Trispectrum and higher order spectra for non-Gaussian homogenous and isotropic field on the plane

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July 17, 2013

Abstract

In this paper we study the non-Gaussian homogenous and isotropic field on the plane in frequency domain. The trispectrum and higher order spectra of such a field are described in terms of Bessel functions. Poisson formulae are given for the spectrum and for the bispectrum. Some particular integrals of Bessel functions are considered as well.

Keywords: Homogenous fields, Isotropic fields, Non-Gaussian field, Tricovariances, Trispectrum, Higher order cumulants, Higher order spectra, 2D, Isotropic process on circle.

1 Introduction

Non-Gaussian observations appear in several fields of sciences like geophysics, astrophysics and so on, see [NY97], [OT96] for instance. Characterization of a Gaussian phenomenon is possible by the covariances or equivalently by the spectrum, but one needs either all finite dimensional distributions or higher order spectra in non-Gaussian cases. Although the theory of Gaussian fields and non-Gaussian processes are well developed, see [Yag87], [Yad83], [Adl10] and [Bri01], [Ros00], [Ros85], [Pri88], [Leo89], [BH86], [LS12], [Mok07], [Mok08] the particular spectral properties of non-Gaussian isotropic fields on the plane is not known.

In this paper we study the non-Gaussian homogenous and isotropic field on the plane in frequency domain. The trispectrum and higher order spectra of such a field are described in terms of Bessel functions. Some particular integrals of Bessel functions are given in terms of sides and angles of a multilateral. The connection between the tricovariance and the trispectrum is similar the one between the bicovariance and the bispectrum except the inversion formula is missing. I strongly believe that some more careful parametrization would solve this problem. A homogenous and isotropic field restricted on a circle becomes isotropic in a natural manner. Since the cumulants on the circle can be considered as the restriction of the cumulants of the fields on the plane therefore the spectra on the circle is expressed in terms of spectra on the plane, these formulae are called Poisson formulae and will be given for the spectrum and for the bispectrum.

A homogenous real valued stochastic field $X(\underline{x})$, $\underline{x} \in \mathbb{R}^2$, which is continuous (in mean square sense), has spectral representation

$$X(\underline{x}) = \int_{\mathbb{R}^2} e^{i\underline{x}\cdot\underline{\omega}} Z(d\underline{\omega}), \quad \underline{\omega}, \underline{x} \in \mathbb{R}^2, \tag{1}$$

with $EX(\underline{x}) = 0$ and spectral measure $E|Z(d\underline{\omega})|^2 = F_0(d\underline{\omega})$. Homogeneity is meant by in strict sense i.e. all the finite dimensional distributions of $X(\underline{x})$ are translation invariant, see

[Yag87] for details. Rewrite $X(\underline{x})$ in term of polar coordinates

$$X(r,\varphi) = \int_0^\infty \int_0^{2\pi} e^{i\rho r \cos(\varphi-\eta)} Z(\rho d\rho d\eta)$$

and the following notation is applied $\underline{x} = (r,\varphi)$, $\underline{\omega} = (\rho,\eta)$ are polar coordinates, $r = |\underline{x}| = \sqrt{x_1^2 + x_2^2}$, $\rho = |\underline{\omega}|$, $\underline{x} \cdot \underline{\omega} = r\rho\cos(\varphi - \eta)$. This representation provides an isotropic field if $F_0(d\underline{\omega})$ is isotropic, i. e. $F_0(d\underline{\omega}) = E|Z(d\underline{\omega})|^2 = E|Z(\rho d\rho d\eta)|^2 = F(\rho d\rho)\, d\eta$. The isotropy usually is defined through the invariance of the covariance structure under rotations. A rotation g is characterized by an angle γ. We consider rotations about the origin of the coordinate system. If $\underline{x} \in \mathbb{R}^2$ is given in polar coordinates $\underline{x} = (r,\varphi)$, then $g\underline{x} = (r, \varphi - \gamma)$, and as usual the operator $\Lambda(g)$ acts on functions $f(r,\varphi)$, such that $\Lambda(g) f(r,\varphi) = f\left(g^{-1}(r,\varphi)\right) = f(r, \varphi + \gamma)$.

The invariance of the covariance function is satisfactory for Gaussian cases but for non-Gaussian fields we need invariance of higher order cumulants as well.

Definition 1 *A homogenous stochastic field $X(\underline{x})$ is strictly isotropic if all finite dimensional distributions of $X(\underline{x})$ are invariant under rotations.*

As far as the homogenous field $X(\underline{x})$ is Gaussian the isotropy of the spectral measure $F_0(d\underline{\omega})$, i.e. in polar coordinates $F_0(d\underline{\omega}) = F(\rho d\rho)\, d\eta$, implies

$$\operatorname{Cov}\left(\Lambda(g) X(\underline{x}_1), \Lambda(g) X(\underline{x}_2)\right) = \operatorname{Cov}\left(X(\underline{x}_1), X(\underline{x}_2)\right),$$

for each $\underline{x}_1$, $\underline{x}_2$ and for every $g \in SO(2)$. It will be convenient for us later if we assume the existence all the moments of the field $X(\underline{x})$, in this way from the isotropy follows that all higher order moments are also invariant under rotations.

Let us consider a homogenous and isotropic stochastic field $X(\underline{x}) = X(r,\varphi)$, $(r > 0$, $\varphi \in [0, 2\pi))$ on the plane and put it into series representation, see [Ter13]

$$X(r,\varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r)\, Z_\ell(\rho d\rho), \tag{2}$$

where J_ℓ denotes the Bessel function of the first kind, and

$$Z_\ell(\rho d\rho) = \int_0^{2\pi} i^\ell e^{-i\ell\eta}\, Z(\rho d\rho d\eta). \tag{3}$$

Z_ℓ is an array of measures, orthogonal to each other

$$\operatorname{Cov}\left(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)\right) = \delta_{\ell_1-\ell_2} F(\rho d\rho).$$

Note that the spectral measure $F(\rho d\rho)$ of the stochastic spectral measure $Z_\ell(\rho d\rho)$ does not depend on ℓ. In representation (2) $e^{i\ell\varphi}$ plays a role of spherical harmonics of degree ℓ with complex values on the plane. It follows that an isotropic random field $X(\underline{x})$ can be decomposed into a countable number of mutually uncorrelated spectral measures with a one dimensional parameter.

The isotropy of $X(r,\varphi)$ implies that the distribution of $X(r,\varphi)$ does not change under rotations $g \in SO(2)$

$$\begin{aligned}\Lambda(g) X(r,\varphi) &= \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r)\, e^{i\ell\gamma} Z_\ell(\rho d\rho) \\ &= \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r)\, Z_\ell(\rho d\rho),\end{aligned}$$

hence the distribution of $Z_\ell(\rho d\rho)$ and $e^{i\ell\gamma}Z_\ell(\rho d\rho)$ should be the same. Under isotropy assumption we have

$$\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)\right) = e^{i(\ell_1+\ell_2)\gamma}\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)\right),$$

for each γ, hence either $\ell_1+\ell_2=0$, or $\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)\right)=0$. In general, under isotropy assumption we have

$$\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), \ldots, Z_{\ell_p}(\rho_p d\rho_p)\right) = e^{i(\ell_1+\ell_2\cdots+\ell_p)\gamma}\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), \ldots, Z_{\ell_p}(\rho_p d\rho_p)\right),$$

that is either $\ell_1+\ell_2\cdots+\ell_p=0$, or $\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), \ldots, Z_{\ell_p}(\rho_p d\rho_p)\right)=0$ should be fulfilled. In turn, if this assumption fulfils for each p, then the cumulants $\operatorname{Cum}\left(Z_{\ell_1}(\rho_1 d\rho_1), \ldots, Z_{\ell_p}(\rho_p d\rho_p)\right)$ are invariant under rotations and if in addition the distributions of $X(r,\varphi)$ are determined by the moments then the field is isotropic.

