A generalization of Wigner's unitary–antiunitary theorem to Hilbert modules

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Let $\mathcal{H}$ be a Hilbert C^*-module over a matrix algebra A. It is proved that any function $T:\mathcal{H}\rightarrow\mathcal{H}$ which preserves the absolute value of the (generalized) inner product is of the form $Tf=\varphi(f)Uf$ $(f\in\mathcal{H})$, where φ is a phase-function and U is an A-linear isometry. The result gives a natural extension of Wigner's classical unitary–antiunitary theorem for Hilbert modules. © *1999 American Institute of Physics.* [S0022-2488(99)01611-4]

I. INTRODUCTION AND STATEMENT OF THE RESULT

Wigner's unitary–antiunitary theorem reads as follows. Let H be a complex Hilbert space and let $T:H\rightarrow H$ be a bijective function (linearity or continuity is not assumed) with the property that

$$|\langle Tx,Ty\rangle|=|\langle x,y\rangle| \quad (x,y\in H).$$

Then T is of the form

$$Tx=\varphi(x)Ux \quad (x\in H),$$

where $U:H\rightarrow H$ is either a unitary or an antiunitary operator and $\varphi:H\rightarrow\mathbb{C}$ is a so-called phase-function which means that its values are of modulus 1. This celebrated result plays a very important role in quantum mechanics and in representation theory in physics.

In our recent paper[1] we presented a new, algebraic approach to this theorem. Our idea turned out to be strong enough to give a natural generalization of Wigner's theorem for Hilbert C^*-modules over matrix algebras. However, in the main result [Ref. 1, Theorem 1] we supposed that our map is surjective and, in addition, a condition was imposed on the underlying module which was proved to be equivalent to that its so-called modular dimension is high enough. In the present paper, refining and modifying our argument quite significantly, we obtain our Wigner-type result in full generality, that is, neither the surjectivity of the transformation in question nor the high dimensionality of the Hilbert module is assumed.

First, we clarify the concepts and notation that we are going to use throughout. For a bit more detailed discussion we refer to the introduction of Ref. 1. Let A be a C^*-algebra. Let $\mathcal{H}$ be a left A-module with a map $[.,.]:\mathcal{H}\times\mathcal{H}\rightarrow A$ satisfying

(i) $[f+g,h]=[f,h]+[g,h]$;
(ii) $[af,g]=a[f,g]$;
(iii) $[g,f]=[f,g]^*$;
(iv) $[f,f]\geq0$ and $[f,f]=0$ if and only if $f=0$

for every $f,g,h\in\mathcal{H}$ and $a\in A$. If $\mathcal{H}$ is complete with respect to the norm $f\mapsto\|[f,f]\|^{1/2}$, then we say that $\mathcal{H}$ is a Hilbert A-module or a Hilbert C^*-module over A with generalized inner product $[.,.]$. Nowadays, Hilbert modules over C^*-algebras play a very important role in many parts of functional analysis such as, for example, in the K-theory of C^*-algebras. There is another concept of Hilbert modules due to Saworotnow.[2] These are modules over H^*-algebras. The only formal difference in the definition is that in the case of Saworotnow's modules, the generalized inner

product takes its values in the trace-class of the underlying H^*-algebra and the norm with respect to which we require completeness is $f \mapsto (\mathrm{tr}[f,f])^{1/2}$. Saworotnow's modules appear naturally when dealing with multivariate stochastic processes and they have applications in Clifford analysis and hence in some parts of mathematical physics.

If the underlying C^*-algebra A is the algebra $M_d(\mathbb{C})$ of all $d \times d$ complex matrices, then, A being finite dimensional, the norms on A are all equivalent. Therefore, the Hilbert C^*-modules over the C^*-algebra $M_d(\mathbb{C})$ are the same as Saworotnow's Hilbert modules over the H^*-algebra $M_d(\mathbb{C})$. We emphasize this fact since, in general, the behavior of Saworotnow's Hilbert modules is much nicer and we shall use several results concerning them. Finally, we note that it seems to be more common to use right modules instead of left ones. Of course, this is not a real difference, only a question of taste.

Now we are in a position to formulate the main result of the paper. Recall that in any C^*-algebra A, the element $|a|$ denotes the square root of a^*a $(a \in A)$.

Theorem: *Let $\mathcal{H}$ be a Hilbert C^*-module over the matrix algebra $A = M_d(\mathbb{C})$, $d > 1$. Let $T:\mathcal{H} \to \mathcal{H}$ be a function with the property that*

$$|[Tf,Tf']| = |[f,f']| \quad (f,f' \in \mathcal{H}). \tag{1}$$

Then there exists an A-isometry $U:\mathcal{H} \to \mathcal{H}$ and a phase-function $\varphi:\mathcal{H} \to \mathbb{C}$ such that

$$Tf = \varphi(f)Uf \quad (f \in \mathcal{H}).$$

Here, A-isometry means that $U:\mathcal{H} \to \mathcal{H}$ is a linear map with $U(af) = aUf$ and $[Uf,Uf'] = [f,f']$ $(a \in A, f, f' \in \mathcal{H})$.

The corresponding result for the case $d = 1$, that is, when $\mathcal{H}$ is a Hilbert space, can be found in Refs. 3 and 4 (for a recent paper also see Ref. 5). As we shall see in the proof, the nonappearance of A-anti-isometries in the above result is the consequence of the noncommutativity of the underlying algebra A.

