

Short thesis for the degree of doctor of philosophy
(PhD)

ON THE HOLONOMY
OF FINSLER MANIFOLDS

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Contents

1	Introduction	
2	Preliminaries	2
3	Finsler manifold with maximal holonomy	5
3.1	Maximal holonomy	5
3.2	Spherically symmetric projective Finsler metrics	6
4	Randers surfaces with constant curvature	9
4.1	Infinitesimal holonomy algebras	10
4.2	Holonomy groups	12
5	Natural parallelism	16
5.1	Parallel translation associated to navigation data	16
5.2	Connection associated to the natural parallelism	19
	Bibliography	22

1 Introduction

The end of the nineteenth century and the beginning of the twentieth century witnessed a profound development in mathematical thought, as new directions emerged aiming to extend the classical framework of Riemannian geometry. Although Bernhard Riemann had already indicated in his celebrated 1854 habilitation lecture the possibility of generalizing the notion of a metric beyond one arising from an inner product, this idea did not crystallize into an independent field until the work of Paul Finsler. His dissertation, written under the influence of the calculus of variations, laid the methodological foundations of what is now known as Finsler geometry by broadening the concept of length so that it depends not only on position but also on direction, thereby providing a more flexible and comprehensive geometric framework than the traditional Riemannian model. At the beginning of the twentieth century, the intensive study of Finsler metrics was further stimulated by problems arising in the calculus of variations and optimal transport theory. Following Finsler's foundational work [23], the theory developed through the contributions of several outstanding mathematicians, including C. Carathéodory, H. Busemann, É. Cartan, L. Berwald, and S. S. Chern. In particular, ideas from the calculus of variations and from Riemannian geometry—such as affine connections, Jacobi fields, and curvature notions—were gradually adapted to this broader setting. Although the theory has evolved significantly, many structural questions remain open. In recent decades, increasing attention has also been directed toward applications of Finsler geometry in the natural sciences, ranging from general relativity and seismic ray modeling to wildfire spread and quantum mechanics [38, 2]. In particular, the investigations of B. Russell and S. Stepney [53, 54] have opened new perspectives in quantum information processing by exploiting the theorem of D. Bao, C. Robles, and Z. Shen on the one-to-one correspondence between solutions of Zermelo's navigation problem and Randers metrics [6].

One important geometric feature that still requires a deeper understanding is the holonomy structure, which forms one of the central themes of this thesis. The holonomy group is a natural geometric invariant associated with a manifold endowed with a connection, and it reflects how the local geometric data encoded by the connection influences the global structure of the manifold. The subject of this thesis is the investigation of the holonomy structure of Finsler manifolds and some related aspects of Finsler geometry.

The notion of holonomy has its historical roots in classical mechanics, where it appeared at the end of the nineteenth century. It was Heinrich Hertz who introduced the terms *holonomic* and *non-holonomic*. This sentence needs revision. constraints in his posthumously published work *Die Prinzipien der Mechanik, in neuen Zusammenhängen dargestellt* (*The Principles of Mechanics Presented in a New Form*), which appeared in 1895. In its modern mathematical meaning, however, holonomy emerged only later as a genuinely geometric concept.

In the mathematical setting, the notion of holonomy of an affine connection and the corresponding parallel transport seems to have first appeared in the work of É. Cartan [13, 14, 15]. Working with the Levi–Civita connection of a Riemannian manifold M , Cartan showed that the corresponding holonomy group is a subgroup of the orthogonal group. He also proved that, if M is simply connected, then the holonomy group is connected. Furthermore, he observed that for any two points $p, q \in M$ on a connected manifold, the holonomy groups $\mathcal{H}ol_p(M)$ and $\mathcal{H}ol_q(M)$ are conjugate via parallel translation along any curve joining p to q . Therefore, the holonomy group $\mathcal{H}ol(M) \subset Gl(n, \mathbb{R})$ is well defined up to conjugation.

The geometric meaning of holonomy is that it captures important information about the global structure of a manifold. Informally, the holonomy group describes the fact that parallel transport may depend on the chosen path: when a geometric object is transported along a closed curve, it may fail to return to its original position. This is one of the basic effects of curvature, and it appears not only in differential geometry but also in physics and classical mechanics.

To explain this more precisely, we first recall the notion of parallel transport. In Euclidean space, vectors at different points can be compared directly, since the tangent spaces are naturally identified. On a curved manifold, however, such a comparison requires additional structure, namely a connection. In the Riemannian case, the natural choice is the Levi–Civita connection ∇ . Let γ be a curve joining p and q . A vector field V along γ

is said to be parallel if it satisfies

$$\nabla_{\dot{\gamma}}V = 0. \tag{1.1}$$

For such a curve, the parallel transport is the map

$$\mathcal{P}_{\gamma}: T_pM \rightarrow T_qM \tag{1.2}$$

defined as follows: for a given vector $v \in T_pM$, let V be the parallel vector field along γ with initial value $V(0) = v$. Then the parallel transport of v along γ is given by $\mathcal{P}_{\gamma}(v) = w$, where $w = V(1)$. The map \mathcal{P}_{γ} is an isomorphism from T_pM onto T_qM . In general, however, this isomorphism depends on the chosen curve γ , and there is no reason why parallel transports along different curves should coincide. This dependence on the path is described by the holonomy group, which is one of the main objects of our investigation. More precisely, the holonomy group at a point $p \in M$ is the group of all automorphisms to (1.2). In the Riemannian case, the holonomy groups have been extensively studied and now their complete classification is known, due to the work of excellent mathematicians. In 1952, A. Borel and A. Lichnerowicz showed that the holonomy group of a simply connected n -dimensional Riemannian manifold is a closed Lie subgroup of the orthogonal group [9]. In that same year, de Rham [21] established the result now known as the de Rham decomposition theorem. It states that when the holonomy group of a Riemannian manifold is reducible, the manifold locally splits into a product of Riemannian manifolds. Shortly afterwards, W. Ambrose and M. Singer clarified the close relationship between holonomy and curvature [1]. A few years later, M. Berger in his doctoral thesis established his celebrated list of all possible holonomy groups of Riemannian manifolds [8]. Since then, each group on Berger's list has been realized as the holonomy group of a suitable Riemannian manifold. In 1998, S. Merkulov and L. J. Schwachhöfer classified all irreducible holonomy groups of torsion-free connections [37, 57]. This classification revealed new symplectic holonomies, namely holonomy groups associated with symplectic manifolds carrying a parallel symplectic form. After Bryant's first example [10], further work produced an infinite family of such connections [19, 20]. In the Finslerian setting, some early contributions to holonomy in Finsler geometry already appeared in the classical literature. In particular, Barthel's paper [7] on nonlinear connections and their holonomy groups is a fundamental work in this direction; see also Kozma's survey [34] for a clear summary. Barthel also refers to an earlier

paper of V.V. Wagner [62] on two-dimensional Finsler spaces with finite and continuous holonomy groups.

In Finsler geometry, the holonomy structure may be very different from that of the Riemannian case. The main reason is that the canonical connection of a Finsler manifold is, in general, neither linear nor compatible with a metric in the usual sense. As a result, parallel transport in Finsler geometry can behave differently: it is only positively homogeneous of degree one and preserves the norm instead of the metric tensor. Parallel transport acts naturally on the indicatrices, and the holonomy group can therefore be viewed as a subgroup of the diffeomorphism group of the indicatrix. Nevertheless, our knowledge of Finslerian holonomy groups remains rather limited, and this topic continues to be an active area of modern geometric research. The first significant contributions to the study of holonomy in Finsler geometry were made by Z. Szabó [58] and later by L. Kozma [35]. Their work established fundamental results for certain special classes of Finsler manifolds and showed that, in these cases, holonomy theory still exhibits strong similarities with the classical Riemannian setting.

