On the Derived Length of Lie Solvable Group Algebras

Doktori (PhD) értekezés

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Chapter 1

Introduction

1.1 Preliminaries

At the beginning of the last century an exciting algebraic construction consisting of a group and a field, appeared in the works of G. Frobenius which he used for the observation of the representations of finite groups. This is the construction that was named group algebra in the '30s by E. Noether. After this, through decades the group algebras were not so much observed for themselves but for the possibilities of their applications in representation theory and algebraic topology, etc. For example, W. Magnus proved his famous theorem on the lower central series of free groups at that time with the aid of group algebras, which theorem was demonstrated using pure group theoretical methods only after a long time. But in the beginning of the '60s the stress was laid upon the group algebras of infinite groups as a result of I. Kaplansky's problems in ring theory. The characterization theorems born in this period have led to newer directions in research. Several survey articles were published which brought the formation of this new branch of science. The study of the ring theoretical properties of group algebras is now at the most advanced level, but significant results are known in their unit groups as well. The unit group of group algebras aroused the researchers' interest for the first time because of their topological applications, and then again, after the description of simple groups, as special finite p-groups. The study of the unit group of modular group algebras was started by S.A. Jennings in the '40s, but since the solution of almost every single problem required the elaboration of a new method, the results came very slowly. In the more interesting cases the group of units have such a high order that not even the computers of present day have a capacity strong enough to deal with them. The more important results and open questions of this area are surveyed by the article A.A. Bódi's [9].

The investigation of the Lie properties of group algebras as a special polinomial identity was started after the description of group algebras satisfying a polinomial identity, but the invention of the relation between the property of unit group and the associated Lie algebra of group algebras led to an extended intensity of the observation in the '80s. Under general conditions it is not easy even to decide whether an element is a unit or not, so to determine of its inverse would be extremely difficult, such as the computation of the group commutators. However, the so-called Lie commutators can be calculated without the knowledge of the elements' inverses. Considering the results connected to the series which are constructed with the help of Lie commutators we can have conclusions for the corresponding series of the group of units, for example, derived series, upper and lower central series, etc. This method was first applied by A.A. Bódi and I.I. Khripta [6] who obtained that the unit group of group algebras is solvable if and only if the group algebra is Lie solvable, provided that the characteristic p of the field is greater than three and the basic group is a nonabelian and if it is a nontorsion group, then its p-Sylow subgroup is infinite. Furthermore, Lie methods were used by C. Bagiński [1] and J. Kurdics [17, 18] for the investigation of the derived length, the nilpotency class and the Engel length of the group of units. The harmony between the unit groups and the associated Lie algebras is also illustrated by the following theorem of A.A. Bódi [5]: the unit group of the modular group algebra FG of characteristic p is a bounded Engel group if and only if FG is a bounded Engel algebra. Additional results on the Lie structure of group algebras may be found in [2, 7, 8, 10, 11, 13, 19, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

Our goal in this thesis is the investigation of the derived length and the upper Lie nilpotency index of group algebras. Before we present the new results we give a short survey of the basic concepts and notations. For details we refer the reader to the book of A.A. Bódi [3].

Let G be a group and let F be a field. Denote by FG all the formal sums $\sum_{g\in G}\alpha_g g$, where only finitely many coefficients $\alpha_g \in F$ are nonzero. Clearly, two formal sums are equal if and only if all corresponding coefficients of group elements are equal. Let us define the sum of $x = \sum_{g\in G}\alpha_g g \in FG$ and $y = \sum_{g\in G}\beta_g g \in FG$ as

$$x + y = \sum_{g\in G}(\alpha_g + \beta_g)g$$

and the product of $\beta \in F$ and x as

$$\beta \cdot x = x \cdot \beta = \textstyle\sum_{g\in G}(\beta\alpha_g)g.$$

Then FG can be considered as a vector space over F and the elements of G form an F-basis for FG. The multiplication of formal sums are defined as follows:

$$xy = \sum_{g\in G}\left(\sum_{h\in G}\alpha_h\beta_{h^{-1}g}\right)g.$$

With these operations FG is an algebra over the field F which is called *group algebra* (of the group G over the field F).

In the special case when F is a field of characteristic $\mathrm{char}(F) = p$ and G contains an element of order p, FG is called *modular group algebra*.

Let $x = \sum_{g\in G}\alpha_g g$ be a nonzero element of the group algebra FG. The subset $\{g \in G \mid \alpha_g \neq 0\}$ of the group G is said to be the *support* of x.

The function ε mapping FG into F given by

$$\varepsilon\Big(\sum_{g\in G}\alpha_g g\Big) = \sum_{g\in G}\alpha_g.$$

is the so-called *augmentation map*, and its kernel

$$\omega(FG) = \{x \in FG \mid \varepsilon(x) = 0\}$$

the *augmentation ideal* of the group algebra FG. It is well-known that $\omega(FG)$ is nilpotent if and only if G is a finite p-group and $\mathrm{char}(F) = p$. Furthermore, if $t_N(G)$ denotes the nilpotency index of $\omega(FG)$ and G has order p^n, then $1 + n(p-1) \le t_N(G) \le p^n$. For example, if $G = \langle a_1\rangle \times \cdots \times \langle a_n\rangle$ and the order of a_i is p^{m_i}, then $t_N(G) = 1 + \sum_{i=1}^{n}(p^{m_i} - 1)$.

For any normal subgroup H of G the set

$$\mathfrak{I}(H) = \{(h-1)x \mid h \in H,\ x \in FG\}$$

is a two-sided ideal of FG. Clearly, $\mathfrak{I}(G)$ coincides with $\omega(FG)$ and $\mathfrak{I}(H) = \omega(FH)FG$. Let $T(G/H)$ be a transversal of the normal subgroup H in G. Then all the elements of the form $(h-1)u$, where $1 \ne h \in H$ and $u \in T(G/H)$ form a basis of the vector space $\mathfrak{I}(H)$. The isomorphism $FG/\mathfrak{I}(H) \cong F(G/H)$ is valid, which is called the *isomorphism theorem* of group algebras.

We shall use the following notations. For $x, y, x_1, x_2, \dots, x_n \in G$ let $x^y = y^{-1}xy$, $(x, y) = x^{-1}x^y$, and the commutator $(x_1, x_2, \dots, x_n)$ is defined inductively to be $\big((x_1, x_2, \dots, x_{n-1}), x_n\big)$. We shall use freely the commutator identities

$$(a, bc) = (a, c)(a, b)^c = (a, c)(a, b)(a, b, c);$$
$$(ab, c) = (a, c)^b(b, c) = (a, c)(a, c, b)(b, c),$$

and the Hall-Witt identity

$$(a, b^{-1}, c)^b(b, c^{-1}, a)^c(c, a^{-1}, b)^a = 1,$$

for any $a, b, c \in G$.

Our group theoretical notation is mostly standard. By $\zeta(G)$ we mean the center, by $\mathrm{aut}(G)$ the automorphism group of the group G, by $\mathrm{Syl}_p(G)$ the p-Sylow subgroup, by $\exp(G)$ the exponent and by $\gamma_n(G)$ the n-th term of the lower central series of G with $\gamma_1(G) = G$.

Evidently, $\gamma_2(G)$ coincides with the commutator subgroup G'. For $H \subseteq G$ we will denote by $C_G(H)$ the centralizer of the subset H in G. Furthermore, denote by C_n the cyclic group of order n and by C_g the conjugacy class including the element $g \in G$.

The upper integral part of a real number r is denoted by $\lceil r \rceil$.

1.2 Associated Lie algebra of group algebras

Let $(L, +)$ be a vector space over the field F and assume that a second binary operation $[a, b]$ is defined in L and the identities

- $\alpha[a, b] = [\alpha a, b] = [a, \alpha b]$;
- $[a + b, c] = [a, c] + [b, c]$ and $[a, b + c] = [a, b] + [a, c]$;
- $[a, a] = 0$;
- $\big[[a, b], c\big] + \big[[b, c], a\big] + \big[[c, a], b\big] = 0$

hold for all $a, b, c \in L$ and $\alpha \in F$. Then we say that L is a *Lie algebra* over the field F.

Let A be an associative algebra over the field F and $x, y \in A$. The element $[x, y] = xy - yx$ will be called the *Lie commutator* of x and y. Let us introduce in A the new operation $[x, y] = xy - yx$. Then A is a Lie algebra with respect to the operations $+$ and $[\ ,\ ]$, which is said to be the *associated Lie algebra* of A.

For the sequence (x_i) of elements of A we define the *left n-normed Lie commutator* by induction as

$$[x_1, x_2, \dots, x_n] = \big[[x_1, x_2, \dots, x_{n-1}], x_n\big].$$

We shall use freely the identities

$$[x, yz] = [x, y]z + y[x, z], \qquad [xy, z] = x[y, z] + [x, z]y,$$

and for units a, b that

$$[a,b] = ba\big((a,b)-1\big) = \big((a^{-1},b^{-1})-1\big)ba.$$

For the subsets $X, Y \subseteq A$ we denote by $[X, Y]$ the additive subgroup generated by all Lie commutators $[x, y]$ with $x \in X$ and $y \in Y$. It is easy to check that

$$\big[X,Y\big] = \big[Y,X\big],$$

$$[X,YZ] \subseteq [X,Y]Z + Y[X,Z]$$

and

$$[XY,Z] \subseteq X[Y,Z] + [X,Z]Y,$$

for any $X, Y, Z \subseteq A$.

1.2.1 Series in the associated Lie algebra

As before, let A be an associative algebra over the field F. Define the *Lie derived series* of A as follows: let $\delta^{[0]}(A) = A$ and for $n \geq 0$ let

$$\delta^{[n+1]}(A) = \big[\delta^{[n]}(A), \delta^{[n]}(A)\big].$$

Clearly,

$$A = \delta^{[0]}(A) \supseteq \delta^{[1]}(A) \supseteq \cdots \supseteq \delta^{[m]}(A) \supseteq \cdots.$$

We introduce similarly the *strong Lie derived series* of A: let $\delta^{(0)}(A) = A$ and for $n \geq 0$ let $\delta^{(n+1)}(A)$ be the ideal of A generated by all Lie commutators $[x, y]$ with $x, y \in \delta^{(n)}(A)$, that is

$$\delta^{(n+1)}(A) = \big[\delta^{(n)}(A), \delta^{(n)}(A)\big]A.$$

Evidently,

$$A = \delta^{(0)}(A) \supseteq \delta^{(1)}(A) \supseteq \cdots \supseteq \delta^{(m)}(A) \supseteq \cdots.$$

It is easy to check that $\delta^{[n]}(A) \subseteq \delta^{(n)}(A)$ for any n, but the equality does not always hold.

We say that A is *Lie solvable* if there exists $m \in \mathbb{N}$ such that $\delta^{[m]}(A) = 0$ and the number $\mathrm{dl}_L(A) = \min\{m \in \mathbb{N} \mid \delta^{[m]}(A) = 0\}$ is called the *Lie derived length* of A. Similarly, the algebra A is said to be *strongly Lie solvable* of derived length $\mathrm{dl}^L(A) = m$ if $\delta^{(m)}(A) = 0$ and $\delta^{(m-1)}(A) \neq 0$.

Let a and b be elements of the unit group $U(A)$ of A. Then by the equality $(a, b) = 1 + a^{-1}b^{-1}[a, b]$ the m-th term of the derived series $U(A)$ is contained in $1 + \delta^{(m)}(A)$. Thus to the investigation of the derived length of $U(A)$ the strong Lie derived series is a useful tool. Now we establish further two series, which can be applied similarly for the study of the nilpotency class of $U(A)$.

Let $A^{[1]} = A$ and for $n > 1$ let $A^{[n]}$ be the ideal of A generated by all the Lie commutators $[x_1, \dots, x_n]$ with $x_1, \dots, x_n \in A$. Then the ideal $A^{[n]}$ is the *n-th lower Lie power* and the series

$$A = A^{[1]} =\supseteq A^{[2]} \supseteq \cdots \supseteq A^{[n]} \supseteq \cdots$$

is called the *lower Lie power series* of the associative algebra A.

By induction, we define the *n-th upper Lie power* $A^{(n)}$ of A as the ideal generated by all the Lie commutators $[x, y]$, where $x \in A^{(n-1)}$, $y \in A$ and $A^{(1)} = A$. The series

$$A = A^{(1)} \supseteq A^{(2)} \supseteq \cdots \supseteq A^{(n)} \supseteq \cdots$$

is *the upper Lie power series* of the associative algebra A. Clearly, $A^{[n]} \subseteq A^{(n)}$ for all n.

The algebra A is called *Lie nilpotent* if there exists n such that $A^{[n]} = 0$ and the least integer of this kind is called the *Lie nilpotency index* of A and it is denoted by $t_L(A)$. Similarly, A is said to be *upper Lie nilpotent* and its *upper Lie nilpotency index* is $t^L(A) = m$ if $A^{(m)} = 0$ but $A^{(m-1)} \neq 0$.

Evidently, $\delta^{(0)}(A) = A^{(1)}$ and assume that $\delta^{(n)}(A) \subseteq A^{(2^n)}$ for some $n \geq 0$. By the well-known inclusion $[A^{(i)}, A^{(j)}] \subseteq A^{(i+j)}$ we obtain

$$\begin{aligned}\delta^{(n+1)}(A) &= \big[\delta^{(n)}(A), \delta^{(n)}(A)\big]A \subseteq \big[A^{(2^n)}, A^{(2^n)}\big]A \\ &\subseteq A^{(2^{n+1})}A = A^{(2^{n+1})}.\end{aligned}$$

We have just shown that

$$\delta^{(n)}(A) \subseteq A^{(2^n)} \quad \text{for all} \quad n \geq 0,$$

which is an analog of a standard fact from group theory.

1.2.2 Lie derived lengths of group algebras

M. Sahai [24] proved the relation

$$\mathfrak{I}(G')^{2^n-1} \subseteq \delta^{(n)}(FG) \subseteq \mathfrak{I}(G')^{2^{n-1}} \quad \text{for all } n > 0, \tag{1.1}$$

from which it follows that a group algebra FG is strongly Lie solvable if and only if either G is abelian or the ideal $\mathfrak{I}(G')$ is nilpotent, that is G' is a finite p-group and $\mathrm{char}(F) = p$. The description of the Lie solvable group algebras is due to I.B.S. Passi, D.S. Passman and S.K. Sehgal [22]: a group algebra FG is Lie solvable if and only if one of the following conditions holds: (i) G is abelian; (ii) G' is a finite p-group and $\mathrm{char}(F) = p$; (iii) G has a subgroup of index two whose commutator subgroup is a finite 2-group and $\mathrm{char}(F) = 2$.

Clearly, $\mathrm{dl}_L(FG)$ and $\mathrm{dl}^L(FG)$ are equal to one if and only if G is an abelian group. The group algebras of Lie derived length (strong Lie derived length) two, that is the so-called *Lie metabelian* (*strongly Lie metabelian*) group algebras were described by F. Levin and G. Rosenberger [19]. For odd characteristic the complete list of the strongly Lie solvable group algebras of strong Lie derived length three can be found in M. Sahai's paper [24]. Moreover, he also showed that the statements $\delta^{[3]}(FG) = 0$ and $\delta^{(3)}(FG) = 0$ are equivalent, provided that $\mathrm{char}(F) \geq 7$. All the other cases the question is still open.

In general, we have very little information about the Lie derived length of group algebras. The first and, at the same time, the more significant results on this topic can be found in papers [27] and [29] of A. Shalev.

Throughout this part by FG we always mean a strongly Lie solvable group algebra.

