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Mixed-integer linear-fractional programming model and it's linear analogue for reducing inconsistency of pairwise comparison matrices☆

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ABSTRACT

Pairwise comparison matrices are often used in multicriteria decision making (MCDM). The most critical part of this technique is the inconsistency, which emerges for logical reasons but can cause significant problems during the decision making process. Hence it is necessary to keep inconsistency below an acceptable threshold. In order to support the decision maker (DM) in making a rational decision, we must keep in mind the following: Our suggestion should be as close to the DM's original result as possible, moreover it should have as low inconsistency as possible. We have studied various linear programming (LP) models that are used for reducing the inconsistency of pairwise comparison matrices (PCM) (Bozóki et al., 2011, 2015). These models, however aim at fulfilling only one of the previously mentioned two goals at a time. Therefore, the optimal solutions given may differ from each in the other respect, which is not taken as an objective but as a constraint in the model. So, they cannot be considered as equally good optimal solutions from a wider perspective. Based on our experiences concerning these models, we have developed a mixed-integer linear-fractional programming (MILFP) model that takes both mentioned goals as objectives by combining them into a linear-fractional objective function. We also provide the linear analogue (LA) of our MILFP model using an appropriate adaptation and combination of the Charnes-Cooper transformation and Glover's linearization scheme.

1. Introduction

The pairwise comparison technique is a fundamental tool in MCDM for making judgements about alternatives. Generally, it is a process of comparing entities in pairs to judge which entity in each pair is preferred [8]. The design of effective and precise mathematical tools for prioritizing among competing alternatives in MCDM problems is a critical component in many modern applications connected to human activity, including manufacturing, service industry, research, surveys etc.

The most popular methodology for MCDM based on pairwise comparison is the analytic hierarchy process (AHP) [20]. It has been widely used for structuring, measurement, and synthesis for example in financial decisions, system manufacturing, allocation problems and other business processes [12]. The idea of using paired comparisons instead of direct allocation when determining weights and evaluations appeared long before the AHP [24]. The 1–9 evaluation scale was also suggested

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based on psychological observations [23]. Here we have to mention that pairwise comparison is a widely used technique in decision making and several professional algorithms use it as a key component in their decision process, such as Potentially All Pairwise RanKings of all possible Alternatives (PAPRIKA) [11]. It is also widely used in survey software (e.g.: [18,16].

A lack of transitivity in human decision making is a common phenomenon which leads to inconsistent judgments. It could be attributed to factors like disturbance in concentration [19], features of human behaviour when trying to make rational decisions [15], the threshold effect or errors in the input data [9]. Inconsistency makes it hard to determine accurate performance weights based on the comparison of alternative-pairs. Therefore, consistency of judgements should be measured before creating performance weight vectors and it is also crucial to support users in making consistent comparisons. There are several consistency indices that can be used and almost all of them offer a threshold for acceptance. For more details on these measures and their comparisons we refer to [1].

We can give suggestions to the DMs how they should modify their obtained matrix of judgements in order for it to be less inconsistent. Here we have to consider the following two goals:

- Stay close to the original concept of DM, i.e. minimize the number of modified elements in the matrix.
- Reach smallish inconsistency, i.e. minimize the inconsistency of the suggested matrix.

Recent studies provide optimization models to determine the most "similar", consistent or acceptably inconsistent pairwise comparison matrix to the inconsistent one that was created by the DM [4]. Other models aim at finding the matrix with the lowest possible inconsistency level within a limited distance from the DM's matrix [3]. Our experiments show that these two goals should both be considered as goals at the same time or else obtained optimal solutions obtained cannot be considered equally good choices. As we will present later, the optimal solutions provided by the model described in [4] obviously have the same minimal distance from the original matrix, but even so they may differ from each other in their inconsistency level, so one of them could be regarded as a better solution from a wider perspective. On the other hand, the model in [3] gives optimal solutions with the same minimal inconsistency measure, but they differ from each other in their distance from the original matrix.

After revealing this phenomenon we set out to create a model that incorporates the two mentioned goals at the same time. For this we chose linear-fractional objectives so that the ratio of improvement in inconsistency and the number of modified elements is optimized. In this paper we present our MILFP model that suggests a close and less inconsistent PCM than the non-acceptable PCM of the DM. Unlike the previously mentioned optimization models, this one considers both goals, the number of modified elements (i.e. distance) and the level of inconsistency, simultaneously as an objective.

At first we give a short overview of features of pairwise comparison matrices and the importance of consistency.

Then we briefly discuss two different methods for measuring the inconsistency of pairwise comparison matrices, namely Saaty's consistency ratio (CR) in Section 2.1.1 and Koczkodaj's consistency measure (CM) in Section 2.1.3.

After that, we show LP models developed by Bozóki et al. first to make pairwise comparison matrices consistent with an upper bound on number of modified elements [3], then to find the matrix closest to the original one with an upper bound on the inconsistency level of the suggested matrix [4]. We point out practical efficiencies and difficulties of these models, and then we propose our MILFP model (in Section 4) for reducing inconsistency of pairwise comparison matrices that fixes the revealed problems of previous LP models. Since solving a MILFP may be computationally difficult and requires nonlinear solvers, we have also created the mixed-integer linear programming (MILP) equivalent of our model, so that it could be globally optimized with MILP methods.

2. Preliminaries

A DM compares pairs of elements (E_i and $E_j, \forall i,j \in \{1,2,\dots n\}$) and expresses their opinion about the relation of corresponding elements through a numerical value. So an $A = \|a_{ij}\|_{n\times n}$ pairwise comparison matrix is defined.

This kind of matrices have the following features:

- $a_{ij} \in \mathbb{R}^+, \forall i,j \in \{1,2,\dots n\}$,
- reciprocal matrix, i.e. $a_{ij} = \frac{1}{a_{ji}}, \forall i,j \in 1,2,\dots n$,
- $a_{ii} = 1, \forall i \in \{1,2,\dots n\}$,
- consistent, i.e. $a_{ij} = a_{ik}a_{kj},\ \forall i,j,k \in 1,2,\dots n$,
- $rank(A) = 1$,
- $\lambda_{max} = n$,

where λ_{max} is the largest absolute eigenvalue of the matrix A.

Consistency is of great interest to us. If this feature is not satisfied the matrix is said to be inconsistent. The level of inconsistency can be measured with various formulas, and it is a key component in decision making methods that use the pairwise comparison technique. The authors of [6] established a relation between consistency indices and levels of coherence (e.g. reciprocity, transitivity, weak consistency). Large inconsistency makes it very difficult to assign an appropriate priority weight vector for entities $E_1, E_2, \ldots, E_n$ from the matrix. Therefore, it is very important to support DMs in achieving an acceptable (very low) inconsistency level when making comparisons.

