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SOME COMPARISON INEQUALITIES FOR GINI AND STOLARSKY MEANS

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(*communicated by J. E. Pečarić*)

Abstract. We study the following inequality

$$G_{a,b}(x,y) \leqslant S_{c,d}(x,y) \quad (x,y \in \mathbb{R}_+),$$

where $G_{a,b}(x,y)$ stands for the so-called Gini mean of the positive variables x and y, and $S_{c,d}(x,y)$ is the Stolarsky mean of them. We give some necessary conditions for the parameters a,b,c,d, then, distinguishing the different positions of the Gini-parameters, we present some sufficient conditions for the inequality above to hold for variables indicated. In some special cases it turns out that these conditions coincide.

1. Introduction

There is an extended literature concerning the so-called Gini and Stolarsky means. These two variable homogenous means play important roles both in the theory of means and in the application of inequalities in various branches of mathematics.

We recall now the definition of these means. If x,y are real numbers, then their Gini mean is defined by:

$$G_{a,b}(x,y) = \begin{cases} \left(\dfrac{x^a+y^a}{x^b+y^b}\right)^{\frac{1}{a-b}} & \text{if } a \neq b, \\ \exp\left(\dfrac{x^a \log x + y^a \log y}{x^a+y^a}\right) & \text{if } a = b, \end{cases}$$

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while their Stolarsky mean is the following:

$$S_{a,b}(x,y)=\begin{cases}\left(\dfrac{b(x^a-y^a)}{a(x^b-y^b)}\right)^{\frac{1}{a-b}} & \text{if } (a-b)ab\neq 0,\\ \exp\left(-\dfrac{1}{a}+\dfrac{x^a\log x-y^a\log y}{x^a-y^a}\right) & \text{if } a=b\neq 0,\\ \left(\dfrac{x^a-y^a}{a(\log x-\log y)}\right)^{\frac{1}{a}} & \text{if } a\neq 0,\ b=0,\\ \sqrt{xy} & \text{if } a=b=0.\end{cases}$$

By taking particular choices of the parameters a,b, one can see that the power means are included in both classes of means. More surprisingly, as it has recently been proved by Alzer and Ruscheweyh [1], the class of power means forms exactly the intersection of the classes of Gini and Stolarsky means.

The comparison problem for the means of the same kind, but with different parameters has played an important role in the research concerning Gini and Stolarsky means. This problem, for the Stolarsky means, was solved by Leach and Sholander [6] (cf. also [8]) and for the Gini means by Páles [9]. The case when the variables x,y run in a subinterval of $\mathbb{R}_+$ was treated for both classes of means by Páles [10]. In three recent papers [2], [3], [4] the authors have restated and completed these theorems (covering also the cases of equal parameters) offering new (but equivalent) necessary and sufficient conditions.

These results, however, describe only the cases where at the two sides of the comparison inequality there stand *means of the same kind*. The aim of the present paper is to state necessary/sufficient conditions for the comparison of Gini *and* Stolarsky means. The first results in this direction are due to Neuman and Páles [7] who investigated the comparison of Gini and Stolarsky means of equal parameters and proved that, for given real numbers a,b, the comparison inequality

$$G_{a,b}(x,y)\ \underset{(\geqslant)}{\leqslant}\ S_{a,b}(x,y)\qquad (x,y\in\mathbb{R}_+)$$

holds if and only if $a+b\ \underset{(\geqslant)}{\leqslant}\ 0$. Another problem, the so-called strong comparison problem (which is equivalent to the monotonicity property of the ratios of the means in question), has recently been investigated by Hästö [5].

In the next section we recall the main known comparison theorems for the Gini and Stolarsky means. Then some necessary conditions are obtained. In Section 4 we formulate two propositions that offer necessary and sufficient conditions for the comparison problem in a particular setting. These results will play an important role in the last section when stating the sufficient conditions for the comparison of Gini and Stolarsky means.

In the sequel, we restrict our attention to the inequality $G_{a,b}\leqslant S_{c,d}$ only because the analogous inequality $S_{a,b}\leqslant G_{c,d}$ is equivalent to $G_{-c,-d}\leqslant S_{-a,-b}$, therefore the results for the second type of the comparison inequality can easily be derived from what we will obtain.

2. Preliminary results. Comparison inequalities

The next result offers a necessary and sufficient condition for the comparison of Gini means (cf. [9], [2], [4]).