From (1.1) it follows immediately that

$$\lceil \log_2(t_N(G') + 1) \rceil \leq \mathrm{dl}^L(FG) \leq \lceil \log_2(2t_N(G')) \rceil \tag{1.2}$$

and so

$$\mathrm{dl}_L(FG) \leq \lceil \log_2(2t_N(G')) \rceil.$$

Since there is no similarly valuable lower bound on $\mathrm{dl}_L(FG)$, the computation of its value is more difficult than of the value of $\mathrm{dl}^L(FG)$.

Applying the method used A. Shalev in the proof of Lemma 2.2 in [27] we have that if G is nilpotent of class two, then

$$\mathrm{dl}_L(FG) \leq \mathrm{dl}^L(FG) = \lceil \log_2(t_N(G') + 1) \rceil.$$

In particular, if G is an abelian-by-cyclic p-group with $p > 2$ then

$$\mathrm{dl}_L(FG) = \lceil \log_2(t_N(G') + 1) \rceil,$$

as it was stated in [29].

The purpose in the second chapter of this thesis will be to extend these results above to a larger class of groups, namely, it is enough to assume that $\gamma_3(G) \subseteq (G')^p$, instead of the condition G is of class two.

The investigation of the next chapter was motivated by the following result of A. Shalev [27]: if G is nilpotent of class two with cyclic commutator subgroup of order p^n, then $\mathrm{dl}_L(FG) = \lceil \log_2(p^n + 1) \rceil$. We generalize this result and determine both the Lie derived length and the strong Lie derived length of group algebras in the case when the commutator subgroup of the basic group is cyclic of odd order.

For characteristic two, when G is a nilpotent group with (not necessary cyclic) commutator subgroup of order 2^n, in the fourth chapter we obtain the description of the group algebras FG which have the highest possible value of $\mathrm{dl}_L(FG)$, namely, $n + 1$.

By the inclusion $\delta^{[n]}(FG) \subseteq \delta^{(n)}(FG)$, a strongly Lie solvable group algebra FG is Lie solvable too and $\mathrm{dl}_L(FG) \leq \mathrm{dl}^L(FG)$. It would be also interesting to know when the equality $\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG)$ does hold, but this question is still open. As a consequence, we get a necessary and sufficient condition for $\mathrm{dl}_L(FG)$ to coincide with $\mathrm{dl}^L(FG)$, provided that G' is cyclic.

According to (1.2), $\lceil \log_2(p + 1) \rceil \leq \mathrm{dl}^L(FG)$, where p is the characteristic of F. The characterization of the group algebras of minimal strong Lie derived length is also a consequence of our result.

We finish the study of the Lie derived length with the description of the group algebras FG of Lie derived length three, in the case when G' is cyclic and a possibility of the application of this result will be shown. This results are contained in Chapter 5.

1.2.3 Lie nilpotency indices of group algebras

The object of the sixth chapter is the investigation of the Lie nilpotency indices of Lie nilpotent group algebras. For the noncommutative modular group algebra FG the next theorem from A.A. Bódi and I.I. Khripta [7] is well-known: The following statements are equivalent: (i) FG is Lie nilpotent; (ii) FG is upper Lie nilpotent; (iii) G is a nilpotent group whose commutator subgroup is a finite p-group and $\mathrm{char}(F) = p$.

Since $(FG)^{[n]} \subseteq (FG)^{(n)}$ for all n, $t_L(FG) \leq t^L(FG)$. Moreover, for characteristic $p > 3$ the equality is also guaranteed by a result of A.K. Bhandari and I.B.S. Passi [2], but the problem is still open otherwise.

According to [32], if FG is Lie nilpotent and G' has order p^n, then

$$t_L(FG) \leq t^L(FG) \leq p^n + 1.$$

A. Shalev in [25] began to study the question when a Lie nilpotent group algebra has the maximal upper Lie nilpotency index, but the complete description of such group algebras was given by V. Bódi and E. Spinelli in [13]. Joining this research we determine the group algebras whose upper Lie nilpotency index is 'almost maximal', that is, it takes the next highest possible value, namely $p^n - p + 2$, where p^n is the order of the commutator subgroup of the basic group.

Chapter 2

An extension of a result of A. Shalev

As it was proved in [27, 29] by A. Shalev, if G is a nilpotent group of class two and $\mathrm{char}(F) = p$, then $\mathrm{dl}_L(FG) \leq \lceil \log_2(t_N(G')+1) \rceil$ and, in particular, if G is an abelian-by-cyclic p-group with $p > 2$ then the equality holds. Our goal is to extend this result to groups G for which $\gamma_3(G) \subseteq (G')^p$ holds. We note that these groups are nilpotent, according to a result of A.G.R. Stewart [33].

2.1 Preliminary results

We will refer to the following statements in the proofs and the examples.

Proposition 2.1.1 (M. Sahai [24]). *For all $n \geq 1$*

$$\mathfrak{I}(G')^{2^n-1} \subseteq \delta^{(n)}(FG) \subseteq \mathfrak{I}(G')^{2^{n-1}}.$$

Proposition 2.1.2 (F. Levin and G. Rosenberger [19]). *The group algebra FG is Lie metabelian if and only if one of the following statements is satisfied:*

(i) $p = 3$ and G' is central of order 3;

(ii) *$p = 2$ and G' is central elementary abelian subgroup of order dividing 4.*

Moreover, $\mathrm{dl}_L(FG) = 2$ *if and only if* $\mathrm{dl}^L(FG) = 2$.

Proposition 2.1.3 (M. Sahai [24])**.** *Let G be a group and let F be a field of characteristic $p > 2$. Then $\delta^{(3)}(FG) = 0$ if and only if one of the following conditions holds:*

(i) *G is abelian;*

(ii) *$p = 7$, $G' = C_7$ and $\gamma_3(G) = 1$;*

(iii) *$p = 5$, $G' = C_5$ and either $\gamma_3(G) = 1$ or $\gamma_n(G) = G'$ for all $n \geq 3$ with $x^g = x^{-1}$ for all $x \in G'$ and $g \notin C_G(G')$;*

(iv) *$p = 3$ and G' is a group of one of the following types:*

a) *$G' = C_3$;*

b) *$G' = C_3 \times C_3$ and either $\gamma_3(G) = 1$ or $\gamma_3(G) = C_3$, $\gamma_4(G) = 1$ or $\gamma_n(G) = G'$ for all $n \geq 3$ with $x^g = x^{-1}$ for all $x \in G'$ and $g \notin C_G(G')$;*

c) *$G' = C_3 \times C_3 \times C_3$ and $\gamma_3(G) = 1$.*

Proposition 2.1.4 (A.A. Bódi and J. Kurdics [8])**.** *Let G be a nilpotent group whose commutator subgroup is a finite p-group and let* $\mathrm{char}(F) = p$. *If $\gamma_3(G) \subseteq (G')^p$ then $(FG)^{(n)} = \mathfrak{I}(G')^{n-1}$ for all $n \geq 2$.*

2.2 The generalized result

We need the following lemma.

Lemma 2.2.1. *Let G be a group with commutator subgroup of order* p^n, $\mathrm{char}(F) = p$ *and assume that $\gamma_3(G) \subseteq (G')^p$. Then for all $m \geq 1$*

$$\big[\omega^m(FG'), \omega(FG)\big] \subseteq \mathfrak{I}(G')^{m+p-1}.$$

Moreover, if G' is abelian, then for all $m, k \geq 1$

$$\big[\mathfrak{I}(G')^m, \mathfrak{I}(G')^k\big] \subseteq \mathfrak{I}(G')^{m+k+1}.$$

Proof. We use induction on m. For every $y \in G'$ and $g \in G$ we have

$$[y-1, g-1] = [y, g] = gy\big((y,g)-1\big) \in \mathfrak{I}\big(\gamma_3(G)\big) \subseteq \mathfrak{I}(G')^p.$$

This shows that the statement holds for $m = 1$, because all elements of the form $g - 1$ with $g \in G$ constitute an F-basis of $\omega(FG)$.

Now, assume that $\big[\omega^m(FG'), \omega(FG)\big] \subseteq \mathfrak{I}(G')^{m+p-1}$ for some m. Then

$$\begin{aligned}
&\big[\omega^{m+1}(FG'), \omega(FG)\big] \\
&\qquad \subseteq \omega^m(FG')\big[\omega(FG'), \omega(FG)\big] + \big[\omega^m(FG'), \omega(FG)\big]\omega(FG') \\
&\qquad \subseteq \omega^m(FG')\mathfrak{I}(G')^p + \mathfrak{I}(G')^{m+p-1}\omega(FG') \subseteq \mathfrak{I}(G')^{m+p},
\end{aligned}$$

and the proof of the first assertion is complete. The second one is a consequence of the first one, because

$$\mathfrak{I}(G') = \omega(FG)\omega(FG') + \omega(FG').$$

Indeed,

$$\begin{aligned}
\big[\mathfrak{I}(G'), \mathfrak{I}(G')\big] &\subseteq \big[\omega(FG)\omega(FG'), \omega(FG)\omega(FG')\big] \\
&\qquad + \big[\omega(FG)\omega(FG'), \omega(FG')\big] \\
&\subseteq \omega(FG)\big[\omega(FG), \omega(FG')\big]\omega(FG') \\
&\qquad + \big[\omega(FG), \omega(FG)\big]\omega^2(FG') \\
&\subseteq \mathfrak{I}(G')^3,
\end{aligned}$$

hence by induction on m we have

$$\begin{aligned}
\big[\mathfrak{I}(G')^{m+1}, \mathfrak{I}(G')\big] &\subseteq \mathfrak{I}(G')\big[\mathfrak{I}(G')^m, \mathfrak{I}(G')\big] + \big[\mathfrak{I}(G'), \mathfrak{I}(G')\big]\mathfrak{I}(G')^m \\
&\subseteq \mathfrak{I}(G')^{m+3}.
\end{aligned}$$

Now one can finish the proof similarly by an induction on k. $\square$

Similar inclusions were proved by C. Bagiński in [1] for finite p-groups.

Theorem 2.2.2. *Let G be a nilpotent group whose commutator subgroup is a finite p-group,* $\mathrm{char}(F) = p$ *and assume that* $\gamma_3(G) \subseteq (G')^p$. *Then*

$$\mathrm{dl}_L(FG) \leq \mathrm{dl}^L(FG) = \lceil \log_2(t_N(G') + 1) \rceil,$$

and if G is an abelian-by-cyclic p-group with $p > 2$ then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2 t_N(G') + 1 \rceil.$$

Proof. Recall that $\delta^{(n)}(FG) \subseteq (FG)^{(2^n)}$ for all $n \geq 0$. For $n \geq 1$ Proposition 2.1.1 yields that

$$\mathfrak{I}(G')^{2^n-1} \subseteq \delta^{(n)}(FG)$$

and since $\gamma_3(G) \subseteq (G')^p$, by Proposition 2.1.4, we have

$$(FG)^{(2^n)} = \mathfrak{I}(G')^{2^n-1}.$$

Combining this facts we get

$$\mathfrak{I}(G')^{2^n-1} \subseteq \delta^{(n)}(FG) \subseteq (FG)^{(2^n)} = \mathfrak{I}(G')^{2^n-1}.$$

Now it is easy to see that $\delta^{(n)}(FG) = 0$ if and only if $2^n - 1 \geq t_N(G')$, therefore $n \geq \log_2(t_N(G') + 1)$, which implies the statement.

Let now G be an abelian-by-cyclic p-group with $p > 2$. Then $G = \langle A, x \rangle$ for some abelian normal subgroup A of G and $x \in G$. Clearly, $G' = (A, x)$ is abelian and all $z \in G'$ can be written in the form $z = (a_1, x) \cdots (a_s, x)$ for some $a_1, \dots, a_s \in A$.

We are going to show that if $c \in A$, $z_1, z_2, \dots, z_{2^n-1} \in G'$ and j is not divisible by p, then there exists $\varrho \in \mathfrak{I}(G')^{2^n}$ such that

$$x^j c(1 - z_1)(1 - z_2) \cdots (1 - z_{2^n-1}) + \varrho \in \delta^{[n]}(FG).$$

We use induction on n. Let first $n = 1$ and let us choose an integer k such that $2k \equiv j$ modulo the order of x. Then $G' = (A, x^k)$ and $z_1 = (a_1, x^k) \cdots (a_s, x^k)$ for some $a_1, \dots, a_s \in A$. Applying the identity

$$1 - \alpha\beta = (1 - \alpha) + (1 - \beta) - (1 - \alpha)(1 - \beta)$$

a sufficient number times, we have

$$(2.1) \qquad x^j c(1-z_1) \equiv \sum_{i=1}^{s} x^j c\big(1-(a_i,x^k)\big) \pmod{\mathfrak{I}(G')^2}.$$

Since p is an odd prime, we can choose the elements u_i, v_i such that $u_i^2 = ca_i^{-1}$ and $v_i^2 = ca_i$. Then $u_i, v_i \in A$, $(u_iv_i)^2 = c^2$ and $(u_i^{-1}v_i)^2 = a_i^2$ which implies $u_iv_i = c$ and $u_i^{-1}v_i = a_i$. Setting $w_i = x^k u_i (x^k)^{-1}$ we have

$$\begin{aligned}[x^k w_i, x^k v_i] &= x^j(w_i^{x^k} v_i - w_i v_i^{x^k}) \\ &= x^j w_i^{x^k} v_i\big(1-(w_i^{-1}v_i, x^k)\big) = x^j c\big(1-(a_i,x^k)\big),\end{aligned}$$

because $(w_i^{-1}v_i, x^k) = (u_i^{-1}v_i, x^k) = (a_i, x^k)$. Now by (2.1) it follows that

$$(2.2) \qquad x^j c(1-z_1) \equiv \sum_{i=1}^{s} [x^k w_i, x^k v_i] \pmod{\mathfrak{I}(G')^2},$$

which proves our statement for $n = 1$.

Now, assume that $j, c, z_1, z_2, \dots, z_{2^n-1}$ have already been given, and let $2k \equiv j$ modulo the order of x. We can apply the method above to find elements $w_i, v_i \in A$ such that the congruence (2.2) holds. Set

$$f_i = x^k w_i (1-z_2)\cdots(1-z_{2^{n-1}})$$

and

$$g_i = x^k v_i (1-z_{2^{n-1}+1})\cdots(1-z_{2^n-1}).$$

for $1 \le i \le s$. By the induction hypothesis there exist $\varrho_1^{(i)}, \varrho_2^{(i)} \in \mathfrak{I}(G')^{2^{n-1}}$ such that $f_i + \varrho_1^{(i)}, g_i + \varrho_2^{(i)} \in \delta^{[n-1]}(FG)$. Evidently,

$$\begin{aligned}\big[f_i + \varrho_1^{(i)}, g_i + \varrho_2^{(i)}\big] = [f_i, g_i] + [f_i, \varrho_2^{(i)}] \\ + [\varrho_1^{(i)}, g_i] + [\varrho_1^{(i)}, \varrho_2^{(i)}] \in \delta^{[n]}(FG).\end{aligned}$$

According to Lemma 2.2.1 the last three summands are in $\Im(G')^{2^n}$. Furthermore,

$$\begin{aligned}
&\big[f_i, g_i\big] \\
&\quad = x^k w_i\big[(1-z_2)\cdots(1-z_{2^{n-1}}), x^k v_i\big](1-z_{2^{n-1}+1})\cdots(1-z_{2^n-1}) \\
&\quad + x^k v_i\big[x^k w_i, (1-z_{2^{n-1}+1})\cdots(1-z_{2^n-1})\big](1-z_2)\cdots(1-z_{2^{n-1}}) \\
&\quad + \big[x^k w_i, x^k v_i\big](1-z_2)\cdots(1-z_{2^n-1})
\end{aligned}$$

and the first two summands on the right-hand side belong to $\Im(G')^{2^n}$ by Lemma 2.2.1. So,

$$\begin{aligned}
&\big[f_i + \varrho_1^{(i)}, g_i + \varrho_2^{(i)}\big] \\
&\qquad\qquad \equiv \big[x^k w_i, x^k v_i\big](1-z_2)\cdots(1-z_{2^n-1}) \pmod{\Im(G')^{2^n}},
\end{aligned}$$

for all $1 \le i \le s$. Summing this for all possible i, we get

$$x^j c(1-z_1)(1-z_2)\cdots(1-z_{2^n-1}) + \varrho \in \delta^{[n]}(FG),$$

for some $\varrho \in \Im(G')^{2^n}$, as we claimed.