However, the situation changes fundamentally in more general Finslerian setting. In [42], it was established that the holonomy groups of Finsler manifolds need not be compact or finite-dimensional. In [44], explicit two-dimensional examples were exhibited whose holonomy groups are infinite-dimensional and isomorphic to $\mathcal{D}iff_+(\mathbb{S}^1)$, the orientation-preserving diffeomorphism group of the circle. Further systematic investigations have since deepened our understanding. In [31], it was shown that the holonomy group of a simply connected, non-Riemannian, projectively flat Finsler two-manifold of constant nonzero flag curvature is maximal and isomorphic to $\mathcal{D}iff_+(\mathbb{S}^1)$. These results suggest that infinite-dimensional holonomy is not an anomaly but rather a common phenomenon among Finsler metrics. The results of [29] showed that infinite-dimensional holonomy is the typical situation in Finsler geometry: on every manifold, there exists an open and dense subset of Finsler metrics whose holonomy groups are infinite-dimensional. Consequently, finite-dimensional holonomy appears only in exceptional cases. These include flat metrics, as well as Riemannian, Berwald, and Landsberg-type Finsler metrics [9, 58, 35]. In addition, for locally projectively flat Randers manifolds, [43] proved that finite-dimensional holonomy occurs exactly in the flat or Riemannian case.

This thesis is devoted to further results on the holonomy theory of Finsler manifolds. In the preliminary Chapter 2, we introduce the basic notions and concepts of spray geometry and Finsler geometry that will be

used throughout the thesis.

In Chapter 3, we investigate the holonomy structure of Finsler manifolds with maximal holonomy. In Section 3.1, we establish a general result showing that if the holonomy algebra at a point is dense in the Lie algebra of smooth vector fields on the indicatrix, then the holonomy group is maximal, meaning that its closure is isomorphic to the identity component of the diffeomorphism group of the indicatrix. In Section 3.2, we apply this theorem to spherically symmetric projective Finsler metrics of nonzero constant flag curvature and derive explicit descriptions of their holonomy groups. In particular, this yields the holonomy groups of the standard Funk metric and the Bryant–Shen metrics in arbitrary dimension. These results are generalizations of the results of [44]. This chapter is based on the results of [39].

In Chapter 4, we investigate the holonomy of two-dimensional Randers manifolds with constant flag curvature and provide a complete classification of their holonomy groups. Our starting point is the classification of Randers metrics of constant flag curvature [6] given in Theorem 4.1, which yields a finite family of models. In Section 4.1, we first analyze the infinitesimal holonomy structure by determining the corresponding infinitesimal holonomy algebras, and show that they are finite-dimensional in some cases and infinite-dimensional in others. In Section 4.2, we then turn our attention to the holonomy groups. We prove that when the infinitesimal holonomy algebra is infinite-dimensional, then the holonomy is maximal, that is, its closure is isomorphic to the group of orientation-preserving diffeomorphisms of the circle. On the other hand, when the infinitesimal holonomy algebra is finite-dimensional, we determine explicitly the holonomy groups. In particular, we show that a proper (i.e., one that is neither Riemannian, Berwald, nor Landsberg type) non-flat Finsler metric can have a finite-dimensional holonomy group. This chapter is based on the results of [40].

In Chapter 5, we introduce a natural parallelism associated with navigation data (h, W) . D. Bao, C. Robles, and Z. Shen proved in [6] that the Zermelo’s navigation problem is equivalent to considering geodesics of Randers-type Finsler metrics. The construction of the metric structure associated to the navigation data is easy to understand (the sets of unit vectors, called indicatrices, are blown away by the wind W), however, the affine structure (the parallel translation) is not so easy or natural to understand [51]. Moreover, the holonomy group can be very large even in cases when the metric structure is relatively simple [30]. For this reason,

we consider the geometric setting of navigation data and introduce a natural parallel translation using the Riemannian parallelism. The geometry obtained in this way has some nice and natural features: a natural parallel translation is homogeneous (but in general nonlinear), preserves the Randers type Finslerian norm constituted by the navigation data, and the holonomy group is finite-dimensional. This chapter is based on the results of [41].

2 Preliminaries

Let M be a C^∞ smooth manifold. $\mathfrak{X}(M)$ denotes the $C^\infty(M)$ -module and \mathbb{R} -vector space of smooth vector fields on M and $\mathcal{D}iff(M)$ is the group of all smooth diffeomorphisms of M . The first and the second tangent bundles of M are denoted by (TM, π, M) and (TTM, τ, TM) , respectively. Local coordinates (x^i) on M induce local coordinates (x^i, y^i) on TM .

Finsler manifolds, connection

A *Finsler manifold* is a pair (M, \mathcal{F}) , where the norm function $\mathcal{F}: TM \rightarrow \mathbb{R}_+$ is continuous, smooth on $\hat{TM} := TM \setminus \{0\}$, its restriction $\mathcal{F}_x = \mathcal{F}|_{T_x M}$ is a positively homogeneous function of degree one and the symmetric bilinear form

$$g_{x,y}: (u, v) \mapsto g_{ij}(x, y)u^i v^j = \frac{1}{2} \frac{\partial^2 \mathcal{F}_x^2(y + su + tv)}{\partial s \partial t} \Big|_{t=s=0} \quad (2.1)$$

is positive definite at every $y \in \hat{T}_x M$. The hypersurface of $T_x M$ defined by

$$\mathcal{I}_x = \{y \in T_x M \mid \mathcal{F}(x, y) = 1\}, \quad (2.2)$$

is called the *indicatrix* at $x \in M$. We note that at any point $x \in M$ the indicatrix is diffeomorphic to the $(n - 1)$ -dimensional sphere. *Geodesics* of (M, \mathcal{F}) are determined by a system of 2nd order ordinary differential equations

$$\ddot{x}^i + 2G^i(x, \dot{x}) = 0, \quad i = 1, \dots, n, \quad (2.3)$$

in a local coordinate system (x^i, y^i) of TM , where $G^i(x, y)$ are given by

$$G^i(x, y) := \frac{1}{4} g^{il}(x, y) \left(2 \frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y) \right) y^j y^k. \quad (2.4)$$

A vector field $V(t) = V^i(t) \frac{\partial}{\partial x^i}$ along a curve $\gamma(t)$ is said to be parallel with respect to the associated *homogeneous (nonlinear) connection* if it satisfies

$$D_{\dot{\gamma}}V := \left(\frac{dV^i}{dt} + G_j^i(\gamma, V) \cdot \dot{\gamma}^j \right) \frac{\partial}{\partial x^i} = 0, \quad (2.5)$$

where $G_j^i = \frac{\partial G^i}{\partial y^j}$. The *horizontal Berwald covariant derivative* $\nabla_X \xi$ of $\xi(x, y) = \xi^i(x, y) \frac{\partial}{\partial y^i}$ by the vector field $X(x) = X^i(x) \frac{\partial}{\partial x^i}$ is expressed locally by

$$\nabla_X \xi = \left(\frac{\partial \xi^i(x, y)}{\partial x^j} - G_j^k(x, y) \frac{\partial \xi^i(x, y)}{\partial y^k} + G_{jk}^i(x, y) \xi^k(x, y) \right) X^j \frac{\partial}{\partial y^i}, \quad (2.6)$$

where we denote $G_{jk}^i(x, y) := \frac{\partial G_j^i(x, y)}{\partial y^k}$. The *curvature tensor* field

$$R = R_{jk}^i(x, y) dx^j \otimes dx^k \otimes \frac{\partial}{\partial x^i}, \quad (2.7)$$

has the expression

$$R_{jk}^i = \frac{\partial G_j^i}{\partial x^k} - \frac{\partial G_k^i}{\partial x^j} + G_j^m G_{km}^i - G_k^m G_{jm}^i. \quad (2.8)$$

