\mu(u)d\mu(z) \le c\|f\|_1.\end{aligned}$$

This completes the proof of Lemma 7. □

Lemma 8 $\|H_n\|_1 \le c.$

Proof Let $a < |d_n|$.

It is easy to see that

$$\begin{aligned}\|M_{n,a+1}\|_1 \le & \int_{\overline{I_a}} M_{n,a+1}(z)d\mu(z)+\int_{I_a} M_{n,a+1}(z)d\mu(z)\\ =:& E_1+E_2\end{aligned}$$

and

$$\begin{aligned}\|H_{n,|d_n|}\|_1 &\leq \int_{\overline{I_{|d_n|}}} H_{n,|d_n|}(z)d\mu(z) + \int_{I_{|d_n|}} H_{n,|d_n|}(z)d\mu(z)\\ &=: F_1 + F_2.\end{aligned}$$

Lemma 5 gives

$$E_1 \leq c\sum_{v=0}^{a-1} \frac{2^{v/2}}{d_n^{1/2}} \leq c\frac{2^{a/2}}{d_n^{1/2}}$$

and

$$F_1 \leq c\sum_{v=0}^{|d_n|-1} \frac{2^{v/2}}{d_n^{1/2}} \leq c.$$

Moreover, using conditions (13), (14), and the Cauchy–Bunyakovsky–Schwarz inequality, we obtain

$$\begin{aligned}E_2 &= \int_{I_a} \left|\sum_{l=2^a+1}^{2^{a+1}} t_{l,d_n} D_l(z)\right| d\mu(z)\\ &\leq c\sum_{l=2^a+1}^{2^{a+1}} t_{l,d_n} \leq c2^{a/2}\left(\sum_{l=2^a+1}^{2^{a+1}} t_{l,d_n}^2\right)^{1/2}\\ &\leq c2^{a/2}d_n^{-1/2}\end{aligned}$$

and

$$\begin{aligned}F_2 &= \int_{I_{|d_n|}} \left|\sum_{l=2^{|d_n|}+1}^{d_n} t_{l,d_n} D_l(z)\right| d\mu(z)\\ &\leq c\sum_{l=2^{|d_n|}+1}^{d_n} t_{l,d_n} \leq c.\end{aligned}$$

Thus, we have $\|M_{n,a+1}\|_1 \leq c2^{a/2}d_n^{-1/2}$ and $\|H_{n,|d_n|}\|_1 \leq c$.

Finally,

$$
\begin{aligned}
\|H_n\|_1 &= \left\| \sum_{l=1}^{d_n} t_{l,d_n} D_l \right\|_1 \\
&\le c \sum_{a=0}^{|d_n|-1} \left\| \sum_{l=2^a+1}^{2^{a+1}} t_{l,d_n} D_l \right\|_1 + c \left\| \sum_{l=2^{|d_n|}+1}^{d_n} t_{l,d_n} D_l \right\|_1 \\
&= c \sum_{a=0}^{|d_n|-1} \left\| M_{n,a+1} \right\|_1 + c \left\| H_{n,|d_n|} \right\|_1 \\
&\le c \sum_{a=0}^{|d_n|-1} \frac{2^{a/2}}{d_n^{1/2}} + c \le c.
\end{aligned}
$$

This completes the proof of Lemma 8. □

The next step in proving Theorem 4 uses Lemma 7 to show that the operator $h_*(f) = \sup_n |f * H_n|$ is of weak type (L_1, L_1).

Lemma 9 *The operator h_* is of weak type (L_1, L_1) and of strong type (L_p, L_p) for each $1 < p \le \infty$.*

Proof First, we show that the operator h_* is of strong type (L_∞, L_∞). Essentially, this property of the operator h_* is a trivial consequence of Lemma 8, the fact that the kernel functions H_n are uniformly bounded in L_1. That is, the operator h_* is of strong type (L_∞, L_∞) and since it is σ-sublinear, then by a standard argument the fact that it is quasi-local (Lemma 7) gives that it is also of weak type (L_1, L_1). Finally, applying the Marcinkiewicz interpolation theorem for sublinear operators completes the proof of Lemma 9. □

Let $\sigma_*^T(f) := \sup_n \left|\sigma_{d_n}^T(f)\right|$. We continue to assume only that the $(t_{l,d_n}, 1 \le l \le d_n)$ fulfills properties (13) and (14) (no monotonicity of any kind is supposed).