Let the abelian p-group G' be presented as $G' = \langle c_1\rangle \times \cdots \times \langle c_r\rangle$, and assume that c_i is of order p^{m_i}. The product

$$(1-c_1)^{i_1}(1-c_2)^{i_2}\cdots(1-c_r)^{i_r}$$

with $0 \le i_j \le p^{m_j}$ has $s = i_1 + i_2 + \cdots + i_r$ factors and, according to Jennings' theory, it belongs to $\omega^s(FG') \setminus \omega^{s+1}(FG')$. It follows that if $2^n - 1 < t_N(G')$ then the elements $z_1, z_2, \ldots, z_{2^n-1}$ of G' can be chosen such that

$$x^j c(1-z_1)(1-z_2)\cdots(1-z_{2^n-1}) \in \Im(G')^{2^n-1} \setminus \Im(G')^{2^n}.$$

We have that $\delta^{[n]}(FG)$ has nonzero elements while $2^n - 1 < t_N(G')$, and therefore

$$\mathrm{dl}_L(FG) \ge \lceil \log_2 t_N(G') + 1 \rceil.$$

The converse inequality follows immediately from the first part of the theorem. □

As the following examples show, Theorem 2.2.2 breaks down without the condition $\gamma_3(G) \subseteq (G')^p$.

- Let G be a group with $G' = C_2 \times C_2$ such that $\gamma_3(G) \neq 1$ and let $\mathrm{char}(F) = 2$. Then $\gamma_3(G) \nsubseteq (G')^2$ and, by Proposition 2.1.2, $\mathrm{dl}^L(FG) > 2$. So $\mathrm{dl}^L(FG) \neq \lceil \log_2(t_N(G') + 1) \rceil$, because now $\lceil \log_2(t_N(G') + 1) \rceil = 2$.

- Let G be a group with $G' = C_3 \times C_3 \times C_3$ such that $\gamma_3(G) \neq 1$ and let $\mathrm{char}(F) = 3$. Then $\gamma_3(G) \nsubseteq (G')^3$ and, by Proposition 2.1.3, $\mathrm{dl}^L(FG) > 3$. It follows that $\mathrm{dl}^L(FG) \neq \lceil \log_2(t_N(G') + 1) \rceil$, because $\lceil \log_2(t_N(G') + 1) \rceil = 3$.

Chapter 3

Lie derived lengths of group algebras of groups with cyclic commutator subgroup

In this chapter we determine the Lie derived length and the strong Lie derived length of group algebras in the case when the commutator subgroup of the basic group is cyclic of odd order.

We distinguish two cases according as the basic group is nilpotent or not.

3.1 The basic group is nilpotent

A. Shalev proved the following

Proposition 3.1.1 (A. Shalev [27]). *Let G be a nilpotent group of class two with cyclic commutator subgroup of order p^n and let* $\mathrm{char}(F) = p$. *Then*

$$\mathrm{dl}_L(FG) = \lceil \log_2(p^n + 1) \rceil.$$

Evidently, when G' is cyclic and G is nilpotent, the condition $\gamma_3(G) \subseteq (G')^p$ holds, therefore Theorem 2.2.2 ensures that the integer $\lceil \log_2(p^n + 1) \rceil$ is an upper bound on $\mathrm{dl}_L(FG)$. It is shown here that this bound is always achieved for odd p, that is, Proposition 3.1.1 is valid for arbitrary nilpotent groups with cyclic commutator subgroup, provided that p is odd.

Theorem 3.1.2. *Let G be a nilpotent group with cyclic commutator subgroup of order p^n, where p is an odd prime, and let* $\mathrm{char}(F) = p$. *Then*

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2(p^n+1) \rceil.$$

Proof. Let $G' = \langle x \mid x^{p^n} = 1 \rangle$ and let us choose $a, b \in G$ such that $x = (a, b)$. First of all, we claim that

$$[b^l a^m, b^s a^t] \equiv (ms - lt) b^{l+s} a^{m+t}(x-1) \pmod{\mathfrak{I}(G')^2} \tag{3.1}$$

for every $l, s, m, t \in \mathbb{Z}$. Indeed, an easy computation yields

$$\begin{aligned} \big[b^l a^m, b^s a^t\big] &= b^s a^t b^l a^m \big((b^l a^m, b^s a^t) - 1\big) \\ &= b^{l+s} a^{m+t} (a^t, b^l)^{a^m} \big((b^l a^m, b^s a^t) - 1\big) \\ &\equiv b^{l+s} a^{m+t} \big((b^l a^m, b^s a^t) - 1\big) \pmod{\mathfrak{I}(G')^2}, \end{aligned} \tag{3.2}$$

and since now $\gamma_3(G) \subseteq (G')^p$,

$$\begin{aligned} (b^l a^m, b^s a^t) &\equiv (b^l, a^t)(a^m, b^s) \\ &\equiv (b,a)^{lt}(a,b)^{ms} \equiv x^{ms-lt} \pmod{(G')^p}. \end{aligned}$$

Thus $(b^l a^m, b^s a^t) = x^{ms-lt+pi}$ for some i. In view of the identity

$$uv - 1 = (u-1)(v-1) + (u-1) + (v-1),$$

we have

$$\begin{aligned} (b^l a^m, b^s a^t) - 1 &\equiv (ms - lt + pi)(x-1) \\ &\equiv (ms - lt)(x-1) \pmod{\mathfrak{I}(G')^2} \end{aligned}$$

and putting this into (3.2) we obtain (3.1).

Now, let $k \geq 1$, $l, m, s, t \in \mathbb{Z}$, $z_1, z_2 \in \mathfrak{I}(G')^{2^k}$ and set

$$f_k(l, m, s, t, z_1, z_2) = \big[b^l a^m (x-1)^{2^k - 1} + z_1, b^s a^t (x-1)^{2^k-1} + z_2\big].$$

We shall show that

$$f_k(l,m,s,t,z_1,z_2) \equiv (ms-lt)b^{l+s}a^{m+t}(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}}. \tag{3.3}$$

Lemma 2.2.1 ensures that the elements

$$\big[b^l a^m (x-1)^{2^k-1}, z_2\big], \big[z_1, z_2\big] \quad \text{and} \quad \big[z_1, b^s a^t (x-1)^{2^k-1}\big]$$

belong to $\mathfrak{I}(G')^{2^{k+1}}$, thus

$$f_k(l,m,s,t,z_1,z_2) \equiv \big[b^l a^m (x-1)^{2^k-1}, b^s a^t (x-1)^{2^k-1}\big] \pmod{\mathfrak{I}(G')^{2^{k+1}}}.$$

Furthermore, for $p>2$ the inclusions

$$\big[b^l a^m, (x-1)^{2^k-1}\big], \big[(x-1)^{2^k-1}, b^s a^t\big] \in \mathfrak{I}(G')^{2^k+1}$$

is guaranteed by Lemma 2.2.1 and it implies that

$$f_k(l,m,s,t,z_1,z_2) \equiv [b^l a^m, b^s a^t](x-1)^{2^{k+1}-2} \pmod{\mathfrak{I}(G')^{2^{k+1}}}.$$

This congruence, together with (3.1), proves (3.3).

Define the following three series of elements of FG inductively by:

$$u_0 = a, \quad v_0 = b, \quad w_0 = b^{-1}a^{-1},$$

and, for $k \geq 0$,

$$u_{k+1} = [u_k, v_k], \quad v_{k+1} = [u_k, w_k], \quad w_{k+1} = [w_k, v_k].$$

Obviously, the k-th elements of these series belong to $\delta^{[k]}(FG)$. By induction on k we show for odd k that

$$\begin{aligned} u_k &\equiv \pm ba(x-1)^{2^k-1} \pmod{\mathfrak{I}(G')^{2^k}}; \\ v_k &\equiv \pm b^{-1}(x-1)^{2^k-1} \pmod{\mathfrak{I}(G')^{2^k}}; \\ w_k &\equiv \pm a^{-1}(x-1)^{2^k-1} \pmod{\mathfrak{I}(G')^{2^k}}, \end{aligned} \tag{3.4}$$

and if k is even then

$$(3.5)\qquad \begin{aligned} u_k &\equiv \pm a(x-1)^{2^k-1} \pmod{\mathfrak{I}(G')^{2^k}};\\ v_k &\equiv \pm b(x-1)^{2^k-1} \pmod{\mathfrak{I}(G')^{2^k}};\\ w_k &\equiv \pm b^{-1}a^{-1}(x-1)^{2^k-1} \pmod{\mathfrak{I}(G')^{2^k}}. \end{aligned}$$

Evidently, $u_1 = [a,b] = ba(x-1)$, and from (3.1) it follows that

$$v_1 = [a, b^{-1}a^{-1}] \equiv -b^{-1}(x-1) \pmod{\mathfrak{I}(G')^2},$$

and $w_1 = [b^{-1}a^{-1}, b] \equiv -a^{-1}(x-1) \pmod{\mathfrak{I}(G')^2}$. Therefore (3.4) holds for $k=1$.
Now, assume that (3.4) is true for some odd k. According to (3.3) the congruences

$$\begin{aligned} u_{k+1} &= \pm f_k(1,1,-1,0,{u_k}',{v_k}')\\ &\equiv \pm(-1)a(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}};\\ v_{k+1} &= \pm f_k(1,1,0,-1,{u_k}',{v_k}')\\ &\equiv \pm b(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}};\\ w_{k+1} &= \pm f_k(0,-1,-1,0,{u_k}',{v_k}')\\ &\equiv \pm b^{-1}a^{-1}(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}} \end{aligned}$$

hold, where ${u_k}', {v_k}', {w_k}'$ are suitable elements from $\mathfrak{I}(G')^{2^k}$. Similarly, supposing the truth of (3.5) for some even k we see

$$\begin{aligned} u_{k+1} &= \pm f_k(0,1,1,0,{u_k}',{v_k}')\\ &\equiv \pm ba(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}};\\ v_{k+1} &= \pm f_k(0,1,-1,-1,{u_k}',{v_k}')\\ &\equiv \pm(-1)b^{-1}(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}};\\ w_{k+1} &= \pm f_k(-1,-1,1,0,{u_k}',{v_k}')\\ &\equiv \pm(-1)a^{-1}(x-1)^{2^{k+1}-1} \pmod{\mathfrak{I}(G')^{2^{k+1}}}. \end{aligned}$$

So, (3.4) and (3.5) are valid for any $k > 0$.

Assume that $k < \lceil \log_2(p^n+1) \rceil$. Then $2^k - 1 < p^n$ and the elements u_k, v_k, w_k are nonzero in $\delta^{[k]}(FG)$, thus $\mathrm{dl}_L(FG) \geq \lceil \log_2(p^n+1) \rceil$.

At the same time, Theorem 2.2.2 says that

$$\mathrm{dl}^L(FG) \leq \lceil \log_2(p^n+1) \rceil.$$

The proof is complete. □

3.2 The basic group is not nilpotent

Let G be a group with commutator subgroup $G' = \langle x \mid x^{p^n} = 1 \rangle$ of odd order, and let C denote its centralizer in G. As it is well-known, the automorphism group of G' is isomorphic to the unit group $U(\mathbb{Z}_{p^n})$ of $\mathbb{Z}_{p^n}$. Since p is odd, $U(\mathbb{Z}_{p^n})$ is cyclic, so the factor group G/C, which is isomorphic to a subgroup of it, is cyclic, too. In our observation the order of the group G/C will play an important role.

It is easy to check that

$$\mathrm{Syl}_p\big(U(\mathbb{Z}_{p^n})\big) = \big\{\overline{k} \in \mathbb{Z}_{p^n} \mid k \equiv 1 \pmod{p}\big\},$$

from which it follows for $a \in G$ that the automorphism $x \mapsto x^a$ of G' has p-power order if and only if $x^a = x^k$, where $\overline{k} \in \mathrm{Syl}_p\big(U(\mathbb{Z}_{p^n})\big)$. In other words, the element aC has p-power order if and only if $x^a = x^k$ for some $k \equiv 1 \pmod{p}$.

Lemma 3.2.1. *Let G be a group with cyclic commutator subgroup of order p^n, where p is an odd prime. Then G is nilpotent if and only if G/C is a p-group.*

Proof. Assume first that G/C is a p-group. The normal subgroup G' can be written as union of different conjugacy classes of G, moreover, the assumption ensures that every class has p-power order. Since G' has order p^n, there are at least p classes which contain only one element, that is, G' contains nontrivial central elements of G. Consequently, G is nilpotent.

Conversely, assume that G is nilpotent. Set $G' = \langle x \mid x^{p^n} = 1 \rangle$, $G/C = \langle aC \rangle$ and $x^k = x^a$. Then $(x, a) = x^{-1+k}$ belongs to $\gamma_3(G)$ and since $\gamma_3(G) \subseteq (G')^p$, we have that $k \equiv 1 \pmod{p}$. Therefore, aC has p-power order and G/C is a p-group. □

Since we have already closed the nilpotent case, by Lemma 3.2.1 we may assume that G/C has order $2^m p^r$ with $m > 0, r \geq 0$, or it is divisible by some odd prime not equal to p. These two cases will be discussed in the sequel.

Choose the element $\langle aC \rangle$ and the integer k such that $G/C = \langle aC \rangle$ and $x^k = x^a$. We claim that

$$[(x-1)^{2^l}, a] \equiv (k^{2^l} - 1)a(x-1)^{2^l} \pmod{\mathfrak{I}(G')^{2^l+1}}, \tag{3.6}$$

for any $l \geq 0$. Using the well-known equation

$$(x^s - 1) = \sum_{i=1}^{s} \binom{s}{i} (x-1)^i, \tag{3.7}$$

we have

$$\begin{aligned}
[(x-1)^{2^l}, a] &\equiv a\left(\left(\sum_{i=1}^{k} \binom{k}{i} (x-1)^i \right)^{2^l} - (x-1)^{2^l} \right) \\
&\equiv a\big(k^{2^l}(x-1)^{2^l} - (x-1)^{2^l}\big) \\
&\equiv (k^{2^l} - 1)a(x-1)^{2^l} \pmod{\mathfrak{I}(G')^{2^l+1}},
\end{aligned}$$

as we claimed.

