On conformal equivalence of Riemann-Finsler metrics and special Finsler Manifolds

Doktori (Ph.D.) értekezés tézisei

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Debrecen, 2000

INTRODUCTION

In this Ph.D. dissertation first of all we undertake a quite comprehensive survey of general theoretical elements of Finsler geometry. The primary aim of this survey is to present a standard system of notations and terminology built on three pillars: the theory of horizontal endomorphisms, the calculus of vector-valued forms and a "tangent-bundle version" of the method of moving frames. On the other hand we present a systematic treatment of some distinguished Finsler connections and some special Finsler manifolds. In particular we are interested in the conformal theory of Riemann-Finsler metrics and the theory of Wagner connections and Wagner manifolds. As we shall see, they are closely related. Finally, we investigate a special conformal change of the metric proving that its existence implies the Finsler manifold to be Riemannian. (The necessity is clear.)

I

This dissertation is divided into three parts. In part I first of all we present a quite detailed exposition of the conceptual and calculational background. Our main purpose is to insert the theory of Finsler connections and the foundations of special Finsler manifolds into a new approach of Finsler geometry. The first epoch-making steps in this direction were done by J. GRIFONE [10], [11], our work can be considered as a systematic continuation of the program initiated by him. Following Grifone's theory of nonlinear connections (whose role is played in our presentation by the so-called *horizontal endomorphisms*) we use systematically an "intrinsic" calculus based on the Frölicher-Nijenhuis formalism. Technically, we enlarged and – at the same time – simplified the apparatus by using the tools of tangent bundle differential geometry. This means first of all the consistent use of a special frame field, constituted by vertically and completely (or vertically and horizontally) lifted vector fields. Thus the third pillar of our approach is the method of moving frames. It has a decisive superiority in calculations over coordinate methods: the formulation of the concepts and results becomes perfectly transparent, and the proofs have a purely intrinsic character. We believe that the compact, elegant and efficient formulation presented here demonstrate the power of our approach. For example, in section I/4 we present an invariant and axiomatic description of three notable Finsler connection (linear connections associated to a nonlinear one with the help of some conditions of compatibility): the *Berwald, Cartan* and *Chern-Rund connections*. Theorems are organized as follows: the first group of axioms characterizes a unique Finsler connection allowing us to derive the explicit rules of calculations for the corresponding covariant derivatives. Adding further conditions to them, the second group yields the characterization of the three classical Finsler connections. Although these results belong to the foundations they are new. Moreover, we hope that they help in better understanding the role of the different axioms, and open a path for further, essential generalizations. As motivations, we can mention the so-called *Wagner connections* (as *generalized Cartan connections*, i.e. Cartan connections with nonvanishing $(h)h$-torsion) or the associated *Berwald-type* Finsler connections (as *generalized Berwald connections*, i.e. Berwald connections with nonvanishing $(h)h$-torsion).

II

In part II we start with the definition of conformal equivalence of Riemann-Finsler metrics. (This relation is formally the same as that in Riemannian geometry.) We give a modern proof of Knebelman's famous observation which points out that the scale function between conformally equivalent Riemann-Finsler metrics must be independent of the "direction", i.e. it is a vertical lift. We also derive some important conformal invariants and transformation formulas. As an application of the results a well-known classical theorem will be proved intrinsically. It states (in H. Weil's terminology; [30], p. 226) that "*the projective and conformal properties of a Finsler space determine its metric properties uniquely*".

In this part we also demonstrate that the Frölicher-Nijenhuis formalism provides a perfectly adequate conceptual and technical framework for the study even of such complicated objects as Wagner connections. Our intrinsically formulated and proved results not only cover the classical local results but give a much more precise and transparent picture and open new perspectives. First of all we establish an explicit formula between the (canonical) Barthel endomorphism and a Wagner endomorphism (the nonlinear part of a Wagner connection). Then we calculate its tension, weak and strong torsion, i.e. data determining uniquely a nonlinear connection by Grifone's theory. It turns out that the rules of calculation with respect to a Wagner connection are formally the same as those with respect to the classical Cartan connection. These investigations are based on a number of some new (but more or less) technical observations and a fine analysis of the second Cartan tensor belonging to a Wagner endomorphism. Using these results an important classical theorem on the so-called *Landsberg manifolds*, first formulated and proved intrinsically by J. G. Diaz will be generalized. The classical version contains equivalent characterizations of the vanishing of the second Cartan tensor belonging to the Barthel endomorphism (i.e. the canonical nonlinear connection of the Finsler manifold). In his thesis [8] the author gives a coordinate-free proof of the theorem using several explicit relations between the classical Cartan tensors and curvatures (or their lowered tensors) of the Cartan connection. We managed to reduce the number of these relations to some of fundamental ones and the theorem is proved in generality of Wagner connections and Wagner manifolds. *Techniques we need to discuss them are suitable to reproduce lots of classical results as well*. We found this observation very useful.

Finally, after a new intrinsic definition as well as several tensorial characterizations of Wagner manifolds we present coordinate-free proofs of Hashiguchi-Ichijyō's theorems to clarify the geometrical meaning of this special class of Finsler manifolds. In the classical terminology: "The condition that a Finsler space be conformal to a Berwald space is that the space becomes a Wagner space with respect to a gradient $\alpha_i(x)$", (see [16], Theorem B).