2.1. Inconsistency

A pairwise comparison matrix $A = \|a_{ij}\|_{n \times n}$ is inconsistent if $a_{ij} = a_{ik}a_{kj}, \ \forall i, j, k \in \{1, 2, \ldots n\}$, is not satisfied.

Here we briefly discuss two methods that are used to measure the inconsistency of pairwise comparison matrices. These are the Consistency Ratio provided by Saaty [20] and the Consistency Measure defined by Koczkodaj [13]. The authors of [2,4] pointed out that optimization models containing consistency measures in a general form are non-convex nonlinear optimization models. They also provided an other convex analogue of these optimization models by applying the trick of logarithmic mapping. Therefore, we will give a brief overview of the calculations of these inconsistency measures in a logarithmic space as well.

2.1.1. Consistency Ratio

Saaty showed that $\lambda_{max} \geqslant n$ and $\lambda_{max} = n$ if and only if A is a consistent pairwise comparison matrix. Then he gave the following measure of consistency of pairwise comparison matrices, called the Consistency Index (CI) [21].

$$CI(A) = \frac{\lambda_{max} - n}{n - 1}.$$

Saaty's method entails the comparison of CI with the Random Index (RI) [21]. RI estimates for positive reciprocal matrices of different orders from the integer set 1 to 9 (and their reciprocals) can be seen below.

Size	1	2	3	4	5	6	7	8	9	10
RI	0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49

The Consistency Ratio is the ratio of $CI(A)$ and RI from same size.

$$CR(A) = \frac{CI(A)}{RI_n}$$

Saaty suggested an acceptable tolerance of 10%, i.e. a pairwise comparison matrix A with consistency ratio less than 0.1 ($CR(A) \leqslant 0.1$) can be accepted and used for calculating the priority weight vector. Later Vargas gave a statistical interpretation of this 10% suggestion in [25]. Recent practices suggest an acceptance ratio that fits the size of the corresponding matrix. So in the case of smaller matrices the threshold could be smaller than for larger matrices [22].

2.1.2. Calculating CR in logarithmic space

Using an appropriate adaptation of the Frobenius theorem, Bozóki et al. [4] showed that $CR(A)$ can be defined as follows. An optimal solution of the following convex optimization problem gives the λ_{max}:

$$\min \lambda \tag{1}$$

s.t.

$$\sum_{j=1}^{n} e^{\bar{a}_{ij} + z_j - z_i} \leqslant \lambda, \ i = 1, \ldots, n,$$

where $\bar{a}_{ij} = ln(a_{ij}), \ i = 1, 2, \ldots, n, \ j = 1, 2, \ldots, n$ and $\lambda, z_i, i = 1, 2, \ldots, n$ are variables. λ^{opt}, the optimal objective of problem offers the consistency ratio of A so that $CR(A) = \frac{\lambda^{opt} - n}{RI_n(n-1)}$ and the elements of the priority weight vector can be defined as $w_i = e^{z_i}, i = 1, 2, \ldots, n$.

2.1.3. Consistency Measure

Koczkodaj highlighted the weakness of Saaty's CR as it does not point to the location of inconsistency in a pairwise comparison matrix. Koczkodaj proposed another method [13] which can detect the critical values.

As an illustration of Koczkodaj's consistency measure let us introduce a 3×3 pairwise comparison matrix B with the following elements:

$$B = \begin{bmatrix} 1 & a & b \\ \frac{1}{a} & 1 & c \\ \frac{1}{b} & \frac{1}{c} & 1 \end{bmatrix}$$

Definition 1. Koczkodaj introduces a consistency measure for 3×3 matrices as follows:

$$CM(a,b,c) = min\left\{\frac{1}{a}\left|a - \frac{b}{c}\right|, \frac{1}{b}|b - ac|, \frac{1}{c}\left|c - \frac{b}{a}\right|\right\}$$

$CM(a,b,c)$ is the smallest relative deviation between the user defined elements and the values that are expected with consistency.

Definition 2. Generalization of consistency measure for $n \times n$ matrices. Let $A = \|a_{ij}\|_{n\times n}$, where $n > 2$.

$$CM(A) = max\{CM(a,b,c)\}\text{foreach}(a,b,c)\text{triadin}A.$$

Triad (a^*, b^*, c^*) in a matrix A with the largest consistency measure (i.e. (a^*, b^*, c^*) where $CM(a^*, b^*, c^*) = CM(A)$) identifies the critical values in the PCM.

2.1.4. Calculating CM in logarithmic space

Bozóki et. al. [4,5] showed that the following convex linear programming problem provides $CM(A)$ as follows:

$$\min z \tag{2}$$

s.t.

$$\bar{a}_{ij} + \bar{a}_{jk} + \bar{a}_{ki} \leqslant z \quad 1 \leqslant i < j < k \leqslant n,$$

$$-\left(\bar{a}_{ij} + \bar{a}_{jk} + \bar{a}_{ki}\right) \leqslant z \quad 1 \leqslant i < j < k \leqslant n.$$

$CM(A) = 1 - \frac{1}{e^{z^{opt}}}$, where z^{opt} is the objective value of problem (2) and $\bar{a}_{ij} = ln(a_{ij}),\ i = 1,2,\ldots,n,\ j = 1,2,\ldots,n.$

3. Known models - Pros & Cons

Here we briefly overview two LP models for optimizing inconsistency from papers [3,4]. We have implemented these models and conducted computational tests using a set of empirical pairwise comparison matrices collected by A. Poesz [17]. This collection contains 137 matrices ranging in size from 3×3 to 16×16.

Let ϕ_n be one of the inconsistency measures defined in the previous sections, $\mathcal{P}_n$ is the set of $n \times n$ pairwise comparison matrices and α_n is an acceptance threshold of inconsistency.

$$\mathcal{A}(\phi_n, \alpha_n) = \{A \in \mathcal{P}_n | \phi_n(A) \leqslant \alpha_n\}$$

I.e. $\mathcal{A}(\phi_n, \alpha_n)$ is the set of $n \times n$ pairwise comparison matrices that do not exceed the predefined acceptance tolerance α_n.

Let $A, \widehat{A} \in \mathcal{P}_n$ and

$$d\left(A, \widehat{A}\right) = |\{(i,j) : 1 \leqslant i < j \leqslant n, a_{ij} \neq \hat{a}_{ij}\}|$$

denotes the number of matrix elements above the diagonal where these matrices differ from each other.

We discussed, in Sections 1 and 2, the convex calculation of CM and CR values in logarithmic space. This paper is confined to the presentation of our results concerning LP models, so here we only discuss CM, because it can be defined by linear expressions.

