



Characterizations of second-order differential operators

Włodzimierz Fechner¹ · Eszter Gselmann² ·
Aleksandra Świątczak-Kolenda¹

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Abstract

Let N, k be positive integers with $k \geq 2$, and $\Omega \subset \mathbb{R}^N$ be a domain. By the well-known properties of the Laplacian and the gradient, we have

$$\nabla^2(f \cdot g) = g \nabla^2 f + f \nabla^2 g + 2\langle \nabla f, \nabla g \rangle$$

for all $f, g \in \mathcal{C}^k(\Omega, \mathbb{R})$. H. König and V. Milman showed that the converse is also true, i.e. this operator equation characterizes the Laplacian and the gradient under some assumptions. Thus the main aim of this paper is to provide an extension of this result and to study the corresponding equation

$$T(f \cdot g) = fT(g) + T(f)g + 2B(A(f), A(g)) \quad (f, g \in P),$$

where Q and R are commutative rings, P is a subring of Q and $T: P \rightarrow Q$ and $A: P \rightarrow R$ are additive, while $B: R \times R \rightarrow Q$ is a symmetric and bi-additive. Related identities with one function will also be considered.

Keywords Leibniz rule · Laplacian · Differential operator · Additive operator · Generalized polynomial

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Włodzimierz Fechner and Eszter Gselmann have contributed equally to this work

✉ Eszter Gselmann
gselmann@science.unideb.hu
Włodzimierz Fechner
wlodzimierz.fechner@p.lodz.pl
Aleksandra Świątczak-Kolenda
aleswi97@gmail.com

¹ Institute of Mathematics, Lodz University of Technology, al. Politechniki 8, 93-590 Łódź, Poland

² Department of Analysis, Institute of Mathematics, University of Debrecen, Egyetem tér 1., Debrecen 4032, Hungary

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1 Introduction

Let Q and R be commutative rings and P be a subring of Q . In this paper we will study additive mappings $T: P \rightarrow Q$ and $A: P \rightarrow R$ for which we have

$$T(f \cdot g) = fT(g) + T(f)g + 2B(A(f), A(g)) \quad (\Delta)$$

for all $f, g \in P$ with an appropriate symmetric and bi-additive mapping $B: R \times R \rightarrow Q$.

As we will see Eq. (Δ) plays an important role in the characterization of the Laplacian. Indeed, if N is a positive integer and $\Omega \subset \mathbb{R}^N$ is a domain (by a domain we mean a nonempty, open and connected set) and we set $P = \mathcal{C}^2(\Omega, \mathbb{R})$, $Q = \mathcal{C}(\Omega, \mathbb{R})$, $R = \mathcal{C}(\Omega, \mathbb{R}^N)$ (by $\mathcal{C}^k(\Omega, \mathbb{R})$ with k being a nonnegative integer we mean the space of real-valued k -times continuously differentiable functions on Ω). Further, if

$$T(f)(x) = \nabla^2 f(x) = \sum_{i=1}^N \frac{\partial^2 f}{\partial x_i^2}(x) \quad \left(f \in \mathcal{C}^2(\Omega, \mathbb{R}), x \in \Omega \right)$$

and

$$A(f)(x) = f'(x) = \nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_N}(x) \end{pmatrix} \quad \left(f \in \mathcal{C}^2(\Omega, \mathbb{R}), x \in \Omega \right)$$

and

$$B(u, v) = \langle u, v \rangle = \sum_{i=1}^N u_i v_i \quad \left(u, v \in \mathbb{R}^N \right),$$

then we have

$$\begin{aligned} T(f \cdot g)(x) &= \nabla^2(f \cdot g)(x) = g(x)\nabla^2 f(x) + f(x)\nabla^2 g(x) + 2\langle \nabla f(x), \nabla g(x) \rangle \\ &= f(x)T(g)(x) + T(f)(x)g(x) + 2B(A(f)(x), A(g)(x)) \end{aligned}$$

for all $f, g \in \mathcal{C}^2(\Omega, \mathbb{R})$ and $x \in \Omega$.