Hilbert spaces over algebras different from $\mathbb{R}$ and $\mathbb{C}$ do appear in mathematical physics (see, for example, Ref. 6 for a Wigner-type theorem concerning Hilbert spaces over the skew-field of quaternions). We believe that our present result may also have physical interpretation.

II. PROOF

We give some additional definitions and notation that we shall use in the proof of our theorem. As mentioned in the introduction, Saworotnow's modules have many convenient properties which are familiar in the theory of Hilbert spaces (we refer to Ref. 2). First of all, if $\mathcal{H}$ is a Hilbert module over an H^*-algebra, then $\mathcal{H}$ is a Hilbert space with the inner product $\langle .,. \rangle = \mathrm{tr}[.,.]$. If $\mathcal{M} \subset \mathcal{H}$ is a closed submodule, then its orthogonal complement with respect to $\langle .,. \rangle$ and $[.,.]$ are the same. A linear operator T on $\mathcal{H}$ which is bounded with respect to the Hilbert space norm defined above is called an A-linear operator if $T(af) = aTf$ holds true for every $f \in \mathcal{H}$ and $a \in A$. Every A-linear operator T is adjointable, namely, the adjoint T^* of T in the Hilbert space sense is A-linear and we have $[Tf,g] = [f,T^*g]$ $(f,g \in \mathcal{H})$. Consequently, the collection of all A-linear operators forms a C^*-subalgebra in the full operator algebra on the Hilbert space $\mathcal{H}$. This will be denoted by $\mathcal{B}(\mathcal{H})$ while the notation of the full operator algebra over a Hilbert space H is $B(H)$.

In the case of a Hilbert module $\mathcal{H}$ over an H^*-algebra, the natural equivalent of the Hilbert basis is the so-called modular basis.[7] An element $f \in \mathcal{H}$ is called a modular unit vector, if $[f,f]$ is a nonzero minimal projection in A. A family $\{f_\alpha\}_\alpha \subset \mathcal{H}$ is said to be modular orthonormal if

(a) $[f_\alpha, f_\beta] = 0$ if $\alpha \neq \beta$,
(b) f_α is a modular unit vector for every α.

A maximal modular orthonormal family of vectors in $\mathcal{H}$ is called a modular basis. The common cardinality of modular bases in $\mathcal{H}$ is called the modular dimension of $\mathcal{H}$ (see Ref. 7, Theorem 2).

Now, we define operators which are the natural equivalent of the finite rank operators in the case of Hilbert spaces. If $f, g \in \mathcal{H}$, then let $f \odot g$ denote the A-linear operator defined by

$$(f \odot g)h = [h, g]f \quad (h \in \mathcal{H}).$$

It is easy to see that for every A-linear operator S we have

$$S(f \odot g) = (Sf) \odot g, \quad (f \odot g)S = f \odot (S^*g)$$

and

$$(f \odot g)(f' \odot g') = ([f', g]f) \odot g' = f \odot ([g, f']g').$$

Define

$$\mathcal{F}(\mathcal{H}) = \left\{ \sum_{k=1}^{n} f_k \odot g_k : f_k, g_k \in \mathcal{H}(k = 1, \ldots, n), n \in \mathbb{N} \right\}$$

which is a $*$-ideal in the C^*-algebra of all A-linear operators. Observe that if $\mathcal{H}$ is a Hilbert module over $M_d(\mathbb{C})$, then the range of every element of $\mathcal{F}(\mathcal{H})$ has finite linear dimension, but there can be finite rank operators on the Hilbert space $\mathcal{H}$ which do not belong to $\mathcal{F}(\mathcal{H})$. In general, if the underlying H^*-algebra is infinite dimensional, then these two classes of operators have nothing to do with each other.

We begin with some auxiliary results that we shall need in the proof of our theorem.

Lemma 1: Let $A = M_d(\mathbb{C})$, $d \in \mathbb{N}$. If $\mathcal{H}$ is a Hilbert A-module, then every projection in $\mathcal{B}(\mathcal{H})$ is of the form $P = \sum_{\alpha} f_{\alpha} \odot f_{\alpha}$, where $\{f_{\alpha}\}_{\alpha} \subset \mathcal{H}$ is a modular orthonormal basis in the range of P (the range of an A-linear projection is a closed submodule).

If $\{f_{\alpha}\}_{\alpha} \subset \mathcal{H}$ is a modular orthonormal set, then for the orthogonal projection onto the closed submodule generated by $\{f_{\alpha}\}_{\alpha}$ (which is an A-linear projection) we have $P = \sum_{\alpha} f_{\alpha} \odot f_{\alpha}$.

Proof: Let first $P \in \mathcal{B}(\mathcal{H})$ be a projection and let $\{f_{\alpha}\}_{\alpha}$ denote a modular orthonormal basis in the closed submodule rng P. By Ref. 7 Theorem 1, we have

$$f = \sum_{\alpha} [f, f_{\alpha}]f_{\alpha} \quad (f \in \text{rng } P).$$

Since $Pf = 0$ and $[f, f_{\alpha}] = 0$ for $f \in \text{rng } P^{\perp}$, we obtain $P = \sum_{\alpha} f_{\alpha} \odot f_{\alpha}$.