III

In part III we deal with a special conformal change of Riemann-Finsler metrics introduced by M. Hashiguchi [14]. The point of the *C-conformality* is that we require the vanishing of one of conformal invariants. Under this hypothesis the gradient vector field of the scale function becomes independent of the "direction", i.e. it will be a vertically lifted vector field. (Vector fields with such a property is called *concurrent* too; see e.g. [14], [28] and [37].)

In his cited work [14] Hashiguchi proved for some special Finsler manifolds (in his terminology: two-dimensional spaces, C-reducible spaces, spaces with (α, β)-metric etc.) that the existence of a C-conformal change of the metric implies that the manifold is Riemannian (at least locally). Here we show that Hashiguchi's result is valid without any extra condition. In terms of our characterization this means that the vanishing of some conformal invariants, like the conformal invariant first Cartan tensor, can be interpreted as a sufficient condition for a Finsler manifold to be Riemannian. (The necessity is clear.) Our result is based on a usual, but relatively "rigid" definition of Finsler manifolds: the differentiability is required at *all* nonzero tangent vectors, i.e. there is *no* singularities except for the zero vectors of tangent spaces. Actually, the main points are the homogeneity and continuity of the Riemann-Finsler metric along the gradient vector field of the scale function which depends only on the "position" in case of a C-conformal change. Weakening the condition of differentiability new perspectives open to investigate the C-conformality. As an illustration we shall cite some valuable fragments from Hashiguchi's original ideas in one of the last remarks.

I. A NEW LOOK AT FINSLER CONNECTIONS AND SPECIAL FINSLER MANIFOLDS

In part I first of all we present a quite detailed exposition of the conceptual and calculational background. Although it means a practical summary (the troublesome details will be omitted) it seems to be enough to make our work self-contained as far as possible. As the next step we come to the overview of the fundamental facts and constructions concerning a Finsler manifold.

3.1. Definitions. Let a function $E : TM \to \mathbb{R}$ be given. The pair (M, E), or simply M, is said to be a *Finsler manifold*, if the following conditions are satisfied:

(3.1a) $\forall a \in \mathcal{T}M : E(a) > 0;\ E(0) = 0.$

(3.1b) E is of class C^1 on TM and smooth over $\mathcal{T}M$.

(3.1c) $CE = 2E$; i.e. E is homogeneous of degree 2.

(3.1d) The *fundamental form* $\omega := dd_J E \in \Omega^2(\mathcal{T}M)$ is nondegenerate.

The function E is called the *energy function* of the Finsler manifold. A horizontal endomorphism on M is said to be *conservative* if $d_h E = 0$.

In conformity with the demands of Finsler geometry, the smoothness is not required or assured a priori on the whole tangent manifold TM. (It is well-known that Finsler structures without singularities are just Riemannian). With the help of the so-called *energy function* E we can introduce a (pseudo-) Riemannian metric

$$g : v \in TM \setminus \{0\} \to g_v, \quad g_v : T^v{}_v TM \times T^v{}_v TM \to R$$

(the so-called *Riemann-Finsler metric*) on the vertical subbundle of the "tangent bundle" TTM. This means that in such the geometry all of objects depend on both "position" and "direction". As it is usual in case of a Riemannian manifold we also can associate a canonical horizontal endomorphism (the so-called *Barthel endomorphism*) to the function E together with lots of important tensors and further geometrical structures such as Cartan tensors and the canonical almost complex structure on the Finsler manifold.

4.2. Theorem (Fundamental lemma of Finsler geometry)**.** *Let (M, E) be a Finsler manifold. There exists a unique horizontal endomorphism h on M,called the Barthel endomorphism, such that*

(a) *h is conservative (i.e., $d_h E = 0$),*

(b) *h is homogeneous (i.e., $H = [h, C] = 0$),*

(c) *h is torsion-free (i.e., $t = [J, h] = 0$).*

Explicitly,

$$h = \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J, S]\right),$$

where S is the canonical spray.

The result is due to J. Grifone [10].

Among others we pay a particular attention to the so-called *first* and *second Cartan tensors*. The second Cartan tensor is introduced in a more general situation as usual. It means that this tensor is associated to an arbitrary horizontal endomorphism instead of just the canonical one. In particular we investigate the connection between the symmetry properties of the tensor and the characteristic data of the horizontal endomorphism.

3.9. Proposition. *Let (M, E) be a Finsler manifold. If h is a conservative, torsion-free horizontal endomorphism then the lowered second Cartan tensor is totally symmetric.*

Although our results belong to the foundations, they are new. The reason of this careful investigation is that the first and second Cartan tensors play an essential role in Finsler geometry as it will be demonstrated in section I/4. Here we present an invariant and axiomatic description of three notable Finsler connections (linear connections associated to a nonlinear one with the help of some conditions of compatibility): the *Berwald, Cartan* and *Chern-Rund connections.* We hope that our approach helps in better understanding the role of the different axioms, and open a path for further, essential generalizations. Theorems are organized as follows: the first group of the axioms characterizes a unique Finsler connection allowing us to derive the explicit rules of calculations for the corresponding covariant derivatives. Adding further conditions to them the second group yields the characterization of the three classical Finsler connections.

