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ON THE CONNECTIONS OF SUB-FINSLERIAN GEOMETRY

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Abstract. A sub-Finslerian manifold is, roughly speaking, a manifold endowed with a Finsler type metric which is defined on a k-dimensional smooth distribution only, not on the whole tangent manifold. Our purpose is to construct a generalized non-linear connection for a sub-Finslerian manifold, called $\mathcal{L}$-connection by the Legendre transformation which characterizes normal extremals of a sub-Finsler structure as geodesics of this connection. We also wish to investigate some of its properties like normal, adapted, partial and metrical.

1. Introduction

A sub-Riemannian geometry is a smooth manifold M of dimension n endowed with a subbundle $\mathcal{H}$ of the tangent bundle TM (i.e. a smooth distribution of constant rank) and a Riemannian metric h on the distribution $\mathcal{H}$. Sub-Riemannian structures and related concepts arise naturally in many areas of pure mathematics (algebra, geometry, analysis) and applied mathematics (mechanics, control theory, mathematical physics), as well as in applications (e.g. robotics), that was the reason this subject has received much attention in recent years. Moreover, sub-Riemannian geometry has been discussed extensively in the literature, (see for instance [10], [9], [7]).

Lopez and Martínez introduced the notion of sub-Finsler geometry first in 2000 as a natural generalization of sub-Riemannian geometry ([8]), one motivation for the generalization comes from control theory. Here, instead of the sub-Riemannian metric h, a more general, called sub-Finslerian metric $F : \mathcal{H} \to [0, \infty)$ is considered, where $\mathcal{H}$ is a subbundle of the tangent bundle on M as above. As in Finsler geometry we suppose that F is regular, positively homogeneous, and strictly convex. Practically it gives a norm in all horizontal subspaces $\mathcal{H}_x, x \in M$.

In this paper, we develop some steps to study the connections of sub-Finslerian geometry on a manifold, particularly, a generalized non-linear connection over a bundle mapping will be constructed on the cotangent bundle of $\mathcal{H}$ by the Legendre transformation. For the construction we apply the extension of F to the whole tangent manifold, being compatible with a chosen projection operator. This step is different from the sub-Riemannian case of Langerock ([7]), where an arbitrary Riemannian extension is taken, and then the orthogonal projection played the same role as ours. Using Barthel (canonical) non-linear connection on the manifold we can construct a generalized non-linear $\mathcal{L}$-connection. Its properties such as normal, adapted, partial and metrical can be analyzed in an analogue manner. In the sub-Riemannian case an abnormal extremal is characterized by the existence of a section in the annihilator bundle parallel with respect to an adapted connection,

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while normal extremals tangent to the distribution are just geodesics of the extended Riemannian metric.

In Section 2, we define the sub-Finsler metrics and introduce some notions and some basic properties of sub-Finsler manifolds. Here we provide the famous example of the Heisenberg group that meets the sub-Finsler property. Also, we shall take a look at the horizontal path-connectedness of the manifold, especially for the Chow's theorem, [4]. Stepping to the analytical considerations in Section 3 we give some basics of generalized connections in sub-Riemannian geometry, with which the normal and abnormal extremal can be characterized in the sub-Riemannian case (see the local version in [10], and the coordinate free version in [7]). Afterwards, in Section 4 we give a brief description of the Legendre transformation, we show the relationship between the sub-Finsler geometry with Lagrange spaces as well as the Hamiltonian spaces. Our main results are given in Section 5. We introduce the symmetric bracket associated to sub-Finsler geometry, and the symmetric product of an $\mathcal{L}$-connection. They are coincident if and only if the $\mathcal{L}$-connection is normal. With the help of the generalization of the Bott connection for involutive distribution we can characterize $\mathcal{H}$-adapted and normal connections, resp. In a forthcoming paper we plan to figure out how these properties of an $\mathcal{L}$-connection may characterize the normal and abnormal extremals of the sub-Finslerian structure.

2. Definitions and some properties of sub-Finsler manifolds

In this section we define the sub-Finsler metrics and we introduce some notions which will play an essential role in this paper.

We denote the set of smooth vector fields by $\mathfrak{X}(M)$ and for its dual by $\mathfrak{X}^*(M)$.

Definition 1. Let M be an n–dimensional manifold. A *sub-Finsler manifold* on the manifold M consists of a distribution of rank k, which is a (vector) subbundle $\mathcal{H} \subset TM$ of the tangent bundle of M and a function $F : \widetilde{\mathcal{H}} \to \mathbb{R}$, where $\widetilde{\mathcal{H}} = \mathcal{H}\backslash\{0\}$, called a *sub-Finsler metric*, which satisfies the following properties:

- $F(v) > 0$ for all $v \in \widetilde{\mathcal{H}}$;
- F is C^∞ on $\widetilde{\mathcal{H}}$;
- $F(\lambda v) = \lambda F(v)$ for all $v \in \widetilde{\mathcal{H}}$ and $\lambda \in \mathbb{R}_+$;
- The Hessian matrix of F^2 with respect to the vector variables is positive definite.

Let $\mathcal{H}_x \subset T_xM$ be the fiber over $x \in M$. The last condition means that the matrix $\frac{\partial^2 F^2}{\partial v^i \partial v^j}(v)$ is positive definite for all $v = (v^1, \ldots, v^k) \in \mathcal{H}_x \subset T_xM$. Equivalently, the corresponding indicatrix

$$I_x = \{v \,|\, v \in \mathcal{H}_x, F(v) = 1\}$$

is strictly convex.