Finslerian parallel translation and holonomy

Let (M, \mathcal{F}) be a Finsler manifold. The notion of *parallel translation* along a curve $\gamma: [0, 1] \rightarrow M$ is described by a smooth mapping

$$\mathcal{P}_\gamma: T_{\gamma(0)}M \rightarrow T_{\gamma(1)}M, \quad (2.9)$$

determined by parallel vector fields along γ . Since this map preserves the Finsler norm, it also induces a smooth map between the corresponding indicatrices. The *holonomy group* $\mathcal{H}ol_x(\mathcal{F})$ at a point $x \in M$ is the subgroup of transformations generated by parallel translation along all piece-wise smooth closed loops based at $x \in M$. Due to the norm-preserving property of parallel translation, each holonomy transformation can be regarded as a diffeomorphism of the indicatrix. Consequently, the holonomy group can be realized as a subgroup of the diffeomorphism group of the indicatrix:

$$\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}iff(\mathcal{I}_x). \quad (2.10)$$

The *holonomy algebra* $\mathfrak{hol}_x(\mathcal{F})$, introduced in [31], is the tangent space of the holonomy group at its unit element, and is a Lie subalgebra of the Lie algebra of vector fields on the indicatrix \mathcal{I}_x :

$$\mathfrak{hol}_x(\mathcal{F}) \subset \mathfrak{X}(\mathcal{I}_x). \quad (2.11)$$

A key class of vector fields in $\mathfrak{hol}_x(\mathcal{F})$ arises from the curvature tensor (2.7): the curvature vector fields

$$y \mapsto R_{(x,y)}(X_1, X_2), \quad (2.12)$$

along with their iterated covariant derivatives, generate the *infinitesimal holonomy algebra* at a point $x \in M$:

$$\mathfrak{hol}_x^*(\mathcal{F}) := \left\langle \nabla_{X_k} \dots \nabla_{X_3} R(X_1, X_2) \Big|_{\mathcal{I}_x} \mid X_1, \dots, X_k \in \mathfrak{X}(M) \right\rangle_{\text{Lie}} \quad (2.13)$$

At any point x of the manifold M the infinitesimal holonomy algebra is a Lie subalgebra of the holonomy algebra:

$$\mathfrak{hol}_x^*(\mathcal{F}) \subset \mathfrak{hol}_x(\mathcal{F}), \quad (2.14)$$

therefore the dimension of the infinitesimal holonomy algebra gives the lower estimate

$$\dim(\mathfrak{hol}_x^*(\mathcal{F})) \leq \dim(\mathcal{Hol}_x(\mathcal{F})), \quad (2.15)$$

for the dimension of the holonomy group.

3 Finsler manifold with maximal holonomy

This chapter studies the holonomy of n -dimensional Finsler manifolds via holonomy algebras, proves that the density of the holonomy algebra implies maximal holonomy, and applies this result to spherically symmetric projective Finsler metrics of nonzero constant flag curvature, showing that their holonomy groups are isomorphic to $\mathcal{D}iff_o(\mathbb{S}^{n-1})$. This chapter is based on the article [39].

3.1 Maximal holonomy

Before proving the main theorem of this section, we establish a fundamental topological property of the holonomy group in the simply connected case.

3.1 Lemma. *Let (M, \mathcal{F}) be an n -dimensional simply connected Finsler manifold, and let $x \in M$. Then the holonomy group at x is contained in the connected component of the identity in the diffeomorphism group of the indicatrix:*

$$\mathcal{H}ol_x(\mathcal{F}) \subset \mathcal{D}iff_o(\mathcal{I}_x). \quad (3.1)$$

3.2 Lemma. *Let \mathcal{I} be a n -dimensional manifold and set*

$$G := \langle \exp(\mathfrak{X}(\mathcal{I})) \rangle_{\text{group}} \subset \mathcal{D}iff_o(\mathcal{I}). \quad (3.2)$$

Then G is a non-trivial normal subgroup of $\mathcal{D}iff_o(\mathcal{I})$. In particular, if $\mathcal{D}iff_o(\mathcal{I})$ is simple, then $G = \mathcal{D}iff_o(\mathcal{I})$.

We are now in a position to state the main theorem of this section, which provides a sufficient condition for the holonomy group to be maximal in terms of the holonomy algebra in the n -dimensional case.

3.3 Theorem. *Let (M, \mathcal{F}) be an n -dimensional simply connected Finsler manifold and let $x \in M$. If the holonomy algebra $\mathfrak{hol}_x(\mathcal{F})$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_x)$ of smooth vector fields on the indicatrix \mathcal{I}_x , then the holonomy group at x is maximal. More precisely,*

$$\overline{\text{Hol}_x(\mathcal{F})} \cong \text{Diff}_o(\mathbb{S}^{n-1}), \quad (3.3)$$

where $\text{Diff}_o(\mathbb{S}^{n-1})$ denotes the connected component of the identity in the diffeomorphism group of the $(n-1)$ -dimensional sphere.

In the following section, we apply this criterion to spherically symmetric projective Finsler metrics.

3.2 Spherically symmetric projective Finsler metrics

Z. Shen in his paper [56] investigated projective Finsler metrics with constant flag curvature, and give a complete classification in the x -analytical case. In particular, he showed that a projective Finsler metrics \mathcal{F} with constant flag curvature is completely determined – using an x_0 centered coordinate system – by the functions $\psi(y) = \mathcal{F}(0, y)$ and $\phi(y) = \mathcal{P}(0, y)$, where \mathcal{P} is the projective factor. In this section, we focus on the geometrically natural situation in which these data are spherically symmetric. More precisely at some particular point $x_o \in M$, the Finsler function and the projective factor are both a multiple of the Euclidean norm

$$\mathcal{F}(x_o, y) = c_1 \|y\|, \quad \mathcal{P}(x_o, y) = c_2 \|y\|, \quad (3.4)$$

with $c_1, c_2 \neq 0$. We normalize the metric by choosing $c_1 = 1$. Thus, without loss of generality, we assume that

$$\mathcal{F}(x_o, y) = \|y\|, \quad \mathcal{P}(x_o, y) = c \|y\|, \quad (3.5)$$

where $c \neq 0$ is a constant. Under this normalization, the indicatrix at x_o is \mathbb{S}^{n-1} , the $(n-1)$ -dimensional Euclidean sphere. We introduce a multi-index notation. For a multi-index $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$ we define its length by $\ell(\mathbf{m}) = m_1 + \dots + m_n$, and set

$$\mathbf{y}^{\mathbf{m}} := \prod_{k=1}^n (y^k)^{m_k} = (y^1)^{m_1} \dots (y^n)^{m_n}.$$

For each integer $p \geq 0$, we define the real vector space

$$\mathcal{A}_p := \text{span}_{\mathbb{R}} \left\{ \frac{\mathbf{y}^{\mathbf{m}}}{\|y\|^{\ell(\mathbf{m})}} \xi_{ij} \Big|_{\widehat{T}_{x_0} M} \mid 1 \leq i < j \leq n, |\ell(\mathbf{m})| = p \right\}, \quad (3.6)$$

and we introduce the Lie algebra

$$\mathcal{A} := \bigoplus_{p=0}^{\infty} \mathcal{A}_p. \quad (3.7)$$

3.4 Remark. The elements of \mathcal{A} can be interpreted as 1-homogeneous vector fields on the slit tangent space $\widehat{T}_{x_0} M$. Equivalently, after restriction to the indicatrix $\mathcal{I}_{x_0} \simeq \mathbb{S}^{n-1}$, they can be regarded as vector fields on \mathbb{S}^{n-1} with polynomial coefficients. Indeed, in (3.6) the factor $\|y\|^{-|\mathbf{m}|}$ is identically equal to 1 on the indicatrix, since $\|y\| = 1$ on \mathcal{I}_{x_0} .