Now, let $\{f_{\alpha}\}_{\alpha} \subset \mathcal{H}$ be a modular orthonormal set and denote $\mathcal{M}$ the closed submodule generated by this set. We show that $\{f_{\alpha}\}_{\alpha}$ is a modular basis in $\mathcal{M}$. Since this collection is a modular orthonormal family, if this was not maximal, then we could find a nonzero element $f \in \mathcal{M}$ which is modular orthogonal to $\{f_{\alpha}\}_{\alpha}$, that is, $[f, f_{\alpha}] = 0$ for every α. But this is a contradiction, since every element of $\mathcal{M}$ can be approximated by finite sums of the form $a_1 f_{\alpha_1} + \cdots + a_n f_{\alpha_n}$ $(a_i \in A)$ and hence we would obtain that f is modular orthogonal to itself. By the first part of the proof we obtain that the orthogonal projection onto $\mathcal{M}$ is equal to $\sum_{\alpha} f_{\alpha} \odot f_{\alpha}$, so this operator is an A-linear projection. $\square$

Lemma 2: Let $A = M_d(\mathbb{C})$, $d \in \mathbb{N}$ and let $\mathcal{H}$ be a Hilbert A-module. Suppose that $\mathcal{M} \subset \mathcal{H}$ is a closed submodule and $\{f_{\alpha}\}_{\alpha}$ is a modular orthonormal system generating $\mathcal{M}$. Then for every $g, h \in \mathcal{M}$ we have

(i) $g = \sum_{\alpha} [g, f_{\alpha}]f_{\alpha}$,
(ii) $[g, h] = \sum_{\alpha} [g, f_{\alpha}][f_{\alpha}, h]$.

Moreover, the vector $k \in \mathcal{H}$ belongs to $\mathcal{M}$ if and only if

$$[k,k] = \sum_{\alpha} [k,f_{\alpha}][f_{\alpha},k].$$

Proof: See Ref. 7, Theorem 2, and its proof.   □

Proposition 3: Let $A = M_d(\mathbb{C})$, $d \in \mathbb{N}$. If $\mathcal{H}$ is a Hilbert A-module, then $\mathcal{B}(\mathcal{H})$ is a type I von Neumann factor. If the modular dimension of $\mathcal{H}$ is greater than 2, then $\mathcal{B}(\mathcal{H})$ is not isomorphic to $M_2(\mathbb{C})$.

Proof: It is clear that $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra since it is the commutant of the set $\{L_a : L_a f = af (f \in \mathcal{H}, a \in A)\}$ in the full operator algebra over $\mathcal{H}$ as a Hilbert space. To show that $\mathcal{B}(\mathcal{H})$ is a factor, it is sufficient to verify that the central projections in $\mathcal{B}(\mathcal{H})$ are all trivial. Let $P \in \mathcal{B}(\mathcal{H})$ be a nonzero central projection. Let f be a modular unit vector in rng P. For any $a,b \in A$ we have

$$P \cdot g \odot (af) = g \odot (af) \cdot P = g \odot P(af) = g \odot (af).$$

This implies that

$$P(b[f,f]a^*g) = P((g \odot (af))bf) = (g \odot (af))bf = b[f,f]a^*g.$$

The element $[f,f]$ is a rank-one projection. Hence, every element of A is the sum of $b[f,f]a^*$-type elements and hence we obtain that $Pg = g$ for every $g \in \mathcal{H}$. Thus $P = I$. So, $\mathcal{B}(\mathcal{H})$ is a factor. We next prove that $\mathcal{B}(\mathcal{H})$ is type I. Let $f \in \mathcal{H}$ be a modular unit vector. Since $[f,f]f = f$ (see Ref. 7, Lemma 1), for any $A \in \mathcal{B}(\mathcal{H})$ we compute

$$f \odot f \cdot A \cdot f \odot f = ([Af,f]f) \odot f = ([A([f,f]f),[f,f]f]f \odot f) = ([f,f][Af,f][f,f]f) \odot f = \lambda(f \odot f),$$

where λ is scalar such that $[f,f][Af,f][f,f]f = \lambda f$ (the existence of such a scalar follows from the fact that $[f,f]$ is a rank-one matrix). This shows that the projection $f \odot f$ is Abelian. So, every nonzero central projection in $\mathcal{B}(\mathcal{H})$ contains a nonzero Abelian projection which means that $\mathcal{B}(\mathcal{H})$ is type I.

Suppose that the modular dimension of $\mathcal{H}$ is greater than 2. To see that $\mathcal{B}(\mathcal{H})$ is not isomorphic to $M_2(\mathbb{C})$ it is now enough to show that the linear dimension of $\mathcal{B}(\mathcal{H})$ is greater then 4. Let $\{f_1, f_2, f_3\}$ be a modular orthonormal set in $\mathcal{H}$. Denote $[f_i, f_i] = e_i$. If $d \geqslant 2$, then there are elements $a_i, b_i \in A$ such that $\{e_i a_i, e_i b_i\}$ is independent for every $i = 1,2,3$. It is easy to check that $\{(a_i f_i) \odot f_i, (b_i f_i) \odot f_i : i = 1,2,3\}$ is linearly independent. Therefore, the algebraic dimension of $\mathcal{B}(\mathcal{H})$ is at least 6. If $d = 1$, then the statement is trivial.   □

Let H be a Hilbert space. Recall that if $x,y \in H$, then $x \otimes y$ stands for the operator defined by $(x \otimes y)(z) = \langle z,y \rangle x (z \in H)$. The ideal of all finite rank operators in $B(H)$ is denoted by $F(H)$.