1.1 Spectrum and Bispectrum

It is well known from theory of Gaussian fields that

$$\operatorname{Cov}\left(X(\underline{x}), X(\underline{y})\right) = 2\pi\int_0^\infty J_0(\rho r)\, F(\rho d\rho),$$

where $r=\left|\underline{x}-\underline{y}\right|$, [Yad83], [Yag87], [Bri74]. The covariances and the spectral measure corresponds to each other since the Hankel transform gives the inverse, in particular for absolutely continuos spectral measure $F(\rho d\rho)=2\pi\sigma^2\left|A(\rho)\right|^2\rho d\rho$ we have

$$\begin{aligned}
\mathcal{C}_2(r) &= 2\pi\int_0^\infty J_0(\rho r)\,\sigma^2\left|A(\rho)\right|^2\rho d\rho,\\
\sigma^2\left|A(\rho)\right|^2 &= \frac{1}{2\pi}\int_0^\infty J_0(\rho r)\,\mathcal{C}_2(r)\, r dr,
\end{aligned}$$

where $\mathcal{C}_2(r)=\operatorname{Cov}\left(X(\underline{x}), X(\underline{y})\right)$.

The third order structure of a homogenous and isotropic stochastic field $X(\underline{x})$ is described by either the third order cumulants (bicovariances) in spatial domain or the bispectrum in frequency domain. The bicovariance

$$\operatorname{Cum}\left(X(\underline{x}_1), X(\underline{x}_2), X(\underline{x}_3)\right) = \operatorname{Cum}\left(X(0), X(|\underline{x}_2-\underline{x}_1|\,\underline{n}), X(g(\underline{x}_3-\underline{x}_1))\right)$$

where g denotes the rotation carrying $\underline{x}_2-\underline{x}_1$ into the North pole $\underline{n}=(0,1)$. The bicovariance $\operatorname{Cum}\left(X(0), X(r_2\underline{n}), X(\underline{x}_3)\right)$ depends on the lengths r_2, $r_3=|\underline{x}_3|$ and the angle φ between them, this way a triangle is defined with length of the third side r_1, such that $r_1^2=r_2^2+r_3^2-2r_2r_3\cos(\varphi)$. According to this definition of r_1, we introduce $\mathcal{C}_3(r_1,r_2,r_3)=\operatorname{Cum}\left(X(0), X(r_2\underline{n}), X(\underline{x}_3)\right)$. Similarly the bispectrum S_3 of the homogenous and isotropic stochastic field $X(\underline{x})$ depends on wave numbers (ρ_1,ρ_2,ρ_3) such that ρ_1,ρ_2,ρ_3 should form a triangle. It has been shown, see [Ter13], that

$$\mathcal{C}_3(r_1,r_2,r_3) = 4\pi\iint_0^\infty\int_0^\pi \mathcal{T}_3(\eta,\rho_2,\rho_3|\,\varphi,r_2,r_3)\, S_3(\rho_1,\rho_2,\rho_3)\, d\eta\prod_{k=2}^{3}\rho_k d\rho_k,$$

where the function

$$\mathcal{T}_3(\eta,\rho_2,\rho_3|\,\varphi,r_2,r_3) = \sum_{\ell=-\infty}^{\infty}\cos(\ell\varphi)\, J_\ell(\rho_2 r_2)\, J_\ell(\rho_3 r_3)\cos(\ell\eta), \tag{4}$$

gives the transformation of the bispectrum $S_3(\rho_1, \rho_2, \rho_3)$ into the bicovariance $\mathcal{C}_3(r_1, r_2, r_3)$. Notice that both angles φ and η are equivalent to the third sides ρ_1 and r_1 of the triangles, according to the wave numbers (ρ_1, ρ_2, ρ_3) and distances (r_1, r_2, r_3). The bispectrum is expressed also by the bicovariance

$$S_3(\rho_1, \rho_2, \rho_3) = \frac{1}{4\pi^3} \iint_0^\infty \int_0^\pi \mathcal{T}_3(\eta, \rho_2, \rho_3 | \varphi, r_2, r_3)\, \mathcal{C}_3(r_1, r_2, r_3)\, d\varphi \prod_{k=2}^3 r_k dr_k. \tag{5}$$

2 Trispectrum

Start with the spectral representation of the fourth order cumulant of a homogenous field

$$\mathrm{Cum}\left(X(\underline{x}_1), X(\underline{x}_2), X(\underline{x}_3), X(\underline{x}_4)\right) = \underbrace{\int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2}}_{4} e^{i\left(\Sigma_1^4 \underline{x}_k \cdot \underline{\omega}_k\right)} S_4(\underline{\omega}_{1:4})\, \delta\left(\Sigma_1^4 \underline{\omega}_k\right) \prod_{k=1}^4 d\underline{\omega}_k,$$

where $\underline{\omega}_{1:4} = (\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3, \underline{\omega}_4)$, under isotropy assumption for each $g \in SO(2)$ we have in addition

$$\begin{aligned}\mathrm{Cum}\left(X(g\underline{x}_1), X(g\underline{x}_2), X(g\underline{x}_3), X(g\underline{x}_4)\right) &= \underbrace{\int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2}}_{4} e^{i\left(\Sigma_1^4 \underline{x}_k \cdot \underline{\omega}_k\right)} S_4(g\underline{\omega}_{1:4})\, \delta\left(\Sigma_1^4 \underline{\omega}_k\right) \prod_{k=1}^4 d\underline{\omega}_k \\ &= \mathrm{Cum}\left(X(\underline{x}_1), X(\underline{x}_2), X(\underline{x}_3), X(\underline{x}_4)\right)\end{aligned}$$

hence $S_4(\underline{\omega}_{1:4}) = S_4\left(\underline{\omega}_{1:3}, -\Sigma_1^3 \underline{\omega}_k\right)$ and at the same time $S_4(g\underline{\omega}_{1:4}) = S_4(\underline{\omega}_{1:4})$. Now $\underline{\omega}_{1:4} = (\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3, \underline{\omega}_4)$ is defined by eight coordinates, if $\Sigma_1^4 \underline{\omega}_k = 0$, then these vectors form a quadrilateral, in general. This quadrilateral has invariants under rotations, actually they are the sides $(\rho_1, \rho_2, \rho_3, \rho_4)$ and a diagonal κ, say. It means that the quadrilateral is equivalent to two triangles (ρ_1, ρ_2, κ) and (κ, ρ_3, ρ_4) with common side κ. Actually the quantities (ρ_1, ρ_2, κ) and (κ, ρ_3, ρ_4) are invariant under rotations and translations, in this way they define uniquely the invariants of the quadrilateral under the movements of a rigid body. Hence $S_4(\underline{\omega}_{1:4}) = S_4(\rho_{1:4}, \kappa)$, see Figure 3. Naturally, one may choose an angle, γ_4 say, instead of the diagonal κ for determining of the quadrilateral.

