DIFFERENCES BETWEEN MEAN-VARIANCE AND MEAN-CVAR PORTFOLIO OPTIMIZATION MODELS

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Abstract: *Everybody heard already that one should not expect high returns without high risk, or one should not expect safety without low returns. The goal of portfolio theory is to find the balance between maximizing the return and minimizing the risk. To do so we have to first understand and measure the risk. Naturally a good risk measure has to satisfy several properties - in theory and in practise. Markowitz suggested to use the variance as a risk measure in portfolio theory. This led to the so called mean-variance model - for which Markowitz received the Nobel Prize in 1990. The model has been criticized because it is well suited for elliptical distributions but it may lead to incorrect conclusions in the case of non-elliptical distributions. Since then many risk measures have been introduced, of which the Value at Risk (VaR) is the most widely used in the recent years. Despite of the widespread use of the Value at Risk there are some fundamental problems with it. It does not satisfy the subadditivity property and it ignores the severity of losses in the far tail of the profit-and-loss (P&L) distribution. Moreover, its non-convexity makes VaR impossible to use in optimization problems. To come over these issues the Expected Shortfall (ES) as a coherent risk measure was developed. Expected Shortfall is also called Conditional Value at Risk (CVaR). Compared to Value at Risk, ES is more sensitive to the tail behaviour of the P&L distribution function. In the first part of the paper I state the definition of these three risk measures. In the second part I deal with my main question: What is happening if we replace the variance with the Expected Shortfall in the portfolio optimization process. Do we have different optimal portfolios as a solution? And thus, does the solution suggests to decide differently in the two cases? To answer to these questions I analyse seven Hungarian stock exchange companies. First I use the mean-variance portfolio optimization model, and then the mean-CVaR model. The results are shown in several charts and tables.*

Keywords: risk; Value at Risk; Expected Shortfall; Mean-Variance Portfolio Optimization; Mean-CVaR Portfolio Optimization

JEL classification: G11

1. How to measure risk?

In literature risk and risk measures have no unique definition and usage. However it is often useful to express risk with one number (Emmer, Kratz and Tasche, 2013). The historically most important risk measures in finance are the variance and the standard deviation and it goes back to Markowitz and the modern portfolio theory (Markowitz, 1952; 1968).

In the last years and also these days the most popular risk measure is the Value at Risk (VaR). Value at Risk answers the following question: What is the minimum potential loss that a portfolio can suffer in the $100\alpha\%$ worst cases (in a given time horizon) (Acerbi and Tasche, 2002):

$$VaR_\alpha(X) = -inf\{x \in \mathbb{R}: F_X(x) \geq \alpha\},$$

where F_X is the profit distribution function of X and α $(0 < \alpha < 1)$ is the confidence level. In other words, VaR is the $100\alpha\%$ quantile of F_X. VaR is criticised because of several reasons. The most important are that VaR is not subadditive, which means that portfolio diversification may lead to an increase of risk, then that it does not measure losses exceeding VaR and its non-convexity makes VaR impossible to use in optimization problems (Szegö, 2002).

To solve these problems a coherent risk measure called Expected Shortfall (ES) was introduced:

$$ES_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_u(X)du.$$

If F_X is continuous, the definition of expected shortfall is equivalent to $ES_\alpha(X) = E\big(X: X \geq VaR_\alpha(X)\big)$, hence alternative names like conditional Value at Risk (CVaR) and conditional tail expectation are in use in finance (Embrechts and Hofert, 2014). In this paper I use CVaR and ES as synonyms.

2. Mean-Variance vs. Mean-CVaR Portfolio Optimization

2.1. The data

I work with Hungarian daily stock data between 01.07.2005 and 29.06.2015. The data was downloaded from Budapest Stock Exchange homepage (www.bet.hu). I analyse the daily logarithmic returns of seven stocks, namely FHB, MOL, MTELEKOM, OTP, Pannenergy, Raba and Richter, in this ten years. To analyse the data and to answer to my questions I use the Rmetrics mathematical software (Würtz et al,, 2009). Note that I work with daily data and thus the results (returns, variance, VaR, ES, etc.) are stated on a daily basis.

2.2. Mean-Variance Portfolio Optimization

In the mean-variance (MV) model, return and risk are estimated by the sample mean and the sample variance of the asset returns. The goal is to minimize the risk for a given target return. I work with long-only portfolios only (short selling is not allowed) and therefore the weights are between zero and one after an appropriate normalization. We can see the results of the mean-variance portfolio optimization in Figure 1 and in Table 1.