Let us introduce some notations that will be used here and in what follows (Table 1):

3.1. Optimization model for achieving acceptable inconsistency by replacing a minimum number of elements

The first model of Bozóki et al. [4] can be described in general as follows:

$$\min d\left(A, \widehat{A}\right) \tag{3}$$

s.t.

$$\widehat{A} \in \mathcal{A}(\phi_n, \alpha_n)$$

This model (3) aims to determine the pairwise comparison matrix $\widehat{A}$ closest to the DM's matrix A where $\widehat{A}$ has an acceptable (lower than α_n) consistency measure.

3.1.1. The model in logarithmic space adopting the "Big M" method and using CM

The authors formulated the model in logarithmic space in order to achieve a convex linear programming optimization model as follows:

Table 1
Parameters and descriptions.

Symbol	Description
$A = \|a_{ij}\|_{n\times n}$	PCM, created by DM. It is supposed to be inconsistent, i.e. it needs to be improved. Input matrix of the optimization process.
$\overline{A} = \|\overline{a}_{ij}\|_{n\times n}$	According to [3,4] models formulated in logarithmic space, $\overline{A} = ln(A)$.
$\widehat{A} = \|\hat{a}_{ij}\|_{n\times n}$	Suggested PCM, output of the optimization process.
$X = \|x_{ij}\|_{n\times n}$	$\widehat{A}$ in logarithmic space, $X = ln\left(\widehat{A}\right)$.
$Y = \|y_{ij}\|_{n\times n}$	Binary variables for identifying the positions where A and $\widehat{A}$ differ from each other. It is a symmetrical matrix, we only use its upper triangular portion.
M	Theoretical upper bound for judgement values. It generally comes from the evaluation scale of the pairwise comparison.
$\overline{M}$	$\overline{M} = ln(M)$, logarithmic value of M.
	Additional parameters for distance minimization model (4)
α_n	Predefined acceptance threshold for inconsistency of the output matrix.
z^*	According to the calculation of CM in logarithmic space, the corresponding inconsistency threshold value is $z^* = ln\left(\frac{1}{1-\alpha_n}\right)$.
	Additional parameters for inconsistency minimization model (6)
K	Upper bound for the number of modified elements in the input matrix.
z	Consistency measure of the output matrix in logarithmic space, based on calculations in 2.1.4, where $CM\left(\widehat{A}\right) = 1 - \frac{1}{e^z}$.
	Additional parameters for our proposed model (9)
z^{or}	Consistency measure of the input matrix A in logarithmic space, i.e. $z^{or} = ln\left(\frac{1}{1-CM(A)}\right)$
M_2	Theoretical bound for the number of matrix elements above the diagonal. It is required in linearization.

$$\min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} \tag{4}$$

s.t.

$$\begin{aligned}
x_{ij} + x_{jk} + x_{ki} \leqslant z^* \quad & 1 \leqslant i < j < k \leqslant n, \\
-\left(x_{ij} + x_{jk} + x_{ki}\right) \leqslant z^* \quad & 1 \leqslant i < j < k \leqslant n, \\
x_{ij} = -x_{ji}, \quad & 1 \leqslant i \leqslant j \leqslant n, \\
-\overline{M} \leqslant x_{ij} \leqslant \overline{M}, \quad & 1 \leqslant i < j \leqslant n, \\
-2\overline{M}y_{ij} \leqslant x_{ij} - \overline{a}_{ij} \leqslant 2\overline{M}y_{ij}, \quad & 1 \leqslant i < j \leqslant n, \\
y_{ij} \in \{0,1\}, 1 \leqslant i < j \leqslant n. &
\end{aligned}$$

The first two constraints limit the consistency measure so that $CM(e^X)$ (= $CM\left(\widehat{A}\right)$) $\leqslant \alpha_n$. The third constraint comes from the reciprocity property in logarithmic space. The fourth constraint is about the theoretical limits of evaluations. Finally comes the constraint on the distance between the input and output matrices.

The objective value of (4) gives the minimal number of elements above the main diagonal in matrix A that should be modified to achieve $CM(e^X) \leqslant \alpha_n$.

3.1.2. Practical efficiency of the model

We have implemented two variants of model (4) in Lingo [14] where $\alpha_n = 0.1$ and $\alpha_n = 0$ and tested them on the data set [17].

The following practical problem was revealed during the computational tests. Let X^* denote the optimal solution of (4) and let d^* be the optimal objective value i.e. the minimal distance between the two matrices $A, \widehat{A}$. Because of the first two constraints on the inconsistency threshold, $CM(e^{X^*}) \leqslant \alpha_n$. Let's suppose (4) has multiple optimal solutions, and $X\prime$ denotes another optimal solution of the model. So $X\prime$ is a matrix with the same optimal distance d^* from the original matrix $\overline{A}$, but it may have a smaller consistency measure, $CM(e^{X\prime}) < CM(e^{X^*}) \leqslant \alpha_n$, because this feature appears as an upper bound limit. Considering the decision process this $X\prime$ is a "better" optimal solution than X^*. Even though they have the same optimal distance $d^*, X\prime$ may have smaller inconsistency, which means a better quality PCM.

We have often observed that the number of elements that are required to be modified is the same when α_n is set to be 0.1 or 0. In other words the matrix A can be made consistent by modifying the same number of elements that are required for an acceptable consistency, but putting other appropriate values in these positions. In 105 out of the total 137 test cases matrix can be made consistent by modifying the same number of elements as it was required for an acceptable inconsistency level (Table 2). Although the model with parameter $\alpha_n = 0.1$ may have more optimal solutions (also solutions with 0 consistency measure), optimization typically results in one of them which $CM(e^X) = \alpha_n = 0.1$, i.e. where the constraints on the consistency measure are bounded.

Nevertheless, setting α_n to 0 by default is not a good choice because there are matrices that require a huge number of elements to be changed in order to be consistent. You can also see in the test results, that sometimes more than 30% of the elements need to be replaced in order to make the matrix consistent.

Let us see a numerical example for this phenomenon (Test case $M23$). Let A be a 3×3 pairwise comparison matrix:

$$A = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{5} \\ 3 & 1 & \frac{1}{2} \\ 5 & 2 & 1 \end{pmatrix}$$

$CM(A) = 0.1667$.

Applying the optimization model (4) with parameter $\alpha_3 = 0.1$, we find that at least one item be modified to obtain the following optimal solution by Lingo:

$$\widehat{A} = e^X = \begin{pmatrix} 1 & \mathbf{\frac{9}{25}} & \frac{1}{5} \\ \mathbf{\frac{25}{9}} & 1 & \frac{1}{2} \\ 5 & 2 & 1 \end{pmatrix}$$

The modified elements with respect to A are highlighted in bold. $CM\left(\widehat{A}\right) = 0.1$.