We apply the invariance of the cumulant under the shifts and rotations

$$\begin{aligned}\mathrm{Cum}\left(X(\underline{x}_1), X(\underline{x}_2), X(\underline{x}_3), X(\underline{x}_4)\right) &= \mathrm{Cum}\left(X(0), X(\underline{x}_2 - \underline{x}_1), X(\underline{x}_3 - \underline{x}_1), X(\underline{x}_4 - \underline{x}_1)\right) \\ &= \mathrm{Cum}\left(X(0), X(|\underline{x}_2 - \underline{x}_1|\, \underline{n}), X(g(\underline{x}_3 - \underline{x}_1)), X(g(\underline{x}_4 - \underline{x}_1))\right)\end{aligned}$$

where g denotes the rotation carrying the $\underline{x}_2 - \underline{x}_1$ into the y axis. The general form of cumulants is $\mathrm{Cum}\left(X(0), X(r_2\underline{n}), X(\underline{x}_3), X(\underline{x}_4)\right)$, where $\underline{x}_3$ and $\underline{x}_4$ are arbitrary locations and $\underline{n} = (0, 1)$. The fourth order cumulants of a homogenous and isotropic stochastic field $X(\underline{x})$ are determined by the quantities r_2, $\underline{x}_3$ and $\underline{x}_4$, in other words by r_2, r_3, r_4, φ_3, and φ_4, see Figure 1. We apply the series representation (2) of $X(\underline{x})$, and rewrite it for particular cases

$$X(r\underline{n}) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\pi/2} \int_0^\infty J_\ell(\rho r)\, Z_\ell(\rho d\rho), \tag{6}$$

$$\begin{aligned}X(\underline{0}) &= \int_{\mathbb{R}^2} Z(d\underline{\omega}) \\ &= \int_0^\infty Z_0(\rho d\rho).\end{aligned} \tag{7}$$

We obtain

$$\operatorname{Cum}\left(X\left(0\right),X\left(r_2\underline{n}\right),X\left(\underline{x}_3\right),X\left(\underline{x}_4\right)\right)=\sum_{\ell_2,\ell_3,\ell_4=-\infty}^{\infty}e^{i\left(\ell_2\pi/2+\ell_3\varphi_3+\ell_4\varphi_4\right)}$$

$$\times\iiiint_0^{\infty}J_{\ell_2}\left(\rho_2r_2\right)J_{\ell_3}\left(\rho_3r_3\right)J_{\ell_4}\left(\rho_4r_4\right)\operatorname{Cum}\left(Z_0\left(\rho_1d\rho_1\right),Z_{\ell_2}\left(\rho_2d\rho_2\right),Z_{\ell_3}\left(\rho_3d\rho_3\right),Z_{\ell_4}\left(\rho_4d\rho_4\right)\right)$$

$$=\sum_{\ell_2,\ell_3=-\infty}^{\infty}e^{i\left(\ell_2\left(\pi/2-\varphi_4\right)+\ell_3\left(\varphi_3-\varphi_4\right)\right)}\iiiint_0^{\infty}J_{\ell_2}\left(\rho_2r_2\right)J_{\ell_3}\left(\rho_3r_3\right)J_{-\left(\ell_2+\ell_3\right)}\left(\rho_4r_4\right)$$
$$\times\operatorname{Cum}\left(Z_0\left(\rho_1d\rho_1\right),Z_{\ell_2}\left(\rho_2d\rho_2\right),Z_{\ell_3}\left(\rho_3d\rho_3\right),Z_{-\left(\ell_2+\ell_3\right)}\left(\rho_4d\rho_4\right)\right),$$

in polar coordinates. The fourth order cumulant of the stochastic spectral measure $Z\left(d\underline{\omega}\right)$ according to a homogenous field $X\left(\underline{x}\right)$ fulfils the following equation

$$\operatorname{Cum}\left(Z\left(d\underline{\omega}_1\right),Z\left(d\underline{\omega}_2\right),Z\left(d\underline{\omega}_3\right),Z\left(d\underline{\omega}_4\right)\right)=\delta\left(\Sigma_1^4\underline{\omega}_k\right)S_4\left(\underline{\omega}_{1:4}\right)\prod_{k=1}^{4}d\underline{\omega}_k,$$

and the stochastic spectral measures $Z_\ell\left(\rho d\rho\right)$ are connected to $Z\left(d\underline{\omega}\right)$ by (3) in frequency domain, hence

$$\operatorname{Cum}\left(Z_0\left(\rho_1d\rho_1\right),Z_{\ell_2}\left(\rho_2d\rho_2\right),Z_{\ell_3}\left(\rho_3d\rho_3\right),Z_{-\left(\ell_2+\ell_3\right)}\left(\rho_4d\rho_4\right)\right)$$
$$=\iiiint_0^{2\pi}e^{-i\ell_2\left(\eta_4-\eta_2\right)-i\ell_3\left(\eta_4-\eta_3\right)}\,\delta\left(\Sigma_1^4\rho_k\widehat{\underline{\omega}}_k\right)S_4\left(\rho_{1:4},\kappa\right)\prod_{k=1}^{4}\rho_kd\rho_kd\eta_k. \quad (8)$$

where $\widehat{\underline{\omega}}_k=\underline{\omega}_k/\left|\underline{\omega}_k\right|=\left(\cos\eta_k,\sin\eta_k\right)$ defines the angle η_k. Now, let us substitute $\delta\left(\Sigma_1^4\rho_k\widehat{\underline{\omega}}_k\right)$ by 12 and integrate

$$\operatorname{Cum}\left(Z_0\left(\rho_1d\rho_1\right),Z_{\ell_2}\left(\rho_2d\rho_2\right),Z_{\ell_3}\left(\rho_3d\rho_3\right),Z_{-\left(\ell_2+\ell_3\right)}\left(\rho_4d\rho_4\right)\right)$$
$$=2^3\pi\left(-1\right)^{\ell_2+\ell_3}\int_0^{\pi}\cos\left(\ell_2\alpha_2-\ell_3\gamma_4\right)\cos\left(\ell_2\beta_3\right)\frac{S_4\left(\rho_{1:4},\kappa\right)}{\rho_2\kappa\sin\left(\beta_3\right)}d\gamma_4\prod_{k=1}^{4}\rho_kd\rho_k,$$

see Appendix C for more details. Notice that the cumulants $\operatorname{Cum}\left(X\left(0\right),X\left(r_2\underline{n}\right),X\left(\underline{x}_3\right),X\left(\underline{x}_4\right)\right)=\mathcal{C}_3\left(r_2,r_3,r_4,\varphi_3,\varphi_4\right)$ are given in terms of three distances $\left(r_2,r_3,r_4\right)$ and two angles φ_3 and φ_4, one may replace angles φ_3,φ_4 by the ones $\psi_2=\pi/2-\varphi_4$ and $\psi_3=\varphi_3-\varphi_4$, hence a quadrilateral can be formed for the locations, see Figure 1. The function

$$\operatorname{Cum}\left(X\left(0\right),X\left(r_2\underline{n}\right),X\left(\underline{x}_3\right),X\left(\underline{x}_4\right)\right)$$
$$=\frac{1}{\pi^2}\sum_{\ell_2,\ell_3=-\infty}^{\infty}e^{i\left(\ell_2\psi_2+\ell_3\psi_3\right)}\iint_0^{\pi}\iiint_0^{\infty}J_{\ell_2}\left(\rho_2r_2\right)J_{\ell_3}\left(\rho_3r_3\right)J_{\ell_2+\ell_3}\left(\rho_4r_4\right)$$
$$\times\cos\left(\ell_2\alpha_2-\ell_3\gamma_4\right)\cos\left(\ell_2\beta_3\right)S_4\left(\rho_{1:4},\kappa\right)\prod_{k=2}^{4}\rho_kd\rho_kd\gamma_4d\beta_3,$$

where $\kappa=\sqrt{\rho_3^2+\rho_4^2-2\rho_3\rho_4\cos\gamma_4}$, $\alpha_2=\arccos\left[\left(\rho_3^2-\rho_4^2-\kappa^2\right)/2\kappa\rho_4\right]$, hence α_2 is determined by ρ_3, ρ_4 and γ_4. Introduce

$$\mathcal{C}_4\left(r_2,r_3,r_4,\psi_2,\psi_3\right)=\operatorname{Cum}\left(X\left(0\right),X\left(r_2\underline{n}\right),X\left(\underline{x}_3\right),X\left(\underline{x}_4\right)\right),$$

and the trispectrum $S_4\left(\rho_{1:4},\kappa\right)$ is given on the equivalent domain of variables $\left(\rho_2,\rho_3,\rho_4,\gamma_4,\beta_3\right)$, $S_4\left(\rho_{1:4},\kappa\right)=S_4\left(\rho_2,\rho_3,\rho_4,\gamma_4,\beta_3\right)$.