In Figure 1 the hyperbola is the solution of the Markowitz model. The set inside the hyperbola called the feasible set, the upper border (black points) is the efficient frontier and the lower border is the minimum variance locus (grey points). The efficient frontier plot includes all single assets risk vs. return points (coloured circles), the tangency line (blue solid line), the tangency point for the risk-free rate (blue point), the equal weight portfolio (EWP) (grey square) and the global minimum risk portfolio (red point). The yellow solid line indicates the Sharpe ratio and the range of the Sharpe ratio is printed on the right hand side axis. We can see, that in the case we decide to invest just in one stock, positive return is reachable just with the Raba stock. The MTELEKOM has the lowest while the Richter stock has the highest risk.

First I calculated the EWP, which is a feasible portfolio with given equal weights.

The EWP has quite a small risk (variance=0.0161) but in this case the reachable return is negative (-0.03%). Moreover, one could reach a higher return with the same risk or the same return with a smaller risk. The global minimum risk portfolio is the efficient portfolio with the lowest possible risk and thereby the global minimum risk point separates the efficient frontier from the minimum variance locus. In my case the global minimum variance portfolio has a negative return (-0.02%).
The tangency portfolio is calculated by maximizing the Sharpe ratio for a given risk-free rate. The Sharpe ratio is the slope of the capital allocation line given by the ratio of the target return lowered by the risk-free rate and the portfolio risk. In the case of the Hungarian daily data the risk-free rate is 0.008%. The tangency point coincides with the Raba single asset, so one can reach the highest return (0.02%) with the Raba single asset portfolio. But on this return level the risk is quite high (variance=0.0213). The Sharpe ratio is between -0.00026 and 0.000206.

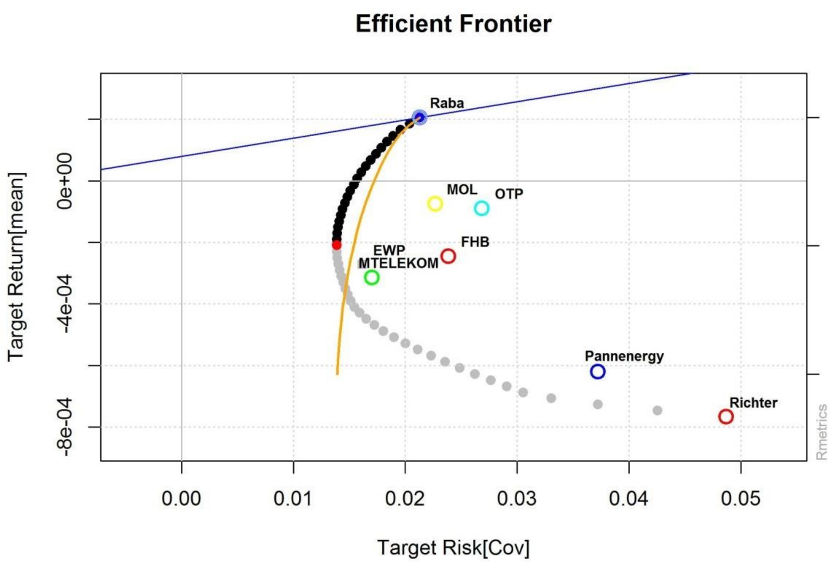

Figure 1: Efficient frontier: solution of the mean-variance portfolio optimization (alpha=0.05)
Source: own calculation, Rmetrics

Table 1: Result of the mean-variance portfolio optimization (alpha=0.05)

Portfolio	**Return**	**Variance**	**VaR**	**ES**
Equal Weight	-0.0003	0.0161	0.0211	0.0374
Minimal Risk (target return=0)	0.0000	0.0156	0.0231	0.0372
Minimal Risk (target return=0.0001)	0.0001	0.0177	0.0257	0.0413
Tangency	0.0002	0.0213	0.0282	0.0482
Global Minimum Risk	-0.0002	0.0139	0.0208	0.0343

Source: own calculation, Rmetrics

We can see the weights of four different portfolios in Figure 2. As I mentioned earlier, in the case of the tangency portfolio the optimal portfolio consists only of the Raba stock. If we invested only in the Raba stock we could reach a 0.02% return, having the following risks: variance=0.0213, VaR=0.0282 and ES=0.0482 (see Table 1). The minimum risk portfolio is an efficient portfolio with the lowest risk for a given target return. With my data a highest reachable return is 0.02% that is why as a first example I calculated the minimum risk portfolio with 0% target return. The optimal portfolio consists of four stocks (from the seven): Raba (51.8%), MTELEKOM (25%), MOL (16.7%) and FHB (6.5%). As a second example I calculated the minimum risk portfolio with 0.01% target return. The composition of the optimal portfolio is the same as in the previous case, but the weights are different: Raba (69.6%), MOL (21.3%), MTELEKOM (7.7%), and FHB (1.4%). The last plot shows the weights in the global minimum optimal portfolio. All the stocks excepting OTP are present: MTELEKOM (45.1%), Raba (23.3%), FHB (11.4%), Pannenergy (8.9%), MOL (8%) and Richter (3.4%).