As in the sub-Riemannian case we call $\mathcal{H}$ the *horizontal distribution.* A curve $\sigma : [0, 1] \to M$ is called *horizontal*, or *admissible* if $\dot\sigma(t) \in \mathcal{H}_{\sigma(t)}$ for all $t \in [0, 1]$, that is $\sigma(t)$ is tangent to $\mathcal{H}$. The length of a smooth horizontal curve σ is defined as usual by

$$\ell(\sigma) = \int_0^1 F(\dot\sigma(t))dt.$$

The length induces a sub-Finslerian distance $d(x_0, x_1)$ between two points x_0 and x_1 as in Finsler geometry:

$$d(x_0, x_1) = \inf \ell(\sigma)$$

where we consider the infimum over all smooth horizontal curves joining x_0 and x_1. The distance is infinite if there is no such a horizontal curve between x_0 and x_1.

The length functional will be studied on the absolutely continuous curves, namely on the curves $\sigma : [0,1] \to M$ which have derivative for almost all $t \in [0,1]$, and the components of the derivative $\dot{\sigma}$ are measurable curves. This property is independent on the coordinate maps. An absolutely continuous curve σ is horizontal if its tangent vector field $\dot{\sigma}(t)$ lies in $\mathcal{H}_{\sigma(t)}$ whenever it exists. The length of an absolutely continuous curve exists, but it can be infinite. The distance $d(x_0, x_1)$ between two points is defined in the same way as for the continuous curves, the infimum being taken on the class of absolutely continuous curves joining x_0 and x_1.

Definition 2. An absolutely continuous horizontal curve between x_0 and x_1 which realizes the distance $d(x_0, x_1)$ is called a *minimizing* geodesic between x_0 and x_1.

Naturally arises the question: given two points x_0 and x_1 in a sub-Finsler manifold, is there a horizontal curve that joins x_0 and x_1?

The answer is not always positive. In the case of an involutive distribution $\mathcal{H}$ the Frobenius theorem asserts that the set of horizontal paths through x_0 form a smooth immersed submanifold, the leaf through x_0, of dimension equal to the rank k of distribution $\mathcal{H}$. In this case, if $\mathcal{H}$ is involutive and x_1 is not contained in the leaf through x_0, there is no horizontal curve joining x_0 and x_1.

A positive answer is given by Chow's theorem in the case of bracket generating distributions, which are the "contrary" of the involutive distributions.

Definition 3. [9] A distribution $\mathcal{H}$ is said to be a *bracket generating* if any local frame X_i of $\mathcal{H}$, together with all of its iterated Lie brackets spans the whole tangent bundle TM.

Theorem 4. *(Chow's theorem)*[4, 9]. *If $\mathcal{H}$ is a bracket generating distribution on a connected manifold M then any two points of $\mathcal{H}$ can be joined by a horizontal path.*

Example 1. The most interesting well known example in $\mathbb{R}^3$ is the sub-Finslerian structure of the Heisenberg group where the group is given as follows:

$$\mathbb{H} = \left\{ \begin{pmatrix} 1 & x_2 & x_3 + \frac{1}{2}x_1x_2 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} : (x_1, x_2, x_3) \in \mathbb{R}^3 \right\}.$$

Then

$$\mathcal{H} = \{dx_3 + \frac{1}{2}(x_1 dx_2 - x_2 dx_1)\}^{\perp}$$

is a rank two distribution, a contact structure on $\mathbb{H}$.

By an appropriate sub-Riemannian metric on $(\mathbb{H}, \mathcal{H})$, one can assume that

$$X_1 = \frac{\partial}{\partial x_1} + \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2} - \frac{x_1}{2}\frac{\partial}{\partial x_3}$$

are orthonormal vectors. It is clear that the frame $\{X_1, X_2\}$ is left invariant and gives a global basis of the horizontal bundle.

Let Σ_1 be the unit circle bundle for this sub-Riemannian structure on $(\mathbb{H}, \mathcal{H})$, and define a coordinate θ on Σ_1 by the condition that

$$u = (\cos\theta)X_1 + (\sin\theta)X_2 \quad \text{for } u \in \Sigma_1.$$

Then any scaling function $r(\theta)$ which depends on θ alone defines a homogeneous sub-Finslerian metric on $\mathbb{H}$:

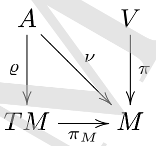
$$F(u) = r(\theta)g^{1/2}(u,u) \quad \text{for any } u \in \mathcal{H},$$

where g denotes the sub-Riemannian metric above, and θ is the angle of $\dfrac{u}{g^{1/2}(u,u)} \in \Sigma_1$ with respect to X_1-axis. The indicatrix bundle of this metric is given as follows: $\Sigma = \{r(\theta)^{-1}u \,|\, u \in \Sigma_1\}$. For more details we refer the reader to [5].

3. Generalized connections in sub-Riemannian geometry

Langerock in [7], gave a coordinate free approach to sub-Riemannian geometry, based on a notion of generalized connection over a bundle map: The main idea was to consider the natural notion of connection on a Lie algebroid and extend it to bundles with an anchor map. It was shown that a generalized connection is associated with a sub-Riemannian structure. In this approach one can study length-minimizing curves and, in particular, give necessary and sufficient conditions for the existence of abnormal extremals.

Let (V,π,M) be a vector bundle, and (A,ν,M) the anchor bundle over the same base manifold M, while there is given a vector bundle map $\varrho : A \longrightarrow TM$, called the anchor map such the diagram is commutative:



Definition 5. The map $D : Sec\,\nu \times Sec\,\pi \longrightarrow Sec\,\pi$, $(s,\sigma) \mapsto D_s\sigma$ is called *a generalized linear connection over the anchor map* ϱ if

- $\mathbb{R}$-linear in s and σ;
- $\mathcal{F}(M)$-linear in s, where $\mathcal{F}(M)$ is the set of (real valued) smooth functions on a manifold M;
- $D_s(f\sigma) = fD_s\sigma + \varrho(s)(f)\sigma$, for any $f \in \mathcal{F}(M)$ and for all $s \in Sec\,\nu$ and $\varrho \in Sec\,\pi$.