Lemma 4: Let H be a Hilbert space. If $\phi : F(H) \to B(H)$ is a *-homomorphism which preserves the rank-one projections, then there is an isometry $U \in B(H)$ such that ϕ is of the form

$$\phi(A) = UAU^* \quad (A \in F(H)).$$

Similarly, if $\psi : F(H) \to B(H)$ is a *-antihomomorphism preserving the rank-one projections, then ψ is of the form

$$\psi(A) = VA^{\mathrm{tr}}V^* \quad (A \in F(H)),$$

where V is an isometry and tr *denotes the transpose with respect to a fixed orthonormal basis in H.*

Proof: Let $y,z \in H$ be such that $\langle \phi(y \otimes y)z, z \rangle = 1$. Define

$$Ux = \phi(x \otimes y)z \quad (x \in H).$$

It is easy to see that U is an isometry and $UA = \phi(A)U(A \in F(H))$. Let $x \in H$ be an arbitrary unit vector. Then $\phi(x \otimes x)$ is a rank-one projection, so it is of the form $\phi(x \otimes x) = x' \otimes x'$ with some unit vector $x' \in H$. Since

$$Ux \otimes x = \phi(x \otimes x)U = x' \otimes U^*x',$$

we obtain that x' is equal to Ux multiplied by a scalar of modulus 1. Therefore, $\phi(x \otimes x) = Ux \otimes Ux = U \cdot x \otimes x \cdot U^*$. Since this holds true for every unit vector $x \in H$, by linearity we have the first assertion of the lemma.

As for the second statement, we can apply a similar argument. Choosing $y, z \in H$ such that $\langle \psi(y \otimes y)z, z \rangle = 1$, define

$$\tilde{V}x = \psi(y \otimes x)z \quad (x \in H).$$

One can verify that $\tilde{V}$ is an anti-isometry (that is, a conjugate-linear isometry), and then prove that $\psi(A) = \tilde{V}A^*\tilde{V}^*(A \in F(H))$. Considering an antiunitary operator J for which $JA^*J^* = A^{\text{tr}}$ and defining $V = \tilde{V}J$, we conclude the proof. $\qquad\square$

Lemma 5: Let (a_n) be a sequence in the Hilbert space H and let $b \in H$ be such that $\Sigma_n a_n \otimes a_n = b \otimes b$ in the trace norm. Then for every n there exists a scalar λ_n such that $a_n = \lambda_n b$.

Proof: Clearly, we may assume that $\|b\| = 1$. Taking traces on both sides of the equality $\Sigma_n a_n \otimes a_n = b \otimes b$, we obtain $\Sigma_n \|a_n\|^2 = 1$. On the other hand, we also have

$$\sum_n |\langle b, a_n \rangle|^2 = \left\langle \left(\sum_n a_n \otimes a_n \right)b, b \right\rangle = 1.$$

By the Schwarz inequality,

$$1 = \sum_n |\langle b, a_n \rangle|^2 \leq \sum_n \|a_n\|^2 = 1.$$

So, there are equalities in the Schwarz inequalities $|\langle b, a_n \rangle| \leq \|a_n\|$. This implies the assertion. $\square$

Proof of Theorem: We define an orthoadditive projection-valued measure μ on the lattice $\mathcal{P}(\mathcal{H})$ of all A-linear projections as follows. If $\{f_\alpha\}_\alpha$ is a modular orthonormal set, then let

$$\mu\left(\sum_\alpha f_\alpha \odot f_\alpha \right) = \sum_\alpha Tf_\alpha \odot Tf_\alpha.$$

Observe that by (1), $\{Tf_\alpha\}_\alpha$ is also modular orthonormal and, hence, by Lemma 1 $\Sigma_\alpha Tf_\alpha \odot Tf_\alpha$ belongs to $\mathcal{P}(\mathcal{H})$. We show that μ is well-defined. Let $\{f_\alpha\}_\alpha$ and $\{g_\beta\}_\beta$ generate the same closed submodule $\mathcal{M}$. We claim that the same holds true for $\{Tf_\alpha\}_\alpha$ and $\{Tg_\beta\}_\beta$. Indeed, if $g \in \mathcal{M}$, then due to the fact that $\{f_\alpha\}_\alpha$ is a modular basis in $\mathcal{M}$ we see that $g = \Sigma_\alpha[g, f_\alpha]f_\alpha$. This implies that

$$[Tg, Tg] = [g, g] = \sum_\alpha [g, f_\alpha][f_\alpha, g] = \sum_\alpha [Tg, Tf_\alpha][Tf_\alpha, Tg],$$

which, by Lemma 2, gives us that Tg belongs to the closed submodule generated by $\{Tf_\alpha\}_\alpha$. It is now obvious that μ is an orthoadditive $\mathcal{P}(\mathcal{H})$-valued measure on $\mathcal{P}(\mathcal{H})$.

