Preserver problems and reflexivity problems on operator algebras and on function algebras

doktori (PhD) értekezés

Győry Máté

Debreceni Egyetem
Természettudományi Kar

Debrecen, 2003

Ezen értekezést a Debreceni Egyetem Matematika- és Számítástudományok Doktori Iskola Matematikai analízis, függvényegyenletek programja keretében készítettem a Debreceni Egyetem TTK doktori (PhD) fokozatának elnyerése céljából.

Debrecen, 2003. június 27.

. . . .
Győry Máté
(jelölt)

Tanúsítom, hogy Győry Máté doktorjelölt 1998 és 2003 között a fent megnevezett Doktori Iskola Matematikai analízis, függvényegyenletek programjának keretében irányításommal végezte munkáját. Az értekezésben foglalt eredményekhez a jelölt önálló alkotó tevékenységével meghatározóan hozzájárult. Az értekezés elfogadását javaslom.

Debrecen, 2003. június 27.

. . . .
Dr. Molnár Lajos
(témavezető)

Köszönetnyilvánítás

Szeretnék köszönetet mondani témavezetőmnek,

dr. Molnár Lajos tanár úrnak

- oktatói-kutatói pályám támogatásáért és
- doktori értekezésem elkészítése során nyújtott önzetlen segítségéért.

Acknowledgement

I wish to thank my supervisor

Professor Lajos Molnár

- for his support to my academic carrier and
- for his generous help while preparing my PhD Thesis.

CONTENTS

1. INTRODUCTION

In our dissertation we present our results on *preserver problems* concerning certain algebraic structures of linear operators or continuous functions, as well as those on *reflexivity problems* concerning certain algebras of functions.

Linear preserver problems concern the question of determining all linear maps on an algebra which leave invariant a given subset, function or relation defined on the underlying algebra. The study of linear preserver problems on matrix algebras represents one of the most active research areas in matrix theory (see e.g. the survey papers [48, 49]). In the last decades considerable attention has also been paid to the infinite dimensional case, i.e. to preserver problems on operator algebras, and the investigations have resulted in several important results (see e.g. the survey paper [14]).

In what follows we mention three of the main groups of linear preserver problems on operator algebras. For brevity, we shall sometimes write **LPPs** for the expression **linear preserver problems**.

The *first group* of LPPs is concerned with the characterization of linear transformations on a linear space of bounded linear operators which preserve a certain given *function*. A well-known example of this kind of LPPs is Frobenius's result [22] from 1897. He gave the general form of all determinant preserving linear maps on a matrix algebra. This result is commonly considered as the first one on linear preserver problems. Another important example is the result of Jafarian and Sourour [37], where they described the general form of surjective linear transformations on the algebra of all bounded linear operators on a Banach space which preserve the spectrum of the operators.

The *second group* of LPPs concerns the characterization of linear transformations which preserve a certain *subset* of operators. As an example we mention Kaplansky's famous question [42] on invertibility preserving maps. Namely, he asked whether every surjective linear transformation between Banach algebras which preserves invertibility in one direction is a Jordan homomorphism. In such a generality the answer turned out to be negative, but his question, modified with the additional assumption of the semi-simplicity of the underlying Banach algebras, is still open and is one of the most important unsolved preserver problems.

The *third group* of LPPs deals with the characterization of linear transformations which preserve a certain *relation* between operators. An important example is the problem of preserving commutativity, which topic is still an active research area (cf.

[77, 78]).

In several cases LPPs on matrix algebras or on operator algebras can be reduced to linear preserver problems which concern rank (see e.g. [37, 87, 91]). Therefore, it is not surprising that there is a vast literature on such problems. We mention just two important finite-dimensional results here: Beasley's result [9] on rank-k preserving linear maps and Loewy's result [50] on rank-k non-increasing linear maps. With regard to the infinite-dimensional case, i.e. to preserver problems of operator algebras, the particular cases of preserving rank-1 operators or preserving operators with rank at most 1 have been treated in the papers [36, 79].

In **Chapter 2** we characterize the rank-k non-increasing linear maps, the rank-k preserving linear maps, and the corank-k preserving linear maps on the algebra of all bounded linear operators on a Hilbert space under a mild continuity condition, and we unify and extend the results mentioned above. (The problem of corank-k preservers occurs obviously in the infinite-dimensional case only.) We obtain that all those preservers are either of the form $\phi(T) = ATB$ or of the form $\phi(T) = AT^{tr}B$, where A and B are bounded linear operators with some additional properties. The content of Chapter 2 was published in our paper [29].

The concept of *linear preserver problems* has so far meant investigations on matrix algebras and on operator algebras, but similar questions can obviously be raised on *arbitrary algebras*. In **Chapter 3** we consider LPPs concerning *function algebras*, in which case the main LPPs considered before have been the characterizations of linear bijections preserving some given norm, or preserving disjointness of the support. We refer to [1, 21, 33, 38, 93, 94] for some of the important and relatively recent papers on such problems. For convenience, we introduce some notation. Let X be a locally compact Hausdorff space and let $C_0(X)$ denote the algebra of all continuous complex valued functions on X which vanish at infinity. The linear bijections of $C_0(X)$ preserving the sup-norm are determined in the famous Banach-Stone theorem. Besides the sup-norm, one of the most natural possibilities is to consider the diameter of its range. In Chapter 3 we are going to characterize all the linear bijections of $C_0(X)$ which *preserve the diameter of the range*, and give a unification of the contents of our papers [23] and [28]. We shall prove that every linear bijection of $C_0(X)$ (X being a first countable locally compact Hausdorff space) which preserves the diameter of the range of any $f \in C_0(X)$ is induced by a fixed homeomorphism of X, a fixed rotation of the complex number field $\mathbb{C}$, and translations in $C_0(X)$ with constant functions depending linearly on f.

Up to now we have considered *linear* preserver problems. It is clear that preserver problems can be raised *without assuming any kind of linearity.* We may consider transformations on some algebraic structure which preserve some property, quantity, relation, etc. There are a great number of results in mathematics which can be interpreted as preserver problems in this general sense. We mention the very simple example of the isometries of a metric space: the isometries can be viewed as transformations which preserve distance. Another example of preserver problems of this general kind is Wigner's famous unitary-antiunitary theorem, which we treat in the dissertation in detail. One of its several formulations characterizes the bijections on a Hilbert space preserving the absolute value of the inner product of any pair of vectors. This result is one of the most important theorems concerning the probabilistic aspects of quantum mechanics.

Several different proofs have been given for Wigner's fundamental theorem mentioned above. In **Chapter 4** we are going to present a further, elementary proof which is based on a completely new approach.

Wigner's theorem has been generalized in (at least) three directions. *First*, Uhlhorn [92] generalized Wigner's result by requiring only the preservation of *orthogonality* instead of that of the absolute value of the inner product, and he was able to achieve the same conclusion for the case in which the underlying space is at least 3 dimensional. Uhlhorn's result has a serious impact in physics. *Secondly*, Bargmann [7] and Sharma and Almeida [89] obtained results similar to Wigner's *without the assumption of bijectivity*. As for the *third* direction, we recall that sometimes Wigner's theorem is formulated as the characterization of bijections of the set of all 1-dimensional subspaces of a Hilbert space which preserve the angle between those subspaces. Molnár [65] extended Wigner's result in this respect for transformations on the set of all *n-dimensional subspaces* (n being fixed) which preserve the so-called principle angles between the subspaces. (For other generalizations of Wigner's theorem see e.g. [55, 57, 59, 62]). In **Chapter 5** we shall extend Wigner's theorem in *all the three directions* mentioned above, by obtaining results on the structure of orthogonality preserving transformations on the set of all n-dimensional subspaces of a Hilbert space under various conditions.

In the **second part** of the dissertation we deal with the problem of ***reflexivity*** of the automorphism group and the isometry group of certain algebras of functions. The study of reflexive linear subspaces of the algebra $B(H)$ of all bounded linear operators on a Hilbert space H represents one of the most active research areas in operator theory (see [30] for a nice general view of reflexivity of this kind). In the last decades, similar questions concerning certain important sets of transformations acting on Banach algebras rather than on Hilbert spaces have also attracted considerable attention. The initiators of the research in this direction are Kadison, Larson and Sourour. Kadison [41] studied *local derivations* from a von Neumann algebra $\mathcal{R}$ into a dual $\mathcal{R}$-bimodule $\mathcal{M}$. He called a continuous linear map from $\mathcal{R}$ into $\mathcal{M}$ a local derivation, if it agrees with a derivation at each point in the algebra $\mathcal{R}$ (the derivation may differ from point to point). This investigation of Kadison was motivated by some problems concerning the Hochschild cohomology of operator algebras. The main result, Theorem A, in [41] states that in the above setting, every local derivation is a derivation. Independently, Larson and Sourour [46] proved that the same conclusion holds for the local derivations of $B(X)$ (the definition is clear), where X is a Banach space. Since then, a considerable amount of work has been done concerning local derivations of various algebras. See, for example, [11, 18, 32, 40, 76, 85, 95, 97, 98, 99, 100]. Besides derivations, there are at least two other very important classes of transformations on operator algebras which certainly deserve attention. Namely, the group of automorphisms and the group of surjective isometries. Larson [45, Some concluding remarks (5), p. 298] initiated the study of *local automorphisms* (the definition is self-explanatory) of Banach algebras. In his joint paper with Sourour [46] that we have already mentioned they proved that if X is an infinite dimensional Banach space, then every surjective local automorphism of $B(X)$ is an automorphism (see also the paper [11] of Brešar and Šemrl). For a separable infinite dimensional Hilbert space H, Brešar and Šemrl [12] showed that the above conclusion holds without the assumption of surjectivity, i.e. every local automorphism of $B(H)$ is an automorphism. For further results on local automorphisms, we refer to [72, 85]. The common feature of all those results is that they show that the local derivations, local automorphisms, local isometries, etc. of the underlying structures are (global) derivations, automorphisms, isometries, etc., respectively. Clearly, this is a remarkable property of the underlying structure. As for function algebras, results of this kind were obtained by Cabello Sanchez and Molnár [84], and by Molnár and Zalar [71].

We now define our concept of *reflexivity*. Let X be a Banach space (in fact, in the cases we are interested in, X is usually a Banach-algebra) and for any subset $\mathcal{E} \subset B(X)$ let

$$\mathrm{ref}_{alg}\,\mathcal{E} = \{T \in B(X) : Tx \in \mathcal{E}x \text{ for all } x \in X\}$$

and

$$\mathrm{ref}_{top}\,\mathcal{E} = \{T \in B(X) : Tx \in \overline{\mathcal{E}x} \text{ for all } x \in X\},$$

where bar denotes norm-closure. The above sets are called the *algebraic reflexive closure* and the *topological reflexive closure* of $\mathcal{E}$, respectively. The collection $\mathcal{E}$ of transformations is called *algebraically reflexive* if $\mathrm{ref}_{alg}\,\mathcal{E} = \mathcal{E}$, and *topologically reflexive* if $\mathrm{ref}_{top}\,\mathcal{E} = \mathcal{E}$.

In this terminology, the main result in the paper [12] can be reformulated by saying that the automorphism group of $B(H)$ is algebraically reflexive. Similarly, Theorem 1.2 in [46] states that the Lie algebra of all generalized derivations on $B(X)$ is algebraically reflexive. For some further results on the algebraic reflexivity of the automorphism and isometry groups, we refer to [61, 71, 84].

Obviously, the topological reflexivity is a stronger property than the algebraic reflexivity. Shulman [90] showed that the derivation algebra of any C^*-algebra is topologically reflexive. Hence, not only the local derivations are derivations in this case, but even every bounded linear map which agrees with the limit of some sequence of derivations at each point (this sequence may differ from point to point). For the topological reflexivity of derivation algebras, automorphism groups and isometry groups we refer to [8, 39, 54, 56].

For the automorphism group or the isometry group of C^*-algebras, such a general result as in [90] does not hold. If $\mathcal{A}$ is a Banach algebra, then denote by $\mathrm{Aut}(\mathcal{A})$ and $\mathrm{Iso}(\mathcal{A})$ the group of automorphisms (i.e. multiplicative linear bijections) and the group of surjective linear isometries of $\mathcal{A}$, respectively. Now, if X is an uncountable discrete topological space, then it is not difficult to verify that the groups $\mathrm{Aut}\big(C_0(X)\big)$ and $\mathrm{Iso}\big(C_0(X)\big)$ of the C^*-algebra $C_0(X)$ of all continuous complex valued functions on X vanishing at infinity are not reflexive even algebraically. With regard to topological reflexivity, there are even von Neumann algebras whose automorphism and isometry groups are not topologically reflexive. For example, Batty and Molnár [8] showed that the infinite dimensional commutative von Neumann algebras acting on a separable Hilbert space have this nonreflexivity property. However, Molnár [54] proved that if H is a separable infinite dimensional Hilbert space, then $\mathrm{Aut}\big(B(H)\big)$ and $\mathrm{Iso}\big(B(H)\big)$ are topologically reflexive. For an interesting result, we refer to [61], where Molnár studied the reflexivity of the automorphism and isometry groups of C^*-algebras appearing in the famous Brown-Douglas-Fillmore theory, which are extensions of the C^*-algebra of all compact operators on H by commutative separable unital C^*-algebras. He proved that the groups Aut and Iso are algebraically reflexive in the case of every such extension, but, for example, in the probably most important case of extensions by $C(\mathbb{T})$ (i.e. the so called Toeplitz extensions) those groups are *not* topologically reflexive.

In **Chapter 6** we deal with the reflexivity of the automorphism group and the isometry group of the suspension of $B(H)$. The concept of the suspension of a C^*-algebra plays a very important role in the K-theory of operator algebras. If $\mathcal{A}$ is a C^*-algebra then its suspension is the C^*-tensor product $C_0(\mathbb{R}) \otimes \mathcal{A}$, which is well-known to be isomorphic to $C_0(\mathbb{R}, \mathcal{A})$, the algebra of all continuous functions from $\mathbb{R}$ into $\mathcal{A}$ which vanish at infinity. We know that the automorphism group and the isometry group of $B(H)$ are topologically reflexive [54]. We shall see that $\mathrm{Aut}\big(C_0(\mathbb{R})\big)$ and $\mathrm{Iso}\big(C_0(\mathbb{R})\big)$ are algebraically (but not topologically) reflexive. The main result of Chapter 6 is that the automorphism group and the isometry group of the suspension $C_0(\mathbb{R}) \otimes B(H)$ of $B(H)$ are algebraically (but not topologically) reflexive. The content of Chapter 6 was published in our paper [69]. The referee of the manuscript put the question whether it is possible to describe the topological reflexive closures of $\mathrm{Aut}\big(C_0(\mathbb{R}) \otimes B(H)\big)$ and $\mathrm{Iso}\big(C_0(\mathbb{R}) \otimes B(H)\big)$. **Chapter 7** is devoted to answer this question.

In the concept of local derivations, local automorphisms, etc. we supposed that the transformations under consideration are linear and they equal a derivation, automorphism, etc., respectively, at every single point of the underlying algebra. If we drop the condition of linearity, it is easy to see that the obtained concept is so general (because the assumption is so weak) that it is practically useless. Motivated by the result of Kowalski and Slodkowski [44] on a non-linear characterization of the characters of commutative Banach algebras, Šemrl [88] introduced the concept of

2-locality. For example, we say that a (not necessarily linear) transformation ϕ of a Banach algebra is called a 2-local automorphism, if for any pair x, y of points in the Banach algebra under consideration we have an automorphism $\phi_{x,y}$ (depending on x and y) such that $\phi(x) = \phi_{x,y}(x)$ and $\phi(y) = \phi_{x,y}(y)$. The definition of 2-local derivations, 2-local isometries, etc. are similar. Šemrl [88] proved that every 2-local automorphism of $B(H)$ (H being an infinite dimensional separable Hilbert space) is an automorphism, and every 2-local derivation of $B(H)$ is a derivation.

This concept of 2-locality has the advantage that it can be considered in relation with any algebraic structure as we do not assume any kind of linearity. Clearly, it is a remarkable property of the underlying algebraic structure if its 2-local automorphisms, 2-local (surjective linear) isometries, etc. are (global) automorphisms, isometries, etc., respectively. In fact, this means that the automorphisms, isometries, etc. are determined by their local actions on the 2-point subsets. For some recent results on 2-local derivations, 2-local automorphisms and 2-local isometries, we refer to [6, 35, 43, 64, 66, 67, 70].

Molnár [66] studied 2-local isometries of operator algebras. He proved that every such transformation of a C^*-subalgebra of $B(H)$ which contains the compact operators and the identity is a (surjective linear) isometry. Moreover, he raised the problem of considering similar questions concerning function algebras. In **Chapter 8** we obtain such a result, namely we show that if X is a first countable σ-compact Hausdorff space then every 2-local isometry of $C_0(X)$ is a surjective linear isometry. The content of Chapter 8 appeared in our paper [24].

Part I

SOME PRESERVER PROBLEMS

2. RANK AND CORANK PRESERVING LINEAR MAPS ON $B(H)$ AND AN APPLICATION TO *-SEMIGROUP ISOMORPHISMS OF OPERATOR IDEALS

2.1 Introduction and Statement of the Results

As was mentioned in the Introduction, linear preserver problems concerning *rank* represent one of the most important classes of LPPs. A frequently used method to solve a particular LPP on a matrix algebra is to reduce the problem to that of characterizing linear preserving maps concerning rank (see e.g. [37, 87, 91]). So, it is not surprising that there is a vast literature on linear maps preserving rank. We mention here two results in connection with our theorems presented below: Beasley's result [9] on rank-k preserving linear maps and Loewy's theorems [50] on rank-k non-increasing linear maps of matrix algebras. As for operator algebras, linear transformations preserving rank-1 operators or preserving operators with rank at most 1 have been treated in the papers [36, 79]. Hou [36] described the general form of all weakly continuous linear maps on the whole operator algebra of a Banach space which preserve rank-1 operators, which is probably the most fundamental result concerning rank preservers on operator algebras.

We note that *additive* preserver problems on matrix algebras and on operator algebras (i.e. when we assume that the transformations under consideration are merely additive, not necessarily linear) have also been studied recently. To mention a few corresponding papers, we refer to [5, 52, 79, 80]. In what follows, we obtain results both on linear preservers and on additive preserves.

In this chapter we characterize rank-k non-increasing linear maps, rank-k preserving linear maps, and corank-k preserving linear maps on $B(H)$, the algebra of all bounded linear operators on a Hilbert space H, thus unifying and extending the above mentioned results in [9, 36, 50]. Here, it is also natural to ask about the structure of the set of all linear maps preserving both infinite rank and infinite corank. As we shall see later, the set of such maps is much larger than that of the previous ones. The content of this chapter was published in our paper [29].

Before stating our results we fix some notation. For an arbitrary pair of vectors x, y from a Hilbert space H we denote their scalar product by y^*x, while xy^* denotes the rank one operator defined by $(xy^*)z = (y^*z)x$. Note that every operator of rank one can be written in this form. Let M be a (not necessarily closed) linear subspace of H. We say that M is of finite codimension if $\dim M^\perp < \infty$. In this case we define the codimension of M by codim $M = \dim M^\perp$. An operator $A \in B(H)$ has finite corank if its range rng A is of finite codimension. In this case we define corank A = codim rng A. Obviously, corank $A = k$ if and only if $\dim \operatorname{Ker} A^* = k$. For any positive integer k we denote by $B_k(H)$, $B_{\leq k}(H)$ and $B_{-k}(H)$, the set of all operators of rank k, of rank at most k and of corank k, respectively. We say that a linear map $\phi : B(H) \to B(H)$ is a rank-k preserver (a rank-k non-increasing map) if $A \in B_k(H)$ implies $\phi(A) \in B_k(H)$ ($A \in B_k(H)$ implies $\phi(A) \in B_{\leq k}(H)$). Similarly, ϕ is said to preserve corank k in both directions provided that $A \in B_{-k}(H)$ if and only if $\phi(A) \in B_{-k}(H)$.

Our first result unifies and extends Loewy's result [50] on linear maps on matrix algebras which are rank-k non-increasing and Hou's result [36] on rank-1 non-increasing linear maps in the infinite-dimensional case. In our proof we will use both of these results.

Theorem 2.1. *Let k be a positive integer, and H be a Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a rank-k non-increasing linear map which is weakly continuous on norm bounded sets. Then either the image of ϕ is a linear space consisting of operators of rank at most k, or there exist $A, B \in B(H)$ such that either $\phi(T) = ATB$ for all $T \in B(H)$, or $\phi(T) = AT^{tr}B$ for all $T \in B(H)$, where T^{tr} denotes the transpose of T relative to an arbitrary but fixed orthonormal basis of H.*

Note that in the above result the weak continuity assumption is essential. Namely, without this assumption nothing can be said about the behaviour of ϕ outside $\mathcal{F}(H)$, the ideal of all bounded linear operators of finite rank. Of course, this assumption is automatically fulfilled in the finite-dimensional case.

Next, we will generalize Beasley's result [9] on linear rank-k preservers on matrix algebras and Hou's result [36] on linear rank-1 preservers in the infinite-dimensional case.

Theorem 2.2. *Let k be a positive integer, and H be a Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a rank-k preserving linear map which is weakly continuous on norm bounded sets. Assume also that the image of ϕ is not contained in $B_k(H)$. Then there exists an injective operator $A \in B(H)$ and an operator $B \in B(H)$ with dense image such that either $\phi(T) = ATB$ for all $T \in B(H)$, or $\phi(T) = AT^{tr}B$ for all $T \in B(H)$.*

The finite-dimensional analogue of this result holds without the assumption that the image of ϕ contains an operator of rank greater than k (see [9]). However, this assumption is indispensable in the infinite-dimensional case. To see this let k be a positive integer and consider a separable infinite-dimensional Hilbert space with orthonormal basis $\{e_n : n = 1, 2, \ldots\}$. Define a family $T_n \in B(H)$, $n = 1, 2, \ldots$, by

$$T_n e_j = \begin{cases} e_{j-n+1} & \text{if } n \le j \le n+k-1, \\ 0 & \text{otherwise.} \end{cases}$$

For every $A \in B(H)$ order the countable set $\{e_j^* A e_i : i, j = 1, 2, \ldots\}$ into a sequence $(a_n)_{n=1}^{\infty}$. Use the same ordering for all operators and define $\phi : B(H) \to B(H)$ by

$$\phi(A) = \sum_{n=1}^{\infty} \frac{a_n}{n^2} T_n.$$

If A is nonzero then at least one a_n is nonzero, and hence $\phi(A)$ has rank k. Therefore, ϕ is rank-k preserving linear map weakly continuous on norm bounded sets, but is not of one of the forms described in the above theorem.

In the case of corank-k preserving maps we shall need stronger assumptions than in the case of rank-k preserving maps. Namely, we shall get our result under the stronger assumptions of bijectivity and preserving corank k in both directions.

Theorem 2.3. *Let k be a positive integer, and H be an infinite-dimensional Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a bijective linear map weakly continuous on norm bounded sets which preserves corank-k operators in both directions. Then there exist invertible operators $A, B \in B(H)$ such that $\phi(T) = ATB$ for all $T \in B(H)$.*

We have already mentioned that many linear preserver problems were solved by reducing them to rank preserver problems. Here is another example. Following Hestenes [34] we say that two operators $T, S \in B(H)$ are orthogonal ($T \perp S$), if $T^*S = TS^* = 0$. In the following theorem we characterize additive maps preserving orthogonality.

Theorem 2.4. *Let H be a Hilbert space with $\dim H > 1$, and $\mathcal{A} \subset B(H)$ an ideal. Assume that $\phi : \mathcal{A} \to \mathcal{A}$ is an additive bijection which preserves orthogonality in both directions. Then there exists a nonzero constant c and unitary or antiunitary operators $U, V \in B(H)$ such that either $\phi(T) = cUTV$ for all $T \in B(H)$, or $\phi(T) = cUT^{tr}V$ for all $T \in B(H)$.*

This result is related to Theorem 2 in [52], where additve maps preserving a stronger orthogonality relation were considered.

In our final theorem we solve an open problem raised in [73] concerning *-identities on operator ideals. Let ϕ be a bijective function (no additivity or continuity is assumed) on an operator ideal fulfilling the n-variable *-identity

$$\phi\big(\tau_1(T_1)\tau_2(T_2)\ldots\tau_n(T_n)\big) = \tau_1\big(\phi(T_1)\big)\tau_2\big(\phi(T_2)\big)\ldots\tau_n\big(\phi(T_n)\big)$$

for all T_j's, where every τ_j is fixed and is either the identity or the adjoint operation. We prove that if at least one τ_j is the adjoint operation, then ϕ is an additive triple automorphism. An additive function ψ is called a triple homomorphism if it satisfies

$$\psi(TS^*R) = \psi(T)\psi(S)^*\psi(R)$$

for all T, S, R. Consequently, we obtain the surprising result that on operator ideals, the most general n-variable *-identity is that of the triple automorphisms. We think that this result is interesting even in the case of matrix algebras.

It should be mentioned that the concept of linear triple isomorphisms plays an essential role in the theory of infinite-dimensional holomorphy as well as in the study of isometries of associative and Jordan operator algebras (see, for example, [19] and the references therein).

As for the 'additivity' part of our theorem below, we remark that the problem of additivity of *-semigroup isomorphisms between operator algebras was raised by K. Saitô and S. Sakai, and was treated in a series of papers by J. Hakeda (see [31] and the references therein). Semigroup isomorphisms of standard operator algebras were considered in [86]. For some further related results see [60, 63].

Theorem 2.5. *Let H be a Hilbert space with $\dim H > 1$, and $\mathcal{A} \subset B(H)$ an ideal. Let $n \geq 2$ be an integer, and for every $1 \leq j \leq n$ let τ_j be either the identity or the adjoint operation on $\mathcal{A}$. Suppose that $\phi : \mathcal{A} \to \mathcal{A}$ is a bijective function satisfying the identity*

$$\phi\big(\tau_1(T_1)\tau_2(T_2)\dots\tau_n(T_n)\big) = \tau_1\big(\phi(T_1)\big)\tau_2\big(\phi(T_2)\big)\dots\tau_n\big(\phi(T_n)\big)$$

for all $T_1, T_2, \dots, T_n \in \mathcal{A}$.

If there is an index j such that τ_j is the adjoint operation, then ϕ is an additive triple automorphism of $\mathcal{A}$.

If τ_j is the identity for all j, then ϕ is equal to an additive ring automorphism of $\mathcal{A}$ multiplied by an $(n-1)$th root of unity.

Note that every operator ideal is self-adjoint (closed under taking adjoints) and hence the expression $\phi\big(\tau_1(T_1)\tau_2(T_2)\dots\tau_n(T_n)\big)$ is well-defined even in the case in which all τ_j's are the adjoint operation.

2.2 Proofs

Proof of Theorem 2.1. Assume first that ϕ maps all finite rank operators in $B(H)$ into $B_{\leq k}(H)$. It is easy to see that $B_{\leq k}(H)$ is closed in the weak operator topology. Let T be any operator from $B(H)$. Then we can find a bounded net of finite rank operators weakly converging to T. It follows that $\phi(T)$ belongs to $B_{\leq k}(H)$. So, if there is an operator S of rank greater than k in the image of ϕ, then there is a finite rank operator R such that the rank of $\phi(R)$ is also greater than k. We will assume from now on that this is the case.

We shall show that ϕ is rank-1 non-increasing. First note, that if P is any projection (self-adjoint idempotent) of finite rank p, then the algebra $PB(H)P$ is isomorphic to M_p, the algebra of all $p \times p$ complex matrices. Assume that there is a rank one operator W such that $\phi(W)$ has rank greater than one. Then we can find two finite rank projections $P, Q \in B(H)$ such that $PWP = W$, $PRP = R$, the rank of $Q\phi(W)Q$ is greater than one, and the rank of $Q\phi(R)Q$ is greater than k. By enlaring P or Q, if necessary, we can assume that the ranks of P and Q are the same, say p. Composing the natural isomorphism between M_p and $PB(H)P$, as well as the one between $QB(H)Q$ and M_p, with our map ϕ in the way

$$M_p \to PB(H)P \xrightarrow{\phi} QB(H)Q \to M_p$$

we get a rank-k non-increasing linear map from M_p into itself. The image of this map obviously contains a matrix with rank greater than k, so, by the theorem of Loewy [50] it is also rank-1 non-increasing. This contradicts the fact that the rank of $\phi(W)$ is greater than one.

Hence, ϕ is rank-1 non-increasing and we can apply a result of Hou (Corollary 1.1 in [36]) to complete the proof. Two minor remarks should be added here. Namely, when characterizing rank-1 non-increasing linear maps Hou used the slightly stronger assumption that ϕ is weakly continuous on the whole $B(H)$. It is easy to see that his proof works also under our assumption of weak continuity on norm bounded sets only. The other remark is that he formulated his result for general Banach spaces. So, he had to use the adjoint operator (here, the adjoint is meant in the Banach space sense) where we use the transpose. □

Proof of Theorem 2.2. This is a direct consequence of Theorem 2.1. □

Proof of Theorem 2.3. Our proof is based on the following characterization of rank one operators. A nonzero operator $T \in B(H)$ has rank one if and only if for every corank-k operator S one of the following possibilities holds: either $\alpha T + S \in B_{-k}(H)$ for all but at most one complex number α, or $\alpha T + S \notin B_{-k}(H)$ for all nonzero complex number α.

So, assume temporarily that we have already proved this characterization. Then, clearly, ϕ preserves operators of rank one. Applying Theorem 2.2, ϕ must be either of the form $\phi(T) = ATB$, or of the form $\phi(T) = AT^{tr}B$ for some $A, B \in B(H)$. Since ϕ is bijective, the operators A and B must be invertible. If T^{tr} denotes the transpose relative to the orthonormal basis $\{e_\alpha : \alpha \in J\}$, then $T^{tr} = UT^*U$, where U is an antiunitary operator defined by

$$U\left(\sum_{\alpha \in J} \langle x, e_\alpha \rangle e_\alpha\right) = \sum_{\alpha \in J} \langle e_\alpha, x \rangle e_\alpha.$$

It is well-known that in general $\dim \operatorname{Ker} T^* = k$ does not imply $\dim \operatorname{Ker} T = k$. So, the map $T \mapsto T^{tr}$ does not preserve corank k, and hence, the second possibility cannot occur.

It remains to prove our characterization of rank one operators. Assume first that $\operatorname{rank} T = 1$ and $S \in B_{-k}(H)$. So, T is of the form $T = xy^*$ for some nonzero $x, y \in H$. Let W denote $S^{*-1}(\operatorname{span}\{y\})$. The condition $\alpha T + S \in B_{-k}(H)$ is equivalent to $\dim \operatorname{Ker}(\overline{\alpha}T^* + S^*) = k$. Since the kernel of $\overline{\alpha}T^* + S^*$ is contained in W, we have

$$\operatorname{Ker}(\overline{\alpha}T^* + S^*) = \operatorname{Ker}\left(\overline{\alpha}T^*_{|W} + S^*_{|W}\right).$$

Clearly, the restrictions $T^*_{|W}$ and $S^*_{|W}$ can be considered as linear maps from W to the linear span of y. Therefore, they can be represented as row matrices.

We have to distinguish two cases. Assume first that y belongs to the image of S^*. Then it follows from $S \in B_{-k}(H)$ that $\dim W = k+1$. So, the restrictions $T^*_{|W}$ and $S^*_{|W}$ have matrix representations $[t_1, \ldots, t_{k+1}]$ and $[s_1, \ldots, s_{k+1}] \neq 0$, respectively. The operator $\alpha T^* + S^*$ does not belong to $B_{-k}(H)$ if and only if $\alpha t_1 + s_1 = \ldots = \alpha t_{k+1} + s_{k+1} = 0$. This can happen for at most one complex number α.

In the remaining case in which y does not belong to the image of S^*, we have $\dim W = k$ and $S^*_{|W} = 0$. In the case in which $T^*_{|W} = 0$ we have $\alpha T + S \in B_{-k}(H)$ for every complex number α, while $T^*_{|W} \neq 0$ implies that $\alpha T + S \notin B_{-k}(H)$ for every nonzero α.

To prove the converse assume that $\operatorname{rank} T > 1$. Once again we consider two cases. First, assume that $\dim \operatorname{Ker} T^* \geq k-1$. Then we can find invertible operators $P, Q \in B(H)$ such that T^* has the following matrix representation

$$PT^*Q = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix},$$

where T_1 is a $(k+1) \times (k+1)$-diagonal matrix $T_1 = \operatorname{Diag}(1, 1, 0, 0, \ldots, 0)$. Define $S \in B_{-k}(H)$ by

$$PS^*Q = \begin{bmatrix} S_1 & 0 \\ 0 & \delta I \end{bmatrix},$$

where S_1 is a $(k+1) \times (k+1)$-diagonal matrix $S_1 = \operatorname{Diag}(1, 0, 0, \ldots, 0)$ and $|\delta| > \|T_3\|$. It is easy to see that $-T + S$ and S belong to $B_{-k}(H)$, while $\alpha T + S \notin B_{-k}(H)$ for every real $\alpha \in (0, 1)$.

Now, if $\dim \operatorname{Ker} T^* < k-1$, then we can find invertible operators $P, Q \in B(H)$ such that T^* has the following matrix representation

$$PT^*Q = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix},$$

where T_1 is a $2k \times 2k$ identity matrix. Define $S \in B_{-k}(H)$ by

$$PS^*Q = \begin{bmatrix} S_1 & 0 \\ 0 & \delta I \end{bmatrix},$$

where S_1 is a $2k \times 2k$-diagonal matrix having first k diagonal elements equal to zero and the rest of them equal to one. Let $|\delta| > \|T_3\|$ be just as in the previous case. It is easy to see that $-T+S$ and S belong to $B_{-k}(H)$, while $\alpha T + S \notin B_{-k}(H)$ for every real $\alpha \in (0,1)$. This completes the proof. □

Proof of Theorem 2.4. Without further mentioning we will use the fact that every nontrivial ideal contains $F(H)$.

We assert that ϕ is a rank-1 preserving map. To this end take any T of rank one and denote $S = \phi(T)$. Assume on the contrary that S has rank greater than one. If the spectrum of $|S|$ consists of one point only, then $|S|$ is a scalar multiple of the identity, and consequently, it can be written as the orthogonal sum of two nonzero positive operators $R_1, R_2 \in \mathcal{A}$. Using spectral theory we can decompose $|S|$ into the sum of two nonzero positive operators $R_1, R_2 \in \mathcal{A}$ also when the spectrum of $|S|$ is not a singleton. Note, that because of the ideal structure of $\mathcal{A}$, R_1, R_2 can be chosen to belong to $\mathcal{A}$. Now, let $R_3 = UR_1$ and $R_4 = UR_2$, where U is the partial isometry in the polar decomposition of S. Then, clearly $S = R_3 + R_4$ with $R_3, R_4 \in \mathcal{A}$ being orthogonal. Hence, $T = \phi^{-1}(S)$ is a sum of two orthogonal nonzero operators $\phi^{-1}(R_3)$ and $\phi^{-1}(R_4)$. Because of the orthogonality these two operators have orthogonal images and orthogonal kernels. Consequently, T has rank greater than one. This contradiction shows that ϕ preservers rank one operators. As ϕ preserves orthogonality in both directions it must preserve also rank one operators in both directions. Hence, the restriction of ϕ to $F(H)$ is a bijective additive map of $F(H)$ onto itself preserving operators of rank one in both directions. It follows from Theorem 3.3 in [79] that there exsist a ring automorphism h of the complex field and bijective h-quasilinear operators $A, B : H \to H$ such that either $\phi(xy^*) = (Ax)(By)^*$ for all $x, y \in H$ or $\phi(xy^*) = (Ay)(Bx)^*$ for all $x, y \in H$. Suppose that ϕ is of the first form above. We claim that h is either the identity or the conjugation. To see this, it is enough to show that h is a real-valued function on the real numbers. If r is a real number, then consider the rank-one operators $T = (e + rf)(e + 2f)^*$ and $S = (-2re + 2f)(2e - f)^*$, where $e, f \in H$ are orthogonal unit vectors. It is trivial to check that T and S are orthogonal. Consequenlty, $\phi(T)^*\phi(S) = 0$. Using the easy fact the A, B maps orthogonal vectors into orthogonal vectors, we obtain $-2h(r) + 2\overline{h(r)} = 0$. Without serious loss of generality, we may suppose that A, B are linear.

Since $\langle x, y\rangle = 0$ if and only if $\langle Ax, Ay\rangle = 0$, by the linearity of A we have $\langle Ax, Ay\rangle = c\langle x, y\rangle$, $x, y \in H$, for some complex constant c. So, both A and B are scalar multiples of unitary operators U and V^*. It follows that we have $\phi(T) = \lambda UTV$ for every finite rank operator T.

After multiplying ϕ from both sides by appropriate operators we can assume with no loss of generality that $\phi(T) = T$ for every finite rank operator T. It remains to show that this holds for every $T \in \mathcal{A}$, too. As ϕ preserves orthogonality in both directions, we have $A \perp T + B$ if and only if $A \perp \phi(T) + B$ for every finite rank operators A and B. Take any $x \in H$. Let P be the projection onto the subspace generated by the vectors x, Tx, T^*x. Define $B = -PTP$. It is obvious that $B \in F(H)$ and $Bx = -Tx$, $B^*x = -T^*x$. Let $A = xx^*$. Plainly, A is orthogonal to $T + B$, and consequently, it must be orthogonal to $\phi(T) + B$, which yields $Bx = -\phi(T)x$. As a consequence we have $\phi(T)x = Tx$. This completes the proof in the case which we have considered.

The remaining cases can be treated in a similar way. □

Proof of Theorem 2.5. We first show that ϕ is additive. We use an argument similar to that in [51]. Let $T \in \mathcal{A}$ such that $\phi(T) = 0$. Then we have

$$\phi(0) = \phi\big(\tau_1(T)\tau_2(0)\dots\tau_n(0)\big) = \tau_1\big(\phi(T)\big)\tau_2\big(\phi(0)\big)\dots\tau_n\big(\phi(0)\big) = 0.$$

It is not hard to see that we can assume that $n \geq 3$. In fact, if our equation is of the form

$$\phi\big(\tau_1(T)\tau_2(S)\big) = \tau_1\big(\phi(T)\big)\tau_2\big(\phi(S)\big),$$

then write $T = ZW$ in the above expression and compute to get an equality in three variables. For examply, if our equation is

$$\phi(TS^*) = \phi(T)\phi(S)^*,$$

then, using the substitution $T = ZW$, we have

$$\phi(ZWS^*) = \phi(ZW)\psi(S)^* = \phi(Z)\phi(W^*)^*\phi(S)^*,$$

which can be rewritten as

$$\phi(ZW^*S^*) = \phi(Z)\phi(W)^*\phi(S)^*.$$

So, let $n \geq 3$. We next assert that we can suppose that there is an index $1 < i < n$ for which $\tau_i(T) = T$ $(T \in \mathcal{A})$. Indeed, if for every $i = 1, \dots, n$ we have $\tau_i(T) = T^*$, then, replacing T_1 by $Z_1 \dots Z_n$ in our equation, we arrive at

$$\begin{aligned}\phi\big((Z_1\dots Z_n)^*T_2^*\dots T_n^*\big) &= \phi(Z_1\dots Z_n)^*\phi(T_2)^*\dots\phi(T_n)^* \\ &= \big(\phi(Z_1^*)^*\dots\phi(Z_n^*)^*\big)^*\phi(T_2)^*\dots\phi(T_n)^* = \phi(Z_n^*)\dots\phi(Z_1^*)\phi(T_2)^*\dots\phi(T_n)^*\end{aligned}$$

which yields

$$\phi(Z_n\dots Z_1T_2^*\dots T_n^*) = \phi(Z_n)\dots\phi(Z_1)\phi(T_2)^*\dots\phi(T_n)^*$$

and this fulfills our requirements. One can follow the same argument if $\tau_1(T) = \dots = \tau_{n-1}(T) = T^*$, $\tau_n(T) = T$ or $\tau_1(T) = T$, $\tau_2(T) = \dots = \tau_{n-1}(T) = T^*$, $\tau_n(T) = T$. Finally, we replace T_n by $Z_1\dots Z_n$ if $\tau_1(T) = T$ and $\tau_2(T) = \dots = \tau_n(T) = T^*$. From now on we assume that $n \geq 3$ and there is an index $1 < i < n$ for which $\tau_i(T) = T$.

Let $T, S \in \mathcal{A}$ be fixed and E be a finite-rank projection. Pick arbitrary projections $P, Q \in \mathcal{A}$ of finite-rank. Since ϕ is a bijection, there is a unique $A \in \mathcal{A}$ for which $\phi(A) = \phi(TE) + \phi\big(S(I-E)\big)$. We obtain

$$\begin{aligned}
\phi(P\dots PAQ\dots Q) =& \tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi(A)\\
&\cdot\tau_{i+1}\big(\phi(\tau_{i+1}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)\\
=& \tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi(TE)\\
&\cdot\tau_{i+1}\big(\phi(\tau_{i+1}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)\\
&+\tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi\big(S(I-E)\big)\\
&\cdot\tau_{i+1}\big(\phi(\tau_{i+1}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)\\
=& \phi\big(P\dots P(TE)Q\dots Q\big) + \phi\big(P\dots P(S(I-E))Q\dots Q\big).
\end{aligned}$$

This implies that $PAQ = P(TE)Q + PS(I-E)Q$ if $Q = E$ or if $Q \perp E$. Since P was arbitrary, these result in $A = TE + S(I-E)$ and we have

$$\phi\big(TE + S(I-E)\big) = \phi(TE) + \phi\big(S(I-E)\big).$$

One can similarly verify that

$$\phi\big(ET + (I-E)S\big) = \phi(ET) + \phi\big((I-E)S\big).$$

Now, let $A \in \mathcal{A}$ be such that $\phi(A) = \phi\big(ET(I-E)\big) + \phi\big(ES(I-E)\big)$. Let $\tilde{T} = ET(I-E)$ and $\tilde{S} = ES(I-E)$. We compute

$$\begin{aligned}
&\phi(P\dots PAQ\dots Q)\\
&\quad= \tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi(A)\\
&\qquad\cdot\tau_{i+1}\big(\phi(\tau_{i+1}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)\\
&\quad= \tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi\big(ET(I-E)\big)\\
&\qquad\cdot\tau_{i+1}\big(\phi(\tau_{i+1}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)\\
&\qquad+\tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi\big(ES(I-E)\big)\\
&\qquad\cdot\tau_{i+1}\big(\phi(\tau_{i+1}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)\\
&\quad= 0 + \phi\big(P(ET(I-E))Q\big) + \phi\big(P(ES(I-E))Q\big)\\
&\quad= \tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi(E)\\
&\qquad\cdot\tau_{i+1}\big(\phi(\tau_{i+1}((I-E)Q))\big)\tau_{i+2}\big(\phi(\tau_{i+2}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)\\
&\qquad+\tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi(\tilde{T})\cdot\\
&\qquad\cdot\tau_{i+1}\big(\phi(\tau_{i+1}((I-E)Q))\big)\tau_{i+2}\big(\phi(\tau_{i+2}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)\\
&\qquad+\tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi(E+\tilde{T})\cdot\\
&\qquad\cdot\tau_{i+1}\big(\phi(\tau_{i+1}(\tilde{S}Q))\big)\tau_{i+2}\big(\phi(\tau_{i+2}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big) = (*).
\end{aligned}$$

Since by what we have proved above we know $\phi(E)+\phi(\tilde{T})=\phi(E+\tilde{T})$, hence we have

$$\begin{aligned}(*) =&\tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi\big(E+\tilde{T}\big)\\ &\cdot\tau_{i+1}\big(\phi(\tau_{i+1}((I-E)Q))\big)\tau_{i+2}\big(\phi(\tau_{i+2}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)\\ &+\tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi\big(E+\tilde{T}\big)\\ &\cdot\tau_{i+1}\big(\phi(\tau_{i+1}(\tilde{S}Q))\big)\tau_{i+2}\big(\phi(\tau_{i+2}(Q))\big)\dots\tau_n\big(\phi(\tau_n(Q))\big)=(**).\end{aligned}$$

We also know that $\phi\big(\tau_{i+1}((I-E)Q)\big)+\phi\big(\tau_{i+1}(\tilde{S}Q)\big)=\phi\big(\tau_{i+1}((I-E)Q+\tilde{S}Q)\big)$, hence we can continue

$$\begin{aligned}(**) =&\tau_1\big(\phi(\tau_1(P))\big)\dots\tau_{i-1}\big(\phi(\tau_{i-1}(P))\big)\phi\big(E+\tilde{T}\big)\\ &\cdot\tau_{i+1}\Big(\phi\big(\tau_{i+1}((I-E)Q+\tilde{S}Q)\big)\Big)\tau_{i+2}\Big(\phi\big(\tau_{i+2}(Q)\big)\Big)\dots\tau_n\Big(\phi\big(\tau_n(Q)\big)\Big)\\ =&\phi\Big(P\dots P(E+\tilde{T})\big((I-E)Q+\tilde{S}Q\big)Q\dots Q\Big)\\ =&\phi\big(P(\tilde{T}+\tilde{S})Q\big).\end{aligned}$$

By the injectivity of ϕ it follows that $PAQ=P(\tilde{T}+\tilde{S})Q$ for every projection $P,Q\in F(H)$. Plainly, this implies that $A=\tilde{T}+\tilde{S}$ which yields $\phi\big(ET(I-E)+ES(I-E)\big)=\phi\big(ET(I-E)\big)+\phi\big(ES(I-E)\big)$.