It follows from the Pontryagin maximum principle that sub-Riemannian minimizers fall into two non-disjoint classes, the "normal" and the "abnormal" extremals. In 1986 Strichartz [10] characterized first the normal and abnormal extremals of the sub-Riemannian structure. "Normal" extremals satisfy differential equations similar to the geodesic equations in Riemannian geometry, and they are smooth. On the other hand, abnormal extremals which happen to be minimizers were not known to actually exist until the first such example was given by R. Montgomery in 1994 (see [9]).

The notion of generalized connections can be applied to sub-Riemannian geometry with the choices: $V = A = T^*M$, $\varrho = g : T^*M \longrightarrow TM$. Then it was proved that there exists a normal $\mathcal{H}$-adapted linear g-connection D of T^*M.

The main theorems of ([7]) imply that normal extremals are the base curves of autoparallel curves of D, and the abnormal extremals are the base curves of parallel sections in the annihilator bundle $\mathcal{H}^0$. The annihilator bundle of the distribution

$\mathcal{H}$ is the set of all covectors which annihilate the vectors in $\mathcal{H}_x$ i.e.

$$\mathcal{H}_x^0 = \{\alpha \in T_x^*M : \alpha(v) = 0 \; \forall \; v \in \mathcal{H}_x\}.$$

4. Legendre transformation of sub-Finslerian geometry

The sub-Lagrange space determined by F is given in the following way:

$$L = \frac{1}{2}F^2.$$

The fiber derivative of L defines the map

$$\mathcal{L}_L : \mathcal{H} \subset TM \longrightarrow \mathcal{H}^* \subset T^*M,$$

$$\mathcal{L}_L(v)(w) = \frac{d}{dt}L_x(v + tw) \text{ such that } v, w \in \mathcal{H}_x,$$

which is known in the literature as the *Legendre transformation* (see [1], [2]). We use the Legendre transformation to carry over the geometrical objects of a sub-Lagrange space from $\mathcal{H}$ onto $\mathcal{H}^*$. Let us denote by (x^i) the coordinates in a neighborhood $U \subset M$ with (x^i, v^a) in $\mathcal{H}|_U \subset TM$ and (x^i, p_a) in $\mathcal{H}^*|_U \subset T^*M$, respectively, where $i = 1, \dots, n$ while $a = 1, \dots, k$. Then the relation of the distribution $\mathcal{H}$ of tangent bundle TM and the distribution $\mathcal{H}^*$ of cotangent bundle T^*M is given by Legendre transformation in local coordinates as follows

$$\mathcal{L}_L(x^i, v^a) = (x^i, \frac{\partial L}{\partial v^a}).$$

Considering the sub-Lagrange metric

$$L : \mathcal{H} \longrightarrow \mathbb{R},$$

the sub-Hamiltonian is given by

$$\eta = \iota_{\mathcal{L}_L^{-1}} - L \circ \mathcal{L}_L^{-1},$$

where $\iota_{\mathcal{L}_L^{-1}}(p) = p(\mathcal{L}_L^{-1})$ for any $p \in \mathcal{H}^*$, or locally,

$$\eta(x^i, p_a) = v^a p_a - L(x^i, v^a), \text{ where } p_a = \frac{\partial L}{\partial v^a}.$$

Introducing $g^{ab}(x,p) = \frac{1}{2}\frac{\partial^2 \eta}{\partial p_a \partial p_b}(x,p)$, we have the form

$$\eta(x,p) = \frac{1}{2}g^{ab}(x,p)p_a p_b.$$

Secondly, for fiber derivative of η, we define the Legendre transformation of the sub-Hamiltonian η in the following way

$$\mathcal{L}_\eta : \mathcal{H}^* \subset T^*M \longrightarrow \mathcal{H} \subset TM.$$

For any $\alpha, \beta \in \mathcal{H}_x^*$, it holds

$$\beta(\mathcal{L}_\eta(\alpha)) = \frac{d}{dt}\eta_x(\alpha + t\beta),$$

which relates the distribution $\mathcal{H}^*$ of the cotangent bundle and the distribution $\mathcal{H}$ of the tangent bundle, locally by the next expression

$$\mathcal{L}_\eta(x^i, p_a) = (x^i, \frac{\partial \eta}{\partial p_a}).$$

We say η is a Hamiltonian if and only if $\mathcal{L}_\eta$ is local diffeormorphism ([1]).

It is clear that $\mathcal{L}_L$ and $\mathcal{L}_\eta$ are inverses of each other, so we have the following relations:

$$\mathcal{L}_\eta \circ \mathcal{L}_L = 1_{\mathcal{H}}, \qquad \mathcal{L}_L \circ \mathcal{L}_\eta = 1_{\mathcal{H}^*},$$
$$\frac{\partial \eta}{\partial x^i} = -\frac{\partial L}{\partial x^i}, \qquad g^{ab} g_{bc} = \delta^a_c,$$
$$\frac{\partial^2 L}{\partial x^i \partial v^b} = -\frac{\partial^2 \eta}{\partial x^i \partial p_a} g_{ab}, \qquad \text{where} \quad g_{ab} = \frac{\partial^2 L}{\partial v^a \partial v^b}.$$

We recall here the basic relations of non-linear connections of Lagrange and Hamiltonian spaces. For a Lagrange space, there exists a canonical non-linear connection given by (see [6])

$$N^i_j = \frac{1}{2}\frac{\partial G^i}{\partial v^j}; \quad G^i = g^{ij}\left(\frac{\partial^2 L}{\partial v^j \partial x^k} v^k - \frac{\partial L}{\partial x^j}\right). \tag{1}$$

In the homogeneous case, i.e. for Finsler manifolds this connection is also called as the Barthel non-linear connection. For Hamiltonian spaces, the non-linear connection is the image of the non-linear connection of the Finsler spaces (Lagrangian spaces) by Legendre transformation.