If we select another value for $a_{12} = \frac{2}{5}$, we get another optimal solution ($d(A, A\prime) = 1$):

$$A\prime = \begin{pmatrix} 1 & \mathbf{\frac{2}{5}} & \frac{1}{5} \\ \mathbf{\frac{5}{2}} & 1 & \frac{1}{2} \\ 5 & 2 & 1 \end{pmatrix}$$

but $CM(A\prime) = 0$.

We did not know for sure if there was another optimal solution with a lower consistency measure or not, only multiply executions can tell whether there are "better" optimal solutions.

3.2. Minimizing inconsistency level by replacing a few elements in pairwise comparison matrices

The second model of Bozóki et al. [3] determines matrix $\widehat{A}$ with minimal inconsistency whose distance from the original matrix A is at most K, and the authors defined three cases for K (1, 2 and 3) (model (5)).

$$\min \alpha_n \tag{5}$$

s.t.

$$d\left(A, \widehat{A}\right) \leqslant K,$$

$$\widehat{A} \in \mathcal{A}(\phi_n, \alpha_n)$$

3.2.1. The model in logarithmic space adopting the "Big M" method and using CM

$$\min z \tag{6}$$

s.t.

$$\begin{aligned}
& x_{ij} + x_{jk} + x_{ki} <= z, \quad 1 \leqslant i < j < k \leqslant n, \\
& -\left(x_{ij} + x_{jk} + x_{ki}\right) <= z, \quad 1 \leqslant i < j < k \leqslant n, \\
& \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} y_{ij} \quad \leqslant K, \\
& x_{ij} = -x_{ji}, \quad 1 \leqslant i \leqslant j \leqslant n, \\
& -\overline{M} \leqslant x_{ij} \leqslant \overline{M}, \quad 1 \leqslant i < j \leqslant n, \\
& -2\overline{M}y_{ij} \leqslant x_{ij} - \overline{a}_{ij} \leqslant 2\overline{M}y_{ij}, \quad 1 \leqslant i < j \leqslant n,
\end{aligned}$$

$$y_{ij} \in \{0, 1\}, 1 \leqslant i < j \leqslant n$$

This inconsistency minimizing model gives the output matrix with the smallest possible inconsistency measure by replacing at most K elements in the input matrix A. $CM\left(\widehat{A}\right) = 1 - \frac{1}{e^{(z^{opt})}}$, where z^{opt} is the optimal objective value.

3.2.2. Practical efficiency of the model

We have implemented three variants of model (6) in Lingo for cases $K = 1, K = 2$ and $K = 3$ and tested on [17].

The following practical issues should be considered here:

- Because the number of modified elements appears as a constraint (instead of being an objective), optimal solutions (with the same *CM*) may differ from each other in their distance from the original matrix.
- The value of *K* should be reconciled with the size of the matrix. Modifying 3 elements above the diagonal in a 3×3 matrix means replacing the whole matrix.
- No preliminary information is available about the possibility to reaching a smaller *CM* by increasing the value of *K*.

Fig. 1 shows the average improvement in CM ($CM\left(\widehat{A}\right) - CM(A)$) grouped by size and separated by the value of parameter *K*. We can see that allowing 3 elements, instead of $K = 2$ or $K = 1$, to be modified can make an enormous improvement in the case of smaller (i.e. $< 8 \times 8$) matrices, moreover this means a decisive change of the input matrix. While in the case of larger (i.e. $> 8 \times 8$) matrices the CM improvement for different *K* values is unremarkable. The test data set contains only one 16×16 matrix.).

We should reconcile the acceptable distance and the size of the matrix and also check if it is worth initiating further changes to get lower CM.

4. Main result - MILFP model for decreasing inconsistency

To avoid the above mentioned practical problems we have revealed in the previous two models, we have developed the following MILFP optimization model where objective function considers both goals at the same time as a ratio of them, thus it maximizes the improvement in CM per unit of distance.

Our LFP model in a general form:

$$\max\ Q\left(\widehat{A}\right) = \frac{\phi_n(A) - \phi_n\left(\widehat{A}\right)}{d\left(A, \widehat{A}\right)} \tag{7}$$

st.

$$\widehat{A} \in \mathcal{P}_n$$

4.1. The model in logarithmic space using the "Big M" method

We adopted the techniques that were used by Bozóki et al. [3,4] in order to transform our model for logarithmic space.

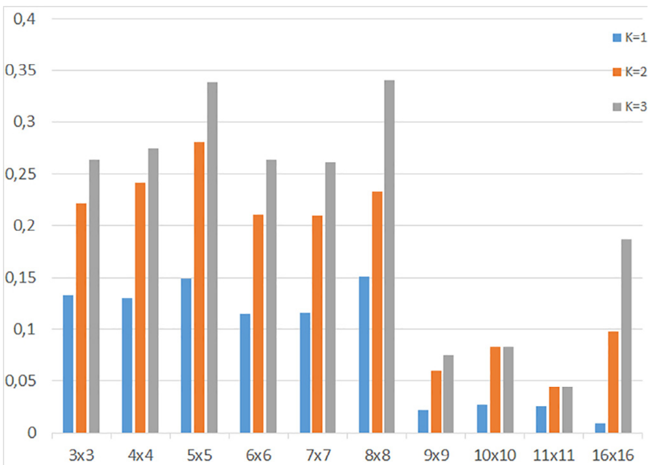

Fig. 1. Average improvement in CM grouped by size and distance.

$$\max \frac{\phi_n(A) - \alpha}{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} y_{ij}} \tag{8}$$

s.t.

$$\begin{aligned} \phi_n\left(e^X\right) &= \alpha, \\ x_{ij} &= -x_{ji}, \quad 1 \leqslant i \leqslant j \leqslant n, \\ -\overline{M} \leqslant x_{ij} &\leqslant \overline{M}, \quad 1 \leqslant i < j \leqslant n, \\ -2\overline{M}y_{ij} \leqslant x_{ij} - \overline{a}_{ij} &\leqslant 2\overline{M}y_{ij}, \quad 1 \leqslant i < j \leqslant n, \\ y_{ij} &\in \{0,1\}, 1 \leqslant i < j \leqslant n. \end{aligned}$$

4.2. Adopting CM into the model

After inserting Koczkodaj's consistency measure into model (8) we obtain the following MILFP model:

$$\max \frac{z^{or} - z}{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} y_{ij}} \tag{9}$$

s.t.