Figure 1: Locations on the plane

Introduce

$$
\begin{aligned}
&\mathcal{T}_4\left(\rho_2, \rho_3, \rho_4, \gamma_4, \beta_3 \middle| r_2, r_3, r_4, \psi_2, \psi_3\right) \\
&= \sum_{\ell_2, \ell_3 = -\infty}^{\infty} \cos\left(\ell_2 \psi_2 + \ell_3 \psi_3\right) J_{\ell_2}\left(\rho_2 r_2\right) J_{\ell_3}\left(\rho_3 r_3\right) J_{\ell_2+\ell_3}\left(\rho_4 r_4\right) \cos\left(\ell_2 \alpha_2 - \ell_3 \gamma_4\right) \cos\left(\ell_2 \beta_3\right),
\end{aligned}
$$

where the angle α_2 is determined by $\rho_2, \rho_3, \rho_4, \gamma_4, \beta_3$, see Figure 3.

Theorem 2 *Let $X\left(\underline{x}\right)$ be a homogenous and isotropic stochastic field on the plane then*

$$
\begin{aligned}
\mathcal{C}_4\left(r_2, r_3, r_4, \psi_2, \psi_3\right) = \frac{1}{\pi^2} \iiint_0^{\infty} \iint_0^{\pi} &\mathcal{T}_4\left(\rho_2, \rho_3, \rho_4, \gamma_4, \beta_3 \middle| r_2, r_3, r_4, \psi_2, \psi_3\right) \\
&\times S_4\left(\rho_2, \rho_3, \rho_4, \gamma_4, \beta_3\right) \prod_{k=2}^{4} \rho_k d\rho_k d\gamma_4 d\beta_3,
\end{aligned}
$$

unless the integral exists. The notations correspond to Figure 1 and Figure 3.

3 Higher order spectra

Let $\beta_p = \gamma_p$, $\rho_{2:p} = \left(\rho_2, \ldots, \rho_p\right)$, $\beta_{3:p} = \left(\beta_3, \ldots, \beta_p\right)$, $r_{2:p} = \left(r_2, \ldots, r_p\right)$, $\psi_{2:p-1} = \left(\psi_2, \ldots, \psi_{p-1}\right)$, and introduce the transformation

$$
\begin{aligned}
&\mathcal{T}_p\left(\rho_{2:p}, \beta_{3:p} \middle| r_{2:p}, \psi_{2:p-1}\right) \\
&\qquad = \sum_{\ell_2, \ldots, \ell_{p-1} = -\infty}^{\infty} J_{\Sigma_1^{p-1} \ell_k}\left(\rho_p r_p\right) \prod_{k=2}^{p-1} e^{i \ell_k \psi_k} J_{\ell_k}\left(\rho_k r_k\right) \cos\left(\alpha_{k-1} \textstyle\sum_{j=2}^{k-1} \ell_j - \ell_k \beta_{k+1}\right),
\end{aligned}
$$

where $\alpha_1 \sum_{j=2}^{1} \ell_j = 0$.

The argument of obtaining higher order spectra is similar to the case of the trispectrum, see Theorem 6.

Theorem 3 *Let $X(\underline{x})$ be a homogenous and isotropic stochastic field on the plane then*

$$\begin{aligned}&\mathcal{C}_p(r_2,\ldots,r_p,\psi_2,\ldots,\psi_{p-1})\\&=\frac{1}{\pi^{p-2}}\int_0^\infty\cdots\int_0^\infty\int_0^\pi\cdots\int_0^\pi\mathcal{T}_p(\rho_{2:p},\beta_{3:p}|\,r_{2:p},\psi_{2:p-1})\,S_p(\rho_{2:p},\beta_{3:p})\prod_{k=2}^p\rho_k d\rho_k\prod_{k=3}^p d\beta_k.\end{aligned}$$

4 Poisson formulae

If we consider the field $X(\underline{x})$ on the circle with fixed radius R, $|\underline{x}|=R$, the field $X(\underline{x})$ becomes isotropic on the circle. A natural question arises what is the connection between the spectrums of the field $X(\underline{x})$ on $\mathbb{R}^2$, and its restriction on the circle. The shorter notation $X(R,\varphi)=X_R(\varphi)$, will be used. The fields on the circle $|\underline{x}|=1$, is given by

$$\begin{aligned}X_R(\varphi)&=\sum_{\ell=-\infty}^{\infty}e^{i\ell\varphi}Z_{R,\ell},\\Z_{R,\ell}&=\int_0^\infty J_\ell(R\rho)\,Z_\ell(\rho d\rho),\end{aligned}$$

with spectrum $f_{R,\ell}=E\left|Z_{R,\ell}\right|^2$, and the covariance between two points on the circle depends on the distance $r=\left|\underline{x}-\underline{y}\right|=R\sqrt{2(1-\cos(\varphi_1-\varphi_2))}$

$$\begin{aligned}\operatorname{Cov}(X_R(\varphi_1),X_R(\varphi_2))&=2\pi\int_0^\infty J_0(\rho r)\,F(\rho d\rho)\\&=2\pi\sum_{m=-\infty}^{\infty}e^{im(\varphi_1-\varphi_2)}\int_0^\infty J_m^2(R\rho)\,F(\rho d\rho),\end{aligned}$$

by the addition formula of Bessel functions

$$J_0(r\rho)=\sum_{m=-\infty}^{\infty}J_m^2(R\rho)\,e^{im(\varphi_1-\varphi_2)},$$

it follows

$$f_{R,\ell}=\int_0^\infty J_\ell^2(R\rho)\,F(\rho d\rho).$$

Let us consider now the bicovariances

$$\begin{aligned}\operatorname{Cum}(X_R(\pi/2),X_R(\varphi_2),X_R(\varphi_3))&=\sum_{\ell_1,\ell_2,\ell_3=-\infty}^{\infty}e^{i(\ell_1\pi/2+\ell_2\varphi_2+\ell_3\varphi_3)}\operatorname{Cum}(Z_{R,\ell_1},Z_{R,\ell_2},Z_{R,\ell_3})\\&=\sum_{\ell_2,\ell_3=-\infty}^{\infty}e^{i(\ell_2(\varphi_2-\pi/2)+\ell_3(\varphi_3-\pi/2))}\operatorname{Cum}(Z_{R,-\ell_2-\ell_3},Z_{R,\ell_2},Z_{R,\ell_3}),\end{aligned}$$

$$\begin{aligned}&\operatorname{Cum}(Z_{-R,\ell_2-\ell_3}(\rho_1 d\rho_1),Z_{R,\ell_2}(\rho_2 d\rho_2),Z_{R,\ell_3}(\rho_3 d\rho_3))\\&=S_3(\rho_1,\rho_2,\rho_3)\iiint_0^{2\pi}e^{i(\ell_2(\eta_2-\eta_1)+\ell_3(\eta_3-\eta_1))}\delta\left(\Sigma_1^3\rho_k\widehat{\underline{\omega}}_k\right)\prod_{k=1}^3\rho_k d\rho_k d\eta_k\\&=S_3(\rho_1,\rho_2,\rho_3)\frac{(-1)^{\ell_2+\ell_3}}{2\pi}\int_0^\infty J_{-\ell_2-\ell_3}(\rho_1\lambda)\,J_{\ell_2}(\rho_2\lambda)\,J_{\ell_3}(\rho_3\lambda)\,\lambda d\lambda\\&=S_3(\rho_1,\rho_2,\rho_3)\frac{1}{2\pi}\frac{\cos(\ell_2\gamma_2-\ell_3\gamma_1)}{\pi\rho_1\rho_2\sin\gamma_2}\end{aligned}$$

$$\mathrm{Cum}\left(Z_{R,-\ell_2-\ell_3}, Z_{R,\ell_2}, Z_{R,\ell_3}\right)$$
$$= \iiint_0^\infty \int_0^\pi J_{\ell_2+\ell_3}\left(\rho_1\right) J_{\ell_2}\left(\rho_2\right) J_{\ell_3}\left(\rho_3\right) S_3\left(\rho_1,\rho_2,\rho_3\right) \cos\left(\ell_2\gamma_2 - \ell_3\gamma_1\right) d\gamma_2 \prod_{k=1}^2 \rho_k d\rho_k.$$