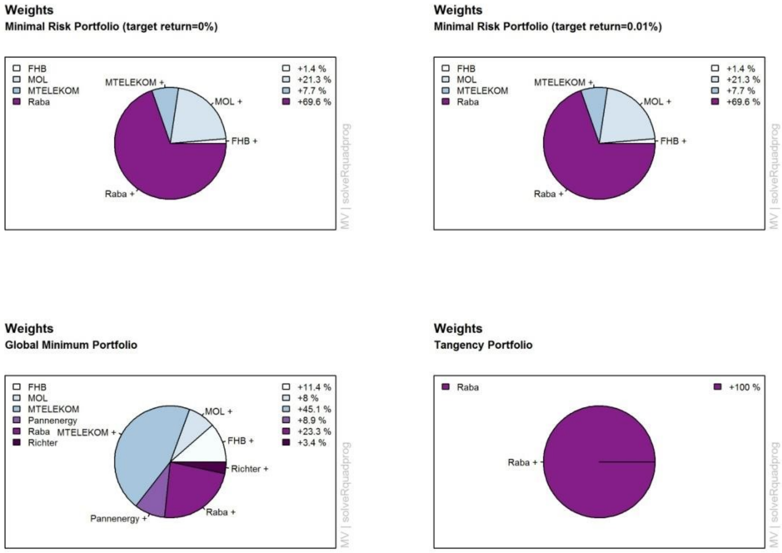

Figure 2: MV portfolio weights: minimal risk portfolio with 0% target return (left up), minimal risk portfolio with 0.01% target return (right up), global minimum portfolio (left down), tangency portfolio (right down), alpha=0.05
Source: own calculation, Rmetrics

2.3. Mean-CVaR Portfolio Optimization

In the mean-CVaR portfolio model the variance is replaced by the CVaR. In contrast to the mean-variance portfolio optimization, the set of assets are no longer restricted to have an elliptical distribution. The results are shown in Figure 3 and Table 2. I use the same notation as in the case of the mean-variance portfolio optimization.

Similarly to the result of the mean-variance optimization all single assets have negative returns except the Raba stock. Moreover Raba is the solution of the tangency portfolio optimization thus this single asset portfolio has the highest Sharpe ratio. With this stock one could reach the maximum 0.02% target return, having variance=0.0213, VaR=0.0282 and ES=0.0482 (see Table 2). The Sharpe ratio is between 0.000532 and 0.000206.

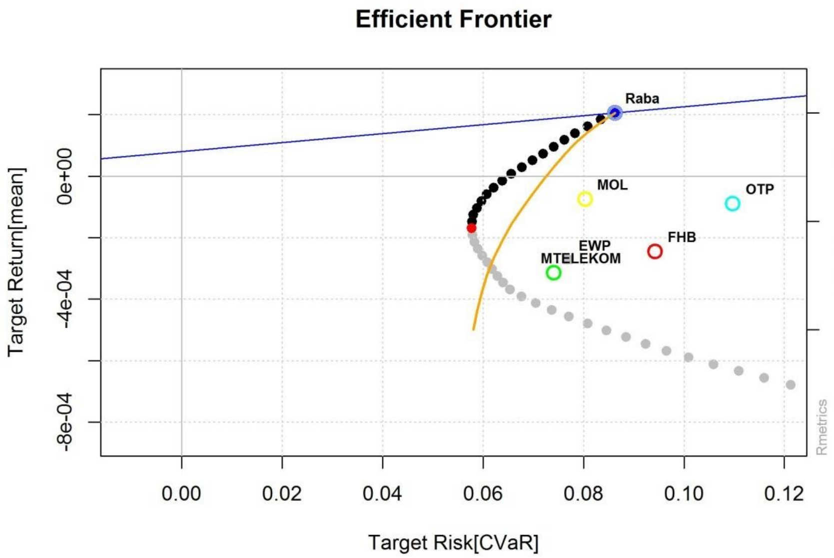

Figure 3: Efficient frontier: the solution of the mean-CVaR portfolio optimization (alpha=0.05)
Source: own calculation, Rmetrics