4.5. Theorem and definition. *Let (M, E) be a Finsler manifold and suppose that h is a conservative torsion-free horizontal endomorphism on M. Let g_h be the prolongation of g along h and $\mathcal{C}'$ the second Cartan tensor belonging to h. There exists a unique Finsler connection (D, h) on M such that*

(4.5a) *D is metrical (i.e. $Dg_h = 0$);*

(4.5b) *the $(v)v$-torsion $\mathbb{S}^1$ of D vanishes;*

(4.5c) *the $(h)h$-torsion $\mathbb{A}$ of D vanishes.*

The covariant derivatives with respect to D can be explicitly calculated by the following formulas: for each vector fields X, Y on $\mathcal{T}M$,

(4.5d) $$D_{JX}JY = J[JX, Y] + \mathcal{C}(X, Y) = \overset{\circ}{D}_{JX}JY + \mathcal{C}(X, Y);$$

(4.5e) $$D_{hX}JY = v[hX, JY] + \mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX}JY + \mathcal{C}'(X, Y);$$

(4.5f) $$D_{JX}hY = h[JX, Y] + F\mathcal{C}(X, Y) = \overset{\circ}{D}_{JX}hY + F\mathcal{C}(X, Y);$$

(4.5g) $$D_{hX}hY = hF[hX, JY] + F\mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX}hY + F\mathcal{C}'(X, Y).$$

Then

$$h^*DC = \frac{1}{2}H$$

where H is the tension of h (2.2b). *Therefore, if in addition to* (4.5a)–(4.5c)

$$h^* DC = 0 \tag{4.5h}$$

is also satisfied, then h is the Barthel endomorphism of the Finsler manifold. In this case (D, h) is called the (classical) Cartan connection *of the Finsler manifold (M, E).*

The idea of the existence proof is immediate. We start from a conservative, torsion-free horizontal endomorphism h (whose existence is clearly guaranteed; see also 4.6. Remarks (c)) and build the second Cartan tensor $\mathcal{C}'$ belonging to h. Then we define a rule of covariant differentation by the formulas (4.5d)–(4.5g). It can be checked by a straightforward calculation that the pair (D, h) obtained in this way is indeed a Finsler connection, and the axioms (4.5a)–(4.5c) are satisfied.

In our subsequent considerations we are going to prove the unicity statement. Assume (D, h) is a Finsler connection on M, satisfying (4.5a)–(4.5c). By the help of a systematic application of the "Christoffel process" we show that the rules of calculation (4.5d)–(4.5g) are valid.

4.6. Remarks.
(a) Observe that axioms (4.5a) and (4.5h) imply for any Finsler connection (D, h) that h is a conservative horizontal endomorphism. Indeed, for each vector field X on M,

$$0 \overset{(4.5a)}{=} (D_{X^h} g_h)(C, C) = X^h g_h(C, C) - 2g_h(D_{X^h} C, C) =$$
$$\overset{(3.3a)}{=} 2X^h E - 2g_h(D_{X^h} C, C) \overset{(4.5h)}{=} 2X^h E = 2(d_h E)(X^h),$$

which means that $d_h E = 0$.

(b) Axioms to characterize the classical Cartan connection were first formulated by M. Matsumoto; for an instructive historical remark see [24], p. 112.

(c) For the sake of completness we sketch an original process to construct torsion-free, conservative horizontal endomorphisms on a Finsler manifold (M, E). Let a function $\beta \in C^\infty(M)$ be given and define a semispray $\widetilde{S}$ by the formula

$$\widetilde{S} = S - \operatorname{grad} \beta^v, \tag{4.6}$$

where S denotes the canonical spray and $\operatorname{grad}\beta^v$ is the gradient of the function $\beta^v := \beta \circ \pi$ (see II.1.1). Then *the horizontal endomophism $\widetilde{h}$ induced by $\widetilde{S}$ is torsion-free and conservative*. Indeed, from the definition

$$\widetilde{h} := \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J, \widetilde{S}]\right) = h - \frac{1}{2}[J, \operatorname{grad}\beta^v]$$

we get immediately that for any vector fields $X, Y \in \mathfrak{X}(M)$,

$$\widetilde{t}(X^c, Y^c) \overset{(2.6b)}{=} [X^{\widetilde{h}}, Y^v] - [Y^{\widetilde{h}}, X^v] - [X, Y]^v =$$
$$= t(X^c, Y^c) - \frac{1}{2}\big[[X^v, \operatorname{grad}\beta^v], Y^v\big] + \frac{1}{2}\big[[Y^v, \operatorname{grad}\beta^v], X^v\big] \overset{\text{Th. 4.2 (c)}}{=}$$
$$= -\frac{1}{2}\big[[X^v, \operatorname{grad}\beta^v], Y^v\big] + \frac{1}{2}\big[[Y^v, \operatorname{grad}\beta^v], X^v\big] = 0$$

using the Jacobi identity. This means that $\widetilde{h}$ is torsion-free.

On the other hand, for any vector field $X \in \mathfrak{X}(TM)$,

$$d_{\widetilde{h}}E(X) = \widetilde{h}(X)E = h(X)E - \frac{1}{2}[J, \operatorname{grad}\beta^v](X)E \overset{\text{Th. 4.2 (a)}}{=}$$
$$= -\frac{1}{2}[J, \operatorname{grad}\beta^v](X)E \overset{\text{II.(1.2d)}}{=} \mathcal{C}(F\operatorname{grad}\beta^v, X)E = dE\big(\mathcal{C}(F\operatorname{grad}\beta^v, X)\big) =$$
$$\overset{(4.1)}{=} -\omega\big(S, \mathcal{C}(F\operatorname{grad}\beta^v, X)\big) = \omega(\mathcal{C}(F\operatorname{grad}\beta^v, X), S) =$$
$$= g(\mathcal{C}(F\operatorname{grad}\beta^v, X), C) = \mathcal{C}_\flat(F\operatorname{grad}\beta^v, X, S) \overset{\text{Lemma 3.8}}{=} 0,$$

i.e. $\widetilde{h}$ is conservative.