Lemma 3.2.2. *Let G be a group with cyclic commutator subgroup of order p^n, where p is an odd prime, and let* $\mathrm{char}(F) = p$. *If G/C has order $2^m p^r$ then*

$$[\mathfrak{I}(G')^{2^m i}, \mathfrak{I}(G')^{2^m j}] \subseteq \mathfrak{I}(G')^{2^m i + 2^m j + 1}.$$

Proof. With the notation introduced above, the assumption ensures the congruence $k^{2^m} \equiv 1 \pmod{p}$, thus we can apply (3.6) to conclude the inclusion

$$[\omega(FG')^{2^m}, FG] \subseteq \mathfrak{I}(G')^{2^m+1}.$$

Hence, for $i = j = 1$ we have

$$
\begin{aligned}
[\mathfrak{I}(G')^{2^m}, \mathfrak{I}(G')^{2^m}] &= [\omega^{2^m}(FG')FG, \omega^{2^m}(FG')FG] \\
&\subseteq \omega^{2^{m+1}}(FG')[FG, FG] + \omega^{2^m}(FG')[\omega^{2^m}(FG'), FG]FG \\
&\subseteq \mathfrak{I}(G')^{2^{m+1}+1},
\end{aligned}
$$

and by induction on i we obtain

$$
\begin{aligned}
[\mathfrak{I}(G')^{2^m i}, \mathfrak{I}(G')^{2^m}] &= \mathfrak{I}(G')^{2^m}[\mathfrak{I}(G')^{2^m(i-1)}, \mathfrak{I}(G')^{2^m}] \\
&\quad + [\mathfrak{I}(G')^{2^m}, \mathfrak{I}(G')^{2^m}]\mathfrak{I}(G')^{2^m(i-1)} \\
&\subseteq \mathfrak{I}(G')^{2^m i + 2^m + 1}.
\end{aligned}
$$

Now we can finish the proof by an easy induction on j. Indeed,

$$
\begin{aligned}
[\mathfrak{I}(G')^{2^m i}, \mathfrak{I}(G')^{2^m j}] &= \mathfrak{I}(G')^{2^m}[\mathfrak{I}(G')^{2^m i}, \mathfrak{I}(G')^{2^m(j-1)}] \\
&\quad + [\mathfrak{I}(G')^{2^m i}, \mathfrak{I}(G')^{2^m}]\mathfrak{I}(G')^{2^m(j-1)} \\
&\subseteq \mathfrak{I}(G')^{2^m i + 2^m j + 1}.
\end{aligned}
$$

□

Definition. *For $m \geq 0$ let us define the series $(s_l^{(m)})$ as follows:*

$$
s_l^{(m)} = \begin{cases} 1 & \textit{if } l = 0; \\ 2s_{l-1}^{(m)} + 1 & \textit{if } s_{l-1}^{(m)} \textit{ is divisible by } 2^m; \\ 2s_{l-1}^{(m)} & \textit{otherwise.} \end{cases}
$$

It is easy to check that for $m = 1$ this series can also be given in the form

$$
s_l^{(1)} = \begin{cases} (2^{l+2} - 1)/3 & \text{if } l \text{ is even;} \\ (2^{l+2} - 2)/3 & \text{if } l \text{ is odd.} \end{cases} \tag{3.8}
$$

Lemma 3.2.3. *Let $G = D_{2p^n} = \langle a, b \mid a^2 = b^{p^n} = 1, ba = ab^{-1} \rangle$ be the dihedral group of order $2p^n$, where p is an odd prime, and let $\mathrm{char}(F) = p$. If d is the minimal integer such that $s_d^{(1)} \geq p^n$, then $\mathrm{dl}_L(FG) \geq d + 1$.*

Proof. Evidently, $x = (a, b)$ is of order p^n, $\quad x^b = x$ and $x^a = x^{-1}$.

Define the following three series of elements of FG inductively by

$$u_0 = a, \quad v_0 = b, \quad w_0 = b^{-1}a,$$

and for $l \geq 0$, by

$$u_{l+1} = [u_l, v_l], \quad v_{l+1} = [u_l, w_l], \quad w_{l+1} = [w_l, v_l].$$

By induction on l we show for odd l that

$$\begin{aligned} u_l &\equiv t_u^{(l)} ba(x^{-1} - 1)^{s_{l-2}}(x - 1)^{s_{l-2}+1} \pmod{\mathfrak{I}(G')^{s_{l-1}+2}};\\ v_l &\equiv t_v^{(l)} b^{-1}(x^{-1} - 1)^{s_{l-2}}(x - 1)^{s_{l-2}+1} \pmod{\mathfrak{I}(G')^{s_{l-1}+2}};\\ w_l &\equiv t_w^{(l)} a(x^{-1} - 1)^{s_{l-2}}(x - 1)^{s_{l-2}+1} \pmod{\mathfrak{I}(G')^{s_{l-1}+2}}, \end{aligned} \tag{3.9}$$

and if l is even then

$$\begin{aligned} u_l &\equiv t_u^{(l)} a(x^{-1} - 1)^{s_{l-2}}(x - 1)^{s_{l-2}} \pmod{\mathfrak{I}(G')^{s_{l-1}+2}};\\ v_l &\equiv t_v^{(l)} b(x^{-1} - 1)^{s_{l-2}+1}(x - 1)^{s_{l-2}} \pmod{\mathfrak{I}(G')^{s_{l-1}+2}};\\ w_l &\equiv t_w^{(l)} b^{-1}a(x^{-1} - 1)^{s_{l-2}}(x - 1)^{s_{l-2}} \pmod{\mathfrak{I}(G')^{s_{l-1}+2}}, \end{aligned} \tag{3.10}$$

where $t_u^{(l)}, t_v^{(l)}, t_w^{(l)}$ are nonzero elements in the field F, and for convenience let us set $s_l = s_l^{(1)}$ with $s_{-1} = 0$. Obviously, $u_1 = [a, b] = ba(x - 1)$,

$$v_1 = [a, b^{-1}a] = b^{-1}\big((x^{-1})^a - 1\big) = b^{-1}(x - 1),$$

and similarly, $w_1 = [b^{-1}a, b] = a(x - 1)$. Therefore (3.9) holds for $l = 1$. Now, assume that (3.9) is true for some odd l. Then by (3.8),

$$\begin{aligned} &u_{l+1}\\ &\equiv t_u^{(l)} t_v^{(l)} [ba(x^{-1} - 1)^{s_{l-2}}(x - 1)^{s_{l-2}+1}, b^{-1}(x^{-1} - 1)^{s_{l-2}}(x - 1)^{s_{l-2}+1}]\\ &\equiv t_u^{(l)} t_v^{(l)} \big(bab^{-1}(x^{-1} - 1)^{2s_{l-2}}(x - 1)^{2s_{l-2}+2}\\ &\qquad\qquad - a(x^{-1} - 1)^{2s_{l-2}+1}(x - 1)^{2s_{l-2}+1}\big)\\ &\equiv (-2) t_u^{(l)} t_v^{(l)} a(x^{-1} - 1)^{s_{l-1}}(x - 1)^{s_{l-1}} \pmod{\mathfrak{I}(G')^{s_l+2}}; \end{aligned}$$

$$\begin{aligned} &v_{l+1} \\ &\quad\equiv t_u^{(l)}t_w^{(l)}[ba(x^{-1}-1)^{s_{l-2}}(x-1)^{s_{l-2}+1}, a(x^{-1}-1)^{s_{l-2}}(x-1)^{s_{l-2}+1}] \\ &\quad\equiv t_u^{(l)}t_w^{(l)}\big(b(x^{-1}-1)^{2s_{l-2}+1}(x-1)^{2s_{l-2}+1} \\ &\qquad\qquad - aba(x^{-1}-1)^{2s_{l-2}+1}(x-1)^{2s_{l-2}+1}\big) \\ &\quad\equiv t_u^{(l)}t_w^{(l)}b(x^{-1}-1)^{s_{l-1}+1}(x-1)^{s_{l-1}} \pmod{\mathfrak{I}(G')^{s_l+2}}, \end{aligned}$$

and

$$\begin{aligned} &w_{l+1} \\ &\quad\equiv t_w^{(l)}t_v^{(l)}[a(x^{-1}-1)^{s_{l-2}}(x-1)^{s_{l-2}+1}, b^{-1}(x^{-1}-1)^{s_{l-2}}(x-1)^{s_{l-2}+1}] \\ &\quad\equiv t_w^{(l)}t_v^{(l)}\big(ab^{-1}(x^{-1}-1)^{2s_{l-2}}(x-1)^{2s_{l-2}+2} \\ &\qquad\qquad - b^{-1}a(x^{-1}-1)^{2s_{l-2}+1}(x-1)^{2s_{l-2}+1}\big) \\ &\quad\equiv t_w^{(l)}t_v^{(l)}(-2)b^{-1}a(x^{-1}-1)^{s_{l-1}+1}(x-1)^{s_{l-1}} \pmod{\mathfrak{I}(G')^{s_l+2}}. \end{aligned}$$

Supposing the truth of (3.10) for some even l, we apply Lemma 3.2.2 (with $m=1$) and (3.8) to get the congruence

$$u_{l+1} \equiv t_u^{(l)}t_v^{(l)}[a(x^{-1}-1)^{s_{l-2}}(x-1)^{s_{l-2}}, b(x^{-1}-1)^{s_{l-2}+1}(x-1)^{s_{l-2}}] \pmod{\mathfrak{I}(G')^{s_l+2}}.$$

Hence

$$\begin{aligned} u_{l+1} &\equiv t_u^{(l)}t_v^{(l)}\big(ab(x^{-1}-1)^{2s_{l-2}+1}(x-1)^{2s_{l-2}} \\ &\qquad - ba(x^{-1}-1)^{2s_{l-2}}(x-1)^{2s_{l-2}+1}\big) \\ &\equiv t_u^{(l)}t_v^{(l)}(-2)ba(x^{-1}-1)^{s_{l-1}}(x-1)^{s_{l-1}+1} \pmod{\mathfrak{I}(G')^{s_l+2}}. \end{aligned}$$

Similar calculations give us the required congruences for v_{l+1} and w_{l+1}.

So, (3.9) and (3.10) hold for all $l \geq 0$.

Let d be the minimal integer such that $s_d \geq p^n$. According to the congruences (3.9) and (3.10), u_d is a nonzero element of $\delta^{[d]}(FG)$, because $s_{d-1} < p^n$. So $\mathrm{dl}_L(FG) > d$, from which the statement follows. □

Let $H = \langle h \mid h^{p^n} = 1\rangle$ be a subgroup of a group G and char(F)=p. Then $\omega(FH)$ is nilpotent and $\omega^{p^n}(FH) = 0$. Evidently, for $g \in G \setminus H$

and $s > 0$ the set $g\omega^s(FH) = \{gy \mid y \in \omega^s(FH)\}$ is a vector space over F, and by [16], the set $\{g(h-1)^{s+j} \mid j \geq 0\}$ forms an F-basis of $g\omega^s(FH)$.

Lemma 3.2.4. *Let $H = \langle h \mid h^{p^n} = 1\rangle$ be a subgroup of a group G and* $\operatorname{char}(F) = p$. *Then for all $g \in G \setminus H$ the vector space $g\omega^s(FH)$ can be spanned by the elements $z_i \in g\omega^{s+i}(FH) \setminus g\omega^{s+i+1}(FH)$, where $0 \leq i \leq p^n - s - 1$.*

Proof. Denote by V the vector space spanned by the z_i's. It is enough to prove that $g(h-1)^{s+j} \in V$ for all $j \geq 0$. This is clear when $j \geq p^n - s$. Let t be an integer such that $0 \leq t < p^n - s$ and assume that $g(h-1)^{s+j} \in V$ for all $j > t$. Since $z_t \in g\omega^{s+t}(FH) \setminus g\omega^{s+t+1}(FH)$, it can be written as

$$z_t = \sum_{r \geq t} \alpha_r^{(z_t)} g(h-1)^{s+r},$$

where $\alpha_r^{(z_t)} \in F$ and $\alpha_t^{(z_t)} \neq 0$. Hence

$$\alpha_t^{(z_t)} g(h-1)^{s+t} = z_t - \sum_{r \geq t+1} \alpha_r^{(z_t)} g(h-1)^{s+r}.$$

According to the inductive hypothesis, the right-hand side of this equality belongs to V, and since $\alpha_t^{(z_t)} \neq 0$, we have $g(h-1)^{s+t} \in V$. The proof is complete. □

As before, let G be a group with commutator subgroup $G' = \langle x \mid x^{p^n} = 1\rangle$, where p is an odd prime, but now assume that G/C has order 2^m, with $m > 1$. Since G/C is cyclic, we can choose an element $a \in G$ such that $\langle aC\rangle = G/C$. Clearly, $\langle a^{-1}C\rangle = \langle aC\rangle = G/C$. Setting $x^k = x^a$ and $x^{k'} = x^{a^{-1}}$, it follows that $k \not\equiv 1 \pmod{p}$, $k' \not\equiv 1 \pmod{p}$, furthermore $k^{2^m} \equiv (k')^{2^m} \equiv 1 \pmod{p^n}$ and $kk' \equiv 1 \pmod{p}$. Now, we show that $k^i \equiv 1 \pmod{p}$ if and only if 2^m divides i. Indeed, if $k^i \equiv 1 \pmod{p}$ then $\overline{k^i} \in \operatorname{Syl}_p\big(U(\mathbb{Z}_{p^n})\big)$ and $(\overline{k})^{ip^r} = \overline{1}$ holds in $U(\mathbb{Z}_{p^n})$ for some r. Since $\overline{k}$ is of order 2^m we have that 2^m divides ip^r and therefore 2^m divides i. The converse statement is trivial.

In the following we shall use freely these notations and facts and the congruence

$$(x^s - 1) \equiv s(x-1) \pmod{\omega^2(FG)},$$

which is a simple consequence of (3.7) and is valid for all $x \in G$ and any integer s.

Lemma 3.2.5. *Let G be a group with commutator subgroup $G' = \langle x \mid x^{p^n} = 1\rangle$ of odd order and let $\mathrm{char}(F) = p$. If G/C has order 2^m with $m > 1$ then*

$$a\omega^{s_l^{(m)}}(FG') \oplus a^{-1}\omega^{s_l^{(m)}}(FG') \subseteq \delta^{[l+1]}(FG)$$

for all $l \geq 0$.

Proof. We shall prove this lemma by induction on l. Evidently, $\delta^{[1]}(FG)$ can be considered as a vector space over F, and according to a result of R. Brauer [14] the element

$$\sum_{g\in G} \alpha_g g \in FG$$

belongs to $\delta^{[1]}(FG)$ if and only if $\sum_{g\in C_h} \alpha_g = 0$ for all $h \in G$. The binomial formula says that the support of $a^i(x-1)^j$ is a subset of the coset a^iG' and it is easy to check that $aG' = C_a$ and $a^{-1}G' = C_{a^{-1}}$. Thus $a\omega(FG'), a^{-1}\omega(FG') \subseteq \delta^{[1]}(FG)$ and the statement follows for $l = 0$.

The next step is to show the lemma for $l = 1$. Clearly, $[a^{-1}x, a] \in \delta^{[1]}(FG)$ and

$$[a^{-1}x, a] = x^k - x = (x^k - 1) - (x-1) = (k-1)(x-1) + w$$

for some $w \in \omega^2(FG')$. Keeping in mind that $\delta^{[1]}(FG)$ is a vector space over F and $k-1$ is an invertible element of F, we have $(x-1)+w' \in \delta^{[1]}(FG)$ for some $w' \in \omega^2(FG')$. Let $z_j = [(x-1)+w', a(x-1)^{1+j}]$ for all $j \geq 0$. Since the assertion is true for $l = 0$, $z_j \in \delta^{[2]}(FG)$. Evidently, $z_j = a(k-1)(x-1)^{j+2} + aw_{j+3}$ for some $w_{j+3} \in \omega^{j+3}(FG')$, so we can apply Lemma 3.2.4 to conclude $a\omega^2(FG') \in \delta^{[2]}(FG)$. Substituting a^{-1} for a and thus k' for k, we can similarly get that $a^{-1}\omega^2(FG') \in \delta^{[2]}(FG)$. Therefore the statement is indeed valid for $l = 1$.

Now, let $l \geq 2$ and assume the truth of the assertion for all $t < l$.