One of our main objectives is to prove the converse of this statement. More precisely, we intend to answer the question whether Eq. (Δ) is appropriate to characterize second-order differential operators. Moreover, if we substitute f in place of g in Eq. (Δ), then

we get

$$T(f^2) = 2fT(f) + 2B(A(f), A(f)). \quad (*)$$

Further, if we combine Eqs. (Δ) and ($*$), then by induction

$$T(f^n) = nf^{n-1}T(f) + nB(A(f), A(f^{n-1})) \quad (\bullet)$$

follows for each fixed positive integer $n \geq 2$. As we will see, the answer to the first question we raised is affirmative, that is, Eq. (Δ) is suitable for characterizing second-order differential operators defined on $\mathcal{C}^k(\Omega)$. After this, the question naturally arises as to whether instead of (Δ) it is sufficient to require the much weaker identities ($*$) or ($\bullet$) for the operators in question.

Here Q and R are assumed to be commutative rings such that $\text{char}(Q) > 2$ and P be a subring of Q , while the unknown mappings $T: P \rightarrow Q$ and $A: P \rightarrow R$ are additive. Further, $B: R \times R \rightarrow Q$ is a given symmetric and bi-additive mapping.

At first, we consider the above-mentioned identities in a purely algebraic setting (i.e., we are not limited to the case when the above rings are function spaces). After that, the obtained results will be applied for $Q = R = \mathcal{C}(\Omega, \mathbb{R})$ and $P = \mathcal{C}^k(\Omega, \mathbb{R})$, providing characterization results for first- and second-order differential operators. Here k is a nonnegative integer.

The antecedents of this topic can be found in the paper [1], where the authors considered a special case of the above problems with $B = 0$. Accordingly, the authors obtained characterization theorems for first-order differential operators. As we will see, these results are consequences of that of this paper.

In Sects. 1.1 and 1.2 we summarize the necessary preliminaries from the theory of functional equation and of operator relations, respectively. Then, the results can be found in Sect. 2.

1.1 Preliminaries from the Theory of Functional Equations

In this subsection, we collect briefly the most important notions and statements related to generalized polynomials, which we will use in the second section. Here we follow the monograph Székelyhidi [9]. In this subsection S is a commutative semigroup and G is a commutative group, both written additively.

For $n \in \mathbb{N}$ a function $A: S^n \rightarrow G$ is termed *n-additive* if in each variable it is a semigroup homomorphism of S into G . If $n = 1$ or $n = 2$ then the function A is called *additive* or *bi-additive*, respectively.

The *diagonalization* or *trace* of an n -additive function $A: S^n \rightarrow G$ is defined as

$$A^*(x) = A(x, \dots, x) \quad (x \in S).$$

The above notion can also be extended for the case $n = 0$ by letting $S^0 = S$ and by calling 0-additive any constant function from S to G .

One of the most important theoretical results concerning multi-additive functions is the so-called *Polarization formula*, which says that every n -additive symmetric

function is *uniquely* determined by its diagonalization under some conditions on the domain and on the range. The action of the *difference operator* Δ on a function $f: S \rightarrow G$ is defined as

$$\Delta_y f(x) = f(x + y) - f(x) \quad (x, y \in S).$$

Polarization formula says that if $A: S^n \rightarrow G$ is a symmetric, n -additive function, then for all $m \in \mathbb{N}$ and $x, y_1, \dots, y_m \in S$ we have

$$\Delta_{y_1, \dots, y_m} A^*(x) = \begin{cases} 0 & \text{if } m > n \\ n! A(y_1, \dots, y_m) & \text{if } m = n. \end{cases}$$

In particular, for all $x, y \in S$ one has $\Delta_y^n A^*(x) = n! A^*(y)$.

The next corollary will be very useful for us in what follows. However, it is a consequence of the above observations.