The proof of Theorem 4 The proof follows immediately from Lemma 9 and the inequality.

$$
\sigma_*^T(f) \le h_*(f)
$$

which gives that the operator σ_*^T is of weak type (L_1, L_1). Moreover, the relation $0 \le t_{k,d_n} \le c/2^{n/2}$ (which comes from (14)) and the almost everywhere convergence property for Walsh polynomials (a set of which is dense in the Lebesgue space of integrable functions) complete the proof of Theorem 4. □

7 Divergence almost everywhere

Theorem 5 *Let $(\lambda_l, l \in \mathbb{P})$ be an arbitrary sequence of real numbers monotonically tending to infinity. Then there exists a strictly increasing sequence $(N_l, l \in \mathbb{P})$ of* *natural numbers and a finite sequence of non-negative numbers* $(t_{k,2^{N_l}}, 1 \le k \le 2^{N_l})$ *with properties*

$$\sum_{k=1}^{2^{N_l}} t_{k,2^{N_l}} = 1 \tag{19}$$

for every $l \in \mathbb{P}$, *and* $f \in L_1(G)$, *such that*

$$\sum_{k=1}^{2^{N_l}} t_{k,2^{N_l}}^2 = O\left(\frac{\lambda_{N_l}}{2^{N_l}}\right) \tag{20}$$

and

$$\sigma_{2^{N_l}}^T(f) \to \infty$$

almost everywhere as $l \to \infty$.

In this section let L be an odd natural number, let $A \in \mathbb{N}$ with $L < A$, and $x \in G$. Define $n_{(A,L)} := \sum_{j=0}^{(L-1)/2} 2^{A-2j-1} = 2^{A-L} + \cdots + 2^{A-3} + 2^{A-1}$.

Let

$$g_{(A,L)}(x) := \sum_{u \in U_{A,L}} \operatorname{sgn} D_{n_{(A,L)}}(x+u),$$

where $U_{A,L} := \left\{u \in G : u = \sum_{j=0}^{A-L-1} e_j u_j\right\}$ and $e_j := (0, \ldots, 0, 1, 0, \ldots)$ (only the jth coordinate is 1, the others are 0). First, we show that $g_{(A,L)}(x) \in \{-1, 0, 1\}$ holds for every $x \in G$. By Lemma 2 $D_{n_{(A,L)}}(x+u)$ can be different from zero only in the case when $x+u \in I_{A-L}$. That is, when $u_j = x_j$ for $j = 0, \ldots, A-L-1$. This means that in the summation $\sum_{u \in U_{A,L}}$ there is exactly one $u \in U_{A,L}$ such that $D_{n_{(A,L)}}(x+u)$ can be different from zero. This immediately implies that $g_{(A,L)}(x) \in \{-1, 0, 1\}$. Define $U'_{A,L} := \left\{u \in G : u = \sum_{j=A-L}^{A-1} e_j u_j\right\}$, which has exactly 2^L elements. Write $U'_{A,L} = \{v_0, \ldots, v_{2^L-1}\}$ in an arbitrary but fixed order.

Lemma 10 *Let* $x \in U_{A,L}$. *Then there exists an absolute constant* c *such that*

$$S_{n_{(A,L)}}\left(g_{(A,L)}; x\right) \ge cL.$$

Proof By definition

$$S_{n_{(A,L)}}\left(g_{(A,L)}; x\right) = \int_G \sum_{u \in U_{A,L}} \operatorname{sgn} D_{n_{(A,L)}}(u+t) D_{n_{(A,L)}}(x+t) d\mu(t).$$

Since $x \in U_{A,L}$, if $j \ge A-L$ then $x_j = 0$. On the other hand, by Lemma 4, if $D_{2^{A-L}+\cdots+2^{A-1}}(x+t) \ne 0$, then $t_j = x_j$, where $j \in \{0, \ldots, A-L-1\}$. Similarly, it is true for $D_{2^{A-L}+\dots+2^{A-1}}(u+t)$, so $t_j = x_j = u_j$, where $j \in \{0, \dots, A-L-1\}$. This implies that in the expression $\sum_{u \in U_{A,L}} \operatorname{sgn} D_{n_{(A,L)}}(u+t) D_{n_{(A,L)}}(x+t)$ there is exactly one u for which the product does not vanish. Since $x+t, u+t \in I_{A-L}$ and $x_j = 0$ if $j \geq A-L$, we get