$$\begin{aligned} x_{ij} + x_{jk} + x_{ki} &\leqslant z \quad 1 \leqslant i \leqslant j \leqslant n, \\ -\left(x_{ij} + x_{jk} + x_{ki}\right) &\leqslant z \quad 1 \leqslant i \leqslant j \leqslant n, \\ x_{ij} &= -x_{ji}, \quad 1 \leqslant i \leqslant j \leqslant n, \\ -\overline{M} \leqslant x_{ij} &\leqslant \overline{M}, \quad 1 \leqslant i < j \leqslant n, \\ -2\overline{M}y_{ij} \leqslant x_{ij} - \overline{a}_{ij} &\leqslant 2\overline{M}y_{ij}, \quad 1 \leqslant i < j \leqslant n, \\ y_{ij} &\in \{0,1\}, \quad 1 \leqslant i < j \leqslant n. \end{aligned}$$

The model requires the consistency measure of the original matrix as an input, i.e. $z^{or} = ln\left(\frac{1}{1-CM(A)}\right)$. Moreover $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} y_{ij}$ is supposed to be larger than 0. Practically it means that changes to the input matrix A are supposed to be needed.

The optimal solution of model (9) is the matrix $X^* = \|x^*_{ij}\|_{n\times n}$ where the amount of improvement in consistency measure relative to the number of modified elements is the largest.

4.3. Linear Analogue of the MILFP model

Here we provide the linear analogue of our MILFP model. Thus, the complexity and the computational effort required in solving a MILFP problem can be reduced.

In the case of model (9) we cannot apply the Charnes-Cooper transformation [7], because it is restricted to cases where variables are continuous. Our model contains binary variables as well, so we have to use a special combination of the Charnes-Cooper transformation and Glover's linearization scheme [10]. This two-step, reformulation–linearization method for MILFPs with binary variables was proposed by Yue et al. in [26]:

Consider a general MILFP with binary variables.

1. Use the Charnes-Cooper transformation except for the binary variables to obtain a Mixed Integer Non Linear Programming (MINLP) model.
2. Use Glover's linearization scheme on the previously obtained MINLP model to transform it into MILP form.

The appropriate adaptation of this method for model (9):

After step 1. (i.e. the Charnes-Cooper transformation) the following MINLP model was obtained:

$$\max z^{or}t_0 - t^z \tag{10}$$

s.t.

$$
\begin{aligned}
t^x_{ij} + t^x_{jk} + t^x_{ki} - t^z \leqslant 0 \qquad & 1 \leqslant i \leqslant j \leqslant n, \\
-\left(t^x_{ij} + t^x_{jk} + t^x_{ki}\right) - t^z \leqslant 0 \qquad & 1 \leqslant i \leqslant j \leqslant n, \\
t^x_{ij} + t^x_{ji} = 0, \quad & 1 \leqslant i \leqslant j \leqslant n, \\
-\overline{M}t_0 \leqslant t^x_{ij} \leqslant \overline{M}t_0, \quad & 1 \leqslant i < j \leqslant n, \\
-2\overline{M}y_{ij}t_0 \leqslant t^x_{ij} - \overline{a}_{ij}t_0 \leqslant 2\overline{M}y_{ij}t_0, \quad & 1 \leqslant i < j \leqslant n, \\
\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} t_0 y_{ij} = 1, \quad & \\
y_{ij} \in \{0,1\}, \quad & 1 \leqslant i < j \leqslant n \\
t_0 \geqslant 0 \quad &
\end{aligned}
$$

where $t_0 = \frac{1}{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} y_{ij}}, t^z = \frac{z}{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} y_{ij}}, t^x_{lk} = \frac{x_{lk}}{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} y_{ij}},\ 1 \leqslant l \leqslant k \leqslant n.$

Here (10) we still have non-linear constraints, so Glover's linearization scheme should be applied. Then, we obtained the Linear Analogue of the original MILFP model (9):

$$\max z^{or}t_0 - t^z \tag{11}$$

s.t.

$$
\begin{aligned}
t^x_{ij} + t^x_{jk} + t^x_{ki} - t^z \leqslant 0 \qquad & 1 \leqslant i \leqslant j \leqslant n, \\
-\left(t^x_{ij} + t^x_{jk} + t^x_{ki}\right) - t^z \leqslant 0 \qquad & 1 \leqslant i \leqslant j \leqslant n, \\
t^x_{ij} + t^x_{ji} = 0, \quad & 1 \leqslant i \leqslant j \leqslant n, \\
-\overline{M}t_0 \leqslant t^x_{ij} \leqslant \overline{M}t_0, \quad & 1 \leqslant i < j \leqslant n, \\
-2\overline{M}t^y_{ij} \leqslant t^x_{ij} - \overline{a}_{ij}t_0 \leqslant 2\overline{M}t^y_{ij}, \quad & 1 \leqslant i < j \leqslant n, \\
\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} t^y_{ij} = 1, \quad & \\
t^y_{ij} \leqslant t_0 \quad & 1 \leqslant i < j \leqslant n, \\
t^y_{ij} \leqslant M_2 q_{ij} \quad & 1 \leqslant i < j \leqslant n, \\
t^y_{ij} \leqslant t_0 - M_2(1 - q_{ij}) \quad & 1 \leqslant i < j \leqslant n, \\
q_{ij} \in \{0,1\}, \quad & 1 \leqslant i < j \leqslant n, \\
t^y_{ij} \geqslant 0 \quad & 1 \leqslant i < j \leqslant n, \\
t_0 \geqslant 0 \quad &
\end{aligned}
$$

where $t^y_{kl} = \frac{y_{lk}}{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} y_{ij}},\ 1 \leqslant i < j \leqslant n$. M_2 is an upper bound that depends on the size of matrix A. Pairwise comparison matrices are typically not large in size so M_2 can be easily defined.

4.4. Practical efficiency of the model

We have implemented the model (11) in Lingo and conducted computational tests with the earlier mentioned test collection.

Although the size of the model is extended by the linearization, it does not cause a significant increase in the runtime, because pairwise comparison matrices are typically small.

The structure of our model does not allows multiple optimal solutions to differ from each other in any of the above mentioned objectives: distance or consistency measure. Let us prove this statement:

Let X^* denote the matrix defined by the optimal solution of model (8).

- It is sure that there does not exist any other matrix with smaller CM than the provided optimal solution X^* with the same distance from the original one.
  Let us suppose $X\prime$ is a PCM where $d(A, e^{X\prime}) = d(A, e^{X^*})$.
  If $CM(e^{X\prime}) < CM(e^{X^*})$, then $\frac{CM(A) - CM(e^{X\prime})}{d(A, e^{X\prime})} > \frac{CM(A) - CM(e^{X^*})}{d(A, e^{X^*})}$, but this cannot be true because of the optimality of X^*, which says $Q(e^X) \leqslant Q(e^{X^*}),\ \forall \widehat{A}(= e^X) \in \mathcal{P}_n$.