The next step is to insert the foundations of special Finsler manifolds such as the so-called *Berwald* and *locally Minkowski manifolds* into the framework has been elaborated by then. To realize our plan we need several technical observations summarized under the title *Basic curvature identities*. In this section we derive some important (partly well-known) relations between the curvature data of the different Finsler connections, these will be indispensable for a tensorial description of the special Finsler manifolds studied in the last two sections. Section I/6 concentrates the characterization of Berwald manifolds; some of them (e.g. 6.5 and 6.7) are new, and all of the proofs are original.

6.5. Definition. A Finsler manifold (M, E) is said to be a *Berwald manifold* if there is a linear connection ∇ on M such that for each vector fields X, Y on M,

$$(\nabla_X Y)^v = [X^h, Y^v],$$

where the horizontal lifting is taken with respect to the Barthel endomorphism.

6.7. Lemma. *A Finsler manifold (M, E) is a Berwald manifold if and only if*

$$\forall X, Y \in \mathfrak{X}(M) : [X^h, Y^v] \quad \textit{is a vertical lift.} \tag{6.7}$$

We believe that the compact, elegant and efficient formulations presented here demonstrate the power of our approach. In the concluding section I/7 the key observation is given in Proposition 7.2; this provides a very simple proof of the classical characterization of locally Minkowski manifolds.

7.2. Proposition. *A Finsler manifold (M, E) is a locally Minkowski manifold if and only if there exists a torsion-free, flat linear connection ∇ on M whose horizontal lift $\overset{h}{\nabla}$ is h-metrical with respect to the horizontal endomorphism arising from ∇.*

Finally, we have to emphasize that these results, more precisely the analogous ones play an important role in the theory of Wagner connections and Wagner manifolds. Since they are generalizations of the usual concepts in Finsler geometry (such as the Cartan connection and the Berwald or locally Minkowski manifolds) we present some of proofs in sections II/4 and II/5 in a more general situation as well. In this consideration one of the most important results is a generalization of a classical theorem (see I.6.3), first formulated and proved intrinsically by J. G. Diaz [8]. It contains equivalent tensorial characterizations of the vanishing of the second Cartan tensor associated to the Barthel endomorphism, i.e. the characterizations of the so-called *Landsberg manifolds*. In his thesis [8] the author gives a coordinate-free proof of this theorem using several explicit relations between the classical Cartan tensors and curvatures (or their lowered tensors) of the Cartan connection. We managed to reduce the number of these relations to some of fundamental ones and the theorem is proved in generality of Wagner connections and Wagner manifolds in section II/5; see Proposition 5.4. *Techniques we need to discuss them are suitable to reproduce lots of classical results as well.* We found this observation very useful.

II. WAGNER CONNECTIONS AND WAGNER MANIFOLDS

In part II we start with the definition of conformal equivalence of Riemann-Finsler metrics. This relation is formally the same as that in Riemannian geometry. Two Riemann-Finsler metrics g and $\widetilde{g}$ are said to be *conformally equivalent* if there exists a function $\varphi : TM \setminus \{0\} \to R^+$ such that

$$\widetilde{g} = \varphi g.$$

It is an immediate consequence of the definition that the so-called *scale* or *proportionality* function φ can be prolonged to a smooth function on the whole tangent bundle, actually it is constant on each tangent space TpM $(p \in M)$. We give a modern proof of this fameous observation due to M. S. KNEBELMAN [19]. In sections II/1 and II/2 we also derive some important conformal invariants and transformation formulas, first of all a key formula describing the change of the canonical spray of a Finsler manifold under a conformal change of the metric.

2.1. Theorem. *Suppose that g and $\widetilde{g}$ are conformally equivalent Riemann – Finsler metrics on M, namely*

$$\widetilde{g} = \varphi g; \quad \varphi = \exp \circ \alpha \circ \pi, \quad \alpha \in C^\infty(M).$$

Then the corresponding canonical sprays satisfy the relation

$$\widetilde{S} = S - \alpha^c C + E \operatorname{grad} \alpha^v. \tag{2.1}$$

As a consequence, we get immediately, how the Barthel endomorphism is changing.

2.2. Corollary. *Under the conditions of Theorem 2.1, the Barthel endomorphisms are related as follows:*

$$\widetilde{h} = h - \frac{1}{2}(\alpha^c J + d\alpha^v \otimes C) + \frac{1}{2} E[J, \operatorname{grad} \alpha^v] + \frac{1}{2} d_J E \otimes \operatorname{grad} \alpha^v. \tag{2.2}$$

Having these results, one can also describe the change of the Berwald and Cartan (and other) connections, etc. A complete summary can be found in Hashiguchi's paper [14] using the classical coordinate methods of calculation. In order to illustrate the problem we derive how the second Cartan tensors are related in case of conformal equivalence of Riemann-Finsler metrics.