A Hamilton space (M, η) for which the Hamiltonian is 2-homogeneous with respect to p_i is called a *Cartan space*. Since η is 2-homogeneous, we have

$$\frac{\partial \eta}{\partial p_i} = g^{ij} p_j, \quad \frac{\partial \eta}{\partial x^k} = \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k} p_i p_j, \quad \frac{\partial^2 \eta}{\partial p_i \partial x^k} = p_j \frac{\partial g^{ij}}{\partial x^k}.$$

Let us consider Γ^i_{jk} the Christoffel symbols (for more details see [2, p. 34]) of g_{ij}:

$$\Gamma^i_{jk} = \frac{1}{2} g^{ih}\left(\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right),$$
$$\Gamma^0_{jk} = p_i \Gamma^i_{jk}, \quad \Gamma^0_{j0} = \frac{1}{2}\Gamma^0_{jk} g^{kl} p_l.$$

Then the coefficients for the non-linear connection of a Cartan space are given by

$$N_{ij} = \Gamma^0_{ij} - \Gamma^0_{k0}\frac{\partial g_{jh}}{\partial p_k}.$$

This non-linear connection can be obtained as the Legendre transformation of the canonical non-linear connection of a Finsler space. Moreover, the canonical non-linear connection of the Hamiltonian is given by [6]

$$N_{ij} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial p_k}\frac{\partial \eta}{\partial x^k} - \frac{\partial g_{ij}}{\partial x^k}\frac{\partial \eta}{\partial p_k}\right) - \frac{1}{2}\left(g_{ik}\frac{\partial^2 \eta}{\partial p_k \partial x^j} + g_{jk}\frac{\partial^2 \eta}{\partial p_k \partial x^i}\right).$$

Now, one can get the coefficients for the above non-linear connection by contraction both sides by p^j(i.e. $p^j = g^{jl} p_l$) and consequently, we have

$$N_{ij} p^j = \frac{1}{2}\frac{\partial g^{kl}}{\partial x^i} p_k p_l \tag{2}$$
$$N_{ij} p^j = \Gamma^{lj}_i p_l p_j. \tag{3}$$

For the subsequent theorem, let us clarify some facts which will play an important role later on.

Definition 6. Let $\xi(t) = (x(t), p(t))$ be a solution of the sub-Hamiltonian system

$$\dot{x}^i = \frac{\partial \eta}{\partial p_i}(x, p),$$
$$\dot{p}_i = -\frac{\partial \eta}{\partial x^i}(x, p), \quad i = 1, \ldots, n.$$

Then its projection $x(t)$ to M is called a *normal extremal*.

One can see that every sufficiently short subarc of the normal extremal $x(t)$ is a minimizer sub-Finslerian geodesic. This subarc is the unique minimizer joining its end points (see [9]). In the sub-Finslerian geometry, not all the sub-Finslerian geodesics are normal (contrary to the Finsler geometry). This is due to the fact that some sub-Finslerian minimizing geodesics might not solve the sub-Hamiltonian system. Those minimizer that are not normal extremal called *singular* or *abnormal* extremal (see [9]). Even in sub-Finslerian case, Pontryagin's maximum principle implies that every minimizer of the arc length of admissible curve is a normal or abnormal extremal. On the other hand, there exists a unique *sub-Hamiltonian vector field* on $\mathcal{H}^*$, denoted by $\vec{H}$, given by

$$\vec{H} = \frac{\partial \eta}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial \eta}{\partial x^i}\frac{\partial}{\partial p_i}.$$

The above vector field determined by the relation

$$\iota_{\vec{H}}\omega = -d\eta,$$

such that ω is the canonical symplectic form on $\mathcal{H}^*$.

Definition 7. A linear transformation $P : TM \longrightarrow \mathcal{H} \subset TM$ is called a *projection operator* onto $\mathcal{H}$, if $P^2 = P$ and $\mathcal{H} = Im(P)$. If P is a projection operator onto $\mathcal{H}$ then

$$TM = Ker(P) \oplus Im(P), \quad \text{and} \quad P = 0_{Ker(P)} \oplus I_{Im(P)},$$

such that $\mathcal{H}^{\perp} := Ker(P)$. Its complement projection is $P^c = id - P$.

For the sake of notation, we shall use the symbols by P^* to denote the projection on T^*M and $(P^*)^c$ to denote its complement, respectively, namely,

$$P^* : T^*M \longrightarrow (\mathcal{H}^{\perp})^0 \subset T^*M,$$

where $P^*(\alpha)(v) = \alpha(P(v))$ for all $\alpha \in T^*M$, $P(v) \in \mathcal{H}$ and v is a vector in TM. Next the projection complement which is given by

$$(P^*)^c : T^*M \longrightarrow \mathcal{H}^0 \subset T^*M,$$

satisfies the condition $(P^*)^c = id - P^*$ such that for all $(P^*)^c(\alpha) \in \mathcal{H}^0$ we have

$$(P^*)^c(\alpha)(v) = \alpha(v - P(v)) = 0, \text{ if } v \in \mathcal{H}.$$

Furthermore,

$$P^*(\alpha) + (P^*)^c(\alpha) = \alpha, \; P^* + (P^*)^c = id_{T^*M}.$$

Now T^*M can be written as the direct sum of $(\mathcal{H}^{\perp})^0$ and $\mathcal{H}^0$.