Now, let $A\in\mathcal{A}$ be such that $\phi(A)=\phi(ETE)+\phi(ESE)$. Moreover, let $\tilde{T}=ETE$ and $\tilde{S}=ESE$. Suppose that τ_1 is the identity on $\mathcal{A}$. Then we have

$$\begin{aligned}\phi\big(AQ(I-E)\big) =&\phi(A)\tau_2\big(\phi(\tau_2(Q))\big)\dots\tau_{n-1}\big(\phi(\tau_{n-1}(Q))\big)\tau_n\big(\phi(\tau_n(Q(I-E)))\big)\\ =&\phi(\tilde{T})\tau_2\big(\phi(\tau_2(Q))\big)\dots\tau_{n-1}\big(\phi(\tau_{n-1}(Q))\big)\tau_n\big(\phi(\tau_n(Q(I-E)))\big)\\ &+\phi(\tilde{S})\tau_2\big(\phi(\tau_2(Q))\big)\dots\tau_{n-1}\big(\phi(\tau_{n-1}(Q))\big)\tau_n\big(\phi(\tau_n(Q(I-E)))\big)\\ =&\phi\big(\tilde{T}Q(I-E)\big)+\phi\big(\tilde{S}Q(I-E)\big)=(***).\end{aligned}$$

But from the previous step we obtain

$$\phi\big(\tilde{T}Q(I-E)\big)+\phi\big(\tilde{S}Q(I-E)\big)=\phi\big(\tilde{T}Q(I-E)+\tilde{S}Q(I-E)\big),$$

and this implies

$$(***)=\phi\big(\tilde{T}Q(I-E)+\tilde{S}Q(I-E)\big)=\phi\big((\tilde{T}+\tilde{S})Q(I-E)\big).$$

By the injectivity of ϕ we have $AQ(I-E)=(\tilde{T}+\tilde{S})Q(I-E)$ for every Q. It is not hard to see that this gives $A=\tilde{T}+\tilde{S}$. If τ_1 is the adjoint operation, then one can argue in a similar way.

To prove the additivity, finally let $A\in\mathcal{A}$ be such that $\phi(A)=\phi(T)+\phi(S)$. Just as above we obtain easily $\phi(PAP)=\phi(PTP)+\phi(PSP)$. By what we already know, it follows $\phi(PTP) + \phi(PSP) = \phi(PTP + PSP) = \phi\big(P(T + S)P\big)$. Consequently, $\phi(PAP) = \phi\big(P(T + S)P\big)$ and this implies $PAP = P(T + S)P$ for every finite rank projection P. Therefore, $A = T + S$ and this gives us $\phi(A + B) = \phi(A) + \phi(B)$.

Suppose that there exists an index i such that $\tau_i(T) = T^*$ for all $T \in \mathcal{A}$. We show that in this case ϕ preserves orthogonality in both directions. If there are indices j, k such that

$$\tau_j(T) = T,\ \tau_{j+1}(T) = T^* \quad \text{and} \quad \tau_k(T) = T^*,\ \tau_{k+1}(T) = T,$$

this follows from our basic equation and the bijectivity of ϕ. If this is not the case, then there are indices j, k such that either

$$\tau_j(T) = T,\ \tau_{j+1}(T) = T^* \quad \text{or} \quad \tau_k(T) = T^*,\ \tau_{k+1}(T) = T$$

(see our assumption and its justification in the beginning of the proof). Without serious loss of generality we can assume that our *-identity is in the form

$$\phi(T_1 \dots T_j S_1^* \dots S_k^*) = \phi(T_1) \dots \phi(T_j) \phi(S_1)^* \dots \phi(S_k)^*$$

where $k + j = n$. We compute

$$\begin{aligned}
&\phi\big(T_1 \dots T_j S_1^* \dots S_{k-1}^*(Z_1 \dots Z_k W_1^* \dots W_j^*)\big) \\
&\qquad = \phi(T_1) \dots \phi(T_j)\phi(S_1)^* \dots \phi(S_{k-1})^* \phi\big((Z_1 \dots Z_k W_1^* \dots W_j^*)^*\big)^* \\
&\qquad = \phi(T_1) \dots \phi(T_j)\phi(S_1)^* \dots \phi(S_{k-1})^* \phi(W_j \dots W_1 Z_k^* \dots Z_1^*)^* \\
&\qquad = \phi(T_1) \dots \phi(T_j)\phi(S_1)^* \dots \phi(S_{k-1})^* \phi(Z_1) \dots \phi(Z_k)\phi(W_1)^* \dots \phi(W_j)^*.
\end{aligned}$$

Because of the form of this equality, we obtain the orthogonality preserving property of ϕ.

Now, apply Theorem 2.4 to have a form of ϕ. Since it satisfies a *-identity, one can check easily that $|c| = 1$. This gives the assertion.

If every τ_j $(1 \le j \le n)$ is the identity, then we can apply the main result in [10] on surjective n-Jordan homomorphisms of prime rings. In our particular case this says that there is an $(n-1)$th root of identity λ and a ring automorphism ψ of $\mathcal{A}$ such that $\phi = \lambda\psi$.

The proof of Theorem 2.5 is complete. □

2.3 Remarks

Theorem 2.1 gives an almost complete characterization of rank-k non-increasing linear maps. To get the complete understanding of the structure of such maps, one must characterize maximal linear subspaces of $B(H)$ consisting of operators of rank not greater than k. This problem seems to be difficult even in the finite-dimensional case and to be of interest in algebra in general [2, 3, 4, 20, 31].

As for the remaining case of linear maps preserving infinite rank as well as infinite corank we define a linear map $\phi : B(H) \to B(H)$ by $\phi(A) = A + \psi(A)$, where ψ is any linear map from $B(H)$ into $F(H)$ with norm strictly less than 1. Then ϕ is bijective and obviously preserves operators of infinite rank and infinite corank in both directions. This example shows that the set of such maps is much larger than the set of rank-k preservers. With regard to the preservation of certain important classes of operators of infinite rank and infinite corank, we refer to Molnárs's results [58].

There are several possibilities how to define corank. Our definition, i.e. corank $A = k$ if and only if $\dim(\text{rng } A)^\perp = \dim \text{Ker } A^* = k$, corresponds to column rank for matrices. Another possible definition, that is, corank $A = k$ if and only if $\dim(\text{rng } A^*)^\perp = \dim \text{Ker } A = k$ corresponds to row rank for matrices. These two definitions do not coincide in the infinite dimensional case. So, we have also the third possibility that corank A is equal to k if and only if $\dim \text{Ker } A = \dim \text{Ker } A^* = k$. Among all three definitions only the last one has the property that corank $A = k$ if and only if corank $A^* = k$. So, it is not surprising that the analogue of Theorem 2.3 corresponding to this definition reads as follows: Let k be a positive integer, and H be an infinite-dimensional Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a bijective linear map weakly continuous on norm bounded sets satisfying $\dim \text{Ker } A = \dim \text{Ker } A^* = k$ if and only if $\dim \text{Ker } \phi(A) = \dim \text{Ker } \big(\phi(A)\big)^* = k$. Then there exist invertible operators $A, B \in B(H)$ such that either $\phi(T) = ATB$ for all $T \in B(H)$, or $\phi(T) = AT^{tr}B$ for all $T \in B(H)$. The proof of this statement is similar to the proof of Theorem 2.3. It is based on the following characterization of rank one operators among all nonzero operators from $B(H)$: a nonzero $T \in B(H)$ has rank one if and only if for every $S \in B'_{-k}(H)$ we have either $\alpha T + S \in B'_{-k}(H)$ for all complex α but at most two, or $\alpha T + S \notin B'_{-k}(H)$ for all nonzero complex α. Here, of course, $B'_{-k}(H)$ stands for the set of all operators from $B(H)$ of corank k with respect to our last definition. As the idea of the proof is almost the same as in the proof of Theorem 2.3, we omit the details.

3. DIAMETER PRESERVING BIJECTIONS OF $C_0(X)$

3.1 Introduction and Statement of the Results

As was mentioned in the Introduction, linear preserver problems can be raised not only on matrix algebras or on operator algebras, but also on *arbitrary algebras*. In this chapter we consider LPPs concerning *function algebras*, in which case the main LPPs considered before have been the characterizations of linear bijections preserving some given norm, or preserving disjointness of the support (these latter maps are also called separating).

Let X be a locally compact Hausdorff space and let $C_0(X)$ denote the algebra of all continuous complex valued functions on X which vanish at infinity. One way of measuring a function $f \in C_0(X)$ is to consider its sup-norm. The linear bijections of $C_0(X)$ preserving the sup-norm are determined in the famous Banach-Stone theorem, and we mention [1, 21, 33, 38, 93, 94] as some of the important and relatively recent papers on similar problems. Besides the sup-norm, another natural way to measure the function in question is by some data which reflect how large its range is. For example, this can be done by considering the diameter of the range. In this chapter we are going to characterize all the linear bijections of $C_0(X)$ which preserve the seminorm $f \mapsto \operatorname{diam}\big(f(X)\big)$. For brevity, we call these maps diameter preserving. Here we unify the contents of our papers [23] and [28] and present a common proof for all the results which appeared there. Under a mild condition on the underlying space X, we completely describe all bijective diameter preserving linear maps on $C_0(X)$.

The main result of this chapter is presented in Theorem 3.1 below. To state the theorem we need to introduce some notation.

For the locally compact Hausdorff space X with topology Λ, let X_0 denote $X \cup \{\infty\}$ if X is not compact, and X if X is compact. Then X_0 endowed with the topology

$$\Lambda_0 = \Lambda \cup \big\{ X_0 \setminus K \mid K \subseteq X \text{ is compact} \big\}$$

is a compact Hausdorff space, and X is a subspace of X_0.

The main result of this chapter is stated in the following theorem.

Theorem 3.1. *Let X be a first countable locally compact Hausdorff space.*

If X is compact, then a bijective linear map $\phi : C_0(X) \to C_0(X)$ is diameter preserving if and only if there exists a complex number τ of modulus 1, a homeomorphism

$\varphi : X \to X$ and a linear functional $t : C_0(X) \to \mathbb{C}$ with $t(1) \neq -\tau$ such that ϕ is of the form

$$\phi(f) = \tau \cdot f \circ \varphi + t(f)1 \qquad \big(f \in C_0(X)\big). \tag{3.1}$$

If X is not σ-compact, then a bijective linear map $\phi : C_0(X) \to C_0(X)$ is diameter preserving if and only if there exists a complex number τ of modulus 1 and a homeomorphism $\varphi : X \to X$ such that ϕ is of the form

$$\phi(f) = \tau \cdot f \circ \varphi \qquad \big(f \in C_0(X)\big). \tag{3.2}$$

If the space X is σ-compact but not compact, then a bijective linear map $\phi : C_0(X) \to C_0(X)$ is diameter preserving if and only if there exists a complex number τ of modulus 1 and a homeomorphism $\varphi : X_0 \to X_0$ such that ϕ is of the form

$$\phi(f) = \tau \cdot f \circ \varphi - \tau f\big(\varphi(\infty)\big)1 \qquad \big(f \in C_0(X)\big), \tag{3.3}$$

where $f(\infty) = 0$ for every $f \in C_0(X)$.

Remark 3.2. We make some remarks concerning our Theorem 3.1. If we have $\varphi(\infty) = \infty$ in (3.3), then ϕ is of the same form as in (3.2). If ϕ is of the form (3.2), then it is obviously a surjective isometry. Theorem 3.1 also holds for the algebra of all continuous real valued functions on X. In this situation we have $\tau = \pm 1$ and in (3.1) $t : C_0(X) \to \mathbb{R}$. In this case the proof is more simple and we omit it.

3.2 Proofs

Proof of Theorem 3.1. It is easy to verify that under the assumptions of Theorem 3.1, a linear map ϕ of the form (3.1), (3.2) and (3.3), respectively, is a diameter preserving linear bijection of $C_0(X)$.

Now suppose that $\phi : C_0(X) \to C_0(X)$ is a linear bijection which preserves the diameter of the ranges of functions in $C_0(X)$.

Because of the natural isomorphism, we shall consider $C_0(X)$ to be subalgebra of $C(X_0)$, defining every function $f \in C_0(X)$ at the point ∞ as $f(\infty) = 0$. We note that $\operatorname{diam}\big(f(X)\big) = \operatorname{diam}\big(f(X_0)\big)$ for any $f \in C_0(X)$.

We introduce the following notation. Let $\mathcal{X}$ denote X_0 if X is σ-compact and X if X is not σ-compact. Let $\tilde{X}$, $\tilde{X_0}$ and $\tilde{\mathcal{X}}$ stand for the collection of all such subsets of X, X_0 and $\mathcal{X}$, respectively, which have exactly two elements. Further, for any $f \in C_0(X)$ let

$$S(f) = \Big\{\{x,y\} \in \tilde{X_0} \,:\, \big|f(x) - f(y)\big| = \operatorname{diam}\big(f(X)\big)\Big\},$$
$$P(f) = \Big\{(x,y) \in X_0 \times X_0 \,:\, \big|f(x) - f(y)\big| = \operatorname{diam}\big(f(X)\big)\Big\},$$

$$T(f) = \Big\{(x,y,u) \in X_0 \times X_0 \times \mathbb{C} \,:\, \big|f(x)-f(y)\big| = \operatorname{diam}\big(f(X)\big),\\ u = f(x)-f(y)\Big\}.$$

Further, for every $\{x,y\} \in \tilde{X}_0$ and $u \in \mathbb{C}$ let

$$\begin{aligned}
\mathcal{S}\big(\{x,y\}\big) &= \Big\{f \in C_0(X) \ :\ \{x,y\} \in S(f)\Big\},\\
\mathcal{S}_s\big(\{x,y\}\big) &= \Big\{f \in C_0(X) \ :\ \big\{\{x,y\}\big\} = S(f)\Big\},\\
\mathcal{T}(x,y,u) &= \Big\{f \in C_0(X) \ :\ (x,y,u) \in T(f)\Big\},\\
\mathcal{T}_s(x,y,u) &= \Big\{f \in C_0(X) \ :\ \big\{(x,y,u),(y,x,-u)\big\} = T(f)\Big\}.
\end{aligned}$$

Finally, we define

$$\begin{aligned}
G\big(\{x,y\}\big) &= \bigcap\Big\{S\big(\phi(f)\big) \ : f \in C_0(X),\, \{x,y\} \in S(f)\Big\},\\
H(x,y,u) &= \bigcap\Big\{T\big(\phi(f)\big) \ : f \in C_0(X),\, (x,y,u) \in T(f)\Big\}.
\end{aligned}$$

Let

$$\mathcal{D} = \Big\{f \in C_0(X) \ : \exists \{x,y\} \in \tilde{\mathcal{X}} :\ \big\{\{x,y\}\big\} = S(f)\Big\}.$$

It is clear that for every function $f \in C_0(X)$, the sets $S(f)$, $P(f)$ and $T(f)$ are nonempty. Since X is first countable, with the aid of Urysohn's lemma it is easy to see that for every distinct $x, y \in X$ there exists a continuous real valued function $f \in C_0(X)$ from X_0 into $[-1,1]$ such that $f(x) = 1, f(y) = -1$ and $-1 < f(z) < 1$ $(z \in X, z \neq x, z \neq y)$. If X is σ-compact, then X_0 is first countable and similarly, for every distinct $x, y \in X_0$ there exists a real valued function $f \in C_0(X)$ from X_0 into $[0,1]$ such that $f(x) = 1, f(y) = 0$ (we may assume that $x \neq \infty$) and $0 < f(z) < 1$ $(z \in X, z \neq x, z \neq y)$. This shows that for any element $\{x,y\} \in \tilde{\mathcal{X}}$ and any non-zero $u \in \mathbb{C}$, the sets $\mathcal{S}_s(\{x,y\})$, $\mathcal{T}_s(x,y,u)$ are also nonempty. It is obvious that the sets $\mathcal{S}(\{x,y\})$, $\mathcal{T}(x,y,u)$ are nonempty for any $\{x,y\} \in \tilde{X}_0$ and any non-zero $u \in \mathbb{C}$.

We begin now the proof of necessity in our statements which will be carried out through a series of steps. The following lemma will be used repeatedly in our proof.

Lemma 3.3. *Let $f_1, \ldots, f_n \in C_0(X)$ be arbitrary functions. Then*

$$\operatorname{diam}\big((f_1 + \ldots + f_n)(X)\big) = \operatorname{diam}\big(f_1(X)\big) + \ldots + \operatorname{diam}\big(f_n(X)\big) \tag{3.4}$$

holds if and only if there exists an $\{x,y\} \in \tilde{X}_0$ and a complex number v of modulus 1 such that $f_i \in \mathcal{T}(x,y,\lambda_i v)$ holds for every $i = 1, \ldots, n$, where $\lambda_i = \operatorname{diam}\big(f_i(X)\big)$ $(i = 1, \ldots, n)$.

Assume that (3.4) holds. Then there exist $\{x,y\} \in \tilde{X}_0$ and a complex number v of modulus 1 such that

$$f_1 + \ldots + f_n \in \mathcal{T}\big(x, y, (\lambda_1 + \ldots + \lambda_n)v\big).$$

Now, we compute

$$\begin{aligned}\lambda_1 + \ldots + \lambda_n &= \big|(f_1 + \ldots + f_n)(x) - (f_1 + \ldots + f_n)(y)\big| \\ &\le \big|f_1(x) - f_1(y)\big| + \ldots + \big|f_n(x) - f_n(y)\big| \le \lambda_1 + \ldots + \lambda_n.\end{aligned}$$

It readily follows that $f_i \in \mathcal{T}(x, y, \lambda_i v)$ $(i = 1, \ldots, n)$. The converse statement of Lemma 3.3 is trivial.

Step 3.1. *For arbitrary $\{x,y\} \in \tilde{X}_0$ and $0 \ne u \in \mathbb{C}$, we have $G\big(\{x,y\}\big) \ne \emptyset$ and $H(x,y,u) \ne \emptyset$.*

Let $\{x,y\} \in \tilde{X}_0$. We first show that

$$\bigcap \Big\{ P\big(\phi(f)\big) \,:\, f \in C_0(X), \{x,y\} \in S(f) \Big\} \ne \emptyset. \tag{3.5}$$

Since the collection of the sets $P(\phi(f))$ in (3.5) consists of closed subsets of the compact Hausdorff space $X_0 \times X_0$, in order to verify (3.5), it is sufficient to show that this collection has the finite intersection property. Accordingly, let $f_1, \ldots, f_n \in C_0(X)$ be such that $\{x,y\} \in S(f_1), \ldots, S(f_n)$. Define $u_i = f_i(x) - f_i(y)$ $(i = 1, \ldots, n)$. Then there exist complex numbers μ_i with $|\mu_i| = 1$ for which $\mu_i u_i \ge 0$. Since the diameter of the range of $\mu_i f_i$ is $\mu_i u_i$ and

$$\begin{aligned}&\big|(\mu_1 f_1 + \ldots + \mu_n f_n)(x) - (\mu_1 f_1 + \ldots + \mu_n f_n)(y)\big| \\ &\qquad = \big|\mu_1\big(f_1(x) - f_1(y)\big) + \ldots + \mu_n\big(f_n(x) - f_n(y)\big)\big| \\ &\qquad = |\mu_1 u_1 + \ldots + \mu_n u_n| = \mu_1 u_1 + \ldots + \mu_n u_n = |u_1| + \ldots + |u_n|,\end{aligned}$$

we deduce that the diameter of the range of $\mu_1 f_1 + \ldots + \mu_n f_n$ is $|u_1| + \ldots + |u_n|$. From the diameter preserving property of ϕ it follows that the diameter of the range of $\phi(\mu_1 f_1 + \ldots + \mu_n f_n)$ is $|u_1| + \ldots + |u_n|$ which equals the sum of the diameters of the ranges of $\phi(\mu_1 f_1), \ldots, \phi(\mu_n f_n)$. By Lemma 3.3 we conclude that there exist $\{x_0, y_0\} \in \tilde{X}_0$ and a complex number v of modulus 1 for which $\phi(\mu_i f_i) \in \mathcal{T}(x_0, y_0, |u_i| v)$ $(i = 1, \ldots, n)$. Obviously, we have $(x_0, y_0) \in P(\phi(\mu_i f_i)) = P(\phi(f_i))$. This shows the desired finite intersection property; hence we have (3.5). Since there is a nonconstant function in $\mathcal{S}(\{x,y\})$, every element of the intersection appearing in (3.5) has distinct coordinates. This implies that $G(\{x,y\}) \ne \emptyset$.

We now prove the remaining assertion

$$H(x,y,u) = \bigcap \Big\{ T\big(\phi(f)\big) \,:\, f \in C_0(X),\ (x,y,u) \in T(f) \Big\} \ne \emptyset. \tag{3.6}$$

It is easy to see that for any $f \in C_0(X)$ the set $T\big(\phi(f)\big)$ is a compact subset of $X_0 \times X_0 \times \mathbb{C}$. Therefore, just as above, in order to verify (3.6), it is sufficient to check that the system $\big\{T\big(\phi(f)\big) : f \in C_0(X),\, (x, y, u) \in T(f)\big\}$ has the finite intersection property. Let $f_1, \dots, f_n \in C_0(X)$ be such that $(x, y, u) \in T(f_1), \dots, T(f_n)$. We evidently have $f_1 + \dots + f_n \in \mathcal{T}(x, y, nu)$ and hence it follows that $\operatorname{diam}\big((f_1 + \dots + f_n)(X_0)\big) = n|u|$. From the diameter preserving property of ϕ we deduce that

$$\operatorname{diam}\big(\phi(f_1 + \dots + f_n)(X_0)\big) = \operatorname{diam}\big(\phi(f_1)(X_0)\big) + \dots + \operatorname{diam}\big(\phi(f_n)(X_0)\big).$$

By Lemma 3.3, there exist $\{x_0, y_0\} \in \tilde{X}_0$ and a complex number v of modulus 1 such that $\phi(f_i) \in \mathcal{T}(x_0, y_0, |u|v)$ $(i = 1, \dots, n)$. Plainly, this can be reformulated as $(x_0, y_0, |u|v) \in \cap_{i=1}^n T(\phi(f_i))$, thus verifying the claimed finite intersection property. Hence, we obtain (3.6).

Step 3.2. *If* $\{x_1, y_1\}, \{x_2, y_2\} \in \tilde{X}_0$ *and* $\{x_1, y_1\} \neq \{x_2, y_2\}$*, then we have*

$$G\big(\{x_1, y_1\}\big) \cap G\big(\{x_2, y_2\}\big) = \emptyset.$$

Suppose on the contrary that $G(\{x_1, y_1\}) \cap G(\{x_2, y_2\}) \neq \emptyset$. Clearly, we may assume that $x_1 \neq x_2$. Let $\{x, y\} \in G(\{x_1, y_1\}) \cap G(\{x_2, y_2\})$. By Urysohn's lemma there are functions $f_1 \in \mathcal{T}(x_1, y_1, 1)$ and $f_2 \in \mathcal{T}(x_2, y_2, 1)$ with disjoint supports and with ranges in $[0, 1]$. Let

$$u_1 = \phi(f_1)(x) - \phi(f_1)(y) \quad \text{and} \quad u_2 = \phi(f_2)(x) - \phi(f_2)(y).$$

Then, by the definition of G, we have $|u_1| = |u_2| = 1$ and $(x, y, u_1) \in T(\phi(f_1))$, $(x, y, u_2) \in T(\phi(f_2))$. Let $t \in [-\pi/3, \pi/3]$ be arbitrary and set $\mu_t = e^{it}$. This yields $f_1 + \mu_t f_2 \in \mathcal{T}(x_1, y_1, 1)$ and hence $\{x, y\} \in S\big(\phi(f_1 + \mu_t f_2)\big)$. We compute

$$\begin{aligned} |u_1 + \mu_t u_2| &= \Big|\big(\phi(f_1)(x) - \phi(f_1)(y)\big) + \big(\phi(\mu_t f_2)(x) - \phi(\mu_t f_2)(y)\big)\Big| \\ &= \big|\phi(f_1 + \mu_t f_2)(x) - \phi(f_1 + \mu_t f_2)(y)\big| = 1. \end{aligned} \tag{3.7}$$

Since $|u_1| = |u_2| = 1$ and (3.7) holds for every $t \in [-\pi/3, \pi/3]$, we arrive easily at a contradiction.

Step 3.3. *We have* $f \in \mathcal{D}$ *if and only if* $\phi(f) \in \mathcal{D}$.

Let $f \in \mathcal{D}$. Then there exists $\{x, y\} \in \tilde{\mathcal{X}}$ such that $f \in \mathcal{S}_s(\{x, y\})$. Let $f_0 = \phi^{-1}(f)$ and let $\{x_0, y_0\} \in S(f_0)$ be arbitrary. Then

$$\emptyset \neq G\big(\{x_0, y_0\}\big) \subseteq S\big(\phi(f_0)\big) = S(f) = \big\{\{x, y\}\big\},$$

and so

$$G\big(\{x_0, y_0\}\big) = \big\{\{x, y\}\big\}.$$

The arbitrary choice of $\{x_0, y_0\} \in S(f_0)$ and Step 3.2 imply that $S(f_0)$ has exactly one element, thus $\phi^{-1}(f) \in \mathcal{D}$. Applying this result to the diameter preserving bijection ϕ^{-1} instead of ϕ, the proof of Step 3.3 is complete.

Step 3.4. *For every $\{x,y\} \in \tilde{\mathcal{X}}$, the set $G\big(\{x,y\}\big)$ has exactly one element which is contained in $\tilde{\mathcal{X}}$. The function $G' : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ defined by $\big\{G'(\{x,y\})\big\} = G\big(\{x,y\}\big)$ is a bijection.*

Let $\{x,y\} \in \tilde{\mathcal{X}}$ and $f \in \mathcal{S}_s(\{x,y\})$. Since $f \in \mathcal{D}$, by Step 3.3 we have $\phi(f) \in \mathcal{D}$, thus $S(\phi(f))$ has exactly one element which is in $\tilde{\mathcal{X}}$. Hence from

$$\emptyset \neq G\big(\{x,y\}\big) \subseteq S\big(\phi(f)\big)$$

we deduce that $G(\{x,y\})$ has also exactly one element which is contained in $\tilde{\mathcal{X}}$.

We now prove that the function G' is bijective. In view of Step 3.2, injectivity is obvious. To prove surjectivity, let $\{x,y\} \in \tilde{\mathcal{X}}$ be arbitrary and pick $f \in C_0(X)$ for which $\phi(f) \in \mathcal{S}_s(\{x,y\})$. Then $\phi(f) \in \mathcal{D}$, so by Step 3.3 we infer that $f \in \mathcal{D}$. Thus there exists $\{x_0,y_0\} \in \tilde{\mathcal{X}}$ for which $S(f) = \{\{x_0,y_0\}\}$. Hence we have $G'(\{x_0,y_0\}) \in S(\phi(f)) = \{\{x,y\}\}$, thus $G'(\{x_0,y_0\}) = \{x,y\}$ verifying our claim.

Step 3.5. *Let $\{x,y\} \in \tilde{\mathcal{X}}$ and $f \in C_0(X)$ be arbitrary. If $\phi(f) \in \mathcal{S}_s\big(G'(\{x,y\})\big)$, then $f \in \mathcal{S}_s\big(\{x,y\}\big)$.*

If $\phi(f) \in \mathcal{S}_s\big(G'(\{x,y\})\big)$ and $\{x_0,y_0\} \in S(f)$ then $G'(\{x_0,y_0\}) \in S(\phi(f)) = \{G'(\{x,y\})\}$, thence Step 3.4 implies $\{x_0,y_0\} = \{x,y\}$. Thus $S(f) = \{\{x,y\}\}$.

Step 3.6. *Define the function G'_{-1} corresponding to ϕ^{-1} in the same way as G' corresponding to ϕ was defined in Step 3.4. Then we have $G'_{-1} = (G')^{-1}$.*

Pick an $\{x,y\} \in \tilde{\mathcal{X}}$. Let $G'_{-1}(\{x,y\}) = \{a,b\}$ and $G'(\{a,b\}) = \{x',y'\}$. Applying Step 3.5 to ϕ and ϕ^{-1}, we find that for a function $f \in C_0(X)$ with $\phi(f) \in \mathcal{S}_s(\{x',y'\})$ we have $\phi^{-1}(\phi(f)) = f \in \mathcal{S}_s(\{a,b\}) = \mathcal{S}_s(G'_{-1}(\{x,y\}))$ and then $\phi(f) \in \mathcal{S}_s(\{x,y\})$. Consequently, $\{x,y\} = \{x',y'\} = G'(G'_{-1}(\{x,y\}))$. The assertion is now obvious.

Step 3.7. *If $\{x_1,y_1\},\{x_2,y_2\} \in \tilde{\mathcal{X}}$ and $\{x_1,y_1\} \cap \{x_2,y_2\} \neq \emptyset$, then we have*

$$G'\big(\{x_1,y_1\}\big) \cap G'\big(\{x_2,y_2\}\big) \neq \emptyset.$$

Further, if $\{x_1,y_1\},\{x_2,y_2\} \in \tilde{\mathcal{X}}$ have exactly one element in common, then the same holds for $G'\big(\{x_1,y_1\}\big)$ and $G'\big(\{x_2,y_2\}\big)$.

Let $\{x,y_1\},\{x,y_2\} \in \tilde{\mathcal{X}}$ with $y_1 \neq y_2$, and suppose that

$$G'\big(\{x,y_1\}\big) \cap G'\big(\{x,y_2\}\big) = \emptyset.$$

Then we may assume that $\infty \notin G'(\{x,y_2\})$. Let $K \subseteq X \setminus G'(\{x,y_1\})$ be compact such that $G'(\{x,y_2\}) \subseteq K^o$. Then it follows from the surjectivity of ϕ that there exist functions $f_1, f_2 \in C_0(X)$ with the following properties. The support of $\phi(f_2)$ is a subset of K, the range of $\phi(f_1)$ is included in $[0,1]$, the range of $\phi(f_2)$ is included in $[-\frac{1}{2}, \frac{1}{2}]$, $\phi(f_1)$ is $\frac{1}{2}$ on the set K, and

$$\phi(f_1) \in \mathcal{S}_s\big(G'(\{x,y_1\})\big), \quad \operatorname{diam}\big(\phi(f_1)(X)\big) = 1,$$
$$\phi(f_2) \in \mathcal{S}_s\big(G'(\{x,y_2\})\big), \quad \operatorname{diam}\big(\phi(f_2)(X)\big) = 1.$$

Now $f_1, f_2 \in C_0(X)$ are functions with diameter 1 and by Step 3.5 we infer that $f_1 \in \mathcal{S}_s(\{x,y_1\}), f_2 \in \mathcal{S}_s(\{x,y_2\})$. For every complex number μ with $|\mu| = 1$, the diameter of the range of $\phi(f_1)+\mu\phi(f_2)$ is 1 and hence the same must hold for $f_1+\mu f_2$.

Since $f_2 \in \mathcal{S}_s(\{x,y_2\})$, we have $f_2(x) \neq f_2(y_1)$. Define

$$\mu = \frac{f_1(x) - f_1(y_1)}{f_2(x) - f_2(y_1)}\big|f_2(x) - f_2(y_1)\big|.$$

It follows that $|\mu| = 1$ and

$$\begin{aligned}\big|(f_1+\mu f_2)(x) - (f_1+\mu f_2)(y_1)\big| &= \big|\big(f_1(x) - f_1(y_1)\big) + \mu\big(f_2(x) - f_2(y_1)\big)\big| \\ &= \big|\big(f_1(x) - f_1(y_1)\big)\big(1 + |f_2(x) - f_2(y_1)|\big)\big| = 1 + \big|f_2(x) - f_2(y_1)\big| > 1,\end{aligned}$$

which is untenable, since the diameter of the range of $f_1 + \mu f_2$ is 1. This justifies our assertion. The second statement of Step 3.7 follows now from Step 3.4.

Step 3.8. *Let $\{x_1,y_1\},\{x_2,y_2\} \in \tilde{\mathcal{X}}$. Then $\{x_1,y_1\} \cap \{x_2,y_2\} = \emptyset$ if and only if $G'\big(\{x_1,y_1\}\big) \cap G'\big(\{x_2,y_2\}\big) = \emptyset$.*

Necessity follows from Step 3.7. By Steps 3.6 and 3.7, sufficiency is obvious.

Step 3.9. *Let $x \in \mathcal{X}$. There exists a unique element $g(x) \in \mathcal{X}$ such that $g(x) \in G'\big(\{x,y\}\big)$ for every distinct $x,y \in \mathcal{X}$. Then the function $g : \mathcal{X} \to \mathcal{X}$ is bijective and $\big\{g(x),g(y)\big\} = G'\big(\{x,y\}\big)$ for any $\{x,y\} \in \tilde{\mathcal{X}}$.*

Let $y_1, y_2 \in \mathcal{X}$ such that x, y_1, y_2 are distinct. Denote $g(x)$ the unique element of the set $G'(\{x,y_1\}) \cap G'(\{x,y_2\})$ (see Step 3.7). We shall show that $g(x) \in G'(\{x,y\})$ holds for every $y \in \mathcal{X}$, $y \neq x$. If $\mathcal{X}$ has only three elements, this is obvious. Otherwise, pick $y \in \mathcal{X}$ with $y \neq x, y_1, y_2$, and suppose on the contrary that $g(x) \notin G'(\{x,y\})$. Let $a_1, a_2 \in \mathcal{X}$ such that $G'(\{x,y_1\}) = \{g(x),a_1\}$ and $G'(\{x,y_2\}) = \{g(x),a_2\}$. Then the sets $G'(\{x,y\}) \cap G'(\{x,y_1\})$ and $G'(\{x,y\}) \cap G'(\{x,y_2\})$ are nonempty, and we deduce that $G'(\{x,y\}) = \{a_1,a_2\}$. Similarly, we infer that $G'(\{y,y_1\})$ contains an element from $G'(\{x,y\}) = \{a_1,a_2\}$ in addition to one from $G'(\{x,y_1\}) = \{g(x),a_1\}$. By Step 3.8, $\{x,y_2\} \cap \{y,y_1\} = \emptyset$ implies $G'(\{x,y_2\}) \cap G'(\{y,y_1\}) = \emptyset$, whence $a_2, g(x) \notin G'(\{y,y_1\})$. Therefore, we have $a_1 \in G'(\{y,y_1\})$ and a similar argument can be applied to yield $a_2 \in G'(\{y,y_2\})$. Hence, by Step 3.8 again, we can choose a point $g(x) \neq b \in \mathcal{X}$ such that $G'(\{y,y_1\}) = \{a_1,b\}$ and $G'(\{y,y_2\}) = \{a_2,b\}$. Similarly as above, one can check that $G'(\{y_1,y_2\}) = \{g(x),b\}$. To sum up, we have

$$\begin{aligned}&G'\big(\{x,y_1\}\big) = \big\{g(x),a_1\big\}, \quad G'\big(\{x,y_2\}\big) = \big\{g(x),a_2\big\}, \quad G'\big(\{x,y\}\big) = \big\{a_1,a_2\big\}, \\ &G'\big(\{y,y_1\}\big) = \big\{a_1,b\big\}, \quad G'\big(\{y,y_2\}\big) = \big\{a_2,b\big\}, \quad G'\big(\{y_1,y_2\}\big) = \big\{g(x),b\big\}.\end{aligned}$$

Assume temporarily that $\mathcal{X} = \{x, y, y_1, y_2\}$. Let f be the function which takes the following values: 0 at y, 1 at y_1, $e^{i\pi/3}$ at y_2, and the geometric center $(1 + e^{i\pi/3})/3$ of the triangle $[0, 1, e^{i\pi/3}]$ at x. Let χ denote the characteristic function of the singleton $\{x\}$. Since $\phi(\chi)$ is nonconstant, one of the values $\phi(\chi)(a_1), \phi(\chi)(a_2), \phi(\chi)(g(x))$ differs from $\phi(\chi)(b)$. Let it be $\phi(\chi)(g(x))$. Since $f \in \mathcal{S}(\{y_1, y_2\})$ and $\operatorname{diam}(f(\mathcal{X})) = 1$, using the last equation in the second line in (3.2), we have $|\phi(f)(b) - \phi(f)(g(x))| = 1$. Pick a nonzero $\mu \in \mathbb{C}$ of modulus small enough to guarantee that $\mu + (1 + e^{i\pi/3})/3$ is still inside the triangle $[0, 1, e^{i\pi/3}]$ and for which

$$\mu\big(\phi(\chi)(b) - \phi(\chi)(g(x))\big) = \lambda\big(\phi(f)(b) - \phi(f)(g(x))\big)$$

with some positive λ. Now $f + \mu\chi \in \mathcal{S}(\{y_1, y_2\})$, $\operatorname{diam}(f + \mu\chi) = 1$ and (3.2) imply

$$\big|\phi(f + \mu\chi)(b) - \phi(f + \mu\chi)(g(x))\big| = 1.$$

On the other hand, we can compute

$$\begin{aligned}&\big|\phi(f + \mu\chi)(b) - \phi(f + \mu\chi)(g(x))\big| \\ &\qquad = \big|\big(\phi(f)(b) - \phi(f)(g(x))\big) + \mu\big(\phi(\chi)(b) - \phi(\chi)(g(x))\big)\big| = 1 + \lambda > 1,\end{aligned}$$

which is a contradiction. If $\mathcal{X}$ has at least five points, then by Step 3.7 we deduce that for any point $z \in \mathcal{X}$ with $z \neq x, y, y_1, y_2$, the set $G'(\{x, z\})$ has one common element with each of the sets $G'(\{x, y_1\})$, $G'(\{x, y_2\})$, $G'(\{x, y\})$. From examination of the first line in (3.2), one can see that this again is a contradiction. Thus we have proved that $g(x) \in G'(\{x, y\})$ is valid for every $x \neq y \in \mathcal{X}$.

In view of Step 3.7, the uniqueness of $g(x)$ is obvious.

We next prove that g is injective. Suppose temporarily, that $\mathcal{X}$ has exactly three points x, y, z. If $g(x) = g(y) = a$, say, then we have

$$a \in G'\big(\{x, y\}\big) \cap G'\big(\{x, z\}\big) \cap G'\big(\{y, z\}\big).$$

Now, using the fact that each of the sets $G'(\{x, y\})$, $G'(\{x, z\})$, $G'(\{y, z\})$ has two distinct elements, we arrive easily at a contradiction. In the general case (when $\mathcal{X}$ has at least four points), we argue as follows. Let $x_1 \neq x_2$ and suppose on the contrary that $g(x_1) = g(x_2)$. Choose $\{y_1, y_2\} \in \tilde{\mathcal{X}}$ such that $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$. Then clearly $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$, whence Step 3.8 implies $G'(\{x_1, y_1\}) \cap G'(\{x_2, y_2\}) = \emptyset$. By $g(x_1) \in G'(\{x_1, y_1\})$ and $g(x_2) \in G'(\{x_2, y_2\})$ now we obtain $g(x_1) \neq g(x_2)$.

We now show that g is surjective. Let $x_0 \in \mathcal{X}$ be arbitrary and $y_0 \in \mathcal{X}$ with $y_0 \neq x_0$. By the surjectivity of G', there exists an $\{x, y\} \in \tilde{\mathcal{X}}$ for which $\{g(x), g(y)\} = G'(\{x, y\}) = \{x_0, y_0\}$. Therefore, x_0 is in the range of g, thus verifying its surjectivity.

Step 3.10. *There exists a complex number τ of modulus 1 such that for every $\{x, y\} \in \tilde{\mathcal{X}}$, $0 \neq u \in \mathbb{C}$ and $f \in \mathcal{T}(x, y, u)$ we have $\phi(f) \in \mathcal{T}\big(g(x), g(y), \tau u\big)$.*

Let $\{x,y\} \in \tilde{X}$ be arbitrary and let $f_0 \in \mathcal{T}(x,y,1) \subseteq \mathcal{D}$. By Step 3.3 now $\phi(f_0) \in \mathcal{D}$. Let

$$\tau(x,y) = \phi(f_0)\big(g(x)\big) - \phi(f_0)\big(g(y)\big).$$

Then we have

$$H(x,y,1) \subseteq T\big(\phi(f_0)\big) = \Big\{\big(g(x),g(y),\tau(x,y)\big),\big(g(y),g(x),-\tau(x,y)\big)\Big\}.$$

By the definition of H and its non-emptiness (Step 3.1), we obtain

$$H(x,y,1) = \big\{\big(g(x),g(y),\tau(x,y)\big),\big(g(y),g(x),-\tau(x,y)\big)\big\}.$$

It is now easy to see that the implication

$$f \in \mathcal{T}(x,y,u) \Longrightarrow \phi(f) \in \mathcal{T}\big(g(x),g(y),\tau(x,y)u\big) \tag{3.8}$$

holds for every $u \in \mathbb{C}$. It remains to show that τ does not depend on x and y.

Let $x, y_1, y_2 \in X$ be distinct points. Pick functions $f_1 \in \mathcal{T}(x,y_1,-1)$ and $f_2 \in \mathcal{T}(x,y_2,-1)$ with disjoint supports and with ranges in [0,1]. Define

$$f = f_1 + e^{i\pi/3} f_2 \in \mathcal{T}(x,y_1,-1) \cap \mathcal{T}(x,y_2,-e^{i\pi/3}) \cap \mathcal{T}(y_1,y_2,1-e^{i\pi/3}).$$

Then we have

$$\begin{aligned}\phi(f) \in &\mathcal{T}\big(g(x),g(y_1),-\tau(x,y_1)\big)\cap \\ &\mathcal{T}\big(g(x),g(y_2),-e^{i\pi/3}\tau(x,y_2)\big) \cap \mathcal{T}\big(g(y_1),g(y_2),\tau(y_1,y_2)(1-e^{i\pi/3})\big).\end{aligned}$$

Hence, for the complex numbers $\phi(f)\big(g(x)\big)$, $\phi(f)\big(g(y_1)\big)$ and $\phi(f)\big(g(y_2)\big)$, we have

$$\begin{aligned}\big|\phi(f)\big(g(x)\big) - \phi(f)\big(g(y_1)\big)\big| &= \big|\phi(f)\big(g(x)\big) - \phi(f)\big(g(y_2)\big)\big| \\ &= \big|\phi(f)\big(g(y_1)\big) - \phi(f)\big(g(y_2)\big)\big| = 1.\end{aligned}$$

It is easy to see that for any complex numbers a,b of modulus 1, if $|a-b|=1$, then we have $a = e^{\pm i\pi/3}b$. Therefore, we deduce that

$$\begin{aligned}-\tau(x,y_1) &= \phi(f)\big(g(x)\big) - \phi(f)\big(g(y_1)\big) \\ &= e^{\pm i\pi/3}\Big(\phi(f)\big(g(x)\big) - \phi(f)\big(g(y_2)\big)\Big) = e^{\pm i\pi/3}\big(-e^{i\pi/3}\tau(x,y_2)\big),\end{aligned}$$

whence $\tau(x,y_1) = e^{\pm i\pi/3}e^{i\pi/3}\tau(x,y_2)$. Applying the same argument to the function $f_1 + e^{-i\pi/3}f_2$, we obtain $\tau(x,y_1) = e^{\pm i\pi/3}e^{-i\pi/3}\tau(x,y_2)$. Comparing these two equalities, we conclude that $\tau(x,y_1) = \tau(x,y_2)$. It is now clear that τ is constant on $X \times X \setminus \{(x,x) | x \in X\}$. Let the same symbol τ denote this constant value.

Suppose that X is σ-compact but not compact, $x \in X$ and $f \in \mathcal{T}_s(x,\infty,1)$. Let $z_n \in X$ with $z_n \to \infty$ and $z_n \neq x$. Since X_0 is compact and $g : X_0 \to X_0$ is a bijection, we may assume that there exists $y \in X_0$ for which $g(z_n) \to g(y)$. It is easy to see that there exist $f_n \in \mathcal{T}(x, z_n, 1)$ such that $f_n \to f$. Hence $\phi(f_n) \in \mathcal{T}\big(g(x), g(z_n), \tau\big)$. Since X is not compact and ϕ is continuous, thus from $f, \phi(f) \in C_0(X)$, $\phi(f_n)(g(x)) - \phi(f_n)(g(z_n)) = \tau$, $g(z_n) \to g(y)$ and $f_n \to f$ we deduce that

$$\phi(f)\big(g(x)\big) - \phi(f)\big(g(y)\big) = \tau.$$

Thus from $|\tau| = 1 = |\tau(x, \infty)|$ and $\phi(f) \in \mathcal{T}_s\big(g(x), g(\infty), \tau(x, \infty)\big)$ we infer that $g(y) = g(\infty)$, so

$$\tau(x, \infty) = \phi(f)\big(g(x)\big) - \phi(f)\big(g(\infty)\big) = \phi(f)\big(g(x)\big) - \phi\big(g(y)\big) = \tau.$$

Hence τ is constant on $\mathcal{X} \times \mathcal{X} \setminus \{(x, x) | x \in \mathcal{X}\}$, and the assertion follows from (3.8).