2.8. Proposition. *The second Cartan tensor associated with the Barthel endomorphism h changes by the formula*

$$\begin{aligned} \widetilde{\mathcal{C}}'(X,Y) = \mathcal{C}'(X,Y) - \frac{1}{2}\Big(&\alpha^c \mathcal{C}(X,Y) + \\ &+ JX(E)\mathcal{C}(F \operatorname{grad} \alpha^v, Y) + JY(E)\mathcal{C}(F \operatorname{grad} \alpha^v, X) + \\ &+ \mathcal{C}_\flat(F \operatorname{grad} \alpha^v, X, Y) C\Big) - E(D_{\operatorname{grad} \alpha^v} \mathcal{C})(X,Y), \end{aligned} \tag{2.8}$$

where $\varphi := \exp \circ \alpha^v$ *is the scale function.*

An application of our results is also given in this section: we present an intrinsic proof of the classical theorem which (roughly speaking) states that in case of a simultaneous conformal and projective change the scale function is constant, i.e. the conformal change must be homothetic (see [30], p. 226).

After these "preliminaries" and illustrations we define the notion of Wagner connections. Such kind of Finsler connections were first constructed and used by V. V. WAGNER [40]. With the help of this seemingly strange connection Wagner introduced the notion of generalized Berwald manifolds (especially – in present day terminology – *Wagner manifolds*) and he showed that this class of special Finsler manifolds contains any two-dimensional Finsler manifold with cubic metric. The next important steps in the extension of the theory of Wagner connections and generalized Berwald manifolds were taken by M. HASHIGUCHI [13]. He successfully carried over Wagner's ideas to the arbitrary finite dimensional case, characterizing the Wagner connections by an elegant system of axioms (cf. section II/3). One of the most important observations, due to M. HASHIGUCHI and Y. ICHIJYŌ [16] is that Wagner connections are at the heart of the theory of conformal change of Riemann-Finsler metrics. Among others it turned out that the class of Wagner manifolds is closed under a conformal change of the metric. These results confirm Matsumoto's remarkable principle: "there should be existing a *best* Finsler connection for every theory of Finsler spaces" (see [25]).

3.1. Definition. Let (M, E) be a Finsler manifold. The triplet $(\overline{D}, \overline{h}, \alpha)$ is said to be a *Wagner connection* on M if it satisfies the following conditions:

(3.1a) $(\overline{D}, \overline{h})$ is a Finsler connection on $M, \alpha \in C^\infty(M)$;

(3.1b) $\overline{D}$ is metrical with respect to $g_{\overline{h}} : \overline{D} g_{\overline{h}} = 0$;

(3.1c) the $(v)v$-torsion $\overline{\mathbb{S}}^1$ of $\overline{D}$ vanishes: $\overline{\mathbb{S}}^1 = 0$;

(3.1d) $\overline{D}$ is $(h)h$-semisymmetric, i.e. the $(h)h$-torsion $\overline{\mathbb{A}}$ of $\overline{D}$ has the following form:

$$\overline{\mathbb{A}} = d\alpha^v \otimes \overline{h} - \overline{h} \otimes d\alpha^v;$$

(3.1e) the h-deflection $\overline{h}^*(DC)$ vanishes: $\overline{h}^*(DC) = 0$.

Then $\overline{h}$ is called a *Wagner endomorphism* on M.

In this part we demonstrate that the Frölicher-Nijenhuis formalism provides a perfectly adequate conceptual and technical framework for the study even of such complicated objects as Wagner connections. Our intrinsically formulated and proved results not only cover the classical local results but give a much more precise and transparent picture and open new perspectives. For example, we calculate the tension, the weak and strong torsion of a so-called *Wagner endomorphism* (the "nonlinear part" of a Wagner connection), i.e. data determining uniquely a nonlinear connection by Grifone's theory.

3.2. Proposition. *Any Wagner endomorphism is a conservative horizontal endomorphism, i.e.* $d_{\overline{h}}E = 0$.

(cf. I.4.6. Remarks (a))

3.3. Theorem. *The Wagner endomorphism* $\overline{h}$ *and the Barthel endomorphism* h *of a Finsler manifold are related as follows:*

$$\overline{h} = h + \alpha^c J - E[J, \operatorname{grad} \alpha^v] - d_J E \otimes \operatorname{grad} \alpha^v. \tag{3.3a}$$

The *proof* is based on a twofold Christoffel process applying the metrical character of the Wagner and the (classical) Cartan connection.

3.4. Corollary. *The tension of a Wagner endomorphism vanishes.*

3.5. Corollary. *The weak torsion and the strong torsion of a Wagner endomorphism can be given as follows:*

$$\overline{t} = d\alpha^v \otimes J - J \otimes d\alpha^v, \qquad \overline{T} = \alpha^c J - d\alpha^v \otimes C.$$

As one of the main results we conclude that the rules of calculation with respect to a Wagner connection are formally the same as those with respect to the classical Cartan connection. These investigations are realized with the help of a number of new (but more or less) technical observations and a fine analysis of the second Cartan tensor belonging to a Wagner endomorphism.