Let us suppose that the modular dimension of $\mathcal{H}$ is greater than 2. By Proposition 3 we can apply a deep result of Bunce and Wright [Ref. 8, Theorem A]. It states that every bounded finitely orthoadditive, Banach space valued measure on the set of all projections in a von Neumann algebra without a summand isomorphic to $M_2(\mathbb{C})$ can be uniquely extended to a bounded linear transformation defined on the whole algebra. Let $\phi:\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ denote the transformation

corresponding to μ. Since it sends projections to projections, it is a standard argument to verify that ϕ is a Jordan *-endomorphism of $\mathcal{B}(\mathcal{H})$, that is, we have $\phi(T)^2 = \phi(T^2)$, $\phi(T)^* = \phi(T^*)$ $(T \in \mathcal{B}(\mathcal{H}))$ (see, for example, the proof of Ref. 9, Theorem 2).

We prove that $\phi(f \odot f) = Tf \odot Tf$ for every $f \in \mathcal{H}$. Let $[f,f] = \Sigma_i \lambda_i^2 e_i$, where λ_i's are non-negative real numbers and e_i's are pairwise orthogonal rank-one projections. Define $f_i = (1/\lambda_i)e_i f$. We have $[f_i,f_i] = e_i$ and $[f_i,f_j] = 0$ if $i \neq j$, that is, $\{f_i\}_i$ is modular orthonormal. Then $f = \Sigma_i \lambda_i f_i = \Sigma_i e_i f$ since

$$\sum_i [f,f_i][f_i,f] = \sum_i \lambda_i^2 e_i = [f,f]$$

implies that $f = \Sigma_i [f,f_i]f_i = \Sigma_i e_i f$ (see Lemma 2). So, we have

$$\phi(f \odot f) = \sum_{i,j} \phi(e_i f \odot e_j f).$$

But $(e_i f) \odot (e_j f) = 0$ if $i \neq j$. Indeed, we compute $[g,e_j f]e_i f = [g,f]e_j e_i f = 0$ for every $g \in \mathcal{H}$. Hence,

$$\phi(f \odot f) = \sum_i \phi(e_i f \odot e_i f) = \sum_i \lambda_i^2 \phi(f_i \odot f_i) = \sum_i \lambda_i^2 \mu(f_i \odot f_i) = \sum_i \lambda_i^2 Tf_i \odot Tf_i.$$

So, the question is that whether the equality $Tf \odot Tf = \Sigma_i \lambda_i^2 Tf_i \odot Tf_i$ holds true. Clearly, $\{Tf_i\}$ is modular orthonormal. We compute

$$[Tf,Tf] = [f,f] = \sum_i [f,f_i][f_i,f] = \sum_i [Tf,Tf_i][Tf_i,Tf]$$

which, by Lemma 2, implies that $Tf = \Sigma_i [Tf,Tf_i]Tf_i$. We know that $|[Tf,Tf_i]| = |[f,f_i]| = \lambda_i e_i$. Similarly, $|[Tf_i,Tf]| = |[f_i,f]| = \lambda_i e_i$. Since e_i is a rank-one projection, we obtain that $[Tf,Tf_i]$ is also rank-one. Furthermore, as $|[Tf,Tf_i]| = |[Tf_i,Tf]|$ is a scalar multiple of e_i we can infer that $[Tf,Tf_i] = \mu_i \lambda_i e_i$, where μ_i is a scalar of modulus 1. Therefore, we have

$$Tf \odot Tf = \sum_{i,j} \mu_i \bar{\mu}_j (\lambda_i e_i Tf_i \odot \lambda_j e_j Tf_j).$$

But similarly as above, for $i \neq j$ we have

$$(e_i Tf_i \odot e_j Tf_j)g = [g,e_j Tf_j]e_i Tf_i = [g,Tf_j]e_j e_i Tf_i = 0.$$

Therefore

$$Tf \odot Tf = \sum_{i,j} \mu_i \bar{\mu}_j (\lambda_i e_i Tf_i \odot \lambda_j e_j Tf_j)$$

$$= \sum_i \mu_i \bar{\mu}_i (\lambda_i e_i Tf_i \odot \lambda_i e_i Tf_i) = \sum_i \lambda_i e_i Tf_i \odot \lambda_i e_i Tf_i = \sum_i \lambda_i^2 (e_i Tf_i \odot e_i Tf_i).$$

But $(e_i Tf_i \odot e_i Tf_i) = Tf_i \odot Tf_i$. Indeed, since Tf_i is a modular unit vector, we have $e_i Tf_i = [f_i,f_i]Tf_i = [Tf_i,Tf_i]Tf_i = Tf_i$ (see Ref. 7, Lemma 1). Consequently, we obtain $Tf \odot Tf = \Sigma_i \lambda_i^2 Tf_i \odot Tf_i$ and this was to be proved. So, we get $\phi(f \odot f) = Tf \odot Tf$ for every $f \in \mathcal{H}$.

