# REGULARITY OF WEAKLY SUBQUADRATIC FUNCTIONS

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Abstract. Related to the theory of convex and subadditive functions, we investigate weakly subquadratic mappings, that is, solutions of the inequality

$$f(x+y)+f(x-y)\le 2f(x)+2f(y) \qquad (x,y\in G)$$

for real valued functions defined on a topological group $G=(G,+)$. Especially, we study the lower and upper hulls of such functions and we prove Bernstein–Doetsch type theorems for them.

## 1. Introduction

In the present paper, we investigate weakly subquadratic functions, that is, solutions of the inequality

$$f(x+y)+f(x-y)\le 2f(x)+2f(y) \qquad (x,y\in G), \tag{1}$$

in the case when $f$ is a real valued function defined on a group $G=(G,+)$. Our aim is to prove regularity theorems for functions of this type. Our studies have been motivated by classical results of the regularity theory of convex and subadditive functions.

A fundamental result of the regularity theory of convex functions is the theorem of F. Bernstein and G. Doetsch [13] which states that if a real valued Jensen-convex function defined on an open interval is locally bounded from above at one point in its domain, then it is continuous (cf. also [16] and [17]). It is easy to prove and is well-known that, in the case of subadditive functions, local boundedness does not imply continuity. However, as R. A. Rosenbaum proved in [19], if a subadditive function defined on $\mathbb{R}^n$ is locally bounded from above at one point in $\mathbb{R}^n$ then it is locally bounded everywhere in $\mathbb{R}^n$. Furthermore, if a subadditive function $f\colon \mathbb{R}^n\to\mathbb{R}$ is continuous at 0 and $f(0)=0$ then it is continuous everywhere in $\mathbb{R}^n$ (see [16] and [17], too).

Strongly related to these results, we investigate regularity properties of weakly subquadratic functions. After giving some hints on the terminology we use, in the third part of the paper, we present some examples and basic properties of weakly subquadratic functions. In Part 4, we investigate the lower and upper hulls of weakly subquadratic functions, which play a key role in the theory of convex and subadditive functions. We prove, that, similarly to the case of convex and subadditive functions, the lower and upper hulls of a weakly subquadratic function, under quite general conditions, are weakly subquadratic, too. In the last part of the paper we prove Bernstein–Doetsch type results for weakly subquadratic functions. We show some theorems concerning the local boundedness of weakly subquadratic functions.

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As a consequence of these, we obtain an analogous theorem to Rosenbaum's main regularity result for subadditive functions mentioned above.

## 2. Terminology

The subquadraticity concept we use in this paper is related to the notion of subadditivity. A real valued function defined on a group $G$ is called additive if it satisfies the equation

$$f(x+y) = f(x) + f(y) \qquad (x, y \in G),$$

it is said to be subadditive, if it fulfils

$$f(x+y) \leq f(x) + f(y) \qquad (x, y \in G) \tag{2}$$

and it is superadditive, if

$$f(x+y) \geq f(x) + f(y) \qquad (x, y \in G).$$

It is easy to see that a function $f\colon G \to \mathbb{R}$ is superadditive if and only if $-f$ is subadditive, therefore, it is enough to investigate one of these types of functions.
Analogously to these concepts, we may define subquadratic and superquadratic functions. Our definition is based on the well-known concept of quadratic functions: a real valued function defined on a group $G$ is called quadratic, if it satisfies the quadratic (or parallelogram or square-norm or Jordan–von Neumann) equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \qquad (x, y \in G).$$

A function $f\colon G \to \mathbb{R}$ is called weakly subquadratic if it satisfies inequality (1), it is said to be weakly superquadratic if inequality (1) is valid in the opposite direction. Obviously, between weakly subquadratic and weakly superquadratic functions there is a similar connection as between subadditive and superadditive functions, therefore, it is enough to consider one of these concepts, too. Weakly subquadratic functions, in this sense, were studied, among others in the papers [15], [20] and [21].
We note that, recently, another concept of subquadraticity has also been investigated. In their paper [7], S. Abramovich, G. Jameson and G. Sinnamon introduced this concept, calling a function $f\colon [0, \infty[ \to \mathbb{R}$ subquadratic if, for each $x \geq 0$, there exists a constant $c_x \in \mathbb{R}$ such that the inequality