Hence

$$\mathrm{Cum}\left(X_R\left(\pi/2\right), X_R\left(\varphi_2\right), X_R\left(\varphi_3\right)\right) = 4\pi \iiint_0^\infty \int_0^\pi \mathcal{T}_{R,3}\left(\eta,\rho_2,\rho_3 \middle| \varphi_2,\varphi_3\right) S_3\left(\rho_1,\rho_2,\rho_3\right) d\gamma_2 \prod_{k=1}^2 \rho_k d\rho_k,$$

where

$$\mathcal{T}_{R,3}\left(\eta,\rho_2,\rho_3 \middle| \varphi_2,\varphi_3\right)$$
$$= \sum_{k_2,k_3=-\infty}^{\infty} \cos\left(k_2\left(\varphi_2-\pi/2\right)+k_3\left(\varphi_3-\pi/2\right)\right) J_{k_2+k_3}\left(R\rho_1\right) J_{k_2}\left(R\rho_2\right) J_{k_3}\left(R\rho_3\right) \cos\left(k_2\gamma_2-k_3\gamma_1\right).$$

One might derive this formula based on the general connection between the bicovariance and bispectrum, see Appendix D for details. Let us note here that if we are given a bispectrum on the circle with radius R for each $R>0$, then it defines a bispectrum for a homogenous and isotropic field on the plane. Indeed for every three points on the plane defines a circle all of them belongs to, this circle can be shifted to the origin and the bicovariance is calculated according these points.

A Integral of Bessel functions

One of the key formula deriving the bispectrum, see [Ter13], is the integral

$$\int_0^\infty J_0\left(\rho_1\lambda\right) J_\ell\left(\rho_2\lambda\right) J_\ell\left(\rho_3\lambda\right) \lambda d\lambda = \frac{\cos\left(\ell \arccos\left(R\right)\right)}{\pi\rho_2\rho_3\sqrt{1-R^2}}$$
$$= \frac{\cos\left(\ell\gamma_3\right)}{\pi\rho_2\rho_3\sin\gamma_3},$$

where $\rho_1^2=\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos\gamma_3$ and $R=\left(\rho_2^2+\rho_3^2-\rho_1^2\right)/\left(2\rho_2\rho_3\right)=\cos\gamma_3$, see [PBM86] Tom. II, 2.12.41.16. This formula is a particular case of the following result.

Lemma 4 *Let $\rho_k>0$, $|\rho_2-\rho_3|\le\rho_1\le\rho_2+\rho_3$, and $\rho_1^2=\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos\gamma_3$, then*

$$\int_0^\infty J_{\ell_1}\left(\rho_1\lambda\right) J_{\ell_2}\left(\rho_2\lambda\right) J_{\ell_1+\ell_2}\left(\rho_3\lambda\right) \lambda d\lambda = \frac{\cos\left(\ell_1\gamma_1-\ell_2\gamma_3\right)}{\pi\rho_2\rho_3\sin\gamma_3},$$

otherwise if ρ_1, ρ_2 and ρ_3 do not form a triangle then the integral is zero.

Proof. Formulae of integrals of Bessel functions require care and attention, for instance formula [Vil68] p224 and the Addition Theorem [Kor02], p27 are wrong. The assumptions imply that a triangle Figure 2 can be formed. Notice that the usual notations when the angles γ_1, γ_2 and γ_3 are opposite to the sides ρ_1, ρ_2 and ρ_3 has been changed in Figure 2, the reason is that we are going to generalize our results for multilateral. Formally the following relations are valid; $\rho_1^2=\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos\gamma_3$, $\rho_2-\rho_3\cos\gamma_3=\rho_1\cos\gamma_2$, $\rho_3\sin\gamma_3=\rho_1\sin\gamma_2$, [EMOT81], T2, p54 , equivalently, $\sqrt{4\rho_2^2\rho_3^2-\left(\rho^2-\rho_2^2-\rho_3^2\right)^2}=2\rho_2\rho_3\sin\gamma_3$, $\gamma_3\in(0,\pi)$. Let us start with the Graf's Addition Theorem

$$e^{i\ell_1\gamma_1} J_{\ell_1}\left(\rho_1\right) = \sum_{m=-\infty}^{\infty} J_m\left(\rho_2\right) J_{m+\ell_1}\left(\rho_3\right) e^{im\gamma_3}.$$

Figure 2: Triangle

The system $e^{im\gamma}$ is orthogonal on the $[0,2\pi]$, but the angle γ_3 is changing on interval $[0,\pi]$, therefore we consider the integral

$$\int_0^\pi e^{i(m-\ell_2)\varphi} d\varphi = \begin{cases} \pi & \text{if} \quad m=\ell_2, \\ \frac{i}{m-\ell_2}\left(1-(-1)^{m-\ell_2}\right) & \text{if} \quad m \neq \ell_2, \end{cases}$$

hence

$$\begin{aligned} \int_0^\pi e^{i\ell_1\gamma_1} J_{\ell_1}(\rho_1) e^{-i\ell_2\gamma_3} d\gamma_3 &= \int_0^\pi \sum_{m=-\infty}^{\infty} J_m(\rho_2) J_{m+\ell_1}(\rho_3) e^{i(m-\ell_2)\gamma_3} d\gamma_3 \\ &= \pi J_{\ell_2}(\rho_2) J_{\ell_1+\ell_2}(\rho_3) + 2i \sum_{k=-\infty}^{\infty} \frac{1}{2k+1} J_{2k+1+\ell_2}(\rho_2) J_{2k+1+\ell_1+\ell_2}(\rho_3). \end{aligned}$$

The real part of this equality provides

$$\int_0^\pi \cos(\ell_1\gamma_1 - \ell_2\gamma_3) J_{\ell_1}(\lambda\rho_1) d\gamma_3 = \pi J_{\ell_2}(\lambda\rho_2) J_{\ell_1+\ell_2}(\lambda\rho_3).$$

Now, integrate by $\lambda d\lambda$ and apply the formula (9)

$$\begin{aligned} &\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_1+\ell_2}(\rho_3\lambda) \lambda d\lambda \\ &\quad = \int_0^\infty J_{\ell_1}(\rho_1\lambda) \frac{1}{\pi} \int_0^\pi \cos(\ell_1\gamma_1 - \ell_2\gamma) J_{\ell_1}(\rho\lambda) d\gamma \lambda d\lambda \\ &\quad = \frac{1}{\pi} \int_{|\rho_2-\rho_3|}^{\rho_2+\rho_3} \int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_1}(\rho\lambda) \lambda d\lambda \frac{\cos(\ell_1\gamma_1 - \ell_2\gamma) \rho d\rho}{\rho_2\rho_3 \sin\gamma} \\ &\quad = \frac{1}{\pi} \int_{|\rho_2-\rho_3|}^{\rho_2+\rho_3} \frac{\cos(\ell_1\gamma_1 - \ell_2\gamma)}{\rho_2\rho_3\sin\gamma} \frac{\delta(\rho_1-\rho)}{\rho_1} \rho d\rho \\ &\quad = \frac{\cos(\ell_1\gamma_1 - \ell_2\gamma_3)}{\pi\rho_2\rho_3\sin\gamma_3}. \end{aligned}$$

The integral is zero if inequality $|\rho_2-\rho_3| \leq \rho_1 \leq \rho_2+\rho_3$ does not fulfils, [Vil68] p224. ∎

Figure 3: Quadrilateral

We consider a quadrilateral according to the wave numbers $(\rho_1, \rho_2, \rho_3, \rho_4)$ defined by two triangles (ρ_1, ρ_2, κ) and (κ, ρ_3, ρ_4) where $\kappa = |\underline{\kappa}|$ is the diagonal and $\rho_j = |\underline{\omega}_j|$, see Figure 3. In other words $(\underline{\omega}_1, \underline{\omega}_2, \underline{\kappa})$ and $(\underline{\omega}_3, \underline{\omega}_4, -\underline{\kappa})$ are triangulars and their sides (ρ_1, ρ_2, κ) and (κ, ρ_3, ρ_4) fulfil the triangle relation.