3.5 Lemma. *The Lie algebra \mathcal{A} satisfies $\mathcal{A} \subset \mathfrak{hol}_{x_0}^*(\mathcal{F})$.*

The infinitesimal holonomy algebra $\mathfrak{hol}_{x_0}^*(\mathcal{F})$ contains the spaces \mathcal{A}_p , $p \in \mathbb{N}$, that is by using (2.14), they are in the holonomy algebra $\mathfrak{hol}_{x_0}(\mathcal{F})$. In particular, this implies that $\mathfrak{hol}_{x_0}(\mathcal{F})$ contains all polynomial-type vector fields on the indicatrix \mathcal{I}_{x_0} obtained from curvature vector fields by multiplication with polynomial coefficients. These vector fields form a dense subalgebra of $\mathfrak{X}(\mathcal{I}_{x_0})$. Consequently, $\mathfrak{hol}_{x_0}(\mathcal{F})$ is dense in $\mathfrak{X}(\mathcal{I}_{x_0})$:

3.6 Proposition. *Let (M, \mathcal{F}) be a projectively flat, spherically symmetric Finsler manifold of constant flag curvature $\lambda \neq 0$, and let $x_0 \in M$ be a point at which condition (3.5) is satisfied. Then the holonomy algebra $\mathfrak{hol}_{x_0}^*(\mathcal{F})$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_{x_0})$ of smooth vector fields on the indicatrix \mathcal{I}_{x_0} .*

We now pass from the holonomy algebra to the holonomy group. The following maximality result is a direct application of Theorem 3.3 from the previous section.

3.7 Theorem. *Let (M, \mathcal{F}) be a simply connected, projectively flat, spherically symmetric Finsler manifold of constant curvature $\lambda \neq 0$, and let $x_0 \in M$ be a point at which condition (3.5) is satisfied. Then the holonomy group $\text{Hol}_{x_0}(\mathcal{F})$ is maximal; that is, its closure is isomorphic to $\text{Diff}_o(\mathbb{S}^{n-1})$, the connected component of the identity in the group of smooth diffeomorphisms of the $(n-1)$ -dimensional sphere.*

We conclude this section by presenting two classical examples which illustrate the applicability of Theorem 3.7. In both cases, the metrics under

consideration are projectively flat and have constant flag curvature, and the assumptions of the theorem can be verified explicitly at a fixed point. These examples show how the general theory applies to well-known Finsler metrics.

3.8 Example. (P. Funk [26, 27]) The standard Funk metric on \mathbb{D}^n is projectively flat with constant flag curvature $\lambda = -\frac{1}{4}$ [26, 27]. At the origin,

$$\mathcal{F}(x_o, y) = |y|, \quad \mathcal{P}(x_o, y) = \pm \frac{1}{2}|y|.$$

Hence the assumptions of Theorem 3.7 are satisfied, and $\mathcal{H}ol_{x_o}(\mathcal{F}) \cong \mathcal{D}iff_o(\mathbb{S}^{n-1})$.

3.9 Example. The Bryant–Shen metrics form a family of projectively flat Finsler metrics with constant flag curvature $\lambda = 1$ [11, 12, 56]. At the origin,

$$\mathcal{F}_\alpha(0, y) = |y| \cos \alpha, \quad \mathcal{P}_\alpha(0, y) = |y| \sin \alpha, \quad |\alpha| < \frac{\pi}{2}.$$

Therefore the assumptions of Theorem 3.7 hold, and $\mathcal{H}ol_{x_o}(\mathcal{F}_\alpha) \cong \mathcal{D}iff_o(\mathbb{S}^{n-1})$.

4 Holonomy of Randers surfaces with constant flag curvature

This chapter gives a classification of the holonomy groups of Randers surfaces of constant curvature, which are among the most illustrative two-dimensional Finsler structures arising from navigation data. Both finite- and infinite-dimensional holonomy groups occur. The classification shows that the two negatively curved classes have essentially different holonomy behavior, and proves that, although generic Finsler holonomy groups are infinite-dimensional, there exist genuine Finsler metrics with finite-dimensional holonomy groups. To study these holonomy groups, we use the classification of Randers metrics of constant flag curvature obtained by Bao, Robles, and Shen [6], which we recall below. This chapter is based on [40].

4.1 Theorem ([6, Proposition 5.and 6]). *Let $\mathcal{F}(x, y)$ be a Randers metric with constant flag curvature K defined by the navigation data (h, W) on the n -dimensional manifold M . Then, up to local isometry, (h, W) must belong to one of the following families:*

- $K = 0$, then h is the Euclidean metric, $W = Qx + C$, with $(Qx + C) \cdot (Qx + C) < 1$;
- $K > 0$, then h is $1/K$ times the standard metric on the unit n -sphere, and
 $W = Qx + C + (x \cdot C)x$, with $\frac{1}{1+(x \cdot x)}((Qx + C) \cdot (Qx + C) + (x \cdot C)^2) < K$,
- $K < 0$, there are two possibilities:
 - Case (\mathcal{K}_1^-) : h is the Klein metric of sectional curvature K on the unit ball \mathbb{B}^n , and $W = Qx + C - (x \cdot C)x$, where
 $\frac{1}{1-(x \cdot x)}((Qx + C) \cdot (Qx + C) - (x \cdot C)^2) < |K|$,
 - Case (\mathcal{K}_2^-) : h is the Euclidean metric, and $W = Qx + C - \frac{\sigma}{2}x$ satisfies

$$(Qx + C) \cdot (Qx + C) + \sigma x \cdot \left(\frac{\sigma}{4}x - C\right) < 1, \text{ where} \\ \sigma = \pm\sqrt{|K|},$$

where $Q = (Q_j^i)$ is a constant skew-symmetric matrix and $C = (C_i)$ is a constant vector, $Qx = (Q_j^i x^j)$, $x = (x^i)$, and “ \cdot ” denotes the standard Euclidean dot product.

4.1 Infinitesimal holonomy algebra of Randers surfaces with constant flag curvature

In this section, we study the infinitesimal holonomy algebra of simply connected Randers surfaces of constant flag curvature arising from navigation data. According to Theorem 4.1, we consider separately the cases $K = 0$, $K > 0$, and $K < 0$. By the classification [6, Section 6], we may assume without loss of generality that

$$Q = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}, \quad C = (c, 0),$$

where $q, c \in \mathbb{R}$.

4.1.1 Randers surfaces of vanishing curvature ($K = 0$)

In the flat case, the curvature tensor vanishes identically, and therefore all curvature vector fields and their covariant derivatives also vanish. Hence the infinitesimal holonomy algebra $\mathfrak{hol}_x^*(\mathcal{F})$ is trivial at every point $x \in M$.

4.1.2 Randers surface of positive constant curvature

We first recall a lemma which gives a sufficient condition for the infinitesimal holonomy algebra to be one-dimensional.

4.2 Lemma. *If the curvature tensor of a 2-dimensional Finsler manifold (M, \mathcal{F}) is recurrent with respect to the horizontal Berwald derivative (2.6), then $\dim(\mathfrak{hol}_x^*) = 1$ at any point $x \in M$.*

In the positive curvature case, up to a local isometry and a rescaling—which do not change the holonomy group—we may consider as model the Randers metric corresponding to the navigation data (h, W) , where $h = h_{ij}dx^i \otimes dx^j$ is the standard round metric on \mathbb{S}^2 given by

$$h_{ij} = \frac{(1 + (x^1)^2 + (x^2)^2)\delta_j^i - x^i x^j}{(1 + (x^1)^2 + (x^2)^2)^2}, \quad (4.1)$$

and the wind vector field W is defined by

$$W(x^1, x^2) = (c(x^1)^2 + qx^2 + c) \frac{\partial}{\partial x^1} + (cx^1x^2 - qx^1) \frac{\partial}{\partial x^2}. \quad (4.2)$$

Using Lemma 4.2, we obtain the following proposition.

4.3 Proposition. *Let (M, \mathcal{F}) be a two-dimensional Randers manifold of constant positive curvature. Then the infinitesimal holonomy algebra $\mathfrak{hol}_x^*(\mathcal{F})$ is one-dimensional at every point $x \in M$.*

4.1.3 Randers surface of negative constant curvature

Up to local isometry, two distinct model families describe two-dimensional Randers manifolds of constant negative curvature. These models correspond precisely to cases (\mathcal{K}_1^-) and (\mathcal{K}_2^-) in Theorem 4.1.