Firstly, suppose that $s_{l-1}^{(m)}$ is divisible by 2^m. According to the inductive hypothesis the element

$$z = [a(x-1)^{s_{l-2}^{(m)}}, a^{-1}(x-1)^{s_{l-2}^{(m)}+1}]$$

belongs to $\delta^{[l]}(FG)$. Clearly,

$$\begin{aligned} z &= (x^{k'}-1)^{s_{l-2}^{(m)}}(x-1)^{s_{l-2}^{(m)}+1} - (x^k-1)^{s_{l-2}^{(m)}+1}(x-1)^{s_{l-2}^{(m)}} \\ &= \big((k')^{s_{l-2}^{(m)}} - k^{s_{l-2}^{(m)}+1}\big)(x-1)^{s_{l-1}^{(m)}+1} + w \end{aligned}$$

for some $w \in \omega^{s_{l-1}^{(m)}+2}(FG')$, and

$$\begin{aligned} (k')^{s_{l-2}^{(m)}} - k^{s_{l-2}^{(m)}+1} &= (k')^{s_{l-2}^{(m)}}(1 - k^{2s_{l-2}^{(m)}+1}) \\ &= (k')^{s_{l-2}^{(m)}}(1 - k^{s_{l-1}^{(m)}+1}) \neq 0. \end{aligned}$$

This implies that $(x-1)^{s_{l-1}^{(m)}+1} + w' \in \delta^{[l]}(FG)$ for suitable $w' \in \omega^{s_{l-1}^{(m)}+2}(FG')$. Let

$$z_j = [(x-1)^{s_{l-1}^{(m)}+1} + w', a(x-1)^{s_{l-1}^{(m)}+j}]$$

for all $j \geq 0$. By the inductive hypothesis $z_j \in \delta^{[l+1]}(FG)$. At the same time,

$$\begin{aligned} z_j &= a(k^{s_{l-1}^{(m)}+1} - 1)(x-1)^{2s_{l-1}^{(m)}+1+j} + aw_{s_l^{(m)}+1+j} \\ &= a(k^{s_{l-1}^{(m)}+1} - 1)(x-1)^{s_l^{(m)}+j} + aw_{s_l^{(m)}+1+j} \end{aligned}$$

for some $w_{s_l^{(m)}+1+j} \in \omega^{s_l^{(m)}+1+j}(FG')$, and since $k^{s_{l-1}^{(m)}+1} - 1 \neq 0$ we have that $z_j \in a\omega^{s_l^{(m)}+j}(FG') \setminus a\omega^{s_l^{(m)}+1+j}(FG')$ for all $j \geq 0$. Applying Lemma 3.2.4 we get $a\omega^{s_l^{(m)}}(FG') \subseteq \delta^{[l+1]}(FG)$, and we can similarly verify the inclusion $a^{-1}\omega^{s_l^{(m)}}(FG') \subseteq \delta^{[l+1]}(FG)$ too. So, the statement is proved when $s_{l-1}^{(m)}$ is divisible by 2^m.

Now, suppose that $s_{l-1}^{(m)}$ is equal to $2^r q$, where $r < m$ and q is an odd integer. We claim that

$$(3.11) \qquad (x-1)^{s_{l-1}^{(m)}} + w \in \delta^{[l]}(FG) \quad \text{for some} \quad w \in \omega^{s_{l-1}^{(m)}+1}(FG').$$

First assume that $r \neq 0$. By the inductive hypothesis

$$[a(x-1)^{s_{l-2}^{(m)}}, a^{-1}(x-1)^{s_{l-2}^{(m)}}] \in \delta^{[l]}(FG),$$

furthermore

$$\begin{aligned}[a(x-1)^{s_{l-2}^{(m)}}, &a^{-1}(x-1)^{s_{l-2}^{(m)}}] \\ &= (x^{k'}-1)^{s_{l-2}^{(m)}}(x-1)^{s_{l-2}^{(m)}} - (x^k-1)^{s_{l-2}^{(m)}}(x-1)^{s_{l-2}^{(m)}} \\ &= \big((k')^{s_{l-2}^{(m)}} - k^{s_{l-2}^{(m)}}\big)(x-1)^{s_{l-1}^{(m)}} + w'\end{aligned}$$

for some $w' \in \omega^{s_{l-1}^{(m)}+1}(FG')$. Since

$$(k')^{s_{l-2}^{(m)}} - k^{s_{l-2}^{(m)}} = (k')^{s_{l-2}^{(m)}}(1 - k^{2s_{l-2}^{(m)}}) = (k')^{s_{l-2}^{(m)}}(1 - k^{s_{l-1}^{(m)}}) \neq 0,$$

(3.11) is true. Now suppose $r = 0$. Clearly,

$$\begin{aligned}[a(x-1)^{s_{l-2}^{(m)}}, &a^{-1}(x-1)^{s_{l-2}^{(m)}+1}] \\ &= (x^{k'}-1)^{s_{l-2}^{(m)}}(x-1)^{s_{l-2}^{(m)}+1} - (x^k-1)^{s_{l-2}^{(m)}+1}(x-1)^{s_{l-2}^{(m)}} \\ &= \big((k')^{s_{l-2}^{(m)}} - k^{s_{l-2}^{(m)}+1}\big)(x-1)^{s_{l-1}^{(m)}} + w'\end{aligned}$$

for some $w' \in \omega^{s_{l-1}^{(m)}+1}(FG')$ and

$$(k')^{s_{l-2}^{(m)}} - k^{s_{l-2}^{(m)}+1} = (k')^{s_{l-2}^{(m)}}(1 - k^{2s_{l-2}^{(m)}+1}) = (k')^{s_{l-2}^{(m)}}(1 - k^{s_{l-1}^{(m)}}) \neq 0,$$

so (3.11) follows again from the inductive hypothesis.
For $j \geq 0$ let $z_j = [(x-1)^{s_{l-1}^{(m)}} + w, a(x-1)^{s_{l-1}^{(m)}+j}]$. By the inductive

hypothesis and (3.11), $z_j \in \delta^{[l+1]}(FG)$ and

$$\begin{aligned} z_j &= a(k^{s_{l-1}^{(m)}} - 1)(x-1)^{2s_{l-1}^{(m)}+j} + aw_{s_l^{(m)}+1+j} \\ &= a(k^{s_{l-1}^{(m)}} - 1)(x-1)^{s_l^{(m)}+j} + aw_{s_l^{(m)}+1+j} \end{aligned}$$

for some $w_{s_l^{(m)}+1+j} \in \omega^{s_l^{(m)}+1+j}(FG')$. Since $k^{s_{l-1}^{(m)}} \neq 1$, we have that $z_j \in a\omega^{s_l^{(m)}+j}(FG') \setminus a\omega^{s_l^{(m)}+j+1}(FG')$ for all $j \geq 0$. According to Lemma 3.2.4, $a\omega^{s_l^{(m)}}(FG') \subseteq \delta^{[l+1]}(FG)$ and we can similarly verify that $a^{-1}\omega^{s_l^{(m)}}(FG') \subseteq \delta^{[l+1]}(FG)$. The proof is complete. □

Lemma 3.2.6. *Let G be a group with cyclic commutator subgroup of order p^n and let* $\operatorname{char}(F) = p$. *Assume that one of the following conditions holds:*

(i) the order of G/C is divisible by an odd prime $q \neq p$;

(ii) p can be written in the form $2^r + 1$ for some $r > 1$, $n = 1$ and G/C has order 2^r.

Then $\operatorname{dl}_L(FG) \geq \lceil \log_2(2p^n) \rceil$.

Proof. Let $G' = \langle x \mid x^{p^n} = 1 \rangle$. Assume first that condition (i) is satisfied. Let us choose an element $bC \in G/C$ of order q and set $x^k = x^b$. Evidently, $k^{2^m} \not\equiv 1 \pmod p$ for all m. Under condition (ii) let bC be of order 2^r and let again $x^k = x^b$. Then $k^{2^m} \not\equiv 1 \pmod p$ for all $0 \leq m \leq r-1$.

Set $H = \langle b, C \rangle$. In both cases $x^{k-1} = (x, b) \in H'$ is of order p^n, so H' too has order p^n. Since $H' = (b, C)$ and the map $c \mapsto (b, c)$ is an epimorphism of C onto H', we can choose c from C such that $(b, c) = x$. Define the following three series in FG:

$$u_0 = b, \quad v_0 = c, \quad w_0 = c^{-1}b^{-1},$$

and, for $l > 0$,

$$u_{l+1} = [u_l, v_l], \quad v_{l+1} = [u_l, w_l], \quad w_{l+1} = [w_l, v_l].$$

By induction on l we show for odd l that

$$
\begin{aligned}
u_l &\equiv t_u^{(l)} cb(x-1)^{2^{l-1}} \pmod{\mathfrak{I}(G')^{2^{l-1}+1}};\\
v_l &\equiv t_v^{(l)} c^{-1}(x-1)^{2^{l-1}} \pmod{\mathfrak{I}(G')^{2^{l-1}+1}};\\
w_l &\equiv t_w^{(l)} b^{-1}(x-1)^{2^{l-1}} \pmod{\mathfrak{I}(G')^{2^{l-1}+1}},
\end{aligned}
\tag{3.12}
$$

and if l is even then

$$
\begin{aligned}
u_l &\equiv t_u^{(l)} b(x-1)^{2^{l-1}} \pmod{\mathfrak{I}(G')^{2^{l-1}+1}};\\
v_l &\equiv t_v^{(l)} c(x-1)^{2^{l-1}} \pmod{\mathfrak{I}(G')^{2^{l-1}+1}};\\
w_l &\equiv t_w^{(l)} c^{-1}b^{-1}(x-1)^{2^{l-1}} \pmod{\mathfrak{I}(G')^{2^{l-1}+1}},
\end{aligned}
\tag{3.13}
$$

where $t_u^{(l)}, t_v^{(l)}, t_w^{(l)}$ are nonzero elements in the field F while $2^{l-1} < p^n$. Evidently, $u_1 = [b,c] = cb(x-1)$, and applying (3.7) we have

$$
\begin{aligned}
v_1 &= [b, c^{-1}b^{-1}] = c^{-1}\big((x^{-1})^{b^{-1}} - 1\big)\\
&= c^{-1}(x^{-k'} - 1) \equiv -k'c^{-1}(x-1) \pmod{\mathfrak{I}(G')^2},
\end{aligned}
$$

and similarly, $w_1 = [c^{-1}b^{-1}, c] \equiv -k'b^{-1}(x-1) \pmod{\mathfrak{I}(G')^2}$, where $x^{k'} = x^{b^{-1}}$. Therefore (3.12) holds for $l = 1$. Now assume that (3.12) is true for some odd l. Then, using the congruences (3.6) and $kk' \equiv 1 \pmod{p}$, we have

$$
\begin{aligned}
u_{l+1} &\equiv t_u^{(l)} t_v^{(l)} [cb(x-1)^{2^{l-1}}, c^{-1}(x-1)^{2^{l-1}}]\\
&\equiv -t_u^{(l)} t_v^{(l)} [(x-1)^{2^{l-1}}, b](x-1)^{2^{l-1}}\\
&\equiv -t_u^{(l)} t_v^{(l)} (k^{2^{l-1}} - 1) b (x-1)^{2^l} \pmod{\mathfrak{I}(G')^{2^l+1}},
\end{aligned}
$$

$$
\begin{aligned}
v_{l+1} &\equiv t_u^{(l)} t_w^{(l)} [cb(x-1)^{2^{l-1}}, b^{-1}(x-1)^{2^{l-1}}]\\
&\equiv t_u^{(l)} t_w^{(l)} \big(- b^{-1}c[(x-1)^{2^{l-1}}, b](x-1)^{2^{l-1}}\\
&\qquad\qquad + cb[(x-1)^{2^{l-1}}, b^{-1}](x-1)^{2^{l-1}}\big)\\
&\equiv t_u^{(l)} t_w^{(l)} k^{2^{l-1}} (k'^{2^l} - 1) c (x-1)^{2^l} \pmod{\mathfrak{I}(G')^{2^l+1}}
\end{aligned}
$$

and

$$\begin{aligned} w_{l+1} &\equiv t_w^{(l)} t_v^{(l)} [b^{-1}(x-1)^{2^{l-1}}, c^{-1}(x-1)^{2^{l-1}}] \\ &\equiv -t_w^{(l)} t_v^{(l)} c^{-1}[(x-1)^{2^{l-1}}, b](x-1)^{2^{l-1}} \\ &\equiv -t_u^{(l)} t_v^{(l)} (k'^{2^{l-1}} - 1)c^{-1}b^{-1}(x-1)^{2^l} \pmod{\mathfrak{I}(G')^{2^l+1}}. \end{aligned}$$

The assumption on k (see at the beginning of the proof) ensures that the coefficients of the element u_{l+1}, v_{l+1} and w_{l+1} are nonzero in the field F, provided $2^l < p^n$. Supposing that (3.13) is true for some even l we can similarly get the required congruences.

So, (3.12) and (3.13) are valid for any $l > 0$.

Assume that $l < \lceil \log_2(2p^n) \rceil$. Then $2^{l-1} < p^n$ and the elements u_l, v_l, w_l are nonzero in $\delta^{[l]}(FH)$, thus

$$\mathrm{dl}_L(FG) \geq \mathrm{dl}_L(FH) \geq \lceil \log_2(2p^n) \rceil.$$

□

The main result of this part is

Theorem 3.2.7. *Let G be a non-nilpotent group with commutator subgroup $G' = \langle x \mid x^{p^n} = 1 \rangle$ and let* $\mathrm{char}(F) = p$. *Then*

$$\lceil \log_2(3p^n/2) \rceil \leq \mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) \leq \lceil \log_2(2p^n) \rceil,$$

furthermore,

(i) if the order of $G/C_G(G')$ is divisible by some odd prime $q \neq p$, then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2(2p^n) \rceil;$$

(ii) if $G/C_G(G')$ has order $2^m p^r$ with $m > 0$, then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = d + 1,$$

where d is the minimal integer for which $s_d^{(m)} \geq p^n$ holds.

Proof. The upper bound $\lceil \log_2(2p^n) \rceil$ on $\mathrm{dl}^L(FG)$ is a consequence of Proposition 2.1.1.

The statement (i) follows directly from Lemma 3.2.6. In order to prove the inequality

$$\lceil \log_2(3p^n/2) \rceil \le \mathrm{dl}_L(FG) \le \mathrm{dl}^L(FG) \le \lceil \log_2(2p^n) \rceil,$$

it remains to show that if G/C has order $2^m p^r$, then $\lceil \log_2(3p^n/2) \rceil \le \mathrm{dl}_L(FG)$. Since G is not nilpotent, Lemma 3.2.1 ensures $m > 0$, so there is an element $bC \in G/C$ of order 2. Let $H = \langle x, b \rangle$. Clearly, $b^2 \in \zeta(H)$ and $x^b = x^{-1}$, therefore the factor group $\overline{H} = H/\zeta(H)$ is isomorphic to the dihedral group of order $2p^n$, so by Lemma 3.2.3, if d is the minimal integer such that $s_d^{(1)} \ge p^n$ then we have

$$d + 1 \le \mathrm{dl}_L(F\overline{H}) \le \mathrm{dl}_L(FH) \le \mathrm{dl}_L(FG).$$

At the same time, by (3.8) we have $(2^{d+2} - 1)/3 \ge s_d^{(1)} \ge p^n$, whence $d + 1 \ge \lceil \log_2(3p^n/2 + 1/2) \rceil$ follows. Since $\lceil \log_2(3p^n/2 + 1/2) \rceil = \lceil \log_2(3p^n/2) \rceil$, the the required inequality is guaranteed.