Step 3.11. *For every $f \in C_0(X)$, the function $\phi(f) \circ g - \tau \cdot f$ is constant on $\mathcal{X}$.*

Let $\{x, y\} \in \tilde{\mathcal{X}}$ and $f \in \mathcal{T}(x, y, 1)$. By Step 3.10 now $\phi(f) \in \mathcal{T}\big(g(x), g(y), \tau\big)$. Thus we have

$$\phi(f)\big(g(x)\big) - \phi(f)\big(g(y)\big) = \tau = \tau\big(f(x) - f(y)\big),$$

which implies

$$\phi(f)\big(g(y)\big) - \tau f(y) = \phi(f)\big(g(x)\big) - \tau f(x). \tag{3.9}$$

Let $z \in X$ be such that $z \neq x, y$. Set $u = f(x) - f(z)$. Clearly, $|u| \leq 1$. If $u = 0$, then $f \in \mathcal{T}(z, y, 1)$, which further implies $\phi(f) \in \mathcal{T}(g(z), g(y), \tau)$. Analogously to the derivation of (3.9), we find that

$$\phi(f)\big(g(z)\big) - \tau f(z) = \phi(f)\big(g(y)\big) - \tau f(y) = \phi(f)\big(g(x)\big) - \tau f(x).$$

Suppose now that $u \neq 0$. Define

$$U = \left\{ p \in X \setminus \{y\} \;:\; f(p) \neq f(x) \text{ and } \left| \frac{f(x) - f(p)}{|f(x) - f(p)|} - \frac{u}{|u|} \right| < \frac{1}{2} \right\}.$$

Since f is continuous, U is an open neighbourhood of the point z. By Urysohn's lemma, there exists a function $f_0 \in C(X)$ with range in $[0, 1]$ and support in U for which $f_0(z) = 1$ and $f_0(y) = 0$. For any $p \in X$ let

$$f_1(p) = \frac{|u| f_0(p)}{\max\big\{|f(x) - f(p)|, |u|\big\}} \quad \text{and} \quad f_2(p) = f_1(p)\big(f(x) - f(p)\big).$$

Clearly $f_1, f_2 \in C_0(X)$ and the support of f_2 is included in that of f_0, the latter being a subset of U. By the definition of U and f_2, it is now easy to check that $\operatorname{diam}(f_2(X)) \leq |u|$. On the other hand, $f_2(z) - f_2(x) = u - 0$ and $f_2(z) - f_2(y) = u - 0$. Consequently, we have

$$f_2 \in \mathcal{T}(z, x, u) \cap \mathcal{T}(z, y, u). \tag{3.10}$$

For an arbitrary $w \in X$ we have

$$(f + f_2)(w) = f(w) + f_1(w)\big(f(x) - f(w)\big) = \big(1 - f_1(w)\big)f(w) + f_1(w)f(x).$$

Since the range of f_1 is a subset of $[0,1]$, this shows that $(f + f_2)(w)$ belongs to the convex hull of $f(X)$, the diameter of which equals $\operatorname{diam}(f(X))$. Therefore, we have

$$\operatorname{diam}\big((f + f_2)(X)\big) \le \operatorname{diam}\big(f(X)\big) = 1.$$

On the other hand, we infer

$$\begin{aligned}(f + f_2)(z) - (f + f_2)(y) &= \big(f(z) - f(y)\big) + \big(f_2(z) - f_2(y)\big)\\ &= \big(f(z) - f(y)\big) + u = \big(f(z) - f(y)\big) + \big(f(x) - f(z)\big) = f(x) - f(y) = 1.\end{aligned}$$

Consequently,

$$f + f_2 \in \mathcal{T}(z, y, 1). \tag{3.11}$$

By (3.10) and (3.11) we obtain that

$$\phi(f_2) \in \mathcal{T}\big(g(z), g(x), \tau u\big) \cap \mathcal{T}\big(g(z), g(y), \tau u\big) \quad \text{and} \quad \phi(f + f_2) \in \mathcal{T}\big(g(z), g(y), \tau\big).$$

Hence, we compute

$$\begin{aligned}&\phi(f)\big(g(x)\big) - \phi(f)\big(g(z)\big)\\ &\quad= \Big(\phi(f)\big(g(x)\big) - \phi(f)\big(g(y)\big)\Big) - \Big(\phi(f + f_2)\big(g(z)\big) - \phi(f + f_2)\big(g(y)\big)\Big)\\ &\qquad+ \Big(\phi(f_2)\big(g(z)\big) - \phi(f_2)\big(g(y)\big)\Big) = \tau - \tau + \tau u = \tau\big(f(x) - f(z)\big).\end{aligned}$$

Thus we have proved that for every $z \in X$ we have

$$\phi(f)\big(g(z)\big) - \tau f(z) = \phi(f)\big(g(x)\big) - \tau f(x). \tag{3.12}$$

If $\mathcal{X} = X$, then we are ready. Suppose that X is σ-compact but not compact. In the proof of Step 3.10 we showed that then there exist $z_n \in X$, $z_n \to \infty$ such that $g(z_n) \to g(\infty)$. Thus from (3.12) we infer that

$$\phi(f)\big(g(\infty)\big) - \tau f(\infty) = \lim_{n\to\infty}\Big(\phi(f)\big(g(z_n)\big) - \tau f(z_n)\Big) = \phi(f)\big(g(x)\big) - \tau f(x),$$

which proves the statement of Step 3.11.

Now we can complete the proof of Theorem 3.1 as follows. By the linearity of ϕ, there exists a linear functional $t : C_0(X) \to \mathbb{C}$ such that

$$\phi(f) \circ g - \tau \cdot f = t(f)1 \qquad \big(f \in C_0(X)\big).$$

Since $g : \mathcal{X} \to \mathcal{X}$ is a bijection, with the notation $\varphi = g^{-1}$ we have

$$\phi(f) - \tau \cdot f \circ \varphi = t(f)1 \qquad \big(f \in C_0(X)\big). \tag{3.13}$$

It follows from (3.13) that $f \circ \varphi$ is continuous for every $f \in C_0(X)$. Using Urysohn's lemma, we deduce that φ is continuous. If X is σ-compact, then $\mathcal{X}$ is compact, so φ is a continuous bijection between compact Hausdorff spaces, thus φ is a homeomorphism. Let us consider the case in which X is not σ-compact. Suppose on the contrary that there exist $x_n \in X$ $(n \in \mathbb{N})$ such that $x_n \to \infty$ and $x_n \to y \in X$. Then there exists $y_0 \in X$ such that $\varphi(y_0) = y$. Now by (3.13) we have

$$\begin{aligned}\phi(f)(y_0) - \tau \cdot f\big(\varphi(y_0)\big) &= \phi(f)(x_n) - \tau \cdot f\big(\varphi(x_n)\big) \\ &\to \phi(f)(\infty) - \tau \cdot f\big(\varphi(y_0)\big) = -\tau \cdot f\big(\varphi(y_0)\big),\end{aligned}$$

thus $\phi(f)(y_0) = 0$ for every $f \in C_0(X)$, which is a contradiction. So for any $x_n \in X$ $(n \in \mathbb{N})$ with $x_n \to \infty$ we have $\varphi(x_n) \to \infty$. Now, defining φ at the point ∞ as $\varphi(\infty) = \infty$, $\varphi : X_0 \to X_0$ is a continuous bijection between compact Hausdorff spaces, thus $\varphi : X \to X$ is a homeomorphism.

If X is compact, then $t(1) \neq -\tau$ is obvious and $\mathcal{X} = X$, and so we are ready.

If X is not σ-compact, then for any $z_n \in X$ with $z_n \to \infty$ we have $\varphi(z_n) \to \infty$. Thus, by (3.13), we have

$$t(f) = \phi(f)(z_n) - \tau \cdot f\big(\varphi(z_n)\big) \to \phi(f)(\infty) - \tau \cdot f(\infty) = 0$$

for every $f \in C_0(X)$, which completes the proof.

Finally, suppose that X is σ-compact but not compact. Then $\varphi : X_0 \to X_0$ is a homeomorphism and by (3.13) we deduce that

$$t(f) = \phi(f)(\infty) - \tau \cdot f\big(\varphi(\infty)\big) = -\tau f\big(\varphi(\infty)\big)$$

for every $f \in C_0(X)$. The proof of Theorem 3.1 is now complete. □

3.3 Remarks

Linear preserver problems on operator algebras concerning the spectrum or the spectral radius belong to the most important preserver problems. For two fundamental results of this kind we refer to the papers [37, 13]. Therefore, we believe that it would be interesting to study the problem of the present chapter for the operator algebra $B(H)$ of all bounded linear operators acting on a complex Hilbert space H, i.e. to characterize those linear bijections of $B(H)$ which preserve the diameter of the spectrum. (Observe that if X is compact then the spectrum of an element of the unital Banach algebra $C_0(X)$ is equal to its range.) This problem is open, but we mention that Molnár and Barczy [68] characterized the space of all bounded self-adjoint operators on a Hilbert space which preserve the diameter of the spectrum.

4. A NEW PROOF OF WIGNER'S THEOREM

4.1 Introduction and Statement of the Result

As was mentioned in the Introduction, there are several formulations of Wigner's famous unitary-antiunitary theorem ([96] pp. 251-254). One of them characterizes the bijections on a Hilbert space preserving the absolute value of the inner product of any pair of vectors. This result belongs to the most important theorems concerning the probabilistic aspects of quantum mechanics.

In his book [96] appeared in 1931 Wigner did not give a rigorous mathematical proof of his result. In fact, the first such proofs were published only in the 1960's by Bargman [7] and Lemont and Mendelson [47]. Later, further proofs were given for that fundamental theorem. To mention some of them, we refer to the papers of Sharma and Almeida [89], Rätz [82] and Casinelli, de Vito, Lahti and Levrero [15], in chronological order. A common feature of the proofs given earlier is that they manipulate in the underlying Hilbert space. Molnár [53] proved Wigner's theorem by using a different, algebraic approach, which allowed him to generalize Wigner's result for several other structures. In this chapter we are going to present a short, elementary proof which is based on a completely new approach.

We now give the formulation of Wigner's theorem which we consider. Observe that, as in [7] and [89], we do not assume bijectivity.

Theorem 4.1. *Let H be a complex Hilbert space and $T : H \to H$ an arbitrary function. Then*

$$\big|\langle Tx, Ty\rangle\big| = \big|\langle x, y\rangle\big| \tag{4.1}$$

holds for any $x, y \in H$ if and only if there exists a function $\varphi : H \to \mathbb{C}$ with $|\varphi| = 1$, and a linear or conjugate linear isometry $U : H \to H$ such that

$$T = \varphi \cdot U. \tag{4.2}$$

4.2 Proof

The basic idea of our proof of Theorem 4.1 is as follows. We pick an orthonormal basis in the Hilbert space H, and first we prove that there exist φ and U (with the properties specified in Theorem 4.1) for the set F of all vectors with *real coordinates.*

To see this, we show that on an arbitrary subset $G \subseteq F$, the elements of which are not orthogonal to a given vector, our transformation T is of the form (4.2) with φ and U depending on G. To prove this we consider all the subsets of $G \times G$ for the elements of which (4.2) holds with (not necessarily the same) adequate φ and U, and we show that those subsets satisfy the conditions of *Zorn's lemma*. So there is a maximal subset in $G \times G$ with this property, which turns out to be the whole $G \times G$. Then it is easy to show that there are φ and U for which T is of the form (4.2) on the *whole Hilbert space*.

Our proof consists of several steps. Let e_γ $(\gamma \in \Gamma)$ be an orthonormal basis of H and let F denote the set of elements with real coordinates. Then $\langle x, y\rangle \in \mathbb{R}$ for any $x, y \in F$. For an arbitrary $A : H \to H$, put

$$\alpha(A) = \big\{(x, y) \in H \times H \big| \langle Ax, Ay\rangle = \langle x, y\rangle\big\}.$$

Let $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and

$$\mathcal{S} = \Big\{S : H \to H \Big| \text{ either } S(x + iy) = Sx + iSy \quad (x, y \in F) \\ \text{or } S(x + iy) = Sx - iSy \quad (x, y \in F)\Big\}.$$

For any $0 \neq u \in F$, let

$$G_u = \{z \in F : \langle z, u\rangle \neq 0\}.$$

Define $T_0(0) = 0$. By the axiom of choice, there is a set L which contains exactly one element from each of the disjoint sets $\{\lambda x | \lambda \in \mathbb{C}, \lambda \neq 0\}$ $(x \in H)$. For any $0 \neq x \in H$ there exist uniquely determined $y \in L$ and $\lambda \in \mathbb{C}$ such that $x = \lambda y$. Define $T_0(x) = \lambda T_0(y) = \lambda T(y)$. Then $\|T(\lambda y)\| = \|\lambda y\| = |\lambda|\|y\| = |\lambda|\|Ty\|$ and

$$|\langle T(\lambda y), Ty\rangle| = |\langle \lambda y, y\rangle| = |\lambda| \cdot \|y\|^2 = \|\lambda y\| \cdot \|y\| = \|T(\lambda y)\| \cdot \|Ty\|.$$

Thus there exists a complex number α of modulus 1 such that $T(x) = T(\lambda y) = \alpha\lambda T(y) = \alpha T_0(x)$. Now $T_0 : H \to H$ is well-defined, homogeneous and (4.1) holds for T_0 if and only if it holds for T. Therefore, we may assume that T *is homogeneous.*

Step 4.1. *For any $x, y \in H$ there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ such that*

$$T(x + y) = \alpha_1 Tx + \alpha_2 Ty.$$

If $y = \lambda x$ for some $\lambda \in \mathbb{C}$ then

$$T(x + y) = T\big((1 + \lambda)x\big) = (1 + \lambda)Tx = Tx + \lambda Tx = Tx + T(\lambda x) = Tx + Ty.$$

Assume that $|\langle x, y\rangle| \neq \|x\| \cdot \|y\|$ and let $u, v \in H$ be an orthonormal basis of $\operatorname{span}(x, y)$, i.e. the linear subspace generated by x and y. Then for any $\lambda_1, \lambda_2 \in \mathbb{C}$ we have

$$\begin{aligned}\big\|\big\langle T(\lambda_1 u+\lambda_2 v), Tu\big\rangle Tu + \big\langle T(\lambda_1 u+\lambda_2 v), Tv\big\rangle Tv\big\|^2 \\ =&\big|\big\langle T(\lambda_1 u+\lambda_2 v), Tu\big\rangle\big|^2 + \big|\big\langle T(\lambda_1 u+\lambda_2 v), Tv\big\rangle\big|^2 \\ =&\big|\big\langle (\lambda_1 u+\lambda_2 v), u\big\rangle\big|^2 + \big|\big\langle (\lambda_1 u+\lambda_2 v), v\big\rangle\big|^2 \\ =&|\lambda_1|^2+|\lambda_2|^2 = \|\lambda_1 u+\lambda_2 v\|^2 = \big\|T(\lambda_1 u+\lambda_2 v)\big\|^2,\end{aligned}$$

whence

$$T(\lambda_1 u+\lambda_2 v) = \big\langle T(\lambda_1 u+\lambda_2 v), Tu\big\rangle Tu + \big\langle T(\lambda_1 u+\lambda_2 v), Tv\big\rangle Tv.$$

Thus we have $Tx, Ty, T(x+y) \in \operatorname{span}(Tu, Tv)$. Since

$$\big|\langle Tx, Ty\rangle\big| = |\langle x, y\rangle| \neq \|x\|\cdot\|y\| = \|Tx\|\cdot\|Ty\|,$$

now $\operatorname{span}(Tx, Ty) = \operatorname{span}(Tu, Tv)$, and so $T(x+y) \in \operatorname{span}(Tx, Ty)$.

Lemma 4.2. *Let $w \in F$ be arbitrary with $w \neq 0$, and let $G = G_w$. Then there exists a function $\varphi : H \to \mathbb{D}$ and a linear isometry $U_w : H \to H$ such that*

$$Tx = \varphi_w(x) U_w x \ (x \in G).$$

Proof of Lemma 4.2. Let

$$\begin{aligned}\mathcal{A} = \Big\{K \subseteq G\times G \;\Big|\; &\exists A : H \to H, \exists \varphi : H \to \mathbb{D} : T = \varphi A, \\ &A(\lambda x) = \lambda Ax \ (x \in H, \lambda \in \mathbb{C}) \text{ and } \alpha(A) \cap G \times G = K\Big\}.\end{aligned}$$

Further, let $K \in \mathcal{A}$ be arbitrary with the corresponding functions A and φ. For any $x, y \in H$ and $0 \neq \lambda \in \mathbb{C}$ we get $\varphi(\lambda x) = \varphi(x)$ and $|\langle Ax, Ay\rangle| = |\langle Tx, Ty\rangle| = |\langle x, y\rangle|$.

Step 4.2. *If $(x, y) \in K$ with $\langle x, y\rangle \neq 0$, then there exists $\lambda_{x,y} \in \mathbb{D}$ for which*

$$A(x+y) = \lambda_{x,y}(Ax+Ay).$$

We may assume that $\langle x, y\rangle \neq \|x\|\cdot\|y\|$. By Step 4.1 there exist $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $A(x+y) = \alpha_1 Ax + \alpha_2 Ay$. Then

$$\begin{aligned}\|\alpha_1 x+\alpha_2 y\|^2 =&\alpha_1\|x\|^2+\alpha_2\|y\|^2+\alpha_1\overline{\alpha_2}\langle x, y\rangle+\overline{\alpha_1}\alpha_2\langle y, x\rangle \\ =&\alpha_1\|Ax\|^2+\alpha_2\|Ay\|^2+\alpha_1\overline{\alpha_2}\langle Ax, Ay\rangle+\overline{\alpha_1}\alpha_2\langle Ay, Ax\rangle \\ =&\|\alpha_1 Ax+\alpha_2 Ay\|^2 = \|A(x+y)\|^2 = \|x+y\|^2.\end{aligned} \tag{4.3}$$

Further, there exist $\lambda_1, \lambda_2 \in \mathbb{D}$ for which

$$\begin{aligned}\lambda_1\langle x+y, x\rangle =&\langle A(x+y), Ax\rangle = \langle \alpha_1 Ax+\alpha_2 Ay, Ax\rangle \\ =&\alpha_1\|Ax\|^2+\alpha_2\langle Ay, Ax\rangle = \alpha_1\|x\|^2+\alpha_2\langle y, x\rangle \\ =&\langle \alpha_1 x+\alpha_2 y, x\rangle,\end{aligned}$$

and similarly

$$\lambda_2\langle x+y,y\rangle = \langle A(x+y),Ay\rangle = \langle \alpha_1 x+\alpha_2 y,y\rangle.$$

Hence

$$\begin{aligned}\alpha_1 =\ & \lambda_1\|y\|^2\frac{\|x\|^2+\langle x,y\rangle}{\|x\|^2\|y\|^2-\langle x,y\rangle^2} - \lambda_2\langle x,y\rangle\frac{\|y\|^2+\langle x,y\rangle}{\|x\|^2\|y\|^2-\langle x,y\rangle^2},\\ \alpha_2 =\ & -\lambda_1\langle x,y\rangle\frac{\|x\|^2+\langle x,y\rangle}{\|x\|^2\|y\|^2-\langle x,y\rangle^2} + \lambda_2\|x\|^2\frac{\|y\|^2+\langle x,y\rangle}{\|x\|^2\|y\|^2-\langle x,y\rangle^2}.\end{aligned} \tag{4.4}$$

If $\|x\|^2+\langle x,y\rangle = 0$ or $\|y\|^2+\langle x,y\rangle = 0$ then (4.4) implies $\alpha_1=\alpha_2=\lambda_2\in\mathbb{D}$ or $\alpha_1=\alpha_2=\lambda_1\in\mathbb{D}$, and so we are ready. Suppose that $\|x\|^2+\langle x,y\rangle\neq 0$ and $\|y\|^2+\langle x,y\rangle\neq 0$. By (4.3) and (4.4), we obtain after a short calculation that

$$\begin{aligned}0 =& \|\alpha_1 x+\alpha_2 y\|^2-\|x+y\|^2\\ =& -\left(\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}-2\right)\langle x,y\rangle\frac{\left(\|x\|^2+\langle x,y\rangle\right)\left(\|y\|^2+\langle x,y\rangle\right)}{\|x\|^2\|y\|^2-\langle x,y\rangle^2},\end{aligned}$$

which implies $\lambda_1=\lambda_2$, and hence Step 4.2 is proved.

Step 4.3. *If $(x,y),(y,z)\in K$ with $\langle x,y\rangle,\langle y,z\rangle\neq 0$, then $(x,z)\in K$.*

We may assume that $\langle x,z\rangle\neq 0$. Now $\langle Ax,Az\rangle=\lambda\langle x,z\rangle$ for some $\lambda\in\mathbb{D}$, therefore

$$\begin{aligned}|\langle x+y,z\rangle| =& \left|\langle A(x+y),Az\rangle\right| = \left|\lambda_{x,y}\langle Ax+Ay,Az\rangle\right|\\ =& \left|\langle Ax,Az\rangle+\langle Ay,Az\rangle\right| = \left|\lambda\langle x,z\rangle+\langle y,z\rangle\right|.\end{aligned}$$

We infer that there exists $\alpha\in\mathbb{D}$ for which $\alpha\langle x+y,z\rangle=\lambda\langle x,z\rangle+\langle y,z\rangle$. Hence

$$(\alpha-1)\langle y,z\rangle=(\lambda-\alpha)\langle x,z\rangle,$$

thus, by $\langle y,z\rangle,\langle x,z\rangle\in\mathbb{R}\setminus\{0\}$ and $|\alpha|=|\lambda|=1$, we have $\lambda=1$. So $\langle Ax,Az\rangle=\langle x,z\rangle$, whence $(x,z)\in K$.

Step 4.4. *$(\mathcal{A},\subseteq)$ contains a maximal element M. Let the corresponding functions be $U: H\to H$ and $\varphi: H\to\mathbb{D}$, i.e. U is homogeneous, $T=\varphi\cdot U$ and $\alpha(U)\cap G\times G=M$.*

It is enough to prove that $(\mathcal{A},\subseteq)$ satisfies the conditions of Zorn's lemma. Let $K_\gamma\in\mathcal{A}(\gamma\in\Gamma)$ be a chain in $\mathcal{A}$, with the corresponding functions A_γ and φ_γ $(\gamma\in\Gamma)$. Further, let $K=\cup_{\gamma\in\Gamma}K_\gamma$ and

$$\begin{aligned}\mathcal{K}=\big\{(x,y)\in G\times G \;\big|\; &\exists n\in\mathbb{N}:\exists x_0=x,\dots,x_n=y\in G:\\ &(x_i,x_{i+1})\in K,\langle x_i,x_{i+1}\rangle\neq 0\ (i=0,\dots,n-1)\big\}.\end{aligned}$$

It is easy to see that $\mathcal{K}$ is an equivalence relation on G. For any $x\in G$ let M_x denote its equivalence class. By the axiom of choice there is a set $B\subseteq G$ for which $B\cap M_x$ is singleton for any $x\in G$.

Let $y \in G$ be arbitrary and $M_y \cap B = \{x\}$. Then there exist $x_0, x_1, \dots, x_n \in G$ such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in K$, $\langle x_i, x_{i+1} \rangle \neq 0$ $(i = 0, \dots, n-1)$. By the chain property, $(x_i, x_{i+1}) \in K_\gamma$ $(i = 0, \dots, n-1)$ for some $\gamma \in \Gamma$. Let

$$\varphi(y) = \frac{\varphi_\gamma(y)}{\varphi_\gamma(x)}.$$

Suppose that γ_1, γ_2 are such elements of Γ for which $(x_i, x_{i+1}) \in K_{\gamma_1}, K_{\gamma_2}$ $(i = 0, \dots, n-1)$. Then

$$\begin{aligned}
\frac{\varphi_{\gamma_1}(x_{i+1})}{\varphi_{\gamma_1}(x_i)}\langle Tx_i, Tx_{i+1}\rangle &= \Big\langle \frac{Tx_i}{\varphi_{\gamma_1}(x_i)}, \frac{Tx_{i+1}}{\varphi_{\gamma_1}(x_{i+1})} \Big\rangle = \big\langle A_{\gamma_1}x_i, A_{\gamma_1}x_{i+1} \big\rangle \\
&= \big\langle x_i, x_{i+1} \big\rangle = \big\langle A_{\gamma_2}x_i, A_{\gamma_2}x_{i+1} \big\rangle \\
&= \Big\langle \frac{Tx_i}{\varphi_{\gamma_2}(x_i)}, \frac{Tx_{i+1}}{\varphi_{\gamma_2}(x_{i+1})} \Big\rangle = \frac{\varphi_{\gamma_2}(x_{i+1})}{\varphi_{\gamma_2}(x_i)}\langle Tx_i, Tx_{i+1}\rangle.
\end{aligned}$$

By $|\langle Tx_i, Tx_{i+1}\rangle| = |\langle x_i, x_{i+1}\rangle| \neq 0$ $(i = 1, \dots, n-1)$, for $i = 1, \dots, n-1$ we deduce that $\frac{\varphi_{\gamma_1}(x_{i+1})}{\varphi_{\gamma_1}(x_i)} = \frac{\varphi_{\gamma_2}(x_{i+1})}{\varphi_{\gamma_2}(x_i)}$, whence $\frac{\varphi_{\gamma_1}(y)}{\varphi_{\gamma_1}(x)} = \frac{\varphi_{\gamma_2}(y)}{\varphi_{\gamma_2}(x)}$. So φ does not depend on the choice of γ. Let $Ay = \frac{Ty}{\varphi(y)}$. Then $A : H \to H$ and $\varphi : H \to \mathbb{D}$ are well-defined.

Now let $(u, v) \in K$ be arbitrary and $M_u \cap B = M_v \cap B = \{z\}$. Then, by the definition above, for some $\gamma \in \Gamma$ we have $(u, v) \in K_\gamma$ and

$$\begin{aligned}
\big\langle Au, Av \big\rangle &= \Big\langle \frac{Tu}{\varphi(u)}, \frac{Tv}{\varphi(v)} \Big\rangle = \Big\langle \varphi_\gamma(z)\frac{Tu}{\varphi_\gamma(u)}, \varphi_\gamma(z)\frac{Tv}{\varphi_\gamma(v)} \Big\rangle \\
&= \big\langle \varphi_\gamma(z)A_\gamma u, \varphi_\gamma(z)A_\gamma v \big\rangle = \big\langle A_\gamma u, A_\gamma v \big\rangle = \big\langle u, v \big\rangle.
\end{aligned}$$

Thus for any $(u, v) \in K$ we have $\langle Au, Av\rangle = \langle u, v\rangle$. Then we obtain that $K_\gamma \subseteq K \subseteq \big(\alpha(A) \cap G \times G\big) \in \mathcal{A}$ for any $\gamma \in \Gamma$, so $\mathcal{A}$ satisfies the conditions of Zorn's lemma.

Step 4.5. *For any $x, y \in G$ we have $\langle Ux, Uy\rangle = \langle x, y\rangle$.*

Suppose on the contrary that there exists $x \in G$ such that $(x, w) \notin M$. Then for any $y \in H$ define

$$\varphi_0(y) = \begin{cases} \frac{\langle Uw, Ux\rangle}{\langle w, x\rangle}\varphi(y) & \text{if } (w, y) \in M, \\ \varphi(y) & \text{if } (w, y) \notin M, \end{cases}$$

and let $U_0 y = \frac{Ty}{\varphi_0(y)}$. Now, by $(w, x) \notin M$, we have

$$\langle U_0 w, U_0 x\rangle = \Big\langle \frac{\langle w, x\rangle}{\langle Uw, Ux\rangle}\frac{Tw}{\varphi(w)}, \frac{Tx}{\varphi(x)} \Big\rangle = \Big\langle \frac{\langle w, x\rangle}{\langle Uw, Ux\rangle}Uw, Ux \Big\rangle = \langle w, x\rangle.$$

Let $(u, v) \in M$ be arbitrary with $\langle u, v\rangle \neq 0$. Since $u, v \in G = G_w$, Step 4.3 yields that either $(u, w), (v, w) \in M$ or $(u, w), (v, w) \notin M$ holds. In both cases $\langle U_0 u, U_0 v\rangle =$

$\langle Uu, Uv\rangle = \langle u, v\rangle$. Since $U_0 : H \to H$ is homogeneous, we obtain $M \subsetneqq \alpha(U_0)\cap G\times G \in \mathcal{A}$, which contradicts the maximal property of M in Step 4.4.

Let $a, b \in G$ be arbitrary. Now $(a, w), (b, w) \in M$, hence by $\langle a, w\rangle, \langle b, w\rangle \neq 0$ and Step 4.3 we have $(a, b) \in M$. Therefore $M = G \times G$, and Step 4.5 is proved.

Step 4.6. *The function U is linear on G, thus there is a linear isometry $U_w : H \to H$ for which*

$$U_w x = Ux \quad (x \in G_w).$$

Assume that $n \in \mathbb{N}$, $x_i \in G$, $\lambda_i \in \mathbb{C}$ $(i = 1, \dots, n)$ and $\sum_{i=1}^n \lambda_i x_i \in G$. Then for any $z \in G$ we obtain

$$\begin{aligned}\Big\langle U\Big(\sum_{i=1}^n \lambda_i x_i\Big), Uz\Big\rangle =& \Big\langle \sum_{i=1}^n \lambda_i x_i, z\Big\rangle = \sum_{i=1}^n \lambda_i \langle x_i, z\rangle \\ =& \sum_{i=1}^n \lambda_i \langle Ux_i, Uz\rangle = \Big\langle \sum_{i=1}^n \lambda_i Ux_i, Uz\Big\rangle.\end{aligned}$$

Substituting z by $x_1, \dots, x_n$ and $\sum_{i=1}^n \lambda_i x_i$, we conclude that

$$U\Big(\sum_{i=1}^n \lambda_i x_i\Big) = \sum_{i=1}^n \lambda_i Ux_i.$$

Denote $\varphi_w = \varphi$. Now, by Steps 4.5 and 4.6, the proof of Lemma 4.2 is complete. □

Lemma 4.3. *There exists a function $\varphi : H \to \mathbb{D}$ and a linear isometry $U : H \to H$ such that for any $x \in F$ we have*

$$Tx = \varphi(x)Ux.$$

Proof of Lemma 4.3. Let $0 \neq x \in F$ be fixed and $0 \neq y \in F$ be arbitrary. By Lemma 4.2 there are linear isometries $U_x, U_y : H \to H$ and functions $\varphi_x, \varphi_y : H \to \mathbb{D}$ such that $Tu = \varphi_x(u)U_x u$ $(u \in G_x)$ and $Tu = \varphi_y(u)U_y u$ $(u \in G_y)$. Let $U = U_x$. Further, let $v \in G_x \cap G_y$ be fixed and set $\lambda = \frac{\varphi_x(v)}{\varphi_y(v)}$. Let $u \in G_x \cap G_y \cap G_v$ be arbitrary. Then

$$\begin{aligned}\frac{\varphi_x(v)}{\varphi_x(u)}\langle Tu, Tv\rangle =& \Big\langle \frac{Tu}{\varphi_x(u)}, \frac{Tv}{\varphi_x(v)}\Big\rangle = \langle Uu, Uv\rangle = \langle u, v\rangle \\ =& \langle U_y u, U_y v\rangle = \Big\langle \frac{Tu}{\varphi_y(u)}, \frac{Tv}{\varphi_y(v)}\Big\rangle = \frac{\varphi_y(v)}{\varphi_y(u)}\langle Tu, Tv\rangle,\end{aligned}$$

hence by $|\langle Tu, Tv\rangle| = |\langle u, v\rangle| \neq 0$ we have $\varphi_x(u) = \lambda\varphi_y(u)$. Thus $U_y u = \lambda Uu$ $(u \in G_x \cap G_y \cap G_v)$. Since $G_x \cap G_y \cap G_v$ is a generating system of F and the operators U, U_y are linear, we infer that $U_y = \lambda U$ on F. Thus

$$Ty = \varphi_y(y)U_y(y) = \lambda\varphi_y(y)Uy.$$

So for any $0 \neq y \in F$ there is a $\varphi(y) \in \mathbb{D}$ for which $Ty = \varphi(y)Uy$. $\square$

Proof of Theorem 4.1.

Step 4.7. *For any $x, y \in F$ there are $\alpha, \mu \in \mathbb{D}$ such that*

$$T(x+iy) = \alpha U(x+\mu iy).$$

Moreover, if $\langle x, y\rangle \neq 0$ then $\mu = 1$ or $\mu = -1$.

If $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ then the proof is trivial. So assume that $|\langle x, y\rangle| \neq \|x\| \cdot \|y\|$. By Step 4.1 there are $\beta_1, \beta_2 \in \mathbb{C}$ such that

$$T(x+iy) = \beta_1 Tx + \beta_2 T(iy) = \beta_1 Tx + \beta_2 iTy = \alpha(Ux + \mu iUy),$$

where $\alpha = \frac{\beta_1}{\varphi(x)}$ and $\mu = \frac{\beta_2 \varphi(y)}{\alpha}$. Now

$$\begin{aligned} \|x\|^2 + \|y\|^2 =&\|x+iy\|^2 = \|T(x+iy)\|^2 = \|\alpha U(x+i\mu y)\|^2 \\ =&|\alpha|^2\|x+i\mu y\|^2 = |\alpha|^2\|x\|^2 + |\alpha\mu|^2\|y\|^2 + i|\alpha|^2(\mu - \bar{\mu})\langle x, y\rangle. \end{aligned} \tag{4.5}$$

First assume that $\langle x, y\rangle \neq 0$. Then (4.5) implies $\mu \in \mathbb{R}$, thus

$$\begin{aligned} \|x\|^4 + \langle x, y\rangle^2 =&\big|\langle x+iy, x\rangle\big|^2 = \big|\big\langle T(x+iy), Tx\big\rangle\big|^2 = \big|\big\langle \alpha U(x+\mu iy), Ux\big\rangle\big|^2 \\ =&|\alpha|^2\big|\langle x+\mu iy, x\rangle\big|^2 = |\alpha|^2\big|\|x\|^2 + i\mu\langle y, x\rangle\big|^2 \\ =&|\alpha|^2\|x\|^4 + |\alpha\mu|^2\langle x, y\rangle^2, \end{aligned}$$

and similarly $\|y\|^4 + \langle x, y\rangle^2 = |\alpha|^2\langle x, y\rangle^2 + |\alpha\mu|^2\|y\|^4$. If $|\alpha| \neq 1$ or $|\mu| \neq 1$ then now $|\langle x, y\rangle| = \|x\| \cdot \|y\|$, which is a contradiction. Thus $|\alpha| = |\mu| = 1$, and so $\mu \in \{1, -1\}$.

Next assume that $\langle x, y\rangle = 0$. Then

$$\begin{aligned} \|x\|^4 =&\big|\langle x+iy, x\rangle\big|^2 = \big|\big\langle T(x+iy), Tx\big\rangle\big|^2 \\ =&\big|\big\langle \alpha U(x+\mu iy), Ux\big\rangle\big|^2 = |\alpha|^2\big|\langle x+\mu iy, x\rangle\big|^2 = |\alpha|^2\|x\|^4. \end{aligned}$$

So $|\alpha| = 1$, whence, by (4.5), we obtain that $|\mu| = 1$.

Step 4.8. *For any $x, y \in F$ with $x, y \neq 0$, there exist uniquely determined numbers $\alpha \in \mathbb{D}$ and $\mu \in \{1, -1\}$ such that*

$$T(x+iy) = \alpha U(x+\mu iy),$$

and let $\alpha(x+iy) = \alpha$, $\mu(x+iy) = \mu$ and $S(x+iy) = U(x+\mu iy)$. Then $\alpha : H \setminus (F \cup iF) \to \mathbb{D}$, $\mu : H \setminus (F \cup iF) \to \{1, -1\}$ and $S : H \setminus (F \cup iF) \to H$ are well-defined and $T = \alpha \cdot S$ on $H \setminus (F \cup iF)$.

If $\langle x, y\rangle \neq 0$ then, by Step 4.7, we are done. So assume that $\langle x, y\rangle = 0$. By Step 4.7 there are $\alpha, \mu \in \mathbb{D}$ such that $T(x+iy) = \alpha(Ux + \mu iUy)$. Let $u = \frac{x}{\|x\|^2} + \frac{y}{\|y\|^2}$. Then $u \in F$, $\langle x, u\rangle = \langle y, v\rangle = 1$, and for any $0 \neq \lambda \in \mathbb{R}$ we have $\langle x + \lambda u, y - \lambda u\rangle \neq 0$. Now, by Step 4.7, for any $0 \neq \lambda \in \mathbb{R}$ there exist $\alpha_\lambda \in \mathbb{D}$ and $\mu_\lambda \in \{1, -1\}$ for which

$$T\big((x + \lambda u) + i(y - \lambda u)\big) = \alpha_\lambda U\big((x + \lambda u) + \mu_\lambda i(y - \lambda u)\big).$$

Let λ_n be a sequence with $0 < \lambda_n \to 0$. Since $\mu_{\lambda_n} \in \{1, -1\}$, we may also assume that $\mu_{\lambda_n} = \mu_{\lambda_1}$ $(n \in \mathbb{N})$. For brevity, put $\alpha_n = \alpha_{\lambda_n}$ $(n \in \mathbb{N})$ and $\mu_1 = \mu_{\lambda_1}$. Then

$$\begin{aligned}
\big(\|x\|^2 + \|y\|^2\big)^2 + 4\lambda_n^2 =& \big|\big\langle x + iy, (x + \lambda_n u) + i(y - \lambda_n u)\big\rangle\big|^2 \\
=& \big|\big\langle T(x+iy), T\big((x + \lambda_n u) + i(y - \lambda_n u)\big)\big\rangle\big|^2 \\
=& \big|\big\langle \alpha U(x + \mu iy), \alpha_n U\big((x + \lambda_n u) + \mu_1 i(y - \lambda_n u)\big)\big\rangle\big|^2 \\
=& \big|\big\langle x + \mu iy, (x + \lambda_n u) + \mu_1 i(y - \lambda_n u)\big\rangle\big|^2 \\
=& \big|\|x\|^2 + \mu\mu_1\|y\|^2 + \lambda_n(1 + \mu_1 i + \mu i - \mu\mu_1)\big|^2
\end{aligned}$$

holds for any $n \in \mathbb{N}$. Now $\lambda_n \to 0$ implies that

$$\big(\|x\|^2 + \|y\|^2\big)^2 = \big|\|x\|^2 + \mu\mu_1\|y\|^2\big|^2,$$

whence, by $x, y \neq 0$, $|\mu| = 1$ and $\mu_1 \in \{1, -1\}$, we obtain $\mu = \frac{1}{\mu_1} = \mu_1 \in \{1, -1\}$.

Step 4.9. *The function $\mu : H \setminus (F \cup iF) \to \{1, -1\}$ is constant. Denote this constant by $\mu \in \{1, -1\}$, and for any $x, y \in F$ let $\alpha(x) = \varphi(x)$, $S(x) = Ux$, $\alpha(iy) = \frac{\varphi(y)}{\mu}$ and $S(iy) = \mu iUy$. Then $\alpha : H \to \mathbb{D}$ and $S : H \to H$ are well-defined, $S \in \mathcal{S}$ and $T = \alpha \cdot S$. By this the proof of Theorem 4.1 is complete.*

Suppose on the contrary that $\alpha(x_1 + iy_1) \neq \alpha(x_2 + iy_2)$ for some $x_1, x_2, y_1, y_2 \in F \setminus \{0\}$. There clearly exist $u, v \in F$ such that $u, v \neq 0$ and $\langle u, x_1\rangle\langle v, y_1\rangle \neq \langle v, x_1\rangle\langle u, y_1\rangle$ and $\langle u, x_2\rangle\langle v, y_2\rangle \neq \langle v, x_2\rangle\langle u, y_2\rangle$. For brevity, let $\alpha_1 = \alpha(x_1 + iy_1)$, $\alpha_2 = \alpha(x_2 + iy_2)$, $\alpha = \alpha(u + iv)$, $\mu_1 = \mu(x_1 + iy_1)$, $\mu_2 = \mu(x_2 + iy_2)$ and $\mu = \mu(u + iv)$. Now

$$\begin{aligned}
&\langle x_1, u\rangle^2 + \langle y_1, v\rangle^2 + \langle y_1, u\rangle^2 + \langle x_1, v\rangle^2 + 2\big(\langle x_1, u\rangle\langle y_1, v\rangle - \langle y_1, u\rangle\langle x_1, v\rangle\big) \\
&= \big|\langle x_1 + iy_1, u + iv\rangle\big|^2 = \big|\big\langle T(x_1 + iy_1), T(u + iv)\big\rangle\big|^2 \\
&= \big|\big\langle \alpha_1 U(x_1 + i\mu_1 y_1), \alpha U(u + i\mu v)\big\rangle\big|^2 = \big|\langle x_1 + i\mu_1 y_1, u + i\mu v\rangle\big|^2 \\
&= \langle x_1, u\rangle^2 + \langle y_1, v\rangle^2 + \langle y_1, u\rangle^2 + \langle x_1, v\rangle^2 + 2\mu_1\mu\big(\langle x_1, u\rangle\langle y_1, v\rangle - \langle y_1, u\rangle\langle x_1, v\rangle\big),
\end{aligned}$$

hence $\mu_1 = \frac{1}{\mu} = \mu$ and similarly $\mu_2 = \mu$, which implies that $\mu_1 = \mu_2$.

For any $u, v \in F$ we have $S(u + iv) = Uu + \mu iUv = Su + \mu iSv$, thus $S \in \mathcal{S}$.

Let $x, y \in F$ be arbitrary. Then $T(x) = \varphi(x)Ux = \alpha(x)Sx$ and

$$T(iy) = iTy = i\varphi(y)Uy = \alpha(iy)\mu iUy = \alpha(iy)S(iy).$$

Our proof is now completed. □

5. TRANSFORMATIONS ON THE SET OF ALL N-DIMENSIONAL SUBSPACES OF A HILBERT SPACE PRESERVING ORTHOGONALITY

5.1 *Introduction and Statement of the Results*

As was remarked before, Wigner's classical unitary-antiunitary theorem has several formulations. We considered one of them in Chapter 4. There is another formulation which describes the bijections on the set of all 1-dimensional subspaces of a Hilbert space which preserve the angles between those subspaces. This fundamental result has been extended in (at least) three directions:

- if the underlying Hilbert space is at least three-dimensional, then, keeping the condition of bijectivity, the assumption of preserving angles can be replaced by the rather mild condition of *preserving orthogonality in both directions* (cf. Uhlhorn's paper [92]),

- keeping the condition of preserving angles, the assumption of bijectivity can be omitted (in this case the transformation is induced by a linear or conjugate linear isometry instead of a unitary or antiunitary operator; see [7, 89]),

- Molnár [65] extended Wigner's result to higher dimensional subspaces, namely he obtained the following result. If n is a positive integer, H is a Hilbert space with dimension not less than n and $n = 1$ or $\dim H \neq 2n$ then any transformation ϕ on the set of all n-dimensional subspaces of H, which preserves the so-called principal angles (see the definition below) between those subspaces, is of the form $\phi(M) = V(M)$, where V is a linear or conjugate linear isometry on H. Moreover, if H is an infinite dimensional Hilbert space, then a surjective transformation ϕ on the set of all infinite dimensional subspaces of H, which preserves the principal angles between those subspaces, is of the form $\phi(P) = UP$, where U is a unitary operator or antiunitary operator on H.

For further generalizations see e.g. [55, 57, 59, 62]. In this chapter we are going to extend Wigner's theorem in all the three directions above.

We introduce some concepts and notation. Let H be a (real or complex) Hilbert space and set

$$H_n = \big\{V \subseteq H \mid \dim V = n,\ \operatorname{codim} V \geq n,\ V \text{ is closed}\big\} \quad \big(n \in \mathbb{N} \cup \{\infty\}\big),$$
$$H_{(n)} = \big\{V \subseteq H \mid \dim V \geq n,\ \operatorname{codim} V \geq n,\ V \text{ is closed}\big\} \quad \big(n \in \mathbb{N} \cup \{\infty\}\big).$$

We say that a transformation $\phi : H_n \to H_n$ *preserves orthogonality in both directions*, if for any $M, N \in H_n$ we have

$$M \perp N \;\Leftrightarrow\; \phi(M) \perp \phi(N).$$

For any subspace $M \subseteq H$, let P_M denote the orthogonal projection to M. Following Molnár [65] we say that ϕ *preserves principal angles* if for any $K, L \in H_n$ the positive operators $P_K P_L P_K$ and $P_{\phi(K)} P_{\phi(L)} P_{\phi(K)}$ are unitarily equivalent. It is clear that if ϕ preserves principal angles then it also preserves orthogonality in both directions.

We now present our results. In Theorem 5.1 we characterize the transformations ϕ on H_n which preserve orthogonality in both directions under certain natural conditions. Our basic idea is to show that ϕ is induced by a transformation acting on the 1-dimensional subspaces of H, and then we apply Uhlhorn's result [92].

To formulate Theorem 5.1, we remark that if $\dim H < 2n$ then there do not exist two orthogonal n-dimensional subspaces of H, thus in this case the condition of preserving orthogonality has no meaning. In Proposition 5.4 we show that the case in which $\dim H = 2n \in \mathbb{N}$ is singular in a certain sense. Further, if $n = \infty$ then subspaces of finite codimension are clearly not orthogonal to any infinite dimensional subspace, and so the property of preserving orthogonality cannot imply anything for them. Therefore, in the case when $n = \infty$, we consider subspaces of infinite dimension and infinite codimension. These justify the assumption (5.1) below.

Theorem 5.1. *Let H be a Hilbert space and $n \in \mathbb{N} \cup \{\infty\}$ with*

$$\begin{cases} \dim H > 2n & \text{if } n \in \mathbb{N}, \\ \dim H = \infty & \text{if } n = \infty, \end{cases} \tag{5.1}$$

and let $\phi : H_n \to H_n$.

If $\phi : H_n \to H_n$ is ***surjective****, then ϕ preserves orthogonality in both directions if and only if there exists a unique bijection $\psi : H_1 \to H_1$ which preserves orthogonality in both directions, and for any $K \in H_n$ we have*

$$\phi(K) = \operatorname{span}\big\{\psi(X) | X \in H_1, X \subseteq K\big\}, \tag{5.2}$$

where span *denotes the generated linear subspace.*

Thus, by Uhlhorn's theorem, if $n \in \mathbb{N} \cup \{\infty\}$ is such that (5.1) *holds and $\phi : H_n \to H_n$ is surjective, then ϕ preserves orthogonality in both directions if and only if there* *exists a unitary or antiunitary operator U on H such that for any $K \in H_n$ we have*

$$\phi(K) = U(K). \tag{5.3}$$

*If H is **finite dimensional**, then ϕ preserves orthogonality in both directions (surjectivity is not assumed) if and only if there exists a unique transformation $\psi : H_1 \to H_1$ which preserves orthogonality in both directions and for any $K \in H_n$ (5.2) holds. Moreover, if ϕ preserves principal angles then ψ also preserves angles, thus in this case ϕ is of the form (5.3) with a unitary or antiunitary operator U on H.*

Remark 5.2. We make some short remarks.

- We learn from [65] that if ϕ preserves principal angles then ϕ is of the form (5.2) even in the case in which $n \leq \dim H < 2n$.
- As for the case in which $\dim H = 2n \in \mathbb{N}$, observe that the bijection ϕ defined by $\phi(K) = K^\perp$ ($K \in H_n$) preserves principal angles, but is not of the form (5.3) (cf. [65]).
- We mention that Theorem 5.1 implies Molnár's result [65] in the case in which H is finite dimensional.
- Theorem 5.1 implies that surjective transformations on H_n which preserve orthogonality in both directions are the same as surjective transformations which preserve principle angles. In the case in which $n = 1$ this is a trivial consequence of Uhlhorn's theorem.

Proposition 5.3 below shows that if H is infinite dimensional then surjectivity in Theorem 5.1 is a necessary condition.

Proposition 5.3. *If H is infinite dimensional then for any $n \in \mathbb{N}$ there exists a (non-surjective) transformation $\phi : H_n \to H_n$ which preserves even principal angles, but there is no $\psi : H_1 \to H_1$ for which (5.2) holds.*

Proposition 5.4 shows that Theorem 5.1 is not valid if $\dim H = 2n \in \mathbb{N}$.