THEOREM 1. *Let a,b,c,d be positive numbers. Then the inequality*

$$G_{a,b}(x,y) \leqslant G_{c,d}(x,y) \quad (x,y \in \mathbb{R}_+)$$

holds if and only if the conditions

$$a+b \leqslant c+d \tag{1}$$

and

$$\mathcal{M}(a,b) \leqslant \mathcal{M}(c,d), \quad \mu(a,b) \leqslant \mu(c,d) \tag{2}$$

are valid, where

$$\mathcal{M}(u,v) = \begin{cases} \min\{u,v\}, & \text{if } u,v \geqslant 0\,, \\ 0, & \text{if } uv < 0\,, \\ \max\{u,v\}, & \text{if } u,v \leqslant 0\,. \end{cases} \quad \text{and} \quad \mu(u,v) := \begin{cases} \dfrac{|u|-|v|}{u-v}, & \text{if } u \neq v\,, \\ \operatorname{sgn}(u), & \text{if } u = v\,. \end{cases}$$

The comparison problem for Stolarsky means is answered by the following theorem ([6], [8], [3]).

THEOREM 2. *Let a,b,c,d be positive numbers. Then the inequality*

$$S_{a,b}(x,y) \leqslant S_{c,d}(x,y) \quad (x,y \in \mathbb{R}_+)$$

holds if and only if the conditions

$$a+b \leqslant c+d \tag{3}$$

and

$$\mathcal{L}(a,b) \leqslant \mathcal{L}(c,d), \quad \mu(a,b) \leqslant \mu(c,d) \tag{4}$$

are valid, where

$$\mathcal{L}(u,v) = \begin{cases} \dfrac{u-v}{\log(u/v)}, & \text{if } 0 < uv \text{ and } u \neq v\,, \\ u, & \text{if } 0 < uv \text{ and } u = v\,, \\ 0, & \text{otherwise.} \end{cases}$$

3. Necessary conditions

In this section we derive conditions that are necessary for the mixed comparison inequality of Gini and Stolarsky means.

THEOREM 3. *Suppose that the inequality*

$$G_{a,b}(x,y) \leqslant S_{c,d}(x,y) \tag{5}$$

holds for any positive x,y. Then

(i)

$$3(a+b) \leqslant c+d, \tag{6}$$

(ii)

$$\min\{a,b\} \leqslant 0. \tag{7}$$

If $\min\{a,b\} = 0 < \max\{a,b\}$ *then*

$$\max\{a,b\} \leqslant \log 2 \cdot \mathcal{L}(c,d). \tag{8}$$

(iii) *Finally,*

$$\mu(a,b) \leqslant \mu(c,d). \tag{9}$$

Proof. Applying the homogeneity of the means, and denoting the ratio x/y by t, (5) is equivalent to the following:

$$g_{a,b}(t) \leqslant s_{c,d}(t) \quad (t \in \mathbb{R}_+), \tag{10}$$

where $g_{a,b}(t)$ stands for $G_{a,b}(t,1)$ and $s_{c,d}(t)$ for $S_{c,d}(t,1)$. It follows from this inequality that

$$\frac{g_{a,b}(t) - \frac{t+1}{2}}{(t-1)^2} \leqslant \frac{s_{c,d}(t) - \frac{t+1}{2}}{(t-1)^2} \quad (t \in \mathbb{R}_+ \setminus \{1\}).$$

Using l'Hospital's rule twice for the left hand side, four times for the right hand side, and performing the limit $t \to 1$, we get that

$$\frac{a+b-1}{8} \leqslant \frac{c+d-3}{24},$$

i.e., (6) holds.

Starting from (10) again, we obtain that

$$\lim_{t\to 0} g_{a,b}(t) \leqslant \lim_{t\to 0} s_{c,d}(t). \tag{11}$$

On the other hand, we have that

$$\lim_{t\to 0} g_{a,b}(t) = \begin{cases} 0, & \text{if } \min\{a,b\} < 0 \text{ or } (a,b) = (0,0), \\ 2^{-\frac{1}{\max\{a,b\}}}, & \text{if } \min\{a,b\} = 0 \text{ and } \max\{a,b\} > 0, \\ 1, & \text{if } \min\{a,b\} > 0, \end{cases}$$

and

$$\lim_{t\to 0} s_{c,d}(t) = \begin{cases} 0, & \text{if } \min\{c,d\} \leqslant 0, \\ \left(\frac{d}{c}\right)^{\frac{1}{c-d}}, & \text{if } \min\{c,d\} > 0. \end{cases}$$

In this way, we have obtained that $\lim_{t\to 0} s_{c,d}(t) < 1$, therefore $\min\{a,b\} > 0$ is not possible, that is, (7) must be valid.