After all, one can imagine a picture from above through which one has a complete conception of generating a Finsler metric from sub-Finsler one by using the upcoming technique. Starting with a sub-Finsler metric F in the subbundle $\mathcal{H}$, we choose an arbitrary $\widetilde{F}$ which is defined in the orthogonal complement $\mathcal{H}^{\perp}$. Now if we take the sum of both metrics we will obtain a *Finsler metric* $\hat{F}$ in TM, specifically

$$\hat{F}^2(v) = F^2(P(v)) + \widetilde{F}^2(P^c(v)) \text{ for all } v \in TM.$$

Comparing the Legendre transformations of the sub-Finsler metric F and the extended Finsler metric $\hat{F}$ one can easily see that the following relations hold for the natural injection $i : \mathcal{H} \longrightarrow TM$,

$$\begin{aligned} \mathcal{L}_L = \mathcal{L}_{\hat{L}}|_{\mathcal{H}}, &\quad P^* \circ \mathcal{L}_L = \mathcal{L}_{\hat{L}} \circ i, \\ \mathcal{L}_\eta = \mathcal{L}_{\hat{\eta}}|_{\mathcal{H}^*}, &\quad \mathcal{L}_\eta \circ i = P^* \circ \mathcal{L}_{\hat{\eta}}. \end{aligned}$$

5. Connections for sub-Finslerian geometry

Suppose that $(M, \mathcal{H}, F)$ is a sub-Finsler geometry such that M is a smooth manifold of dimension n equipped with distribution $\mathcal{H} \subset TM$ and F is a sub-Finsler metric on $\mathcal{H}$. A distribution is also completely characterized by its annihilator, i.e. giving $\mathcal{H}$ is equivalent to specifying the subbundle $\mathcal{H}^0$ of the cotangent T^*M whose fibre over $x \in M$ consists of all covector at x which annihilates all vectors in the subspace $\mathcal{H}_x$ of T_xM. With a sub-Finsler structure one can associate a smooth mapping, defined by

$$E : T^*M \longrightarrow TM, \quad E(\alpha_x) = i(\mathcal{L}_\eta(i^*(\alpha_x))) \in TM, \tag{4}$$

where $i^* : T^*M \longrightarrow \mathcal{H}^*$ is the adjoint mapping of i, i.e. for any $\alpha_x \in T_x^*M$, $i^*(\alpha_x)$ is determined by

$$\langle X_x, i^*(\alpha_x)\rangle = \langle i(X_x), \alpha_x\rangle \text{ for all } X_x \in \mathcal{H}_x,$$

such that $\langle v, \alpha\rangle := \alpha(v)$ for all $v \in \mathcal{H}, \alpha \in \mathcal{H}^*$.

Clearly, E is a bundle mapping whose image set is precisely the subbundle $\mathcal{H}$ of TM and whose kernel is the annihilator $\mathcal{H}^0$ of $\mathcal{H}$. To simplify notations we shall often identify an arbitrary vector in $\mathcal{H}$ with its image in TM under i and smooth section of $\mathcal{H}$ (i.e. element of $\Gamma(\mathcal{H})$) will often be regarded as a vector field on M. Also, it is convenient to associate E with a section $\bar{E}$ of $TM \otimes TM \longrightarrow M$ according to

$$\begin{aligned} \bar{E}(x)(\alpha_x, \beta_x) &= \langle E(\alpha_x), \beta_x\rangle \text{ for all } x \in M \text{ and } \alpha_x, \beta_x \in T_x^*M \\ &= \langle \mathcal{L}_\eta(i^*(\alpha_x)), i^*(\beta_x)\rangle. \end{aligned}$$

Recall that a curve $\sigma : [0, 1] \to M$ is called *horizontal*, or *admissible* if $\dot{\sigma}(t) \in \mathcal{H}_{\sigma(t)}$ for all $t \in [0, 1]$. Let us consider a curve $\xi : [0, 1] \longrightarrow T^*M$ in the cotangent bundle and put $\sigma = \pi_M \circ \xi$, with $\pi_M : T^*M \longrightarrow M$ the natural cotangent bundle projection. Then, we say that ξ is a *E-admissible* if $E(\xi(t)) = \dot{\sigma}(t)$, for all $t \in [0, 1]$.

One can check the invariance of $\bar{E}$ under the projection for all $\alpha \in \mathfrak{X}^*(M)$, more explicitly

$$\begin{aligned} (P^* \bar{E})(\alpha, \alpha) =& \langle E(\alpha(P)), \alpha(P)\rangle \\ =& \langle E(P^*(\alpha)), P^*(\alpha)\rangle \\ =& \langle E(\alpha - (P^*)^c(\alpha)), \alpha - (P^*)^c(\alpha)\rangle \\ =& \langle E(\alpha), \alpha\rangle - \langle E((P^*)^c(\alpha)), \alpha\rangle - \langle E(\alpha), (P^*)^c(\alpha)\rangle \\ &+ \langle E((P^*)^c(\alpha)), (P^*)^c(\alpha)\rangle \\ =& \langle E(\alpha), \alpha\rangle = \bar{E}(\alpha, \alpha), \qquad \text{for all } (P^*)^c(\alpha) \in \Gamma(\mathcal{H}^0). \end{aligned} \tag{5}$$

In order to construct a connection for sub-Finsler geometry, we need to introduce the concept of generalized non-linear connection over an anchor map in the same vector bundle setting as in Section 3:

Definition 8. The map $\nabla : Sec\,\nu \times Sec\,\pi \longrightarrow Sec\,\pi, \quad (s,\sigma) \mapsto \nabla_s\sigma$ is called *a generalized non-linear connection over the anchor map* ϱ if

- $\mathbb{R}$-linear in s and σ;
- additive in s;
- $\nabla_s(f\sigma) = f\nabla_s\sigma + \varrho(s)(f)\sigma$.

This will be applied to sub-Finslerian geometry with the following choices: $V = A = T^*M, \varrho = E : T^*M \longrightarrow TM$.