Lemma 5 *Assume* $\kappa^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\gamma_2$, $\gamma_2 \in (0, \pi)$ *and* $(\rho_1, \rho_2, \kappa, \rho_3, \rho_4)$ *defines a quadrilateral, then*

$$\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_3}(\rho_3\lambda) J_{\ell_1+\ell_2+\ell_3}(\rho_4\lambda)\lambda d\lambda$$
$$= \frac{1}{\pi^2}\int_0^\pi \cos\left((\ell_1+\ell_2)\alpha_2 - \ell_3\gamma_4\right) \frac{\cos(\ell_1\alpha_1 - \ell_2\beta_3)}{\rho_2\kappa\sin(\beta_3)} d\gamma_4,$$

where the notations correspond to the Figure 3.

Proof. The Addition Theorem and Lemma 4 give

$$J_{\ell_3}(\rho_3\lambda) J_{\ell_1+\ell_2+\ell_3}(\rho_4\lambda) = \frac{1}{\pi}\int_{|\rho_4-\rho_3|}^{\rho_3+\rho_4} J_{\ell_1+\ell_2}(\kappa\lambda) \frac{\cos\left((\ell_1+\ell_2)\alpha_2 - \ell_3\gamma_4\right)\kappa d\kappa}{\rho_3\rho_4\sin\gamma_4},$$

$$\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_1+\ell_2}(\kappa\lambda)\lambda d\lambda = \frac{\cos(\ell_1\alpha_1 - \ell_2\beta_3)}{\pi\rho_2\kappa\sin\beta_3},$$

hence

$$\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_3}(\rho_3\lambda) J_{\ell_1+\ell_2+\ell_3}(\rho_4\lambda)\lambda d\lambda$$
$$= \frac{1}{\pi^2}\int_{|\rho_4-\rho_3|}^{\rho_4+\rho_3} \frac{\cos\left((\ell_1+\ell_2)\alpha_2 - \ell_3\gamma_4\right)}{\rho_3\rho_4\sin\gamma_4} \int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_1+\ell_2}(\kappa\lambda)\lambda d\lambda \kappa d\kappa$$
$$= \frac{1}{\pi^2}\int_{|\rho_4-\rho_3|}^{\rho_4+\rho_3} \frac{\cos\left((\ell_1+\ell_2)\alpha_2 - \ell_3\gamma_4\right)}{\rho_3\rho_4\sin\gamma_4} \frac{\cos(\ell_1\alpha_1 - \ell_2\beta_3)}{\rho_2\kappa\sin\beta_3}\kappa d\kappa$$
$$= \frac{1}{\pi^2}\int_0^\pi \cos\left((\ell_1+\ell_2)\alpha_2 - \ell_3\gamma_4\right) \frac{\cos(\ell_1\alpha_1 - \ell_2\beta_3)}{\rho_2\kappa\sin\beta_3} d\gamma_4,$$

Figure 4: Multilateral, $p = 5$

where $\gamma_4 = \pi - \eta_4 + \eta_3$, $\beta_3 = \eta_2 - \eta_1 - \alpha_1$, $\gamma_2 = \pi - \eta_2 + \eta_1$, $\sqrt{\left(2\rho_3\rho_4\right)^2 - \left(\kappa^2 - \rho_3^2 - \rho_4^2\right)^2} = 2\rho_3\rho_4 \sin\gamma_4$, $\kappa d\kappa = \rho_3\rho_4 \sin\left(\gamma_4\right) d\gamma_4$, $\rho_1 \sin\gamma_2 = \kappa \sin\beta_3$, see Figure 3. ■

For further generalization of Lemma 4 we consider multilateral on the plane. A multilateral of order 5, say, has 5 vertices, 2 diagonals, see Figure 4. Invariants under the motion of a rigid body are the angles the length of the sides and diagonals. The multilateral will be well defined if the length of the sides and diagonals are given, one may replace the diagonals by the angle against it. For instance the $\kappa_2 = |\underline{\kappa}_2|$ and angle γ_5 are equivalent in deterring the triangle (A_1, A_4, A_5) together with sides $\rho_4 = |\underline{\omega}_4|$ and $\rho_5 = |\underline{\omega}_5|$.

Theorem 6 *Let $p \geq 4$ and consider a multilateral of order p then*

$$\int_0^\infty J_{\Sigma_1^{p-1}\ell_k}\left(\rho_p\lambda\right)\prod_{k=1}^{p-1} J_{\ell_k}\left(\rho_k\lambda\right)\lambda d\lambda$$
$$= \frac{1}{\pi^{p-2}}\int_0^\pi \cdots \int_0^\pi \prod_{k=1}^{p-2} \cos\left(\alpha_k \textstyle\sum_{j=1}^k \ell_j - \ell_{k+1}\beta_{k+2}\right)\frac{d\beta_{k+2}}{\rho_2\kappa_1 \sin\left(\beta_3\right)},$$

where $\beta_p = \gamma_p$, see Figure 4 for the notations.

Proof. A multilateral can be split up $p-2$ triangles, see Figure 4. We show that from $p = 4$ follows $p = 5$, such that the pattern of general induction shows up. By the Addition Theorem we have

$$J_{\ell_4}\left(\rho_4\lambda\right) J_{\ell_1+\ell_2+\ell_3+\ell_4}\left(\rho_5\lambda\right) = \frac{1}{\pi}\int_0^\pi J_{\ell_1+\ell_2+\ell_3}\left(\kappa_2\lambda\right)\cos\left(\left(\ell_1+\ell_2+\ell_3\right)\alpha_3 - \ell_4\gamma_5\right) d\gamma_5,$$

and the result of Lemma 5 provides

$$\int_0^\infty J_{\ell_1}\left(\rho_1\lambda\right) J_{\ell_2}\left(\rho_2\lambda\right) J_{\ell_3}\left(\rho_3\lambda\right) J_{\ell_1+\ell_2+\ell_3}\left(\kappa_2\lambda\right)\lambda d\lambda$$
$$= \frac{1}{\pi^2}\int_0^\pi \cos\left(\left(\ell_1+\ell_2\right)\alpha_2 - \ell_3\beta_4\right)\cos\left(\ell_1\alpha_1 - \ell_2\beta_3\right)\frac{d\beta_4}{\rho_2\kappa_1 \sin\left(\beta_3\right)}$$

hence we obtain

$$\int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_3}(\rho_3\lambda) J_{\ell_4}(\rho_4\lambda) J_{\ell_1+\ell_2+\ell_3+\ell_4}(\rho_5\lambda) \lambda d\lambda$$
$$= \frac{1}{\pi^3} \int_0^\infty J_{\ell_1}(\rho_1\lambda) J_{\ell_2}(\rho_2\lambda) J_{\ell_3}(\rho_3\lambda) \int_0^\pi J_{\ell_1+\ell_2+\ell_3}(\kappa_2\lambda) \cos((\ell_1+\ell_2+\ell_3)\alpha_3 - \ell_4\gamma_5) d\gamma_5 \lambda d\lambda$$
$$= \frac{1}{\pi^3} \int_0^\pi \int_0^\pi \cos(\ell_1\alpha_1 - \ell_2\beta_3) \cos((\ell_1+\ell_2)\alpha_2 - \ell_3\beta_4) \frac{\cos((\ell_1+\ell_2+\ell_3)\alpha_3 - \ell_4\gamma_5) d\gamma_5 d\beta_4}{\rho_2\kappa_1 \sin(\beta_3)}.$$

■

B Dirac-function in spherical coordinates

$\delta(\rho-\kappa)$ denotes the Dirac 'function', more precisely $\delta(\rho-\kappa)$ is a distribution, for instance the integral of Bessel functions provides Dirac function

$$\int_0^\infty J_\ell(\rho r) J_\ell(\kappa r) r dr = \frac{\delta(\rho-\kappa)}{\rho}, \tag{9}$$

see [AW01] Sect 11. p691.