Corollary 1 *Let $n \in \mathbb{N}$ and suppose that the multiplication by $n!$ is surjective in the commutative semigroup S or injective in the commutative group G . Then for any symmetric, n -additive function $A: S^n \rightarrow G$, $A^* \equiv 0$ implies that $A \equiv 0$, as well.*

A function $p: S \rightarrow G$ is called a *generalized polynomial* if it has a representation as the sum of diagonalizations of symmetric multi-additive functions. In other words, p is a generalized polynomial of degree at most n if and only if, it has the representation

$$p = \sum_{k=0}^n A_k^*,$$

where n is a nonnegative integer and $A_k: S^k \rightarrow G$ is a symmetric, k -additive function for each $k = 0, 1, \dots, n$. Moreover, functions $p_k: S \rightarrow G$ of the form $p_k = A_k^*$, where $A_k: S^n \rightarrow G$ are symmetric and k -additive are called *generalized monomials of degree k* . Obviously, generalized monomials of degree 0 are constant functions and generalized monomials of degree 1 are additive functions.

Now, assume that X and Y are linear-topological spaces, Y is partially ordered, D is a nonvoid, open and convex subset of X , $C \subset X$ is a proper cone in X , i.e., a subset of X that is closed under positive scalar multiplication and $n \in \mathbb{N}$. A map $f: D \rightarrow Y$ is termed *n -convex* if $\Delta_h^{n+1} f(x) \geq 0$, for all $x \in D$ and $h \in C$ such that $x + ih \in D$ for all $i = 1, 2, \dots, n + 1$. It is clear that every generalized polynomial of degree at most n is an n -convex function.

The following theorem is a special case of a result of R. Ger [2, Theorem 6].

Theorem 1 (R. Ger) *Assume that X and Y are linear-topological spaces and Y is partially ordered. If $f: D \rightarrow Y$ is an n -convex function defined on an open and convex subset of X which is bounded on a set of the second category with the Baire property. Then f is continuous on D .*

The above theorem applies a fortiori to generalized polynomials. In particular, if the set of zeros of a generalized polynomial is of the second category with the Baire

property, then it is closed. Consequently, it has a nonempty interior and one can apply a result of L. Székelyhidi [9], which says that if a generalized polynomial vanishes on a nonempty open set, then it vanishes everywhere. Thus, we have the following corollary.

Corollary 2 *Assume that X and Y are linear-topological spaces and Y is partially ordered. If $f: X \rightarrow Y$ is a generalized polynomial which is equal to zero on a set of the second category with the Baire property, then $f \equiv 0$ on X .*

1.2 Preliminaries from the Theory of Operator Relations

In this subsection we use the notation and the terminology of the monograph König–Milman [8], see also [3–7].

Definition 1 Let k be a nonnegative integer, N and m be positive integers and $\Omega \subset \mathbb{R}^N$ be an open set. An operator $A: \mathcal{C}^k(\Omega, \mathbb{R}) \rightarrow \mathcal{C}(\Omega, \mathbb{R}^m)$ is *non-degenerate* if for each nonvoid open subset $U \subset \Omega$ and all $x \in U$, there exist functions $g_1, \dots, g_{m+1} \in \mathcal{C}^k(\Omega, \mathbb{R})$ with supports in U such that the vectors $(g_i(x), Ag_i(x)) \in \mathbb{R}^{m+1}$, $i = 1, \dots, m+1$ are linearly independent in $\mathbb{R}^{m+1}$.

Definition 2 Let k and N be positive integers with $k \geq 2$ and $\Omega \subset \mathbb{R}^N$ be an open set. We say that the operator $A: \mathcal{C}^k(\Omega, \mathbb{R}) \rightarrow \mathcal{C}(\Omega, \mathbb{R})$ *depends non-trivially on the derivative* if there exists $x \in \Omega$ and there are functions $f_1, f_2 \in \mathcal{C}^k(\Omega, \mathbb{R})$ such that

$$f_1(x) = f_2(x) \quad \text{and} \quad Af_1(x) \neq Af_2(x)$$

holds.

In the second section we will utilize Theorem 7.1 from [8] which is the statement below.