3.7. Proposition. *The second Cartan tensor* $\overline{\mathcal{C}}'$ *of a Wagner endomorphism* $\overline{h}$ *has the following properties:*

(3.7a) *it is semibasic,*

(3.7b) *its lowered tensor* $\overline{\mathcal{C}}'_b$ *is totally symmetric,*

(3.7c) $\overline{\mathcal{C}}'^{\circ} := i_{S_0}\overline{\mathcal{C}}' = 0$ *(S_0 is an arbitrary semispray on M).*

Basic curvature identities concerning a Wagner connection, including Bianchi identities are also derived. Using these results an important classical theorem on Landsberg manifolds will be generalized in section II/5 (cf. remarks at the end of part I):

5.4. Proposition. *Let* $(\overline{D}, \overline{h}, \alpha)$ *be a Wagner connection. Then the following assertions are equivalent:*

- *(a) the hv-curvature tensor* $\overline{\mathbb{P}}$ *of* $\overline{D}$ *vanishes:* $\overline{\mathbb{P}} = 0$.
- *(b) The second Cartan tensor* $\overline{\mathcal{C}}'$ *of* $\overline{h}$ *vanishes:* $\overline{\mathcal{C}}' = 0$.
- *(c)* $\forall X, Y, Z \in \mathfrak{X}(TM) : \left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y, Z) = \left(\overline{D}_{\overline{h}Z}\mathcal{C}\right)(X, Y)$.
- *(d)* $\forall X, Y, Z \in \mathfrak{X}(\mathcal{T}M) : \overset{\circ}{\overline{\mathbb{P}}}(X, Y)Z = -\left(\overline{D}_{\overline{h}X}\mathcal{C}\right)(Y, Z)$.

Finally, after a new intrinsic definition as well as several tensorial characterizations of Wagner manifolds, we present an intrinsic formulation and coordinate-free proofs for Hashiguchi-Ichijyō's theorems. In their joint work [16] the authors have explored the significance of Wagner manifolds relating them to the conformal changes of Riemann-Finsler metrics. Simply put, for any conformal change of the metric we can construct a special Finsler connection, the so-called *Wagner connection* with the help of the scale function. We can say that a Wagner connection is a *Cartan connection with non-vanishing* $(h)h$*-torsion*; i.e. it is a *generalized Cartan connection.* (The $(h)h$-torsion has a special semisymmetric form; cf. section II/3; Definition 3.1.) Then Wagner manifolds can be introduced on the model of classical Berwald manifolds. This means that the Wagner endomorphism, i.e. the nonlinear part of the Wagner connection is induced by a linear connection on the underlying manifold M. (Or, equivalently, the Wagner endomorphism is smooth on the *whole* tangent manifold TM.)

5.1. Definition. Let (M, E) be a Finsler manifold endowed with a Wagner connection $(\overline{D}, \overline{h}, \alpha)$. (M, E) is said to be a *Wagner manifold* (with respect to $(\overline{D}, \overline{h}, \alpha)$) if there is a linear connection ∇ on M such that

$$\forall X, Y \in \mathfrak{X}(M) : \overline{D}_{X^{\overline{h}}} Y^v = (\nabla_X Y)^v . \tag{5.1}$$

Then ∇ is called the *linear connection of the Wagner manifold.*

5.3. Theorem. *Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection on the Finsler manifold (M, E). Then the following assertions are equivalent:*

(a) (M, E) is a Wagner manifold (with respect to $(\overline{D}, \overline{h}, \alpha)$).

(b) The hv-curvature tensor $\overset{\circ}{\overline{\mathbb{P}}}$ of the Finsler connection $(\overset{\circ}{\overline{D}}, \overline{h})$ vanishes.

5.5. Theorem. *Let $(\overline{D}, \overline{h}, \alpha)$ be a Wagner connection on the Finsler manifold (M, E). Then the following assertions are equivalent:*

(a) (M, E) is a Wagner manifold (with respect to $(\overline{D}, \overline{h}, \alpha)$).

(b) $\forall X, Y, Z \in \mathfrak{X}(TM) : \left(\overline{D}_{\overline{h}X} \mathcal{C}\right)(Y, Z) = 0$.

In his paper [14], Hashiguchi suggested and (in some sense!) solved the problem: under what conditions does a Finsler manifold become conformal to a Berwald (or locally Minkowski) manifold. “These conditions were, however, given in terms of very complicated systems of differential equation, for which appropriate geometrical meanings have been wanted”, he wrote a year later in [16]. As it was shown these “appropriate geometrical meanings” were hidden in the notion of Wagner manifolds, sketched by the following diagram:

Wagner manifolds (with integrable Wagner endomorphism)

$\nwarrow \quad \nearrow$

$\searrow \quad \swarrow$

Conformal change of the metric

$\nearrow \quad \nwarrow$

$\swarrow \quad \searrow$

Berwald manifolds $- - - - - - - \dashrightarrow$ Locally Minkowski manifolds

Integrability condition

Namely, in the classical terminology: "The condition that a Finsler space be conformal to a Berwald space is that the space becomes a Wagner space with respect to a gradient $\alpha_i(x)$" ([16], Theorem B.). The key observation is the following

6.1. Theorem. *Let (M, E) be a Wagner manifold with respect to $(\overline{D}, \overline{h}, \alpha)$ and let us consider the conformal change $\widetilde{g} = \varphi g$ $(\varphi = \exp \circ \beta^v)$ of the metric g. Then the Finsler manifold $(M, \widetilde{E})$ is also a Wagner manifold with respect to the Wagner connection induced by $\frac{1}{2}\beta + \alpha \in C^\infty(M)$.*

It is now possible for us to complete our investigations by two important results (due to Hashiguchi and Ichijyō) of the theory of Wagner manifolds:

6.3. Theorem. *A Finsler manifold is conformal to a Berwald manifold if and only if it is a Wagner manifold.*

Proof. Let us suppose that the Finsler manifold (M, E) is conformal to a Berwald manifold, i.e., there is a conformal change $\widetilde{g} = \varphi g$ $(\varphi = \exp \circ \beta^v)$ such that $(M, \widetilde{E})$ is a Berwald manifold. Since the Berwald manifolds are, in particular, Wagner manifolds (cf. Prop. 3.12), in view of Theorem 6.1, the conformal change $g = \frac{1}{\varphi}\widetilde{g}$ yields a Wagner manifold with respect to the Wagner connection induced by $-\frac{1}{2}\beta \in C^\infty(M)$.