$$f(y) - f(x) \leq c_x(y-x) + f\big(|y-x|\big)$$

is valid for all nonnegative $y$. (More precisely, in the paper above, superquadratic functions were considered, but here is an analogous relation between the concepts as above.) Subquadratic (or superquadratic) functions have been investigated by several authors in this sense (cf., e.g, [1], [2], [4], [5], [6], [8], [9], [10], [11], [12], [18]). As a result of the study of the connection between the two different concepts of subquadraticity (cf. [3] and [14]), it turned out that if a function $f\colon [0, \infty[ \to \mathbb{R}$ is subquadratic in the sense of Abramovich, Jameson and Sinnamon, then its even extension $f\colon \mathbb{R} \to \mathbb{R}$ satisfies inequality (1). On the other hand, there are solutions $f\colon [0, \infty[ \to \mathbb{R}$ of inequality (1) which are not subquadratic functions in the other sense. This is the reason why we use the notion of 'weakly subquadratic function' for a solution of inequality (1).

## 3. Examples and basic properties

**Examples 3.1.**

1. It is easy to see that if $B : G \times G \to \mathbb{R}$ is a biadditive function and $b$ is a nonnegative real number, then the function $f : G \to \mathbb{R}$
$$f(x) = B(x, x) + b \qquad (x \in G)$$
satisfies (1).
As a special case of the example above, we obtain that if $a_1 : G \to \mathbb{R}$ and $a_2 : G \to \mathbb{R}$ are additive function, $c$ is an arbitrary and $b$ is a nonnegative real constant, then the function $f : G \to \mathbb{R}$
$$f(x) = c\ a_1(x)a_2(x) + b \qquad (x \in G)$$
solves (1), too.
In the class of continuous real functions, the example above gives the weakly subquadratic functions $f : \mathbb{R} \to \mathbb{R}$
$$f(x) = cx^2 + b$$
where $c$ is an arbitrary, $b$ is a nonnegative real constant.
2. A simple calculation yields that if $a : G \to \mathbb{R}$ is an additive function and $b$ and $d$ are nonnegative constants then the function $f : G \to \mathbb{R}$
$$f(x) = b\ |a(x)| + d \qquad (x \in G)$$
is weakly subquadratic.
3. The function $f : G \to \mathbb{R}$
$$f(x) = \begin{cases} b & \text{if } x \neq 0 \\ d & \text{if } x = 0, \end{cases}$$
where $b$ and $d$ are nonnegative constants such that $d \leq 3b$, is weakly subquadratic.
4. The function $f\colon \mathbb{R} \to \mathbb{R}$
$$f(x) = \begin{cases} b & \text{if } x \in [-1, 1] \\ d & \text{otherwise}, \end{cases}$$
where $b$ and $d$ are nonnegative constants such that $\frac{b}{3} \leq d \leq 3b$, is weakly subquadratic.
5. An arbitrary function $f : G \to \mathbb{R}$ satisfying the inequality
$$\sup_{x \in G} f(x) \leq 2 \inf_{x \in G} f(x)$$
is weakly subquadratic.
6. The function $f : \mathbb{R}^n \to \mathbb{R}$
$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}^n \\ b & \text{otherwise}, \end{cases}$$
with an arbitrary nonnegative constant $b$, is weakly subquadratic.

**Remarks 3.2.**

1. The structure of inequality (1) shows that a linear combination of its solutions with nonnegative real coefficients also yields a solution. Therefore, such linear combinations of the examples above are weakly subquadratic functions, too.

2. It is easy to see, that an even and subadditive function is weakly subquadratic. To show this statement, let $f$ be a real valued subadditive function defined on a group $G$ and suppose that $f$ is even, that is, it satisfies $f(x) = f(-x)$ for each $x \in G$. Writing $-y$ instead of $y$ in inequality (2) we obtain
$$f(x-y) \le f(x) + f(-y) \qquad (x, y \in G).$$
Adding this inequality and (2) side by side and using the evenness of $f$, we obtain the statement.
3. It is also obvious that a non-positive, weakly subquadratic function defined on a 2-divisible abelian group is Jensen-concave. In fact, writing $u = x + y$ and $v = x - y$ in (1), we obtain
$$f(u) + f(v) \le 2f\left(\frac{u+v}{2}\right) + 2f\left(\frac{u-v}{2}\right) \qquad (u, v \in G),$$
which, using the non-positivity of $f$, implies
$$\frac{f(u)+f(v)}{2} \le f\left(\frac{u+v}{2}\right) \qquad (u, v \in G),$$
that is, the defining inequality of Jensen-concavity.