Theorem 2 (H. König, V. Milman) *Let N be a positive integer, k be a nonnegative integer and $\Omega \subset \mathbb{R}^N$ be a domain. Assume that $T, A: \mathcal{C}^k(\Omega, \mathbb{R}) \rightarrow \mathcal{C}(\Omega, \mathbb{R})$ satisfy*

$$T(f \cdot g) = T(f) \cdot g + f \cdot T(g) + 2A(f) \cdot A(g) \quad \left(f, g \in \mathcal{C}^k(\Omega, \mathbb{R}) \right) \quad (1)$$

and that A is non-degenerate and depends non-trivially on the derivative. Then there are continuous functions $a \in \mathcal{C}(\Omega, \mathbb{R})$ and $b, c \in \mathcal{C}(\Omega, \mathbb{R}^N)$ such that we have

$$\begin{aligned} T(f)(x) &= \langle f''(x)c(x), c(x) \rangle + R(f)(x) \\ A(f)(x) &= \langle f'(x), c(x) \rangle \end{aligned} \quad \left(f \in \mathcal{C}^k(\Omega, \mathbb{R}), x \in \Omega \right),$$

where

$$R(f)(x) = \langle f'(x), b(x) \rangle + a(x)f(x) \ln(|f(x)|) \quad \left(f \in \mathcal{C}^k(\Omega, \mathbb{R}) \right).$$

If $k = 1$, then necessarily $c \equiv 0$. Further, if $k = 0$, then necessarily $b \equiv 0$ and $c \equiv 0$. Conversely, these operators satisfy the above second-order Leibniz rule.

Corollary 3 *Under the assumptions of the previous theorem, suppose that the operators $T, A: \mathcal{C}^k(\Omega, \mathbb{R}) \rightarrow \mathcal{C}(\Omega, \mathbb{R})$ are additive. Then $a \equiv 0$ in the above theorem. In other words, there are continuous functions $b, c \in \mathcal{C}(\Omega, \mathbb{R}^N)$ such that we have*

$$\begin{aligned} T(f)(x) &= \langle f''(x)c(x), c(x) \rangle + \langle f'(x), b(x) \rangle \\ A(f)(x) &= \langle f'(x), c(x) \rangle \end{aligned} \quad \left(f \in \mathcal{C}^k(\Omega, \mathbb{R}), x \in \Omega \right). \quad (2)$$

If $k = 1$, then necessarily $c \equiv 0$. Further, if $k = 0$, then necessarily $b \equiv 0$ and $c \equiv 0$.

Conversely, these operators are additive and also satisfy the above second-order Leibniz rule.

The assumption that A is non-degenerate and depends non-trivially on the derivative, is essential in Theorem 2 and also in Corollary 3. Our example below is a slight modification of the example on page 114 of [8]. To see the necessity of the above-mentioned conditions, suppose that $\Omega = \mathbb{R}^N$ and let $h \in \mathbb{R}^N$ be an arbitrary nonzero vector. Consider the difference operators $T, A: \mathcal{C}^k(\Omega, \mathbb{R}) \rightarrow \mathcal{C}(\Omega, \mathbb{R})$ defined by $Tf = \Delta_h f$ and $Af = -\frac{\sqrt{2}}{2} \Delta_h f$ for all $f \in \mathcal{C}^k(\Omega, \mathbb{R})$. Then we have

$$\begin{aligned} T(f \cdot g)(x) &= f(x+h)g(x+h) - f(x)g(x) \\ &= f(x)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x)g(x) \\ &\quad + f(x)g(x) - g(x+h)f(x) - f(x+h)g(x) + g(x+h)f(x+h) \\ &= f(x) \cdot [g(x+h) - g(x)] + [f(x+h) - f(x)] \cdot g(x) \\ &\quad + 2 \frac{\sqrt{2}}{2} [f(x) - f(x+h)] \cdot \frac{\sqrt{2}}{2} [g(x) - g(x+h)] \\ &= f(x) \cdot T(g)(x) + T(f)(x) \cdot g(x) + 2A(f)(x) \cdot A(g)(x) \end{aligned}$$

for all $f, g \in \mathcal{C}^k(\Omega, \mathbb{R})$ and $x \in \Omega$. At the same time, the operators T and A are obviously not of the form as stated in Theorem 2. This is caused by the fact that in this example A is not non-degenerate (and neither does T). Indeed, since $h \neq 0$, then we will find an open set U around each point $x \in \Omega$ such that for every mapping $g \in \mathcal{C}^k(\Omega, \mathbb{R})$ with support contained in U one has $Tg = -g$ and $Ag = \frac{\sqrt{2}}{2}g$. A similar example is discussed in [8, Chapter 8, pages 141–142].