Jacobi-Anger expansion on the plane

$$e^{i\rho r\cos(\varphi-\eta)} = \sum_{\ell=-\infty}^{\infty} i^\ell J_\ell(\rho r) e^{i\ell(\varphi-\eta)}, \tag{10}$$

In order to understand the influence of the Dirac 'function' in polar coordinates we express it by the integral through the Jacobi-Anger expansion (10) and obtain

$$\delta(\Sigma_1^p \rho_k \underline{\widehat{\omega}}_k) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(\underline{\lambda}\cdot\Sigma_1^p \underline{\omega}_k)} d\underline{\lambda}$$
$$= \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} \prod_{k=1}^p \sum_{m_k=-\infty}^{\infty} i^{m_k} J_{m_k}(\rho_k\lambda) e^{im_k(\eta_k-\xi)} \lambda d\lambda d\xi$$
$$= \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} \sum_{m_{1:p}=-\infty}^{\infty} i^{\Sigma_1^p m_k} e^{i\Sigma_1^p m_k(\eta_k-\xi)} \prod_{k=1}^p J_{m_k}(\rho_k\lambda) \lambda d\lambda d\xi$$

If $p=2$,

$$\delta(\Sigma_1^p \rho_k \underline{\widehat{\omega}}_k) = \frac{1}{2\pi} \int_0^\infty \sum_{m=-\infty}^{\infty} e^{im(\eta_2-\eta_1)} J_m(\rho_1\lambda) J_{-m}(\rho_2\lambda) \lambda d\lambda$$
$$= \frac{1}{2\pi} \frac{\delta(\rho_1-\rho_2)}{\rho_1} \sum_{m=-\infty}^{\infty} e^{im(\eta_2+\pi-\eta_1)}$$
$$= \frac{\delta(\rho_1-\rho_2)}{\rho_2} \delta(\eta_1+\eta_2+\pi),$$

hence the integral is taken according to the subspace $\rho_1 = \rho_2$ and $\eta_1 = -(\eta_2+\pi)$, since $\eta_2+\pi$ corresponds to $-\underline{\omega}_2$, this subspace is the one $\underline{\omega}_1 = -\underline{\omega}_2$, what is expected.

For $p=3$, we apply Lemma 4

$$\delta(\Sigma_1^3 \rho_k \underline{\widehat{\omega}}_k) = \frac{1}{2\pi} \int_0^\infty \sum_{m_{2:3}=-\infty}^{\infty} e^{i\Sigma_2^3 m_k(\eta_k-\eta_1)} J_{-m_2-m_3}(\rho_1\lambda) \prod_{k=2}^3 J_{m_k}(\rho_k\lambda) \lambda d\lambda \tag{11}$$
$$= \frac{1}{2\pi^2} \sum_{m_{2:3}=-\infty}^{\infty} (-1)^{m_2+m_3} e^{i\Sigma_2^3 m_k(\eta_k-\eta_1)} \frac{\cos(m_2\gamma_2 - m_3\gamma_1)}{\rho_1\rho_2 \sin\gamma_2},$$

where the notations of Figure 2 are used. Here the integral is concentrated on the subspace when (ρ_1, ρ_2, ρ_3) forms a triangle.

Similarly, for $p = 4$, we have

$$\begin{aligned}
&\delta\left(\Sigma_1^4 \rho_k \widehat{\underline{\omega}}_k\right) \qquad (12)\\
&= \frac{1}{2\pi}\int_0^\infty \sum_{m_{1:3}=-\infty}^{\infty} e^{i\Sigma_1^3 m_k(\eta_k-\eta_4)} J_{-m_1-m_2-m_3}(\rho_4\lambda)\prod_{k=1}^{3} J_{m_k}(\rho_k\lambda)\,\lambda d\lambda\\
&= \frac{1}{2\pi^3}\sum_{m_{1:3}=-\infty}^{\infty} e^{i\Sigma_1^3 m_k(\eta_k-\eta_4)}(-1)^{m_1+m_2+m_3}\\
&\times \int_0^\pi \cos(m_1\alpha_1 - m_2\beta_3)\frac{\cos((m_1+m_2)\alpha_2 - m_3\gamma_4)\,d\gamma_4}{\rho_2\kappa\sin(\beta_3)},
\end{aligned}$$

see Lemma 5 and Figure 3 for this case.

C Cumulants

$$\begin{aligned}
&\operatorname{Cum}\left(Z_0(\rho_1 d\rho_1), Z_\ell(\rho_2 d\rho_2), Z_{-\ell}(\rho_3 d\rho_3)\right)\\
&\qquad = 4\pi(-1)^\ell\,\delta(\rho\triangle)\frac{\cos(\ell\arccos(R))}{\rho_2\rho_3\sqrt{1-R^2}} S_3(\rho_1,\rho_2,\rho_3)\prod_{k=1}^{3}\rho_k d\rho_k
\end{aligned}$$

where $\delta(\rho\triangle) = \delta\left(\rho_2^2+\rho_3^2-2\rho_2\rho_3\cos\eta-\rho_1^2\right)$ implies that the wave numbers ρ_1, ρ_2, and ρ_3 should satisfy the triangle relation.

Consider

$$\begin{aligned}
&\operatorname{Cum}\left(Z_0(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), Z_{\ell_3}(\rho_3 d\rho_3), Z_{-(\ell_2+\ell_3)}(\rho_4 d\rho_4)\right) =\\
&\qquad \iiiint_0^{2\pi} e^{-i\ell_2(\eta_4-\eta_2)-i\ell_3(\eta_4-\eta_3)}\,\delta\left(\Sigma_1^4\rho_k\widehat{\underline{\omega}}_k\right) S_4(\rho_{1:4},\kappa)\prod_{k=1}^{4}\rho_k d\rho_k d\eta_k,
\end{aligned}$$

replace the Dirac-function by (12), use the orthogonality of the 'spherical harmonics'

$$\iiiint_0^{2\pi} e^{-i\ell_2(\eta_4-\eta_2)-i\ell_3(\eta_4-\eta_3)}\; e^{i\Sigma_1^3 m_k(\eta_k-\eta_4)}\prod_{k=1}^{4} d\eta_k = \delta_{m_1}\delta_{m_2+\ell_2}\delta_{m_3+\ell_3}(2\pi)^4,$$

and a particular case of Lemma 5

$$\int_0^\infty J_0(\rho_1\lambda)J_{\ell_2}(\rho_2\lambda)J_{\ell_3}(\rho_3\lambda)J_{\ell_2+\ell_3}(\rho_4\lambda)\,\lambda d\lambda = \frac{1}{\pi^2}\int_0^\pi \cos(\ell_2\alpha_2-\ell_3\gamma_4)\frac{\cos(\ell_2\beta_3)\,d\gamma_4}{\rho_2\kappa\sin(\beta_3)}.$$