**Lemma 3.3.** *If $G$ is a group and $f : G \to \mathbb{R}$ is a weakly subquadratic function, then we have*
$$f(kx) \le k^2 f(x) \qquad (x \in G)$$
*for each positive integer $k$.*

*Proof.* The statement was proved by Z. Kominek and K. Troczka in [15] in the case when the domain of the function considered is a linear space. Essentially the same argumentation yields the validity of the Theorem if the domain is a group. □

## 4. Lower and upper hulls

The lower and upper hulls play a very important role in the regularity theory of convex and subadditive functions (cf., e.g., [13], [19], [16] and [17]). Motivated by and related to these results, in this section, we will investigate the lower and upper hulls of weakly subquadratic functions in a general setting. We will consider real valued functions defined on a topological group.

In a topological space $(X, \mathcal{O})$, we denote by $\mathcal{U}(x)$ the family of all neighborhoods of an element $x \in X$ (that is, the class of all open sets containing $x$). For a function $f : X \to \mathbb{R}$, the mapping $m_f : X \to [-\infty, \infty[$
$$m_f(x) = \sup_{U \in \mathcal{U}(x)} \inf_{u \in U} f(u) \qquad (x \in X)$$
is called the *lower hull*, while $M_f : X \to ]-\infty, \infty]$
$$M_f(x) = \inf_{U \in \mathcal{U}(x)} \sup_{u \in U} f(u) \qquad (x \in X)$$
is said to be the *upper hull* of $f$.

A topological group is a group endowed with a topology such that the group operation as well as taking inverses are continuous functions.

**Theorem 4.1.** *The lower hull of a real valued weakly subquadratic function defined on a topological group is weakly subquadratic, too.*

*Proof.* Let $G$ be a topological group, $f : G \to \mathbb{R}$ be a weakly subquadratic function and let $x_0, y_0 \in G$ be fixed. Obviously, if $m_f(x_0+y_0) = -\infty$ or $m_f(x_0-y_0) = -\infty$, then the desired inequality holds. Thus, we may assume that $m_f(x_0 + y_0) > -\infty$ and $m_f(x_0 - y_0) > -\infty$. Let, in this case, $\alpha, \beta \in \mathbb{R}$ be such that

$$\alpha < m_f(x_0 + y_0) \quad \text{and} \quad \beta < m_f(x_0 - y_0). \tag{3}$$

By the definition of the lower hull, there exist open sets $W_1 \in \mathcal{U}(x_0+y_0)$ and $W_2 \in \mathcal{U}(x_0-y_0)$ such that

$$\alpha < \inf_{s\in W_1} f(s) \quad \text{and} \quad \beta < \inf_{t\in W_2} f(t).$$

There exist neighborhoods $U \in \mathcal{U}(x_0)$ and $V \in \mathcal{U}(y_0)$ with the properties $U + V \subseteq W_1$ and $U - V \subseteq W_2$ and, since $f$ is weakly subquadratic, we have

$$\alpha + \beta < f(u + v) + f(u - v) \le 2f(u) + 2f(v)$$

for arbitrary elements $u \in U$ and $v \in V$. Therefore,

$$\alpha + \beta \le 2 \inf_{u\in U} f(u) + 2 \inf_{v\in V} f(v),$$

thus,

$$\alpha + \beta \le 2 \sup_{U\in\mathcal{U}(x_0)} \inf_{u\in U} f(u) + 2 \sup_{V\in\mathcal{U}(y_0)} \inf_{v\in V} f(v),$$

that is,

$$\alpha + \beta \le 2m_f(x_0) + 2m_f(y_0).$$

Since $\alpha$ and $\beta$ have been arbitrarily chosen in (3), the theorem is proved. □

**Theorem 4.2.** *The upper hull of a real valued weakly subquadratic function defined on a topological abelian group uniquely divisible by* 2 *is weakly subquadratic, too.*