We assert that ϕ is either a *-homomorphism or a *-antihomomorphism. By Lemma 1 the minimal projections in $\mathcal{H}$ are exactly the operators of the form $f \odot f$, where $f \in \mathcal{H}$ is a modular unit vector. Clearly, ϕ sends minimal projections to minimal projections. By Ref. 1, Lemma 2, the

linear space generated by the minimal projections in $\mathcal{B}(\mathcal{H})$ is $\mathcal{F}(\mathcal{H})$. Since $\mathcal{B}(\mathcal{H})$ is a type I factor, it is isomorphic to the full operator algebra $B(H)$ on a Hilbert space H. Since *-isomorphisms preserve the minimal projections, $\mathcal{F}(\mathcal{H})$ corresponds to the ideal $F(H)$ of all finite rank operators in $B(H)$. Under this identification, we obtain a Jordan *-homomorphism $\tilde{\phi}$ on $F(H)$ corresponding to $\phi_{|\mathcal{F}(\mathcal{H})}$ which sends rank-one projections to rank-one projections. Since $F(H)$ is a local matrix algebra, by Ref. 10, Theorem 8, we obtain that $\tilde{\phi}$ is the sum of a *-homomorphism and a *-antihomomorphism. As $\tilde{\phi}$ preserves the rank-one projections, from the simplicity of the ring $F(H)$ it follows that $\tilde{\phi}$ is either a *-homomorphism or a *-antihomomorphism. Obviously, the same holds for $\phi_{|\mathcal{F}(\mathcal{H})}$.

Let us suppose that the modular dimension of $\mathcal{H}$ is greater than d. By Ref. 1, Remark 2, there are vectors $g,h \in \mathcal{H}$ such that $[g,h]=I$. The map $\phi_{|\mathcal{F}(\mathcal{H})}$ is either a *-homomorphism or a *-antihomomorphism. First consider this latter case. Referring to Lemma 4 we have an operator $U \in \mathcal{B}(\mathcal{H})$ with $U^*U=I$ and a *-antiautomorphism ψ of $\mathcal{F}(\mathcal{H})$ such that $\phi(A)=U\psi(A)U^*(A \in \mathcal{F}(\mathcal{H}))$.

We define

$$Vf = \psi(g \odot f)U^*Th \quad (f \in \mathcal{H}),$$

where $g,h \in \mathcal{H}$ are fixed and such that $[g,h]=I$. Clearly, V is a conjugate-linear operator. We have

$$VAf = \psi(g \odot (Af))U^*Th = \psi(g \odot fA^*)U^*Th = \psi(A)^*\psi(g \odot f)U^*Th = \psi(A)^*Vf,$$

that is, $VA = \psi(A)^*V(A \in \mathcal{F}(\mathcal{H}))$. We compute

$$
\begin{aligned}
[Vf,Vf] &= [\psi(g \odot f)U^*Th, \psi(g \odot f)U^*Th] \\
&= [\psi(g \odot f \cdot f \odot g)U^*Th, U^*Th] \\
&= [U\psi(g \odot f \cdot f \odot g)U^*Th, Th] \\
&= [\phi(g \odot f \cdot f \odot g)Th, Th] \\
&= [\phi(\sqrt{[f,f]}g \odot \sqrt{[f,f]}g)Th, Th] \\
&= [(T(\sqrt{[f,f]}g) \odot T(\sqrt{[f,f]}g))Th, Th] \\
&= [Th, T(\sqrt{[f,f]}g)][T(\sqrt{[f,f]}g), Th] \\
&= [h, \sqrt{[f,f]}g][\sqrt{[f,f]}g, h] = [h,g][f,f][g,h] = [f,f].
\end{aligned}
$$

Since V is conjugate-linear, by polarization we obtain

$$[Vf,Vf'] = [f',f] \quad (f,f' \in \mathcal{H}).$$

We show that rng $T \subset$ rng U which will imply $UU^*T=T$ (UU^* is the projection onto the range of U). Let $f \in \mathcal{H}$. In the previous part of the proof we have learned that $Tf \odot Tf$ is a linear combination of operators of the form $Tf_b \odot Tf_b$, where f_b's are modular unit vectors. We have

$$Tf_b \odot Tf_b = \phi(f_b \odot f_b) = U\psi(f_b \odot f_b)U^*$$

and, ψ being a *-antiautomorphism, $\psi(f_b \odot f_b)$ is a minimal projection. Therefore, $\psi(f_b \odot f_b) = f_b' \odot f_b'$ with some modular unit vector f_b' and hence $Tf_b \odot Tf_b = Uf_b' \odot Uf_b'$. Now let $Tf = g' + g''$, where $g' \in$ rng U and $g'' \in$ rng $U^\perp$. We have

$$[g'',g'']^2 = [g'',Tf][Tf,g''] = [(Tf \odot Tf)g'',g''] = 0.$$

This gives us that $g''=0$ which shows that $Tf \in \text{rng } U$.

We next prove that V is surjective. Let $f \in \mathcal{H}$ be arbitrary. Since ψ is a *-antiautomorphism of $\mathcal{F}(\mathcal{H})$, we can find an operator $R \in \mathcal{F}(\mathcal{H})$ such that $\psi(R)^*=f \odot U^*Th$. We compute

$$VRg = \psi(R)^*Vg$$
$$= \psi(R)^*\psi(g \odot g)U^*Th$$
$$= \psi(R)^*U^*\phi(g \odot g)Th$$
$$= \psi(R)^*U^*(Tg \odot Tg)Th$$
$$= [Th,Tg]\psi(R)^*U^*Tg$$
$$= [Th,Tg][U^*Tg,U^*Th]f$$
$$= [Th,Tg][UU^*Tg,Th]f$$
$$= [Th,Tg][Tg,Th]f = [h,g][g,h]f = f.$$

Since f was arbitrary, we have the surjectivity of V.