Explicitly, the Wagner endomorphism $\overline{h}$ and the Barthel endomorphism $\widetilde{h}$ of the Berwald manifold $(M, \widetilde{E})$ are related as follows:

$$\overline{h} = \widetilde{h} + \frac{1}{2} d\beta^v \otimes C.$$

Conversely, let us suppose that (M, E) is a Wagner manifold with respect to $(\overline{D}, \overline{h}, \alpha)$. Then, in view of Theorem 6.1, the conformal change $\widetilde{g} = \varphi g$ $(\varphi := \exp \circ \beta^v, \beta := -2\alpha)$ yields a Wagner manifold whose Wagner connection is induced by the function $\frac{1}{2}\beta + \alpha = -\alpha + \alpha = 0$. Therefore (cf. Prop. 3.12) $(M, \widetilde{E})$ is a Berwald manifold. The Barthel endomorphism $\widetilde{h}$ and the Wagner endomorphism $\overline{h}$ of the Wagner manifold (M, E) are related as follows:

$$\widetilde{h} = \overline{h} + d\alpha^v \otimes C. \qquad \square$$

6.4. Theorem. *A Finsler manifold is conformal to a locally Minkowski manifold if and only if it is a Wagner manifold and one (therefore all) of the conditions*

$$\text{(a)}\quad \overline{\Omega} = 0, \qquad \text{(b)}\quad \overline{\mathbb{R}} = 0, \qquad \text{(c)}\quad \overset{\circ}{\overline{\mathbb{R}}} = 0$$

are satisfied.

III. $\mathcal{C}$-CONFORMALITY

In part III we deal with a special conformal change of Riemann-Finsler metrics introduced by M. HASHIGUCHI [14]. The point of the so-called *C-conformality* is that we require the vanishing of one of conformal invariants described in section II/1; cf. Proposition 1.12.

2.1. Definition. Consider a Finsler manifold (M, E). A conformal change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v$, $\alpha \in C^\infty(M)$) is said to be *$\mathcal{C}$-conformal* at a point $p \in M$ if the following conditions are satisfied:

(2.1a) $$(d\alpha)_p \neq 0, \text{i.e., } \alpha \text{ is regular at the point } p;$$

(2.1b) $$[J, \operatorname{grad} \alpha^v] = 0.$$

(J is the vertical endomorphism or, in equivalent terminology, the canonical almost tangent structure of the tangent bundle $\pi : TM \to M$.)

Under this hypothesis the gradient vector field of the scale function becomes independent of the "direction", i.e. it will be a vertically lifted vector field. (Vector fields with such a property is called *concurrent* too; see e.g. [14], [28] and [37].)

2.2. Proposition. *Let (M, E) be a Finsler manifold and $\alpha \in C^\infty(M)$. Then the following assertions are equivalent:*

(a) $[J, \operatorname{grad} \alpha^v] = 0$.

(b) $\imath_{F \operatorname{grad} \alpha^v} \mathcal{C} = 0$.

(c) $\operatorname{grad} \alpha^v$ *is a vertical lift, i.e., there exists a vector field $X \in \mathfrak{X}(M)$ such that*

(2.2) $$\operatorname{grad} \alpha^v = X^v.$$

In his cited work [14] Hashiguchi proved for some special Finsler manifolds (in his terminology: two-dimensional spaces, C-reducible spaces, spaces with (α, β)-metric etc.) that the existence of a C-conformal change of the metric implies that the manifold is Riemannian (at least locally; cf. the condition (2.1a)). Here we show that Hashiguchi's result is valid without any extra condition.

2.3. Lemma and definition. *Consider a Finsler manifold (M, E) and let us suppose that the change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v$, $\alpha \in C^\infty(M)$) is $\mathcal{C}$-conformal at a point $p \in M$. Let $\sigma \in \mathfrak{X}(M)$ be an arbitrary vector field with the property $\sigma(p) \neq 0$ which obviously implies that σ is nonvanishing over a connected open neighbourhood U of p. Then the mapping*

(2.3a) $$\langle\, , \,\rangle : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U),$$

$$(Y, Z) \to \langle\, , \,\rangle(Y, Z) =: \langle Y, Z \rangle := g(Y^v, Z^v) \circ \sigma$$

is a (pseudo-) Riemannian metric. This metric is called the osculating Riemannian metric *along σ.*

If, in addition, $\operatorname{grad}_U \alpha \in \mathfrak{X}(U)$ *is the gradient of the function α with respect to $\langle\, , \,\rangle$ then*

(2.3b) $$(\operatorname{grad}_U \alpha)^v = \operatorname{grad} \alpha^v.$$

2.4. Remark. In the sequel we shall fix the vector field X determined by the formula (2.2) as σ in Lemma 2.3. (Note that the regularity property (2.1a) implies that $X(p) \neq 0$.)