Proposition 5.4. *If $2 \leq n \in \mathbb{N}$ and $\dim H = 2n$, then there exists a bijection $\phi : H_n \to H_n$ which preserves orthogonality in both directions but is not of the form (5.2).*

5.2 Proofs

Proof of Theorem 5.1. For any $M \in H_{(n)}$ let

$$\psi(V) = \overline{\operatorname{span}\{\phi(K) | K \in H_n, K \subseteq V\}}. \tag{5.4}$$

Since $V \in H_{(n)}$, there exists $M \in H_n$ with $M \perp V$. Now for any $K \in H_n$ with $K \subseteq V$, we have $M \perp K$ which implies $\phi(M) \perp \phi(K)$. Thus $\phi(M) \in H_n$ and $\phi(M) \perp \psi(V)$, hence $\psi(V) \in H_{(n)}$. Therefore $\psi : H_{(n)} \to H_{(n)}$.

Let $n \in \mathbb{N} \cup \{\infty\}$ such that (5.1) holds, and let $\phi : H_n \to H_n$ be an operator which preserves orthogonality in both directions. As in the statement of Theorem 5.1, *in the case in which* $\dim H = \infty$ *we also assume that* ϕ *is surjective*. Clearly, if $n = \infty$ then $\dim H = \infty$.

Our theorem will be proved in several steps.

Step 5.1. *For any* $V_1, V_2 \in H_{(n)}$ *we have*

$$V_1 \subseteq V_2 \;\Leftrightarrow\; \psi(V_1) \subseteq \psi(V_2) \quad \textit{and} \quad V_1 \perp V_2 \;\Leftrightarrow\; \psi(V_1) \perp \psi(V_2).$$

Now ψ *is clearly injective.*

Moreover, if $n < \infty$*, then for any* $K \in H_n$ *obviously* $\psi(K) = \phi(K)$ *holds.*

Let $V_1, V_2 \in H_{(n)}$. If $V_1 \subseteq V_2$ then, by (5.4), it is trivial that $\psi(V_1) \subseteq \psi(V_2)$.

If $V_1 \not\subseteq V_2$ then, by $\operatorname{codim} V_2 \geq n$, there exists $M \in H_n$ for which $M \perp V_2$ and $M \not\perp V_1$. Now there exists $N \in H_n$ with $N \subseteq V_1$, $M \not\perp N$. So $\phi(M) \not\perp \phi(N) \subseteq \psi(V_1)$, thus $\phi(M) \not\perp \psi(V_1)$. For any $K \in H_n$, $K \subseteq V_2$ we have $M \perp K$, which implies $\phi(M) \perp \phi(K)$. Hence, by (5.4), $\phi(M) \perp \psi(V_2)$. Thus $\psi(V_1) \not\perp \phi(M) \perp \psi(V_2)$, which yields $\psi(V_1) \not\subseteq \psi(V_2)$.

If $V_1 \perp V_2$ then for any $M, N \in H_n$, $M \subseteq V_1$, $N \subseteq V_2$ we have $\phi(M) \perp \phi(N)$ which gives $\psi(V_1) \perp \psi(V_2)$.

If $V_1 \not\perp V_2$ then there exist $M, N \in H_n$ with $M \subseteq V_1$, $N \subseteq V_2$ such that $M \not\perp N$. Now $\psi(V_1) \supseteq \phi(M) \not\perp \phi(N) \subseteq \psi(V_2)$, thus $\psi(V_1) \not\perp \psi(V_2)$.

Step 5.2. *For any* $V \in H_{(n)}$ *we have*

$$\dim \psi(V) \geq \dim V, \tag{5.5}$$

$$\operatorname{codim} \psi(V) \geq \operatorname{codim} V. \tag{5.6}$$

Hence for any $m \in \mathbb{N} \cup \{\infty\}$ *for which* (5.1) *holds, we have* $\psi : H_{(m)} \to H_{(m)}$*. Moreover, for any* $K \in H_n$ *we have* $\phi(K) = \psi(K)$*, which implies that* ϕ *is also injective.*

If $\dim V = \infty$ then (5.5) is trivial. Assume that $n \leq \dim V < \infty$. Let $V_k \in H_k$ ($n \leq k \leq \dim V$) with $V_n \subsetneqq V_{n+1} \subsetneqq \cdots \subsetneqq V_{\dim V} = V$. By Step 5.1, we have $\psi(V_n) \subsetneqq \psi(V_{n+1}) \subsetneqq \cdots \subsetneqq \psi(V_{\dim V})$, whence $n = \dim \psi(V_n) < \dim \psi(V_{n+1}) < \cdots < \dim \psi(V_{\dim V}) = \dim V$, which implies (5.5).

Now, applying (5.5) to $V^\perp$, we obtain that $\dim \psi(V^\perp) \geq \dim V^\perp$. By Step 5.1, we have $\psi(V) \perp \psi(V^\perp)$, whence $\operatorname{codim} \psi(V) = \dim \psi(V)^\perp \geq \dim \psi(V^\perp) \geq \dim V^\perp = \operatorname{codim} V$ (5.6).

If $\dim H < \infty$ then $\phi(K) = \psi(K)$ is now trivial.

Finally, let $\dim H = \infty$ and let $K \in H_n$ be arbitrary. Now ϕ is surjective by assumption, and, by (5.4), we have $\phi(K) \subseteq \psi(K)$. Suppose on the contrary that $\phi(K) \subsetneqq \psi(K)$. Then there exists $L \in H_n$ with $\phi(L) \not\perp \psi(K)$ and $\phi(L) \perp \phi(K)$. Hence by (5.4) there exists $M \in H_n$, $M \subseteq K$ with $\phi(M) \not\perp \phi(L)$. Now Step 5.1 yields $K \supseteq M \not\perp L \perp K$, which is a contradiction. Therefore $\phi(K) = \psi(K)$.

Step 5.3. *If ϕ is a bijection then define ψ^{-1} for ϕ^{-1} as ψ was defined above for ϕ. Now for any $V \in H_{(n)}$ we have $\psi^{-1}\big(\psi(V)\big) = \psi\big(\psi^{-1}(V)\big) = V$. Thus, if ϕ is a bijection, then so is ψ.*

Let $V \in H_{(n)}$ and $K \in H_n$ be arbitrary with $K \subseteq \psi(V)$. Suppose on the contrary that $\phi^{-1}(K) \not\subseteq V$. Then there exists $L \in H_n$ with $L \not\perp \phi^{-1}(K)$, $L \perp V$. Now $\phi(L) \not\perp K \subseteq \psi(V) \perp \phi(L)$, which is a contradiction. Thus for any $K \in H_n$ with $K \subseteq \psi(V)$ we have $\phi^{-1}(K) \subseteq V$. This yields $\psi^{-1}\big(\psi(V)\big) \subseteq V$. For any $K \in H_n$ with $K \subseteq V$ we have $\phi(K) \subseteq \psi(V)$, which implies $K = \phi^{-1}\big(\phi(K)\big) \subseteq \psi^{-1}\big(\psi(V)\big)$. Hence $V \subseteq \psi^{-1}\big(\psi(V)\big)$, and thus $\psi^{-1}\big(\psi(V)\big) = V$.

Step 5.4. *For any $V \in H_{(n)}$ we have $\dim \psi(V) = \dim V$ and $\psi(V^\perp) = \psi(V)^\perp$.*

If $\dim H < \infty$ then by Steps 5.1 and 5.2 we are done.

If ϕ is bijective then, applying (5.5) to ψ and ψ^{-1}, by Step 5.3 we obtain that $\dim V = \dim \psi^{-1}\big(\psi(V)\big) \geq \dim \psi(V) \geq \dim V$, which gives $\dim \psi(V) = \dim V$. Moreover,

$$\psi^{-1}\big(\psi(V^\perp)^\perp\big) \perp \psi^{-1}\big(\psi(V^\perp)\big) = V^\perp,$$

whence $\psi^{-1}\big(\psi(V^\perp)^\perp\big) \subseteq V$. Steps 5.1 and 5.3 now yield that $\psi(V^\perp)^\perp \subseteq \psi(V)$. This implies $\psi(V^\perp) \supseteq \psi(V)^\perp \supseteq \psi(V^\perp)$.

Step 5.5. *For any $X \in H_1$ and $V \in H_{(n)}$ with $V \perp X$, let*

$$\psi_V(X) = \psi(V)^\perp \cap \psi(V + X).$$

Then $\dim \psi_V(X) = 1$, and $\psi_V(X)$ does not depend on V. Now let $\psi(X) = \psi_V(X)$.

Let $V \in H_{(n)}$ be arbitrary such that $V \perp X$. Step 5.1 implies $\psi(V) \subsetneqq \psi(V + X)$, whence $\dim \psi_V(X) \geq 1$. If $\dim H < \infty$ then, by Step 5.4, we infer that $\dim \psi(V + X) = \dim (V + X) = \dim (V) + 1 = \dim \psi(V) + 1$, whence $\dim \psi_V(X) = 1$. Suppose temporarily that $\dim H = \infty$. Then ϕ and ψ are bijections. If $\dim \psi_V(X) > 1$ then there exists $U \in H_{(n)}$ such that $\psi(V) \subsetneqq U \subsetneqq \psi(V + X)$. By Step 5.1 now we get $V \subsetneqq \psi^{-1}(U) \subsetneqq V + X$, which is a contradiction. Thus

$$\dim \psi_V(X) = 1 \tag{5.7}$$

for any $V \in H_{(n)}$ with $V \perp X$.

We now show that for any $V_1, V_2 \in H_{(n)}$ with $V_1 \subseteq V_2$ we have $\psi_{V_1}(X) = \psi_{V_2}(X)$. Suppose on the contrary that $\psi_{V_1}(X) \neq \psi_{V_2}(X)$. By Step 5.1 we have

$$\begin{aligned} \psi(V_2)^{\perp} \cap \psi(V_1 + X) \subseteq& \psi(V_1)^{\perp} \cap \psi(V_1 + X) = \psi_{V_1}(X), \\ \psi(V_2)^{\perp} \cap \psi(V_1 + X) \subseteq& \psi(V_2)^{\perp} \cap \psi(V_2 + X) = \psi_{V_2}(X), \end{aligned}$$

whence

$$\psi(V_2)^{\perp} \cap \psi(V_1 + X) \subseteq \psi_{V_1}(X) \cap \psi_{V_2}(X) = \{0\}.$$

Thus $\psi(V_2) \perp \psi(V_1 + X)$. Then, using Step 5.1, we deduce that $V_1 \subseteq V_2 \perp V_1 + X$ which is a contradiction. Consequently, $\psi_{V_1}(X) = \psi_{V_2}(X)$.

Assume that $\dim H = \infty$ and let $U, V \in H_{(n)}$ be arbitrary with $U, V \perp X$. It is easy to see that there exist $U_1, V_1 \in H_{(n)}$ such that $U_1 \subseteq U$, $V_1 \subseteq V$ and $U_1 + V_1 \in H_{(n)}$. Then we have $\psi_U(X) = \psi_{U_1}(X) = \psi_{U_1+V_1}(X) = \psi_{V_1}(X) = \psi_V(X)$.

Now assume that $\dim H < \infty$ and let $U, V \in H_{(n)}$ be arbitrary with $U, V \perp X$. From (5.1) it follows easily that there exist $U_0, \ldots, U_s \in H_{(n)}$ such that $U_0 = U$, $U_s = V$, $U_0, \ldots, U_s \perp X$ and $U_i \subseteq U_{i+1}$ or $U_i \supseteq U_{i+1}$ $(0 \leq i \leq s-1)$. Now applying the above results, we obtain that $\psi_U(X) = \psi_{U_0}(X) = \psi_{U_1}(X) = \cdots = \psi_{U_s}(X) = \psi_V(X)$.

Step 5.6. *For any $X \in H_1$ and $V \in H_{(n)}$, $X \subseteq V$ implies that $\psi(X) \subseteq \psi(V)$.*

Let $X \in H_1$ and $V \in H_{(n)}$ with $X \subseteq V$. If $\dim V > n$ or $n = \infty$ then there exists $V_0 \in H_{(n)}$ for which $V_0 \subseteq V \cap X^{\perp}$, whence, by Step 5.5, we obtain that

$$\psi(X) = \psi_{V_0}(X) \subseteq \psi(V_0 + X) \subseteq \psi(V).$$

Now assume that $n < \infty$ and $\dim V = n$. For any $U \in H_n$ with $U \subseteq V^{\perp}$, we have $\psi(X) = \psi_U(X) = \phi(U)^{\perp} \cap \psi(U + X) \perp \phi(U)$, and thus $\psi(X) \perp \psi(V^{\perp})$. Step 5.4 now yields $\psi(X) \subseteq \psi(V^{\perp})^{\perp} = \psi(V)$.

Step 5.7. *For any $X, Y \in H_1$, we have*

$$X \perp Y \;\Leftrightarrow\; \psi(X) \perp \psi(Y).$$

Suppose that $X \perp Y$. Then there are $M, N \in H_n$ such that $M \perp X$, $M \perp Y$, $N \perp X$, $N \perp M$ and $Y \subseteq N$. By Steps 5.1 and 5.6, we obtain that $\psi(X) \subseteq \psi(X + M) \perp \phi(N) \supseteq \psi(Y)$, whence $\psi(X) \perp \psi(Y)$.

Now suppose that $X \not\perp Y$ and suppose on the contrary that $\psi(X) \perp \psi(Y)$. Then there exists an $M \in H_n$ with $M \perp X$, $M \perp Y$. By Step 5.6, it is easy to see that $\phi(M) \perp \psi(X)$ and $\phi(M) \perp \psi(Y)$. Thus $\psi(M + X) = \psi(X) \oplus \phi(M) \perp \psi(Y)$. Let $K_X, N \in H_n$ such that $X \subseteq K_X \subseteq M + X$ and $Y, K_X \perp N$. Then $\phi(K_X) \perp \phi(N)$ and $\phi(K_X) \subseteq \psi(M + X) \perp \psi(Y)$. Hence $\psi(Y + N) = \psi(Y) \oplus \phi(N) \perp \phi(K_X)$. Now let $K_Y \in H_n$ with $Y \subseteq K_Y \subseteq Y + N$. Then $\phi(K_Y) \subseteq \psi(Y + N) \perp \phi(K_X)$, which implies $\phi(K_X) \perp \phi(K_Y)$. This leads to $X \subseteq K_X \perp K_Y \supseteq Y$, thus $X \perp Y$, which is a contradiction. Therefore $\psi(X) \not\perp \psi(Y)$ indeed.

Step 5.8. *If ϕ is a bijection then $\psi : H_1 \to H_1$ is also a bijection.*

Let ϕ be a bijection. Now there exists $\psi^{-1} : H_1 \to H_1$ corresponding to ϕ^{-1} as ψ corresponds to ϕ. We shall apply the above results also to ψ^{-1}. Let $V \in H_1$. Then there exist $V_1, V_2 \in H_n$ for which $V = V_1 \cap V_2$. Hence $\psi^{-1}(V) \subseteq \phi^{-1}(V_1) \cap \phi^{-1}(V_2)$, thus $\psi\big(\psi^{-1}(V)\big) \subseteq \phi\big(\phi^{-1}(V_1)\big) = V_1$ and $\psi\big(\psi^{-1}(V)\big) \subseteq \phi\big(\phi^{-1}(V_2)\big) = V_2$. This implies that $\{0\} \neq \psi\big(\psi^{-1}(V)\big) \subseteq V_1 \cap V_2 = V$. By $\dim V = 1$, we obtain that $\psi\big(\psi^{-1}(V)\big) = V$. Similarly, $\psi^{-1}\big(\psi(V)\big) = V$. Now it is already clear that ψ is a bijection.

Step 5.9. *For any $V \in H_{(n)}$, we have $\psi(V) = \operatorname{span}\big\{\psi(X) | X \in H_1, X \subseteq V\big\}$.*

By Step 5.6, it is clear that $\psi(V) \supseteq \operatorname{span}\{\psi(X) | X \in H_1, X \subseteq V\}$.

If $\dim H < \infty$ then, by Steps 5.4 and 5.7, we are ready.

Now let ϕ be surjective and let $Y \in H_1$ be arbitrary with $Y \subseteq \psi(V)$. Then, by Step 5.8, ψ is also a bijection. Applying Steps 5.3 and 5.6 to ψ^{-1}, we deduce that $\psi^{-1}(Y) \subseteq \psi^{-1}\big(\psi(V)\big) = V$. Hence $Y = \psi\big(\psi^{-1}(Y)\big) \subseteq \operatorname{span}\{\psi(X) | X \in H_1, X \subseteq V\}$. Thus $\psi(V) \subseteq \operatorname{span}\{\psi(X) | X \in H_1, X \subseteq V\}$, which completes the proof.

Step 5.10. *If ϕ preserves principal angles then ψ also preserves angles.*

Let $X, Y \in H_1$ be arbitrary, and let $N_X, N_Y \in H_{(n-1)}$ be orthogonal subspaces which are also orthogonal both to X and to Y, and for which $K_X = X \oplus N_X \in H_n$ and $K_Y = Y \oplus N_Y \in H_n$. Let e_α $(\alpha \in A)$ be pairwise orthogonal 1-dimensional subspaces of N_X with $N_X = \operatorname{span}\{e_\alpha | \alpha \in A\}$, and similarly, let f_β $(\beta \in B)$ be pairwise orthogonal 1-dimensional subspaces of N_Y with $N_Y = \operatorname{span}\{f_\beta | \beta \in B\}$.

By Steps 5.7 and 5.9, we have

$$\begin{aligned} P_{K_X} &= P_X + \sum_{\alpha \in A} P_{e_\alpha}, \quad P_{\phi(K_X)} = P_{\psi(X)} + \sum_{\alpha \in A} P_{\psi(e_\alpha)}, \\ P_{K_Y} &= P_Y + \sum_{\beta \in B} P_{f_\alpha}, \quad P_{\phi(K_Y)} = P_{\psi(Y)} + \sum_{\beta \in B} P_{\psi(f_\beta)}, \end{aligned}$$

and so

$$\begin{aligned} &P_{\psi(X)} P_{\psi(Y)} P_{\psi(X)} \\ &= \Big(P_{\psi(X)} + \sum_{\alpha \in A} P_{\psi(e_\alpha)}\Big)\Big(P_{\psi(Y)} + \sum_{\beta \in B} P_{\psi(f_\beta)}\Big)\Big(P_{\psi(X)} + \sum_{\alpha \in A} P_{\psi(e_\alpha)}\Big) \\ &= P_{\phi(K_X)} P_{\phi(K_Y)} P_{\phi(K_X)} = U_{K_X,K_Y} P_{K_X} P_{K_Y} P_{K_X} U^*_{K_X,K_Y} \\ &= U_{K_X,K_Y}\Big(P_X + \sum_{\alpha \in A} P_{e_\alpha}\Big)\Big(P_Y + \sum_{\beta \in B} P_{f_\alpha}\Big)\Big(P_X + \sum_{\alpha \in A} P_{e_\alpha}\Big) U^*_{K_X,K_Y} \\ &= U_{K_X,K_Y} P_X P_Y P_X U^*_{K_X,K_Y}. \end{aligned}$$

Now, by Steps 5.7, 5.8, 5.9 and 5.10, the proof of Theorem 5.1 is complete. □

Proof of Proposition 5.3. Let $L \in H_n$ be fixed and $e_i \in H$ $(i \in \mathbb{N})$ be pairwise orthogonal vectors of norm 1, for which $L = \operatorname{span}\{e_1, \dots, e_n\}$ and $\dim\big(\operatorname{span}\{e_i | i \in \mathbb{N}\}^\perp\big) = \infty$. Set $E = \operatorname{span}\{e_i | i \in \mathbb{N}\}$. Let $S(e_i) = e_{i+n}$ $(i \in \mathbb{N})$ and $S(u) = u$ $(u \in E^\perp)$. Then the linear transformation $S : H \to H$ is well-defined, $S^*S = 1$ and $P_L S = S^* P_L = 0$. For any $K \in H_n$, let

$$\phi(K) = \begin{cases} S(K) & \text{if } K \neq L, \\ (S+1)(L) & \text{if } K = L. \end{cases}$$

Then $P_{\phi(K)} = SP_K S^*$ for any $K \in H_n$, $K \neq L$, and $P_{\phi(L)} = SP_L S^* + P_L$. Now for any $K_1, K_2 \in H_n$ with $K_1, K_2 \neq L$, we have

$$P_{\phi(K_1)} P_{\phi(K_2)} P_{\phi(K_1)} = SP_{K_1} S^* SP_{K_2} S^* SP_{K_1} S^* = SP_{K_1} P_{K_2} P_{K_1} S^*,$$

and

$$\begin{aligned} P_{\phi(K_1)} P_{\phi(L)} P_{\phi(K_1)} &= SP_{K_1} S^* (SP_L S^* + P_L) SP_{K_1} S^* \\ &= SP_{K_1} P_L P_{K_1} S^* + SP_{K_1} S^* P_L SP_{K_1} S^* = SP_{K_1} P_L P_{K_1} S^*. \end{aligned}$$

Let $K_1, K_2 \in H_n$ be arbitrary, let $f_i \in K_1^\perp \cap E^\perp$ $(i \in \mathbb{N})$ be pairwise orthogonal vectors of norm 1, and let $F = \operatorname{span}\{f_i | i \in \mathbb{N}\}$. Define the operator U_{K_1,K_2} as follows. Let $U^*_{K_1,K_2}(e_i) = f_i$ $(1 \leq i \leq n)$, $U^*_{K_1,K_2}(e_i) = e_{i-n} = S^* e_i$ $(i > n)$, $U^*_{K_1,K_2}(f_i) = f_{i+n} \in K_1^\perp$ $(i \in \mathbb{N})$ and $U^*_{K_1,K_2}(u) = u = S^* u$ $(u \in E^\perp \cap F^\perp)$. Then the operator U_{K_1,K_2} is well-defined. It is easy to see that U_{K_1,K_2} is a unitary operator and $P_{K_1} U^*_{K_1,K_2} = P_{K_1} S^*$, whence $U_{K_1,K_2} P_{K_1} = SP_{K_1}$. Now

$$P_{\phi(K_1)} P_{\phi(L)} P_{\phi(K_1)} = SP_{K_1} P_{K_2} P_{K_1} S^* = U_{K_1,K_2} P_{K_1} P_{K_2} P_{K_1} U^*_{K_1,K_2}.$$

Thus ϕ preserves principal angles, but for any $K \neq L$ we have $\phi(K) \cap \phi(L) = \{0\}$. The proof of Proposition 5.3 is now complete. □

Proof of Proposition 5.4. Let

$$\mathcal{A} = \big\{\{K, K^\perp\} | K \in H_n\big\}.$$

By the axiom of choice, there exists a set A which contains exactly one element of each set in $\mathcal{A}$. Let $\xi : A \to A$ be an arbitrary bijection, and define $\phi : H_n \to H_n$ by

$$\phi(K) = \begin{cases} \xi(K) & \text{if } K \in A, \\ \xi(K^\perp)^\perp & \text{otherwise.} \end{cases}$$

It is clear that $\phi : H_n \to H_n$ is a bijection which preserves orthogonality in both directions. If $n \geq 2$ then it is easy to see that we may choose a bijection ξ such that ϕ is not of the form (5.2). □

Part II

SOME REFLEXIVITY PROBLEMS ON FUNCTION ALGEBRAS

6. ON THE REFLEXIVITY OF THE AUTOMORPHISM AND ISOMETRY GROUPS OF THE SUSPENSION OF $B(H)$

6.1 Introduction and Statement of the Results

In this chapter we study the reflexivity of the automorphism group and the isometry group of the suspension of $B(H)$. The suspension $S\mathcal{A}$ of a C^*-algebra $\mathcal{A}$ is the tensor product $S\mathcal{A} = C_0(\mathbb{R}) \otimes \mathcal{A}$, which is well-known to be isomorphic to the C^*-algebra $C_0(\mathbb{R}, \mathcal{A})$ of all continuous functions from $\mathbb{R}$ to $\mathcal{A}$ which vanish at infinity. The concept of suspension plays a very important role in the K-theory of C^*-algebras, since the K_1-group of $\mathcal{A}$ is the K_0-group of $S\mathcal{A}$. In Corollary 6.5 below, we obtain that the automorphism and isometry groups of the supsension of $B(H)$ are algebraically reflexive. In fact, here we consider more general C^*-algebras of the form $C_0(X) \otimes B(H) \cong C_0\big(X, B(H)\big)$, where X is a locally compact Hausdorff space. The content of this chapter appeared in our paper [69].

From now on, let H stand for an infinite dimensional separable Hilbert space. Our first result describes the form of the automorphisms and the surjective linear isometries of the suspension of $B(H)$.

Theorem 6.1. *Let X be a locally compact Hausdorff space. A linear map $\phi : C_0\big(X, B(H)\big) \to C_0\big(X, B(H)\big)$ is an automorphism if and only if there exists a function $\tau : X \to \mathrm{Aut}\big(B(H)\big)$ and a bijection $\varphi : X \to X$ such that*

$$\phi(f)(x) = [\tau(x)]\Big(f\big(\varphi(x)\big)\Big) \qquad \Big(f \in C_0\big(X, B(H)\big), x \in X\Big). \tag{6.1}$$

Similarly, a linear map $\phi : C_0\big(X, B(H)\big) \to C_0\big(X, B(H)\big)$ is a surjective isometry if and only if there exists a function $\tau : X \to \mathrm{Iso}\big(B(H)\big)$ and a bijection $\varphi : X \to X$ such that ϕ is of the form (6.1)*.*

Moreover, if the linear map $\phi : C_0\big(X, B(H)\big) \to C_0\big(X, B(H)\big)$ is an automorphism or a surjective isometry, then for the maps τ, φ appearing in (6.1)*, the mappings $x \mapsto \tau(x)$ and $x \mapsto \tau(x)^{-1}$ are strongly continuous, and $\varphi : X \to X$ is a homeomorphism.*

The following two results show that in the case of $C_0(X)$ the algebraic reflexivity of its automorphism and isometry groups implies the algebraic reflexivity of $\mathrm{Aut}\big(C_0(X) \otimes B(H)\big)$ and $\mathrm{Iso}\big(C_0(X) \otimes B(H)\big)$, respectively.

Theorem 6.2. *Let X be a locally compact Hausdorff space. If the automorphism group of $C_0(X)$ is algebraically reflexive then the automorphism group of $C_0\big(X, B(H)\big)$ is also algebraically reflexive.*

Theorem 6.3. *Let X be a σ-compact locally compact Hausdorff space. If the isometry group of $C_0(X)$ is algebraically reflexive then so is the isometry group of $C_0\big(X, B(H)\big)$.*

To obtain the algebraic reflexivity of the automorphism and isometry groups of the suspension of $B(H)$, we prove the following assertion.

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^n$ be an open convex set. The automorphism and isometry groups of $C_0(\Omega)$ are algebraically reflexive.*

The proof of this result will show how difficult it might be to treat our reflexivity problem for tensor product of general C^*-algebras or even for the suspension of any C^*-algebra with algebraically reflexive automorphism and isometry groups.

Finally, we obtain the following corollary, which can be considered as the main result of this chapter.

Corollary 6.5. *The automorphism and isometry groups of the suspension of $B(H)$ are algebraically reflexive.*

As for the natural question of whether the groups above are topologically reflexive, we give an immediate negative answer as follows.

Example 6.6. *Let (φ_n) be a sequence of homeomorphisms of $\mathbb{R}$ which converges uniformly to a non-injective function φ. Define linear maps ϕ_n, ϕ on $C_0\big(\mathbb{R}, B(H)\big)$ by*

$$\phi_n(f) = f \circ \varphi_n \quad \text{and} \quad \phi(f) = f \circ \varphi \qquad \Big(f \in C_0\big(\mathbb{R}, B(H)\big), n \in \mathbb{N}\Big).$$

Then ϕ_n is an isometric automorphism of $C_0\big(\mathbb{R}, B(H)\big)$, the sequence $\big(\phi_n(f)\big)$ converges to $\phi(f)$ for every $f \in C_0\big(\mathbb{R}, B(H)\big)$, but ϕ is not surjective.

The topological reflexive closures of the isometry group and the group of *-automorphisms of the suspension of $B(H)$ are determined in Chapter 7; this answers a question raised by the referee of our paper [69].

6.2 Proofs

We begin with the following lemma on a characterization of certain closed ideals in $C_0\big(X, B(H)\big)$.

Lemma 6.7. *Let X be a locally compact Hausdorff space. A closed ideal $\mathcal{I}$ in $C_0\big(X, B(H)\big)$ is of the form*

$$\mathcal{I} = \mathcal{I}_{x_0} = \big\{f \in C_0\big(X, B(H)\big) \,:\, f(x_0) = 0\big\}$$

for some point $x_0 \in X$ *if and only if* $\mathcal{J}$ *is a proper subset of a maximal ideal* $\mathcal{J}_m$ *in* $C_0\big(X, B(H)\big)$*, there is no closed ideal properly between* $\mathcal{J}$ *and* $\mathcal{J}_m$*, and* $\mathcal{J}$ *is not the intersection of two distinct maximal ideals in* $C_0\big(X, B(H)\big)$*.*

Proof of Lemma 6.7. The structure of closed ideals in Banach algebras of vector valued functions is well-known. See, for example, [75, Remark on p. 342]. Using this result, $\mathcal{J}$ is a closed ideal in $C_0\big(X, B(H)\big)$ if and only if it is of the form

$$\mathcal{J} = \big\{f \in C_0\big(X, B(H)\big) \,:\, f(x) \in \mathcal{I}_x\big\},$$

where every $\mathcal{I}_x$ is a closed ideal of $B(H)$, i.e., by the separability of H, every $\mathcal{I}_x$ is either $\{0\}$ or $\mathcal{C}(H)$ or $B(H)$. By the help of Urysohn's lemma on the construction of continuous functions on X with compact support, one can readily verify that the maximal ideals in $C_0\big(X, B(H)\big)$ are exactly those ideals which are of the form

$$\mathcal{J} = \big\{f \in C_0\big(X, B(H)\big) \,:\, f(x_0) \in \mathcal{C}(H)\big\}$$

for some point $x_0 \in X$. Now, the statement of Lemma 6.7 follows quite easily. □

Proof of Theorem 6.1. First we prove the statement on isometries. Let ϕ be a surjective linear isometry of $C_0\big(X, B(H)\big)$. As a consequence of a deep result due to Kaup (see, for example, [19]), we obtain that every surjective linear isometry ϕ between C^*-algebras $\mathcal{A}$ and B has a certain algebraic property, namely ϕ is a triple isomorphism, i.e. it satisfies the equality

$$\phi(ab^*c) + \phi(cb^*a) = \phi(a)\phi(b)^*\phi(c) + \phi(c)\phi(b)^*\phi(a)$$

for every $a, b, c \in \mathcal{A}$. This implies that ϕ preserves the closed ideals in both directions. Indeed, if $\mathcal{I} \subset \mathcal{A}$ is a closed ideal, then, by $\mathcal{I} = \mathcal{I}^*$, we have

$$\phi(a)\phi(b)^*\phi(c) + \phi(c)\phi(b)^*\phi(a) \in \phi(\mathcal{I}) \qquad (a, c \in \mathcal{A}, b \in \mathcal{I}).$$

Let $\mathcal{I}' = \phi(\mathcal{I})$. We infer that $a'\mathcal{I}'^*c' + c'\mathcal{I}'^*a' \in \mathcal{I}'$ $(a', c' \in B)$. Since $\mathcal{I}'$ is a closed linear subspace of B, if c' runs through an approximate identity, we deduce

$$a'\mathcal{I}'^* + \mathcal{I}'^*a' \in \mathcal{I}' \qquad (a' \in B). \tag{6.2}$$

If now a' runs through an approximate identity, then we have

$$\mathcal{I}'^* \subset \mathcal{I}'. \tag{6.3}$$

We infer from (6.2) and (6.3) that $a'\mathcal{I}' + \mathcal{I}'a' \subset \mathcal{I}'$ $(a' \in B)$, i.e. $\mathcal{I}'$ is a closed Jordan ideal of B. It is well-known that in the case of C^*-algebras, every closed Jordan ideal is an (associative) ideal (see e.g. [17, Theorem 5.3]) and hence the same is true for $\mathcal{I}'$.

In view of Lemma 6.7, we infer that our map ϕ preserves the ideals

$$\mathcal{J}_x = \big\{f \in C_0\big(X, B(H)\big) \,:\, f(x) = 0\big\} \qquad (x \in X)$$

in both directions. This gives us that there exists a bijection $\varphi : X \to X$ for which

$$\phi(f)(x) = 0 \Longleftrightarrow f\big(\varphi(x)\big) = 0 \tag{6.4}$$

holds for every $f \in C_0\big(X, B(H)\big)$ and $x \in X$. For any $x \in X$, define $\tau(x)$ by

$$[\tau(x)]\Big(f\big(\varphi(x)\big)\Big) = \phi(f)(x) \qquad \Big(f \in C_0\big(X, B(H)\big)\Big). \tag{6.5}$$

Because of (6.4), we obtain that $\tau(x)$ is a well-defined injective linear map on $B(H)$. Since ϕ is surjective, we have the surjectivity of $\tau(x)$. Now, we compute

$$\begin{aligned}[\tau(x)]\Big(f\big(\varphi(x)\big)g\big(\varphi(x)\big)^* f\big(\varphi(x)\big)\Big) &= \phi(fg^*f)(x) = \phi(f)(x)\phi(g)(x)^*\phi(f)(x)\\ &= [\tau(x)]\Big(f\big(\varphi(x)\big)\Big)\Big([\tau(x)]\Big(g\big(\varphi(x)\big)\Big)\Big)^*[\tau(x)]\Big(f\big(\varphi(x)\big)\Big)\end{aligned}$$

for every $f, g \in C_0\big(X, B(H)\big)$. This implies that $\tau(x)$ is a triple automorphism of $B(H)$. Since the triple homomorphisms preserve the partial isometries and every operator with norm less than 1 is the average of unitaries, it follows that $\tau(x)$ is a contraction. Applying the same argument to the inverse of $\tau(x)$, we get $\tau(x) \in \mathrm{Iso}\big(B(H)\big)$. This proves that ϕ is of the form (6.1) as requested.

Let now $\phi : C_0\big(X, B(H)\big) \to C_0\big(X, B(H)\big)$ be a linear map of the form

$$\phi(f)(x) = [\tau(x)]\Big(f\big(\varphi(x)\big)\Big) \qquad \Big(f \in C_0\big(X, B(H)\big), x \in X\Big), \tag{6.6}$$

where $\tau : X \to \mathrm{Iso}\big(B(H)\big)$, and $\varphi : X \to X$ is a bijection. The function φ is continuous. Indeed, this follows easily from the equality $\big\|f\big(\varphi(x)\big)\big\| = \|\phi(f)(x)\|$ and from Urysohn's lemma. To see the strong continuity of $\tau : X \to \mathrm{Iso}\big(B(H)\big)$, let (x_α) be a net in X converging to $x \in X$. Let $y_\alpha = \varphi(x_\alpha)$ and $y = \varphi(x)$. We may suppose that every y_α belongs to a fixed compact neighbourhood of y. If $f \in C_0(X)$ is identically 1 on this neighbourhood, then, for every operator $A \in B(H)$, we have

$$\begin{aligned}[\tau(x_\alpha)](A) = [\tau(x_\alpha)]\Big(f\big(\varphi(x_\alpha)\big)A\Big) &= \phi(fA)(x_\alpha)\\ &\longrightarrow \phi(fA)(x) = [\tau(x)]\Big(f\big(\varphi(x)\big)A\Big) = [\tau(x)](A).\end{aligned}$$

Next, from the equality

$$\begin{aligned}\big\|[\tau(x_\alpha)^{-1}](A) - [\tau(x)^{-1}](A)\big\| &= \big\|[\tau(x_\alpha)^{-1}\tau(x)\tau(x)^{-1}](A) - [\tau(x)^{-1}](A)\big\|\\ &= \big\|[\tau(x)]\big([\tau(x)^{-1}](A)\big) - [\tau(x_\alpha)]\big([\tau(x)^{-1}](A)\big)\big\|,\end{aligned}$$

we get the strong continuity of the map $x \mapsto \tau(x)^{-1}$. We prove that φ^{-1} is also continuous. Since ϕ maps into $C_0\big(X, B(H)\big)$, it is quite easy to see from (6.6) that

$f \circ \varphi \in C_0(X)$ for every $f \in C_0(X)$. If $K \subset X$ is an arbitrary compact set and $f \in C_0(X)$ is a function which is identically 1 on K, then, by $f \circ \varphi \in C_0(X)$, there exists a compact set $K' \subset X$ for which $\varphi(x) \in K^c$ holds for all $x \in K'^c$. Thus, we have $K \subset \varphi(K')$. Let (x_α) be a net in X such that $\big(\varphi(x_\alpha)\big)$ converges to some $\varphi(x)$. Obviously, we may suppose that every $\varphi(x_\alpha)$ belongs to a compact neighbourhood K of $\varphi(x)$. By what we have just seen, there exists a compact set $K' \subset X$ which contains the net (x_α) and the point x as well. Since K' is compact, the net (x_α) has a convergent subnet. Because of the continuity of the bijection φ, it is easy to see that the limit of this subnet is x. The continuity of φ^{-1} is now obvious. Finally, one can verify quite readily that ϕ is a surjective linear isometry of $C_0\big(X, B(H)\big)$.

We now turn to the proof of our statement concerning automorphisms. So, let ϕ be an automorphism of $C_0\big(X, B(H)\big)$. Every automorphism Ψ of a C^*-algebra $\mathcal{A}$ is continuous and its norm equals the norm of its inverse. This follows, for example, from [83, 4.1.12. Lemma], and from the proof of [83, 4.1.13. Proposition], where it is proved that $\|a\|/\|\Psi\| \leq \|\Psi(a)\|$ $(a \in \mathcal{A})$ which implies $\|\Psi^{-1}\| \leq \|\Psi\|$. Using these facts, one can get the form (6.1) in a similar way as in the case of isometries. Let now $\phi : C_0\big(X, B(H)\big) \to C_0\big(X, B(H)\big)$ be a linear map of the form

$$\phi(f)(x) = [\tau(x)]\Big(f\big(\varphi(x)\big)\Big) \qquad \Big(f \in C_0\big(X, B(H)\big), x \in X\Big), \tag{6.7}$$

where $\tau : X \to \operatorname{Aut}\big(B(H)\big)$, and $\varphi : X \to X$ is a bijection. We show that φ is continuous. Let (x_α) be a net in X converging to $x \in X$. By (6.7), we have

$$f\big(\varphi(x_\alpha)\big)I = [\tau(x_\alpha)]\Big(f\big(\varphi(x_\alpha)\big)I\Big) \longrightarrow [\tau(x)]\Big(f\big(\varphi(x)\big)I\Big) = f\big(\varphi(x)\big)I$$

for every $f \in C_0(X)$. Referring again to Urysohn's lemma, we infer that $\varphi(x_\alpha) \to \varphi(x)$. This verifies the continuity of φ. We claim that the function τ is bounded. In fact, by the principle of uniform boundedness, in the opposite case we would obtain that there exists an operator $A \in B(H)$ for which $[\tau(.)](A)$ is not bounded. Then there is a sequence (x_n) in X with the property that $\|[\tau(x_n)](A)\| > n^3$ $(n \in \mathbb{N})$. Using Urysohn's lemma, it is an easy task to construct a nonnegative function $f \in C_0(X)$ for which $f\big(\varphi(x_n)\big) \geq 1/n^2$. Indeed, for every $n \in \mathbb{N}$ let $f_n : X \to [0,1]$ be a continuous function with compact support such that $f_n\big(\varphi(x_n)\big) = 1$, and define $f = \sum_n (1/n^2) f_n$. We have $\|\phi(fA)(x_n)\| = \big\|f\big(\varphi(x_n)\big)[\tau(x_n)](A)\big\| > n$ $(n \in \mathbb{N})$ which contradicts the boundedness of the function $\phi(fA)$. The strong continuity of τ can be proved as in the case of isometries. Using the inequality

$$\begin{aligned}
\big\|[\tau(x_\alpha)^{-1}](A) - [\tau(x)^{-1}](A)\big\| \\
&= \big\|[\tau(x_\alpha)^{-1}\tau(x)\tau(x)^{-1}](A) - [\tau(x)^{-1}](A)\big\| \\
&\leq \big\|\tau(x_\alpha)^{-1}\big\| \cdot \Big\|[\tau(x)]\big([\tau(x)^{-1}](A)\big) - [\tau(x_\alpha)]\big([\tau(x)^{-1}](A)\big)\Big\| \\
&= \|\tau(x_\alpha)\| \cdot \Big\|[\tau(x)]\big([\tau(x)^{-1}](A)\big) - [\tau(x_\alpha)]\big([\tau(x)^{-1}](A)\big)\Big\|
\end{aligned}$$

and the boundedness of τ, we get the strong continuity of the map $x \mapsto \tau(x)^{-1}$. The proof can be completed as in the case of isometries. □

The following two lemmas are needed in the proof of Theorem 6.2.

Lemma 6.8. *Let τ, τ_1 and τ_2 be automorphisms of $B(H)$, and let $\lambda \in \mathbb{C}$, $0 \neq \lambda_1, \lambda_2 \in \mathbb{C}$ be scalars such that*

$$\lambda\tau(A) = \lambda_1\tau_1(A) + \lambda_2\tau_2(A) \qquad \big(A \in B(H)\big).$$

Then we have $\tau_1 = \tau_2$.

Proof of Lemma 6.8. The automorphisms of $B(H)$ are all spatial (see, for example, [16, 3.2. Corollary]), hence there exist invertible operators $T, T_1, T_2 \in B(H)$ such that

$$\lambda TAT^{-1} = \lambda_1 T_1 A {T_1}^{-1} + \lambda_2 T_2 A {T_2}^{-1} \qquad \big(A \in B(H)\big). \tag{6.8}$$

It is clear that if $a, b, x, y, u, v \in X$ and

$$a \otimes b = x \otimes y + u \otimes v,$$

then either $\{x, u\}$ or $\{y, v\}$ is linearly dependent. Using this elementary observation and putting $A = x \otimes y$ into (6.8), we infer that either $\{T_1 x, T_2 x\}$ is linearly dependent for all $x \in H$, or $\{{T_1}^{-1^*} y, {T_2}^{-1^*} y\}$ is linearly dependent for all $y \in H$. In both cases we have the linear dependence of $\{T_1, T_2\}$ which results in $\tau_1 = \tau_2$. □

In the proof of the next lemma we need the concept of Jordan homomorphisms. A linear map ϕ between algebras $\mathcal{A}$ and B is called a Jordan homomorphism if

$$\phi(A)^2 = \phi(A^2) \qquad (A \in \mathcal{A}).$$

If, in addition, $\mathcal{A}$ and B have involutions and

$$\phi(A)^* = \phi(A^*) \qquad (A \in \mathcal{A}),$$

then we say that ϕ is a Jordan *-homomorphism.

Lemma 6.9. *Let $\phi : B(H) \to B(H)$ be a bounded linear map with the property that for every $A \in B(H)$ there exists a number $\lambda_A \in \mathbb{C}$ and an automorphism $\tau_A \in \mathrm{Aut}\big(B(H)\big)$ such that $\phi(A) = \lambda_A \tau_A(A)$. Then there exists a number $\lambda \in \mathbb{C}$ and an automorphism $\tau \in \mathrm{Aut}\big(B(H)\big)$ such that $\phi(A) = \lambda\tau(A)$ $\big(A \in B(H)\big)$.*

Proof of Lemma 6.9. First suppose that $\phi(I) = 0$. Assume that there exists a projection $0, I \neq P \in B(H)$ for which $\phi(P) \neq 0$. Applying an appropriate transformation, we may suppose that $\phi(P) = P$. Then we have $\phi(I - P) = -P$. If ϵ, δ are distinct nonzero numbers, then, by our assumption, we infer that $\phi\big(\epsilon P + \delta(I - P)\big)$ is a scalar multiple of an invertible operator which, on the other hand, equals $(\epsilon - \delta)P$. This clearly implies that $\epsilon = \delta$, which is a contradiction. Hence, we obtain that $\phi(P) = 0$ holds for every projection $P \in B(H)$. Using the spectral theorem and the continuity of ϕ, we conclude that $\phi = 0$.

Next suppose that $\phi(I) \neq 0$. Obviously, we may assume that $\phi(I) = I$. Using the linearity of ϕ, for an arbitrary projection $0, I \neq P \in B(H)$

$$I = \phi(I) = \phi(P) + \phi(I - P) = \lambda_P Q + \lambda_{I-P} R$$

follows, where Q, R are idempotents different from $0, I$. Taking squares on both sides in the equality $\lambda_P Q = I - \lambda_{I-P} R$, we infer that

$$\lambda_P^2 Q = I + \lambda_{I-P}^2 R - 2\lambda_{I-P} R.$$

But we also have

$$\lambda_P^2 Q = \lambda_P (I - \lambda_{I-P} R).$$

Comparing these equalities and using $R \neq 0, I$, we deduce that $\lambda_P = 1$. This means that $\phi(P)$ is an idempotent. Therefore, ϕ sends projections to idempotents. Now, a standard argument shows that ϕ is a Jordan endomorphism of $B(H)$ (see, for example, the proof of [54, Theorem 2]). Clearly, the range of ϕ contains a rank-one operator (e.g. a rank-one idempotent) and an operator with dense range (e.g. the identity). Using [54, Theorem 1], we infer that ϕ is either an automorphism or an antiautomorphism. This latter concept means that ϕ is a bijective linear map with the property that $\phi(AB) = \phi(B)\phi(A)$ $\big(A, B \in B(H)\big)$. But ϕ cannot be an antiautomorphism. In fact, in this case we would obtain that the image $\phi(S)$ of a unilateral shift S has a right inverse. But, on the other hand, since ϕ is locally a scalar multiple of an automorphism of $B(H)$, it follows that $\phi(S)$ is not right invertible. This contradiction justifies our assertion. □

Before proving Theorem 6.2, we recall that the automorphisms of the function algebra $C_0(X)$ are of the form $f \mapsto f \circ \varphi$, where $\varphi : X \to X$ is a homeomorphism.