$$\begin{aligned}
S_{n_{(A,L)}}\left(g_{(A,L)}; x\right) &= \int_{I_{A-L}(x)} \operatorname{sgn}\left(D_{n_{(A,L)}}\left(\sum_{j=A-L}^{A-1}(u_j+t_j)e_j\right)\right) \\
&\quad D_{n_{(A,L)}}\left(\sum_{j=A-L}^{A-1}(x_j+t_j)e_j\right) d\mu(t) \\
&= \int_G \left| D_{n_{(A,L)}}\left(\sum_{j=A-L}^{A-1} t_j e_j\right)\right| d\mu(t) \\
&= \int_G \left| D_{1+\dots+2^{L-3}+2^{L-1}}\left(\sum_{j=0}^{L-1} t_j e_j\right)\right| d\mu(t) \geq cL.
\end{aligned}$$

In the last equality we used Lemma 4 and the fact that

$$\int_G |D_{A2^k}(t)|dt = \int_{I_k} 2^k |D_A(t2^k)|dt = \int_G |D_A(t)|dt$$

and the inequality comes from Lemma 1. □

Let

$$R_{k,A,L}(x) := r_{A+k}(x) g_{(A,L)}(x+v_k),$$

where $k \in \{0, 1, \dots, 2^{L-1}\}$ and let

$$Q_{A,L} := \prod_{k=0}^{2^L-1}(1+R_{k,A,L}).$$

These are Walsh-Kolmogorov-like polynomials. It is known (see for example [20]), that

$$Q_{A,L} = \sum_{m=0}^{2^{2^L}-1} W_{m,A,L}, \text{ where } W_{m,A,L} := \prod_{k=0}^{2^L-1} R_{k,A,L}^{m_k}. \tag{21}$$

We define the spectrum of a function $f \in L_1(G)$ as

$$\operatorname{sp}(f) := \{k \in \mathbb{N} : \hat{f}(k) \neq 0\}.$$

Lemma 11 *For every odd natural number L and $A \in \mathbb{N}$ with $L < A$ we have*

$$\mathrm{sp}(Q_{A,L}) \subseteq \left[0, 2^{A+2^L}\right).$$

Proof The function $g_{(A,L)}(x)$ (and thus $g_{(A,L)}^{m_k}(x)$) is $\mathcal{A}_A$ measurable (depends only on the first A coordinates of x). It follows $\mathrm{sp}\left(g_{(A,L)}^{m_k}\right) \subseteq [0, 2^A)$ and from this $\mathrm{sp}\left(\prod_{k=0}^{2^L-1} g_{(A,L)}^{m_k}(.+v_k)\right) \subseteq [0, 2^A)$. It is easy to see, that $\mathrm{sp}\left(r_i r_j\right) = \{2^i + 2^j\}$ and $\mathrm{sp}\left(r_j^i\right) = \{2^j i\}$. From these we obtain

$$\mathrm{sp}\left(\prod_{k=0}^{2^L-1} r_{A+k}^{m_k}\right) = \left\{\sum_{k=0}^{2^L-1} 2^{A+k} m_k\right\} = \left\{2^A \sum_{k=0}^{2^L-1} 2^k m_k\right\} = \left\{2^A m\right\}.$$

Summarising these facts, we obtain

$$\mathrm{sp}\left(R_{k,A,L}^{m_k}\right) \subseteq [m2^A, (m+1)2^A),$$

thus by the definition of $W_{m,A,L}$ we have

$$\mathrm{sp}(W_{m,A,L}) \subseteq [m2^A, (m+1)2^A).$$

Combining this with the definition of $Q_{A,L}$, we obtain the statement of this lemma. □

Lemma 12 *For every odd natural number L and $A \in \mathbb{N}$ with $L < A$ we have*

$$\|Q_{A,L}\|_1 = 1.$$

Proof From the definition of $g_{(A,L)}$ we get $R_{k,A,L} \geq -1$. So $Q_{A,L} = \prod_{k=0}^{2^L-1}(1 + R_{k,A,L}) \geq 0$. On the other hand, Lemma 11 yields, that $\int_G W_{m,A,L} d\mu \neq 0$ only if $m_0 = \cdots = m_{2^L-1} = 0$, that is $m = 0$. Considering the equality (21), this means that

$$\begin{aligned}\|Q_{A,L}\|_1 &= \int_G Q_{A,L} d\mu \\ &= \sum_{m=0}^{2^{2^L}-1} \int_G W_{m,A,L} d\mu \\ &= \int_G W_{0,A,L} d\mu = 1.\end{aligned}$$