2 Results

In this section, we first examine Eqs. (Δ), (*) and • from a purely algebraic point of view. The main point of Lemma 1 below is that Eqs. (*) and • are separately equivalent to Eq. (Δ) for additive operators. Next, we prove some extension theorems. Finally, by using these findings and the results of König–Milman [8], we obtain characterization theorems for first- and second-order differential operators.

Lemma 1 *Let $n \geq 2$ be a positive integer, Q and R be commutative rings such that $\text{char}(Q) > n!$, and P be a subring of Q . Let further $T: P \rightarrow Q$ and $A: P \rightarrow R$ be additive mappings for which Eq. (•) holds for all $f \in P$ with a symmetric and*

bi-additive mapping $B: R \times R \rightarrow Q$. If $n = 2$ or $A(1) = 0$, then Eq. (Δ) is satisfied for all $f, g \in P$.

Proof Under the assumptions of the lemma, let us consider the mapping $\mathcal{A}_n: P^n \rightarrow Q$ defined through

$$\begin{aligned} \mathcal{A}_n(f_1, \dots, f_n) &= T(f_1 \cdots f_n) - [f_2 \cdots f_n T(f_1) + f_1 f_3 \cdots f_n T(f_2) + \cdots \\ &\quad + f_1 \cdots f_{n-1} T(f_n)] \\ &\quad - \left[\sum_{i=1}^n B(A(f_i), A(f_1 \cdots f_{i-1} \cdot f_{i+1} \cdots f_n)) \right] \\ &\quad (f_1, \dots, f_n \in P). \end{aligned} \quad (3)$$

Due to the additivity of the operators T and A and the bi-additivity of B , the mapping $\mathcal{A}_n$ is symmetric and n -additive. Next, by Eq. • its trace is identically zero. Since $\text{char}(Q) > n!$, the multiplication by $n!$ is injective on Q . Thus Corollary 1 implies that $\mathcal{A}_n$ is identically zero. If $n = 2$, then we arrive at

$$0 = \mathcal{A}_2(f_1, f_2) = T(f_1 \cdot f_2) - f_1 T(f_2) - f_2 T(f_1) - 2B(A(f_1), A(f_2)),$$

for all $f_1, f_2 \in P$, which completes the proof in this case. In case of arbitrary $n > 2$ take $f \in P$ and substitute $f_1 = f$ and $f_i = 1$ for $i = 2, \dots, n$ in (3),

$$T(f) = T(f) + (n-1)fT(1) + nB(A(f), A(1)),$$

i.e.

$$(n-1)T(1)f + nB(A(f), A(1)) = 0.$$

Since $A(1) = 0$ and due to the bi-additivity of B , we obtain

$$B(A(f), A(1)) = B(A(f), 0) = 0.$$

From this $T(1) = 0$ follows. Again, since the mapping $\mathcal{A}_n$ is identically zero, if $f, g \in P$ and we substitute $f_1 = f, f_2 = g$ and $f_i = 1$ for $i = 3, \dots, n$ in (3), we get that

$$\begin{aligned} T(fg) &= fT(g) + gT(f) + \\ &\quad (n-2)fgT(1) + [2B(A(f), A(g)) + (n-2)B(A(fg), A(1))] \\ &= fT(g) + gT(f) + 2B(A(f), A(g)) \end{aligned}$$

is fulfilled for all $f, g \in P$. □

As a consequence of this lemma and its proof, now we show that if the rings Q and R are topological rings, then if Eq. (•) is fulfilled only on a subset $U \subset P$ with a nonempty interior (however mappings T and A are defined on the whole ring P), then this also implies that Eq. (Δ) is satisfied everywhere on P .