Case (\mathcal{K}_1^-)

Using Theorem 4.1, up to a local isometry and a rescaling, the corresponding navigation data (h, W) is given by the Klein metric $h = h_{ij}dx^i \otimes dx^j$ on the unit disk \mathbb{B}^2 :

$$h_{ij} = \frac{(1 - (x^1)^2 - (x^2)^2)\delta_j^i + x^i x^j}{(1 - (x^1)^2 - (x^2)^2)^2}, \quad (4.3)$$

and a wind vector field W of the form

$$W(x^1, x^2) = (-c(x^1)^2 + qx^2 + c) \frac{\partial}{\partial x^1} - (cx^1x^2 + qx^1) \frac{\partial}{\partial x^2}, \quad (4.4)$$

where $c, q \in \mathbb{R}$ are constants. Applying Lemma 4.2, we obtain the following proposition.

4.4 Proposition. *Let (M, \mathcal{F}) be a two-dimensional Randers manifold of constant negative curvature whose navigation data (h, W) is given by the Klein metric (4.3) and the wind field (4.4). Then the infinitesimal holonomy algebra $\mathfrak{hol}_x^*(\mathcal{F})$ is one-dimensional at every point $x \in M$.*

Case (\mathcal{K}_2^-)

Using 4.1, up to a local isometry, the navigation data (h, W) is given by the Euclidean metric defined by the coefficients $h_{ij} = \delta_j^i$, by the wind W given by:

$$W(x^1, x^2) = (qx^2 + c - \frac{\sigma}{2}x^1) \frac{\partial}{\partial x^1} - (\frac{\sigma}{2}x^2 + qx^1) \frac{\partial}{\partial x^2}. \quad (4.5)$$

In this case, the infinitesimal holonomy algebra is described by the following proposition

4.5 Proposition. *Let (M, \mathcal{F}) be a two-dimensional Randers manifold of constant negative curvature arising from the navigation data (h, W) , where h is the Euclidean metric and W is the wind vector field given by (4.5). Then the infinitesimal holonomy algebra $\mathfrak{hol}_{x_0}^*(\mathcal{F})$ is dense in the Lie algebra $\mathfrak{X}(\mathcal{I}_{x_0})$ of vector fields on the indicatrix at the origin x_0 .*

Notes. We summarize the results of this section as follows. Let (M, \mathcal{F}) be a 2-dimensional Randers manifold of constant curvature K , associated with the navigation data (h, W) . Then the following possibilities occur:

- $\dim(\mathfrak{hol}_x^*) = 0$ when $K = 0$,
- $\dim(\mathfrak{hol}_x^*) = 1$ when either $K > 0$, or $K < 0$ and \mathcal{K}_1^- is satisfied,
- $\dim(\mathfrak{hol}_x^*) = \infty$ when $K < 0$ and \mathcal{K}_2^- is satisfied.

4.2 Holonomy groups of Randers surfaces with constant flag curvature

In this section, we study the holonomy groups of simply connected Randers surfaces of constant curvature.

4.2.1 Trivial holonomy ($K = 0$)

In the case of a simply connected two-dimensional Randers manifold of vanishing curvature, the holonomy group is trivial. In fact, this holds in a more general setting.

4.6 Theorem. *Let (M, \mathcal{F}) be a simply connected two-dimensional Randers manifold of vanishing curvature. Then its holonomy group is trivial, that is,*

$$\text{Hol}_x(\mathcal{F}) = \{Id\}. \quad (4.6)$$

4.2.2 Finite-dimensional nontrivial holonomy

To prove that the holonomy group is finite-dimensional, we have to study its holonomy algebra $\mathfrak{hol}_x(\mathcal{F})$, and show that it is finite dimensional. As shown in Section 4.1.2, for two-dimensional Randers manifolds of constant nonzero curvature, the infinitesimal holonomy algebra $\mathfrak{hol}_x^*(\mathcal{F})$ is one-dimensional in the positive curvature case and also when $K < 0$ in the case \mathcal{K}_1^- . This suggests that the holonomy group may also be finite-dimensional. However, $\mathfrak{hol}_x^*(\mathcal{F})$ provides only local information and gives only a lower bound for the dimension of the holonomy group. Since holonomy is a global object, parallel translation and conjugation may produce additional holonomy elements. The following lemma gives a sufficient condition under which the holonomy group is one-dimensional.

4.7 Lemma. *Let (M, \mathcal{F}) be a Finsler surface and assume that the indicatrix bundle \mathcal{IM} admits a coordinate system with global α as a fiber coordinate. Suppose that there exist smooth functions Φ on \mathcal{IM} , and Ψ on TM respectively, such that for every curve $\gamma(t)$, the parallelism of unit vector fields along γ is described by a differential equation*

$$\frac{d}{dt}(\Phi(\gamma(t), \alpha(t))) = \Psi(\gamma(t), \dot{\gamma}(t)). \quad (4.7)$$

Then the holonomy group at any point $x \in M$ is one-dimensional.

The coordinate α will be called the *angular parameter* on the indicatrix.

Finite-dimensional holonomy with positive curvature ($K > 0$)

We consider the navigation data (h, W) from Theorem 4.1. If $c = q = 0$, the Finsler norm reduces to the Riemannian case on \mathbb{S}^2 , so we assume at least one of c or q is nonzero. Introducing suitable adapted coordinates, one obtains explicit expressions that simplify the Randers metric. This formulation allows a convenient description of parallel translation and enables the application of Lemma 4.7, leading to the main following result.

4.8 Theorem. *Let (M, \mathcal{F}) be a simply connected two-dimensional Randers manifold of constant positive curvature. Then, for any point $x \in M$, the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ is one-dimensional.*

Finite-dimensional holonomy with negative curvature (with \mathcal{K}_1^-)

In this case the navigation data (h, W) is given, up to an isometry and a rescaling, by the Klein metric (4.3) on the unit disk $\mathbb{B}^2 \subset \mathbb{R}^2$, with the vector field (4.4). Proceeding as in the previous case, we obtain a description of the parallel translation satisfying the assumptions of Lemma 4.7. This leads to the following theorem.

4.9 Theorem. *Let (M, \mathcal{F}) be a simply connected two-dimensional Randers manifold of constant negative curvature with \mathcal{K}_1^- . Then, for any point $x \in M$, the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ is one-dimensional.*

Isomorphism between Finsler and Riemannian holonomy groups

Let (M, \mathcal{F}) be a simply connected two-dimensional Randers manifold of constant curvature arising from navigation data (h, W) with either $K > 0$ or $K < 0$ in the case \mathcal{K}_1^- . The following theorem characterizes these holonomy groups.

4.10 Theorem. *Let (M, \mathcal{F}) be a simply connected two-dimensional Randers manifold of constant curvature K associated with the navigation data (h, W) where either $K > 0$ or $K < 0$ with \mathcal{K}_1^- . Then at any $x \in M$ the Finsler holonomy group $\mathcal{H}ol(\mathcal{F})$ and the Riemannian holonomy group $\mathcal{H}ol(h)$ are isomorphic:*

$$\mathcal{H}ol_x(\mathcal{F}) \cong \mathcal{H}ol_x^R(h). \quad (4.8)$$

4.11 Corollary. *Under the assumptions of Theorem 4.10, the holonomy group satisfies*

$$\mathcal{H}ol_x(\mathcal{F}) \cong SO(2). \quad (4.9)$$

4.12 Remark. In the cases $K > 0$ and $K < 0$ with \mathcal{K}_1^- , the wind field W is not parallel with respect to h , so the Randers metric \mathcal{F} is not Berwaldian [5]. Moreover, its Landsberg tensor does not vanish, hence \mathcal{F} is not Landsbergian. Therefore, these metrics are genuine non-Riemannian Finsler metrics with finite-dimensional holonomy groups.