Proof of Theorem 6.2. Let $\phi : C_0\big(X, B(H)\big) \to C_0\big(X, B(H)\big)$ be a local automorphism of $C_0\big(X, B(H)\big)$, i.e. a bounded linear map which agrees with some automorphism at each point in $C_0\big(X, B(H)\big)$. By Theorem 6.1, for every $f \in C_0\big(X, B(H)\big)$

$$\phi(f)(x) = [\tau_f(x)]\Big(f\big(\varphi_f(x)\big)\Big) \qquad (x \in X)$$

holds for some homeomorphism $\varphi_f : X \to X$ and function $\tau_f : X \to \mathrm{Aut}\big(B(H)\big)$. It follows that for every $f \in C_0(X)$ there exists a homeomorphism $\psi_f : X \to X$ for which $\phi(fI) = (f \circ \psi_f)I$. By assumption, the automorphism group of $C_0(X)$ is reflexive, thus we obtain that there is a homeomorphism $\varphi : X \to X$ for which

$$\phi(fI) = (f \circ \varphi)I \qquad \big(f \in C_0(X)\big). \tag{6.9}$$

Let $f \in C_0(X)$ and $x \in X$. Consider the linear map $\Psi : A \mapsto \phi(fA)(x)$ on $B(H)$. From the form (6.1) of the automorphisms of $C_0\big(X, B(H)\big)$, it follows that Ψ has the property that for every $A \in B(H)$ there exists a number λ_A and an automorphism $\tau_A \in \mathrm{Aut}\big(B(H)\big)$ such that

$$\Psi(A) = \lambda_A \tau_A(A).$$

Now, Lemma 6.9 tells us that there exist functions $\tau_f : X \to \mathrm{Aut}\big(B(H)\big)$ and $\lambda_f : X \to \mathbb{C}$ such that

$$\phi(fA)(x) = [\tau_f(x)]\big(\lambda_f(x)A\big) \qquad \big(f \in C_0(X), A \in B(H), x \in X\big).$$

From (6.9), we obtain that $\lambda_f = f \circ \varphi$, and hence we have

$$\phi(fA)(x) = [\tau_f(x)]\Big(f\big(\varphi(x)\big)A\Big) \qquad \big(f \in C_0(X), A \in B(H), x \in X\big). \tag{6.10}$$

Let $x \in X$ be temporarily fixed. Pick functions $f, g \in C_0(X)$ with the property that $f\big(\varphi(x)\big), g\big(\varphi(x)\big) \neq 0$. Because of linearity, we get

$$\begin{aligned} &[\tau_f(x)]\Big(f\big(\varphi(x)\big)A\Big) + [\tau_g(x)]\Big(g\big(\varphi(x)\big)A\Big) \\ &\quad = \phi(fA)(x) + \phi(gA)(x) = \phi\big((f+g)A\big)(x) \\ &\quad = [\tau_{f+g}(x)]\Big(f\big(\varphi(x)\big)A + g\big(\varphi(x)\big)A\Big) \end{aligned} \qquad \Big(A \in B(H)\Big).$$

Using Lemma 6.8, we infer that $\tau_f(x) = \tau_g(x)$. By the formula (6.10), it follows readily that there is a function $\tau : X \to \mathrm{Aut}\big(B(H)\big)$ for which

$$\phi(fA)(x) = [\tau(x)]\Big(f\big(\varphi(x)\big)A\Big) \qquad \big(f \in C_0(X), A \in B(H), x \in X\big). \tag{6.11}$$

Since the linear span of the set of functions fA $\big(f \in C_0(X), A \in B(H)\big)$ is dense in $C_0\big(X, B(H)\big)$ (see, for example, [74, 6.4.16. Lemma]), the equality in (6.11) gives us

$$\phi(f)(x) = [\tau(x)]\Big(f\big(\varphi(x)\big)\Big) \qquad (x \in X)$$

for every $f \in C_0\big(X, B(H)\big)$. In view of Theorem 6.1, the proof is complete. □

The next lemma that we shall use in the proof of Theorem 6.3 states that every bounded linear map on $B(H)$ which is locally a scalar multiple of a surjective isometry equals globally a scalar multiple of a surjective isometry. For the proof we recall the folk result (in fact this is a consequence of a theorem of Kadison) that every surjective linear isometry of $B(H)$ is either of the form

$$A \longmapsto UAV \ \ \big(A \in B(H)\big) \qquad \text{or} \qquad A \longmapsto UA^{tr}V, \ \ \big(A \in B(H)\big),$$

where U, V are unitary operators and ^{tr} denotes the transpose with respect to an arbitrary but fixed complete orthonormal system in H. In what follows, $\mathcal{P}(H)$ and $\mathcal{U}(H)$ denote the set of all projections and all unitaries on H, respectively.

Lemma 6.10. *Let $\phi : B(H) \to B(H)$ be a bounded linear map with the property that for every $A \in B(H)$ there exists a number $\lambda_A \in \mathbb{C}$ and a surjective linear isometry $\tau_A \in \operatorname{Iso}\big(B(H)\big)$ such that $\phi(A) = \lambda_A \tau_A(A)$. Then there exists a number $\lambda \in \mathbb{C}$ and a surjective linear isometry $\tau \in \operatorname{Iso}\big(B(H)\big)$ for which $\phi(A) = \lambda\tau(A)$ $\big(A \in B(H)\big)$.*

Proof of Lemma 6.10. Just as in the proof of Lemma 6.9, first suppose that $\phi(I) = 0$. Assume that there exists a projection $0, I \neq P \in B(H)$ for which $\phi(P) \neq 0$. Clearly, we may suppose that $\phi(P) = P$. Then we have $\phi(I - P) = -P$. Since for any distinct nonzero numbers $\epsilon, \delta \in \mathbb{C}$, the operator $\epsilon P + \delta(I - P)$ is invertible, we obtain that $(\epsilon - \delta)P = \phi\big(\epsilon P + \delta(I - P)\big)$ is a scalar multiple of an invertible operator, which is a contradiction. Hence we have $\phi(P) = 0$ for every projection P. This implies $\phi = 0$.

So, suppose that $\phi(I) \neq 0$. We may clearly assume that $\phi(I) = I$, and the constants λ_A are all nonnegative. Let $P \neq 0, I$ be a projection. Let λ, μ be nonnegative numbers, and let U, V be partial isometries for which $\phi(P) = \lambda U, \phi(I - P) = \mu V$. We have

$$\lambda U + \mu V = I \quad \text{and} \quad \epsilon\lambda U + \delta\mu V \in \mathbb{C}\mathcal{U}(H) \quad \big(|\epsilon| = |\delta| = 1\big). \tag{6.12}$$

Since $P \neq 0, I$, it follows that $\lambda, \mu > 0$. Choose distinct ϵ and δ with $|\epsilon| = |\delta| = 1$. By (6.12), it follows that the operator

$$\delta I + (\epsilon - \delta)\lambda U = \epsilon\lambda U + \delta(I - \lambda U) = \epsilon\lambda U + \delta\mu V$$

is normal. Thus we obtain that U and then V are both normal partial isometries. Therefore, U has a matrix representation

$$U = \begin{bmatrix} U_0 & 0 \\ 0 & 0 \end{bmatrix},$$

where U_0 is unitary on a proper closed linear subspace H_0 of H. In accordance with (6.12), we have the following matrix representation of V

$$V = \begin{bmatrix} (I - \lambda U_0)/\mu & 0 \\ 0 & I/\mu \end{bmatrix}.$$

Using the characteristic property $VV^*V = V$ of partial isometries, we get that $\mu = 1$ and, by symmetry, that $\lambda = 1$. Taking the matrix representations above into account, it is easy to see that $I - U_0$ is a normal partial isometry and $\epsilon U_0 + \delta(I - U_0)$ is a scalar multiple of a unitary operator for every $\epsilon, \delta \in \mathbb{C}$ with $|\epsilon| = |\delta| = 1$. Since $I - U_0$ is a normal partial isometry, the spectrum of U_0 must consist of such numbers c of modulus one, for which either $1 - c$ has modulus one or $1 - c = 0$. This gives us that $\sigma(U_0) \subset \{1, e^{i\pi/3}, e^{-i\pi/3}\}$. Let P_1, P_2, P_3 denote the projections onto the subspaces $\ker(U_0 - I), \ker(U_0 - e^{i\pi/3}I), \ker(U_0 - e^{-i\pi/3}I)$ of H_0, respectively. We assert that two of the operators P_1, P_2, P_3 are necessarily zero. In fact, for example, if $P_2, P_3 \neq 0$, then it follows from the second property in (6.12) that

$$\left|\epsilon e^{i\pi/3} + \delta e^{-i\pi/3}\right| = \left|\epsilon e^{-i\pi/3} + \delta e^{i\pi/3}\right|$$

for every ϵ, δ of modulus one. But this is an obvious contradiction. The other cases can be treated in a similar way. Therefore, we have $\phi(P) = U \in \{1, e^{i\pi/3}, e^{-i\pi/3}\}\mathcal{P}(H)$ for every projection P on H. Now let P be a projection of infinite rank and infinite corank. Since in this case P is unitarily equivalent to $I-P$, it follows that P and $I-P$ can be connected by a continuous curve within the set of projections. Consequently, we obtain that $\phi(P)$ and $\phi(I-P)$ have the same nonzero eigenvalue. Since $\phi(I-P) = I - \phi(P)$, it follows that this eigenvalue is 1. Thus we obtain that $\phi(P)$ is a projection. If P is a finite rank projection, then P is the difference of two projections of infinite rank and corank. Then we obtain that $\phi(P)$ is the difference of two projections and consequently $\phi(P)$ is self-adjoint. On the other hand, we have $\phi(P) \in \{1, e^{i\pi/3}, e^{-i\pi/3}\}\mathcal{P}(H)$. These result in $\phi(P) \in \mathcal{P}(H)$, and we deduce that ϕ sends every projection to a projection. It now follows that ϕ is a Jordan *-endomorphism of $B(H)$. Since, by our condition, the range of ϕ contains a rank-one operator and an operator with dense range, using again [54, Theorem 1], we infer that ϕ is either a *-automorphism or a *-antiautomorphism of $B(H)$. In both cases we obtain that ϕ is a surjective isometry of $B(H)$, and this completes the proof. $\square$

Lemma 6.11. *If* $\mathcal{M} \subset \mathbb{C}\mathcal{U}(H)$ *is a linear subspace, then* $\mathcal{M}$ *is either 1-dimensional or 0-dimensional.*

Proof of Lemma 6.11. In the first version of our paper [69] we gave a direct proof of this lemma. The referee kindly pointed out that this is just a simple consequence of [81, Remark iii, p. 691] that asserts that every linear space of normal operators is commutative. Nevertheless, to get a completely elementary and trivial proof one can argue as follows. Let $A, B \in \mathcal{M}$. For every $\lambda \in \mathbb{C}$ we have $(A + \lambda B)^*(A + \lambda B) \in \mathbb{C}I$. Since A^*A, B^*B are scalar, choosing $\lambda = 1$ and then $\lambda = i$, it follows easily that A^*B is also scalar. This clearly gives us that A, B are linearly dependent. $\square$

Lemma 6.12. *Let* X *be a locally compact Hausdorff space. Let* $\mathcal{M} \subset \mathbb{C}^X$ *be a linear subspace containing a nowhere vanishing function* $f_0 \in \mathcal{M}$ *and having the property that* $|f| \in C_0(X)$ *for every* $f \in \mathcal{M}$*. Then there is a function* $t : X \to \mathbb{C}$ *of modulus one such that* $t\mathcal{M} \subset C_0(X)$.

Proof of Lemma 6.12. We know that the function $|f + f_0|^2 - |f|^2 - |f_0|^2$ is continuous for every $f \in \mathcal{M}$. Hence $f\overline{f_0}$ is continuous for every $f \in \mathcal{M}$. Let $t = |f_0|/f_0$. Then $|t| = 1$, and the function $(tf)|f_0| = (tf)\left(\overline{tf_0}\right) = f\overline{f_0}$ is continuous. Thus we obtain $tf \in C_0(X)$. $\square$

For the proof of Theorem 6.3 we recall the well-known Banach-Stone theorem stating that the surjective isometries of the function algebra $C_0(X)$ are all of the form $f \mapsto \tau \cdot f \circ \varphi$, where $\tau : X \to \mathbb{C}$ is a continuous function of modulus one and $\varphi : X \to X$ is a homeomorphism.

Proof of Theorem 6.3. Let $\phi : C_0\big(X, B(H)\big) \to C_0\big(X, B(H)\big)$ be a local surjective isometry. Pick a function $f \in C_0(X)$ and a point $x \in X$, and consider the linear map

$$\Psi : A \longmapsto \phi(fA)(x) \in B(H) \qquad \big(A \in B(H)\big).$$

It follows from Theorem 6.1 that for every $A \in B(H)$ there exists a number λ_A and a surjective isometry $\tau_A \in \mathrm{Iso}\big(B(H)\big)$ such that $\Psi(A) = \lambda_A \tau_A(A)$. By Lemma 6.10, we infer that there exists a nonnegative number $\lambda_{f,x}$ and a surjective linear isometry $\tau_{f,x} \in \mathrm{Iso}\big(B(H)\big)$ for which

$$\phi(fA)(x) = \lambda_{f,x}\tau_{f,x}(A) \tag{6.13}$$

holds for every $f \in C_0(X)$, $A \in B(H)$ and $x \in X$. Now, let $U \in B(H)$ be a unitary operator and $x \in X$. The linear map

$$f \longmapsto \phi(fU)(x)$$

maps $C_0(X)$ into $\mathbb{C}\mathcal{U}(H)$. Since the range of this map is a linear subspace, by Lemma 6.11 we infer that it is either 1-dimensional or 0-dimensional. Thus there is a linear functional $F_{U,x} : C_0(X) \to \mathbb{C}$ and a unitary operator $[\tau(x)](U)$ such that

$$\phi(fU)(x) = F_{U,x}(f)[\tau(x)](U) \qquad \big(f \in C_0(X), U \in \mathcal{U}(H), x \in X\big).$$

Clearly, the map $F_U : C_0(X) \to \mathbb{C}^X$ defined by $F_U(f)(x) = F_{U,x}(f)$ is linear, and

$$\phi(fU)(x) = F_U(f)(x)[\tau(x)](U) \qquad \big(f \in C_0(X), U \in \mathcal{U}(H), x \in X\big) \tag{6.14}$$

holds. Since ϕ is a local surjective isometry of $C_0\big(X, B(H)\big)$, it follows from Theorem 6.1 that for every $f \in C_0(X)$ there exists a strongly continuous function $\tau_{f,U} : X \to \mathrm{Iso}\big(B(H)\big)$ and a homeomorphism $\varphi_{f,U} : X \to X$ such that

$$\phi(fU)(x) = f\big(\varphi_{f,U}(x)\big)[\tau_{f,U}(x)](U) \qquad (x \in X). \tag{6.15}$$

Obviously, we have $|F_U(f)| = |f| \circ \varphi_{f,U}$. Since X is σ-compact, it is an easy consequence of Urysohn's lemma that there exists a strictly positive function in $C_0(X)$. Therefore, the range of F_U contains a nowhere vanishing function and has the property that the absolute value of every function belonging to this range is continuous. By Lemma 6.12, there exists a function $t : X \to \mathbb{C}$ of modulus one such that the functions $tF_U(f)$ are all continuous $\big(f \in C_0(X)\big)$. Consequently, we may suppose that the map F_U in (6.14) maps $C_0(X)$ into itself. By (6.14) and (6.15) we have

$$F_U(f)(x)[\tau(x)](U) = f\big(\varphi_{f,U}(x)\big)[\tau_{f,U}(x)](U) \qquad (x \in X). \tag{6.16}$$

If $f \in C_0(X)$ vanishes nowhere, then, by the continuity of the functions $F_U(f)$, $f \circ \varphi_{f,U}$ and $[\tau_{f,U}(.)](U)$, it follows that $[\tau(.)](U)$ is also continuous. By (6.16) we have

$$F_U(f) = f\big(\varphi_{f,U}(x)\big)\big([\tau_{f,U}(x)](U)\big)\big([\tau(x)](U)\big)^* \qquad (x \in X).$$

In particular, this implies that the function

$$x \longmapsto [\tau_{f,U}(x)](U)[\tau(x)](U)^*$$

can be considered as a continuous scalar valued function of modulus one. Hence F_U is a local surjective isometry of $C_0(X)$. By our assumption this means that F_U is a surjective isometry, i.e. there exists a continuous function $t_U : X \to \mathbb{C}$ with $|t_U| = 1$ and a homeomorphism $\varphi_U : X \to X$ such that $F_U(f) = t_U \cdot f \circ \varphi_U$ $\big(f \in C_0(X), U \in \mathcal{U}(H)\big)$. From examination of (6.14), we may obviously suppose that ϕ satisfies

$$\phi(fU)(x) = f\big(\varphi_U(x)\big)[\tau(x)](U) \qquad \big(f \in C_0(X), U \in \mathcal{U}(H), x \in X\big),$$

where $[\tau(x)](U)$ is unitary. If $f \in C_0(X)$ is nonnegative, we see from (6.13) that

$$f\big(\varphi_U(x)\big) = \lambda_{f,x} = f\big(\varphi_I(x)\big)$$

and

$$[\tau(x)](U) = \tau_{f,x}(U) \qquad \big(U \in \mathcal{U}(H), x \in X\big).$$

This verifies the existence of a homeomorphism φ of X, and, due to the fact that every operator in $B(H)$ is a linear combination of unitaries, also the existence of a function $\tau : X \to \mathrm{Iso}\big(B(H)\big)$ for which

$$\phi(fU)(x) = f\big(\varphi(x)\big)[\tau(x)](U) \qquad (U \in \mathcal{U}(H), x \in X)$$

holds for every nonnegative function $f \in C_0(X)$. Every function in $C_0(X)$ is the linear combination of nonnegative functions in $C_0(X)$, thus we finally obtain that

$$\phi(fA)(x) = f\big(\varphi(x)\big)[\tau(x)](A) \qquad \big(f \in C_0(X), A \in B(H), x \in X\big).$$

Referring again to the fact that the linear span of the elementary tensors fA $\big(f \in C_0(X), A \in B(H)\big)$ is dense in $C_0\big(X, B(H)\big)$, we arrive at the form

$$\phi(f)(x) = [\tau(x)]\Big(f\big(\varphi(x)\big)\Big) \qquad \Big(f \in C_0\big(X, B(H)\big), x \in X\Big).$$

By Theorem 6.1, the proof is complete. □

We now turn to the proof of Theorem 6.4. The next result describes the form of local surjective isometries of the function algebra $C_0(X)$.

Lemma 6.13. *Let X be a first countable locally compact Hausdorff space. Let $F : C_0(X) \to C_0(X)$ be a local surjective isometry. Then there is a continuous function $t : X \to \mathbb{C}$ with $|\tau| = 1$, and a homeomorphism g of X onto a subspace of X such that*

$$F(f) \circ g = t \cdot f \qquad \big(f \in C_0(X)\big). \tag{6.17}$$

Proof of Lemma 6.13. By Banach-Stone theorem on the form of surjective linear isometries of $C_0(X)$, it follows that for every $f \in C_0(X)$ there exists a homeomorphism $\varphi_f : X \to X$ and a continuous function $\tau_f : X \to \mathbb{C}$ of modulus one such that

$$F(f) = \tau_f \cdot f \circ \varphi_f. \tag{6.18}$$

For any $x \in X$, let $\mathcal{S}_x$ denote the set of all functions $p \in C_0(X)$ which map into the interval $[0,1]$, $p(x) = 1$ and $p(y) < 1$ for every $x \neq y \in X$. By Urysohn's lemma and the first countability of X, it is easy to verify that $\mathcal{S}_x$ is nonempty. Let $p, p' \in \mathcal{S}_x$. By (6.18) there exist $y, y' \in X$ for which $|F(p)| \in \mathcal{S}_y, |F(p')| \in \mathcal{S}_{y'}$. Similarly, since $(p+p')/2 \in \mathcal{S}_x$, there is a point $y'' \in X$ for which $\big|F\big((p+p')/2\big)\big| \in \mathcal{S}_{y''}$. We have $y = y'$ and $F(p)(y) = F(p')(y')$. This shows that there are functions $t : X \to \mathbb{C}$ and $g : X \to X$ such that

$$t(x) = F(p)\big(g(x)\big) \tag{6.19}$$

holds for every $x \in X$ and $p \in \mathcal{S}_x$. Clearly, $|t(x)| = 1$. Pick $x \in X$. It is easy to see that for any strictly positive function $f \in C_0(X)$ with $f(x) = 1$ we have a function $p \in \mathcal{S}_x$ such that $p(y) < f(y)$ $(x \neq y \in X)$. Now, let $f \in C_0(X)$ be an arbitrary nonnegative function. Then there is a positive constant c for which the function $y \mapsto c + f(x) - f(y)$ is positive. Hence we can choose a function $p \in \mathcal{S}_x$ such that $cp(y) < cp(x) + f(x) - f(y)$ $(x \neq y \in X)$. This means that the nonnegative function $cp + f$ takes its maximum only at x. By (6.19) we infer that

$$t(x)\big(cp(x) + f(x)\big) = F(cp + f)\big(g(x)\big).$$

Clearly, we also have

$$t(x)\big(cp(x)\big) = F(cp)\big(g(x)\big).$$

Therefore we obtain

$$t \cdot f = F(f) \circ g \tag{6.20}$$

for every nonnegative f, and then for every function in $C_0(X)$. We prove that g is a homeomorphism of X onto the range of g. To see this, first observe that for every function $p \in \mathcal{S}_y$ and net (y_α) in X, the condition $p(y_\alpha) \to 1$ implies that $y_\alpha \to y$. Let (x_α) be a net in X converging to $x \in X$. Pick $p \in \mathcal{S}_x$. Since F is a local surjective isometry, we have a homeomorphism φ of X for which

$$p = |t \cdot p| = |F(p) \circ g| = p \circ \varphi \circ g.$$

This implies $p\big(\varphi\big(g(x_\alpha)\big)\big) \to 1$, thus $\varphi\big(g(x_\alpha)\big) \to x = \varphi\big(g(x)\big)$, whence $g(x_\alpha) \to g(x)$. So, g is continuous. The injectivity of g follows from (6.20) immediately, using the fact that the nonnegative elements of $C_0(X)$ separate the points of X. As for the continuity of g^{-1} and t, these follow again from (6.20) and from Urysohn's lemma. □

Now, we are in a position to prove our last theorem.

Proof of Theorem 6.4. It is well-known that every open convex subset of $\mathbb{R}^n$ is homeomorphic to the open unit ball B of $\mathbb{R}^n$. Hence, it is sufficient to show that the automorphism and isometry groups of $C_0(B)$ are algebraically reflexive. Furthermore, by the form of the automorphisms and surjective isometries of the function algebra $C_0(X)$, we are certainly done if we prove the statement only for the isometry group. So, let $F : C_0(B) \to C_0(B)$ be a local surjective isometry. Then F is of the form (6.17). The only thing that we have to verify is that the function g appearing in this form is surjective. Consider the function $f \in C_0(B)$ defined by $f(x) = 1/\big(1 + \|x\|\big)$. Clearly, we may assume that $F(f) = f$. By (6.17), we infer that

$$\frac{1}{1+\|x\|} = \frac{1}{1+\|g(x)\|} \qquad (x \in S).$$

Therefore the continuous function g maps the surface S_r of the closed ball $r\overline{B}$ $(0 \leq r < 1)$ into itself. If $g(S_r)$ is a proper closed subset of S_r, then it is homeomorphic to a subset of $\mathbb{R}^{n-1}$, and, by Borsuk-Ulam theorem, g takes the same value at some antipodal points of S_r, which contradicts the injectivity of g. Thus the range of g contains every set S_r $(0 \leq r < 1)$, whence g is bijective. This completes the proof. □

Remark. The proof of Theorem 6.4 shows how difficult it might be to treat our reflexivity problem for the suspension of arbitrary C^*-algebras. We mean the role of the use of Borsuk-Ulam theorem in the above argument. To reinforce this opinion, let us consider only the particular case of commutative C^*-algebras. Let X be a locally compact Hausdorff space and suppose that the automorphism and isometry groups of $C_0(X)$ are algebraically reflexive. If $F : C_0(\mathbb{R} \times X) \to C_0(\mathbb{R} \times X)$ is a local surjective isometry, then Lemma 6.13 gives the form of F. The problem is to verify that the function g appearing in (6.17) is surjective. This would be easy if there were an injective nonnegative function in $C_0(\mathbb{R} \times X)$. Unfortunately, this is not the case even when X is a singleton. Anyway, if $n \geq 3$, there is no injective function in $C_0(\mathbb{R}^n)$ at all. Therefore, to attack the problem of the surjectivity of g, we had to invent a different approach which was the use of Borsuk-Ulam theorem. To mention another point, it is easy to see that in general the automorphisms as well as surjective isometries of the tensor product $C_0(X_1) \otimes C_0(X_2) \cong C_0(X_1 \times X_2)$ have nothing to do with the automorphisms and surjective isometries of $C_0(X_1)$ and $C_0(X_2)$, respectively. However, according to Theorem 6.1, in the case of the tensor product $C_0(X) \otimes B(H)$ every automorphism as well as surjective isometry is an easily identifiable mixture of a "functional algebraic" and an "operator algebraic" part. This observation was of fundamental importance when verifying the result in Corollary 6.5. These might justify the suspicion why we feel our reflexivity problem really difficult even for the supsension of general commutative C^*-algebras.

7. ON THE TOPOLOGICAL REFLEXIVITY OF THE ISOMETRY GROUP OF THE SUSPENSION OF $B(H)$

7.1 Introduction and Statement of the Results

The referee of our paper [69], whose contents were presented in Chapter 6, asked for the description of the topological reflexive closures of the automorphism group and the isometry group of the suspension $C_0(\mathbb{R}) \otimes B(H)$ of $B(H)$. Here we present the required description for the topological reflexive closures of the isometry group and the group of *-automorphisms of the suspension of $B(H)$.

We introduce the following notation. If X is a Banach space, let $\mathrm{iso}(X)$ denote the set of all linear (not necessarily surjective) isometries of X. The isometry group, i.e. the group of all surjective linear isometries of X, is denoted by $\mathrm{Iso}(X)$. If $\mathcal{A}$ is a $*$-algebra then let $\mathrm{Aut}^*(\mathcal{A})$ denote the group of all $*$-automorphisms of $\mathcal{A}$. For an arbitrary Banach space X, let $B(X)$ be the Banach algebra of all bounded linear operators of X, and for arbitrary $\mathcal{E} \subseteq B(X)$ let

$$\mathrm{ref}_{alg}\, \mathcal{E} = \big\{T \in B(X) : Tx \in \mathcal{E}x \text{ for all } x \in X\big\},$$
$$\mathrm{ref}_{top}\, \mathcal{E} = \big\{T \in B(X) : Tx \in \overline{\mathcal{E}x} \text{ for all } x \in X\big\},$$

where $\overline{\mathcal{E}x}$ denotes the norm-closure of $\mathcal{E}x$. The above sets are called the *algebraic reflexive closure* and the *topological reflexive closure* of $\mathcal{E}$, respectively. The set $\mathcal{E}$ is called *algebraically reflexive* if $\mathrm{ref}_{alg}\, \mathcal{E} = \mathcal{E}$, and *topologically reflexive* if $\mathrm{ref}_{top}\, \mathcal{E} = \mathcal{E}$. Topological reflexivity is clearly a stronger property than algebraic reflexivity. In Chapter 6 we proved that

$$\mathrm{ref}_{alg}\big(\mathrm{Aut}\big(C_0(\mathbb{R}) \otimes B(H)\big)\big) = \mathrm{Aut}\big(C_0(\mathbb{R}) \otimes B(H)\big)$$

and

$$\mathrm{ref}_{alg}\big(\mathrm{Iso}\big(C_0(\mathbb{R}) \otimes B(H)\big)\big) = \mathrm{Iso}\big(C_0(\mathbb{R}) \otimes B(H)\big).$$

Here we shall describe the elements of the sets $\mathrm{ref}_{top}\big(\mathrm{Aut}^*\big(C_0(\mathbb{R}) \otimes B(H)\big)\big)$ and $\mathrm{ref}_{top}\big(\mathrm{Iso}\big(C_0(\mathbb{R}) \otimes B(H)\big)\big)$.

The proof of the main result of this chapter is based on the following auxiliary theorem.

Theorem 7.1. *Let X be a Banach space, $\mathcal{S} \subseteq \operatorname{iso}(X)$ a topologically reflexive subset and $\phi : C_0(\mathbb{R}, X) \to C_0(\mathbb{R}, X)$ a linear map. For any $f \in C_0(\mathbb{R}, X)$ there exist homeomorphisms $\varphi_n : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ and functions $\tau_n : \mathbb{R} \to \mathcal{S}$ $(n \in \mathbb{N})$ with*

$$\tau_n f \circ \varphi_n \to \phi(f),$$

if and only if there exists an open interval $U \subseteq \mathbb{R}$, a surjective, monotone, continuous function $\varphi : U \to \mathbb{R}$, and a function $\tau : U \to \mathcal{S}$ such that for any $f \in C_0(\mathbb{R}, X)$ we have

$$\phi(f)(y) = \begin{cases} \tau(y)\Big(f\big(\varphi(y)\big)\Big) & \text{if } y \in U, \\ 0 & \text{if } y \in \mathbb{R} \setminus U. \end{cases} \tag{7.1}$$

Moreover, if ϕ is of the form (7.1), then $\tau : U \to \mathcal{S}$ is strongly continuous.

The $*$-automorphisms of $C_0\big(\mathbb{R}, B(H)\big)$ are both automorphisms and surjective linear isometries. Theorem 6.1 provides the forms of the surjective linear isometries and the automorphisms of $C_0\big(\mathbb{R}, B(H)\big)$. In view of the topological reflexivity of the isometry group and the automorphism group of $B(H)$ (see [54]), Theorem 7.1 implies immediately the main result of this chapter which reads as follows.

Corollary 7.2. *Let $\phi : C_0\big(\mathbb{R}, B(H)\big) \to C_0\big(\mathbb{R}, B(H)\big)$ be a linear map. We have $\phi \in \operatorname{ref}_{top} \operatorname{Iso}\big(C_0\big(\mathbb{R}, B(H)\big)\big)$ resp. $\phi \in \operatorname{ref}_{top} \operatorname{Aut}^*\big(C_0\big(\mathbb{R}, B(H)\big)\big)$, if and only if there exists an open interval $U \subseteq \mathbb{R}$, a surjective, monotone, continuous function $\varphi : U \to \mathbb{R}$, and $\tau : U \to \operatorname{Iso}\big(B(H)\big)$ resp. $\tau : U \to \operatorname{Aut}^*\big(B(H)\big)$, such that for any $f \in C_0\big(\mathbb{R}, B(H)\big)$, ϕ is of the form (7.1).*

Moreover, if ϕ is of the form (7.1), then τ is strongly continuous.

In Chapter 6 we proved that the isometry group of $C_0\big(\mathbb{R}, B(H)\big)$ is algebraically reflexive. We show that this conclusion can be deduced relatively easily from Theorem 7.1 as well. We first prove the following auxiliary result, which turns to be an easy corollary of Theorem 7.1.

Theorem 7.3. *Let X be a Banach space, $\mathcal{P} \subseteq \operatorname{iso}(X)$ an algebraically reflexive subset and $\phi : C_0(\mathbb{R}, X) \to C_0(\mathbb{R}, X)$ a linear map. If for any $f \in C_0(\mathbb{R}, X)$ there exists a homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ and a function $\tau : \mathbb{R} \to \mathcal{P}$ such that*

$$\phi(f) = \tau f \circ \varphi, \tag{7.2}$$

then there exists a homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ and a function $\tau : \mathbb{R} \to \mathcal{P}$ such that for any $f \in C_0(\mathbb{R}, X)$ we have

$$\phi(f) = \tau f \circ \varphi,$$

and in this case $\tau : \mathbb{R} \to \mathcal{P}$ is strongly continuous.

Now we obtain immediately the following corollary, which was also stated in our main result (Corollary 6.5) in Chapter 6.

Corollary 7.4. *The groups* $\mathrm{Iso}\big(C_0(\mathbb{R}, B(H))\big)$ *and* $\mathrm{Aut}^*\big(C_0(\mathbb{R}, B(H))\big)$ *are algebraically reflexive.*

7.2 Proofs

Proof of Theorem 7.1. First suppose that ϕ is of the form (7.1), where $U =]u_1, u_2[$ with $u_1, u_2 \in \mathbb{R} \cup \{-\infty, \infty\}$, $\varphi : U \to \mathbb{R}$ and $\tau : U \to \mathcal{S}$ are as in Theorem 7.1. Extend $\tau : U \to \mathcal{S}$ to the function $\tau : \mathbb{R} \to \mathcal{S}$. Let $u_1 < a_n \in U$ and $u_2 > b_n \in U$ be real sequences with $a_n \to u_1$ and $b_n \to u_2$. Now for any $n \in \mathbb{N}$ there exists a homeomorphism $\varphi_n : \mathbb{R} \to \mathbb{R}$ for which $|\varphi(y) - \varphi_n(y)| < \frac{1}{n}$ for any $y \in [a_n, b_n]$, and $\varphi\big([a_n, b_n]\big) \subseteq \varphi_n\big([a_n, b_n]\big)$.

We show that for any $f \in C_0(\mathbb{R}, X)$

$$\tau f \circ \varphi_n \to \phi(f).$$

holds. Let $f \in C_0(\mathbb{R}, X)$ and $\varepsilon > 0$ be arbitrary. Then there exists a compact set $K \subseteq \mathbb{R}$ such that $\|f(y)\| < \frac{\varepsilon}{2}$ $(y \in \mathbb{R} \setminus K)$. Since $a_n \to u_1$, $b_n \to u_2$ and $\varphi : U \to \mathbb{R}$ is surjective, there exists $n_1 \in \mathbb{N}$ for which $K \subseteq \varphi\big([a_n, b_n]\big)^\circ$ for any $n \geq n_1$. Since $f \in C_0(\mathbb{R}, X)$ is uniformly continuous, there exists $\delta > 0$ such that for any $x, y \in K$, $|x - y| < \delta$ implies $\|f(x) - f(y)\| < \varepsilon$. Let $n_2 \in \mathbb{N}$ with $\frac{1}{n_2} < \delta$ and $n_0 = \max(n_1, n_2)$. Further, let $y \in U$ and $n \geq n_0$ be arbitrary. If $y \in [a_n, b_n]$ then $|\varphi(y) - \varphi_n(y)| < \frac{1}{n} < \delta$, and thus $\|f(\varphi(y)) - f(\varphi_n(y))\| < \varepsilon$. If $y \in U \setminus [a_n, b_n]$ then, by the monotonicity of φ_n and φ, we have $\varphi_n(y) \notin \varphi_n\big([a_n, b_n]\big)^\circ \supseteq \varphi\big([a_n, b_n]\big)^\circ \supseteq K$ and $\varphi(y) \notin \varphi\big([a_n, b_n]\big)^\circ \supseteq K$, so $\|f(\varphi(y))\|, \|f(\varphi_n(y))\| < \frac{\varepsilon}{2}$, which implies $\|f(\varphi(y)) - f(\varphi_n(y))\| < \varepsilon$. Then, by $\tau(y) \in \mathrm{iso}(X)$, for any $y \in U$ and $n \geq n_0$ we have $\|(\tau f \circ \varphi_n)(y) - \phi(f)(y)\| = \|\tau(y) f(\varphi_n(y)) - \tau(y) f(\varphi(y))\| = \|f(\varphi_n(y)) - f(\varphi(y))\| < \varepsilon$. For any $y \in \mathbb{R} \setminus U$ we obtain that $y \notin [a_n, b_n]$, thus $\varphi_n(y) \notin \varphi_n\big([a_n, b_n]\big)^\circ \supseteq \varphi\big([a_n, b_n]\big)^\circ \supseteq K$, which implies $\|(\tau f \circ \varphi_n)(y) - \phi(f)(y)\| = \|(\tau f \circ \varphi_n)(y)\| = \|f(\varphi_n(y))\| < \frac{\varepsilon}{2}$. Hence $\tau f \circ \varphi_n \to \tau f \circ \varphi = \phi(f)$, which completes the proof in one direction.

Consider now the other direction. Suppose that for any $f \in C_0(\mathbb{R}, X)$ there are homeomorphisms $\varphi_{f,n} : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ and functions $\tau_{f,n} : \mathbb{R} \to \mathcal{S}$ $(n \in \mathbb{N})$ such that

$$\tau_{f,n} f \circ \varphi_{f,n} \to \phi(f).$$

In what follows, for brevity, $\varphi_{f,n}$ and $\tau_{f,n}$ will denote the functions corresponding to $f \in C_0(\mathbb{R}, X)$. Now ϕ is obviously an isometry. Introduce the following notation. For any $x \in \mathbb{R}$ and $A \in X$ with $A \neq 0$, set

$$S_0(x, A) = \Big\{ f \in C_0(\mathbb{R}, X) \; : \; \|f(x)\| = \|f\| > 0, \frac{f(x)}{\|f(x)\|} = \frac{A}{\|A\|}, \\ \forall y \in \mathbb{R}, y \neq x : \|f(y)\| < \|f\| \Big\},$$

$$S(x, A) = \Big\{ f \in C_0(\mathbb{R}, X) \; : \; \|f(x)\| = \|f\| > 0, \frac{f(x)}{\|f(x)\|} = \frac{A}{\|A\|} \Big\},$$

which are clearly non-empty sets.

The real sequences converging to $-\infty$ or $+\infty$ will be considered convergent. Further, for any $f \in C_0(\mathbb{R}, X)$ define $f(-\infty) = 0$ and $f(+\infty) = 0$.

Step 7.1. *For any $x \in \mathbb{R}$ and $A \in X$ with $A \neq 0$, the set*

$$G(x, A) = \bigcap_{f \in S(x,A)} \Big\{ y \in \mathbb{R} : \|\phi(f)(y)\| = \|f\| \Big\} \tag{7.3}$$

is non-empty and compact.

Let $f \in S(x, A)$ be arbitrary. Then, by $\phi(f) \in C_0(\mathbb{R}, X)$, the set $\{y \in \mathbb{R} : \|\phi(f)(y)\| = \|f\|\}$ is compact, so to prove that $G(x, A) \neq \emptyset$, it is sufficient to show that the system of these sets satisfies the finite intersection property.

Let $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in S(x, A)$ be arbitrary. Then

$$\begin{aligned} \|f_1 + \cdots + f_n\| \geq & \|(f_1 + \cdots + f_n)(x)\| = \|f_1(x) + \cdots + f_n(x)\| \\ = & \left\| \frac{\|f_1(x)\|}{\|A\|} \cdot A + \cdots + \frac{\|f_n(x)\|}{\|A\|} \cdot A \right\| \\ = & \left\| \big(\|f_1\| + \cdots + \|f_n\|\big) \frac{1}{\|A\|} \cdot A \right\| = \|f_1\| + \cdots + \|f_n\|, \end{aligned}$$

thus

$$\|f_1 + \cdots + f_n\| = \|f_1\| + \cdots + \|f_n\|.$$

By $\phi(f_1 + \cdots + f_n) \in C_0(\mathbb{R}, X)$, there exists $z \in \mathbb{R}$ for which $\|\phi(f_1 + \cdots + f_n)(z)\| = \|\phi(f_1 + \cdots + f_n)\|$. Hence

$$\begin{aligned} \|\phi(f_1)(z)\| + \cdots + \|\phi(f_n)(z)\| \geq & \|\phi(f_1 + \cdots + f_n)(z)\| = \|\phi(f_1 + \cdots + f_n)\| \\ = & \|f_1 + \cdots + f_n\| = \|f_1\| + \cdots + \|f_n\| \\ = & \|\phi(f_1)\| + \cdots + \|\phi(f_n)\| \\ \geq & \|\phi(f_1)(z)\| + \cdots + \|\phi(f_n)(z)\|, \end{aligned}$$

which implies

$$\|\phi(f_i)(z)\| = \|\phi(f_i)\| = \|f_i\| \quad (1 \leq i \leq n),$$

thus

$$z \in \bigcap_{i=1}^{n} \Big\{ y \in \mathbb{R} : \|\phi(f_i)(y)\| = \|f_i\| \Big\}.$$

Consequently, $G(x, A)$ is indeed a non-empty compact set.

We note that for any $x \in \mathbb{R}$, $A \in X$ with $A \neq 0$ and $\lambda > 0$, we obviously have

$$G(x, \lambda A) = G(x, A). \tag{7.4}$$

Step 7.2. *If $x \in \mathbb{R}$, $A \in X$, $A \neq 0$, $f \in C_0(\mathbb{R}, X)$ and $f(x) = \frac{\|f(x)\|}{\|A\|} \cdot A$ then we have*

$$\|\phi(f)(y)\| \geq \|f(x)\| \qquad \big(y \in G(x, A)\big).$$

By (7.4), we may assume that $\|A\| = 1$. Let $y \in G(x, A)$ be arbitrary. If $\|f(x)\| = \|f\|$ then by Step 7.1 we are ready. So we may also assume that $\|f(x)\| < \|f\|$. Now it is easy to verify that there exists an $f_0 \in S(x, A)$ such that $\|f_0\| = \|f\| - \|f(x)\| > 0$, $f + f_0 \in S(x, A)$ and $\|f + f_0\| = \|f\|$. Let $h = f + f_0$. By Step 7.1, we have

$$\|\phi(h)(y)\| = \|h\| = \|f\|,$$

which implies

$$\begin{aligned} \|f\| =& \|\phi(h)(y)\| = \|\phi(f)(y) + \phi(f_0)(y)\| \\ \leq& \|\phi(f)(y)\| + \|\phi(f_0)\| = \|\phi(f)(y)\| + (\|f\| - \|f(x)\|), \end{aligned}$$

thus $\|\phi(f)(y)\| \geq \|f(x)\|$ holds.

Step 7.3. *For any $x \in \mathbb{R}$, $A \in X$ with $A \neq 0$, and for any $f \in S_0(x, A)$, the set*

$$\big\{ y \in \mathbb{R} : \|\phi(f)(y)\| = \|f\| \big\}$$

is a compact interval which contains $G(x, A)$.

Let $f \in S_0(x, A)$ be arbitrary, and let

$$y_1 = \inf\big\{ y \in \mathbb{R} : \|\phi(f)(y)\| = \|f\| \big\} \quad \text{and} \quad y_2 = \sup\big\{ y \in \mathbb{R} : \|\phi(f)(y)\| = \|f\| \big\}.$$

Now

$$\lim_{n \to \infty} \big\| f\big(\varphi_{f,n}(y_1)\big) \big\| = \|f\| \quad \text{and} \quad \lim_{n \to \infty} \big\| f\big(\varphi_{f,n}(y_2)\big) \big\| = \|f\|. \tag{7.5}$$

Let u_n be an arbitrary subsequence of one of the sequences $\varphi_{f,n}(y_1)$ and $\varphi_{f,n}(y_2)$. By (7.5) and $0 \neq f \in C_0(\mathbb{R}, X)$, the sequence u_n has an accumulation point $y_0 \in \mathbb{R}$. Then (7.5) implies $\|f(y_0)\| = \|f\|$. By $f \in S_0(x, A)$, now we have $y_0 = x$. So every subsequence of $\varphi_{f,n}(y_1)$ or $\varphi_{f,n}(y_2)$ has a subsequence converging to x, which implies

$$\lim_{n \to \infty} \varphi_{f,n}(y_1) = \lim_{n \to \infty} \varphi_{f,n}(y_2) = x. \tag{7.6}$$

Let $y \in [y_1, y_2]$ be arbitrary. Since $\varphi_{f,n} : \mathbb{R} \to \mathbb{R}$ $(n \in \mathbb{N})$ is a homeomorphism (which is clearly monotone), by (7.6) and $y_1 \leq y \leq y_2$, we have $\varphi_{f,n}(y) \to x$, thus

$$\|\phi(f)(y)\| = \lim_{n\to\infty} \|\tau_{f,n}(y) f(\varphi_{f,n}(y))\| = \lim_{n\to\infty} \|f(\varphi_{f,n}(y))\| = \|f(x)\| = \|f\|.$$

Hence $\{y \in \mathbb{R} : \|\phi(f)(y)\| = \|f\|\} = [y_1, y_2]$, and we are done.

Step 7.4. *If $x \in \mathbb{R}$ and $A \in X$ with $A \neq 0$, then $G(x, A) \subset \mathbb{R}$ is a compact interval.*

Let $y \in [\inf G(x, A), \sup G(x, A)]$ and $f \in S(x, A)$ be arbitrary. Then there exist functions $f_n \in S_0(x, A)$ $(n \in \mathbb{N})$ with $f_n \to f$. For any $f_n \in S_0(x, A)$ we have $G(x, A) \subseteq \{z \in \mathbb{R} : \|\phi(f_n)(z)\| = \|f_n\|\}$, thus, by Step 7.3, we obtain $[\inf G(x, A), \sup G(x, A)] \subseteq \{z \in \mathbb{R} : \|\phi(f_n)(z)\| = \|f_n\|\}$. Hence $\|\phi(f)(y)\| = \lim_{n\to\infty} \|\phi(f_n)(y)\| = \lim_{n\to\infty} \|f_n\| = \|f\|$. Thus $y \in G(x, A)$, and so we are done.