Corollary 4 *Let $n \geq 2$ be a positive integer, Q and R be commutative topological rings such that $\text{char}(Q) > n!$, let further P be a subring of Q and $U \subset P$ be a nonvoid open set. Let further $T: P \rightarrow Q$ and $A: P \rightarrow R$ be additive mappings for which Eq. (•) holds for all $f \in U$ with a symmetric and bi-additive mapping $B: R \times R \rightarrow Q$. If $n = 2$ or $A(1) = 0$, then Eq. (Δ) is satisfied for all $f, g \in P$.*

Proof Under the assumptions of the corollary, consider the mapping $\mathcal{A}_n: P \times P \rightarrow Q$ defined as in the proof of Lemma 1. Repeating the calculations from the proof of Lemma 1 we get that the diagonalization of $\mathcal{A}_n$ vanishes on the open subset U of P . In view of Székelyhidi [9] this is possible only if this map is identically zero on P , yielding in turn that $\mathcal{A}_n$ is identically zero on $P \times P$. So, repeating the respective parts of the proof of Lemma 1, we derive that (Δ) is satisfied for all $f, g \in P$. $\square$

We can join this corollary with Corollary 2 to relax the assumptions upon the set of zeros if additionally P and R are linear-topological spaces. Therefore, we assume that P and R are topological algebras over $\mathbb{R}$. Note that any real algebra trivially satisfies the assumptions of injectivity and surjectivity of scalar multiplication.

Corollary 5 *Let Q and R be topological algebras over $\mathbb{R}$, Q being partially ordered. Further, P be a subring of Q and $U \subset P$ be a set of the second category in Q with the Baire property. Let further $T: P \rightarrow Q$ and $A: P \rightarrow R$ be additive mappings for which Eq. (•) holds for all $f \in U$ with a symmetric and bi-additive mapping $B: R \times R \rightarrow Q$. If $n = 2$ or $A(1) = 0$, then Eq. (Δ) is satisfied for all $f, g \in P$.*

Note that the technique described above is suitable also for characterizing first-order differential operators. Indeed, from Lemma 1 we immediately deduce with the choice $B \equiv 0$ the following statement:

Let Q be a commutative ring such that $\text{char}(Q) > 2$ and P be a subring of Q . Let further $T: P \rightarrow Q$ be an additive mapping for which we have

$$T(f^n) = nf^{n-1} \cdot T(f) \quad (4)$$

for all $f \in P$ with a symmetric and bi-additive mapping $B: R \times R \rightarrow Q$. Then equation the mapping T also fulfils

$$T(f \cdot g) = f \cdot T(g) + T(f) \cdot g$$

for all $f, g \in P$.

Further, if we combine this statement with [8, Theorem 3.5] we obtain the following characterization of first-order differential operators. In fact, the following proposition was also proved in [1] with a different method.

Proposition 1 Let $N \in \mathbb{N}$, k be a nonnegative integer and $\Omega \subset \mathbb{R}^N$ be a nonempty open set. If the additive operator $T: \mathcal{C}^k(\Omega, \mathbb{R}) \rightarrow \mathcal{C}(\Omega, \mathbb{R})$ satisfies (4) for all $f \in \mathcal{C}^k(\Omega)$, then there exists a continuous function $b \in \mathcal{C}(\Omega, \mathbb{R}^N)$ such that

$$T(f)(x) = \langle b(x), f'(x) \rangle \quad (f \in \mathcal{C}^k(\Omega, \mathbb{R})).$$

For $k = 0$, the function b is necessarily identically zero.

In what follows, we will apply the above results by choosing P , Q and R to be a given function space, thus obtaining characterization theorems for second-order differential operators.

Theorem 3 Let N be a positive integer, $n \geq 2$ be a positive integer, k be a non-negative integer and $\Omega \subset \mathbb{R}^N$ be a domain. Assume that the additive operators $T, A: \mathcal{C}^k(\Omega, \mathbb{R}) \rightarrow \mathcal{C}(\Omega, \mathbb{R})$ satisfy

$$T(f^n) = nf^{n-1}T(f) + nA(f)A(f^{n-1}) \quad (f \in \mathcal{C}^k(\Omega, \mathbb{R})) \quad (5)$$

and that A is non-degenerate, depends non-trivially on the derivative and annihilates the identically 1 function. If $n = 2$ or $A(1) = 0$, then there are continuous functions $b, c \in \mathcal{C}(\Omega, \mathbb{R}^N)$ such that formulas (2) are satisfied. Further, if $k = 1$, then necessarily $c \equiv 0$. Further, if $k = 0$, then necessarily $b \equiv 0$ and $c \equiv 0$.