*Proof.* Let $f : G \to \mathbb{R}$ be weakly subquadratic and let $x_0, y_0 \in G$ be fixed. Analogously to the first step in the proof of the previous theorem, if $M_f(x_0) = \infty$ or $M_f(y_0) = \infty$, then the desired inequality holds, so we may consider the situation when $M_f(x_0) < \infty$ and $M_f(y_0) < \infty$. Let now $\alpha, \beta \in \mathbb{R}$ satisfy the properties

$$2M_f(x_0) < \alpha \quad \text{and} \quad 2M_f(y_0) < \beta. \tag{4}$$

According to the definition of the upper hull, there exist open sets $U \in \mathcal{U}(x_0)$ and $V \in \mathcal{U}(y_0)$ such that

$$2\sup_{u\in U} f(u) < \alpha \quad \text{and} \quad 2\sup_{v\in V} f(v) < \beta.$$

Since $G$ is uniquely divisible by 2, there exist neighborhoods $W_1 \in \mathcal{U}(x_0 + y_0)$ and $W_2 \in \mathcal{U}(x_0 - y_0)$ such that

$$\frac{W_1 + W_2}{2} \subseteq U \quad \text{and} \quad \frac{W_1 - W_2}{2} \subseteq V.$$

Being $f$ weakly subquadratic and $G$ commutative, the substitutions $x = \frac{s+t}{2}$ and $y = \frac{s-t}{2}$ in inequality (1) give

$$f(s) + f(t) \le 2f\left(\frac{s+t}{2}\right) + 2f\left(\frac{s-t}{2}\right) < \alpha + \beta$$

for all elements $s \in W_1$ and $t \in W_2$.
An analogous argumentation as in the proof of the previous theorem yields that

$$M_f(x_0 + y_0) + M_f(x_0 - y_0) \le \alpha + \beta.$$

Thus, since $\alpha$ and $\beta$ have been chosen arbitrarily in (4), we obtain our statement. □

**Theorem 4.3.** *Let $G$ be a topological group, $f : G \to \mathbb{R}$ be a weakly subquadratic function and let $m_f$ and $M_f$ denote the lower and upper hull of $f$, respectively. Then we have*

$$M_f(x) - m_f(x) \leq 2M_f(0) \tag{5}$$

*for all $x \in G$ for which $m_f(x) \neq -\infty$.*

*Proof.* Let $x_0 \in G$ be fixed. It is easy to see that, by the definitions of the upper and the lower hulls of $f$, for each fixed positive $\varepsilon$, there exist neighborhoods $U \in \mathfrak{U}(x_0)$ and $V \in \mathfrak{U}(0)$ such that

$$f(v) \leq M_f(0) + \varepsilon \qquad (v \in V)$$

and

$$f(u - v) \geq m_f(x_0) - \varepsilon \qquad (u \in U,\ v \in V).$$

Obviously, $(x_0 - V) \in \mathfrak{U}(x_0)$, thus, $(x_0 - V) \cap U \in \mathfrak{U}(x_0)$. Therefore, by the definition of the lower hull of $f$, there exists a $u_0 \in (x_0 - V) \cap U$, such that

$$f(u_0) \leq m_f(x_0) + \varepsilon.$$

In this case, $(u_0 + V) \in \mathfrak{U}(x_0)$, so the definition of the upper hull of $f$ implies the existence of a $v_0 \in V$, for which

$$f(u_0 + v_0) \geq M_f(x_0) - \varepsilon.$$

Since $f$ is weakly subquadratic, we get that

$$f(u_0 + v_0) + f(u_0 - v_0) \leq 2f(u_0) + 2f(v_0).$$

Combining the 5 inequalities derived in this proof, we obtain

$$M_f(x_0) - \varepsilon + m_f(x_0) - \varepsilon \leq 2m_f(x_0) + 2\varepsilon + 2M_f(0) + 2\varepsilon,$$

that is,

$$M_f(x_0) - m_f(x_0) \leq 2M_f(0) + 6\varepsilon.$$

Since $x_0 \in G$ and $\varepsilon > 0$ can be chosen arbitrarily, our statement is proved. □

**Remark 4.4.** Obviously, inequality (5) is also valid if we write constants $c \geq 2$ instead of 2 on the right hand side of the inequality. However, as the functions in Example 4 show, $c = 2$ is an "optimal" constant here, i.e., $c = 2$ is the smallest number for which inequality (5) is generally valid.