Step 7.5. *Let $f \in C_0(\mathbb{R}, X)$ be arbitrary for which there exist $\varepsilon \in]0, \frac{1}{100}[$, elements $A_1, A_2, A_3 \in X$ of norm 1, real numbers $p < x_1 < z_1 < x_2 < z_2 < x_3 < z_3 < x_4$ and disjoint closed intervals $J_0 < I_1 < J_1 < I_2 < J_2 < I_3 < J_3 < I_4$ with $J_0 =]-\infty, p]$, $x_i \in I_i$ $(1 \leq i \leq 4)$, $z_i \in J_i$ $(1 \leq i \leq 3)$, $f(x_1) = A_1$, $f(x_2) = A_2$, $f(x_3) = A_3$, $\|f(x_4)\| = \frac{1}{2}$, $f(z_1) = 0$, $\|f(z_2)\| < \varepsilon$, $\|f(z_3)\| < \varepsilon$, $\|f\| = 1$, and with*

$$\|f(x)\| \in \begin{cases} [0, 4\varepsilon] & \text{if } x \in J_0 \cup J_1 \cup J_2 \cup J_3, \\ ]4\varepsilon, 1 - 4\varepsilon[& \text{if } x \in]\sup J_0, \inf J_3[\setminus (I_1 \cup J_1 \cup I_2 \cup J_2 \cup I_3), \\ [1 - 4\varepsilon, 1] & \text{if } x \in I_1 \cup I_2 \cup I_3, \\ ]4\varepsilon, \frac{1}{2} - 4\varepsilon[& \text{if } \sup J_3 < x < \inf I_4, \\ [\frac{1}{2} - 4\varepsilon, \frac{1}{2}] & \text{if } x \in I_4, \\ [0, \frac{1}{2} - 4\varepsilon[& \text{if } \sup I_4 < x. \end{cases} \tag{7.7}$$

Then there exists $y \in [\inf G(x_1, A_1), \sup G(x_3, A_3)] \cup [\inf G(x_3, A_3), \sup G(x_1, A_1)]$ such that $\phi(f)(y) = 0$. Moreover, we have

$$G(x_1, A_1) < G(x_2, A_2) < G(x_3, A_3) \qquad \text{or} \qquad G(x_1, A_1) > G(x_2, A_2) > G(x_3, A_3).$$

Let

$$K_1 = \Big\{y \in \mathbb{R} : \nexists a, b \in \mathbb{R} : a < b < y, \|\phi(f)(a)\| \geq \frac{1}{2} - 2\varepsilon, \|\phi(f)(b)\| \leq 2\varepsilon,$$
$$\exists a, b \in \mathbb{R} : y < a < b, \|\phi(f)(a)\| \leq 2\varepsilon, \|\phi(f)(b)\| \geq 1 - 2\varepsilon\Big\},$$

$$K_1' = \Big\{y \in \mathbb{R} : \nexists a, b \in \mathbb{R} : a < b < y, \|\phi(f)(a)\| \geq 1 - 2\varepsilon, \|\phi(f)(b)\| \leq 2\varepsilon,$$
$$\exists a, b \in \mathbb{R} : a < b < y, \|\phi(f)(a)\| \geq \frac{1}{2} - 2\varepsilon, \|\phi(f)(b)\| \leq 2\varepsilon,$$
$$\exists a, b \in \mathbb{R} : y < a < b, \|\phi(f)(a)\| \leq 2\varepsilon, \|\phi(f)(b)\| \geq 1 - 2\varepsilon\Big\},$$

$$K_2 = \Big\{ y \in \mathbb{R} : \exists a_1, a_2, b_1, b_2 \in \mathbb{R} : a_1 < a_2 < y < b_1 < b_2,$$
$$\|\phi(f)(a_1)\|, \|\phi(f)(b_2)\| \geq 1 - 2\varepsilon, \|\phi(f)(a_2)\|, \|\phi(f)(b_1)\| \leq 2\varepsilon \Big\},$$
$$K_3 = \Big\{ y \in \mathbb{R} : \nexists a, b \in \mathbb{R} : y < a < b, \|\phi(f)(a)\| \leq 2\varepsilon, \|\phi(f)(b)\| \geq \frac{1}{2} - 2\varepsilon,$$
$$\exists a, b \in \mathbb{R} : a < b < y, \|\phi(f)(a)\| \geq 1 - 2\varepsilon, \|\phi(f)(b)\| \leq 2\varepsilon \Big\},$$
$$K_3' = \Big\{ y \in \mathbb{R} : \nexists a, b \in \mathbb{R} : y < a < b, \|\phi(f)(a)\| \leq 2\varepsilon, \|\phi(f)(b)\| \geq 1 - 2\varepsilon,$$
$$\exists a, b \in \mathbb{R} : y < a < b, \|\phi(f)(a)\| \leq 2\varepsilon, \|\phi(f)(b)\| \geq \frac{1}{2} - 2\varepsilon,$$
$$\exists a, b \in \mathbb{R} : a < b < y, \|\phi(f)(a)\| \geq 1 - 2\varepsilon, \|\phi(f)(b)\| \leq 2\varepsilon \Big\}.$$

It is easy to see that the sets K_1, K_1', K_2, K_3 and K_3' are pairwise disjoint intervals. Let $f_1 \in S_0(x_1, A_1), f_2 \in S_0(x_2, A_2), f_3 \in S_0(x_3, A_3)$ be functions with disjoint supports for which $\|f_1\|, \|f_2\|, \|f_3\| = \varepsilon$ and $f_i(z_j) = 0$ $(i = 1, 2, 3;\ j = 1, 2)$. Then $f + f_2 \in S_0(x_2, A_2)$ and $\|f + f_2\| = 1 + \varepsilon$. We may assume that there are $y_1, y_2, y_3, u_1, u_2 \in \mathbb{R} \cup \{+\infty, -\infty\}$ for which

$$\varphi^{-1}_{f+f_2,n}(x_1) \to y_1, \quad \varphi^{-1}_{f+f_2,n}(x_2) \to y_2, \quad \varphi^{-1}_{f+f_2,n}(x_3) \to y_3,$$
$$\varphi^{-1}_{f+f_2,n}(z_1) \to u_1, \quad \varphi^{-1}_{f+f_2,n}(z_2) \to u_2.$$

Then $\tau_{f+f_2,n} \cdot (f + f_2) \circ \varphi_{f+f_2,n} \to \phi(f + f_2)$ implies

$$\begin{aligned}
&\|\phi(f + f_2)(y_1)\| = \|(f + f_2)(x_1)\| = 1, \\
&\|\phi(f + f_2)(u_1)\| = \|(f + f_2)(z_1)\| = 0, \\
&\|\phi(f + f_2)(y_2)\| = \|(f + f_2)(x_2)\| = 1 + \varepsilon, \\
&\|\phi(f + f_2)(u_2)\| = \|(f + f_2)(z_2)\| < \varepsilon, \\
&\|\phi(f + f_2)(y_3)\| = \|(f + f_2)(x_3)\| = 1.
\end{aligned} \tag{7.8}$$

Thus $y_1, y_2, y_3 \in \mathbb{R}$. By the monotonicity of $\varphi_{f+f_2,n}$ $(n \in \mathbb{N})$, now we have $y_1 < u_1 < y_2 < u_2 < y_3$ or $y_3 < u_2 < y_2 < u_1 < y_1$. Thence $\|\phi(f_2)\| = \|f_2\| = \varepsilon$ and (7.8) imply

$$\|\phi(f)(y_1)\| \geq 1 - \varepsilon, \quad \|\phi(f)(y_2)\| \geq 1, \quad \|\phi(f)(y_3)\| \geq 1 - \varepsilon,$$
$$\|\phi(f)(u_1)\| \leq \varepsilon, \qquad \|\phi(f)(u_2)\| \leq 2\varepsilon.$$

Thus $y_2 \in K_2$. Now $f + f_2 \in S_0(x_2, A_2)$ and Step 7.3 imply that $\{y \in \mathbb{R} : \|\phi(f + f_2)(y)\| = \|f + f_2\| = 1 + \varepsilon\}$ is a compact interval which contains y_2. Then, by the definition of K_2, $y_2 \in K_2$ clearly yields

$$\Big\{ y \in \mathbb{R} : \|\phi(f + f_2)(y)\| = \|f + f_2\| = 1 + \varepsilon \Big\} \subseteq K_2,$$

which implies

$$G(x_2, A_2) \subseteq K_2.$$

We have $f + f_1 \in S_0(x_1, A_1)$ and $\|f + f_1\| = 1 + \varepsilon$. We may assume that there are $y_1, y_2, u_1 \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that

$$\varphi_{f+f_1,n}^{-1}(x_1) \to y_1, \quad \varphi_{f+f_1,n}^{-1}(x_2) \to y_2, \quad \varphi_{f+f_1,n}^{-1}(z_1) \to u_1. \tag{7.9}$$

Now, similarly as above, we deduce that

$$\begin{aligned} \|\phi(f + f_1)(y_1)\| &= \|(f + f_1)(x_1)\| = 1 + \varepsilon, \\ \|\phi(f + f_1)(u_1)\| &= \|(f + f_1)(z_1)\| = 0, \\ \|\phi(f + f_1)(y_2)\| &= \|(f + f_1)(x_2)\| = 1, \end{aligned} \tag{7.10}$$

and either $y_1 < u_1 < y_2$ or $y_2 < u_1 < y_1$. Hence, by (7.10), we have $y_1, y_2, u_2 \in \mathbb{R}$. Again, similarly as above, $\|\phi(f_1)\| = \|f_1\| = \varepsilon$ and (7.10) imply

$$\|\phi(f)(y_1)\| \geq 1, \quad \|\phi(f)(u_1)\| \leq \varepsilon, \quad \|\phi(f)(y_2)\| \geq 1 - \varepsilon. \tag{7.11}$$

Suppose that $y_1 < u_1 < y_2$. Further, suppose on the contrary that there are $a, b \in \mathbb{R}$ such that $a < b < y_1$, $\|\phi(f)(a)\| \geq \frac{1}{2} - 2\varepsilon$ and $\|\phi(f)(b)\| \leq 2\varepsilon$. We may assume that there exist $u_a, u_b \in \mathbb{R} \cup \{+\infty, -\infty\}$ for which

$$\varphi_{f+f_1,n}(a) \to u_a \quad \text{and} \quad \varphi_{f+f_1,n}(b) \to u_b.$$

Then $\|f_1\| = \varepsilon$ implies

$$\|(f + f_1)(u_a)\| = \|\phi(f + f_1)(a)\| \geq \Big| \|\phi(f)(a)\| - \|\phi(f_1)(a)\| \Big| \geq \frac{1}{2} - 2\varepsilon - \varepsilon = \frac{1}{2} - 3\varepsilon$$

and

$$\|(f + f_1)(u_b)\| = \|\phi(f + f_1)(b)\| \leq \|\phi(f)(b)\| + \|\phi(f_1)(b)\| \leq 2\varepsilon + \varepsilon = 3\varepsilon,$$

thus

$$\|f(u_a)\| \geq \frac{1}{2} - 3\varepsilon - \varepsilon = \frac{1}{2} - 4\varepsilon \quad \text{and} \quad \|f(u_b)\| \leq 3\varepsilon + \varepsilon = 4\varepsilon. \tag{7.12}$$

By (7.9) and $a < b < y_1 < y_2$, except for a finite number of $n \in \mathbb{N}$, we have $a < b < \varphi_{f+f_1,n}^{-1}(x_1) < \varphi_{f+f_1,n}^{-1}(x_2)$, which implies that $\varphi_{f+f_1,n}^{-1}$ is monotone increasing. Thus $u_a \leq u_b \leq x_1$, which contradicts (7.12), (7.7) and $x_1 \in I_1$. This means that there do not exist $a, b \in \mathbb{R}$ such that $a < b < y_1$, $\|\phi(f)(a)\| \geq \frac{1}{2} - 2\varepsilon$ and $\|\phi(f)(b)\| \leq 2\varepsilon$. Hence $y_1 < u_1 < y_2$ and (7.11) imply $y_1 \in K_1$. Similarly, if $y_2 < u_1 < y_1$ then $y_1 \in K_3$. Consequently, $y_1 \in K_1$ or $y_1 \in K_3$. By $f + f_1 \in S_0(x_1, A_1)$ and Step 7.3, the set $\{y \in \mathbb{R} : \|\phi(f + f_1)(y)\| = \|f + f_1\| = 1 + \varepsilon\}$ is a compact interval which contains y_1. By the definitions of K_1 and K_3, it is now clear that

$$\big\{y \in \mathbb{R} : \|\phi(f + f_1)(y)\| = \|f + f_1\| = 1 + \varepsilon\big\} \subseteq K_1$$

or

$$\{y \in \mathbb{R} : \|\phi(f+f_1)(y)\| = \|f+f_1\| = 1+\varepsilon\} \subseteq K_3,$$

which implies

$$G(x_1, A_1) \subseteq K_1 \quad \text{or} \quad G(x_1, A_1) \subseteq K_3.$$

It can be proved in a similar way that

$$G(x_3, A_3) \subseteq K_1' \quad \text{or} \quad G(x_3, A_3) \subseteq K_3'.$$

Now let $v_1 \in G(x_1, A_1)$ and $v_3 \in G(x_3, A_3)$. Suppose on the contrary that $G(x_1, A_1) \subseteq K_1$ and $G(x_3, A_3) \subseteq K_1'$. Then $v_1 \in K_1$, $v_3 \in K_1'$, and

$$\|\phi(f)(v_1)\| = \|\phi(f)(v_3)\| = 1.$$

Suppose that $v_1 < v_3$. Then, by $v_3 \in K_1'$, there exist $a < b < v_3$ for which $\|\phi(f)(a)\| \geq \frac{1}{2} - 2\varepsilon$ and $\|\phi(f)(b)\| \leq 2\varepsilon$. By $\|\phi(f)(v_1)\| = 1$, the inequality $v_1 < b < v_3$ would contradict $v_3 \in K_1'$. Thus $a < b < v_1$, hence $\|\phi(f)(a)\| \geq \frac{1}{2} - 2\varepsilon$ and $\|\phi(f)(b)\| \leq 2\varepsilon$ contradicts $v_1 \in K_1$. We get similarly a contradiction in the case in which $v_3 < v_1$. Thus $G(x_1, A_1) \subseteq K_1$ and $G(x_3, A_3) \subseteq K_1'$ cannot hold simultaneously. It can be shown in a similar way that $G(x_1, A_1) \subseteq K_3$ and $G(x_3, A_3) \subseteq K_3'$ cannot hold at the same time. Thus

$$G(x_1, A_1) \subseteq K_1 \;\text{ and }\; G(x_3, A_3) \subseteq K_3' \quad \text{or} \quad G(x_1, A_1) \subseteq K_3 \;\text{ and }\; G(x_3, A_3) \subseteq K_1'.$$

It is easy to see that $K_1 < K_2 < K_3'$ and $K_1' < K_2 < K_3$, whence we have

$$G(x_1, A_1) < G(x_2, A_2) < G(x_3, A_3) \quad \text{or} \quad G(x_1, A_1) > G(x_2, A_2) > G(x_3, A_3).$$

Finally, we may assume that there are $y_1, y_2, y_3, u_1, u_2 \in \mathbb{R} \cup \{+\infty, -\infty\}$ with

$$\varphi_{f,n}^{-1}(x_i) \to y_i \;(1 \leq i \leq 3) \quad \text{and} \quad \varphi_{f,n}^{-1}(z_j) \to u_j \;(1 \leq j \leq 2).$$

Now $\tau_{f,n} \cdot f \circ \varphi_{f,n} \to \phi(f)$ and the monotonicity of $\varphi_{f,n}$ $(n \in \mathbb{N})$ imply $\|\phi(f)(y_1)\| = \|f(x_1)\| = 1$, $\|\phi(f)(y_2)\| = \|f(x_2)\| = 1$, $\|\phi(f)(u_1)\| = \|f(z_1)\| = 0$ and $y_1 < u_1 < y_2$ or $y_2 < u_1 < y_1$. Then $y_1, y_2 \in \mathbb{R}$, and so $u_1 \in \mathbb{R}$. Moreover, we obtain that $u_1 \in K_2$. Then, by $K_1 < K_2 < K_3'$ and $K_1' < K_2 < K_3$, we are ready.

Step 7.6. *For any $A, B \in X$ with $A, B \neq 0$, and for $x, y \in \mathbb{R}$ with $x \neq y$, we have*

$$G(x, A) \cap G(y, B) = \emptyset.$$

Moreover, G is 'monotone' in the sense that one of the inequalities

$$G(x_1, A) < G(x_2, B) \text{ for every } x_1, x_2 \in \mathbb{R} \text{ with } x_1 < x_2,$$
$$G(x_1, A) > G(x_2, B) \text{ for every } x_1, x_2 \in \mathbb{R} \text{ with } x_1 < x_2$$

holds, where the relations '$<$' and '$>$' stand pointwise.

Let x_1, x_2, x_3 and $A_1, A_2, A_3 \in X$ be arbitrary with $x_1 < x_2 < x_3$ and $\|A_1\| = \|A_2\| = \|A_3\| = 1$. It is easy to see that there exists a function $f \in C_0(\mathbb{R}, X)$ such that f satisfies the conditions of Step 7.5. Then, by Step 7.5, we are done.

Step 7.7. *G is 'continuous' in the following sense: for any $x_n \to x_0$ and $A \in X$ we have $G(x_n, A) \to G(x_0, A)$, i.e.*

$$\sup\big\{d\big(y, G(x_0, A)\big) : y \in G(x_n, A)\big\} \to 0,$$

where

$$d\big(y, G(x_0, A)\big) = \inf\big\{|y - z| : z \in G(x_0, A)\big\}.$$

Let $x_n \in \mathbb{R}$ be a monotone decreasing sequence with $x_n \to x_0 \in \mathbb{R}$. (If x_n is increasing then the proof is similar.) For simplicity, assume that G is monotone increasing. Let

$$y_0 = \lim_{n\to\infty} \sup G(x_n, A) \geq \sup G(x_0, A)$$

and let $f \in S(x_0, A)$ be arbitrary. Then there exist functions $f_n \in S(x_n, A)$ $(n \in \mathbb{N})$ for which $\|f_n\| = \|f\|$ $(n \in \mathbb{N})$ and $f_n \to f$. Since $\sup G(x_n, A) \to y_0$ and $\phi(f)$ is continuous, we have

$$\big\|\phi(f)(y_0)\big\| = \lim_{n\to\infty} \big\|\phi(f_n)\big(\sup G(x_n, A)\big)\big\| = \lim_{n\to\infty} \|f_n\| = \|f\|.$$

Hence $y_0 \in G(x_0, A)$, thus $y_0 = \sup G(x_0, A)$. Since G is monotone, now we are ready.

Step 7.8. *Let $G(x) = G(x, I)$ for any $x \in \mathbb{R}$, where $I \in X$ is fixed. Then for any $A \in X$ with $A \neq 0$, we have*

$$G(x) = G(x, A).$$

Let $A, B \in X$ be arbitrary with $A, B \neq 0$, and let x_n be a monotone decreasing sequence with $x < x_n \to x$. For simplicity, assume that G is monotone increasing. Then Step 7.6 and Step 7.7 imply that

$$\sup G(x, A) < \inf G(x_n, B) \to \sup G(x, B),$$

thus

$$\sup G(x, A) \leq \sup G(x, B).$$

We get in a similar way that $\sup G(x, B) \leq \sup G(x, A)$, thus $\sup G(x, A) = \sup G(x, B)$. Similarly $\inf G(x, A) = \inf G(x, B)$, and, by Step 7.4, we are done.

Step 7.9. *Let*

$$U = \bigcup_{x\in\mathbb{R}} G(x).$$

Then U is an open interval. For any $u \in U$ denote by $\varphi(u) \in \mathbb{R}$ the uniquely determined real number for which $u \in G(\varphi(u))$. Then $\varphi : U \to \mathbb{R}$ is surjective, continuous and monotone. Moreover, we have

$$\big\|\phi(f)(y)\big\| \geq \big\|f\big(\varphi(y)\big)\big\| \qquad (y \in U). \tag{7.13}$$

By Steps 7.2 and 7.8 and the definition of φ, we get immediately (7.13).

The definition of φ and Step 7.6 imply that φ is monotone. It is clear that φ is surjective. We show that φ is also continuous. Let $u_n \in U$ $(n \in \mathbb{N})$ and $u \in U$ such that $u_n \to u$. Further, let

$$x_1 = \liminf_{n\to\infty} \varphi(u_n) \quad \text{and} \quad x_2 = \limsup_{n\to\infty} \varphi(u_n).$$

Then $x_1, x_2 \in \mathbb{R}$, and there exists a subsequence v_n of the sequence u_n for which

$$x_1 = \lim_{n\to\infty} \varphi(v_n).$$

Then, by Step 7.7 and the definition of φ, we have

$$v_n \in G\big(\varphi(v_n)\big) \to G(x_1),$$

whence

$$u = \lim_{n\to\infty} v_n \in G(x_1).$$

Similarly

$$u \in G(x_2).$$

Hence $x_1 = x_2 = \varphi(u)$, so $\varphi(u_n) \to \varphi(u)$. Thus φ is continuous indeed.

We show that U is an interval. Suppose on the contrary that there exist $a_0, b_0 \in U$ and $z \in \mathbb{R} \setminus U$ such that $a_0 < z < b_0$. Let

$$z_1 = \sup\big(]-\infty, z[\cap U\big) \quad \text{and} \quad z_2 = \inf\big(]z, \infty[\cap U\big).$$

Now there exists a monotone increasing sequence $a_n \in]-\infty, z[\cap U$ $(n \in \mathbb{N})$ and a monotone decreasing sequence $b_n \in]z, \infty[\cap U$ $(n \in \mathbb{N})$ such that $a_n \to z_1$ and $b_n \to z_2$. Since φ is monotone and $a_1 \le a_n < z < b_n \le b_1$ $(n \in \mathbb{N})$, now there exist $a, b \in \mathbb{R}$ for which $\varphi(a_n) \to a$ and $\varphi(b_n) \to b$. By the 'continuity' of G, we have $a_n \in G(\varphi(a_n)) \to G(a)$ and $b_n \in G(\varphi(b_n)) \to G(b)$, from which we obtain that

$$z_1 = \lim_{n\to\infty} a_n \in G(a) \quad \text{and} \quad z_2 = \lim_{n\to\infty} a_n \in G(b).$$

If $a \ne b$ then there is a $z_0 \in]a, b[$, thus $G(a) < G(z_0) < G(b)$ or $G(b) < G(z_0) < G(a)$, whence $U \supseteq G(z_0) \subseteq]z_1, z_2[$. So $U \cap]z_1, z_2[\ne \emptyset$, which is a contradiction. Hence $a = b$, which implies $z_1, z_2 \in G(a)$. Therefore $z \in [z_1, z_2] \subseteq G(a) \subseteq U$. This again is a contradiction, so U is an interval indeed. Hence, by the definition of U and the monotonicity of G, we obtain that U is an open interval, which completes the proof of Step 7.9.

Step 7.10. *Let $f \in C_0(\mathbb{R}, X)$ and $z_1 \in \mathbb{R}$ be arbitrary such that f satisfies the conditions of Step 7.5 and $f^{-1}(0) = \{z_1\}$. Then there exists $y \in G(z_1)$ for which*

$$\phi(f)(y) = f(z_1) = 0.$$

By Steps 7.5 and 7.9, there exists $y \in U$ such that $\phi(f)(y) = 0$. Then Step 7.2 implies $\|f(\varphi(y))\| \leq \|\phi(f)(y)\| = 0$. By $f^{-1}(0) = \{z_1\}$, we obtain that $\varphi(y) = z_1$, thus $y \in G(z_1)$ and $\phi(f)(y) = f(z_1) = 0$.

Step 7.11. *Let $f \in C_0(\mathbb{R}, X)$ and $z_1 \in \mathbb{R}$ such that f satisfies the conditions of Step 7.5 and $f(z_1) = 0$. Then for any $y \in G(z_1)$ we have*

$$\phi(f)(y) = f(z_1) = 0.$$

Let $a = \inf G(z_1)$, $b = \sup G(z_1)$, and let $x \in [a, b]$ and $\varepsilon \in]0, \varepsilon_0[$ be arbitrary. By the continuity of f, there is a $\delta > 0$ such that for any $y \in \mathbb{R}$ with $|y - z_1| \leq \delta$ we have $\|f(y)\| < \varepsilon_0$. Let $z_a, z_b \in \mathbb{R}$ with $z_1 - \delta < z_a < z_1 < z_b < z + \delta$. Then there exists $f_0 \in C_0(\mathbb{R}, X)$ such that $\|f_0\| < \varepsilon_0$ and $f_1 = f - f_0$ vanishes exactly at the points z_a, z_b and z_1. It is easy to see that there are functions $g_n \in C_0(\mathbb{R}, X)$ $(n \in \mathbb{N})$ which satisfy the conditions of Step 7.5 and for which $g_n \to f_1$ and $g_n^{-1}(0) = \{z_1\}$ $(n \in \mathbb{N})$. There are functions $h_{n,a} \in C_0(\mathbb{R}, X)$ $(n \in \mathbb{N})$ which satisfy the conditions of Step 7.5 but with z_a instead of z_1, for which $h_{n,a} \to f_1$ and $h_{n,a}^{-1}(0) = \{z_a\}$ $(n \in \mathbb{N})$. Similarly, there are functions $h_{n,b} \in C_0(\mathbb{R}, X)$ $(n \in \mathbb{N})$ which satisfy the conditions of Step 7.5 but with z_b instead of z_1, for which $h_{n,b} \to f_1$ and $h_{n,b}^{-1}(0) = \{z_b\}$ $(n \in \mathbb{N})$. Then, by Step 7.10, for any $n \in \mathbb{N}$ there are $r_n \in G(z_1)$, $s_{n,a} \in G(z_a)$ and $s_{n,b} \in G(z_b)$ such that $\phi(g_n)(r_n) = 0$, $\phi(h_{n,a})(s_{n,a}) = 0$ and $\phi(h_{n,b})(s_{n,b}) = 0$. Since $G(z_1)$, $G(z_a)$ and $G(z_b)$ are compact intervals, we may assume that there are $y_1 \in G(z_1)$, $y_a \in G(z_a)$ and $y_b \in G(z_b)$ for which $r_n \to y_1$, $s_{n,a} \to y_a$ and $s_{n,b} \to y_b$. Using $g_n \to f_1$, $h_{n,a} \to f_1$ and $h_{n,b} \to f_1$, we deduce that $\phi(f_1)(y_1) = \phi(f_1)(y_a) = \phi(f_1)(y_b) = 0$. For simplicity, assume that G is monotone increasing. Then $y_a < a \leq y_1 \leq b < y_b$.

We may assume that there are $u_a, u, u_b \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that $\varphi_{f_1,n}(y_a) \to u_a$, $\varphi_{f_1,n}(x) \to u$ and $\varphi_{f_1,n}(y_b) \to u_b$. Now $\|f_1(\varphi_{f_1,n}(y_a))\| \to \|\phi(f_1)(y_a)\| = 0$ and $\|f_1(\varphi_{f_1,n}(y_b))\| \to \|\phi(f_1)(y_b)\| = 0$. Hence $f_1(u_a) = f_1(u_b) = 0$, which implies $u_a, u_b \in \{z_a, z_1, z_b\} \subseteq [z_a, z_b]$. Then $y_a < x < y_b$ and the monotonicity of $\varphi_{f_1,n}$ $(n \in \mathbb{N})$ imply $u \in [u_a, u_b] \subseteq [z_a, z_b] \subseteq [z_1 - \delta, z_1 + \delta]$, thus $\|f(u)\| < \varepsilon_0$, and so $\|f_1(u)\| < 2\varepsilon_0$. Hence $f_1(\varphi_{f_1,n}(x)) \to f_1(u)$ and $\|f_1(\varphi_{f_1,n}(x))\| \to \|\phi(f_1)(x)\|$ imply $\|\phi(f_1)(x)\| < 2\varepsilon_0$, from which we infer that $\|\phi(f)(x)\| < 3\varepsilon_0$. Now, by the arbitrary choice of $\varepsilon_0 \in]0, \varepsilon[$, we obtain $\phi(f)(x) = 0$. Thus $\phi(f)$ is 0 on the interval $[a, b] = G(z_1)$ indeed.

Step 7.12. *Let $f \in C_0(\mathbb{R}, X)$ and $x \in \mathbb{R}$ be arbitrary. Then for any $y \in G(x)$ we have*

$$\big\|\phi(f)(y)\big\| = \big\|f(x)\big\|.$$

It is clear that there exists a function $f_0 \in C_0(\mathbb{R}, X)$ such that $(f - f_0)(x) = 0$ and $\|f_0\| = \|f_0(x)\| = \|f(x)\|$. Let $g = f - f_0$. It is not difficult to see that there exist $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}$ and $f_n \in C_0(\mathbb{R}, X)$ $(1 \leq i \leq n)$ such that $g = \lambda_1 f_1 + \cdots + \lambda_n f_n$, where the functions $f_1, \ldots, f_n$ satisfy the conditions of Step 7.5 with $z_1 = x$. By Step 7.11, for any $y \in G(x) = G(z_1)$ we have $\phi(f_i)(y) = 0$, whence $\phi(g)(y) = \sum_{i=1}^{n} \lambda_i \phi(f_i)(y) = 0$, and thus $\|\phi(f)(y)\| = \|\phi(f_0)(y)\| = \|f_0(x)\| = \|f(x)\|$.

Step 7.13. *For any $y \in \mathbb{R} \setminus U$ we have $\phi(f)(y) = 0$.*

Let $u_1 = \inf U$, $u_2 = \sup U$, and let $f \in C_0(\mathbb{R}, X)$ be arbitrary which vanishes nowhere. First let $y \in \mathbb{R}$ such that $\|f(y)\| = \|f\|$ and let $x \in G(y)$. Then $\|f(\varphi(x))\| = \|f(y)\| = \|f\|$. By Step 7.12, we obtain easily that $\phi(f)(u_1) = \phi(f)(u_2) = 0$. Thus $\|f(\varphi_{f,n}(u_1))\| \to 0$ and $\|f(\varphi_{f,n}(u_2))\| \to 0$, hence $f(y) \neq 0$ $(y \in \mathbb{R})$ implies

$$|\varphi_{f,n}(u_1)| \to \infty \quad \text{and} \quad |\varphi_{f,n}(u_2)| \to \infty.$$

If $\varphi_{f,n}(u_1) \to \infty$ and $\varphi_{f,n}(u_2) \to \infty$, then $u_1 < x < u_2$ and the monotonicity of $\varphi_{f,n}$ $(n \in \mathbb{N})$ imply $\varphi_{f,n}(x) \to \infty$, from which we deduce that $\phi(f)(x) = 0$. Then $0 = \|\phi(f)(x)\| = \|f(\varphi(x))\| = \|f\| > 0$, which is a contradiction. Similarly, we obtain a contradiction if $\varphi_{f,n}(u_1) \to -\infty$ and $\varphi_{f,n}(u_2) \to -\infty$. Thus either

$$\varphi_{f,n}(u_1) \to -\infty \text{ and } \varphi_{f,n}(u_2) \to \infty \qquad \text{or} \qquad \varphi_{f,n}(u_2) \to -\infty \text{ and } \varphi_{f,n}(u_1) \to \infty.$$

Now let $y \in \mathbb{R} \setminus]u_1, u_2[$ be arbitrary. Then the monotonicity of $\varphi_{f,n}$ $(n \in \mathbb{N})$ implies $\varphi_{f,n}(y) \notin]\varphi_{f,n}(u_1), \varphi_{f,n}(u_2)[$. Hence $|\varphi_{f,n}(y)| \to \infty$, and so

$$\|\phi(f)(y)\| = \lim_{n\to\infty} \|f(\varphi_{f,n}(y))\| = 0.$$

Now let $f \in C_0(\mathbb{R}, X)$ be arbitrary. Then there exist functions $f_n \in C_0(\mathbb{R}, X)$ $(n \in \mathbb{N})$ vanishing nowhere with $f_n \to f$. It follows from the above results that

$$\phi(f)(y) = \lim_{n\to\infty} \phi(f_n)(y) = 0.$$

By the arbitrary choice of $y \in \mathbb{R} \setminus U$, the proof of Step 7.13 is complete.

Step 7.14. *For any $f \in C_0(\mathbb{R}, X)$, we have*

$$\|\phi(f)(y)\| = \begin{cases} \|f(\varphi(y))\| & \text{if } y \in U, \\ 0 & \text{if } y \in \mathbb{R} \setminus U. \end{cases}$$

The above statement is a consequence of Steps 7.9, 7.12 and 7.13.

Step 7.15. *There exists a strongly continuous function $\tau : U \to \mathcal{S}$ such that*

$$\phi(f)(y) = \tau(y)\Big(f\big(\varphi(y)\big)\Big) \qquad (y \in U).$$

Let $y \in U$ and $A \in X$ be arbitrary. Further, let $f \in C_0(\mathbb{R}, X)$ with $f\big(\varphi(y)\big) = A$, and let

$$\tau(y)(A) = \phi(f)(y).$$

We show that $\tau(y)$ is well-defined. Let $f_1, f_2 \in C_0(\mathbb{R}, X)$ for which $f_1(\varphi(y)) = f_2(\varphi(y)) = A$. Then, by Step 7.14, we have $\|\phi(f_1 - f_2)(y)\| = \|(f_1 - f_2)(\varphi(y))\| = 0$, thus $\phi(f_1)(y) = \phi(f_2)(y)$. Now $\tau(y) : X \to X$ is clearly a linear isometry.

Let $y \in U$ and $A \in X$ be arbitrary with $A \neq 0$. Further, let $f \in S_0(\varphi(y), A)$ such that $f(\varphi(y)) = A$ and $f(x) = \frac{\|f(x)\|}{\|A\|} \cdot A$ $(x \in \mathbb{R})$. By $\tau_{f,n} f \circ \varphi_{f,n} \to \phi(f)$, we have

$$\tau_{f,n}(y) f\big(\varphi_{f,n}(y)\big) \to \phi(f)(y) = \tau(y)(A),$$

thus

$$\tau_{f,n}(y) \left(\frac{\|f(\varphi_{f,n}(y))\|}{\|A\|} \cdot A \right) \to \tau(y)(A).$$

Since $\tau_{f,n}(y)$ and $\tau(y)$ are isometries, now $\|f(\varphi_{f,n}(y))\| \to \|A\|$ and so

$$\tau_{f,n}(y)(A) \to \tau(y)(A).$$

By the topological reflexivity of $\mathcal{S}$, we obtain that $\tau(y) \in \mathcal{S}$.

We show that τ is strongly continuous. Let $A \in X$, $x \in \mathbb{R}$ and $x_n \in \mathbb{R}$ $(n \in \mathbb{N})$ be arbitrary with $A \neq 0$ and $x_n \to x$. Then the continuity of φ implies $\varphi(x_n) \to \varphi(x) \in \mathbb{R}$, thus there exists $f \in C_0(\mathbb{R}, X)$ for which $f(\varphi(x_n)) = f(\varphi(x)) = A$ $(n \in \mathbb{N})$. Then the continuity of $\phi(f)$ yields

$$\tau(x_n)(A) = \tau(x_n)\Big(f\big(\varphi(x_n)\big)\Big) = \phi(f)(x_n) \to \phi(f)(x) = \tau(x)(A).$$

Theorem 7.1 is now a consequence of Steps 7.9, 7.13 and 7.15. □

Proof of Theorem 7.3. Assume that ϕ satisfies the conditions in Theorem 7.3, and let $\mathcal{S} = \mathrm{iso}(X)$. Then ϕ also satisfies the the conditions in Theorem 7.1. Thus, by Theorem 7.1, there is an open interval $U \subseteq \mathbb{R}$, a monotone, continuous, surjective function $\varphi : U \to \mathbb{R}$, and a strongly continuous function $\tau : U \to \mathrm{iso}(X)$ such that for any $f \in C_0(\mathbb{R}, X)$ the equation (7.1) holds. Let $x \in \mathbb{R}$ be arbitrary, and let $f \in C_0(\mathbb{R}, X)$ be a function which vanishes nowhere and for which $\{x\} = \{y \in \mathbb{R} : \|f(y)\| = \|f\|\}$. Since ϕ satisfies the conditions in Theorem 7.3, we obtain that $\phi(f)$ vanishes nowhere and the set $\big\{y \in \mathbb{R} : \|\phi(f)(y)\| = \|\phi(f)\| = \|f(x)\|\big\}$ is a singleton. By (7.1), we obtain that $U = \mathbb{R}$ and the set $\varphi^{-1}(x)$ is also a singleton. Hence, by the arbitrary choice of $x \in \mathbb{R}$, we obtain the injectivity of φ. Thus $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous bijection, and so it is a homeomorphism.

Now let $y \in \mathbb{R}$ and $A \in B(X)$ be arbitrary, and let $f \in S_0\big(\varphi(y), A\big)$ with $f\big(\varphi(y)\big) = A$. By the conditions of Theorem 7.3, there is a homeomorphism $\varphi_0 : \mathbb{R} \to \mathbb{R}$ and a function $\tau_0 : \mathbb{R} \to \mathcal{P}$ for which (7.2) holds. Then

$$\big\|f\big(\varphi(y)\big)\big\| = \big\|\phi(f)(y)\big\| = \big\|f\big(\varphi_0(y)\big)\big\|,$$

which together with $f \in S_0\big(\varphi(y), A\big)$ imply $\varphi(y) = \varphi_0(y)$. Thus

$$\begin{aligned} \tau(y)(A) =& \tau(y)\big(f\big(\varphi(y)\big)\big) = \phi(f)(y) \\ =& \tau_0(y)\big(f\big(\varphi_0(y)\big)\big) = \tau_0(y)\big(f\big(\varphi(y)\big)\big) = \tau_0(y)(A). \end{aligned}$$

Now the algebraic reflexivity of $\mathcal{P}$ implies $\tau(y) \in \mathcal{P}$, which completes the proof. □

8. 2-LOCAL ISOMETRIES OF $C_0(X)$

8.1 Introduction and Statement of The Results

In the present chapter we deal with 2-local isometries of certain function algebras. A (not necessarily linear) mapping $\phi : \mathcal{X} \to \mathcal{X}$ ($\mathcal{X}$ being a Banach space) is called a *2-local isometry*, if for any $x, y \in \mathcal{X}$ there is a surjective linear isometry $\phi_{x,y} : \mathcal{X} \to \mathcal{X}$ such that $\phi(x) = \phi_{x,y}(x)$ and $\phi(y) = \phi_{x,y}(y)$. As was already noted in the Introduction, it is a remarkable property of a Banach space if its 2-local isometries are surjective linear isometries.

Molnár [66] studied 2-local isometries of certain operator algebras. He proved that every 2-local isometry of a C^*-subalgebra of $B(H)$ which contains the compact operators and the identity is a surjective linear isometry. He initiated the study of similar questions concerning function algebras, as well.

In this chapter we present some results concerning one of the most important function algebras, namely $C_0(X)$, the algebra of all continuous complex valued functions on the locally compact Hausdorff space X which vanish at infinity. Our main result shows that, under some not too restrictive conditions on X, every 2-local isometry of $C_0(X)$ is a surjective linear isometry.

In what follows, let X be a locally compact Hausdorff space. If X is first countable then X is separable. We shall frequently use the following fact. By Urysohn's lemma, for any $x \in X$ there is a function $f \in C_0(X)$, $f : X \to [0, 1]$, for which $f(x) = 1$. If X is first countable then we may also assume that $f^{-1}(1) = \{x\}$.

For brevity, in this chapter *isometry stands for surjective linear isometry.* By the famous Banach-Stone theorem, if $\phi : C_0(X) \to C_0(X)$ is an isometry then there exists a homeomorphism $\varphi : X \to X$ and a continuous function $\tau : X \to \mathbb{C}$ with $|\tau| = 1$ for which $\phi(f) = \tau \cdot f \circ \varphi$. Thus, if ϕ is a 2-local isometry then for any $f_1, f_2 \in C_0(X)$ there exists a homeomorphism $\varphi_{f_1,f_2} : X \to X$ and a continuous function $\tau_{f_1,f_2} : X \to \mathbb{C}$, $|\tau_{f_1,f_2}| = 1$, such that

$$\phi(f) = \tau_{f_1,f_2} \cdot f \circ \varphi_{f_1,f_2}. \tag{8.1}$$

Consequently, every 2-local isometry preserves the distance between functions. Thus the main problem is to prove that the 2-local isometries are linear and surjective.

In Theorem 8.1 we give the form of the 2-local isometries of $C_0(X)$.

Theorem 8.1. *Let X be a locally compact Hausdorff space and ϕ a 2-local isometry of $C_0(X)$. Then there exists a subset $X_0 \subseteq X$, a continuous function $\mu : X_0 \to \mathbb{C}$ with $|\mu| = 1$, and a surjective continuous function $\psi : X_0 \to X$ such that*

$$\big(\phi(f)\big)_{|X_0} = \mu \cdot (f \circ \psi)$$

holds for every $f \in C_0(X)$.

When X is first countable and σ-compact, by using Theorem 8.1 we prove our main result which can be stated as follows.

Theorem 8.2. *If X is a first countable σ-compact Hausdorff space then every 2-local isometry of $C_0(X)$ is a (surjective linear) isometry.*

Finally, we show that for arbitrary locally compact Hausdorff spaces the above statement does not hold.

Proposition 8.3. *Let X be a non-countable discrete topological space. Then there is an (even linear) 2-local isometry which is not an isometry.*

8.2 Proofs

We introduce the following notation. For any $x \in X$, let

$$\begin{aligned}
A_{x,f} &= \big\{(y,\nu) \in X \times \mathbb{C} \mid \ |\nu| = 1,\ \phi(f)(y) = \nu f(x)\big\} \qquad \big(f \in C_0(X)\big),\\
\mathcal{A}_x &= \big\{A_{x,f} \mid f \in C_0(X)\big\},\\
A_x &= \bigcap \mathcal{A}_x,\\
B_x &= \big\{y \in X \mid \exists \nu \in \mathbb{C} : (y,\nu) \in A_x\big\},\\
\mathcal{D}_x &= \big\{f \in C_0(X) \mid f : X \to [0,1], f(x) = 1\big\},\\
\mathcal{D}'_x &= \big\{f \in C_0(X) \mid f : X \to [0,1], f^{-1}(1) = \{x\}\big\},\\
I_n &= \big\{1, \dots, n\big\} \ \ (n \in \mathbb{N}),
\end{aligned}$$

and for any $f \in C_0(X)$ let

$$\operatorname{supp} f = \{x \in X \mid f(x) \neq 0\}.$$

We shall use $]a,b[$ for the open interval (a,b), where $a, b \in \mathbb{R}$, and for a set $A \subseteq X$ we denote by $\operatorname{cl}(A)$ and ∂A the *closure* of A and the *border* of A, respectively.

If ϕ is a 2-local isometry then, by (8.1), it is clear that $A_{x,f} \neq \emptyset$ ($x \in X, f \in C_0(X)$). By Urysohn's lemma we have $\mathcal{D}_x \neq \emptyset$ ($x \in X$), and, if X is first countable then the sets $\mathcal{D}'_x$ ($x \in X$) are also non-empty.

Proof of Theorem 8.1. It is easy to show that ϕ is *homogeneous*. Let $\lambda \in \mathbb{C}$ and $f \in \mathbb{C}_0(X)$ be arbitrary. For any $x \in X$ we have

$$\phi(\lambda f)(x) = \tau_{f,\lambda f}(x) \cdot (\lambda f)\big(\varphi_{f,\lambda f}(x)\big) = \lambda \tau_{f,\lambda f}(x) \cdot f\big(\varphi_{f,\lambda f}(x)\big) = \lambda \phi(f)(x).$$

Let $x \in X$ be arbitrary. We show that A_x is also non-empty. Since $\mathcal{A}_x$ is a set of closed subsets of a Hausdorff space and for any $f \in \mathcal{D}_x$ the set $A_{x,f}$ is compact, it is sufficient to verify that $\mathcal{A}_x$ satisfies the finite intersection property.

Let $f_1, \ldots, f_n \in C_0(X)$ be arbitrary and let $f_0 \in \mathcal{D}_x$. Furthermore, define the functions

$$g = \big(2\|f_1\| - |f_1 - f_1(x)|\big) + \cdots + \big(2\|f_n\| - |f_n - f_n(x)|\big) + 1 > 0 \tag{8.2}$$

and

$$f = f_0 \cdot g \geq 0.$$

Since g is continuous and bounded, in view of $f_0 \in C_0(X)$ it is clear that $f \in C_0(X)$. Let $(y, \nu) \in A_{x,f}$ be arbitrary. Now for any $i \in I_n$ we have $|f(x)| = |\phi(f)(y)| = |f(\varphi_{f,f_i}(y))|$. This and $f \geq 0$ imply that $f(\varphi_{f,f_i}(y)) = f(x)$. Since the functions f_0, g have maximum at x, by $0 \leq f_0, g$ and

$$(f_0 \cdot g)\big(\varphi_{f,f_i}(y)\big) = f\big(\varphi_{f,f_i}(y)\big) = f(x) = (f_0 \cdot g)(x),$$

we obtain that g and f_0 have maximum both at x and at $\varphi_{f,f_i}(y)$, thus $g(\varphi_{f,f_i}(y)) = g(x)$. Hence (8.2) implies $f_i(\varphi_{f,f_i}(y)) = f_i(x)$. By $f(\varphi_{f,f_i}(y)) = f(x)$, for any $i \in I_n$ we have

$$\nu f(x) = \phi(f)(y) = \tau_{f,f_i}(y) f\big(\varphi_{f,f_i}(y)\big) = \tau_{f,f_i}(y) f(x),$$

and so, by $f(x) \neq 0$, we obtain that $\nu = \tau_{f,f_i}(y)$. Thus

$$\phi(f_i)(y) = \tau_{f,f_i}(y) f_i\big(\varphi_{f,f_i}(y)\big) = \nu f_i(x),$$

whence $(y, \nu) \in A_{x,f_i}$. Therefore $\emptyset \neq A_{x,f} \subseteq A_{x,f_i}$ for any $i \in I_n$, which implies that the set $\mathcal{A}_x$ satisfies the finite intersection property. Hence A_x is indeed non-empty as was claimed above. Then B_x is clearly also non-empty.

It is easy to show that for any $x, y \in X$ with $x \neq y$ the sets B_x, B_y are disjoint. Indeed, let $x, y \in X$, $x \neq y$, be arbitrary elements and $f \in \mathcal{D}_x$ with $f(y) = 0$. Then $|\phi(f)|_{|B_x} = |f(x)| = 1$ and $|\phi(f)|_{|B_y} = |f(y)| = 0$, so we have $B_x \cap B_y = \emptyset$.

Let $X_0 = \cup_{y \in X} B_y$, and for any $x \in X_0$ denote by $\psi(x)$ the unique point of X for which $x \in B_{\psi(x)}$. Then $\psi : X_0 \to X$ is obviously surjective. We show that ψ is continuous. Let $U \neq \emptyset$ be an arbitrary open subset of X. By Urysohn's lemma there exists a function $f \in C_0(X)$ with $\emptyset \neq \operatorname{supp} f \subseteq U$. Let $V = \varphi_{f,f}^{-1}(\operatorname{supp} f)$. For any $y \in V$ we have $\varphi_{f,f}(y) \in \operatorname{supp} f$, so $|f(\psi(y))| = |\phi(f)(y)| = |f(\varphi_{f,f}(y))| \neq 0$, whence $\psi(y) \in \operatorname{supp} f \subseteq U$. Hence $\psi(V) \subseteq U$, which completes the proof of the continuity of ψ.