- It is also sure that there is not any pairwise comparison matrix with the same consistency measure as $CM(e^{X^*})$ which is closer to the original matrix than the obtained optimal solution.

  Let us suppose $X\prime$ is a PCM where $CM(e^{X\prime}) = CM(e^{X^*})$ and $d(A, e^{X\prime}) < d(A, e^{X^*})$. Then, $\frac{CM(A)-CM(e^{X\prime})}{d(A,e^{X\prime})} > \frac{CM(A)-CM(e^{X^*})}{d(A,e^{X^*})}$, which conflicts with the fact that X^* is an optimal solution. So for any other $\widehat{A} = e^X \in \mathcal{P}_n$ feasible solution where $CM(e^X) = CM(e^{X^*})$, it is sure that $\frac{CM(A)-CM(e^X)}{d(A,e^X)} <= \frac{CM(A)-CM(e^{X^*})}{d(A,e^{X^*})}$ i.e. $d(A, e^X) \geqslant d(A, e^{X^*})$.

5. Conclusion

AHP is one of the most commonly used decision support algorithm. Pairwise comparison is a basic tool in this method and in several other expert systems in MCDM. It is known that the inconsistency of pairwise comparison matrices highly influences the success of the decision process. We have studied linear optimization models that propose methods to make a matrix consistent or acceptably inconsistent within a predefined neighbourhood of the original matrix. We have highlighted the critical points of these models as they consider only one of the following two goals at a time:

- Minimizing the distance from the original matrix to obtain an inconsistency level below a predefined limit.
- Minimizing the inconsistency level to remain within a predefined distance from the original matrix.

This results in that multiple optimal solutions with the same, optimal inconsistency level may differ from each other in their distance in an inconsistency minimization problem and similarly they may differ in their consistency measure in a distance minimization problem where minimal distance is what they have in common.

In contrast with the models described above, our MILFP model allows us to consider both mentioned goals at the same time, as it formulates a linear fractional objective function from these two linear objectives. Moreover, we have created a linear programming equivalent of our model, thus it can be solved by linear solvers, when the situation requires so.

We have conducted computational experiments on a collection of pairwise comparison matrices [17], the main results are shown in Table 2 and in Figs. 2–4, where test cases are ordered and grouped by size.

Fig. 2 compares the percentages of matrix elements that were suggested to be modified by models (4) and (7). We can see that a large percentage of matrix elements were required to be changed in order to satisfy the constraint $CM(\widehat{A}) <= \alpha$, where $\alpha = 0.1$ or 0. In the case of model (6), the number of modified elements are regularly fixed by the constraint $d(\widehat{A}, A) \leqslant K$, therefore the results corresponding to this model are not listed in this figure.

The obtained improvement in *CM* is shown in Fig. 3. We can see that the larger the modification the higher the improvement can be. The question is whether it is worth changing a significant part of the matrix in order to obtain an almost consistent variant of it.

Finally, Fig. 4 presents the ratio of improvement in consistency measure and the number of modified elements. Considering the improvement in *CM* per unit of distance, no optimal suggestion is given by the different variants of models (6) and (4). It is model (7) that results in the highest ratio in each test case.

Finally, we have to note that the numerical experiments and analysis of optimality criteria clearly prove that the problem concerning multiple optimal solutions with different distance or inconsistency can be avoided by applying our model.

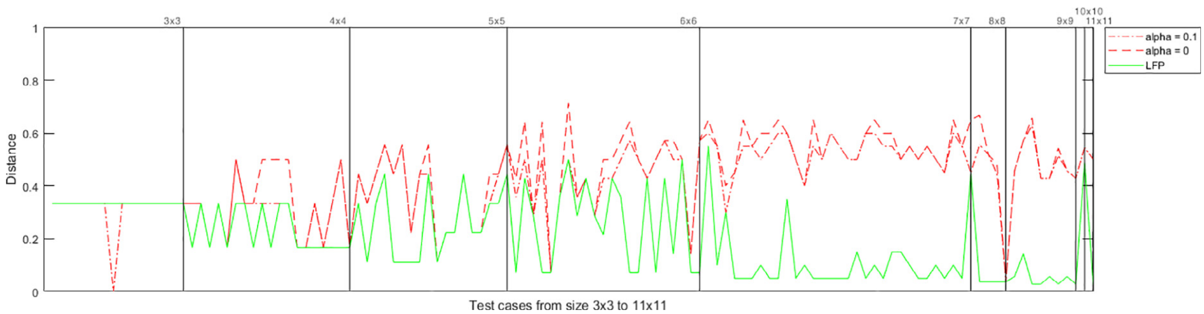

Fig. 2. Number of modified elements.

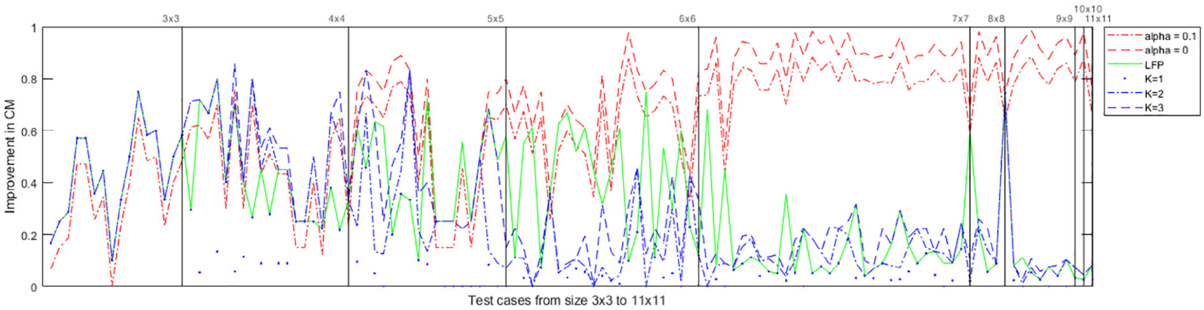

Fig. 3. Obtained improvement in CM value.

Fig. 4. Improvement per distance.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

2

Table 2
Test result.