## 5. Bernstein–Doetsch type theorems

In the following, we present regularity theorems for weakly subquadratic functions. As several Examples in 3.1 show (e.g., 6) the local boundedness (or even boundedness) of a weakly subquadratic functions does not imply its continuity. Therefore, a 'literal' analogue of the Bernstein–Doetsch theorem is not valid here. However, we can prove regularity theorems which are similar to Rosenbaum's results on subadditive functions (cf. [19]).

**Theorem 5.1.** *Let $G$ be a uniquely 2-divisible topological abelian group which is generated by any neighborhood of $0 \in G$. If a weakly subquadratic function $f : G \to \mathbb{R}$ is locally bounded from above at one point in $G$, then it is locally bounded from above at every point in $G$.*

*Proof.* At first we prove that the local boundedness of $f$ from above at a point $x_0 \in G$ implies its local boundedness from above at 0. Due to the local boundedness of $f$ from above at $x_0 \in G$, there exist a neighborhood $U \in \mathcal{U}(x_0)$ and a real number $K$ such that

$$f(u) \le K \qquad (u \in U). \tag{6}$$

Since $G$ is (uniquely) 2-divisible, we may replace $x$ by $\frac{2x_0+v}{2}$ and $y$ by $\frac{2x_0-v}{2}$ in (1). Using the commutativity of $G$, we obtain that

$$f(2x_0) + f(v) \le 2f\left(\frac{2x_0+v}{2}\right) + 2f\left(\frac{2x_0-v}{2}\right) \qquad (v \in G),$$

that is,

$$f(v) \le 2f\left(\frac{2x_0+v}{2}\right) + 2f\left(\frac{2x_0-v}{2}\right) - f(2x_0) \qquad (v \in G). \tag{7}$$

Obviously, the sets $V_1 = 2U - 2x_0$ and $V_2 = -2U + 2x_0$ are open sets containing 0, thus, $V = V_1 \cap V_2$ also has these properties, that is, $V \in \mathcal{U}(0)$. If $v \in V$, by the definition of $V$, we have

$$\frac{2x_0+v}{2} \in U \quad \text{and} \quad \frac{2x_0-v}{2} \in U.$$

Therefore, using (6) and (7), we obtain

$$f(v) \le 2K + 2K - f(2x_0)$$

for $v \in V$, which implies that $f$ is locally bounded from above at 0.
Since $G$ is generated by any neighborhood of 0, Lemma 3.3 implies that the local boundedness of $f$ from above at 0 gives its local boundedness from above everywhere in $G$. □

**Theorem 5.2.** *Let $G$ be a uniquely 2-divisible topological abelian group which is generated by any neighborhood of $0 \in G$ and let $f\colon G \to \mathbb{R}$ be a weakly subquadratic function. If $f$ is locally bounded from above at one point in $G$ and locally bounded from below at one point in $G$ then it is locally bounded at every point in $G$.*

*Proof.* According to Theorem 5.1, the local boundedness of $f$ from above at one point in $G$ implies its local boundedness from above everywhere in $G$. Therefore, we have to prove that the assumptions of the Theorem yield the local boundedness of $f$ from below at each point in $G$. Similarly to the proof of Theorem 5.1, in the first step, we show that the local boundedness of $f$ from below at a point $x_0 \in G$ implies its local boundedness from below at 0. The local boundedness of $f$ from below at $x_0$ means that there exist a $U \in \mathcal{U}(x_0)$ and a real number $K$ such that

$$f(u) \ge K \qquad (u \in U). \tag{8}$$

Writing $x = x_0$ in (1), we obtain

$$f(x_0 + y) + f(x_0 - y) \le 2f(x_0) + 2f(y) \qquad (y \in G),$$

that is,

$$f(x_0 + y) + f(x_0 - y) - 2f(x_0) \le 2f(y) \qquad (y \in G). \tag{9}$$

It is easy to see that, for $H = U - x_0$, we have $H \in \mathcal{U}(0)$. By one of the basic properties of topological groups, there exists a symmetric neighborhood $V \in \mathcal{U}(0)$ such that $V \subseteq H$. Let $V$ be a set satisfying this property. If $y \in V$, then $x_0 + y \in U$ and $x_0 - y \in U$, thus, by (8) and (9)

$$\frac{1}{2}(K + K - 2f(x_0)) \le f(y)$$

for each $y \in V$, which yields the local boundedness of $f$ from below at 0.