Let $y \in X_0$ be arbitrary. Since $y \in B_{\psi(y)}$, there exists a complex number $\mu(y)$ with $|\mu(y)| = 1$ for which $\big(y, \mu(y)\big) \in A_{\psi(y)}$. We show that $\mu(y)$ is unique. Suppose that $\nu_1, \nu_2 \in \mathbb{C}$ such that $(y, \nu_1), (y, \nu_2) \in A_{\psi(y)}$, and let $f \in \mathcal{D}_{\psi(y)}$. Then $\nu_1 = \nu_1 f(\psi(y)) = \phi(f)(y) = \nu_2 f(\psi(y)) = \nu_2$. Thus the function $\mu : X_0 \to \mathbb{C}$ is well-defined and $|\mu| = 1$.

Finally, we show that μ is continuous. Let $x \in X_0$ be arbitrary and $f \in \mathcal{D}_{\psi(x)}$. Now $\operatorname{supp}(f \circ \psi)$ is an open neighbourhood of x (in the subspace X_0) and for any $y \in \operatorname{supp}(f \circ \psi)$ we have $0 \neq \mu(y) f(\psi(y)) = \phi(f)(y)$. Therefore $\mu_{|\operatorname{supp}(f\circ\psi)} = \frac{\phi(f)}{f \circ \psi}$, so μ is continuous on $\operatorname{supp}(f \circ \psi)$. Hence μ is continuous, and thereby Theorem 8.1 is proved. □

To prove Theorem 8.2, we shall need the following lemma which seems to be interesting in itself.

Lemma 8.4. *Let X be a first countable σ-compact Hausdorff space and $R = \{r_n \in X | n \in \mathbb{N}\}$ with $r_i \neq r_j$ $(i \neq j)$ a countable subset of X. Then there exist functions $f, g \in C_0(X)$ with $0 < f, g : X \to [0,1]$ such that f has a strict local maximum at every point of R and $(f,g)^{-1}\big(f(r_n), g(r_n)\big) = \{r_n\}$ $(n \in \mathbb{N})$.*

Proof of Lemma 8.4. Let $R_n = \{r_1, \ldots, r_n\}$ $(n \in \mathbb{N})$. First we show that there exist functions $f_n \in C_0(X)$ $(n \in \mathbb{N})$ and positive numbers t_n $(n \in \mathbb{N})$ with the following properties:

(i) $f_n \in \mathcal{D}'_{r_n} \quad (n \in \mathbb{N})$,

(ii) $f_{n+1}^{-1}(0) = R_n \quad (n \in \mathbb{N})$,

(iii) $0 < t_n \leq 2^{-n+1} \quad (n \in \mathbb{N})$ and $t_1 = 1$,

(iv) $\begin{cases} \text{the numbers } (t_1 f_1 + \cdots + t_n f_n)(r_1), \ldots, (t_1 f_1 + \cdots + t_n f_n)(r_n) \\ \text{are pairwise distinct,} \end{cases}$

(v) $t_1 f_1 + \cdots + t_{n-1} f_{n-1} + t_n \sqrt{f_n}$ has a strict local maximum at r_n,

(vi) $\begin{cases} (t_1 f_1 + \cdots + t_n f_n)(x) < (t_1 f_1 + \cdots + t_{i-1} f_{i-1} + t_i \sqrt{f_i})(x) \\ \text{for any } x \in X \setminus R_i,\ i, n \in \mathbb{N}, \end{cases}$

(vii) $\begin{cases} (t_1 f_1 + \cdots + t_n f_n) \leq (t_1 f_1 + \cdots + t_{i-1} f_{i-1} + t_i \sqrt{f_i}) \\ \text{for any } i, n \in \mathbb{N}. \end{cases}$

Let $t_1 = 1$. Since X is σ-compact and first countable, there exists a function $f_1 \in \mathcal{D}'_{r_1}$ with $0 < f_1$. It is easy to see that t_1 and f_1 satisfy the properties (i)-(vii). Now let $n \in \mathbb{N}$, and assume that there exist functions $f_1, \ldots, f_n$ and numbers

$t_1, \dots, t_n$ with the properties (i)-(vii). Then put

$$(8.3) \quad s = \min\Big(2^{-(n+1)+1}, \min_{1\le i\le n}\Big(\big(t_1f_1+\cdots+t_{i-1}f_{i-1}+t_i\sqrt{f_i}\big)(r_{n+1}) - \big(t_1f_1+\cdots+t_nf_n\big)(r_{n+1})\Big)\Big).$$

Since $r_{n+1} \in X \setminus R_n \subseteq X \setminus R_i$ $(i \in I_n)$, by (vi) we have $s > 0$. Let

$$(8.4) \qquad t_{n+1} \in]0, s[\setminus \big\{(t_1f_1+\cdots+t_nf_n)(r_i) - (t_1f_1+\cdots+t_nf_n)(r_{n+1}) \big| i \in I_n\big\}$$

be arbitrary, and for any $x \in X$ let

$$(8.5) \quad u_0(x) = \min\Big(t_{n+1}, \min_{1\le i\le n}\Big(\big(t_1f_1+\cdots+t_{i-1}f_{i-1}+t_i\sqrt{f_i}\big)(x) - \big(t_1f_1+\cdots+t_nf_n\big)(x)\Big)\Big).$$

By (vii) we have $u_0 \geq 0$. Now (8.3), (8.5) and $t_{n+1} < s$ imply $u_0(r_{n+1}) = t_{n+1}$. It is easy to see that, by Urysohn's lemma, there exists a function $u_1 \in \mathcal{D}'_{r_{n+1}}$ with $u_1^{-1}(0) = R_n$ for which the function $(t_1f_1+\cdots+t_nf_n) + t_{n+1}u_1$ has a strict local maximum at r_{n+1}. Let $u = u_0 \cdot u_1$. By $u_1 \in \mathcal{D}'_{r_{n+1}}$, now $0 \leq u \leq u_0$. Since u_0 is a bounded continuous function and $u_1 \in C_0(X)$, the function $u = u_0 \cdot u_1$ is in $C_0(X)$. If $x \in X \setminus R_n$ then, by (vi), we have $u_0(x) > 0$, so by $u_1(x) > 0$ we obtain $u(x) > 0$. Thus $u_{|R_n} = 0$ implies $u^{-1}(0) = R_n$ and $u \geq 0$. Now $u(r_{n+1}) = u_0(r_{n+1}) \cdot u_1(r_{n+1}) = t_{n+1} \cdot 1 = t_{n+1}$, and for any $x \in X$, $x \neq r_{n+1}$ we have $u(x) = u_0(x) \cdot u_1(x) \leq t_{n+1} \cdot u_1(x) < t_{n+1} \cdot 1 = t_{n+1} = u(r_{n+1})$. Hence $u \in t_{n+1} \cdot \mathcal{D}'_{r_{n+1}}$. Define the function

$$f_{n+1} = \left(\frac{u}{t_{n+1}}\right)^2.$$

We show that $f_1, \dots, f_{n+1}$ and $t_1, \dots, t_{n+1}$ satisfy the properties (i)-(vii). It is clear that $f_{n+1}^{-1}(0) = u^{-1}(0) = R_n$ (property (ii)). By $u \in t_{n+1} \cdot \mathcal{D}'_{r_{n+1}}$, it is obvious that $f_{n+1} \in \mathcal{D}'_{r_{n+1}}$ (property (i)). We also have $0 < t_{n+1} < s \leq 2^{-(n+1)+1}$ (property (iii)). For any $i \in I_n$, by (8.4) and $f_{n+1}(r_i) = 0$ we have

$$\begin{aligned}(t_1f_1+\cdots+t_{n+1}f_{n+1})(r_{n+1}) =& (t_1f_1+\cdots+t_nf_n)(r_{n+1}) + t_{n+1}\\ \neq& (t_1f_1+\cdots+t_nf_n)(r_{n+1}) + \big((t_1f_1+\cdots+t_nf_n)(r_i)\\ &- (t_1f_1+\cdots+t_nf_n)(r_{n+1})\big)\\ =& (t_1f_1+\cdots+t_nf_n)(r_i)\\ =& (t_1f_1+\cdots+t_{n+1}f_{n+1})(r_i).\end{aligned}$$

Since (iv) holds for $f_1, \dots f_n$ and $t_1, \dots, t_n$, by $f_{n+1}^{-1}(0) = R_n$, we also obtain for any

$i, j \in I_n$, $i \neq j$ that

$$\begin{aligned}(t_1 f_1 + \cdots + t_{n+1} f_{n+1})(r_i) =& (t_1 f_1 + \cdots + t_n f_n)(r_i) \\ \neq& (t_1 f_1 + \cdots + t_n f_n)(r_j) \\ =& (t_1 f_1 + \cdots + t_{n+1} f_{n+1})(r_j),\end{aligned}$$

which now implies property (iv). For any $x \in X$ we have

$$\begin{aligned}\big(t_1 f_1 + \cdots + t_n f_n + t_{n+1}\sqrt{f_{n+1}}\big)(x) =& (t_1 f_1 + \cdots + t_n f_n)(x) + u(x) \\ \leq& (t_1 f_1 + \cdots + t_n f_n)(x) + t_{n+1} \cdot u_1(x).\end{aligned} \tag{8.6}$$

Further,

$$\begin{aligned}&\big(t_1 f_1 + \cdots + t_n f_n + t_{n+1}\sqrt{f_{n+1}}\big)(r_{n+1}) \\ &\qquad = \big(t_1 f_1 + \cdots + t_n f_n + t_{n+1} u_1\big)(r_{n+1}).\end{aligned} \tag{8.7}$$

Since the function $(t_1 f_1 + \cdots + t_n f_n) + t_{n+1} \cdot u_1$ has a strict local maximum at r_{n+1}, by (8.6) and (8.7) we obtain that the function $t_1 f_1 + \cdots + t_n f_n + t_{n+1}\sqrt{f_{n+1}}$ also has a strict local maximum at r_{n+1} (property (v)).

Now we show property (vi). If $i \geq n+1$ then, by (i) and $0 < f_i(x) < 1$, we have for any $x \in X \setminus R_i$ that

$$\begin{aligned}(t_1 f_1 + \cdots + t_{n+1} f_{n+1})(x) \leq& (t_1 f_1 + \cdots + t_i f_i)(x) \\ <& (t_1 f_1 + \cdots + t_{i-1} f_{i-1} + t_i\sqrt{f_i})(x).\end{aligned}$$

If $i \in I_n$ and $x \in X \setminus R_{n+1}$ then $0 < f_{n+1}(x) < 1$, $0 < u(x) \leq u_0(x)$, which together with $t_{n+1} > 0$ and (8.5) imply

$$\begin{aligned}(t_1 f_1 +& \cdots + t_{n+1} f_{n+1})(x) \\ &< \big(t_1 f_1 + \cdots + t_n f_n + t_{n+1}\sqrt{f_{n+1}}\big)(x) \\ &= (t_1 f_1 + \cdots + t_n f_n)(x) + u(x) \leq (t_1 f_1 + \cdots + t_n f_n)(x) + u_0(x) \\ &\leq (t_1 f_1 + \cdots + t_n f_n)(x) \\ &\qquad + \Big(\big(t_1 f_1 + \cdots + t_{i-1} f_{i-1} + t_i\sqrt{f_i}\big)(x) - (t_1 f_1 + \cdots + t_n f_n)(x)\Big) \\ &= \big(t_1 f_1 + \cdots + t_{i-1} f_{i-1} + t_i\sqrt{f_i}\big)(x).\end{aligned}$$

If $i \in I_n$ and $x = r_{n+1}$ then, by using $t_{n+1} < s$ and (8.3) we obtain that

$$\begin{aligned}(t_1 f_1 +& \cdots + t_{n+1} f_{n+1})(r_{n+1}) \\ &= (t_1 f_1 + \cdots + t_n f_n)(r_{n+1}) + t_{n+1} \\ &< (t_1 f_1 + \cdots + t_n f_n)(r_{n+1}) \\ &\qquad + \Big(\big(t_1 f_1 + \cdots + t_{i-1} f_{i-1} + t_i\sqrt{f_i}\big)(r_{n+1}) - (t_1 f_1 + \cdots + t_n f_n)(r_{n+1})\Big) \\ &= \big(t_1 f_1 + \cdots + t_{i-1} f_{i-1} + t_i\sqrt{f_i}\big)(r_{n+1}).\end{aligned}$$

If $i \in I_n$ and $x = r_j$ with $i < j \leq n$ then, since $f_{n+1}(r_j) = 0$ and property (vi) holds for $f_1, \ldots, f_n$, $t_1, \ldots, t_n$, we have

$$\begin{aligned}(t_1 f_1 + \cdots + t_{n+1} f_{n+1})(r_j) =&(t_1 f_1 + \cdots + t_n f_n)(r_j)\\ <&\big(t_1 f_1 + \cdots + t_{i-1} f_{i-1} + t_i \sqrt{f_i}\big)(r_j).\end{aligned}$$

Thus property (vi) is also satisfied. Now properties (ii), (vi) and (i) imply property (vii).

Define the functions $f_0 = \sum_{n=1}^{\infty} t_n f_n$ and $f = \frac{1}{\|f_0\|} f_0$. Then $f \in C_0(X)$, $f > \frac{f_1}{\|f_0\|} > 0$, $f : X \to [0,1]$, and, by (ii) and (iv), the numbers $f(r_n)$ $(n \in \mathbb{N})$ are pairwise distinct. Let $n_0 \in \mathbb{N}$ be arbitrary. By (vii) we obtain $f_0 \leq t_1 f_1 + \cdots + t_{n_0-1} f_{n_0-1} + t_{n_0} \sqrt{f_{n_0}}$. By (v) the latter function has a strict local maximum at r_{n_0} and (by $f_{n_0}(r_{n_0}) = 1$) $f_0(r_{n_0}) = (t_1 f_1 + \cdots + t_{n_0-1} f_{n_0-1} + t_{n_0} \sqrt{f_{n_0}})(r_{n_0})$, therefore the function f_0 also has a strict local maximum at r_{n_0}. So f has a strict local maximum at the points r_n $(n \in \mathbb{N})$.

We show that there exist functions $g_n \in C_0(X)$ $(n \in \mathbb{N})$ such that

(a) $g_{i+1} \geq g_i > 0$ $(i \in \mathbb{N})$,

(b) g_i has a strict local maximum at r_j $(i, j \in \mathbb{N})$,

(c) $\|g_i - g_{i+1}\| < 2^{-i}$ $(i \in \mathbb{N})$,

(d) $g_i(r_j) \notin g_i\big(f^{-1}(f(r_j)) \setminus \{r_j\}\big)$ $(i, j \in \mathbb{N}, j \leq i)$,

(e) the sets $g_i\big(f^{-1}(f(r_j))\big)$ are finite $(i, j \in \mathbb{N})$,

(f) $g_i = g_j$ on the set $f^{-1}\big(f(r_j)\big)$ $(i, j \in \mathbb{N}, j \leq i)$.

Let $g_1 = f$. Using (vi) with $i = 1$, we get $f_0 \leq \sqrt{f_1}$. Since $f_0(r_1) = t_1 f_1(r_1) = 1 = \sqrt{f_1}(r_1)$ and $f_1 \in \mathcal{D}'_{r_1}$, the function f_0 and so f has a strict global maximum at r_1, thus $f^{-1}(f(r_1)) = \{r_1\}$, so $f^{-1}(f(r_1)) \setminus \{r_1\} = \emptyset$. Now it is easy to see that the set $\{g_1\}$ satisfies the properties (a)-(f). Assume that the functions $g_1, \ldots, g_n$ satisfy the properties (a)-(f). Since $f(r_{n+1}) \neq f(r_j)$ $(j \in I_n)$ and the functions f, g_n have a strict local maximum at r_{n+1}, it is clear that there exists an open neighbourhood U_0 of r_{n+1} for which $f^{-1}(f(r_j)) \cap U_0 = \emptyset$ $(j \in I_n)$ and f, g_n have a strict maximum on $\mathrm{cl}(U_0)$ at r_{n+1}. Let $r = \max_{x \in \partial U_0} g_n(x) \in]0, g_n(r_{n+1})[$ if $\partial U_0 \neq \emptyset$, and 0 otherwise. Let $t \in]r, g_n(r_{n+1})[$ with $t \neq f(r_j), g_n(r_j)$ $(j \in \mathbb{N})$, and let $U = g_n^{-1}\big(]t, \infty[\big) \cap U_0$. Now $\mathrm{cl}\big(g_n^{-1}\big(]t, \infty[\big)\big)$ is compact, and, by $r < t$, we have $\mathrm{cl}\big(g_n^{-1}\big(]t, \infty[\big)\big) \cap \partial U_0 \subseteq g_n^{-1}\big([t, \infty[\big) \cap \partial U_0 = \emptyset$, whence $\partial U = \partial\big(g_n^{-1}\big(]t, \infty[\big)\big) \cap U_0 \subseteq g_n^{-1}(t)$. Thus $g_n = t$ on ∂U. So U is an open neighbourhood of r_{n+1}, and the functions f, g_n have a strict maximum on U at the point r_{n+1}. Further, $f(r_j), g_n(r_j) \neq t \in]0, g_n(r_{n+1})[$ $(j \in \mathbb{N})$, $g_n > t > 0$ on U, $g_n = t$ on ∂U, $\mathrm{cl}(U)$ is compact, and

$$f^{-1}\big(f(r_j)\big) \cap U = \emptyset \quad (j \in I_n). \tag{8.8}$$

Then obviously $r_j \notin U$ $(j \in I_n)$. Now $f^{-1}(f(r_{n+1})) \cap U = \{r_{n+1}\}$, thus

$$f^{-1}\big(f(r_{n+1})\big) \setminus \{r_{n+1}\} \subseteq X \setminus U. \tag{8.9}$$

Define the function

$$h(x) = \begin{cases} g_n(x) - t & \text{if } x \in U, \\ 0 & \text{if } x \in X \setminus U. \end{cases}$$

It is clear that $h \in C_0(X)$ and $h : X \to [0, g_n(r_{n+1}) - t]$, $h(r_{n+1}) = g_n(r_{n+1}) - t > 0$ and h has a strict maximum at r_{n+1}. Since the set $g_n\big(f^{-1}(f(r_{n+1}))\big)$ is finite (by (e)), we may choose a number $\varepsilon \in \big]0, \frac{2^{-(n+1)}}{h(r_{n+1})}\big[$ such that

$$(g_n + \varepsilon h)(r_{n+1}) \notin g_n\big(f^{-1}(f(r_{n+1}))\big). \tag{8.10}$$

Let $g_{n+1} = g_n + \varepsilon h \in C_0(X)$.

We show that $g_1, \ldots, g_{n+1}$ satisfy the properties (a)-(f). Now $g_{n+1} \geq g_n > 0$ (which implies property (a)) and

$$g_{n+1}(x) = \begin{cases} (1+\varepsilon)g_n(x) - \varepsilon t & \text{if } x \in U, \\ g_n(x) & \text{if } x \in X \setminus U. \end{cases} \tag{8.11}$$

Let $j \in \mathbb{N}$. Since $g_n(r_j) \neq t$ $(j \in \mathbb{N})$, r_j is an inner point of either U or $X \setminus U$. Thus, by (8.11), the property (b) holds. Property (c) is trivial by the definition of g_{n+1}. By (8.10), (8.9) and (8.11), we obtain that

$$\begin{aligned} g_{n+1}(r_{n+1}) \notin g_n\big(f^{-1}(f(r_{n+1}))\big) \supseteq & g_n\big(f^{-1}(f(r_{n+1})) \setminus \{r_{n+1}\}\big) \\ = & g_{n+1}\big(f^{-1}(f(r_{n+1})) \setminus \{r_{n+1}\}\big). \end{aligned}$$

Now let $j \in I_n$. By $r_j \in X \setminus U$, (8.11), (d) (for $g_1, \ldots, g_n$) and (8.8) we have

$$g_{n+1}(r_j) = g_n(r_j) \notin g_n\big(f^{-1}(f(r_j)) \setminus \{r_j\}\big) = g_{n+1}\big(f^{-1}(f(r_j)) \setminus \{r_j\}\big).$$

Thus property (d) also holds. Now let $j \in \mathbb{N}$. Then (8.11) and (e) imply that the sets

$$g_{n+1}\big(f^{-1}(f(r_j)) \cap U\big) = (1+\varepsilon)g_n\big(f^{-1}(f(r_j)) \cap U\big) - \varepsilon t$$

and

$$g_{n+1}\big(f^{-1}(f(r_j)) \cap (X \setminus U)\big) = g_n\big(f^{-1}(f(r_j)) \cap (X \setminus U)\big)$$

are finite, so the property (e) holds. Finally, by (8.8) and (8.11) we obtain that $g_{n+1} = g_n$ on $f^{-1}(f(r_j))$ for any $j \in I_n$. Since property (f) holds for $g_1, \ldots, g_n$, now it is clear that property (f) also holds for $g_1, \ldots, g_{n+1}$. Thus the functions $g_1, \ldots, g_{n+1}$ satisfy all the properties (a)-(f).

By (c), the sequence g_n is a Cauchy sequence, so the function $g_0 = \lim_{n\to\infty} g_n$ is in $C_0(X)$, and by $g_{n+1} \geq g_n > 0$ $(n \in \mathbb{N})$ we have $g_0 > 0$. Let $g = \frac{g_0}{\|g_0\|}$. Now $0 < g \in C_0(X)$ and $g : X \to [0,1]$. To prove Lemma 8.4, suppose on the contrary that there exist $n \in \mathbb{N}$ and $x \in X \setminus \{r_n\}$ for which $f(r_n) = f(x)$ and $g(r_n) = g(x)$. Then $r_n, x \in f^{-1}(f(r_n))$, so, by (f), we have

$$g_n(r_n) = g(r_n) = g(x) = g_n(x) \in g_n\big(f^{-1}(f(r_n)) \setminus \{r_n\}\big)$$

which contradicts (d). This completes the proof of Lemma 8.4. □

Proof of Theorem 8.2. We use the notation introduced in Theorem 8.1 and in its proof. By assumption, X is first countable. Suppose that $x, y \in X$ with $\psi(x) = \psi(y)$, and let $f \in \mathcal{D}'_{\psi(x)}$. Now

$$\big|f\big(\varphi_{f,f}(x)\big)\big| = \big|\phi(f)(x)\big| = \big|f\big(\psi(x)\big)\big| = 1$$
$$= \big|f\big(\psi(y)\big)\big| = \big|\phi(f)(y)\big| = \big|f\big(\varphi_{f,f}(y)\big)\big|,$$

whence, by $f \in \mathcal{D}'_{\psi(x)}$, we have $\varphi_{f,f}(x) = \psi(x) = \varphi_{f,f}(y)$, which implies $x = y$. So the function $\psi : X_0 \to X$ is a bijection.

If X is finite, then clearly $X_0 = X$, and, by Theorem 8.1, the proof is trivial. Assume that X is not finite. Since X is first countable and σ-compact, there exists a countable dense subset $R = \{r_n | n \in \mathbb{N}\}$ of X, where $r_i \neq r_j$ $(i \neq j)$. Let the functions f, g be as in Lemma 8.4. Now for any $n \in \mathbb{N}$ we have

$$\tau_{f,g}\big(\psi^{-1}(r_n)\big) f\big(\varphi_{f,g}(\psi^{-1}(r_n))\big) = \phi(f)\big(\psi^{-1}(r_n)\big) = \mu\big(\psi^{-1}(r_n)\big) f(r_n),$$

and similarly

$$\tau_{f,g}\big(\psi^{-1}(r_n)\big) g\big(\varphi_{f,g}(\psi^{-1}(r_n))\big) = \phi(g)\big(\psi^{-1}(r_n)\big) = \mu\big(\psi^{-1}(r_n)\big) g(r_n).$$

Since $f, g > 0$ and $\big|\mu\big(\psi^{-1}(r_n)\big)\big| = \big|\tau_{f,g}\big(\psi^{-1}(r_n)\big)\big| = 1$, we infer that

$$\Big(f\big(\varphi_{f,g}(\psi^{-1}(r_n))\big), g\big(\varphi_{f,g}(\psi^{-1}(r_n))\big)\Big) = \big(f(r_n), g(r_n)\big),$$

whence, by Lemma 8.4, we obtain that $\varphi_{f,g}(\psi^{-1}(r_n)) = r_n$. Since R is dense in X and $\varphi_{f,g} : X \to X$ is a homeomorphism, it follows that $X_0 \supseteq \psi^{-1}(R) = \varphi_{f,g}^{-1}(R)$ is also dense in X, therefore X_0 is dense in X.

Let $y \in X$ be arbitrary. Then there exists a sequence $y_n \in X_0$ with $y_n \to y$. Suppose on the contrary that $\big(\psi(y_n)\big)_{n\in\mathbb{N}}$ has no accumulation point in X. Since X is σ-compact, there is a function $f \in C_0(X)$, $f : X \to [0,1]$ with $f(x) \neq 0$ $(x \in X)$. Now the sequence $\psi(y_n)$ has at most finitely many terms in every compact set, thus $f\big(\psi(y_n)\big) \to 0$. Hence $\phi(f)(y_n) = \mu(y_n) f\big(\psi(y_n)\big) \to 0$, from which, by $y_n \to y$, we deduce that $\tau_{f,f}(y) f\big(\varphi_{f,f}(y)\big) = \phi(f)(y) = 0$, whence $f\big(\varphi_{f,f}(y)\big) = 0$, which is a contradiction. Thus there is a point $x \in X$ and a subsequence u_n of the sequence y_n such that $\psi(u_n) \to x$. Now for any $f \in C_0(X)$ we have

$$\phi(f)(u_n) = \mu(u_n) f\big(\psi(u_n)\big) \to \mu(y) f(x)$$

and $\phi(f)(u_n) \to \phi(f)(y)$. Therefore $\phi(f)(y) = \mu(y)f(x)$ for any $f \in C_0(X)$, which implies $y \in B_x$, and hence $y \in X_0$. Thus $X_0 = X$ and so $\psi : X \to X$ is a homeomorphism. The proof of Theorem 8.2 is now complete. □

Proof of Proposition 8.3. Let X be an uncountable space with the discrete topology. A function $f : X \to \mathbb{C}$ is in $C_0(X)$ if and only if there exists a sequence x_n such that $f(x_n) \to 0$ and $\operatorname{supp} f \subseteq \big\{x_n | n \in \mathbb{N}\big\}$. Let Y be a proper subset of X for which there is a bijection $\psi : X \to Y$. Define the map $\phi : C_0(X) \to C_0(X)$ by

$$\phi(f)(x) = \begin{cases} f(\psi^{-1}(x)) & \text{if } x \in Y, \\ 0 & \text{if } x \in X \setminus Y. \end{cases}$$

It is obvious that ϕ is linear and not surjective.

Now let $f, g \in C_0(X)$ be arbitrary, and let x_n be a sequence such that $f(x_n) \to 0$, $g(x_n) \to 0$ and $\operatorname{supp} f, \operatorname{supp} g \subseteq \big\{x_n | n \in \mathbb{N}\big\}$. Denote by $X_{f,g}$ the set $X \setminus \big\{\psi(x_n) | n \in \mathbb{N}\big\}$. It is clear that $\operatorname{supp} \phi(f), \operatorname{supp} \phi(g) \subseteq X \setminus X_{f,g}$. There is a bijection $\psi_{f,g} : X_{f,g} \to X \setminus \big\{x_n | n \in \mathbb{N}\big\}$. Let

$$\varphi_{f,g}(x) = \begin{cases} \psi_{f,g}(x) & \text{if } x \in X_{f,g}, \\ \psi^{-1}(x) & \text{if } x \in X \setminus X_{f,g}. \end{cases}$$

Then $\varphi_{f,g} : X \to X$ is a bijection, so it is a homeomorphism. If $x \in X_{f,g}$ then $\phi(f)(x) = 0$, $\phi(g)(x) = 0$ and $\varphi_{f,g}(x) = \psi_{f,g}(x) \notin \{x_n | n \in \mathbb{N}\}$, which imply $f(\varphi_{f,g}(x)) = 0$ and $g(\varphi_{f,g}(x)) = 0$. If $x \notin X_{f,g}$ then $x \in Y$, and so $\phi(f)(x) = f(\psi^{-1}(x)) = f(\varphi_{f,g}(x))$ and $\phi(g)(x) = g(\psi^{-1}(x)) = g(\varphi_{f,g}(x))$. Thus $\phi(f) = f \circ \varphi_{f,g}$ and $\phi(g) = g \circ \varphi_{f,g}$. Hence ϕ is a linear 2-local isometry which is not surjective. □

SUMMARIES

9. SUMMARY

In our dissertation we present our results on *preserver problems* concerning certain algebraic structures of linear operators or continuous functions, as well as those on *reflexivity problems* concerning certain algebras of functions.

Linear preserver problems are concerned with the determination of all linear maps on an algebra which leave invariant a given subset, function or relation defined on the underlying algebra. For brevity, we shall write **LPPs** for the expression **linear preserver problems**. The study of LPPs on matrix algebras represents one of the most active research areas in matrix theory (see e.g. the survey papers [48, 49]). In the last decades considerable attention has also been paid to the infinite dimensional case, i.e. to preserver problems on operator algebras, and the investigations have resulted in several important results (see e.g. the survey paper [14]).

In several cases LPPs on matrix algebras or on operator algebras can be reduced to linear preserver problems which concern rank (see e.g. [37, 87, 91]). Therefore, it is not surprising that there is a vast literature on such problems. Here we mention just two important finite-dimensional results: Beasley's result [9] on rank-k preserving linear maps and Loewy's result [50] on rank-k non-increasing linear maps. With regard to the infinite-dimensional case, i.e. to preserver problems on operator algebras, the particular cases of preserving rank-1 operators or preserving operators with rank at most 1 have been treated in the papers [36, 79].

In **Chapter 2** we characterize the rank-k non-increasing linear maps, the rank-k preserving linear maps, and the corank-k preserving linear maps on the algebra of all bounded linear operators on a Hilbert space under a mild continuity condition, and we unify and extend the results mentioned above. (The problem of corank-k preservers occurs obviously in the infinite-dimensional case only.) Below we present only one of the theorems of Chapter 2. As one can see in the dissertation, the further preservers under consideration are of similar forms.

Theorem 2.2. *Let k be a positive integer and H a Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a rank-k preserving linear map which is weakly continuous on norm bounded sets. Assume also that the image of ϕ is not contained in $B_k(H)$, where $B_k(H)$ denotes the set of all operators of rank k. Then there is an injective operator $A \in B(H)$ and an operator $B \in B(H)$ with dense image such that either $\phi(T) = ATB$ for all $T \in B(H)$, or $\phi(T) = AT^{tr}B$ for all $T \in B(H)$, where T^{tr} denotes the transpose of T relative to an arbitrary but fixed orthonormal basis of H.*

The content of Chapter 2 was published in our paper [29].

The concept of *linear preserver problems* has so far meant investigations on matrix algebras and on operator algebras, but similar questions can obviously be raised on *arbitrary algebras.* In **Chapter 3** we consider LPPs concerning *function algebras*, in which case the main LPPs considered before have been the characterizations of linear bijections preserving some given norm, or preserving disjointness of the support. We refer to [1, 21, 33, 38, 93, 94] for some of the important and relatively recent papers on such problems. We now introduce some notation. Let X be a locally compact Hausdorff space, and let $C_0(X)$ denote the algebra of all continuous complex valued functions on X which vanish at infinity. The linear bijections of $C_0(X)$ preserving the sup-norm are determined in the famous Banach-Stone theorem. Besides the sup-norm, one of the most natural possibilities to measure a function is to consider the diameter of its range. In Chapter 3 we characterize all the linear bijections of $C_0(X)$ which *preserve the diameter of the range*, and give a unification of the contents of our papers [23] and [28]. Namely, we prove the following theorem.

Theorem 3.1. *Let X be a first countable locally compact Hausdorff space, and let X_0 denote $X \cup \{\infty\}$ if X is not compact, and X if X is compact. Then X_0, endowed with a topology in a natural way, is a compact Hausdorff space.*

If X is compact, then a bijective linear map $\phi : C_0(X) \to C_0(X)$ is diameter preserving if and only if there exists a complex number τ of modulus 1, a homeomorphism $\varphi : X \to X$, and a linear functional $t : C_0(X) \to \mathbb{C}$ with $t(1) \neq -\tau$ such that

$$\phi(f) = \tau \cdot f \circ \varphi + t(f)1 \qquad \big(f \in C_0(X)\big). \tag{3.1}$$

If X is not σ-compact, then a bijective linear map $\phi : C_0(X) \to C_0(X)$ is diameter preserving if and only if there exists a complex number τ of modulus 1 and a homeomorphism $\varphi : X \to X$ such that ϕ is of the form

$$\phi(f) = \tau \cdot f \circ \varphi \qquad \big(f \in C_0(X)\big). \tag{3.2}$$

If the space X is σ-compact but not compact, then a bijective linear map $\phi : C_0(X) \to C_0(X)$ is diameter preserving if and only if there exists a complex number τ of modulus 1 and a homeomorphism $\varphi : X_0 \to X_0$ such that ϕ is of the form

$$\phi(f) = \tau \cdot f \circ \varphi - \tau f\big(\varphi(\infty)\big)1 \qquad \big(f \in C_0(X)\big), \tag{3.3}$$

where $f(\infty) = 0$ for every $f \in C_0(X)$.

Above we have considered *linear* preserver problems. It is clear that preserver problems can be raised *without assuming any kind of linearity.* We may consider transformations on some algebraic structure which preserve some property, quantity, relation, etc. There are a great number of results in mathematics which can be interpreted as preserver problems in this general sense. We mention the very simple example of the isometries of a metric space: the isometries can be viewed as transformations which preserve distance. Another example of preserver problems of this general kind is Wigner's famous unitary-antiunitary theorem, which we treat in the present dissertation in detail. The next theorem presents one of its several formulations which characterizes the transformations on a Hilbert space preserving the absolute value of the inner product of any pair of vectors.

Theorem 4.1. *Let H be a complex Hilbert space and $T : H \to H$ an arbitrary function. Then*

$$|\langle Tx, Ty\rangle| = |\langle x, y\rangle| \tag{4.1}$$

holds for any $x, y \in H$ if and only if there exists a function $\varphi : H \to \mathbb{C}$ with $|\varphi| = 1$, and a linear or conjugate linear isometry $U : H \to H$ such that

$$T = \varphi \cdot U. \tag{4.2}$$

This result is one of the most important theorems concerning the probabilistic aspects of quantum mechanics.

Several different proofs have been given for Wigner's fundamental theorem mentioned above. In **Chapter 4** we present a further, elementary proof which is based on a completely new approach. Our basic idea is as follows. We pick an orthonormal basis in the Hilbert space H, and first we prove that there exist φ and U (as in Theorem 4.1) for the set F of all vectors with *real coordinates*. To see this, we show that on an arbitrary subset $G \subseteq F$, the elements of which are not orthogonal to a given vector, our transformation T is of the form (4.2) with φ and U depending on G. To prove this, we consider all the subsets of $G \times G$ for the elements of which (4.2) holds with (not necessarily the same) adequate φ and U, and we show that those subsets satisfy the conditions of *Zorn's lemma*. So there is a maximal element in $G \times G$ with this property, which turns out to be the whole $G \times G$. Then it is easy to show that there are φ and U for which T is of the form (4.2) on the *whole Hilbert space*.

Wigner's theorem has been generalized in (at least) three directions. *First*, Uhlhorn [92] generalized Wigner's result by requiring only the preservation of *orthogonality* instead of that of the absolute value of the inner product, and he was able to achieve the same conclusion for the case in which the underlying space is at least 3 dimensional. Uhlhorn's result has a serious impact in physics. *Secondly*, Bargmann [7] and Sharma and Almeida [89] obtained results similar to Wigner's *without the assumption of bijectivity*. As for the *third* direction, we recall that sometimes Wigner's theorem is formulated as the characterization of bijections of the set of all 1-dimensional subspaces of a Hilbert space which preserve the angle between those subspaces. Molnár [65] extended Wigner's result in this respect to transformations on the set of all *n-dimensional subspaces* (n being fixed) which preserve the so-called principle angles between the subspaces. (For other generalizations of Wigner's theorem see e.g. [55, 57, 59, 62]). In **Chapter 5** we extend Wigner's theorem in *all the three directions* mentioned above, by obtaining results on the structure of orthogonality preserving transformations on the set of all n-dimensional subspaces of a Hilbert space under various conditions. Namely, we prove the following theorem.

Theorem 5.1. *Let H be a Hilbert space and $n \in \mathbb{N} \cup \{\infty\}$ with*

$$\begin{cases} \dim H > 2n & \text{if } n \in \mathbb{N}, \\ \dim H = \infty & \text{if } n = \infty, \end{cases} \tag{5.1}$$

and let $\phi : H_n \to H_n$, where H_n denotes the set of all n-dimensional closed linear subspaces of H, which are also of infinite codimension if $n = \infty$.

*If $\phi : H_n \to H_n$ is **surjective**, then ϕ preserves orthogonality in both directions if and only if there exists a unique bijection $\psi : H_1 \to H_1$ which preserves orthogonality in both directions and for any $K \in H_n$ we have*

$$\phi(K) = \operatorname{span}\{\psi(X) | X \in H_1, X \subseteq K\}, \tag{5.2}$$

where span *denotes the generated linear subspace.*

Thus, by Uhlhorn's theorem, if $n \in \mathbb{N} \cup \{\infty\}$ is such that (5.1) holds and $\phi : H_n \to H_n$ is surjective, then ϕ preserves orthogonality in both directions if and only if there is a unitary or antiunitary $U \in B(H)$ such that for any $K \in H_n$ we have

$$\phi(K) = U(K). \tag{5.3}$$

*If H is **finite dimensional**, then ϕ preserves orthogonality in both directions (surjectivity is not assumed) if and only if there is a unique transformation $\psi : H_1 \to H_1$ which preserves orthogonality in both directions and for any $K \in H_n$ (5.2) holds. Moreover, if ϕ preserves principal angles then ψ also preserves angles, thus in this case ϕ is of the form (5.3) with a unitary or antiunitary $U \in B(H)$.*

In the **second part** we deal with the problem of ***reflexivity*** of the automorphism and isometry groups of certain algebras of functions. The study of reflexive linear subspaces of the algebra $B(H)$ of all bounded linear operators on a Hilbert space H represents one of the most active research areas in operator theory (see [30] for a nice general view of reflexivity of this kind). In the last decades, similar questions concerning certain important sets of transformations acting on Banach algebras rather than on Hilbert spaces have also attracted considerable attention. The initiators of the research in this direction are Kadison, Larson and Sourour. Kadison [41] studied *local derivations* from a von Neumann algebra $\mathcal{R}$ into a dual $\mathcal{R}$-bimodule $\mathcal{M}$. He called a continuous linear map from $\mathcal{R}$ into $\mathcal{M}$ a local derivation, if it agrees with a derivation at each point in the algebra $\mathcal{R}$ (the derivation may differ from point to point). The main result, Theorem A, in [41] states that in the above setting, every local derivation is a derivation. Besides derivations, there are at least two further very important classes of transformations on operator algebras which certainly deserve attention: the group of automorphisms and the group of surjective isometries.

We now define our concept of *reflexivity.* Let X be a Banach space (in fact, in the cases which we are interested in, X is usually a Banach algebra), and for any $\mathcal{E} \subset B(X)$ let

$$\operatorname{ref}_{alg} \mathcal{E} = \{T \in B(X) : Tx \in \mathcal{E}x \text{ for all } x \in X\},$$
$$\operatorname{ref}_{top} \mathcal{E} = \{T \in B(X) : Tx \in \overline{\mathcal{E}x} \text{ for all } x \in X\},$$

where bar denotes norm-closure. The above sets are called the *algebraic reflexive closure* and the *topological reflexive closure* of $\mathcal{E}$, respectively. The collection $\mathcal{E}$ of transformations is called *algebraically reflexive* if $\operatorname{ref}_{alg} \mathcal{E} = \mathcal{E}$, and *topologically reflexive* if $\operatorname{ref}_{top} \mathcal{E} = \mathcal{E}$.

In this terminology, the algebraic reflexivity of the automorphism group means that every local automorphism is an automorphism. Obviously, topological reflexivity is a stronger property than algebraic reflexivity. Shulman [90] showed that the derivation algebra of any C^*-algebra is topologically reflexive. Hence, not only the local derivations are derivations in this case, but every bounded linear map which agrees with the limit of some sequence of derivations at each point (this sequence may differ from point to point) is a derivation. For the topological reflexivity of derivation algebras, automorphism and isometry groups, we refer to [8, 39, 54, 56].

For the automorphism or isometry groups of C^*-algebras, such a general result as in [90] does not hold. If $\mathcal{A}$ is a Banach algebra, then denote by $\operatorname{Aut}(\mathcal{A})$, $\operatorname{Aut}^*(\mathcal{A})$ and $\operatorname{Iso}(\mathcal{A})$ the group of automorphisms (i.e. multiplicative linear bijections), the group of $*$-automorphisms and the group of surjective linear isometries of $\mathcal{A}$, respectively. If X is an uncountable discrete topological space, then it is easy to verify that the groups $\operatorname{Aut}(C_0(X))$ and $\operatorname{Iso}(C_0(X))$ of the C^*-algebra $C_0(X)$ of all continuous complex valued functions on X vanishing at infinity are not reflexive even algebraically. With regard to topological reflexivity, there are even von Neumann algebras whose automorphism and isometry groups are not topologically reflexive. However, Molnár [54] proved that if H is a separable infinite dimensional Hilbert space, then $\operatorname{Aut}(B(H))$ and $\operatorname{Iso}(B(H))$ are topologically reflexive.

In **Chapter 6** we deal with the reflexivity of the automorphism and isometry groups of the suspension of $B(H)$. The concept of the suspension of a C^*-algebra plays a very important role in the K-theory of operator algebras. If $\mathcal{A}$ is a C^*-algebra then its suspension is the C^*-tensor product $C_0(\mathbb{R}) \otimes \mathcal{A}$, which is well-known to be isomorphic to $C_0(\mathbb{R}, \mathcal{A})$, the algebra of all continuous functions from $\mathbb{R}$ into $\mathcal{A}$ which vanish at infinity. We know that the automorphism and the isometry groups of $B(H)$ are topologically reflexive [54]. We show that $\operatorname{Aut}(C_0(\mathbb{R}))$ and $\operatorname{Iso}(C_0(\mathbb{R}))$ are algebraically (but not topologically) reflexive. In Chapter 6 we obtain several results, the following corollary of which can be considered as the main result of that chapter.

Corollary 6.5. *The automorphism and isometry groups of the suspension of $B(H)$ are algebraically reflexive.*

The content of Chapter 6 was published in our paper [69]. The referee of the manuscript put the question whether it is possible to describe the topological reflexive closures of $\mathrm{Aut}(C_0(\mathbb{R}) \otimes B(H))$ and $\mathrm{Iso}(C_0(\mathbb{R}) \otimes B(H))$. **Chapter 7** is devoted to answer this question. Among others, we obtain the following result.

Corollary 7.2. *Let $\phi : C_0(\mathbb{R}, B(H)) \to C_0(\mathbb{R}, B(H))$ be a linear map. We have $\phi \in \mathrm{ref}_{top} \mathrm{Iso}(C_0(\mathbb{R}, B(H)))$ resp. $\phi \in \mathrm{ref}_{top} \mathrm{Aut}^*(C_0(\mathbb{R}, B(H)))$, if and only if there exists an open interval $U \subseteq \mathbb{R}$, a surjective, monotone, continuous function $\varphi : U \to \mathbb{R}$, and $\tau : U \to \mathrm{Iso}(B(H))$ resp. $\tau : U \to \mathrm{Aut}^*(B(H))$, such that we have*

$$\phi(f)(y) = \begin{cases} \tau(y)\big(f\big(\varphi(y)\big)\big) & \text{if } y \in U, \\ 0 & \text{if } y \in \mathbb{R} \setminus U \end{cases} \tag{7.1}$$

for any $f \in C_0(\mathbb{R}, B(H))$. If ϕ is of the form (7.1), then τ is strongly continuous.

In the concept of local derivations, local automorphisms, etc. we supposed that the transformations under consideration are linear and they equal a derivation, automorphism, etc., respectively, at every single point of the underlying algebra. If we drop the condition of linearity, it is easy to see that the obtained concept is so general (because the assumption is so weak) that it is practically useless. Motivated by the result of Kowalski and Slodkowski [44] on a non-linear characterization of the characters of commutative Banach algebras, Šemrl [88] introduced the concept of ***2-locality***. For example, we say that a (not necessarily linear) transformation ϕ of a Banach algebra is called a 2-local automorphism, if for any pair x, y of points in the Banach algebra under consideration we have an automorphism $\phi_{x,y}$ (depending on x and y) such that $\phi(x) = \phi_{x,y}(x)$ and $\phi(y) = \phi_{x,y}(y)$. The definition of 2-local derivations, 2-local isometries, etc. are similar. Šemrl [88] proved that every 2-local automorphism of $B(H)$ (H being an infinite dimensional separable Hilbert space) is an automorphism, and every 2-local derivation of $B(H)$ is a derivation.

This concept of 2-locality has the advantage that it can be considered in relation with any algebraic structure as we do not assume any kind of linearity. Clearly, it is a remarkable property of the underlying algebraic structure if its 2-local automorphisms, 2-local (surjective linear) isometries, etc. are (global) automorphisms, isometries, etc., respectively. This means that the automorphisms, isometries, etc. are determined by their local actions on the 2-point subsets. For some recent results on 2-local derivations, automorphisms and isometries, we refer to [6, 35, 43, 64, 66, 67, 70].

Molnár [66] studied 2-local isometries of operator algebras. He proved that every such transformation of a C^*-subalgebra of $B(H)$ which contains the compact operators and the identity is a (surjective linear) isometry. Moreover, he raised the problem of considering similar questions concerning function algebras. In **Chapter 8**, the content of which appeared in our paper [24], we obtain such a result for one of the most important types of function algebras, namely for $C_0(X)$.