If $\min\{a,b\} = 0 < \max\{a,b\}$, then $\min\{c,d\}$ must be positive and

$$2^{-\frac{1}{\max\{a,b\}}} \leqslant \left(\frac{d}{c}\right)^{\frac{1}{c-d}},$$

that is, $\max\{a,b\} \leqslant \log 2 \cdot \mathcal{L}(c,d)$, which means that (8) holds true.

When proving (9), the only non-trivial case is when $\max\{a,b\} > 0 > \min\{a,b\}$ and $\max\{c,d\} > 0 > \min\{c,d\}$. We may assume that $b < 0 < a$, $d < 0 < c$. Then, it follows from (10) that

$$\lim_{t\to\infty} \frac{g_{a,b}(t)}{s_{c,d}(t)} \leqslant 1. \tag{12}$$

Denoting this limit with h,

$$\begin{aligned}
\log(h) &= \lim_{t\to\infty} \Big[\frac{1}{a-b}\big(\log(t^a+1) - \log(t^b+1)\big) - \frac{1}{c-d}\big(\log\frac{t^c-1}{c} + \log\frac{d}{t^d-1}\big)\Big] \\
&= \lim_{t\to\infty} \Big[\frac{1}{a-b}\log(t^a+1) - \frac{1}{c-d}\log\frac{t^c-1}{c} - \frac{\log(-d)}{c-d}\Big] \\
&= \lim_{t\to\infty} \Big[\frac{1}{a-b}\log(t^a+1) - \frac{1}{c-d}\log(t^c-1) + \frac{1}{c-d}\log\Big(-\frac{c}{d}\Big)\Big] \\
&= \lim_{t\to\infty} \log\frac{(t^a+1)^{\frac{1}{a-b}}}{(t^c-1)^{\frac{1}{c-d}}} + \log\Big(-\frac{c}{d}\Big)^{\frac{1}{c-d}},
\end{aligned}$$

that is,

$$h = \Big(-\frac{c}{d}\Big)^{\frac{1}{c-d}} \lim_{t\to\infty} \frac{(t^a+1)^{\frac{1}{a-b}}}{(t^c-1)^{\frac{1}{c-d}}} > \Big(-\frac{c}{d}\Big)^{\frac{1}{c-d}} \lim_{t\to\infty} \frac{(t^a)^{\frac{1}{a-b}}}{(t^c)^{\frac{1}{c-d}}} = \Big(-\frac{c}{d}\Big)^{\frac{1}{c-d}} \lim_{t\to\infty} t^{\frac{a}{a-b}-\frac{c}{c-d}}.$$

If, in the last expression the exponent of t were positive, then h would be plus infinity, which is in contradiction with the condition (12). Therefore,

$$\frac{a}{a-b} - \frac{c}{c-d} \leqslant 0,$$

which is equivalent to (9).

4. Particular comparison inequalities

In this section we examine two particular cases of the comparison of Gini and Stolarsky means. These statements will turn out to be useful tools in formulating sufficient conditions for the general comparison problem. In these cases the parameters a,b and c,d of the Gini and Stolarsky means are chosen so that the necessary condition (i) of Theorem 3 hold with equality.

PROPOSITION 1. *The inequality*

$$G_{a,b}(x,y) \leqslant S_{3a,3b}(x,y) \tag{13}$$

holds for all positive x,y *if and only if* $a+b \leqslant 0$, *while the reversed inequality holds if and only if* $a+b \geqslant 0$.

Proof. We deal only with the characterization of the inequality (13), the investigation of the reversed inequality is completely analogous.