Now we prove that the local boundedness of $f$ from below at 0 together with its local boundedness from above at a point in $G$ implies its local boundedness from below at an arbitrary $w_0 \in G$. Since $f$ is locally bounded from below at 0, there exist a neighborhood $S \in \mathcal{U}(0)$ and a constant $c \in \mathbb{R}$ such that

$$f(s) \geq c \qquad (s \in S). \tag{10}$$

Since $f$ is locally bounded from above at one point in $G$, by Theorem 5.1, it is locally bounded from above everywhere in $G$, specifically, it has this property at $w_0$. Therefore, there exist a neighborhood $T \in \mathcal{U}(w_0)$ and a real constant $C$ such that

$$f(t) \leq C \qquad (t \in T). \tag{11}$$

Let us consider the sets $R = 2T - 2w_0$ and $H = S \cap R$. It is obvious from this construction that $H$ is a neighborhood of 0. Let $V \in \mathcal{U}(0)$ be a symmetric subset of $H$. Since $V \subseteq S$, property (10) implies that

$$f(v) \geq c \qquad (v \in V). \tag{12}$$

Since $G$ is uniquely 2-divisible, we may consider the set $W = \frac{V}{2} + w_0$, which is, obviously, a neighborhood of $w_0$ with the property $W \subseteq T$. Thus, by (11), we have

$$f(w) \leq C \qquad (w \in W). \tag{13}$$

The definition of $W$ also gives that if $v \in V$, then $\frac{2w_0 - v}{2} \in W$. Therefore, using inequality (13), we get

$$f\left(\frac{2w_0 - v}{2}\right) \leq C \qquad (v \in V). \tag{14}$$

Since $f$ is weakly subquadratic and $G$ is commutative, writing $x = \frac{2w_0+v}{2}$ and $y = \frac{2w_0-v}{2}$ in (1), we obtain

$$f(2w_0) + f(v) \leq 2f\left(\frac{2w_0 + v}{2}\right) + 2f\left(\frac{2w_0 - v}{2}\right) \qquad (v \in G),$$

that is,

$$f(2w_0) + f(v) - 2f\left(\frac{2w_0 - v}{2}\right) \leq 2f\left(\frac{2w_0 + v}{2}\right) \qquad (v \in G). \tag{15}$$

Inequalities (12), (14) and (15) imply

$$\frac{1}{2}\left(f(2w_0) + c - 2C\right) \leq f\left(\frac{2w_0 + v}{2}\right) \qquad (v \in V).$$

Finally, by the definition of $W$, for an arbitrary $w \in W$, there exists a $v \in V$ such that $w = \frac{2w_0+v}{2}$. Therefore, the last inequality above gives

$$\frac{1}{2}\left(f(2w_0) + c - 2C\right) \leq f(w) \qquad (w \in W)$$

which yields the local boundedness of $f$ at $w_0$. □

**Theorem 5.3.** *Let $G$ be a uniquely 2-divisible topological abelian group which is generated by any neighborhood of $0 \in G$ and let $f : G \to \mathbb{R}$ be a weakly subquadratic function. If $f$ is continuous at 0 and $f(0) = 0$, then $f$ is continuous everywhere in $G$.*

*Proof.* The continuity of $f$ at 0 implies that it is locally bounded at 0, therefore, by Theorem 5.2, $f$ is locally bounded from below everywhere in $G$, which yields that $m_f(x) \neq -\infty$ for every $x \in G$. The property $f(0) = 0$ and the continuity of $f$ at 0 imply that $M_f(0) \leq 0$. Thus, Theorem 4.3 gives that $m_f(x) = M_f(x)$ for all $x \in G$, which yields the statement. □

**Remark 5.4.** It is remarkable that if we omit one of the assumptions for $f$ in the theorem above, it does not remain true. In fact, according to Example 5, there exist weakly subquadratic functions $f : G \to R$, which are continuous at 0 but not continuous on the whole $G$. Obviously, $f(0) = 0$ does not imply any continuity properties for $f$. Furthermore, the assumption of the continuity of the function $f$ above at a point other than 0 does not imply its continuity everywhere (cf., e.g., Example 3).

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