Therefore, the osculating Riemannian metric $\langle\ ,\ \rangle$ will be considered as a mapping

$$\langle\ ,\ \rangle : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U), \tag{2.4}$$
$$(Y, Z) \to \langle\ ,\ \rangle(Y, Z) =: \langle Y, Z\rangle := g(Y^v, Z^v) \circ X,$$

where U is a fixed connected open neighbourhood of the point p such that for any $q \in U$, $X(q) \neq 0$.

2.5. Proposition. *Consider a Finsler manifold (M, E) with the Riemann-Finsler metric g and let us suppose that the change $\widetilde{g} = \varphi g$ ($\varphi = \exp \circ \alpha^v$, $\alpha \in C^\infty(M)$) is $\mathcal{C}$-conformal at a point $p \in M$. If $W \subset TpM$ is a subspace of dimension $n - 1$ such that $TpM = W \oplus \mathcal{L}(X(p))$ then for any tangent vector $w \in W \setminus \{0\}$ and $t \in \mathbb{R}$,*

$$g(Y^v, Z^v)(w + tX(p)) = g(Y^v, Z^v)(w).$$

Consequently, for any vector fields $Y, Z \in \mathfrak{X}(M)$, the function $g(Y^v, Z^v)$ is constant on $TpM \setminus \{0\}$.

The main result is a direct consequence of Proposition 2.5. More precisely, for any vector fields $Y, Z \in \mathfrak{X}(M)$,

$$g(Y^v, Z^v) = \langle Y, Z\rangle \circ \pi = \langle Y, Z\rangle^v,$$

where $\langle\ ,\ \rangle$ is the osculating Riemannian metric defined over U (cf. Theorem 2.6).

In terms of our characterization this means that the vanishing of some conformal invariants, like the conformal invariant first Cartan tensor, can be interpreted as a sufficient condition for a Finsler manifold to be Riemannian. (The necessity is clear.) Our result is based on a usual, but relatively "rigid" definition of Finsler manifolds: the differentiability of the energy function is required at *all* nonzero tangent vector, i.e. there is *no* singularity except for the zero vectors of tangent spaces. The basic idea we use to prove our statement is an observation on homogeneous functions. (Actually, we generalize the following well-known fact: *if a function is homogeneous of degree 0 and it is continuous at the origin, then the function is constant.*)

1.2. Proposition. *Let us select a subspace W of dimension $n - 1$ and a nonzero vector q of $\mathbb{R}^n$ ($n \geq 2$) such that*

$$\mathbb{R}^n = W \oplus \{tq \mid t \in \mathbb{R}\} =: W \oplus \mathcal{L}(q).$$

Suppose that a function $f : \mathbb{R}^n \to \mathbb{R}$ has the following properties:

(i) *it is positive homogeneous of degree* 0*;*
(ii) *it is continuous at the points q, $-q$;*
(iii) *for any point $a \in W \setminus \{0\}$ and scalar $t \in \mathbb{R}$*

$$f(a + tq) = f(a).$$

Then f is constant on $\mathbb{R}^n \setminus \{0\}$.

In other words, the main points are the *homogeneity* and *continuity* of the Riemann-Finsler metric along the gradient vector field of the scale function which depends only on the "position" in case of a C-conformal change. Weakening the condition of differentiability new perspectives open to investigate the C-conformality. As an illustration we shall cite some valuable fragments from Hashiguchi's original ideas in one of the last remarks:

"When I wrote my thesis ... , I imaged the following example as a non-Riemannian Finsler metric L admitting a $\mathcal{C}$-conformal change: Let m be a fixed integer such that $1 < m < n$. Indices a, b and λ, μ are supposed to take the values $1, \dots, m$ and $m+1, \dots, n$, respectively. On $\mathbb{R}^n$ we consider L given by

$$L^2(x^i, y^i) = L_1^2(x^i, y^a) + L_2^2(x^i, y^\lambda),$$

where L_1 is a non-Riemannian Finsler metric, and L_2 is a Riemannian metric ... . Especially, the three-dimensional Finsler metric L on $\mathbb{R}^3$ given by

$$L^2(x^1, x^2, x^3, y^1, y^2, y^3) = x^3 \frac{(y^1)^4}{(y^2)^2} + (y^3)^2$$

admits a $\mathcal{C}$-conformal change $\overline{L} = e^\alpha L$, where $\alpha := -\frac{1}{2}(x^3)^2$, which gives a so-called concurrent vector field α_i ". (Hashiguchi's letter to the author; 2000–01–05).

Indeed, a routine calculation shows that for example the gradient of the function $\alpha^v := \alpha \circ \pi$, $\alpha := -\frac{1}{2}(x^3)^2$ is just the vector field $\left(-x^3 \frac{\partial}{\partial x^3}\right)^v = -(x^3 \circ \pi)\frac{\partial}{\partial y^3}$. Therefore, the elements of the matrix

$$(g_{ij})_{3\times 3} = x^3 \begin{pmatrix} 6\left(\frac{y^1}{y^2}\right)^2 & -4\left(\frac{y^1}{y^2}\right)^3 & 0 \\ -4\left(\frac{y^1}{y^2}\right)^3 & 3\left(\frac{y^1}{y^2}\right)^4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are neither continuous along the vector field $X := -x^3 \frac{\partial}{\partial x^3}$ and even nor defined there.

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