We compute

$$[UVf',Tf][Tf,UVf'] = [(Tf \odot Tf)UVf',UVf']$$
$$= [U^*(Tf \odot Tf)UVf',Vf']$$
$$= [U^*\phi(f \odot f)UVf',Vf']$$
$$= [\psi(f \odot f)Vf',Vf']$$
$$= [(V \cdot f \odot f)f',Vf'] = [f',(f \odot f)f'] = [f',f][f,f'].$$

This gives us that

$$[V^{-1}U^*Tf',f][f,V^{-1}U^*Tf'] = [UVV^{-1}U^*Tf',Tf][Tf,UVV^{-1}U^*Tf']$$
$$= [UU^*Tf',Tf][Tf,UU^*Tf']$$
$$= [Tf',Tf][Tf,Tf'] = [f',f][f,f'].$$

Replacing f by $xf(x \in A)$, we obtain

$$[V^{-1}U^*Tf',f]x^*x[f,V^{-1}U^*Tf'] = [f',f]x^*x[f,f'].$$

Since every element of A is a linear combination of elements of the form x^*x, it follows that

$$[V^{-1}U^*Tf',f]y[f,V^{-1}U^*Tf'] = [f',f]y[f,f']$$

holds for every $y \in A$. This implies that for every $f \in \mathcal{H}$, the matrices $[f,V^{-1}U^*Tf']$ and $[f,f']$ are linearly dependent. It requires only elementary linear algebra to verify the following assertion. If X,Y are vector spaces and $A,B:X \to Y$ are linear operators such that for every $x \in X$, the set $\{Ax,Bx\}$ is linearly dependent, then either A and B have rank at most one or $\{A,B\}$ is linearly dependent. Since the rank of the linear operator $f \mapsto [f,f']$ is clearly greater than 1 if $f' \neq 0$, we have a scalar $\lambda_{f'}$ (depending only on f') such that $[f,V^{-1}U^*Tf'] = \lambda_{f'}[f,f'](f,f' \in \mathcal{H})$. This gives us that there is a function $\varphi:\mathcal{H} \to \mathbb{C}$ such that $V^{-1}U^*Tf' = \varphi(f')f'$ which results in $Tf' = \varphi(f')UVf'$. It follows from the properties of T,U,V that φ is of modulus 1. Finally, we have

$$|[f,f']| = |[Tf,Tf']| = |[UVf,UVf']| = |[Vf,Vf']| = |[f',f]|.$$

Since this must hold true for every $f, f' \in \mathcal{H}$, it follows that for every rank-one matrix $a \in A$ we have $|a| = |a^*|$. But this is an obvious contradiction. Since we have started with assuming that $\phi_{|\mathcal{F}(\mathcal{H})}$ is a *-antihomomorphism, we thus obtain that it is in fact a *-homomorphism.

Pushing the problem from $\mathcal{B}(\mathcal{H})$ to the full operator algebra $B(H)(\cong \mathcal{B}(\mathcal{H}))$, we see that there is an A-isometry $U \in \mathcal{B}(\mathcal{H})$ such that $\phi(A) = UAU^*(A \in \mathcal{F}(\mathcal{H}))$. This gives us that $Tf \odot Tf = Uf \odot Uf$ for every $f \in \mathcal{H}$. Similarly as before, this implies that $\operatorname{rng} T \subset \operatorname{rng} U$ which yields $UU^*Tf = Tf(f \in \mathcal{H})$. We next compute

$$[Uf', Tf][Tf, Uf'] = [(Tf \odot Tf)Uf', Uf']$$
$$= [(Uf \odot Uf)Uf', Uf'] = [Uf', Uf][Uf, Uf'] = [f', f][f, f'],$$

which gives us that

$$[U^*Tf', f][f, U^*Tf'] = [UU^*Tf', Tf][Tf, UU^*Tf'] = [Tf', Tf][Tf, Tf'] = [f', f][f, f'].$$

Just as above, it follows that U^*Tf' is a scalar multiple of f'. Therefore, there exists an A-isometry U and a phase-function $\varphi: \mathcal{H} \to \mathbb{C}$ such that

$$Tf = \varphi(f)Uf \quad (f \in \mathcal{H}).$$

This completes the proof in the case when the modular dimension n of $\mathcal{H}$ is greater than d.