(ii) Let d be the integer such that $s_d^{(m)} \ge p^n$, but $s_{d-1}^{(m)} < p^n$. To prove that $d+1$ is an upper bound on $\mathrm{dl}^L(FG)$ it is sufficient to show that

$$\delta^{(l+1)}(FG) \subseteq \mathfrak{I}(G')^{s_l^{(m)}} \quad \text{for all} \quad l \ge 0.$$

This is clear for $l = 0$, and assuming that $\delta^{(l)}(FG) \subseteq \mathfrak{I}(G')^{s_{l-1}^{(m)}}$, by Lemma 3.2.2 we obtain

$$\begin{aligned}\delta^{(l+1)}(FG) &= [\delta^{(l)}(FG), \delta^{(l)}(FG)]FG \\ &\subseteq [\mathfrak{I}(G')^{s_{l-1}^{(m)}}, \mathfrak{I}(G')^{s_{l-1}^{(m)}}]FG \subseteq \mathfrak{I}(G')^{s_l^{(m)}}.\end{aligned}$$

Therefore, $\mathrm{dl}^L(FG) \le d + 1$. Let us choose an element aC of order 2^m in G/C, set $x^k = x^a$ and consider the group $H = \langle x, a \rangle$. Since $(x, a) = x^{-1+k} \in H'$ and $k \not\equiv 1 \pmod p$, we have that H' has order p^n. Moreover, $H/C_H(H')$ has also order 2^m. If $m = 1$ then, as we have already seen before, $\overline{H} = H/\zeta(H)$ is isomorphic to the dihedral group of order $2p^n$, so the statement is a consequence of Lemma

3.2.3. Finally, let $m > 1$. Since $\omega^{s_{d-1}^{(m)}}(FH) \neq 0$, Lemma 3.2.5 forces $\delta^{[d]}(FH) \neq 0$. Hence $d + 1 \leq \mathrm{dl}_L(FH) \leq \mathrm{dl}_L(FG)$, which completes the proof. □

3.3 Summarized result

The determination of the values of $\mathrm{dl}_L(FG)$ and $\mathrm{dl}^L(FG)$ for the case when G' is cyclic of odd order is a consequence of Theorems 3.1.2 and 3.2.7.

Theorem 3.3.1. *Let G be a group with cyclic commutator subgroup of order p^n, where p is an odd prime, and let F be a field of characteristic p.*

(i) If $G/C_G(G')$ has order p^r (that is G is nilpotent), then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2(p^n + 1) \rceil.$$

(ii) If the order of $G/C_G(G')$ is divisible by some odd prime $q \neq p$, then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2(2p^n) \rceil.$$

(iii) If $G/C_G(G')$ has order $2^m p^r$ with $m > 0$, then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = d + 1,$$

where d is the minimal integer for which $s_d^{(m)} \geq p^n$ holds.

Remarks on the theorem

When (iii) of Theorem 3.3.1 holds for the group G we are able to determine explicitly the values of $\mathrm{dl}_L(FG)$ and $\mathrm{dl}^L(FG)$ in the following cases:

(i) We claim that if G/C has order $2p^r$, then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2(3p^n/2) \rceil.$$

Indeed, according to the part (*iii*) of the theorem, if $l = \mathrm{dl}^L(FG)$ then $s^{(1)}_{l-2} < p^n$. From (3.8) it follows that $(2^l - 1)/3 < p^n$. Hence $l < \log_2(3p^n/2 + 1/2) + 1$, and therefore $l \leq \lceil \log_2(3p^n/2 + 1/2) \rceil$. Since $\lceil \log_2(3p^n/2+1/2) \rceil = \lceil \log_2(3p^n/2) \rceil$, the proof is complete.

(*ii*) Since the order of G/C divides the order of $U(\mathbb{Z}_{p^n})$, which is equal to $p^{n-1}(p-1)$, for primes p of the form $4k-1$ the conditions either (*ii*) of the theorem or (*i*) of this remark is satisfied, so we can easily have the derived length and the strong Lie derived length of FG.

(*iii*) Recall that p is called Fermat prime, if p has the form $2^{2^s} + 1$ for some $s \geq 0$. Let G be a non-nilpotent group with commutator subgroup of order $p > 3$, where p is a Fermat prime, and let $\mathrm{char}(F) = p$. Then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \begin{cases} \lceil \log_2(2p) \rceil & \text{if } G/C \text{ has order } p-1; \\ \lceil \log_2(3p/2) \rceil & \text{otherwise.} \end{cases}$$

Indeed, let us write p in the form $2^r + 1$ $(r > 1)$. If G/C has order $p - 1$ then the statement follows directly from Lemma 3.2.6. In the other case G/C has order 2^m for some $0 < m < r$. Since $\lceil \log_2(3p/2) \rceil = r + 1$, by part (*ii*) of the theorem it is enough to show that $s^{(m)}_r \geq p$. Indeed, from the definition it follows that $s^{(r-1)}_{r-1} = 2^{r-1}$. Furthermore, for $m = r - 1$ we have $s^{(m)}_r = s^{(r-1)}_r = 2s^{(r-1)}_{r-1} + 1 = 2^r + 1 = p$, and if $m < r - 1$ then $s^{(m)}_{r-1} > s^{(r-1)}_{r-1}$, which implies $s^{(m)}_r \geq 2s^{(m)}_{r-1} > 2s^{(r-1)}_{r-1} = 2^r = p-1$. The proof is complete.

Finally, we note that some parts of the theorem can also be proved using the Theorems A and B of [29]. Obviously, these theorems of A. Shalev are not enough to determine the derived length under conditions of our theorem. An example for such group algebra is the following: let

$$G = \langle a, b, c \mid a^2 = b^9 = c^{19} = 1, b^a = b, c^a = c^{18}, c^b = c^7 \rangle$$

and $\mathrm{char}(F) = 19$. Since G/G' has even order, the condition of Theorem B of [29] is not satisfied, but the part (*iii*) of our theorem says $\mathrm{dl}_L(FG) = 6$.

The computations above can easily be verified by LAGUNA [12] as well.

Chapter 4

Group algebras of maximal Lie derived length in characteristic two

According to Proposition 2.1.1, if FG is a Lie solvable group algebra then

$$\mathrm{dl}_L(FG) \leq \lceil \log_2(2t_N(G')) \rceil.$$

For characteristic two, if G is a nilpotent group with commutator subgroup of order 2^n, we obtain here the description of the group algebras FG which have the highest possible value of $\mathrm{dl}_L(FG)$, namely, $\lceil \log_2(2t_N(G')) \rceil = n + 1$.

4.1 Preliminaries

Let G be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1 \rangle$. First of all, note that G is a nilpotent group. Indeed, let $m \geq 1$ and $\gamma_m(G) = \langle x^{2^k} \rangle$. Then the subgroup $\gamma_{m+1}(G)$ is generated by the commutators (x^{2^k}, g), where $g \in G$. Clearly,

$$(x^{2^k}, g) = x^{-2^k}(x^{2^k})^g = x^{-2^k} x^{l2^k} = x^{2^k(-1+l)}$$

for suitable odd l. This implies that $\gamma_{m+1}(G) \subseteq \big(\gamma_m(G)\big)^2$ is a proper subgroup of $\gamma_m(G)$ while $\gamma_m(G) \neq 1$, which proves the assertion.

Assume in the sequel that $n \geq 3$. It is well-known that the automorphism group $\operatorname{aut}(G')$ of G' is a direct product of the cyclic group $\langle\alpha\rangle$ of order 2 and the cyclic group $\langle\beta\rangle$ of order 2^{n-2} where the action of these automorphisms on G' is given by $\alpha(x) = x^{-1}$, $\beta(x) = x^5$. For $g \in G$, let τ_g denote the restriction to G' of the inner automorphism $h \mapsto h^g$ of G. The map $G \to \operatorname{aut}(G)$, $g \mapsto \tau_g$ is a homomorphism whose kernel coincides with the centralizer $C = C_G(G')$. Clearly, the map $\varphi : G/C \to \operatorname{aut}(G')$ given by $\varphi(gC) = \tau_g$ is a monomorphism.

The subset

$$G_\beta = \{g \in G \mid \varphi(gC) \in \langle\beta\rangle\}$$

of G will play an important role in the sequel. It is easy to check that G_β is a subgroup of index not greater than two and $g \in G_\beta$ if and only if $x^g = x^{5^i}$ for some $i \in \mathbb{Z}$.

Lemma 4.1.1. *Let G be a group with cyclic commutator subgroup of order 2^n, where $n \geq 3$ and let $\operatorname{char}(F) = 2$. Then*

(i) $(y, g) \in (G')^4$ *for all* $y \in G'$ *and* $g \in G_\beta$*;*

(ii) $\big[\omega^m(FG'), \omega(FG_\beta)\big] \subseteq \mathfrak{I}(G')^{m+3}$.

Proof. Let $g \in G_\beta$ and $y \in G'$.

(i) Clearly, $(y, g) = y^{-1}y^g = y^{-1+5^i}$ for some $i \geq 0$ and $-1+5^i \equiv 0 \pmod 4$. Therefore, $(y, g) \in (G')^4$.

(ii) Using (i) we have that

$$[y-1, g-1] = [y, g] = gy\big((y, g) - 1\big) \in \mathfrak{I}(G')^4,$$

from which (ii) follows for $m = 1$. Now, assume that

$$\big[\omega^m(FG'), \omega(FG_\beta)\big] \subseteq \mathfrak{I}(G')^{m+3}$$

for some $m \geq 1$. Then

$$\begin{aligned}
&\big[\omega^{m+1}(FG'), \omega(FG_\beta)\big] \\
&\qquad \subseteq \omega^m(FG')\big[\omega(FG'), \omega(FG_\beta)\big] + \big[\omega^m(FG'), \omega(FG_\beta)\big]\omega(FG') \\
&\qquad \subseteq \omega^m(FG')\mathfrak{I}(G')^4 + \mathfrak{I}(G')^{m+4}\omega(FG') \subseteq \mathfrak{I}(G')^{m+4},
\end{aligned}$$

and the proof is complete. □

Lemma 4.1.2. *Let G be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1\rangle$, where $n \geq 3$. Then the following are equivalent:*

(i) $G_\beta = G$.

(ii) G has nilpotency class at most n.

Proof. Recall that G is now a nilpotent group of class at most $n+1$.

$(i) \Rightarrow (ii)$ By Lemma 4.1.1(i), $\gamma_3(G) \subseteq (G')^4$, so $|\gamma_2(G)/\gamma_3(G)| \geq 4$ and the class of G is at most n.

$(ii) \Rightarrow (i)$ Suppose that G has nilpotency class at most n, but $G_\beta \neq G$. We claim that $x^{2^{k-2}} \in \gamma_k(G)$ for all $k \geq 2$. Indeed, this is clear for $k = 2$ and assume its truth for some $k \geq 2$. If $g \in G\setminus G_\beta$ then $(x^{2^{k-2}}, g) \in \gamma_{k+1}(G)$ and $(x^{2^{k-2}}, g) = x^{2^{k-2}(-1-5^i)} = (x^{2^{k-1}})^j$ with some i and odd j. This means that $x^{2^{k-1}} \in \gamma_{k+1}(G)$, as desired. Therefore, $\gamma_{n+1}(G) \neq 1$, which is a contradiction. □

Lemma 4.1.3. *Let G be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1\rangle$, where $n \geq 3$. If G has nilpotency class $n+1$ then $(g, h) \in (G')^2$ for all $g, h \in G_\beta$.*

Proof. If the lemma were not true we could choose the elements $g, h \in G_\beta$ so that $(g, h) = x$. By definition of G_β we may additionally assume that $(g, x) = 1$. Lemma 4.1.2 states that $G\setminus G_\beta \neq \emptyset$; let y be in $G\setminus G_\beta$. Evidently, $(g, y) = x^i$ for some i. Using the equalities

$$g^h = gx, \quad g^{h^{-1}} = g(x^{-1})^{h^{-1}}, \quad g^y = gx^i, \quad g^{y^{-1}} = g(x^{-i})^{y^{-1}}$$

it is easy to check

$$\begin{aligned} g = g^{(h,y)} &= g^{h^{-1}y^{-1}hy} = \big(g(x^{-1})^{h^{-1}}\big)^{y^{-1}hy} = g^{y^{-1}hy}x^{-1} \\ &= \big(g(x^{-i})^{y^{-1}}\big)^{hy}x^{-1} = \big(gx(x^{-i})^{y^{-1}h}\big)^{y}x^{-1} = gx^ix^y(x^{-i})^{y^{-1}hy}x^{-1} \\ &= g(x, y)(x^{-i}, h^y), \end{aligned}$$

which is a contradiction. Indeed, keeping in mind that $y \in G \setminus G_\beta$ and $h \in G_\beta$ we have $(x, y) = x^{-1-5^j} \in \langle x^2\rangle \setminus \langle x^4\rangle$ and $(x^{-i}, h^y) = x^{i(1-5^l)} \in \langle x^4\rangle$, thus $(x, y)(x^{-i}, h^y) \neq 1$. □

Lemma 4.1.4. *Let G be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1\rangle$, where $n \geq 3$ and let* $\mathrm{char}(F) = 2$. *If G has nilpotency class $n+1$ then* $\mathrm{dl}_L(FG) \leq n$.

Proof. Clearly, the set of the Lie commutators $[a, b]$ with $a, b \in G$ spans the F-space $\delta^{[1]}(FG)$. Since $[a, b] = g^h + g$ with $g = ba$ and $h = b$, while of course $g^h + g = [a, b]$ with $a = h^{-1}g$ and $b = h$ whenever $g, h \in G$, this spanning set for $\delta^{[1]}(FG)$ can also be described as the set of the elements $g^h + g$ with $g, h \in G$. It follows that the Lie commutators $[{g_1}^{h_1} + g_1, {g_2}^{h_2} + g_2]$, where $g_1, g_2, h_1, h_2 \in G$, span $\delta^{[2]}(FG)$. We shall compute these Lie commutators. It is easy to check that

(4.1)
$$[{g_1}^{h_1}+g_1, {g_2}^{h_2} + g_2] = g_2g_1\Big(\big((g_1, g_2) + 1\big)\big((g_2, h_2) + 1\big)\big((g_1, h_1) + 1\big) + (g_2, h_2)\big((g_2, h_2, g_1) + 1\big)\big((g_1, h_1) + 1\big) + (g_1, g_2)(g_1, h_1)\big((g_1, h_1, g_2) + 1\big)\big((g_2, h_2) + 1\big)\Big).$$

Firstly, if neither g_1 nor g_2 are in G_β then

$$[g_1^{h_1} + g_2, {g_2}^{h_2} + g_2] = b\varrho_3 \tag{4.2}$$

for some $b \in G_\beta$ and $\varrho_3 \in \omega^3(FG')$. Indeed, it is clear from the definition of G_β that then $g_2g_1 \in G_\beta$. Furthermore, the second factor on the right-hand side of (4.1) always belongs to $\omega^3(FG')$, because $\gamma_3(G) \subseteq (G')^2$.

Secondly, if g_1 or g_2, say g_1, belongs to G_β, then we claim that

$$[g_1^{h_1} + g_1, {g_2}^{h_2} + g_2] = g\varrho_4 \tag{4.3}$$

for some $\varrho_4 \in \omega^4(FG')$ and $g \in G$.

For $g_1 \in G_\beta$, Lemma 4.1.1(i) asserts $(g_2, h_2, g_1) \in (G')^4$, therefore (4.1) can be written as

(4.4)
$$[{g_1}^{h_1} + g_1, {g_2}^{h_2} + g_2] = g_2g_1\Big(\big((g_1, g_2) + 1\big)\big((g_1, h_1) + 1\big) + (g_1, g_2)(g_1, h_1)\big((g_1, h_1, g_2) + 1\big)\Big)\big((g_2, h_2) + 1\big) + g_2g_1\varrho_4$$

for some $\varrho_4 \in \omega^4(FG')$. In order to prove (4.3) it will be sufficient to show that the element

$$\vartheta = \big((g_1, g_2) + 1\big)\big((g_1, h_1) + 1\big) + (g_1, g_2)(g_1, h_1)\big((g_1, h_1, g_2) + 1\big)$$

from the right-hand side of (4.4) belongs to $\omega^3(FG')$.