4.2.3 Infinite-dimensional holonomy ($K < 0$ with \mathcal{K}_2^-)

As shown in Proposition 4.5, the infinitesimal holonomy algebra $\mathfrak{ho}_x^*(\mathcal{F})$ of the class of Randers surfaces of constant curvature considered in Section (4.1.3) is infinite dimensional. It follows from (2.15) that the corresponding holonomy group is infinite dimensional. However, a stronger result holds.

4.13 Theorem. *Let (M, \mathcal{F}) be a 2-dimensional Randers manifold of constant negative curvature with \mathcal{K}_2^- . Then the holonomy group $\mathcal{H}ol_x(\mathcal{F})$ is maximal, its closure is isomorphic to $\mathcal{D}iff_+(\mathbb{S}^1)$.*

Notes. The classification of the holonomy groups of Randers surfaces of constant curvature can be summarized as follows: For a Randers metric of constant flag curvature on a simply connected 2-dimensional manifold, then the following possibilities can occur:

1. if $K = 0$, then the holonomy group is trivial, that is $\mathcal{H}ol = \{id\}$,
2. if $K > 0$, then the holonomy group is 1-dimensional and isomorphic to $SO(2)$,
3. if $K < 0$ with \mathcal{K}_1^- , then the holonomy group is 1-dimensional and isomorphic to $SO(2)$,
4. if $K < 0$ with \mathcal{K}_2^- , then the holonomy group is maximal, that is $\overline{\mathcal{H}ol} \cong \mathcal{D}iff_+(\mathbb{S}^1)$,

5 Natural parallelism associated with navigation data

Navigation data (h, W) define Randers metrics via Zermelo navigation, where geodesics correspond to time-minimizing paths under a wind field [6]. While the metric structure is simple, the associated parallel translation and holonomy can be more involved. In this chapter, we introduce a natural parallel translation associated with (h, W) , which preserves the Randers norm and has finite-dimensional holonomy isomorphic to the Riemannian holonomy of h . We also study the induced connection, covariant derivative, parallel vector fields, autoparallel curves, and the associated spray, and relate these structures to the geometry of h . This chapter is based on [41].

5.1 Parallel translation associated to navigation data

A navigation datum (h, W) consists of a Riemannian metric h and a vector field W with $\|W\|_h < 1$, which defines a Randers metric whose indicatrices are obtained by translating the h -unit spheres. By [6], Zermelo navigation is equivalent to the geodesic problem of this Randers metric. This identification allows us to relate the Finsler unit condition to a shifted h -unit condition. While the metric structure is determined by the indicatrix, the navigation framework suggests a natural parallel translation obtained by subtracting the wind, applying Riemannian parallel transport, and then adding the wind back.

5.1 Definition (Natural parallel translation). Let c be a smooth curve joining p to q . Let $V_p^\circ \in T_pM$ be a unit vector with respect to the Randers norm \mathcal{F} associated with the navigation data (h, W) . The *natural parallel translation* of V_p° along c is defined by

$$\mathcal{P}(V_p^\circ) := \mathcal{P}_R(V_p^\circ - W_p) + W_q, \quad (5.1)$$

where \mathcal{P}_R denotes the Riemannian parallel translation along c with respect to h . For an arbitrary nonzero vector $V_p \in T_pM \setminus \{0\}$ we extend the

definition by homogeneity:

$$\mathcal{P}(V_p) := \mathcal{F}(V_p) \cdot \mathcal{P}\left(\frac{1}{\mathcal{F}(V_p)}V_p\right). \quad (5.2)$$

The basic properties of the natural parallel translation are collected in the following proposition.

5.2 Proposition (Homogeneity, nonlinearity, and norm preservation). *The natural parallel translation \mathcal{P} is homogeneous, but in general it is not additive. Moreover, it preserves the Randers norm function \mathcal{F} :*

$$\mathcal{F}(\mathcal{P}(V_p)) = \mathcal{F}(V_p), \quad V_p \in T_pM \setminus \{0\}. \quad (5.3)$$

5.3 Theorem. *Let (h, W) be a navigation data on the manifold M . If W is parallel along the curve c with respect to the Riemannian metric, then the natural and the Riemannian parallel transports on c coincide.*

We now consider the holonomy of the natural parallel translation. As the following theorem shows, its holonomy group is isomorphic to the Riemannian holonomy group.

5.4 Theorem (Holonomy of the natural parallel translation). *Let (h, W) be navigation data on the manifold M . The holonomy group $\mathcal{H}ol(\mathcal{P})$ associated to the natural parallel translation is isomorphic to the Riemannian holonomy group $\mathcal{H}ol(\mathcal{P}_R)$. In particular, the holonomy group of $\mathcal{H}ol(\mathcal{P})$ is finite dimensional.*

5.5 Remark. It is known from [29] that, in general, homogeneous parallel translations associated with Finsler metrics have infinite-dimensional holonomy groups. Finite-dimensional holonomy appears only in exceptional situations. From this perspective, it is natural to look for examples of homogeneous (but nonlinear) parallel translations whose holonomy groups are still finite-dimensional. The preceding proposition shows that the natural parallel translation belongs to this distinguished class.

We now study the geometric structures induced by the natural parallelism, namely the horizontal distribution, the associated connection, and the covariant derivative. Let $\mathcal{A}_{ij}^k(x)$ be the Christoffel symbols of the Levi-Civita connection of the Riemannian metric h . Along a curve $c(t)$, the natural parallel translation is expressed in terms of the Riemannian parallel translation by

$$\mathcal{P}_t(V_0) = (\mathcal{P}_R)_t(V_0 - W_0) + W(c(t)). \quad (5.4)$$

Differentiating the expression of the natural parallel field and using the Riemannian parallel translation, we obtain

$$\frac{d}{dt}[\mathcal{P}_t(V_0)]^k = -\mathcal{A}_{ij}^k(c(t)) \dot{c}^i(t) (\mathcal{P}_R)_t^j(V_0 - W_0) + \frac{\partial W^k}{\partial x^i}(c(t)) \dot{c}^i(t). \quad (5.5)$$

Let $\Gamma_i^k(x, y)$ be the coefficients of the connection associated with the natural parallelism, and let $\mathfrak{h} : TTM \rightarrow \mathcal{H}$ be the corresponding horizontal projector. Comparing with the parallel transport equations, the connection coefficients are given by

$$\Gamma_i^k(x, y) = \mathcal{A}_{is}^k y^s - \mathcal{F}(y) \mathcal{A}_{is}^k W^s - \mathcal{F}(y) \frac{\partial W^k}{\partial x^i}. \quad (5.6)$$

Accordingly, the horizontal lift can be written in terms of the Riemannian horizontal lift and a vertical term involving $\nabla^R W$, by

$$l_{(x,y)}(X) = l_{(x,y)}^R(X) + \mathcal{F}(x, y) (\nabla_X^R W)^v. \quad (5.7)$$

5.6 Definition. Let (h, W) be navigation data on M , and let \mathfrak{h} be the horizontal projector induced by the natural parallelism. The connection

$$\Gamma := 2\mathfrak{h} - \text{Id} \quad (5.8)$$

is called the *natural connection*. Its coefficients are given by (5.6).

The horizontal distribution induces a covariant derivative associated with the natural parallelism. A vector field $V(t)$ along a curve $c(t)$ is parallel if and only if its lift is horizontal, equivalently,

$$\dot{V}_t = l_{(c, V_t)}(\dot{c}(t)), \quad (5.9)$$

or

$$\frac{\nabla V_t}{dt} = \left(\frac{dV_t^k}{dt} + \Gamma_i^k(c(t), V_t) \dot{c}^i(t) \right) \frac{\partial}{\partial x^k}. \quad (5.10)$$

This leads to the nonlinear covariant derivative

$$\nabla_X V = X^i \left(\frac{\partial V^k}{\partial x^i} + \Gamma_i^k(x, V) \right) \frac{\partial}{\partial x^k}. \quad (5.11)$$

Thus we obtain a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M), \quad (X, Y) \mapsto \nabla_X Y. \quad (5.12)$$

which is $C^\infty(M)$ -linear in the first argument and positively homogeneous, but not additive, in the second.