Definition 9. An $\mathcal{L}$-*connection* ∇ on a sub-Finsler manifold is a generalized non-linear connection over the induced mapping $E : T^*M \longrightarrow TM$ constructed by Legendre transformation $\mathcal{L}_\eta : \mathcal{H}^* \longrightarrow \mathcal{H}$ by (4).

Definition 10. The *symmetric bracket* associated to sub-Finsler geometry is mapping

$$\{.,.\} : \mathfrak{X}^*(M) \times \mathfrak{X}^*(M) \longrightarrow \mathfrak{X}^*(M),$$

$$\{\alpha,\beta\} = \bar{\mathcal{L}}_{E(\alpha)}\beta + \bar{\mathcal{L}}_{E(\beta)}\alpha - d(\bar{E}(\alpha,\beta)) - d(\bar{E}(\beta,\alpha)),$$

where $\bar{\mathcal{L}}_X$ is the *Lie derivative* with respect to $X \in \mathfrak{X}(M)$.

The expression of symmetric bracket can be simplified by using Cartan's magic formula as follows:

$$\bar{\mathcal{L}}_{E(\alpha)}\alpha = \iota_{E(\alpha)}(d\alpha) + d(\iota_{E(\alpha)}\alpha),$$

and $\bar{E}(\alpha,\alpha) = \langle E(\alpha),\alpha\rangle = \alpha(E(\alpha)) = \iota_{E(\alpha)}\alpha$. Moreover

$$d(\bar{E}(\alpha,\alpha)) = d(\iota_{E(\alpha)}\alpha).$$

Then, the symmetric bracket can be written as

$$\frac{1}{2}\{\alpha,\alpha\} = \bar{\mathcal{L}}_{E(\alpha)}\alpha - d(\bar{E}(\alpha,\alpha)) = \iota_{E(\alpha)}(d\alpha) + d(\iota_{E(\alpha)}\alpha) - d(\iota_{E(\alpha)}\alpha) = \iota_{E(\alpha)}(d\alpha).$$

The following proposition extends a result from [7] to the sub-Finsler case, it shows some properties of the above bracket, the first of which justifies the denomination "symmetric bracket". We are going to prove the fourth property while the others are straightforward and left to the reader.

Proposition 11. *For the symmetric bracket the following properties are satisfied for any* $\alpha,\beta \in \mathfrak{X}^*(M)$*:*

1. $\{\alpha,\beta\} = \{\beta,\alpha\}$;
2. *the bracket is* $\mathbb{R}$*-linear;*
3. $\{f\alpha,\beta\} = E(\beta)(f)\alpha + f\{\alpha,\beta\}$;
4. $\{\alpha,\gamma\} = \bar{\mathcal{L}}_{E(\alpha)}\gamma$, *for any* $\gamma \in \Gamma(\mathcal{H}^0)$ *and* $\{\alpha,\gamma\} = 0$ *if both* α *and* γ *belong to* $\Gamma(\mathcal{H}^0)$.

Proof.

(4) For the first case, $\{\alpha,\gamma\} = \bar{\mathcal{L}}_{E(\alpha)}\gamma$, for any $\gamma \in \Gamma(\mathcal{H}^0)$ and $\alpha \in \mathfrak{X}^*(M)$:
By the definition of the symmetric bracket, we have

$$\begin{aligned}\{\alpha,\gamma\} =&\bar{\mathcal{L}}_{E(\alpha)}\gamma + \bar{\mathcal{L}}_{E(\gamma)}\alpha - d(\bar{E}(\alpha,\gamma)) - d(\bar{E}(\gamma,\alpha))\\ =&\bar{\mathcal{L}}_{E(\alpha)}\gamma + \bar{\mathcal{L}}_{E(\gamma)}\alpha - d(\langle E(\alpha),\gamma\rangle) - d(\langle E(\gamma),\alpha\rangle)\\ =&\bar{\mathcal{L}}_{E(\alpha)}\gamma + \bar{\mathcal{L}}_{E(\gamma)}\alpha - d(\gamma(E(\alpha))) - d(\alpha(E(\gamma)))\\ =&\bar{\mathcal{L}}_{E(\alpha)}\gamma.\end{aligned} \tag{6}$$

such that $E(\gamma)=0, \gamma(E(\alpha))=0$ for any $\gamma \in \Gamma(\mathcal{H}^0)$ and $\alpha \in \mathfrak{X}^*(M)$.

The second case, it is obvious by the same way we have $\{\alpha,\gamma\} = 0$ for any $\alpha,\gamma \in \Gamma(\mathcal{H}^0)$. □

The first three properties justify the next definition.

Definition 12. An $\mathcal{L}$-connection ∇ on sub-Finsler manifold $(M,\mathcal{H},F)$ is said to be a *normal connection* if the associated symmetric product equals the symmetric bracket, i.e. if $\langle\alpha,\beta\rangle_\nabla = \{\alpha,\beta\}$ holds for all $\alpha,\beta \in \mathfrak{X}^*(M)$, where the symmetric product of ∇ is given by

$$\langle\alpha,\beta\rangle_\nabla = \nabla_\alpha\beta + \nabla_\beta\alpha, \text{ for all } \alpha,\beta \in \mathfrak{X}^*(M).$$

One can associate a mapping δ to any sub-Finsler manifold $(M,\mathcal{H},F)$ according to

$$\delta : \Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}^0) \longrightarrow \mathfrak{X}^*(M), \quad (X,\gamma) \mapsto \delta_X\gamma = i_X d\gamma.$$

It is clear that δ generalizes the Bott connection of involutive distribution to our general case of non-involutive distribution ([3]).

Definition 13. An $\mathcal{L}$-connection ∇ on sub-Finsler space $(M,\mathcal{H},F)$ is said to be *adapted* to the bundle $\mathcal{H}$ (shortly $\mathcal{H}$-adapted) if $\nabla_\alpha\gamma = \delta_{E(\alpha)}\gamma$ for all $\alpha \in \mathfrak{X}^*(M)$ and $\gamma \in \Gamma(\mathcal{H}^0)$.