This is clear if g_2 also belongs to G_β, because then by Lemma 4.1.3 and Lemma 4.1.1(i) both summands of ϑ are in $\omega^3(FG')$. Furthermore, if $g_2 \notin G_\beta$, then $x^{g_2} = x^{-5^l}$ for some l and we distinguish the following three cases:
Case 1: $(g_1, h_1) \in (G')^2$. Then $(g_1, h_1, g_2) = (g_1, h_1)^{-1-5^l} \in (G')^4$ and $\vartheta \in \omega^3(FG')$.
Case 2: $(g_1, g_2) \in (G')^2$. By the well-known Hall-Witt identity,

$$(g_1, h_1, g_2)^{h_1^{-1}}(h_1^{-1}, g_2^{-1}, g_1)^{g_2}(g_2, g_1^{-1}, h_1^{-1})^{g_1} = 1.$$

Lemma 4.1.1(i) ensures that the second factor on the left-hand side belongs to $(G')^4$ and this is true for the last factor too, because

$$(g_2, g_1^{-1}, h_1^{-1}) = \big((g_1, g_2)^{g_1^{-1}}, h_1^{-1}\big) = \big((g_1, g_2)^{g_1^{-1}}\big)^{-1}(g_1, g_2)^{g_1^{-1}h_1^{-1}} = (g_1, g_2)^{2i}$$

for some i. This means that $(g_1, h_1, g_2) \in (G')^4$, which proves $\vartheta \in \omega^3(FG')$.
Case 3: $(g_1, h_1) \notin (G')^2$ and $(g_1, g_2) \notin (G')^2$. Then $\langle (g_1, h_1)\rangle = \langle (g_1, g_2)\rangle = G'$ and $(g_1, g_2) = (g_1, h_1)^k$ for some odd k. With the notation $y = (g_1, h_1)$ ϑ can be written as

$$\vartheta = (y^k + 1)(y + 1) + y^{k+1}(y^{-5^l-1} + 1) = y^{k-5^l} + 1 + y(y^{k-1} + 1).$$

Of course, if $k \equiv 1 \pmod 4$ then $y^{-5^l-1} + 1$ and $y(y^{k-1} + 1)$ are in $\omega^4(FG')$, therefore $\vartheta \in \omega^4(FG')$. Otherwise, if $k \equiv 3 \pmod 4$ then $y^{k-3} + 1 \in \omega^4(FG')$ which implies that

$$y(y^{k-1} + 1) = y\big((y^{k-3} + 1)(y^2 + 1) + (y^{k-3} + 1) + (y^2 + 1)\big) \equiv y(y^2 + 1) \equiv y^2 + 1 \pmod{\omega^3(FG')}.$$

Similarly, we can obtain that

$$\begin{aligned} y^{k-5^l}+1 &= (y^{k-5^l-2}+1)(y^2+1)+(y^{k-5^l-2}+1)+(y^2+1) \\ &\equiv y^2+1 \pmod{\omega^3(FG')}. \end{aligned}$$

Hence

$$\vartheta = y^{k-5^l}+1+y(y^{k-1}+1) \equiv 2(y^2+1) \equiv 0 \pmod{\omega^3(FG')},$$

which completes the checking of (4.3).

Let S be the additive subgroup generated by all elements of the form $g\varrho_4$ and $b\varrho_3$, where $g \in G$, $b \in G_\beta$ and $\varrho_3 \in \omega^3(FG'), \varrho_4 \in \omega^4(FG')$. We claim that $[S,S] \subseteq \mathfrak{I}(G')^8$. Indeed, the additive subgroup $[S,S]$ can be spanned by some Lie commutators of the forms $[g\varrho_4, h\varrho_3]$ and $[b_1\varrho_3, b_2\eta_3]$ with $g \in G, b_1, b_2 \in G_\beta$, $\varrho_3, \eta_3 \in \omega^3(FG')$, $\varrho_4 \in \omega^4(FG')$. Furthermore, by Lemma 2.2.1,

$$\begin{aligned} [g\varrho_4, h\varrho_3] &= g[\varrho_4, h\varrho_3]+[g, h\varrho_3]\varrho_4 \\ &= g[\varrho_4, h+1]\varrho_3 + hg\big((g,h)+1\big)\varrho_3\varrho_4 \\ &\quad + h[g+1,\varrho_3]\varrho_4 \in \mathfrak{I}(G')^8, \end{aligned}$$

and by Lemma 4.1.1(ii) and Lemma 4.1.3,

$$\begin{aligned} [b_1\varrho_3, b_2\eta_3] =& b_1[\varrho_3, b_2\eta_3]+[b_1, b_2\eta_3]\varrho_3 \\ =& b_1[\varrho_3, b_2+1]\eta_3 + b_2[b_1+1,\eta_3]\varrho_3 + b_1b_2\big((b_1,b_2)+1\big)\eta_3\varrho_3 \end{aligned}$$

also belongs to $\mathfrak{I}(G')^8$. Therefore, $[S,S] \subseteq \mathfrak{I}(G')^8$.

From (4.2) and (4.3) we get $\delta^{[2]}(FG) \subseteq S$, so we have

$$\delta^{[3]}(FG) = \big[\delta^{[2]}(FG), \delta^{[2]}(FG)\big] \subseteq [S,S] \subseteq \mathfrak{I}(G')^8.$$

Now, we use induction on k to show that

$$\delta^{[k]}(FG) \subseteq \mathfrak{I}(G')^{2^k} \quad \text{for all} \quad k \geq 3. \tag{4.5}$$

Indeed, assuming the validity of (5) for some $k \geq 3$ we have

$$\delta^{[k+1]}(FG) = \big[\delta^{[k]}(FG), \delta^{[k]}(FG)\big] \subseteq \big[\mathfrak{I}(G')^{2^k}, \mathfrak{I}(G')^{2^k}\big] \subseteq \mathfrak{I}(G')^{2^{k+1}}$$

and this proves the truth of (4.5) for every $k \geq 3$.

Keeping in mind that G' has order 2^n, (4.5) implies that $\delta^{[n]}(FG) = 0$. Hence $\mathrm{dl}_L(FG) \leq n$ and the proof is complete. □

Lemma 4.1.5. *Let G be a nilpotent group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1 \rangle$, $\mathrm{char}(F) = p$ and assume that $n \geq 3$ and G has nilpotency class at most n. Then*

$$\mathrm{dl}_L(FG) = \lceil \log_2(p^n + 1) \rceil.$$

Proof. Let us repeat the proof of Theorem 3.1.2 with the following modification: to obtain the inclusion

$$\left[b^l a^m, (x-1)^{2^k-1}\right], \left[(x-1)^{2^k-1}, b^s a^t\right] \in \mathfrak{I}(G')^{2^k+1}$$

let us apply Lemma 4.1.1(ii), in view of the fact that now $b^l a^m, b^s a^t \in G_\beta$ by Lemma 4.1.2. □

4.2 The description and some consequences

The main theorem of this chapter can be stated as follows:

Theorem 4.2.1. *Let G be a nilpotent group with commutator subgroup of order 2^n and let F be a field of characteristic two. Then $\mathrm{dl}_L(FG) = n+1$ if and only if one of the following conditions holds:*

(i) G' is the noncyclic group of order 4 and $\gamma_3(G) \neq 1$;

(ii) G' is cyclic of order less than 8;

(iii) G' is cyclic, $n \geq 3$ and G has nilpotency class at most n.

Proof. Assume first that G' is cyclic. For $p = 2$ and $n < 3$ the statement is a consequence of Theorem 2.2.2 and Proposition 2.1.2. In the other cases Lemma 4.1.4 and Lemma 4.1.5 state the required result. Now, assume that G' is noncyclic and $\delta^{[n]}(FG) \neq 0$. We know that FG is Lie nilpotent, and as we have already seen, $\delta^{[n]}(FG) \subseteq (FG)^{(2^n)}$. Thus $(FG)^{(2^n)} \neq 0$ and Proposition 6.1.1 (see in Chapter 6) states that $G' = C_2 \times C_2$ and $\gamma_3(G) \neq 1$. Conversely, if $G' = C_2 \times C_2$ then $t_N(G') = 3$ and $\mathrm{dl}_L(FG) \leq \lceil \log_2 (2 \cdot 3) \rceil = 3$. Furthermore, when $\gamma_3(G) \neq 1$, Proposition 2.1.2 says that $\mathrm{dl}_L(FG) \neq 2$. Therefore $\mathrm{dl}_L(FG) = 3$ and the theorem is proved. □

The following question is still open: under which conditions are the Lie derived length and the strong Lie derived length of group algebras equal? From our results a partial answer follows to this question.

Corollary 4.2.2. *Let G be a group with cyclic commutator subgroup of order p^n, and let F be a field of characteristic p. Then $\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG)$ if and only if one of the following conditions is satisfied:*

(i) p is odd;

(ii) $p = 2$ and $n \leq 2$;

(iii) $p = 2$, $n \geq 3$ and the nilpotency class of G is at most n.

Proof. The statement follows immediately from Theorems 3.3.1 and 4.2.1 □

In the case when $p = 2$ and neither (ii) nor (iii) is satisfied, by Theorem 2.2.2 we know that $\mathrm{dl}^L(FG) = n + 1$, but the exact value of $\mathrm{dl}_L(FG)$ is still unknown. To determine it the results of the next chapter can be useful.

According to Proposition 2.1.1, if FG is strongly Lie solvable of characteristic $p > 0$ and G is nonabelian, then $\mathrm{dl}^L(FG)$ is at least $\lceil \log_2(p+1) \rceil$. Now, we describe the group algebras of minimal strong Lie derived length.

Corollary 4.2.3. *Let FG be a strongly Lie solvable group algebra of characteristic $p > 0$. Then $\mathrm{dl}^L(FG) = \lceil \log_2(p+1) \rceil$ if and only if one of the following conditions holds:*

(i) $p = 2$ and G' is central elementary abelian subgroup of order 4;

(ii) G' is of order p and

a) either G' is central;

b) or $G/C_G(G')$ has order $2^m p^r$ with $m > 0, r \geq 0$, and the minimal integer d such that $s_d^{(m)} \geq p$, satisfies the inequality $2^d - 1 < p$.

Proof. Assume that G' has order p^n. Then, as it is well-known, $t_N(G') \geq 1 + n(p-1)$. Hence, if $p = 2$ and $n \geq 3$, then $t_N(G') \geq 4$ and applying Proposition 2.1.1 we have that

$$\lceil \log_2(p+1) \rceil = 2 < 3 = \lceil \log_2(4+1) \rceil \leq \mathrm{dl}^L(FG).$$

Let now $p > 2$ and $n \geq 2$. Since $t_N(G') \geq 2p - 1$ and there exists i such that $p < 2^i < 2p$, we get as before that

$$\lceil \log_2(p+1) \rceil < \lceil \log_2(2p) \rceil \leq \mathrm{dl}^L(FG).$$

Thus, we have just proved that if $\mathrm{dl}^L(FG) = \lceil \log_2(p+1) \rceil$ then either $G' = C_2 \times C_2$ and $p = 2$ or G' has order p. For $p = 2$ the statement follows from Proposition 2.1.2 and for odd p from Theorem 3.3.1. □

Chapter 5

Group algebras of Lie derived length three

The group algebras of Lie derived length two are described in [19] by F. Levin and G. Rosenberger, but which have Lie derived length three are known only for characteristic not less than seven by M. Sahai [24]. The full characterization of these seems to be a difficult problem. A partial solution can be found here for the case when the commutator subgroup of the basic group is cyclic.

5.1 Preliminaries and the description

We will use the following

Proposition 5.1.1 (A. Shalev [27])**.** *Let G be a group with commutator subgroup of order p^n with $n > 0$ and let* $\mathrm{char}(F) = p$. *Then*

$$\mathrm{dl}_L(FG) \geq \lceil \log_2(p+1) \rceil.$$

First of all, in a special case we give an upper bound on the Lie derived length, which will be useful in the sequel.

Theorem 5.1.2. *Let G be a group and* $\mathrm{char}(F) = 2$. *If H is a subgroup of index two of G whose commutator subgroup H' is a finite*

2-*group, then*

$$\mathrm{dl}_L(FG) \leq \lceil \log_2 t(H') \rceil + 3.$$

Proof. Firstly, suppose that H is an abelian subgroup of index two of G. Then $G = \langle H, b \rangle$ for some b and every $x \in FG$ has a unique representation in the form $x = x_1 + x_2 b$, where $x_1, x_2 \in FH$. It is easy to see that the map $u \mapsto \overline{u} = b^{-1}ub$ $(u \in FH)$ is an automorphism of order 2 of FH and for all $x, y \in FG$

$$\begin{aligned}
[x, y] &= [x_1 + x_2 b, y_1 + y_2 b] \\
&= (x_2\overline{y_2} + \overline{x_2}y_2)b^2 + \big((x_1 + \overline{x_1})y_2 + x_2(\overline{y_1} + y_1)\big)b \\
&\equiv w_1 b \pmod{\zeta(FG)},
\end{aligned}$$

where $w_1 \in FH$ and $\zeta(FG)$ denotes the center of FG. Similarly, for $u, v \in FG$ we have $[u, v] \equiv w_2 b \pmod{\zeta(FG)}$ for some $w_2 \in FH$. Hence

$$\big[[x, y], [u, v]\big] = [w_1 b, w_2 b] = (w_1\overline{w_2} + \overline{w_1}w_2)b^2 \in FH.$$

Since the elements of the form $\big[[x, y], [u, v]\big]$ with $x, y, u, v \in FG$ generate $\delta^{[2]}(FG)$ and FH is a commutative algebra, $\delta^{[3]}(FG) = 0$, as asserted.

Let now H be nonabelian. It is clear that H' is normal in G and H/H' is an abelian subgroup of index two of G/H', so we can use the result proved above to get $\delta^{[3]}\big(F(G/H')\big) = 0$. In view of $F(G/H') \cong FG/\mathfrak{I}(H')$ we have $\delta^{[3]}(FG) \subseteq \mathfrak{I}(H')$. Hence an easy induction on k yields $\delta^{[3+k]}(FG) \subseteq \mathfrak{I}(H')^{2^k}$ for all $k \geq 0$. Consequently, if $2^k \geq t(H')$, that is $k \geq \lceil \log_2 t(H') \rceil$, then $\delta^{[3+k]}(FG) = 0$, which implies the statement. ☐

In the previous chapter we introduced the subset G_β of G. Recall that G_β is a subgroup of index not greater than two. It is shown in Lemma 4.1.2 that $G = G_\beta$ if and only if G has nilpotency class at most n, furthermore under this condition $\mathrm{dl}_L(FG) = n+1$. Combining this fact with Theorem 5.1.2 we obtain the following statement.

Theorem 5.1.3. *Let G be a group with cyclic commutator subgroup of order 2^n and let* $\mathrm{char}(F) = 2$*. If G'_β has order 2^r, then*

$$r + 1 \leq \mathrm{dl}_L(FG) \leq r + 3.$$

Proof. If $G = G_\beta$ then Lemma 4.1.2 and Theorem 4.2.1 say that $\mathrm{dl}_L(FG) = r+1$. Otherwise, G_β is of index two in G and we can apply Theorem 5.1.2 to get $\mathrm{dl}_L(FG) \leq r + 3$. Furthermore, Lemma 4.1.2 and Theorem 4.2.1 ensure that $\mathrm{dl}_L(FG_\beta) = r + 1$. Since $\mathrm{dl}_L(FG_\beta) \leq \mathrm{dl}_L(FG)$, the corollary is true. □

Let $\mathrm{char}(F) = 2$ and $H = \langle x \mid x^{2^n} = 1\rangle$. We claim that if $r > 0$ and the k_j's are odd positive integers for $1 \leq j \leq r$ then the element

$$\varrho = (x^{k_1} + 1)(x^{k_2} + 1)\cdots(x^{k_r} + 1) \in FH$$

is equal to zero if and only if $r \geq 2^n$.