Table 2: Result of the mean-CVaR portfolio optimization (alpha=0.05)

Portfolio	Return	Variance	VaR	ES
Equal Weight	-0.0003	0.0161	0.0211	0.0374
Minimal Risk (target return=0)	0.0000	0.0156	0.0231	0.0372
Minimal Risk (target return=0.0001)	0.0001	0.0177	0.0256	0.0413
Tangency	0.0002	0.0213	0.0282	0.0482
Global Minimum Risk	-0.0002	0.0142	0.0203	0.0340

Source: own calculation, Rmetrics

I also calculated the mean-variance portfolio weights (see Figure 4). The optimal tangency portfolio is consists of only the Raba stock. The minimum risk portfolio with 0% target return consists of four stocks: Raba (51.8%), MTELEKOM (25.2%), MOL (16.8%) and FHB (6.2%). As a second example I calculated again the minimum risk portfolio with 0.01% target return. The composition of the optimal portfolio consist just three stocks: Raba (69.1%), MOL (22.7%) and MTELEKOM (8.2%). So with a little bit of higher expected return the FHB is not any more in the optimal portfolio. The last plot shows the weights in the global minimum optimal portfolio. All the stocks are in this portfolio except OTP: MTELEKOM (43.7%), Raba (21.1%), Pannenergy (15.8%), FHB (12.1%), MOL (6.8%) and Richter (0.5%).

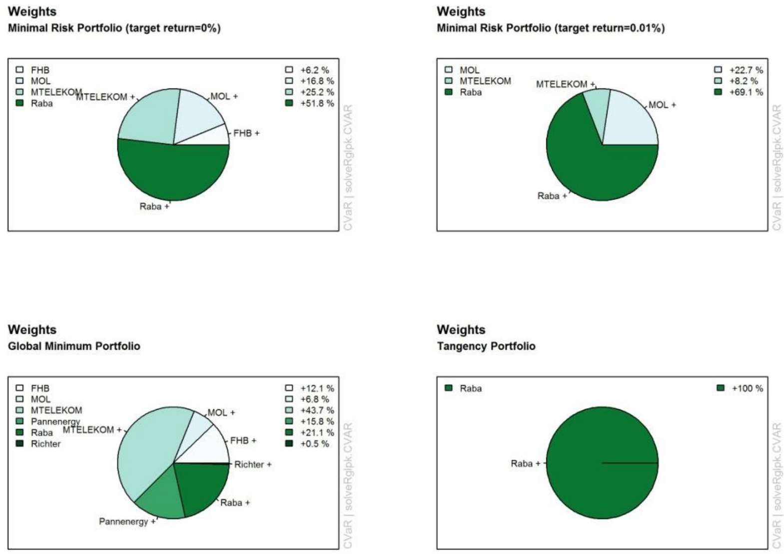

Figure 4: Mean-CVaR portfolio weights: minimal risk portfolio with 0% target return (left up), minimal risk portfolio with 0.01% target return (right up), global minimum portfolio (left down), tangency portfolio (right down), alpha=0.05
Source: own calculation, Rmetrics

3. Comparison and summary

I would like to answer to the question whether we would decide differently using variance or ES as a risk measure. Let us take a closer look of the weights on the efficient frontier in the case of the mean-variance and the mean-CVaR optimization in order to answer to this question. In the Figure 5 we can see the weights on the hyperbola curve. The first plot is the solution of the mean-variance portfolio optimization, the second and the third is the solution of the mean-CVaR portfolio optimization alpha=0.05 and 0.01 respectively. Note that the strong separation line marks the position between the minimum variance locus and the efficient frontier. Target returns are increasing from left to right, whereas target risks are increasing to the left and to right with respect to the separation line.

We can see, that the three plots in Figure 5 show different distributions. Note also that the OTP stock is almost never in the optimal portfolio (independently of the target return and/or the optimization method). For an investor it is more interesting to consider the weights where the target returns are positive.