Theorem 8.2. *If X is a first countable σ-compact Hausdorff space then every 2-local isometry of $C_0(X)$ is a (surjective linear) isometry.*

10. ÖSSZEFOGLALÁS

Értekezésünkben azokat az eredményeinket foglaljuk össze, amelyeket függvény- illetve operátoralgebrákon értelmezett *megőrzési problémák*, valamint függvényalgebrákon értelmezett *reflexivitási problémák* megoldása terén értünk el.

Lineáris megőrzési problémák esetén az a célunk, hogy meghatározzuk egy algebra összes olyan lineáris leképezését, amely egy adott halmazt, függvényt vagy relációt változatlanul hagy. A rövidség kedvéért a **lineáris megőrzési probléma** kifejezés helyett időnként **LPP**-t írunk, amely a *linear preserver problem* angol kifejezés rövidítése. Az LPP-k vizsgálata a mátrix-elmélet egyik legintenzívebben kutatott területe (l. pl. Li és Pierce [48], illetve Li és Tsing [49] összefoglaló cikkét). Az elmúlt évtizedekben jelentős érdeklődés mutatkozott a végtelen-dimenziós eset iránt, vagyis az operátoralgebrákon értelmezett megőrzési problémák irányában is, s e kutatások számos fontos eredményhez vezettek (l. még pl. Brešar és Šemrl [14] összefoglaló cikkét).

Mivel mátrix- és operátoralgebrákon az LPP-k számos esetben visszavezethetők ranggal kapcsolatos LPP-kre (l. pl. [37, 87, 91]), ezért nem meglepő, hogy e problémák irodalma igen bőséges. Két fontos véges-dimenziós eredményt említünk meg, nevezetesen Beasley [9] eredményét a k-rangot megőrző lineáris leképezésekről, valamint Loewy [50] eredményét a k-rangot nem növelő lineáris leképezésekről. Hou [36], valamint Omladič és Šemrl [79] az 1-rangú operátorok, illetve a legfeljebb 1-rangú operátorok megőrzését vizsgálta a végtelen-dimenziós esetben, vagyis operátoralgebrák esetén.

A **2. Fejezetben** enyhe folytonossági feltétel mellett meghatározzuk egy Hilbert-tér korlátos lineáris operátorainak algebráján értelmezett összes olyan lineáris leképezést, amely egy adott k esetén nem növeli a k-rangot, megőrzi a k-rangot, illetve megőrzi a k-korangot, s ezzel egységesítjük és kiterjesztjük a fent említett eredményeket. Alább a fejezet csupán egyetlen tételét emeljük ki. Értekezésünkben láthatjuk, hogy a többi vizsgált megőrző leképezés is hasonló alakú.

Tétel 2.2. *Legyen H Hilbert-tér, k pozitív egész és $\phi : B(H) \to B(H)$ olyan lineáris leképezés, amely gyengén folytonos a normában korlátos halmazokon, megőrzi a k-rangot, és képtere nem része az összes k-rangú operátor $B_k(H)$ halmazának. Ekkor létezik $A \in B(H)$ injektív és $B \in B(H)$ sűrű képterű operátor úgy, hogy vagy minden $T \in B(H)$ esetén $\phi(T) = ATB$, vagy minden $T \in B(H)$ esetén $\phi(T) = AT^{tr}B$, ahol T^{tr} a T-nek H egy tetszőleges, rögzített ortonormált bázisa szerinti transzponáltja.*

A 2. Fejezet eredményeit a [29] cikkünkben publikáltuk.

A *lineáris megőrzési problémák* fogalma eredendően mátrixalgebrákon és operátoralgebrákon való vizsgálatokat jelentett, ám hasonló kérdések nyilvánvalóan *bármely algebrán* felvethetők. A **3. Fejezetben** *függvényalgebrákkal* kapcsolatos LPP-ket vizsgálunk. Korábban e terület legfőbb eredményei olyan lineáris bijekciókat írtak le, amelyek megőriznek egy-egy adott normát, illetve a tartóhalmazok diszjunktságát. Néhány fontos, viszonylag új publikáció ilyen problémákról például [1, 21, 33, 38, 93, 94]. A könnyebb követhetőség kedvéért bevezetünk néhány jelölést. Legyen X lokálisan kompakt Hausdorff-tér, és jelölje $C_0(X)$ az X-en értelmezett összes komplexértékű, végtelenben eltűnő, folytonos függvény algebráját. A híres Banach-Stone-tétel meghatározta a $C_0(X)$ szuprémum-normát megőrző lineáris bijekcióit. A norma mellett egy függvény természetes módon jellemezhető a képtere átmérőjével is. Az alábbi tételben karakterizáljuk $C_0(X)$ összes, *a képtér átmérőjét megőrző* lineáris bijekcióját, és ezzel egyesítjük és egységesítjük a [23] és a [28] publikációnk tartalmát.

Tétel 3.1. *Legyen X első megszámlálható lokálisan kompakt Hausdorff-tér, és jelölje X_0 az $X \cup \{\infty\}$ halmazt, ha X nem kompakt, illetve az X halmazt, ha X kompakt. Ekkor X_0, természetes módon ellátva topológiával, kompakt Hausdorff-tér.*

Ha X kompakt, akkor egy $\phi : C_0(X) \to C_0(X)$ lineáris bijekció pontosan akkor ármérő-tartó, ha létezik τ 1-abszolútértékű komplex szám, $\varphi : X \to X$ homeomorfizmus és $t : C_0(X) \to \mathbb{C}$ lineáris funkcionál úgy, hogy $t(1) \neq -\tau$, és ϕ a következő alakú:

$$\phi(f) = \tau \cdot f \circ \varphi + t(f)1 \qquad \big(f \in C_0(X)\big). \tag{3.1}$$

Ha X nem σ-kompakt, akkor egy $\phi : C_0(X) \to C_0(X)$ lineáris bijekció pontosan akkor átmérő-tartó, ha létezik τ 1-abszolútértékű komplex szám és $\varphi : X \to X$ homeomorfizmus úgy, hogy ϕ a következő alakú:

$$\phi(f) = \tau \cdot f \circ \varphi \qquad \big(f \in C_0(X)\big). \tag{3.2}$$

Ha X σ-kompakt de nem kompakt, akkor egy $\phi : C_0(X) \to C_0(X)$ lineáris bijekció pontosan akkor átmérő-tartó, ha létezik τ 1-abszolútértékű komplex szám és $\varphi : X_0 \to X_0$ homeomorfizmus úgy, hogy ϕ

$$\phi(f) = \tau \cdot f \circ \varphi - \tau f(\varphi(\infty))1 \qquad \big(f \in C_0(X)\big), \tag{3.3}$$

alakú, ahol $f(\infty) = 0$ minden $f \in C_0(X)$ esetén.

A fentiekben *lineáris* megőrzési problémákkal foglalkoztunk, ám megőrzési problémák természetesen felvethetők *bármilyen linearitási feltétel nélkül* is, azaz vizsgálhatunk adott algebrai struktúrán értelmezett, bizonyos tulajdonságot, mennyiséget, relációt, stb. megőrző transzformációkat. Sok matematikai eredmény interpretálható e bővebb értelemben tekintett megőrzési problémaként. A metrikus terek izometriái például egyszerűen távolság-megőrző leképezéseknek tekinthetők. Wigner híres unitér-antiunitér tétele ugyancsak példa ilyen általános értelemben vett megőrzési problémára. E tételnek számos megfogalmazása ismert, amelyek közül az alábbi egy Hilbert-tér olyan leképezéseit írja le, amelyek megőrzik a vektorpárok belsőszorzatának az abszolútértékét.

Tétel 4.1. *Legyen H komplex Hilbert-tér és $T : H \to H$ egy tetszőleges függvény. Pontosan akkor teljesül minden $x, y \in H$ esetén*

$$|\langle Tx, Ty\rangle| = |\langle x, y\rangle|, \tag{4.1}$$

ha létezik egy $\varphi : H \to \mathbb{C}$ 1-abszolútértékű függvény és $U : H \to H$ lineáris vagy konjugált-lineáris izometria úgy, hogy

$$T = \varphi \cdot U. \tag{4.2}$$

A fenti tétel a kvantummechanika valószínűségi aspektusaira vonatkozó egyik legfontosabb eredmény, amelynek számos különböző bizonyítását publikálták. Mi a **4. Fejezetben** a tételt teljesen új megközelítéssel, elemi módon bizonyítjuk be a következőképpen. Vesszük a H Hilbert-tér egy ortonormált bázisát, s először megmutatjuk, hogy létezik a tétel feltételeinek eleget tevő φ és U úgy, hogy az összes *valós koordinátájú* vektor alkotta F halmaz elemeire (4.2) teljesül. Ehhez igazoljuk, hogy F egy tetszőleges, egy adott vektorra ortogonális elemeket nem tartalmazó G részhalmazára a T transzformáció (4.2) alakú G-től függő φ-vel és U-val. Tekintjük ugyanis $G \times G$ összes olyan részhalmazát, melynek elemeire (4.2) teljesül megfelelő (nem szükségképpen azonos) φ-vel és U-val, majd megmutatjuk, hogy e részhalmazok teljesítik a *Zorn-lemma* feltételeit. Így létezik egy fenti tulajdonságú maximális részhalmaz $G \times G$-ben, amelyről belátjuk, hogy a teljes $G \times G$. Ezután könnyű igazolnunk, hogy létezik megfelelő φ és U úgy, hogy T (4.2) alakú *a teljes Hilbert-téren.*

Wigner e tételét (legalább) három irányban általánosították. *Először* Uhlhorn [92] általánosította oly módon, hogy a belsőszorzat abszolútértéke helyett csak az *ortogonalitás* megőrzését követelte meg, s így jutott azonos következtetésre abban az esetben, amikor a tekintett tér legalább 3 dimenziós. Uhlhorn eredményének komoly hatása volt a fizika területén. *Másodszor*, Bargmann [7] és Sharma és Almeida [89] Wigneréhez hasonló eredményekre jutottak a *bijektivitás feltétele nélkül.* A *harmadik* irány ismertetéséhez megjegyezzük, hogy a Wigner-tétel egyik megfogalmazása szerint egy Hilbert-tér összes 1-dimenziós alterén értelmezett azon bijekciókat határozza meg, amelyek megőrzik az alterek által bezárt szöget. Molnár [65] e tekintetben terjesztette ki Wigner eredményét (adott n esetén) az összes n*-dimenziós altér* halmazán értelmezett, az alterek között az ún. principális szögeket megőrző transzformációkra. (Wigner tételének további általánosításai végett l. [55, 57, 59, 62]). Az **5. Fejezetben** kiterjesztjük Wigner tételét a fent említett *mindhárom irányban* oly módon, hogy egy Hilbert-tér összes n-dimenziós alterének halmazán értelmezett, az alterek közötti ortogonalitást megőrző transzformációkra nyerünk eredményeket különböző feltételek mellett. Nevezetesen, igazoljuk a következő tételt.

Tétel 5.1. *Legyen H Hilbert-tér és $n \in \mathbb{N} \cup \{\infty\}$ úgy, hogy*

$$\begin{cases} \dim H > 2n & \text{ha } n \in \mathbb{N}, \\ \dim H = \infty & \text{ha } n = \infty. \end{cases} \tag{5.1}$$

Legyen továbbá $\phi : H_n \to H_n$, ahol H_n a H összes n-dimenziós zárt lineáris altér halmazát jelöli, amelyekről $n = \infty$ esetén feltesszük, hogy végtelen kodimenziósak.

*Ha $\phi : H_n \to H_n$ **szürjektív**, úgy ϕ pontosan akkor őrzi meg mindkét irányban az ortogonalitást, ha egyértelműen létezik egy, az ortogonalitást mindkét irányban megőrző $\psi : H_1 \to H_1$ bijekció, amelyre*

$$\phi(K) = \operatorname{span}\big\{\psi(X) | X \in H_1, X \subseteq K\big\} \tag{5.2}$$

minden $K \in H_n$ esetén, ahol span *a generált lineáris alteret jelöli.*

Ekkor Uhlhorn tétele szerint, ha $n \in \mathbb{N} \cup \{\infty\}$ olyan, hogy (5.1) teljesül és $\phi : H_n \to H_n$ szürjektív, úgy ϕ pontosan akkor őrzi meg mindkét irányban az ortogonalitást, ha létezik egy unitér vagy antiunitér U operátor H-n úgy, hogy minden $K \in H_n$ esetén

$$\phi(K) = U(K). \tag{5.3}$$

*Ha H **véges-dimenziós**, úgy ϕ pontosan akkor őrzi meg mindkét irányban az ortogonalitást (szürjektivitást nem teszünk fel), ha egyértelműen létezik egy, az ortogonalitást mindkét irányban megőrző $\psi : H_1 \to H_1$ transzformáció, amelyre (5.2) minden $K \in H_n$ esetén teljesül. Továbbá, ha ϕ megőrzi a principális szögeket, akkor ψ is megőrzi a szögeket, vagyis ez esetben ϕ (5.3)-alakú egy unitér vagy antiunitér U operátorral H-n.*

Értekezésünk **második részében** bizonyos függvényalgebrák automorfizmus- és izometria-csoportjának ***reflexivitásával*** kapcsolatos problémákat vizsgálunk. Egy H Hilbert-tér összes korlátos lineáris operátorának algebráját jelölje $B(H)$. Az operátoralgebrák elméletének egyik legaktívabban kutatott területe $B(H)$ reflexív lineáris altereinek vizsgálata (a reflexivitás e típusának szép, általános áttekintése végett l. [30]). Az elmúlt évtizedekben figyelemreméltó érdeklődést keltettek Hilbert-terek helyett Banach-algebrákon értelmezett transzformációk bizonyos fontos halmazaival kapcsolatos hasonló kérdések. Az ez irányú kutatások elindítása Kadison, Larson és Sourour nevéhez fűződik. Kadison [41] egy $\mathcal{R}$ Neumann-algebrát egy $\mathcal{M}$ duális $\mathcal{R}$-bimodulusba képező *lokális derivációkat* vizsgált. Lokális derivációnak nevezte azokat az $\mathcal{R}$-et $\mathcal{M}$-be képező folytonos lineáris leképezéseket, amelyek az $\mathcal{R}$ algebra minden egyes pontjában megegyeznek egy (az adott ponttól függő) derivációval. Vizsgálatait operátoralgebrák Hochschild-kohomológiájával kapcsolatos problémák motiválták. Kadison [41] cikkének fő eredménye (Theorem A) szerint a fenti esetben minden lokális deriváció deriváció. A derivációk mellett feltétlenül figyelmet érdemel az operátoralgebrák transzformációinak még legalább két nagyon fontos osztálya, nevezetesen az automorfizmusok illetve a szürjektív izometriák csoportja.

Definiáljuk a *reflexivitás* fogalmát. Legyen X Banach-tér (a bennünket érdeklő esetekben általában Banach-algebra) és tetszőleges $\mathcal{E} \subset B(X)$ részhalmaz esetén legyen

$$\begin{aligned} \mathrm{ref}_{alg}\, \mathcal{E} &= \big\{T \in B(X) : Tx \in \mathcal{E}x \text{ minden } x \in X \text{ esetén}\big\}, \\ \mathrm{ref}_{top}\, \mathcal{E} &= \big\{T \in B(X) : Tx \in \overline{\mathcal{E}x} \text{ minden } x \in X \text{ esetén}\big\}, \end{aligned}$$

ahol a felülvonás a norma szerinti lezártat jelöli. A fenti halmazokat rendre $\mathcal{E}$ *algebrai reflexív lezártjának*, illetve *topologikus reflexív lezártjának* nevezzük. Azt mondjuk, hogy transzformációk egy $\mathcal{E}$ összessége *algebrailag reflexív*, ha $\mathrm{ref}_{alg}\, \mathcal{E} = \mathcal{E}$, és *topologikusan reflexív*, ha $\mathrm{ref}_{top}\, \mathcal{E} = \mathcal{E}$.

E terminológiával élve [12] fő eredménye a $B(H)$ automorfizmus-csoportjának algebrai reflexivitását állítja. A topologikus reflexivitás nyilvánvalóan erősebb tulajdonság mint az algebrai reflexivitás. Shulman [90] megmutatta, hogy minden C^*-algebra deriváció-algebrája topologikusan reflexív, így nem csak a lokális derivációik derivációk, hanem azok a lineáris leképezéseik is, amelyek minden egyes pontban derivációk (az adott ponttól függő) sorozatának határértékével egyenlők. Deriváció-algebrák, automorfizmus-csoportok és izometria-csoportok topologikus reflexivitása tekintetében l. [8, 39, 54, 56].

C^*-algebrák automorfizmus- és izometria-csoportjaira már nem teljesül Shulman [90] eredményéhez hasonló általános állítás. Egy $\mathcal{A}$ Banach-algebra automorfizmusainak (azaz multiplikatív lineáris bijekcióinak) csoportját $\mathrm{Aut}(\mathcal{A})$-val, $*$-automorfizmusainak csoportját $\mathrm{Aut}^*(\mathcal{A})$-val, míg szürjektív lineáris izometriáinak csoportját $\mathrm{Iso}(\mathcal{A})$-val jelöljük. Egy nem megszámlálható X diszkrét topologikus téren értelmezett, végtelenben eltűnő folytonos függvények $C_0(X)$ C^*-algebrájának automorfizmus- és izometria-csoportjairól könnyen belátható, hogy még algebrailag sem reflexívek. A topologikus reflexivitás tekintetében viszont még Neumann-algebrákat is találunk (l. például a szeparábilis Hilbert-téren értelmezett végtelen-dimenziós kommutatív Neumann-algebrákat [8]), amelyek automorfizmus- és izometria-csoportjai nem topologikusan reflexívek. Ám végtelen-dimenziós szeparábilis H Hilbert-tér esetén Molnár [54] igazolta $\mathrm{Aut}(B(H))$ és $\mathrm{Iso}(B(H))$ topologikus reflexivitását.

Értekezésünk **6. Fejezetében** $B(H)$ szuszpenziója automorfizmus- és izometria-csoportjainak reflexivitását vizsgáljuk. Egy C^*-algebra szuszpenziójának fogalma nagyon fontos szerepet játszik az operátoralgebrák K-elméletében. Egy $\mathcal{A}$ C^*-algebra szuszpenziója a $C_0(\mathbb{R}) \otimes \mathcal{A}$ C^*-tenzor szorzat, amely jól ismerten izomorf $C_0(\mathbb{R}, \mathcal{A})$-val, az $\mathbb{R}$-et $\mathcal{A}$-ba képező, végtelenben eltűnő összes folytonos függvény algebrájával. Tudjuk, hogy $B(H)$ automorfizmus- és izometria-csoportja topologikusan reflexív [54]. A 6. Fejezetben megmutatjuk, hogy $\mathrm{Aut}(C_0(\mathbb{R}))$ és $\mathrm{Iso}(C_0(\mathbb{R}))$ algebrailag (de nem topologikusan) reflexív. A fejezet számos eredményének alábbi következménye a fejezet fő eredményének tekinthető.

Következmény 6.1. *A $B(H)$ szuszpenziójának automorfizmus-csoportja és izometria-csoportja algebrailag reflexív.*

E fejezet tartalmát a [69] cikkünkben publikáltuk, melynek lektora azt a kérdést vetette fel, hogy vajon leírható-e $\mathrm{Aut}(C_0(\mathbb{R})\otimes B(H))$ és $\mathrm{Iso}(C_0(\mathbb{R})\otimes B(H))$ topologikus reflexív lezártja. A **7. Fejezetben** e kérdést kívánjuk megválaszolni, s többek között a következő eredményt nyerjük.

Következmény 7.2. *Legyen $\phi : C_0(\mathbb{R}, B(H)) \to C_0(\mathbb{R}, B(H))$ lineáris leképezés. Pontosan akkor teljesül $\phi \in \mathrm{ref}_{top}\,\mathrm{Iso}(C_0(\mathbb{R}, B(H)))$ ill. $\phi \in \mathrm{ref}_{top}\,\mathrm{Aut}^*(C_0(\mathbb{R}, B(H)))$, ha létezik $U \subseteq \mathbb{R}$ nyílt intervallum, $\varphi : U \to \mathbb{R}$ szürjektív, monoton, folytonos függvény és $\tau : U \to \mathrm{Iso}(B(H))$ illetve $\tau : U \to \mathrm{Aut}^*(B(H))$ úgy, hogy ϕ*

$$\phi(f)(y) = \begin{cases} \tau(y)\big(f\big(\varphi(y)\big)\big) & \text{ha } y \in U, \\ 0 & \text{ha } y \in \mathbb{R} \setminus U \end{cases} \tag{7.1}$$

alakú minden $f \in C_0(\mathbb{R}, B(H))$ esetén.

Továbbá, ha ϕ (7.1) alakú, akkor τ erősen folytonos.

A lokális derivációk, lokális automorfizmusok, stb. fogalmában feltettük, hogy a tekintett transzformációk lineárisak és minden egyes pontban megegyeznek az algebra valamely derivációjával, automorfizmusával, stb. Könnyen látható, hogy a linearitás feltételét elhagyva (a feltételek gyengesége miatt) annyira általános fogalmat kapunk, hogy az gyakorlatilag használhatatlan. Kowalski és Slodkowski kommutatív Banach-algebrák karaktereinek nem-lineáris karakterizációjára nyert eredménye [44] által motiválva, Šemrl [88] bevezette a ***2-lokalitás*** fogalmát. Egy Banach-algebra (nem szügségképpen lineáris) ϕ transzformációját 2-lokális automorfizmusnak nevezzük, ha a tekintett Banach-algebra minden x, y pontpárja esetén létezik olyan $\phi_{x,y}$ (x-től és y-tól függő) automorfizmus, amelyre $\phi(x) = \phi_{x,y}(x)$ és $\phi(y) = \phi_{x,y}(y)$. A 2-lokális derivációk, 2-lokális izometriák, stb. hasonlóan definiálhatók. Šemrl [88] igazolta, hogy egy végtelen-dimenziós, szeparábilis H Hilbert-tér esetén $B(H)$ minden 2-lokális automorfizmusa automorfizmus és minden 2-lokális derivációja deriváció.

Mivel semminemű linearitást nem teszünk fel, ezért a 2-lokalitás fogalma bármely algebrai struktúrával kapcsolatban értelmezhető. Egy algebrai struktúrának nyilvánvalóan fontos tulajdonsága, ha 2-lokális automorfizmusai, 2-lokális (szürjektív lineáris) izometriái, stb. (globális) automorfizmusok, izometriák, stb. Ez azt jelenti, hogy az automorfizmusokat, izometriákat, stb. meghatározza a kételemű halmazokon való lokális viselkedésük. Néhány friss publikáció 2-lokális derivációkról, 2-lokális automorfizmusokról és 2-lokális izometriákról például [6, 35, 43, 64, 66, 67, 70].

Molnár [66] igazolta, hogy $B(H)$ minden, az identikus operátort és a kompakt operátorokat tartalmazó C^*-részalgebrájának 2-lokális (szürjektív lineáris) izometriái (szürjektív lineáris) izometriák, s ehhez hasonló kérdéseket vetett fel függvényalgebrák esetén. A **8. Fejezetben**, amelynek tartalma a [24] dolgozatunkban jelent meg, az alábbi tétel igazolásával ilyen eredményt nyerünk a függvényalgebrák egyik legfontosabb típusára, nevezetesen $C_0(X)$-re.

Tétel 8.2. *Ha X első megszámlálható σ-kompakt Hausdorff-tér, akkor $C_0(X)$ minden 2-lokális izometriája (szürjektív lineáris) izometria.*

BIBLIOGRAPHY

[1] J. Araujo, E. Beckenstein, and L. Narici, *Biseparating maps and homeomorphic realcompactifications*, J. Math. Anal. Appl. **192** (1995), 258–265.

[2] M.D. Atkinson, *Primitive spaces of matrices of bounded rank II*, J. Austral. Math. Soc. Ser. A **34** (1983), 306–315.

[3] M.D. Atkinson and S. Lloyd, *Large spaces of matrices of bounded rank*, Quart. J. Math. Oxford Ser. (2) **31** (1980), 253–262.

[4] ——, *Primitive spaces of matrices of bounded rank*, J. Austral. Math. Soc. Ser. A **30** (1980/81), 473–482.

[5] Z. Bai and J. Hou, *Linear maps and additive maps that preserve operators annihilated by a polynomial*, J. Math. Anal. Appl. **271** (2002), 139–154.

[6] M. Barczy and M. Tóth, *Local automorphisms of the sets of states and effects on a Hilbert space*, Rep. Math. Phys. **48** (2001), 289–298.

[7] V. Bargmann, *Note on Wigner's theorem on symmetry operations*, J. Math. Phys. **5** (1964), 862–868.

[8] C.J.K. Batty and L. Molnár, *On topological reflexivity of the groups of *-automorphisms and surjective isometries of $B(H)$*, Arch. Math. **67** (1996), 415–421.

[9] L.B. Beasley, *Linear operators on matrices: the invariance of rank-k matrices*, Linear Algebra Appl. **107** (1988), 161–167.

[10] M. Brešar, W.S. Martindale, and C.R. Miers, *Maps preserving n^{th} powers*, Commun. Algebra **26** (1998), 117–138.

[11] M. Brešar and P. Šemrl, *Mappings which preserve idempotents, local automorphisms, and local derivations*, Canad. J. Math. **45** (1993), 483–496.

[12] ——, *On local automorphisms and mappings that preserve idempotents*, Studia Math. **113** (1995), 101–108.

[13] ———, *Linear maps preserving the spectral radius*, J. Funct. Anal. **142** (1996), 360–368.

[14] ———, *Linear preservers on $B(X)$*, Banach Cent. Publ. **38** (1997), 49–58.

[15] G. Casinelli, E. de Vito, P. Lahti, and A. Levrero, *Symmetry groups in quantum mechanics and the theorem of Wigner on the symmetry transformations*, Rev. Mat. Phys. **8** (1997), 921–941.

[16] P.R. Chernoff, *Representations, automorphisms and derivations of some operator algebras*, J. Funct. Anal. **12** (1973), 275–289.

[17] P. Civin and B. Yood, *Lie and Jordan structures in Banach algebras*, Pacific J. Math. **15** (1965), 775–797.

[18] R.L. Crist, *Local derivations on operator algebras*, J. Funct. Anal. **135** (1996), 76–92.

[19] T. Dang, Y. Friedmann, and B. Russo, *Affine geometric proof of the Banach Stone theorems of Kadison and Kaup*, Rocky Mountain J. Math. **20** (1990), 409–428.

[20] H. Flanders, *On spaces of linear transformations with bounded rank*, J. London Math. Soc. **2** (1962), 10–16.

[21] J.J. Font and S. Hernandez, *On separating maps between locally compact spaces*, Arch. Math. **63** (1994), 158–165.

[22] G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen*, Sitzungsber. Königl. Preuss. Akad. Wiss. Berlin (1897), 994–1015.

[23] M. Győry, *Diameter preserving bijections of $C_0(X)$*, Publ. Math. Debrecen **54** (1999), 207–215.

[24] ———, *2-local isometries of $C_0(X)$*, Acta Sci. Math. (Szeged) **67** (2001), 735–746.

[25] ———, *On the transformations of all n-dimensional subspaces of a Hilbert space preserving orthogonality*, preprint (2001).

[26] ———, *A new elementary proof for Wigner's theorem*, preprint (2002).

[27] ———, *On the reflexivity of the isometry groups of the suspension of $B(H)$*, preprint (2002).

[28] M. Győry and L. Molnár, *Diameter preserving bijections of $C(X)$*, Arch. Math. (Basel) **71** (1998), 301–310.

[29] M. Győry, L. Molnár, and P. Šemrl, *Linear rank and corank preserving maps on $B(H)$ and an application to *-semigroup isomorphisms of operator ideals*, Linear Alg. Appl. **280** (1998), 253–266.

[30] D. Hadwin, *A general view of reflexivity*, Trans. Amer. Math. Soc. **344** (1994), 325–360.

[31] J. Hakeda, *Additivity of *-semigroup isomorphisms among *-algebras*, Bull. London Math. Soc. **18** (1986), 51–56.

[32] D. Han and S. Wei, *Local derivations of nest algebras*, Proc. Am. Math. Soc. **123** (1995), 3095–3100.

[33] S. Hernandez, E. Beckenstein, and L. Narici, *Banach-Stone theorems and separating maps*, Manuscripta Math. **86** (1995), 409–416.

[34] M.R. Hestenes, *Relative Hermitian matrices*, Pacific J. Math. **11** (1961), 225–245.

[35] C. Hou and S. Hou, *Local automorphisms of nest algebras*, Indian J. Pure Appl. Math. **32** (2001), 1667–1678.

[36] J. Hou, *Rank-preserving linear maps on $B(X)$*, Sci. China Ser. A **32** (1989), 929–940.

[37] A.A. Jafarian and A.R. Sourour, *Spectrum-preserving linear maps*, J. Funct. Anal. **66** (1986), 255–261.

[38] J.S. Jeang and N.C. Wong, *Weighted composition operators of $C_0(X)$'s*, J. Math. Anal. Appl. **201** (1996), 981–996.

[39] W. Jing, S. Lu, and G. Han, *On topological reflexivity of the spaces of derivations on operator algebras*, Appl. Math., Ser. B **17** (2002), 75–79.

[40] B.E. Johnson, *Local derivations on C^*-algebras are derivations*, Trans. Am. Math. Soc. **353** (2001), 313–325.

[41] R.V. Kadison, *Local derivations*, J. Algebra **130** (1990), 494–509.

[42] I. Kaplansky, *Algebraic and analytical aspects of operator algebras*, CBMS Regional Conf. Ser. in Math. 1, Amer. Math. Soc., Providence (1970).

[43] S.O. Kim and J.S. Kim, *Local automorphisms and derivations on M_n*, Proc. Amer. Math. Soc., to appear.

[44] S. Kowalski and Z. Slodkowski, *A characterization of multiplicative linear functionals in Banach algebras*, Studia Math. **67** (1980), 215–223.

[45] D.R. Larson, *Reflexivity, algebraic reflexivity and linear interpolation*, Amer. J. Math. **110** (1988), 283–299.

[46] D.R. Larson and A.R. Sourour, *Local derivations and local automorphisms of $B(X)$*, Proc. Sympos. Pure Math. **51** (1990), 187–194.

[47] J.S. Lemont and P. Mendelson, *The Wigner unitary-antiunitary theorem*, Ann. Math. **78** (1963), 548–559.

[48] C.K. Li and S. Pierce, *Linear preserver problems*, Amer. Math. Monthly **108** (2001), 591–605.

[49] C.K. Li and N.K. Tsing, *Linear preserver problems: A brief introduction and some special techniques*, Linear Algebra Appl. **162-164** (1992), 217–235.

[50] R. Loewy, *Linear mappings which are rank-k non-increasing*, Linear and Multilinear Algebra **34** (1993), 21–32.

[51] W.S. Martindale, *When are multiplicative mappings additive?*, Proc. Amer. Math. Soc. **21** (1969), 695–698.

[52] L. Molnár, *Two characterisations of additive $*$-automorphisms of $B(H)$*, Bull. Austral. Math. Soc. **53** (1996), 391–400.

[53] ———, *Wigner's unitary-antiunitary theorem via Herstein's theorem on Jordan homeomorphisms*, J. Nat. Geom. **10** (1996), 137–148.

[54] ———, *The set of automorphisms of $B(H)$ is topologically reflexive in $B(B(H))$*, Studia Math. **122** (1997), 183–193.

[55] ———, *An algebraic approach to Wigner's unitary-antiunitary theorem*, J. Austral Math. Soc. **65** (1998), 354–369.

[56] ———, *A proper standard C^*-algebra whose automorphism and isometry groups are topologically reflexive*, Publ. Math. (Debrecen) **52** (1998), 563–574.

[57] ———, *A generalization of Wigner's unitary-antiunitary theorem to Hilbert modules*, J. Math. Phys. **40** (1999), 5544–5554.

[58] ———, *Some linear preserver problems on $B(H)$ concerning rank and corank*, Lin. Alg. Appl. **286** (1999), 311–321.

[59] ———, *Generalization of Wigner's unitary-antiunitary theorem for indefinite inner product spaces*, Commun. Math. Phys. **201** (2000), 785–791.

[60] ———, *On isomorphisms of standard operator algebras*, Studia Math. **142** (2000), 295–302.

[61] ———, *Reflexivity of the automorphism and isometry groups of C^*-algebras in BDF theory*, Arch. Math. **74** (2000), 120–128.

[62] ———, *A Wigner type theorem on symmetry transformations in type II factors*, Int. J. Theor. Phys. **39** (2000), 1463–1466.

[63] ———, *Jordan maps on standard operator algebras*, Functional Equations - Result and Advances, in Z. Daróczy and Zs. Páles (eds.), Advances in Mathematics, Vol. 3., Kluwer Acad. Publ., Dordrecht (2001), 305–320.

[64] ———, *Local automorphisms of some quantum mechanical strucutres*, Lett. Math. Phys. **58** (2001), 91–100.

[65] ———, *Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving principal angles*, Commun. Math. Phys. **217** (2001), 437–450.

[66] ———, *2-local isometries of some operator algebras*, Proc. Edinb. Math. Soc. **45** (2002), 349–352.

[67] ———, *Local automorphisms of operator algebras on Banach spaces*, Proc. Amer. Math. Soc. **131** (2003), 1867–1874.

[68] L. Molnár and M. Barczy, *Linear maps on the space of all bounded observables preserving maximal deviation*, J. Func. Anal., submitted for publication.

[69] L. Molnár and M. Győry, *Reflexivity of the automorphism and isometry groups of the suspension of $B(H)$*, J. Funct. Anal. **159** (1998), 568–586.

[70] L. Molnár and P. Šemrl, *Local automorphisms of the unitary group and the general linear group on a Hilbert space*, Expo. Math. **18** (2000), 231–238.

[71] L. Molnár and B. Zalar, *Reflexivity of the group of surjective isometries of some Banach spaces*, Proc. Edinburgh Math. Soc. **42** (1999), 17–36.

[72] ———, *On local automorphisms of group algebras of compact groups*, Proc. Am. Math. Soc. **128** (2000), 93–99.

[73] ———, *Three-variable *-identities and ring homomorphisms of operator ideals*, Publ. Math. Debrecen (to appear).

[74] G.J. Murphy, *C^*-algebras and operator theory*, Academic Press, 1990.

[75] M.A. Naimark, *Normed algebras*, Wolters-Noordhoff Publishing, 1972.

[76] A. Nowicki, *On local derivations in the Kadison sense*, Colloq. Math. **89** (2001), 193–198.

[77] M. Omladič, *On operators preseving commutativity*, J. Func. Anal. **66** (1986), 105–122.

[78] M. Omladič, H. Radjavi, and P. Šemrl, *Preserving commutativity*, J. Pure Appl. Algebra **156** (2001), 309–328.

[79] M. Omladič and P. Šemrl, *Additive mappings preserving operators of rank one*, Linear Algebra Appl. **182** (1993), 239–256.

[80] M. Radjabalipour, K. Seddighi, and Y. Taghavi, *Additive mappings on operator algebras preserving abolute values*, Lin. Alg. Appl. **327** (2001), 197–206.

[81] H. Radjavi and P. Rosenthal, *On invariant subspaces and reflexive algebras*, Amer. J. Math. **91** (1969), 683–692.

[82] J. Rätz, *On Wigner's theorem: remarks, complements, comments and corrolaries*, Aequationes Math. **52** (1996), 1–9.

[83] S. Sakai, *C^*-algebras and W^*-algebras*, Springer-Verlag, 1971.

[84] F. Cabello Sánchez and L. Molnár, *Reflexivity of the isometry group of some calssical spaces*, Rev. Mat. Iberoam. **18** (2002), 409–430.

[85] E. Scholz and W. Timmermann, *Local derivations, automorphisms and commutativity preserving maps on $L^+(D)$*, Publ. Res. Inst. Math. Sci. **29** (1993), 977–995.

[86] P. Šemrl, *Isomorphisms of standard operator algebras*, Proc. Amer. Math. Soc. **123** (1995), 1851–1855.

[87] ______, *Linear maps that preserve the nilpotent operators*, Acta Sci. Math. (Szeged) **61** (1995), 523–534.

[88] ______, *Local automorphisms and derivations on $B(H)$*, Proc. Amer. Math. Soc. **125** (1997), 2677–2680.

[89] C.S. Sharma and D.L. Almeida, *A direct proof of Wigner's theorem on maps which preserve transition probabilities between pure states of quantum systems*, Ann. Phys. **197** (1990), 300–309.

[90] V.S. Shulman, *Operators preserving ideals in C^*-algebras*, Studia Math. **109** (1994), 67–72.

[91] A.R. Sourour, *Invertibility preserving linear maps on $L(X)$*, Trans. Amer. Math. Soc. **348** (1996), 13–30.

[92] U. Uhlhorn, *Representation of symmetry transformations in quantum mechanics*, Ark. Fysik **23** (1963), 307–340.

[93] R. Wang, *Linear isometric operators on the $C_0^{(n)}(X)$ type spaces*, Kodai Math. J. **19** (1996), 259–281.

[94] N. Weaver, *Isometries of noncompact Lipschitz spaces*, Canad. Math. Bull. **38** (1995), 242–249.

[95] M. Wiehl, *Local derivations on the Weyl algebra with one pair of generators*, Acta Math. Hung. **92** (2001), 51–59.

[96] E.P. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektrum*, Fredrik Vieweg und Sohn, 1931.

[97] J. Wu, *Local derivations of reflexive algebras*, Proc. Am. Math. Soc. **125** (1997), 869–873.

[98] ______, *Local derivations of reflexive algebras II*, Proc. Am. Math. Soc. **129** (2001), 1733–1737.

[99] Y.H. Yon, *Local derivations of the polynomial ring over a field*, Bull. Korean Math. Soc. **36** (1999), 247–257.

[100] J. Zhu, *Local derivations of nest algebras*, Proc. Am. Math. Soc. **123** (1995), 739–742.

APPENDIX

A. LIST OF PUBLICATIONS TREATED IN THE THESIS

1. M. Győry and L. Molnár, Diameter preserving bijections of $C(X)$,
   Arch. Math. **71** (1998), 301–310.
2. M. Győry, L. Molnár and P. Šemrl, Linear rank and corank preserving maps on $B(H)$ and an application to *-semigroup isomorphisms of operator ideals,
   Linear Alg. Appl. **280** (1998), 253–266.
3. L. Molnár and M. Győry, Reflexivity of the automorphism and isometry groups of the suspension of $B(H)$,
   J. Funct. Anal. **159** (1998), 568-586.
4. M. Győry, Diameter preserving bijections of $C_0(X)$,
   Publ. Math. Debrecen **54** (1999), 207–215.
5. M. Győry, 2-local isometries of $C_0(X)$,
   Acta. Sci. Math. (Szeged) **67** (2001), 735–746.
6. M. Győry, A new elementary proof for Wigner's theorem, in preparation.
7. M. Győry, On the transformations of all n-dimensional subspaces of a Hilbert space preserving orthogonality, in preparation.
8. M. Győry, On the reflexivity of the isometry groups of the suspension of $B(H)$, in preparation.

B. LIST OF INDEPENDENT CITATIONS

1. M. Győry and L. Molnár, Diameter preserving bijections of $C(X)$, *Arch. Math.* **71** (1998), 301–310.

► (1) F. González and V.V. Uspenskij, *On homomorphisms of groups of integer-valued functions*, Extracta Math. **14** (1999), 19–29.

► (2) F. Cabello Sánchez, *Diameter preserving linear maps and isometries*, Arch. Math. **73** (1999), 373–379.

► (3) F. Cabello Sánchez, *Diameter preserving linear maps and isometries II*, Proc. Indian Acad. Sci. **110** (2000), 205–211.

► (4) A. Cabello Sánchez and F. Cabello Sánchez, *Maximal norms on Banach spaces of continous functions*, Math. Proc. Camb. Philos. Soc. **129** (2000), 325–330.

► (5) T.S.S.R.K. Rao and A.K. Roy, *Diameter preserving linear bijections of functions spaces*, J. Austral. Math. Soc. **70** (2001), 323–335.

► (6) B.A. Barnes and A.K. Roy, *Diameter preserving maps on various classes of functions spaces*, Studia Math. **153** (2002), 127–145.

► (7) J.J. Font and M. Sanchis, *A characterization of locally compact spaces with homeomorphic one-point compactifications*, Topology Appl., **121** (2002), 91–104.

► (8) J.J. Font and M. Sanchis, *Extreme points and the diameter norm*, Rocky Mountain J. Math. (to appear)

► (9) A.I. Istratescu and V.I. Istratescu, *Diameter preserving bijections on* Lip_X^a *or* $\mathrm{lip}_X^{a;0}$, (preprint)

► (10) A. I. Istratescu and V. I. Istratescu, *Triangle area preserving maps: I*, (preprint)

2. M. Győry, L. Molnár and P. Šemrl, Linear rank and corank preserving maps on $B(H)$ and an application to *-semigroup isomorphisms of operator ideals, *Linear Alg. Appl.* **280** (1998), 253–266.

► (11) M. Radjabalipour, *Additive mappings on von Neumann algebras preserving absolute values*, Linear Algebra Appl. (to appear)

3. L. Molnár and M. Győry, Reflexivity of the automorphism and isometry groups of the suspension of $B(H)$, *J. Funct. Anal.* **159** (1998), 568-586.

► (12) M. Barczy and M. Tóth, *Local automorphisms of the sets of states and effects on a Hilbert space*, Rep. Math. Phys. **48** (2001), 289–298.

► (13) F. Cabello Sánchez, *The group of automorphisms of L_∞ is algebraically reflexive*, (preprint)

► (14) K. Jarosz and T.S.S.R.K. Rao, *Local isometries of function spaces*, Math. Z. **243** (2003), 449–469.

C. LIST OF CITATIONS BY CO-AUTHORS

1. M. Győry and L. Molnár, Diameter preserving bijections of $C(X)$, *Arch. Math.* **71** (1998), 301–310.

   ► (1) M. Barczy and L. Molnár, *Linear maps on the space of all bounded observables preserving maximal deviation*, J. Funct. Anal. (to appear)

2. M. Győry, L. Molnár and P. Šemrl, Linear rank and corank preserving maps on $B(H)$ and an application to *-semigroup isomorphisms of operator ideals, *Linear Alg. Appl.* **280** (1998), 253–266.

   ► (2) L. Molnár, *Order isomorphisms and triple isomorphisms of operator ideals and their reflexivity,* Arch. Math. **69** (1997), 497–506.

   ► (3) L. Molnár, *Some linear preserver problems on $B(H)$ concerning rank and corank*, Linear Algebra Appl. **286** (1999), 311-321.

   ► (4) L. Molnár, *Some multiplicative preservers on $B(H)$,* Linear Algebra Appl. **301** (1999), 1–13.

3. L. Molnár and M. Győry, Reflexivity of the automorphism and isometry groups of the suspension of $B(H)$, *J. Funct. Anal.* **159** (1998), 568-586.

   ► (5) L. Molnár, *Multiplicative maps on ideals of operators which are local automorphisms*, Acta Sci. Math. (Szeged) **65** (1999), 727–736.

   ► (6) L. Molnár and P. Šemrl, *Local automorphisms of the unitary group and the general linear group on a Hilbert space*, Expo. Math. **18** (2000), 231–238.

   ► (7) L. Molnár, *Reflexivity of the automorphism and isometry groups of C^*-algebras in BDF theory*, Arch. Math. **74** (2000), 120–128.

   ► (8) L. Molnár and B. Zalar, *On local automorphism of group algebras of compact groups*, Proc. Amer. Math. Soc. **128** (2000), 93–99.

   ► (9) L. Molnár, *Local automorphisms of some quantum mechanical structures*, Lett. Math. Phys. **58** (2001), 91–100.

   ► (10) L. Molnár, *A reflexivity problem concerning the C^*-algebra $C(X) \otimes B(H)$*, Proc. Amer. Math. Soc. **129** (2001), 531–537.

   ► (11) F. Cabello Sánchez and L. Molnár, *Reflexivity of the isometry group of some classical spaces*, Rev. Mat. Iberoam. **18** (2002), 409–430.

► (12) L. Molnár, *2-local isometries of some operator algebras*, Proc. Edinb. Math. Soc. **45** (2002), 349–352.

D. LIST OF TALKS AT INTERNATIONAL CONFERENCES

1. Diameter preserving linear bijections of $C(X)$,

   17th International Conference on Operator Theory,

   Temesvár (Romania), 1998

2. Diameter preserving linear bijections of $C_0(X)$,

   Numbers, Functions, Equations '98 International Conference,

   Noszvaj (Hungary), 1998

3. Reflexivity of the automorphism and isometry groups of some operator algebras,

   2nd Workshop on Functional Analysis and its Applications in Mathematical Physics and Optimal Control,

   Nemecka (Slovakia), 1999

4. On the 2-local isometries of $C_0(X)$,

   38th International Symposium on Functional Equations,

   Noszvaj (Hungary), 2000

5. Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving orthogonality,

   3rd Workshop on Functional Analysis and its Applications in Mathematical Physics and Optimal Control,

   Nemecka (Slovakia), 2001

6. Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving orthogonality,

   8th International Conference on Functional Equations and Inequalities,

   Złockie (Poland), 2001

Preserver problems and reflexivity problems on operator algebras and on function algebras

Értekezés a doktori (PhD) fokozat megszerzése érdekében a matematika tudományában.

Írta: Győry Máté okleveles matematikus,
okleveles közgazdász és angol-magyar szakfordító

Készült a Debreceni Egyetem Matematika- és Számítástudományok Doktori Iskola
(Matematikai analízis és függvényegyenletek doktori programja) keretében

Témavezető: Dr. Molnár Lajos

A doktori szigorlati bizottság:

elnök:	Dr. ....
tagok:	Dr. ....
	Dr. ....

A doktori szigorlat időpontja 200... ....

Az értekezés bírálói:

Dr. ....
Dr. ....
Dr. ....

A bírálóbizottság :

elnök:	Dr. ....
tagok:	Dr. ....
	Dr. ....
	Dr. ....
	Dr. ....

Az értekezés védésének időpontja: 200... ....