□

Lemma 13 *Let $x \in U_{A,L}$. Then there exists an absolute constant c, such that*

$$S_{2^{A+k}+n_{(A,L)}}\left(Q_{A,L}; x\right) - S_{2^{A+k}}\left(Q_{A,L}; x\right) \geq cL.$$

Proof From Lemma 10, it follows that if $x + v_k \in U_{A,L}$, then

$$S_{n_{(A,L)}}\left(r_{A+k} R_{k,A,L}; x\right) \geq cL.$$

This means that for every $x \in G$, there exists $k \in \{0, 1, \ldots, 2^{L-1}\}$ such that

$$S_{n_{(A,L)}}\left(g_{(A,L)}(. + v_k); x\right) \geq cL.$$

Based on Lemma 11, we get

$$S_{2^{A+k}+n_{(A,L)}}\left(W_{m,A,L}; x\right) \neq S_{2^{A+k}}\left(W_{m,A,L}; x\right)$$

only if $m = 2^k$. Then $W_{m,A,L}(x) = R_{k,A,L}(x) = r_{A+k}(x) g_{(A,L)}(x + v_k)$, so

$$\begin{aligned} S_{2^{A+k}+n_{(A,L)}}\left(Q_{A,L}; x\right) - S_{2^{A+k}}\left(Q_{A,L}; x\right) \\ =& S_{2^{A+k}+n_{(A,L)}}\left(r_{A+k} g_{(A,L)}(. + v_k); x\right) - S_{2^{A+k}}\left(r_{A+k}(.) g_{(A,L)}(. + v_k); x\right) \\ =& S_{n_{(A,L)}}\left(g_{(A,L)}(. + v_k); x\right) \geq cL. \end{aligned}$$

□

Lemma 14 *Let L_j be odd natural numbers and $A_j \in \mathbb{N}$, where $j \in \mathbb{N}$. Assume that $A_j - L_j,\ L_j \nearrow \infty$ as $j \to \infty$, $A_{j+1} > A_j + 2^{L_j}$, $L_{j+1} \geq (j+1)2^{A_j + 2^{L_j} + 1}$, $\alpha_{j+1} := 2^{-A_j - 2^{L_j} - 1}$ and $f := \sum_{j=1}^{\infty} \alpha_j Q_{A_j, L_j}$. Let $x \in G$ be arbitrary. Then there exist $k \in \{0, \ldots, 2^{L_j - 1}\}$ and an absolute constant c, such that*

$$S_{2^{A_j+k}+n_{\left(A_j,L_j\right)}}(f; x) - S_{2^{A_j+k}}(f; x) \geq cj.$$

Proof Using Lemma 12 we get

$$\|f\|_1 \leq \sum_{j=1}^{\infty} \alpha_j \left\|Q_{A_j,L_j}\right\|_1 \leq \sum_{j=1}^{\infty} \alpha_j \leq 1,$$

thus $f \in L_1(G)$. Based on Lemma 13, there exists an absolute constant c such that

$$S_{2^{A_j+k}+n_{(A_j,L_j)}}\left(Q_{A_j,L_j}; x\right) - S_{2^{A_j+k}}\left(Q_{A_j,L_j}; x\right) \geq cL_j \tag{22}$$

if j is sufficiently large.

We expand the expression as follows:

$$\begin{aligned} S_{2^{A_j+k}+n_{\left(A_j,L_j\right)}}(f; x) - S_{2^{A_j+k}(f;x)} \\ = \sum_{i=1}^{j-1} \alpha_i \left(S_{2^{A_j+k}+n_{(A_j,L_j)}}\left(Q_{A_i,L_i}; x\right) - S_{2^{A_j+k}}\left(Q_{A_i,L_i}; x\right) \right) \end{aligned}$$

$$
\begin{aligned}
&+ \alpha_j \left(S_{2^{A_j+k}+n_{(A_j,L_j)}} \left(Q_{A_j,L_j}; x\right) - S_{2^{A_j+k}} \left(Q_{A_j,L_j}; x\right)\right) \\
&+ \sum_{i=j+1}^{\infty} \alpha_i \left(S_{2^{A_j+k}+n_{(A_j,L_j)}} \left(Q_{A_i,L_i}; x\right) - S_{2^{A_j+k}} \left(Q_{A_i,L_i}; x\right)\right) \\
=:& \; X_1 + X_2 + X_3.
\end{aligned}
$$