We now treat the low dimensional cases, that is, when $n \leq d$. Let H_d denote the d-dimensional complex Euclidean space. Then H_d can be considered as a Hilbert A-module. Here, the module operation is $(a, \xi) \mapsto a(\xi)$ and the generalized inner product is defined by $[\xi, \zeta] = \xi \otimes \zeta$. Clearly, the modular dimension of this module is 1. It now follows from the structure of our Hilbert A-modules (see, for example, Ref. 11) that $\mathcal{H}$ is isomorphic to the n-fold direct sum of H_d with itself. So, we may assume that $\mathcal{H} = \Sigma_{i=1}^{n} \oplus H_d$. The definition of the module operation and that of the inner product on this direct sum is defined as follows:

$$a[\xi_i]_i = [a\xi_i]_i, \quad [[\xi_i]_i, [\zeta_i]_i] = \sum_i \xi_i \otimes \zeta_i.$$

Let us describe the elements of $\mathcal{B}(\mathcal{H})$. Since every element of $\mathcal{B}(\mathcal{H})$ is a linear operator on the direct sum of vector spaces, it can represented by a matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix},$$

where a_{ij}'s are linear operators acting on H_d. Now, A-linearity means that

$$\begin{bmatrix} a_{11}a\xi_1 + \cdots + a_{1n}a\xi_n \\ \vdots \\ a_{n1}a\xi_1 + \cdots + a_{nn}a\xi_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a\xi_1 \\ \vdots \\ a\xi_n \end{bmatrix} = \begin{bmatrix} a(a_{11}\xi_1 + \cdots + a_{1n}\xi_n) \\ \vdots \\ a(a_{n1}\xi_1 + \cdots + a_{nn}\xi_n) \end{bmatrix}$$

holds for every $a \in A$ and $\xi_i \in H_d$. It is easy to see that this is equivalent to $a_{ij}a = aa_{ij}(a \in A)$ which means that a_{ij}'s are scalars. Consequently, $\mathcal{B}(\mathcal{H})$ is isomorphic to $M_n(\mathbb{C})$.

Suppose that $n > 1$. If ζ is any vector in H_d, then let ζ^k denote the element of $\mathcal{H}$ whose coordinates are all 0 except for the kth one which is ζ. Fix a unit vector $\xi \in H_d$. We have

$$\sum_i (T\xi^k)_i \otimes (T\xi^k)_i = [T\xi^k, T\xi^k] = [\xi^k, \xi^k] = \xi \otimes \xi.$$

From Lemma 5 we infer that for every $i=1,...,n$, there is a scalar α_{ik} such that $(T\xi^k)_i = \alpha_{ik}\xi$. Clearly, the columns of the matrix (α_{ik}) are unit vectors. Since $[T\xi^k, T\xi^l]=0$ for $k \neq l$, it follows that the columns of our matrix are pairwise orthogonal as well. So (α_{ik}) is a unitary matrix and hence it defines an A-unitary operator U on $\mathcal{H}$. Considering U^*T instead of T, we can assume that $T\xi^k$ is equal to ξ^k for every $k=1,...,n$. If f is any vector in $\mathcal{H}$, then considering the equality

$$\left|\xi \otimes (Tf)_k\right| = \left|[T\xi^k, Tf]\right| = \left|[\xi^k, f]\right| = \left|\xi \otimes f_k\right|,$$

we obtain

$$(Tf)_k = \mu_k f_k \quad (k=1,...,n) \tag{2}$$

with some scalars μ_k of modulus 1. We claim that all the μ_k's are equal. Fix a $g \in \mathcal{H}$ whose coordinates are pairwise orthogonal unit vectors in H_d (recall that $n \leq d$). It is apparent that if we multiply T from the left by an A-unitary operator whose matrix is diagonal, then the so obtained transformation still has the property (2). So we may assume that $Tg = g$. Let $f \in \mathcal{H}$ be arbitrary. We have

$$\left|\sum_i \mu_i f_i \otimes g_i\right| = \left|[Tf, Tg]\right| = \left|[f, g]\right| = \left|\sum_i f_i \otimes g_i\right|.$$

This implies that

$$\sum_{i,j} \langle \mu_j f_j, \mu_i f_i \rangle g_i \otimes g_j = \sum_{i,j} \langle f_j, f_i \rangle g_i \otimes g_j$$

which gives that

$$\langle \mu_j f_j, \mu_i f_i \rangle = \langle f_j, f_i \rangle.$$

So, if $\langle f_i, f_j \rangle \neq 0$, then we have $\mu_i = \mu_j$. Suppose now that $\langle f_i, f_j \rangle = 0$ but $f_i, f_j \neq 0$. Let $\zeta \in H_d$ be any nonzero vector and consider $\zeta^i + \zeta^j$. By what we have just proved, it follows that $T(\zeta^i + \zeta^j)$ is a scalar multiple of $\zeta^i + \zeta^j$. We compute

$$\left|\zeta \otimes (\mu_i f_i + \mu_j f_j)\right| = \left|[\zeta^i + \zeta^j, Tf]\right| = \left|[T(\zeta^i + \zeta^j), Tf]\right| = \left|[\zeta^i + \zeta^j, f]\right| = \left|\zeta \otimes (f_i + f_j)\right|$$

which clearly gives us that $\mu_i = \mu_j$. Therefore, we obtain that for any vector $f \in \mathcal{H}$, Tf is equal to f multiplied by a complex number of modulus 1. The assertion of the theorem now follows for the case $1 < n \leq d$.

Finally, suppose that $n = 1$, which means that $\mathcal{H} = H_d$. Our problem is to describe those maps $T: H_d \to H_d$ for which $|T\xi \otimes T\zeta| = |\xi \otimes \zeta| (\xi, \zeta \in H_d)$. But this equality clearly implies that $T\zeta$ is equal to ζ multiplied by a scalar of modulus 1.

The proof of the theorem is now complete. $\qquad\square$

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