Input matrix			Distance minimization model (4)				*CM minimization model (6)*			Our model (11)	
Name	Size	$CM(A)$	$\alpha_n = 0.1$		$\alpha_n = 0$		$K = 1$	$K = 2$	$K = 3$		
			$d(A,\hat{A})$	$CM(\hat{A})$	$d(A,\hat{A})$	$CM(\hat{A})$	$CM(\hat{A})$	$CM(\hat{A})$	$CM(\hat{A})$	$d(A,\hat{A})$	$CM(\hat{A})$
M1	7x7	0,837	12	0,1	13	0	0,837	0,833	0,771	11	0,155
M2	6x6	0,667	5	0,1	6	0	0,556	0,556	0,444	1	0,556
M3	6x6	0,775	7	0,1	9	0	0,743	0,667	0,625	6	0,225
M4	6x6	0,611	4	0,1	4	0	0,611	0,611	0,611	4	0
M5	6x6	0,750	7	0,1	9	0	0,679	0,667	0,625	1	0,679
M6	6x6	0,357	1	0,1	1	0	0	0	0	1	0
M7	6x6	0,625	5	0,1	5	0	0,571	0,571	0,556	5	0
M8	6x6	0,700	7	0,1	10	0	0,667	0,625	0,611	7	0,032

Table 2 (*continued*)

Input matrix			Distance minimization model (4)				*CM minimization model (6)*			Our model (11)	
Name	Size	$CM(A)$	$\alpha_n = 0.1$		$\alpha_n = 0$		$K = 1$	$K = 2$	$K = 3$		
			$d(A,\hat{A})$	$CM(\hat{A})$	$d(A,\hat{A})$	$CM(\hat{A})$	$CM(\hat{A})$	$CM(\hat{A})$	$CM(\hat{A})$	$d(A,\hat{A})$	$CM(\hat{A})$
M9	6x6	0,640	5	0,1	5	0	0,571	0,533	0,533	4	0,122
M10	6x6	0,611	6	0,1	6	0	0,556	0,533	0,417	6	0
M11	6x6	0,444	4	0,1	4	0	0,444	0,444	0,444	4	0
M12	8x8	0,981	15	0,1	18	0	0,771	0,75	0,722	1	0,771
M13	8x8	0,889	14	0,1	14	0	0,833	0,75	0,667	1	0,833
M14	5x5	0,762	4	0,1	4	0	0,667	0,533	0,155	3	0,155
M15	5x5	0,833	3	0,1	3	0	0,375	0,167	0	1	0,375
M16	5x5	0,800	4	0,1	4	0	0,75	0,667	0,167	3	0,167
M17	5x5	0,750	5	0,1	5	0	0,75	0,625	0,5	4	0,134
M18	5x5	0,867	4	0,1	4	0	0,667	0,556	0,4	1	0,667
M19	5x5	0,889	5	0,1	5	0	0,533	0,444	0,333	1	0,533
M20	5x5	0,833	2	0,1	2	0	0,5	0	0	1	0,5
M21	5x5	0,500	4	0,1	4	0	0,4	0,286	0,143	1	0,4
M22	4x4	0,714	2	0,1	2	0	0,42	0	0	1	0,42
M23	3x3	0,167	1	0,1	1	0	0	0	0	1	0
M24	3x3	0,250	1	0,1	1	0	0	0	0	1	0
M25	5x5	0,800	4	0,1	5	0	0,714	0,667	0,4	4	0,074
M26	7x7	0,960	11	0,1	11	0	0,933	0,88	0,833	2	0,88
M27	3x3	0,286	1	0,1	1	0	0	0	0	1	0
M28	4x4	0,720	2	0,1	2	0	0,667	0	0	2	0
M29	4x4	0,000	0	0	0	0	0	0	0	4	0
M30	4x4	0,667	1	0,1	1	0	0	0	0	1	0
M31	4x4	0,800	2	0,1	2	0	0,667	0	0	2	0
M32	4x4	0,400	1	0,1	1	0	0	0	0	1	0
M33	4x4	0,857	3	0,1	3	0	0,8	0,155	0	2	0,155
M34	4x4	0,400	2	0,1	2	0	0,286	0	0	2	0
M35	4x4	0,800	2	0,1	2	0	0,533	0	0	1	0,533
M36	4x4	0,533	2	0,1	3	0	0,444	0,083	0	2	0,083
M37	4x4	0,611	2	0,1	3	0	0,333	0,034	0	1	0,333
M38	4x4	0,533	2	0,1	3	0	0,444	0,083	0	2	0,083
M39	4x4	0,533	2	0,1	3	0	0,444	0,083	0	2	0,083
M40	3x3	0,571	1	0,1	1	0	0	0	0	1	0
M41	3x3	0,571	1	0,1	1	0	0	0	0	1	0
M42	3x3	0,000	0	0	0	0	0	0	0	1	0
M43	3x3	0,000	0	0	0	0	0	0	0	1	0
M44	5x5	0,250	1	0,1	1	0	0	0	0	1	0
M45	4x4	0,250	1	0,1	1	0	0	0	0	1	0
M46	4x4	0,250	1	0,1	1	0	0	0	0	1	0
M47	4x4	0,500	2	0,1	2	0	0,25	0	0	1	0,25
M48	5x5	0,250	2	0,1	2	0	0,25	0	0	2	0
M49	5x5	0,250	2	0,1	2	0	0,25	0	0	2	0
M50	5x5	0,556	4	0,1	4	0	0,556	0,333	0,333	4	0
M51	5x5	0,250	2	0,1	2	0	0,25	0	0	2	0
M52	5x5	0,500	2	0,1	2	0	0,5	0	0	2	0
M53	8x8	0,963	12	0,1	13	0	0,875	0,875	0,875	1	0,875
M54	5x5	0,750	3	0,1	4	0	0,667	0,611	0,065	3	0,065
M55	6x6	0,816	6	0,1	7	0	0,775	0,743	0,5	3	0,5
M56	6x6	0,467	6	0,1	7	0	0,444	0,444	0,333	6	0,02
M57	7x7	0,533	6	0,1	8	0	0,533	0,444	0,444	6	0,083
M58	5x5	0,743	4	0,1	4	0	0,743	0,652	0,258	3	0,258
M59	3x3	0,000	0	0	0	0	0	0	0	2	0
M60	3x3	0,000	0	0	0	0	0	0	0	1	0
M61	3x3	0,000	0	0	0	0	0	0	0	2	0
M62	3x3	0,000	0	0	0	0	0	0	0	1	0
M63	3x3	0,000	0	0	0	0	0	0	0	1	0
M64	3x3	0,000	0	0	0	0	0	0	0	1	0
M65	3x3	0,000	0	0	0	0	0	0	0	1	0
M66	3x3	0,357	1	0,1	1	0	0	0	0	1	0
M67	3x3	0,000	0	0	0	0	0	0	0	1	0
M68	3x3	0,000	0	0	0	0	0	0	0	1	0
M69	3x3	0,000	0	0	0	0	0	0	0	1	0
M70	3x3	0,444	1	0,1	1	0	0	0	0	1	0
M71	3x3	0,000	0	0	0	0	0	0	0	1	0
M72	4x4	0,222	1	0,1	1	0	0	0	0	1	0
M73	3x3	0,100	0	0,1	1	0	0	0	0	1	0