5.7 Property. *The coordinate-free expression of the nonlinear covariant derivative (5.12) associated with the natural parallelism is*

$$\nabla_X Y = \nabla_X^R Y - \mathcal{F}(Y) \nabla_X^R W, \quad (5.13)$$

where ∇^R denotes the Levi-Civita connection of the Riemannian metric h .

5.8 Proposition. *The covariant differentiations along the integral curves of the vector field W coincide if and only if the integral curves of W are Riemannian geodesics.*

5.9 Proposition. *The integral curves of W are pre-geodesics (resp. geodesics) of ∇ if and only if they are pre-geodesics (resp. geodesics) of ∇^R .*

5.2 Connection associated to the natural parallelism

In this section, we study the connection associated with the natural parallel translation, focusing on its autoparallel curves and the corresponding differential equation and spray. For navigation data (h, W) , these curves are characterized by the condition

$$\nabla_{\dot{c}} \dot{c} = 0, \quad (5.14)$$

where ∇ is the covariant derivative introduced in the previous section. Substituting the expression (5.13) of the covariant derivative in terms of the Levi-Civita connection ∇^R of the Riemannian metric h , we obtain

$$\nabla_{\dot{c}}^R \dot{c} - \mathcal{F}(\dot{c}) \nabla_{\dot{c}}^R W = 0. \quad (5.15)$$

Thus the acceleration with respect to the Riemannian connection is modified by a term depending on the wind field. In local coordinates this leads to the second-order system

$$\ddot{c}^k + \dot{c}^i \left(\mathcal{A}_{ij}^k \dot{c}^j - \mathcal{F}(\dot{c}) \mathcal{A}_{ij}^k W^j - \mathcal{F}(\dot{c}) \frac{\partial W^k}{\partial x^i} \right) = 0, \quad (5.16)$$

where \mathcal{A}_{ij}^k are the Christoffel symbols of ∇^R . The system (5.16) is generated by the spray

$$S = y^k \frac{\partial}{\partial x^k} - 2G^k(x, y) \frac{\partial}{\partial y^k}, \quad (5.17)$$

whose coefficients are

$$G^k(x, y) = \frac{1}{2} \left(\mathcal{A}_{ij}^k y^i y^j - \mathcal{F}(x, y) y^i \mathcal{A}_{ij}^k W^j - \mathcal{F}(x, y) y^i \frac{\partial W^k}{\partial x^i} \right). \quad (5.18)$$

The spray obtained above determines a natural connection on the tangent bundle. For later reference, we introduce the following terminology.

5.10 Definition. The spray (5.17) with coefficients (5.18) corresponding to the natural parallelism will be called the *natural spray*. The connection

$$\bar{\Gamma} := [J, S] \quad (5.19)$$

generated by this spray will be called the *natural symmetric connection*.

5.11 Remark. The natural connection (5.8) and the natural symmetric connection (5.19) are different in general. Indeed, the torsion of the natural symmetric connection is identically zero, while the torsion of the natural connection is typically nonzero.

The next lemma will be used to describe the relation between these connections.

5.12 Lemma. *Let ρ be a one-form on the base manifold and let φ be a zero-homogeneous function on the tangent bundle. Then the relation*

$$y^i \rho_i^k \circ \pi = \varphi(x, y) y^k \quad (5.20)$$

holds if and only if φ depends only on the base point, that is $\varphi(x, y) = \varphi(x)$, and

$$\rho_j^k(x) = \varphi(x) \delta_j^k. \quad (5.21)$$

5.13 Proposition. *Let (h, W) be navigation data on a manifold M . The natural spray S is projectively related to the quadratic spray S^R of the Levi-Civita connection ∇^R if and only if the vector field W is concircular with respect to the Riemannian metric h .*

We conclude with two examples illustrating the behavior of the natural spray

5.14 Example. Let M be the interior of the Euclidean unit ball in R^n equipped by the standard Euclidean inner product $h_{ij} = \delta_{ij}$. If $W = -x^i \frac{\partial}{\partial x^i}$. Therefore, the natural spray is metrizable as the canonical spray of the Funk metric.

5.15 Example. Let M be the interior of the Euclidean unit ball in \mathbb{R}^2 equipped by the standard Euclidean inner product, and let the wind W be given by the infinitesimal rotation $W = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$, then in this case, the natural spray is not Finsler metrizable.

Bibliography

- [1] W. Ambrose and I. Singer. A Theorem on Holonomy. *Trans. Amer. Math. Soc.*, 75 (3): 428–443, 1953.
- [2] P.L. Antonelli, A. Bóna, M. A. Slawinski, Seismic rays as Finsler geodesics. *Nonlinear Analysis*, 4: 711–722, 2003.
- [3] G. S. Asanov. Finsler space connected by angle in two dimensions. Regular case. *Publ. Math. Debrecen*, 77:245–259, 2010.
- [4] D. Bao, S. Chern, and Z. Shen. An Introduction to Riemann-Finsler Geometry. *Graduate Texts in Mathematics. Springer New York*, 2000.
- [5] D. Bao and C. Robles. Ricci and flag curvatures in Finsler geometry. In *A sampler of Riemann-Finsler geometry*, MSRI Publ. 50, Cambridge Univ. Press, 2004.
- [6] D. Bao, C. Robles, and Z. Shen. Zermelo navigation on Riemannian manifolds. *J. Differential Geom.*, 66(3):377–435, 2004.
- [7] W. Barthel. Nichtlineare Zusammenhänge und deren Holonomiegruppen. *J. Reine Angew. Math.*, 212: 120-149, 1963.
- [8] M. Berger. Sur les groupes d’holonomie homogène des variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83: 279–330, 1955.
- [9] A. Borel and A. Lichnerowicz. Groupes d’holonomie des variétés riemanniennes. *C. R. Acad. Sci. Paris*, 234:1835–1837, 1952.
- [10] R. Bryant. Two exotic holonomies in dimension four, path geometries, and twistor theory. *Complex geometry and Lie theory (Sundance, UT, 1989)*, 33–88. *Proc. Sympos. Pure Math.*, 53. American Mathematical Society, Providence, 1991.

- [11] R. L. Bryant. Finsler structures on the 2-sphere satisfying $K = 1$. In *Finsler geometry (Seattle, 1995)*, Contemp. Math. 196, AMS, 1996.
- [12] R. L. Bryant. Projectively flat Finsler 2-spheres of constant curvature. *Selecta Math.*, 3(2):161–203, 1997.
- [13] E. Cartan. Sur les variétés à connexion affine et la théorie de la relativité généralisée I & II. *Ann. Sci. École Norm. Sup.*, 40:325–412, 1923; 41:1–25, 1924. Also in *Oeuvres complètes*, tome III, 659–746 and 799–824.
- [14] E. Cartan. La géométrie des espaces de Riemann. *Mémoires des Sciences Mathématiques*, 5, Gauthier-Villars, Paris, 1925.
- [15] E. Cartan. Les groupes d’holonomie des espaces généralisés. *Acta Math.*, 48:1–42, 1926. Also in *Oeuvres complètes*, tome III, vol. 2, 997–1038.
- [16] C. Carathéodory. Calculus of Variations and Partial Differential Equations of the First Order. *AMS Chelsea Publishing*, 1999.
- [17] S.-S. Chern and Z. Shen. Riemann-Finsler geometry. volume 6 of *Nankai Tracts in Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.
- [18] X. Cheng and Z. Shen. Randers metrics of scalar flag curvature. *J. Aust. Math. Soc.*, 87:359–370, 2009.
- [19] Q.-S. Chi, S. A. Merkulov, and L. J. Schwachhöfer. On the existence of infinite series of exotic holonomies. *Inventiones Mathematicae*, 126:391–411, 1996.
- [20] Q.-S. Chi, S. A. Merkulov, and L. J. Schwachhöfer. Exotic holonomies E_7^a . *International Journal of Mathematics*, 8:583–594, 1997.
- [21] G. de Rham. Sur la réductibilité d’un espace de Riemann. *Comment. Math. Helv.*, 26:328–344, 1952.
- [22] C. Ehresmann. Les connexions infinitésimales dans un espace fibré différentiable. *Séminaire Bourbaki*, Vol. 1,24:153–168. *Société Mathématique de France*, Paris, 1995