Conversely, these operators are additive and also satisfy Eq. (5).

Proof Consider the rings $P = \mathcal{C}^k(\Omega, \mathbb{R})$, $Q = \mathcal{C}(\Omega, \mathbb{R})$ and $R = \mathcal{C}(\Omega, \mathbb{R})$ with the pointwise addition and multiplication of functions and also the symmetric and bi-additive mapping $B: \mathcal{C}(\Omega, \mathbb{R}) \times \mathcal{C}(\Omega, \mathbb{R}) \rightarrow \mathcal{C}(\Omega, \mathbb{R})$ defined through

$$B(f, g) = f \cdot g \quad (f, g \in \mathcal{C}(\Omega, \mathbb{R})).$$

Then the assumptions of Lemma 1 are fulfilled. This yields that the equality (1) holds for all $f, g \in \mathcal{C}^k(\Omega, \mathbb{R})$. In view of Theorem 2, this implies that there are continuous functions $b, c \in \mathcal{C}(\Omega, \mathbb{R}^N)$ such that formulas (2) are fulfilled. The converse implication is a routine calculation. $\square$

In our last result we will deal with a conditional differential relation that is meaningful for nowhere vanishing mappings. In the proof first we will show that formulas (2) are valid for mappings f such that $f^2 - 1$ is nowhere zero. Then we will apply Corollary 4 to extend (2) to each $f \in \mathcal{C}^k(\Omega, \mathbb{R})$.

Theorem 4 Let N be a positive integer, k be a nonnegative integer and $\Omega \subset \mathbb{R}^N$ be a domain. Let $T, A: \mathcal{C}^k(\Omega, \mathbb{R}) \rightarrow \mathcal{C}(\Omega, \mathbb{R})$ be additive mappings such that $A(1) = 0$ and

$$T\left(\frac{1}{f}\right) = -\frac{1}{f^2}T(f) + \frac{2}{f^3}A(f)^2 \quad (6)$$

holds for all $f \in \mathcal{C}^k(\Omega, \mathbb{R})$ for which $f(x) \neq 0$ for all $x \in \Omega$. Then there are continuous functions $b, c \in \mathcal{C}(\Omega, \mathbb{R}^N)$ such that formulas (2) hold true. Further, if $k = 1$, then necessarily $c \equiv 0$. Further, if $k = 0$, then necessarily $b \equiv 0$ and $c \equiv 0$.

Proof Under the assumptions of the theorem, if we substitute the identically 1 function in place of f in Eq. (6), then

$$T(1) = -T(1) + 2A(1)^2$$

follows. Since $A(1) = 0$, then $T(1) = 0$, as well.

Let now $f \in \mathcal{C}^k(\Omega, \mathbb{R})$ be a function for which we have

$$f(x)^2 \neq 1$$

for all $x \in \Omega$. Since Ω is a connected set and f is continuous, then we have precisely three options: either $f > 1$, $f(x) \in (-1, 1)$ for all $x \in \Omega$, or $f < -1$. If we put $f^2 - 1$ in place of f in equality (6), then

$$T\left(\frac{1}{f^2 - 1}\right) = -\frac{1}{(f^2 - 1)^2}T(f^2 - 1) + \frac{2}{(f^2 - 1)^3}A(f^2 - 1)^2,$$

or after multiplying both sides of this equation with 2:

$$T\left(\frac{2}{f^2 - 1}\right) = -\frac{2}{(f^2 - 1)^2}T(f^2 - 1) + \frac{4}{(f^2 - 1)^3}A(f^2 - 1)^2.$$

At the same time, due to the additivity of the operator T , we have

$$T\left(\frac{2}{f^2 - 1}\right) = T\left(\frac{1}{f - 1} - \frac{1}{f + 1}\right) = T\left(\frac{1}{f - 1}\right) - T\left(\frac{1}{f + 1}\right)$$