Suppose first that $a \neq b$, for example, $a > b$ and $ab \neq 0$. Using the symmetry and homogeneity of the means, setting $t = \log(\sqrt{x/y})$, (13) can be rewritten in the equivalent form

$$\left(\frac{\cosh(at)}{\cosh(bt)}\right)^{\frac{1}{a-b}} \leqslant \left(\frac{\frac{\sinh(3at)}{3a}}{\frac{\sinh(3bt)}{3b}}\right)^{\frac{1}{3a-3b}} \qquad (t \in \mathbb{R}_+),$$

which is also equivalent to

$$\frac{\sinh(3bt)}{bt\cosh^3(bt)} \leqslant \frac{\sinh(3at)}{at\cosh^3(at)} \quad (t \in \mathbb{R}_+). \tag{14}$$

To investigate this inequality, introduce the function

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad x \mapsto \frac{\sinh 3x}{x(\cosh x)^3}.$$

It can immediately be seen that f is even. We also claim that it is decreasing on $\mathbb{R}_+$. We can easily obtain that

$$f'(x) = \frac{6x\cosh 2x - \sinh 4x - \sinh 2x}{2x^2(\cosh x)^4} \quad (x \in \mathbb{R} \setminus \{0\}).$$

Thus, it suffices to show that $6x\cosh 2x - \sinh 4x - \sinh 2x := h(x)$ is negative for any positive x. (Therefore, f' is negative on $\mathbb{R}_+$.)

Expanding the function h into McLaurin series, we get that

$$h(x) = \sum_{i=0}^{\infty}\left(\frac{6\cdot 2^{2i}}{(2i)!} - \frac{4^{2i+1}}{(2i+1)!} - \frac{2^{2i+1}}{(2i+1)!}\right)\cdot x^{2i+1} = \sum_{i=0}^{\infty}(3i+1-2^{2i})\frac{2^{2i+2}}{(2i+1)!}\cdot x^{2i+1}.$$

Here the coefficient $3i+1-2^{2i}$ vanishes for $i = 0$ and $i = 1$ and is negative if $i \geqslant 2$. Therefore $h(x)$ and also $f'(x)$ is negative for all positive x. Thus, the symmetric f is decreasing on the positive half line and increasing on negative reals. It readily follows from this that (14), i.e., $f(bt) \leqslant f(at)$ is valid if and only if $|a| \leqslant |b|$. One can easily observe that this inequality together with $b < a$ holds if and only if $a + b \leqslant 0$.

In the cases $a = b$ or $ab = 0$, the necessity and sufficiency of the condition $a + b \leqslant 0$ can similarly be verified.

REMARK 1. Proposition 1 implies the result of Neuman and Páles [7], since (by Theorem 2) $S_{3a,3b}(x,y) \leqslant S_{a,b}(x,y)$ holds for all positive x, y if and only if $a + b \leqslant 0$.

PROPOSITION 2. *The inequality*

$$G_{a,b}(x,y) \leqslant S_{2a+b,a+2b}(x,y) \tag{15}$$

holds for all positive x, y if and only if $ab(a+b) \leqslant 0$.

Proof. The case $a=b$ is covered by Proposition 1. Moreover, if $ab=0$, then the inequality turns to an identity, since $G_{a,0}(x,y)=S_{2a,a}(x,y)$ for all positive x,y.

In the case $2a+b=0$, $ab\neq 0$, with the notation $t=\log(\sqrt{x/y})$, (15) is equivalent to the inequality

$$\left(\frac{\operatorname{ch}(at)}{\operatorname{ch}(-2at)}\right)^{\frac{1}{3a}} \leqslant \left(\frac{\operatorname{sh}(-3at)}{-3at}\right)^{\frac{1}{-3a}} \quad (t\in\mathbb{R}_+). \tag{16}$$

Thus, we have to show that (16) holds if and only if $a<0$.

For, we will prove that the inequality

$$\frac{\operatorname{ch}(x)}{\operatorname{ch}(-2x)} \geqslant \left(\frac{\operatorname{sh}(-3x)}{-3x}\right)^{-1} \tag{17}$$

holds for all $x\neq 0$. The functions on the two sides of this inequality are even, so we may assume that $x>0$. Then, inequality (17) can be rewritten into the form

$$\operatorname{sh}(4x)+\operatorname{sh}(2x)-6x\operatorname{ch}(2x)\geqslant 0.$$

As we have seen it in the proof of Proposition 1, the left hand side of this inequality is nonnegative. Thus (17) follows for all $x\neq 0$. In view of (17), the inequality (16) holds for all $t>0$ if and only if $a<0$, which completes the proof in this case.

The case $a+2b=0$, $ab\neq 0$ can be treated similarly.