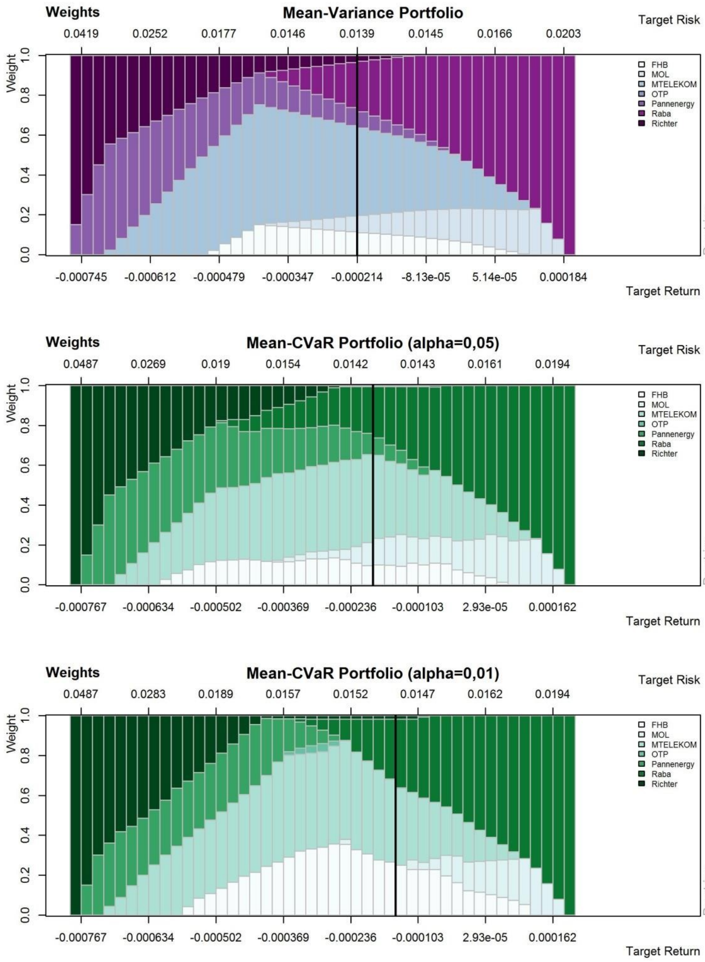

Figure 5: Weights in the stock portfolios
Source: own calculation, Rmetrics

If we fix the target return for example at 0.015% then there is no difference between the methods (see Figure 6 last column): the optimal portfolio consist of the Raba (80%) and the MOL (20%). Secondly we consider a 0.005% fixed target return (see Figure 6 first column). In this case the composition of the three optimal portfolio is the same (FHB, MOL, MTELEKOM, RABA), but the weights are different. This means, that we would conclude differently using one or the other risk measures. There is an even bigger difference for example in the case of the fixed

0.01% target return (Figure 6 middle column) because not only the weights but also the compositions of the portfolios are different. The mean-variance and the mean-CVaR portfolio optimization on the level of 1% suggest four stocks in the optimal portfolio (FHB, MOL, MTALAKOM, Raba) but again in a completely different proportion. The solution of the mean-CVaR portfolio optimization on the level of 5% consists of ‘only’ three stocks: MOL, MTELEKOM, Raba. This third example shows even clearer that we would make a different decision depending on the risk measure.

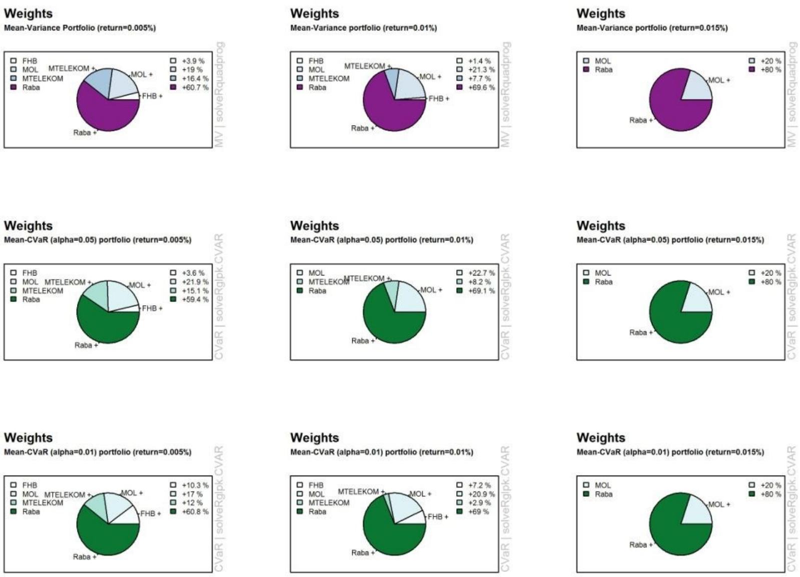

Figure 6: Weights with fixed target returns
Source: own calculation, Rmetrics

In this paper I analysed the difference between the mean-variance and the mean-CVaR portfolio optimization methods by using daily Hungarian stock data and I answered to the question whether we would decide differently concerning the composition of the portfolios in the two cases. The analysis shows clearly – see for example Figure 5 and Figure 6 - that the answer to this question is yes. The decision of an investor depends on the risk measure.

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