The inequality $A_j > A_{j-1} + 2^{L_{j-1}}$ implies $2^{A_j+k} \geq 2^{A_j} > 2^{A_{j-1}+2^{L_{j-1}}}$, where $k \in \mathbb{N}$ and $j \in \mathbb{P}$. So, from Lemma 11 we obtain, that if $1 \leq i < j$ and $l \geq 2^{A_j}$, then $S_l \left(Q_{A_i,L_i}; x\right) = Q_{A_i,L_i}(x)$. This means that $X_1 = \sum_{i=1}^{j-1} \alpha_i \left(Q_{A_i,L_i}(x) - Q_{A_i,L_i}(x)\right) = 0$.

From inequality (22) we get $X_2 \geq c\alpha_j L_j$.

The trivial inequality $|S_n f| \leq n\|f\|_1$ and the definition of α_i yield

$$
\begin{aligned}
X_3 &\leq \sum_{i=j+1}^{\infty} \alpha_i \left(2^{A_j+k} + n_{(A_j,L_j)} + 2^{A_j+k}\right) \\
&\leq 2^{A_j+2^{L_j}+1} \sum_{i=j+1}^{\infty} \alpha_i \leq 2^{A_j+2^{L_j}+1} 2\alpha_{j+1} \leq 2.
\end{aligned}
$$

Summarizing these observations, we obtain

$$
S_{2^{A_j+k}+n_{(A_j,L_j)}}(f; x) - S_{2^{A_j+k}}(f; x) \geq c\alpha_j L_j \geq cj.
$$

□

Proof of Theorem 5 Let A_j, L_j, α_j be as defined in Lemma 14. In this section let $t_{s,2^{N_l}} := T'_{s,2^{N_l}} / T'_{2^{N_l}}$, where $T'_{2^{N_l}} := \sum_{s=1}^{2^{N_l}} T'_{s,2^{N_l}}$. Then condition (19) is satisfied.

Let $(N_l, l \in \mathbb{P})$ be the sequence consisting of the elements of the set $\left\{A_j + k + 1 : k \in \{0, \ldots, 2^{L_j - 1}\}\right\}$ where $j \in \mathbb{N}$ arbitrary and let

$$
T'_{s,2^{N_l}} := \begin{cases} 1, & \text{if } k \in \{0, \ldots, 2^{L_j} - 1\}, s \in \left[2^{A_j+k} + n_{(A_j,L_j)}, 2^{A_j+k} + n_{(A_j,L_j)} + 2^{A_j - L_j}\right) \cap \mathbb{N}, s \leq 2^{N_l}, j \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}
$$

With these assumptions

$$
\sum_{k=1}^{2^{N_l}} t^2_{k,2^{N_l}} = \frac{\sum_{a=1}^{2^{A_j+k+1}} T'^2_{a,2^{A_j+k+1}}}{T'^2_{2^{A_j+k+1}}} = \frac{\sum_{a=2^{A_j+k}+n_{(A_j,L_j)}}^{2^{A_j+k}+n_{(A_j,L_j)}+2^{A_j-L_j}-1} 1}{\left(\sum_{a=1}^{2^{A_j+k+1}} T'_{a,2^{A_j+k+1}}\right)^2} = \frac{2^{A_j-L_j}}{(2^{A_j-L_j})^2} = \frac{2^{L_j}}{2^{A_j}}.
$$

Let $\tilde{\lambda}_{N_j} := \min\left\{\lambda_{N_i} : N_i = 2^{A_j+k+1},\ k \in \{0,\dots,2^{L_j-1}\}\right\} = \lambda_{2^{A_j+1}}$. Let us further assume that

$$A_j \geq \log_2\left(\lambda^{-1}\left(2^{L_j+2^{L_j}}\right)\right) - 1,$$

where λ^{-1} is the inverse of the extension of the strictly increasing series λ to positive real numbers (linear between positive integers). Then $2^{L_j+2^{L_j}} \leq \tilde{\lambda}_{N_j}$, so

$$\sum_{k=1}^{2^{N_l}} t^2_{k,2^{N_l}} = \frac{2^{L_j}}{2^{A_j}} \leq \frac{\tilde{\lambda}_{N_j}}{2^{A_j+2^{L_j}}} \leq \frac{\lambda_{N_l}}{2^{N_l}}.$$

Thus we are ready with equation (20).