We obtain

$$\begin{aligned}
&\iiiint_0^{2\pi} e^{-i\ell_2(\eta_4-\eta_2)-i\ell_3(\eta_4-\eta_3)}\,\delta\left(\Sigma_1^4\rho_k\widehat{\underline{\omega}}_k\right) S_4(\rho_{1:4},\kappa)\prod_{k=1}^{4}\rho_k d\rho_k d\eta_k\\
&\qquad = 2^3\pi(-1)^{\ell_2+\ell_3}\int_0^\pi \cos(\ell_2\beta_3)\cos(\ell_2\alpha_2-\ell_3\gamma_4)\frac{S_4(\rho_{1:4},\kappa)}{\rho_2\kappa\sin(\beta_3)}d\gamma_4\prod_{k=1}^{4}\rho_k d\rho_k\\
&\qquad\quad = 2^3\pi(-1)^{\ell_2+\ell_3}\int_0^\pi \cos(\ell_2\beta_3)\cos(\ell_2\alpha_2-\ell_3\gamma_4)\,S_4(\rho_{1:4},\kappa)\,d\gamma_4 d\beta_3\prod_{k=2}^{4}\rho_k d\rho_k,
\end{aligned}$$

since $\rho_1 d\rho_1 = \rho_2 \kappa \sin(\beta_3) d\beta_3$. Repeat the assumptions

$$\max(|\rho_2 - \rho_1|, |\rho_4 - \rho_3|) < \kappa < \min(\rho_1 + \rho_2, \rho_3 + \rho_4),$$

and $\kappa^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos(\gamma_2) = \rho_3^2 + \rho_4^2 - 2\rho_3\rho_4\cos(\gamma_4)$, see Figure 3.

$$\begin{aligned}&\mathcal{T}_4(\rho_2, \rho_3, \rho_4, \gamma_4, \beta_3 | r_2, r_3, r_4, \psi_2, \psi_3)\\ &= \sum_{\ell_2, \ell_3 = -\infty}^{\infty} e^{i(\ell_2\psi_2 + \ell_3\psi_3)} J_{\ell_2}(\rho_2 r_2) J_{\ell_3}(\rho_3 r_3) J_{\ell_2+\ell_3}(\rho_4 r_4)\cos(\ell_2\beta_3)\cos(\ell_2\alpha_2 - \ell_3\gamma_4)\end{aligned}$$

notice that the terms corresponding to (ℓ_2, ℓ_3) and $(-\ell_2, -\ell_3)$ are conjugates to each other, hence

$$\begin{aligned}&\mathcal{T}_4(\rho_2, \rho_3, \rho_4, \gamma_4, \beta_3 | r_2, r_3, r_4, \psi_2, \psi_3)\\ &= \sum_{\ell_2, \ell_3 = -\infty}^{\infty} \cos(\ell_2\psi_2 + \ell_3\psi_3) J_{\ell_2}(\rho_2 r_2) J_{\ell_3}(\rho_3 r_3) J_{\ell_2+\ell_3}(\rho_4 r_4)\cos(\ell_2\beta_3)\cos(\ell_2\alpha_2 - \ell_3\gamma_4).\end{aligned}$$

D Circle, bispectrum

The bispectrum $B_{\ell_2,\ell_3} = \mathrm{Cum}(Z_{-\ell_2-\ell_3}, Z_{\ell_2}, Z_{\ell_3})$ of $X_R(\varphi)$ on the circle will be given by the bicovariances

$$\mathrm{Cum}(X_R(\underline{n}), X_R(\varphi_2), X_R(\varphi_3)) = 4\pi \iint_0^\infty \int_0^\pi \mathcal{T}(\eta, \rho_2, \rho_3 | \varphi, R, R) S_3(\rho_1, \rho_2, \rho_3) d\eta \prod_{k=2}^{3} \rho_k d\rho_k,$$

where $\varphi = \phi_2 + \phi_3 = (\varphi_3 - \varphi_2)/2 \bmod (2\pi)$, $r_2 = \sqrt{2(1 - \cos(\pi/2 - \varphi_2))}$ and $r_3 = \sqrt{2(1 - \cos(\pi/2 - \varphi_3))}$. We apply Graaf's Addition Theorem

$$e^{i\ell\gamma_1} J_\ell(\rho_1) = \sum_{m=-\infty}^{\infty} J_m(\rho_2) J_{m+\ell}(\rho_3) e^{im\gamma_3},$$

[Kor02], for the expansion of the Bessel functions

$$\begin{aligned}&e^{i\ell(\phi_2+\phi_3)} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3)\\ &= \sum_{k_2=-\infty}^{\infty} J_{k_2}(R\rho_2) J_{k_2+\ell}(R\rho_2) e^{ik_2(\varphi_2 - \pi/2)} \sum_{k_3=-\infty}^{\infty} J_{k_3}(R\rho_3) J_{k_3+\ell}(R\rho_3) e^{ik_3(\pi/2 - \varphi_3)}\\ &= \sum_{k_2, k_3 = -\infty}^{\infty} e^{i(k_2(\varphi_2 - \pi/2) + k_3(\varphi_3 - \pi/2))} J_{k_2}(R\rho_2) J_{-k_3}(R\rho_3) J_{k_2+\ell}(R\rho_2) J_{-k_3+\ell}(R\rho_3)\end{aligned}$$

see Figure 2 for the connections between angles γ_k and sides ρ_k.

$$\begin{aligned}\sum_{\ell=-\infty}^{\infty} J_{k_2+\ell}(R\rho_2) J_{-k_3+\ell}(R\rho_3) e^{i\ell\gamma_3} &= e^{ik_3\gamma_3} \sum_{\ell=-\infty}^{\infty} J_{k_2+k_3+\ell}(R\rho_2) J_\ell(R\rho_3) e^{i\ell\gamma_3}\\ &= e^{i((k_2+k_3)\gamma_2 + k_2\gamma_3)} J_{k_2+k_3}(R\rho_1)\\ &= (-1)^{k_3} e^{i(k_2\gamma_2 - k_3\gamma_1)} J_{k_2+k_3}(R\rho_1)\end{aligned}$$

$$\sum_{\ell=-\infty}^{\infty} J_{k_2+\ell}(R\rho_2) J_{-k_3+\ell}(R\rho_3)\cos(\ell\gamma_3) = (-1)^{k_3}\cos(k_2\gamma_2 - k_3\gamma_1) J_{k_2+k_3}(R\rho_1)$$

Figure 5: Circle

$$
\begin{aligned}
\mathcal{T}_3\left(\eta,\rho_2,\rho_3\right|\varphi,r_2,r_3) &= \sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} J_\ell\left(\rho_2 r_2\right) J_\ell\left(\rho_3 r_3\right)\cos\left(\ell\eta\right) \\
&= \sum_{k_2,k_3=-\infty}^{\infty} e^{i(k_2(\varphi_2-\pi/2)+k_3(\varphi_3-\pi/2))} J_{k_2+k_3}\left(R\rho_1\right) J_{k_2}\left(R\rho_2\right) J_{k_3}\left(R\rho_3\right)\cos\left(k_2\gamma_2-k_3\gamma_1\right) \\
&= \sum_{k_2,k_3=-\infty}^{\infty} \cos\left(k_2\left(\varphi_2-\pi/2\right)+k_3\left(\varphi_3-\pi/2\right)\right) J_{k_2+k_3}\left(R\rho_1\right) J_{k_2}\left(R\rho_2\right) J_{k_3}\left(R\rho_3\right) \\
&\qquad \times\cos\left(k_2\gamma_2-k_3\gamma_1\right) \\
&= \mathcal{T}_{R,3}\left(\eta,\rho_2,\rho_3\right|\varphi_2,\varphi_3).
\end{aligned}
$$

Acknowledgement 7 *The publication was supported by the TÁMOP-4.2.2.C-11/1/KONV-2012-0001 project. The project has been supported by the European Union, co-financed by the European Social Fund.*

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