We define the *Barthel non-linear connection* $\overline{\nabla}^B$ of the cotangent bundle as follows

$$\overline{\nabla}^B_X\alpha(Y) = X(\alpha(Y)) - \alpha(\nabla^B_X Y),$$

where the Berwald connection ∇^B on the tangent bundle was locally given in (1). Recall that Barthel nonlinear connection plays the same role in the positivity homogeneous case as the Levi-Civita connection does in Riemannian geometry.

Definition 14. We call the E-admissible curve $\xi : [0,1] \longrightarrow T^*M$ is an *auto-parallel* curve with regard to an $\mathcal{L}$-connection ∇ if it satisfies $\nabla_\xi\xi(t) = 0$ for all $t \in [0,1]$. The base curve $\sigma = \pi_M \circ \xi$ of E-admissible auto-parallel ξ is then called a geodesic of ∇.

In coordinates, an auto-parallel curve $\xi(t) = (x^i(t), p_i(t))$ satisfies the equations

$$\dot{x}^i(t) = g^{ij}(x(t),p(t))p_j(t), \qquad \dot{p}_i(t) = -\Gamma_i^{jk}(x(t),p(t))p_j(t)p_k(t),$$

such that g^{ij} and Γ_j^{ik} are the local components of the contravariant tensor field of g associated to the sub-Hamiltonian structure and the connection coefficients of ∇, respectively. Indeed, given a non-linear $\mathcal{L}$-connection ∇ we could always introduce a smooth vector field Γ^∇ on T^*M, whose integral curves are auto-parallel curves of ∇. In canonical coordinates, this vector field reads

$$\Gamma^\nabla(x,p) = g^{ij}(x,p)p_j\frac{\partial}{\partial x^i} - \Gamma_j^{ik}(x,p)p_i p_k\frac{\partial}{\partial p_j}.$$

Theorem 15. *Let ∇ be an $\mathcal{L}$-connection, then the following assertions are equivalent:*

(i) *∇ is a normal $\mathcal{L}$-connection;*
(ii) *For any $\alpha \in \mathfrak{X}^*(M)$, ∇ satisfies:*

$$\nabla_\alpha \alpha = \overline{\nabla}^B_{E(\alpha)} P^*(\alpha) + \delta_{E(\alpha)} (P^*)^c(\alpha);$$

(iii) *Every geodesic of ∇ is a normal extremal, and vice versa.*

Proof. (i) $\Longleftrightarrow$ (ii) If we apply the equality between the symmetric product and the symmetric bracket in the definition of normal $\mathcal{L}$-connections, then for all $\alpha \in \mathfrak{X}^*(M)$ we get

$$\langle \alpha, \alpha \rangle_\nabla = \nabla_\alpha \alpha + \nabla_\alpha \alpha = 2\nabla_\alpha \alpha = \{\alpha, \alpha\}$$

which is obviously equivalent to the mentioned definition above. By the definition of the Barthel non-linear connection $\overline{\nabla}^B$ and the definition of the symmetric bracket for all $\alpha \in \mathfrak{X}^*(M)$, we conclude

$$(\overline{\nabla}^B_{E(\alpha)}\ \alpha)(P) = \frac{1}{2}\{\alpha, \alpha\}(P) = \bar{\mathcal{L}}_{E(\alpha)} \alpha(P) - d(\langle E(\alpha(P)), \alpha(P) \rangle). \qquad (7)$$

Taking into account equation (6), we deduce the following

$$(\overline{\nabla}^B_{E(\alpha)}\ \alpha)(P) = \bar{\mathcal{L}}_{E(\alpha)} \alpha(P) - d(\bar{E}(\alpha, \alpha)).$$

Keeping in mind that $P^*(\alpha) = \alpha(P)$ and the Barthel non-linear connection preserves the metric, i.e. $\nabla^B \circ \mathcal{L}_\eta = \mathcal{L}_\eta \circ \overline{\nabla}^B$, we obtain

$$\begin{aligned} \overline{\nabla}^B_{E(\alpha)} P^*(\alpha) &= \bar{\mathcal{L}}_{E(\alpha)} P^*(\alpha) - d(\bar{E}(\alpha, \alpha)) \\ &= \frac{1}{2}\{\alpha, \alpha\} - \bar{\mathcal{L}}_{E(\alpha)} (P^*)^c(\alpha) \\ \frac{1}{2}\{\alpha, \alpha\} = \nabla_\alpha \alpha &= \overline{\nabla}^B_{E(\alpha)} P^*(\alpha) + \bar{\mathcal{L}}_{E(\alpha)} (P^*)^c(\alpha). \end{aligned}$$

Since $(P^*)^c(\alpha) \in \Gamma(\mathcal{H}^0)$ and $E(\alpha) \in \Gamma(\mathcal{H})$. The last term on the right hand side reduces to $\delta_{E(\alpha)}(P^*)^c(\alpha)$, as wanted to be shown. The converse can be easily seen taking into account $P + P^c = id$ and the first equality of (7).