(*continued on next page*)

Table 2 (*continued*)

Input matrix			Distance minimization model (4)				*CM minimization model (6)*			Our model (11)	
Name	Size	$CM(A)$	$\alpha_n = 0.1$		$\alpha_n = 0$		$K = 1$	$K = 2$	$K = 3$		
			$d(A,\hat{A})$	$CM(\hat{A})$	$d(A,\hat{A})$	$CM(\hat{A})$	$CM(\hat{A})$	$CM(\hat{A})$	$CM(\hat{A})$	$d(A,\hat{A})$	$CM(\hat{A})$
M74	5x5	0,800	5	0,1	5	0	0,8	0,729	0,652	4	0,222
M75	4x4	0,667	2	0,1	2	0	0,286	0	0	1	0,286
M76	8x8	0,743	1	0,1	1	0	0	0	0	1	0
M77	5x5	0,000	0	0	0	0	0	0	0	6	0
M78	4x4	0,750	3	0,1	3	0	0,533	0,188	0	1	0,533
M79	9x9	0,880	16	0,1	16	0	0,857	0,8	0,8	2	0,8
M80	9x9	0,944	20	0,1	20	0	0,944	0,933	0,917	5	0,833
M81	9x9	0,987	22	0,1	23	0	0,933	0,914	0,88	1	0,933
M82	9x9	0,914	15	0,1	15	0	0,889	0,889	0,857	1	0,889
M83	9x9	0,875	15	0,1	15	0	0,875	0,8	0,8	2	0,8
M84	9x9	0,939	18	0,1	19	0	0,898	0,898	0,857	1	0,898
M85	9x9	0,960	16	0,1	16	0	0,96	0,857	0,857	2	0,857
M86	9x9	0,889	15	0,1	15	0	0,857	0,816	0,816	1	0,857
M87	7x7	0,875	9	0,1	9	0	0,812	0,8	0,8	1	0,812
M88	7x7	0,944	11	0,1	13	0	0,857	0,792	0,75	1	0,857
M89	6x6	0,810	7	0,1	8	0	0,8	0,767	0,72	5	0,2
M90	11x11	0,978	27	0,1	27	0	0,952	0,933	0,933	1	0,952
M91	3x3	0,333	1	0,1	1	0	0	0	0	1	0
M92	3x3	0,500	1	0,1	1	0	0	0	0	1	0
M93	4x4	0,333	1	0,1	1	0	0	0	0	1	0
M94	6x6	0,978	8	0,1	9	0	0,88	0,8	0,667	1	0,88
M95	16x16	0,987		0,1		0	0,978	0,889	0,8	2	0,889
M96	7x7	0,933	11	0,1	11	0	0,821	0,75	0,733	1	0,821
M97	7x7	0,857	10	0,1	12	0	0,816	0,762	0,743	2	0,762
M98	7x7	0,857	11	0,1	12	0	0,8	0,751	0,743	1	0,8
M99	7x7	0,939	12	0,1	13	0	0,889	0,857	0,816	1	0,889
M100	7x7	0,800	12	0,1	12	0	0,778	0,762	0,75	7	0,444
M101	7x7	0,978	10	0,1	10	0	0,926	0,88	0,8	1	0,926
M102	7x7	0,889	8	0,1	8	0	0,889	0,667	0,667	2	0,667
M103	7x7	0,983	11	0,1	13	0	0,933	0,8	0,8	1	0,933
M104	7x7	0,933	10	0,1	10	0	0,857	0,8	0,8	1	0,857
M105	7x7	0,967	12	0,1	12	0	0,917	0,886	0,733	1	0,917
M106	7x7	0,889	11	0,1	11	0	0,8	0,667	0,667	1	0,8
M107	7x7	0,978	10	0,1	10	0	0,8	0,778	0,733	1	0,8
M108	7x7	0,889	10	0,1	10	0	0,857	0,8	0,571	3	0,571
M109	7x7	0,898	12	0,1	12	0	0,857	0,857	0,8	1	0,857
M110	7x7	0,880	12	0,1	13	0	0,85	0,812	0,8	2	0,812
M111	7x7	0,889	11	0,1	12	0	0,8	0,8	0,667	1	0,8
M112	7x7	0,880	11	0,1	12	0	0,857	0,8	0,72	3	0,72
M113	7x7	0,960	10	0,1	10	0	0,933	0,889	0,667	3	0,667
M114	7x7	0,857	11	0,1	11	0	0,8	0,7	0,667	2	0,7
M115	7x7	0,889	10	0,1	10	0	0,8	0,667	0,667	1	0,8
M116	7x7	0,889	11	0,1	11	0	0,762	0,741	0,667	1	0,762
M117	7x7	0,933	10	0,1	10	0	0,889	0,8	0,8	2	0,8
M118	7x7	0,889	9	0,1	9	0	0,889	0,762	0,667	1	0,8
M119	7x7	0,889	12	0,1	13	0	0,867	0,8	0,8	2	0,8
M120	7x7	0,943	11	0,1	11	0	0,8	0,7	0,7	1	0,8
M121	7x7	0,688	9	0,1	13	0	0,667	0,667	0,583	9	0,087
M122	6x6	0,833	7	0,1	7	0	0,625	0,375	0,375	1	0,625
M123	6x6	0,750	6	0,1	6	0	0,75	0,75	0,625	6	0
M124	6x6	0,778	7	0,1	7	0	0,667	0,611	0,556	1	0,667
M125	6x6	0,833	8	0,1	8	0	0,8	0,741	0,667	6	0,3
M126	6x6	0,800	7	0,1	8	0	0,75	0,5	0,375	2	0,5
M127	6x6	0,600	7	0,1	7	0	0,6	0,583	0,5	7	0
M128	6x6	0,429	2	0,1	2	0	0,2	0	0	1	0,2
M129	6x6	0,833	8	0,1	8	0	0,714	0,714	0,5	1	0,714
M130	3x3	0,750	1	0,1	1	0	0	0	0	1	0
M131	3x3	0,000	0	0	0	0	0	0	0	1	0
M132	3x3	0,583	1	0,1	1	0	0	0	0	1	0
M133	3x3	0,600	1	0,1	1	0	0	0	0	1	0
M134	3x3	0,333	1	0,1	1	0	0	0	0	1	0
M135	3x3	0,500	1	0,1	1	0	0	0	0	1	0
M136	3x3	0,583	1	0,1	1	0	0	0	0	1	0
M137	10x10	0,750	24	0,1	24	0	0,722	0,667	0,667	2	0,667

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