Using Lemma 14

$$\begin{aligned}\sigma^T_{2^{N_l}}(f) &= \sum_{s=2^{A_j+k}+n_{(A_j,L_j)}}^{2^{A_j+k}+n_{(A_j,L_j)}+2^{A_j-L_j}-1} S_s(f;x)\frac{T'_{s,2^{A_j+k+1}}}{T'_{2^{A_j+k+1}}}\\ &= \frac{2^{L_j}}{2^{A_j}}\sum_{s=0}^{2^{A_j-L_j}-1} S_{2^{A_j+k}+n_{(A_j,L_j)}}(f;x) + \frac{2^{L_j}}{2^{A_j}} w_{2^{A_j+k}+n_{(A_j,L_j)}}\\ &\quad \sum_{s=0}^{2^{A_j-L_j}-1} S_s\left(f w_{2^{A_j+k}+n_{(A_j,L_j)}};x\right)\\ &\geq cj - \sigma_{2^{A_j-L_j}}\left(f w_{2^{A_j+k}+n_{(A_j,L_j)}};x\right).\end{aligned}$$

Inequality for the Fejér kernel $K_{2^n} \geq 0$ follows

$$\begin{aligned}\left|\sigma_{2^{A_j-L_j}}\left(f w_{2^{A_j+k}+n_{(A_j,L_j)}};x\right)\right| &\leq \int_G |f(t)|\left|K_{2^{A_j-L_j}}(x+t)\right| d\mu(t)\\ &\leq \int_G |f(t)| K_{2^{A_j-L_j}}(x+t) d\mu(t)\\ &= \sigma_{2^{A_j-L_j}}(|f|;x).\end{aligned}$$

It is known that $\sigma_n(|f|;x) \to |f(x)|$ almost everywhere, as $n \to \infty$. (Recall that the Walsh-Fejér kernels K_{2^j} are nonnegative functions.) Let $x \in G$ be such that convergence holds. Recall that $\|f\|_1 \leq 1$. Then there exists sufficiently large j, that $|f(x)| \leq j^{1/2}$. So $\sigma_{2^{A_j-L_j}}(|f|;x) \leq |f(x)| + 1 \leq 2j^{1/2}$.

Taking these facts into account, we have

$$\sigma^T_{2^{N_l}}(f;x) \geq cj - 2j^{1/2} \to \infty$$

almost everywhere as $j \to \infty$. □

8 Open questions

Question 1 *Let* $(t_{k,n}, 1 \leq k \leq n, k \in \mathbb{P})$ *be a finite sequence of non-negative numbers, where*

$$\sum_{k=1}^{n} t_{k,n} = 1 \tag{23}$$

and

$$\limsup_{n\to\infty} n \sum_{k=1}^{n} t_{k,n}^2 = \infty \tag{24}$$

are satisfied.

Does there exist a function $f \in L_1(G)$ *such that* $\limsup_{n\to\infty} \left\|\sigma_n^T(f)\right\|_1 = \infty$*?*

Question 2 *Let* $(t_{k,n}, 1 \leq k \leq n, k \in \mathbb{P})$ *be a finite sequence of non-negative numbers satisfying conditions (23) and (24). Is it possible that for every* $f \in L_1(G)$*, there exists a subsequence of means* $\sigma_n^T(f)$ *that converges both in the* $L_1(G)$*-norm and almost everywhere, after modifying* f *on a fixed set of arbitrarily small positive measure (independent of the function)?*

Is it possible to achieve both $L_1(G)$ *norm convergence and almost everywhere convergence of a subsequence of* $\sigma_n^T(f)$ *converges for every integrable function, after modifying the function on a set of arbitrarily small positive measure that is independent of the function?*

Remark 3 Question 2 has a positive answer even in the negative-order Cesàro summability methods applied to Fourier series. See Theorem 4 in [8]. See also [12].

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