Moreover, Eq. (6) applied for $f - 1$ and $f + 1$, resp. yield that

$$T\left(\frac{1}{f - 1}\right) = -\frac{1}{(f - 1)^2}T(f - 1) + \frac{2}{(f - 1)^3}A(f - 1)^2$$

and

$$T\left(\frac{1}{f + 1}\right) = -\frac{1}{(f + 1)^2}T(f + 1) + \frac{2}{(f + 1)^3}A(f + 1)^2.$$

Combining the above identities, we get

$$\begin{aligned} & -\frac{2}{(f^2-1)^2}T(f^2-1) + \frac{4}{(f^2-1)^3}A(f^2-1)^2 \\ & = T\left(\frac{2}{f^2-1}\right) = T\left(\frac{1}{f-1}\right) - T\left(\frac{1}{f+1}\right) \\ & = \left[-\frac{1}{(f-1)^2}T(f-1) + \frac{2}{(f-1)^3}A(f-1)^2\right] \\ & \quad - \left[-\frac{1}{(f+1)^2}T(f+1) + \frac{2}{(f+1)^3}A(f+1)^2\right]. \end{aligned}$$

Since $A(1) = 0$ and $T(1) = 0$, after some rearrangement we deduce that

$$\begin{aligned} & -\frac{2}{(f^2-1)^2}T(f^2) + \frac{4}{(f^2-1)^3}A(f^2)^2 \\ & = -\frac{1}{(f-1)^2}T(f) + \frac{2}{(f-1)^3}A(f)^2 + \frac{1}{(f+1)^2}T(f) - \frac{2}{(f+1)^3}A(f)^2 \end{aligned}$$

i.e.,

$$-\frac{2}{(f^2-1)^2}T(f^2) + \frac{4}{(f^2-1)^3}A(f^2)^2 = -\frac{4f}{(f^2-1)^2}T(f) + \frac{12f^2+4}{(f^2-1)^3}A(f)^2.$$

Let us multiply now both sides of this equation with $(f^2-1)^3$ to get that

$$(f^2-1)\left[T(f^2) - 2fT(f) - 2A(f)^2\right] = 2\left[A(f^2)^2 - 4f^2A(f)^2\right].$$

It is clear that one can find at least five distinct rationals $\{r_1, \dots, r_5\}$ such that $r_i^2 f(x)^2 \neq 1$ for all $x \in \Omega$ and each $i = 1, \dots, 5$ and let

$$H = \{r \in \mathbb{Q} : r^2 f(x)^2 \neq 1 \text{ for all } x \in \Omega\}.$$

Next, apply the last formula for f replaced by rf with $r \in H$. Then due to the $\mathbb{Q}$ -homogeneity of the operators T and A , for all $r \in H$ we have

$$\begin{aligned} & r^4 f^2 \left\{ \left[T(f^2) - 2fT(f) - 2A(f)^2 \right] - 2 \left[A(f^2)^2 - 4f^2 A(f)^2 \right] \right\} \\ & + r^2 \left\{ T(f^2) - 2fT(f) - 2A(f)^2 \right\} = 0. \end{aligned}$$

Observe that the left-hand side of this equation is a (classical) polynomial with respect to $r \in H$ and it is identically zero. Since H contains at least five elements, all coefficients of this polynomial vanish. Especially, the coefficient corresponding to r^2 has to

vanish, yielding that $(*)$ holds for any function $f \in \mathcal{C}^k(\Omega, \mathbb{R})$ for which $f^2(x) \neq 1$ for all $x \in \Omega$. Since the set

$$U = \left\{ f \in \mathcal{C}^k(\Omega, \mathbb{R}) \mid f^2(x) \neq 1 \text{ for all } x \in \Omega \right\}$$

is a nonempty and open subset of $\mathcal{C}^k(\Omega, \mathbb{R})$, with the help of Corollaries 3 and 4, the statement of our theorem follows. $\square$

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Declarations

Conflict of interest The authors declare no conflict of interest.

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