We may assume now that $a\neq b$, e.g., $a>b$, $ab\neq 0$, and $(a+2b)(2a+b)\neq 0$. Now (15) can be rewritten in the equivalent form

$$\frac{\operatorname{ch}(at)}{\operatorname{ch}(bt)} \leqslant \frac{\frac{\operatorname{sh}(2a+b)t}{2a+b}}{\frac{\operatorname{sh}(a+2b)t}{a+2b}} \quad (t\in\mathbb{R}_+), \tag{18}$$

or, rearranging this and dividing both sides by the positive t,

$$\frac{\operatorname{ch}(at)\operatorname{sh}(a+2b)t}{(a+2b)t} \leqslant \frac{\operatorname{ch}(bt)\operatorname{sh}(2a+b)t}{(2a+b)t} \quad (t\in\mathbb{R}_+).$$

Finally, applying the product-to-sum formulas and denoting $2t$ by s, our statement is equivalent to the following:

$$\frac{\operatorname{sh}(a+b)s+\operatorname{sh}(bs)}{(a+2b)s} \leqslant \frac{\operatorname{sh}(a+b)s+\operatorname{sh}(as)}{(2a+b)s} \quad (s\in\mathbb{R}_+). \tag{19}$$

Introduce the function $f(x)=1+\frac{x}{3!}+\frac{x^2}{5!}+\frac{x^3}{7!}+\dots$ on $\mathbb{R}_+$. Clearly, f is strictly convex on $\mathbb{R}_+$ and $\operatorname{sh}x=x\cdot f(x^2)$ holds for any real x. Using this notation, (19) transforms to

$$\frac{(a+b)f\left((a+b)^2s^2\right)+bf(b^2s^2)}{a+2b}-\frac{(a+b)f\left((a+b)^2s^2\right)+af(a^2s^2)}{2a+b} \leqslant 0 \quad (s\in\mathbb{R}_+). \tag{20}$$

The left hand side of (20) is, however, the product of $ab(a-b)(a+b)\cdot s^4$ and the 2nd-order divided difference $[(a+b)^2s^2, a^2s^2, b^2s^2; f]$. The function f being strictly convex on $\mathbb{R}_+$, we have that this divided difference is positive if $(a+b)^2s^2$, a^2s^2, and b^2s^2 are pairwise distinct, i.e., if $ab(a+b)(a-b)(a+2b)(2a+b) \neq 0$. Consequently, (20) holds if and only if $ab(a-b)(a+b)$ is nonpositive. With the assumption $a > b$ this yields that (20), that is, (18) is valid if and only if $ab(a+b) \leqslant 0$.

5. Sufficient conditions

In this section, according to the position of the pair $(a,b) \in \mathbb{R}^2$, we give sufficient conditions for the Gini-Stolarsky comparison inequality. These conditions are sometimes (unfortunately not always) also necessary.

THEOREM 4. *Let a,b be positive numbers. Then there are no parameters c,d so that the inequality*

$$G_{a,b}(x,y) \leqslant S_{c,d}(x,y)$$

be valid for all positive numbers x,y.

Proof. This is a direct consequence of (7) in Theorem 3.

THEOREM 5. *Let a,b be real numbers so that $\min\{a,b\} = 0 < \max\{a,b\}$. Then*

$$G_{a,b}(x,y) \leqslant S_{c,d}(x,y)$$

is valid for all positive numbers x,y if and only if
(a) $3a \leqslant c+d$,
(b) $a \leqslant \log 2 \cdot \mathcal{L}(c,d)$.

Proof. The necessity of the condition follows form Theorem 3. For the sufficiency, assume that $0 = b < a$. Then $G_{a,b}(x,y) = G_{a,0}(x,y) = S_{a,2a}(x,y)$, and we can apply Theorem 2 to the inequality $S_{a,2a}(x,y) \leqslant S_{c,d}(x,y)$. Now (a) is equivalent to $a+2a \leqslant c+d$ and (b) yields $\mathcal{L}(a,2a) \leqslant \mathcal{L}(c,d)$. Thus c,d must be positive, whence $\mu(a,0) \leqslant \mu(c,d)$ also follows. Therefore, in view of Theorem 2, (a) and (b) yield that $G_{a,0}(x,y) = S_{a,2a}(x,y) \leqslant S_{c,d}(x,y)$ holds for all positive x,y.