- [23] P. Finsler. Über Kurven und Flächen in allgemeinen Räumen. *PhD thesis. University of Göttingen, Göttingen* 1918. (<https://gdz.sub.uni-goettingen.de/id/PPN321583582>)
- [24] S. G. Elgendi and Z. Muzsnay. Freedom of $h(2)$ -variationality and metrizability of sprays. *Differential Geom. Appl.*, 54:194–207, 2017.
- [25] J.-H. Eschenburg and E. Heintze. Unique decomposition of Riemannian manifolds. *Proc. Amer. Math. Soc.*, 126(10):3075–3078, 1998.
- [26] P. Funk. Über Geometrien, bei denen die Geraden die Kürzesten sind. *Math. Ann.*, 101:226–237, 1929.
- [27] P. Funk. Eine Kennzeichnung der zweidimensionalen elliptischen Geometrie. *Österreich. Akad. Wiss. Math.-Natur.*, 172:251–269, 1963.
- [28] J. Grifone. Structure presque-tangente et connexions, I. *Ann. Inst. Fourier (Grenoble)*, 22(1):287–334, 1972.
- [29] B. Hubicska, V. S. Matveev, and Z. Muzsnay. Almost all Finsler metrics have infinite dimensional holonomy group. *J. Geom. Anal.*, 31(6):6067–6079, 2021.
- [30] B. Hubicska and Z. Muzsnay. The Holonomy groups of projectively flat Randers two-manifolds of constant curvature. *Differential Geom. Appl.*, 73:101677, 2020.
- [31] B. Hubicska and Z. Muzsnay. Tangent Lie Algebra of a Diffeomorphism Group and Application to Holonomy Theory. *J. Geom. Anal.*, 30(1):107–123, Jan. 2020.
- [32] Y. Katznelson. An introduction to Harmonic Analysis. *Cambridge Univ. Press*, 2004.
- [33] S. Kobayashi and K. Nomizu. Foundations of differential geometry. Vol. I. *Interscience publishers*, 1996.
- [34] L. Kozma. Holonomy structures in Finsler geometry. In P. L. Antonelli (ed.), *Handbook of Finsler Geometry*, Vols. 1–2, pages 445–481. Kluwer Academic Publishers, Dordrecht, 2003.
- [35] L. Kozma. On Landsberg spaces and holonomy of Finsler manifolds. In *Finsler geometry (Seattle, 1995)*, AMS, 1996.

- [36] L. Kozma and S. Baran. On metrical homogeneous connections of a Finsler point space. *Publ. Math. Debrecen*, 49:59–68, 1996.
- [37] S. A. Merkulov and L. J. Schwachhöfer. Classification of irreducible holonomies of torsion-free affine connections. *ann. of Math*, 150:77–149, 1999.
- [38] S. Markvorsen. A Finsler geodesic spray paradigm for wildfire spread modelling. *Nonlinear Anal. Real World Appl.*, 28: 208–228, 2016.
- [39] A. Mezrag and Z. Muzsnay. The Holonomy of Spherically Symmetric Projective Finsler Metrics of Constant Curvature. *J. Geom. Anal.*, 34(8), 2024.
- [40] A. Mezrag and Z. Muzsnay. Navigation and holonomy: classification of holonomy groups of Randers surfaces of constant curvature. *submitted.*, 2026.
- [41] A. Mezrag, Z. Muzsnay and Cs. Vincze. Natural parallel translation and connection associated to navigation data. *Diff. Geom. and Appl.*, 102:102328, 2026.
- [42] Z. Muzsnay and P. T. Nagy. Finsler manifolds with non-Riemannian holonomy. *Houston J. Math.*, 38(1):77–92, 2012.
- [43] Z. Muzsnay and P. T. Nagy. Characterization of projective Finsler manifolds of constant curvature having infinite dimensional holonomy group. *Publ. Math. Debrecen*, 84:17–28, 2014.
- [44] Z. Muzsnay and P. T. Nagy. Finsler 2-manifolds with maximal holonomy group of infinite dimension. *Differential Geom. Appl.*, 39:1–9, 2015.
- [45] Z. Muzsnay and P. T. Nagy, Projectively flat Finsler manifolds with infinite dimensional holonomy. *Forum Math.* 27, 2 (2015), 767–786.
- [46] L. Nachbin. Sur les algèbres denses de fonctions différentiables sur une variété. *C. R. Acad. Sci. Paris*, 228:1549–1551, 1949.
- [47] L. Nachbin, A look at approximation theory. In *Approximation theory and functional analysis (Proc. Internat. Sympos. Approximation Theory, Univ. Estadual de Campinas, Campinas, 1977)*, vol. 35 of *Notas de Matemática*, 66; *North-Holland Math. Stud.* North-Holland, Amsterdam-New York, 1979, pp. 309–331.

- [48] T. Okada. On models of projectively flat Finsler spaces with constant negative curvature *Tensor (NS)*, 40: 117–123, 1983.
- [49] H. Omori. Infinite-dimensional Lie groups, vol. 158 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1997. Translated from the 1979 Japanese original and revised by the author.
- [50] G. Randers. On an asymmetrical metric in the four-space of general relativity. *Phys. Rev.*, 59:195–199, 1941.
- [51] C. Robles. Geodesics in Randers spaces of constant curvature. *Transactions of the American Mathematical Society*, 359(4):1633 – 1651, 2007.
- [52] H. Rund. The differential geometry of Finsler spaces. Die Grundlehren der Mathematischen Wissenschaften, Bd. 101. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959.
- [53] B. Russel and S. Stepney. Zermelo navigation and a speed limit to quantum information processing. *Phys. Rev. A*, 90, no. 11, 115303, 29 pp, 2014.
- [54] B. Russel and S. Stepney. Zermelo navigation in the quantum brachistochrone. *Journal of Physics A*, 48, 2015.
- [55] Z. Shen. Differential geometry of spray and Finsler spaces. Kluwer Academic Publishers, Dordrecht, 2001.
- [56] Z. Shen. Projectively flat Finsler metrics with constant flag curvature. *Trans. Amer. Math. Soc.*, 355:1713–1728, 2003.
- [57] L.J. Schwachhöfer. Holonomy groups and algebras. In *Global Differential Geometry*, pages 3–37. Springer Proc. Math., 17, Springer, 2012.
- [58] Z. I. Szabó. Positive definite Berwald spaces. Structure theorems on Berwald spaces *Tensor*, 35:25–39, 1981.
- [59] J. Szilasi, R. L. Lovas, and D. C. Kertész. Connections, Sprays and Finsler Structures. World Scientific Publishing Co. Pte. Ltd., Hackensack, 2014.

- [60] W. Thurston. Foliations and groups of diffeomorphisms. *Bull. Amer. Math. Soc.*, 80(2):304–307, 1974.
- [61] C. Vincze. On Randers manifolds with semi-symmetric compatible linear connections *Indag. Math.*, 26(2):363–379, 2015.
- [62] V.V. Wagner. Les espaces de Finsler à deux dimensions à groupes d'holonomie finis et continus. *C. R. (Doklady) Acad. Sci. USSR (N.S.)*, 39:210–212, 1943.
- [63] E. Zermelo. Über das Navigationsproblem bei ruhender oder veränderlicher Windverteilung *ZAMM - Journal of Applied Mathematics and Mechanics*, 11:114–124, 1931.



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Foreign language scientific articles in international journals (2)

1. **Asma, M.**, Muzsnay, Z., Vincze, C.: Natural parallel translation and connection associated to navigation data.

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2. **Asma, M.**, Muzsnay, Z.: The Holonomy of Spherically Symmetric Projective Finsler Metrics of Constant Curvature.

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