(i) $\Longleftrightarrow$ (iii) Assume that U is a coordinate neighborhood in M and $\pi^{-1}(U)$ in T^*M. The normality of ∇ holds if and only if $\nabla_\alpha \alpha = \frac{1}{2}\{\alpha, \alpha\}$. Consider that

$$\frac{1}{2}\{\alpha, \alpha\} = \bar{\mathcal{L}}_{E(\alpha)} \alpha - d(\bar{E}(\alpha, \alpha)) = \iota_{E(\alpha)}(d\alpha),$$

as was shown before Proposition 11. Calculating locally we get

$$\frac{1}{2}\{\alpha, \alpha\} = \frac{1}{2} \frac{\partial g^{ab}}{\partial x^k} p_a p_b dx^k.$$

So the equality $\nabla_\alpha \alpha = \frac{1}{2}\{\alpha, \alpha\}$ is equivalent to

$$\Gamma^{ab}_k p_a p_b dx^k = \frac{1}{2} \frac{\partial g^{ab}}{\partial x^k} p_a p_b dx^k.$$

The coordinate expression for the sub-Hamiltonian vector field $\vec{H}$ equals:

$$\vec{H}(x,p) = g^{ab}(x,p)p_b\frac{\partial}{\partial x^a} - \frac{1}{2}\frac{\partial g^{ab}}{\partial x^k}(x,p)p_a p_b \frac{\partial}{\partial p_k}.$$

Comparing the latter equation with the definition of Γ^{∇}, it is clear that $\Gamma^{\nabla}(x,p) = \vec{H}(x,p)$ if and only if $\nabla_\alpha \alpha = \frac{1}{2}\{\alpha,\alpha\}$ for each $\alpha \in \mathfrak{X}^*(M)$. □

The above theorem clarifies, in particular, that

$$\nabla_\alpha \beta = \overline{\nabla}^B_{E(\alpha)} P^*(\beta) + \delta_{E(\alpha)}(P^*)^c(\beta),$$

is a non-linear $\mathcal{L}$-connection and it is normal. Moreover, for all $\beta \in \Gamma(\mathcal{H}^0)$ one can verify $\nabla_\alpha \beta = \delta_{E(\alpha)}(\beta)$, i.e. the connection under consideration is also $\mathcal{H}$-adapted. Consequently, we have

Proposition 16. *Given a sub-Finsler structure $(M, \mathcal{H}, F)$, one can always construct a normal and $\mathcal{H}$-adapted $\mathcal{L}$-connection.*

Definition 17. A non-linear $\mathcal{L}$-connection is called *partial* if for any $\alpha \in \mathfrak{X}^*(M)$ and $\beta \in \Gamma(\mathcal{H}^0)$ we have $\nabla_\beta \alpha = 0$.

Proposition 18. *Let ∇ be a normal $\mathcal{L}$-connection. Then ∇ is partial if and only if ∇ is $\mathcal{H}$-adapted.*

Proof. Suppose that ∇ is a normal $\mathcal{L}$-connection, namely

$$\begin{aligned}\{\alpha,\beta\} &= \langle \alpha,\beta\rangle_\nabla \\ &= \nabla_\alpha\beta + \nabla_\beta\alpha.\end{aligned}$$

Let us consider that ∇ is partial. Then for all $\beta \in \Gamma(\mathcal{H}^0)$ the above equalities change as follows

$$\{\alpha,\beta\} = \nabla_\alpha \beta = \bar{\mathcal{L}}_{E(\alpha)}\beta = \delta_{E(\alpha)}\beta,$$

from which it is clear that ∇ satisfies the condition of $\mathcal{H}$-adapted. Conversely, assume that ∇ is normal and $\mathcal{H}$-adapted, then for all $\alpha, \beta \in \mathfrak{X}^*(M)$ ∇ can be written in such a way

$$\nabla_\alpha \beta = \{\alpha,\beta\} - \nabla_\beta \alpha.$$

Moreover, it may be shown that the right hand side of the previous equation is zero, more precisely, for any $\alpha \in \Gamma(\mathcal{H}^0)$ and $\beta \in \mathfrak{X}^*(M)$ one has that $\nabla_\alpha \beta = 0$, which completes the proof. □

Theorem 19. *An $\mathcal{H}$-adapted $\mathcal{L}$-connection is not metrical, supposed that the distribution $\mathcal{H}$ is bracket generating.*

Proof. Assume that ∇ is an $\mathcal{H}$-adapted $\mathcal{L}$-connection. If ∇ were metrical, then ∇ would leave $\bar{E}$ invariant, i.e. for all $\alpha, \beta \in \mathfrak{X}^*(M)$

$$E(\nabla_\alpha \beta) = \nabla_\alpha E(\beta).$$

Then the $\mathcal{H}$-adapted property of ∇ implies that for all $\alpha \in \mathfrak{X}^*(M)$ and $\gamma \in \Gamma(\mathcal{H}^0)$: $E(\delta_{E(\alpha)}\gamma) = 0$. Therefore,

$$\begin{aligned}0 = \langle E(\delta_{E(\alpha)}\gamma), \beta\rangle &= \langle E(\beta), \delta_{E(\alpha)}\gamma\rangle \\ &= -\langle [E(\alpha), E(\beta)], \gamma\rangle,\end{aligned}$$

for every $\alpha, \beta \in \mathfrak{X}^*(M)$ and $\gamma \in \Gamma(\mathcal{H}^0)$, hence $[E(\alpha), E(\beta)] \in \Gamma(\mathcal{H})$. So, $\mathcal{H}$ would be involutive, which contradicts to the assumption of the bracket generating property. □

Finally, let us remark that Langerock ([7]) pointed out the importance of the properties which are assumed on the $\mathcal{L}$-connections defined as a generalized linear connection over an anchor map. E.g. in the sub-Riemannian case, an abnormal extremal is characterized by the existence of a parallel section of the annihilator bundle along the extremal with respect to an adapted connection. Furthermore, if a horizontal curve, i.e. a curve tangent to the given distribution is a normal extremal, then it is a geodesic of the extended Riemannian metric. Our connection defined for a sub-Finsler manifold is a generalized non-linear connection over an anchor map constructed by the Legendre transformation. We extended the sub-Finslerian metric to be defined on the whole tangent bundle by the projection operator. We used this setting of the sub-Finsler manifold to generalize and prove some facts of [7], but this is not completely done yet.

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