THEOREM 6. *Let a,b be real numbers so that $ab < 0$ and $a+b \geqslant 0$. If*
(a) $3(a+b) \leqslant c+d$,
(b) $\mathcal{L}\{a+2b, 2a+b\} \leqslant \mathcal{L}(c,d)$,
(c) $\mu(2a+b, a+2b) \leqslant \mu(c,d)$, *then*

$$G_{a,b}(x,y) \leqslant S_{c,d}(x,y)$$

is valid for all positive numbers x,y.

Proof. By Proposition 2, the condition $ab(a+b) \leqslant 0$ yields that, for any positive x,y,

$$G_{a,b}(x,y) \leqslant S_{a+2b,2a+b}(x,y).$$

Moreover, the conditions guarantee that Theorem 2 can be applied to obtain

$$S_{a+2b,2a+b}(x,y) \leqslant S_{c,d}(x,y).$$

Combining these two inequalities, the desired inequality follows.

THEOREM 7. *Let* a,b *be real numbers so that* $ab<0$ *and* $a+b\leqslant 0$. *Then*

$$G_{a,b}(x,y)\leqslant S_{c,d}(x,y)$$

is valid for all positive numbers x,y *if and only if*
(a) $3(a+b)\leqslant c+d$,
(b) $\mu(a,b)\leqslant\mu(c,d)$.

Proof. The necessity of conditions (a) and (b) is the consequence of Theorem 3.
By Proposition 2, the condition $a+b\leqslant 0$ yields that, for all positive x,y,

$$G_{a,b}(x,y)\leqslant S_{3a,3b}(x,y).$$

To complete the proof of the sufficiency, we will also prove that

$$S_{3a,3b}(x,y)\leqslant S_{c,d}(x,y).$$

For, by Theorem 2, we have to ensure that, in addition to (a), $\mu(3a,3b)\leqslant\mu(c,d)$ and $\mathcal{L}(3a,3b)\leqslant\mathcal{L}(c,d)$ hold. The first inequality trivially follows from (b) since $\mu(3a,3b)\leqslant\mu(a,b)$. On the other hand, $ab<0$, consequently, $\mathcal{L}(3a,3b)=0$. Thus it suffices to show that $\mathcal{L}(c,d)$ is nonnegative. Indeed, in the opposite case we have that $c,d<0$, thus $\mu(c,d)$ is equal to -1, while $\mu(a,b)>-1$ — in contradiction with (b).

REMARK 2. It is clear that the conditions of Theorem 6 and Theorem 7 coincide in the case $ab<0$, $a+b=0$ – that is, $b=-a$, $b\neq 0$.

THEOREM 8. *Let* a,b *be real numbers,* $a,b\leqslant 0$. *If*
(a) $3(a+b)\leqslant c+d$,
(b) $\mathcal{L}(2a+b,a+2b)\leqslant\mathcal{L}(c,d)$, *then*

$$G_{a,b}(x,y)\leqslant S_{c,d}(x,y)$$

is valid for all positive numbers x,y.

Proof. First we check our statement when $(a,b)=(0,0)$. In this case $G_{0,0}(x,y)=\sqrt{xy}=S_{0,0}(x,y)$, so we can use Theorem 2 again. Then (a) and (b) are equivalent to the inequality $0\leqslant c+d$ which, by Theorem 2, results that $S_{0,0}\leqslant S_{c,d}$ is valid.

In the rest of the proof, we assume that $(a,b)\neq(0,0)$.

In view of Proposition 2, it is clear that, for all positive x,y,

$$G_{a,b}(x,y)\leqslant S_{2a+b,a+2b}(x,y).$$

We have that $a,b\leqslant 0$ and $a+b<0$, consequently, $\mu(2a+b,a+2b)=-1$, whence $\mu(2a+b,a+2b)\leqslant\mu(c,d)$ follows. Therefore, using Theorem 2, the conditions (a) and (b) imply that

$$S_{2a+b,a+2b}(x,y)\leqslant S_{c,d}(x,y),$$

which combined with the previous inequality results our statement.

REMARK 3. In the domain $a,b \leqslant 0$ we also have the inequality $G_{a,b} \leqslant S_{3a,3b}$ which could be used to obtain that the inequality in (a) and $\mathcal{L}(3a,3b) \leqslant \mathcal{L}(c,d)$ form also a system of sufficient conditions. However, applying Theorem 2, it easily follows that $S_{2a+b,a+2b} \leqslant S_{3a,3b}$ holds, too. Thus, the sufficient condition obtained this way is essentially weaker than that of Theorem 8.

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