Indeed, $\varrho \in \omega^r(FH)$ and if $r \geq 2^n$ then $\varrho = 0$, because $t(H) = 2^n$. Assume now $r < 2^n$. Applying the identity

$$(x^{k_j} + 1) = (x^{k_j - 1} + 1)(x + 1) + (x^{(k_j-1)/2} + 1)^2 + (x + 1)$$

for every $1 \leq j \leq r$, we can write $\varrho = (x + 1)^r + \varrho_1$, where ϱ_1 is the sum of elements of weight greater than r. Clearly, $(x+1)^r \in \omega^r(FH) \setminus \omega^{r+1}(FH)$ and $\varrho_1 \in \omega^{r+1}(FH)$, hence $\varrho \in \omega^r(FH) \setminus \omega^{r+1}(FH)$ and $\varrho \neq 0$.

In the sequel we shall use freely this fact.

In the proof of the next lemmas we will use that $C' \subseteq G' \cap \zeta(G)$. This inclusion is indeed valid, because for $a, b, c \in G$ the Hall-Witt identity states that

$$(a, b^{-1}, c)^b(b, c^{-1}, a)^c(c, a^{-1}, b)^a = 1.$$

Evidently, if $b, c \in C$ then this formula yields that $(b, c, a) = 1$, which guarantees our statement.

Lemma 5.1.4. *Let G be a group with commutator subgroup $G' = \langle x \mid x^{2^n} = 1\rangle$, where $n > 3$, let $\mathrm{char}(F) = 2$ and assume that $\exp(G/C) \leq 2$. Then $\mathrm{dl}_L(FG) = 3$ if and only if C is abelian and $G/C = \langle aC\rangle$, where $x^a = x^{-1}$.*

Proof. Since $\exp(G/C) \leq 2$, only the following cases are possible:

Case 1: either G/C is trivial or $G/C = \langle bC\rangle$ where $x^b = x^{2^{n-1}+1}$. Clearly, G has nilpotency class at most 3, therefore by Theorem 4.2.1 we have $\mathrm{dl}_L(FG) = n + 1$.

Case 2: $G/C = \langle aC\rangle$, where $x^a = x^{-1}$. Then $C' \subseteq G' \cap \zeta(G) = \langle x^{2^{n-1}}\rangle$. If $C' = \langle 1\rangle$ then C is an abelian subgroup of index two of G and Theorem 5.1.2 implies that $\mathrm{dl}_L(FG) = 3$. Now, let $C' = \langle x^{2^{n-1}}\rangle$. Then we can choose $b, c \in C$ such that

$$(c, a) = x, \qquad (c, b) = x^{2^{n-1}}, \qquad (a, b) \in \langle x^2\rangle.$$

Indeed, let us consider the map $\varphi : C \to G'$, where $\varphi(c) = (c, a)$, which is an epimorphism because $G' = (a, C)$. Of course, $H = \varphi^{-1}\big(\langle x^2\rangle\big)$ is a proper subgroup of C. Let $u \in C \setminus \zeta(C)$ and $c \in C \setminus \big(H \cup C_C(u)\big)$ be such that $(c, a) = x$. Obviously, $(c, u) = x^{2^{n-1}}$. If $(a, u) \in \langle x^2\rangle$ then set $b = u$, otherwise $b = cu$. It is easy to see that the elements b and c satisfy the conditions stated. Then

$$\begin{aligned}
&\Big[\big[[c, a],[c^{-1}a, c]\big],\ \big[[c, a], [c^{-1}ba, c]\big]\Big]\\
&\quad=\big[[ac(x + 1), a(x^{-1} + 1)],\ [ac(x + 1), ba(x^{2^{n-1}-1} + 1)]\big]\\
&\quad=\big[a^2cx^{-1}(x + 1)^3,\ ba^2c\big((b, a)x^{-1} + 1\big)(x^{2^{n-1}+1} + 1)(x + 1)\big]\\
&\quad= a^4bc^2x^{-1}\big((b, a)x^{-1} + 1\big)(x^{2^{n-1}+1} + 1)(x + 1)^{2^{n-1}+4}
\end{aligned}$$

belongs to $\delta^{[3]}(FG)$ and is not equal to zero, thus $\mathrm{dl}_L(FG) > 3$.

Case 3: $G/C = \langle dC\rangle$, where $x^d = x^{2^{n-1}-1}$. Since $G' = (d, C)$, similarly as before, we can choose $c \in C$ such that $(c,d) = x$. Then

$$\begin{aligned}\Big[\big[[c,d],[d^{-1}c,d]\big],\ \big[[c,d],[c,dc]\big]\Big]\\ &= \Big[\big[dc(x+1),c(x+1)\big],\ \big[dc(x+1),dc^2(x+1)\big]\Big]\\ &= \big[dc^2(x+1)^{2^{n-1}+1},\ d^2c^3x(x^{2^{n-1}-1}+1)(x+1)^2\big]\\ &= d^3c^5x(x^{2^{n-1}-1}+1)(x^{2^{n-2}-1}+1)^2(x+1)^{2^{n-1}+2}\end{aligned}$$

is a nonzero element in $\delta^{[3]}(FG)$ so $\mathrm{dl}_L(FG) > 3$.

Case 4: $G/C = \langle aC, bC\rangle$, where $x^a = x^{-1}$ and $x^b = x^{2^{n-1}+1}$. Then

$$G' = \langle (ab,b)\rangle (ab,C)(b,C)C' = \langle (a,b)\rangle (ab,C)(b,C),$$

because $C' \subseteq \langle x^{2^{n-1}}\rangle$. Since G' is cyclic, G' coincides with either $\langle (a,b)\rangle$ or (ab,C) or (b,C).

Assume that $G' = (ab,C)$ and set $H = \langle ab, C\rangle$. Then H satisfies the hypothesis of Case 3 of this lemma, so $\mathrm{dl}_L(FG) \geq \mathrm{dl}_L(FH) > 3$. We get the same result in the case $G' = (b,C)$.

There remains the possibility that $(a,b) = y$ is of order 2^n. Then

$$\begin{aligned}\Big[\big[[a,b],[b^{-1}a,b]\big],\ \big[[a,b],[b,ab]\big]\Big]\\ &= \Big[\big[ba(y+1),a(y+1)\big],\ \big[ba(y+1),ab^2(y^{2^{n-1}-1}+1)\big]\Big]\\ &= \big[ba^2(y^{2^{n-1}-2}+1)(y+1),\ b^3a^2y^{-1}(y^{2^{n-1}-2}+1)(y+1)\big]\\ &= b^4a^4y^{-1}(y^{-1}+1)^4(y^{2^{n-1}+1}+1)(y+1)^{2^{n-1}+1} \neq 0,\end{aligned}$$

and the statement is valid. □

Lemma 5.1.5. *Let G be a group with commutator subgroup $G' = \langle x \mid x^{16} = 1\rangle$ and let $\mathrm{char}(F) = 2$. Then $\mathrm{dl}_L(FG) = 3$ if and only if G has an abelian subgroup of index two.*

Proof. By the previous lemma, the statement is true if $\exp(G/C) \leq 2$. The other possible cases are:

Case 1: $G/C = \langle bC\rangle$, where $x^b = x^5$. Since then $G = G_\beta$, Lemma 4.1.2 and Theorem 4.2.1 state that $\mathrm{dl}_L(FG) = 5$.

Case 2: $G/C = \langle dC \rangle$, where $x^d = x^{-5}$. Then $G' = (d, C)$ and, as before, we can choose $c \in C$ such that $(c, d) = x$ and

$$\begin{aligned}
&\Big[\big[[c,d],[d^{-1}c,d]\big],\ \big[[c,d],[c,dc]\big]\Big] \\
&\qquad = \Big[\big[dc(x+1), c(x+1)\big],\ \big[dc(x+1), dc^2(x^{-5}+1)\big]\Big] \\
&\qquad = \big[dc^2(x^{-4}+1)(x+1),\ d^2c^3(x^{-5}+1)(x+1)^2\big] \\
&\qquad = b^3c^5x^6(x^{-5}+1)(x^9+1)(x+1)^9
\end{aligned}$$

belongs to $\delta^{[3]}(FG)$ and is not zero.

Case 3: $G/C = \langle aC, bC \rangle$, where $x^a = x^{-1}$ and $x^b = x^5$. Then by similar arguments as in the last case of the previous lemma we can restrict ourselves to the case when $(a, b) = x$. Then

$$\begin{aligned}
&\Big[\big[[a,b],[b^{-1}a,b]\big],\ \big[[a,b],[b,ab]\big]\Big] \\
&\qquad = \Big[\big[ba(x+1), a(x+1)\big],\ \big[ba(x+1), ab^2(x^{-5}+1)\big]\Big] \\
&\qquad = \big[ba^2(x^{10}+1)(x+1),\ b^3a^2(x^{10}+1)(x^7+1)\big] \\
&\qquad = b^4a^4x^3(x^5+1)^4(x+1)^6 \neq 0,
\end{aligned}$$

which was to be proved. □

Theorem 5.1.6. *Let G be a group with cyclic commutator subgroup of order p^n and let F be field of characteristic p. Then $\mathrm{dl}_L(FG) = 3$ if and only if one of the following conditions holds:*

(i) $p = 7$, $n = 1$ and G is nilpotent;

(ii) $p = 5$, $n = 1$ and either $x^g = x^{-1}$ for all $x \in G'$ and $g \notin C_G(G')$ or G is nilpotent;

(iii) $p = 3$, $n = 1$ and G is not nilpotent;

(iv) $p = 2$ and one of the following conditions is satisfied:

a) $n = 2$;

b) $n = 3$ *and* G *is of class* 4;

c) G *has an abelian subgroup of index two.*

Proof. Suppose first that $p > 7$. Then Proposition 5.1.1 states that $\mathrm{dl}_L(FG) \geq \lceil \log_2(p+1) \rceil \geq 4$. For odd $p \leq 7$ the statement follows directly from Theorem 3.3.1.

Let $G' = \langle x \mid x^{2^n} = 1 \rangle$. The result follows from Theorem 3.3.1 for $n = 2$ and $n = 3$. For $n > 3$, using induction on n, we shall show that if $\mathrm{dl}_L(FG) = 3$ then C is abelian and $G/C = \langle aC \rangle$, where $x^a = x^{-1}$ (i.e. G has an abelian subgroup of index two). Indeed, by Lemma 5.1.5, this is true for $n = 4$. Let now $n > 4$ and $\mathrm{dl}_L(FG) = 3$ and assume that the statement is true for every group with commutator subgroup of order less than 2^n. Set $H = \langle x^{2^{n-1}} \rangle \subset G'$. Then $\mathrm{dl}_L\big(F(G/H)\big) = 3$ and $(G/H)' = G'/H = \langle xH \rangle$, and by inductive hypothesis we get

$$(xH)^{gH} = x^g H = x^{(-1)^k} H$$

for all $g \in G$. It follows that $x^g = x^i$ with

$$i \in \{-1, 1, 2^{n-1} - 1, 2^{n-1} + 1\},$$

i.e. $\exp(G/C) \leq 2$ and the statement follows from Lemma 5.1.4. □

5.2 Group algebras of 2-groups of order 2^m and exponent 2^{m-2}

The full description of the finite nonabelian 2-group of order 2^m and exponent 2^{m-2} can be found in [20]. These groups are:

- $m \geq 4$
  $G_1 = \langle a, b \mid a^{2^{m-2}} = 1, b^4 = 1, a^b = a^{1+2^{m-3}} \rangle$;
  $G_2 = Q_{2^{m-1}} \times C_2$;
  $G_3 = D_{2^{m-1}} \times C_2$;
  $G_4 = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, ab = ba, ac = ca, b^c = a^{2^{m-3}} b \rangle$;
  $G_5 = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, ab = ba, a^c = ab, bc = cb \rangle$;

- $m \geq 5$
  $G_6 = \langle a, b \mid a^{2^{m-2}} = 1, b^4 = 1, a^b = a^{-1} \rangle$;
  $G_7 = \langle a, b \mid a^{2^{m-2}} = 1, b^4 = 1, a^b = a^{-1+2^{m-3}} \rangle$;
  $G_8 = \langle a, b \mid a^{2^{m-2}} = 1, b^4 = a^{2^{m-3}}, a^b = a^{-1} \rangle$;
  $G_9 = \langle a, b \mid a^{2^{m-2}} = 1, b^4 = 1, b^a = b^{-1} \rangle$;
  $G_{10} = M_{2^{m-1}} \times C_2$;
  $G_{11} = S_{2^{m-1}} \times C_2$;
  $G_{12} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, ab = ba, a^c = a^{-1}, b^c = a^{2^{m-3}} b \rangle$;
  $G_{13} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, ab = ba, a^c = a^{-1} b, bc = cb \rangle$;
  $G_{14} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = a^{2^{m-3}}, ab = ba, a^c = a^{-1} b, bc = cb \rangle$;
  $G_{15} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, a^b = a^{1+2^{m-3}}, a^c = a^{-1+2^{m-3}}, bc = cb \rangle$;
  $G_{16} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, a^b = a^{1+2^{m-3}}, a^c = a^{-1+2^{m-3}}, b^c = a^{2^{m-3}} b \rangle$;
  $G_{17} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, a^b = a^{1+2^{m-3}}, a^c = ab, bc = cb \rangle$;
  $G_{18} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = b, a^b = a^{1+2^{m-3}}, a^c = a^{-1} b \rangle$;

- $m \geq 6$
  $G_{19} = \langle a, b \mid a^{2^{m-2}} = 1, b^4 = 1, a^b = a^{1+2^{m-4}} \rangle$;
  $G_{20} = \langle a, b \mid a^{2^{m-2}} = 1, b^4 = 1, a^b = a^{-1+2^{m-4}} \rangle$;
  $G_{21} = \langle a, b \mid a^{2^{m-2}} = 1, a^{2^{m-3}} = b^4, a^{-1} ba = b^{-1} \rangle$;
  $G_{22} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, ab = ba, a^c = a^{1+2^{m-4}} b, b^c = a^{2^{m-3}} b \rangle$;
  $G_{23} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, ab = ba, a^c = a^{-1+2^{m-4}} b, b^c = a^{2^{m-3}} b \rangle$;
  $G_{24} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = 1, a^b = a^{1+2^{m-3}}, a^c = a^{-1+2^{m-4}} b, bc = cb \rangle$;
  $G_{25} = \langle a, b, c \mid a^{2^{m-2}} = 1, b^2 = 1, c^2 = a^{2^{m-3}}, a^b = a^{1+2^{m-3}}, a^c = a^{-1+2^{m-4}} b, bc = cb \rangle$;

- $G_{26} = \langle a, b, c \mid a^8 = 1, b^2 = 1, c^2 = a^4, a^b = a^5, a^c = ab, bc = cb \rangle$,

where for $m \geq 3$

$$D_{2^m} = \langle a, b \mid a^{2^{m-1}} = 1, b^2 = 1, a^b = a^{-1} \rangle;$$
$$Q_{2^m} = \langle a, b \mid a^{2^{m-1}} = 1, b^2 = a^{2^{m-2}}, a^b = a^{-1} \rangle;$$

and for $m \geq 4$

$$S_{2^m} = \langle a, b \mid a^{2^{m-1}} = 1, b^2 = 1, a^b = a^{-1+2^{m-2}} \rangle;$$
$$M_{2^m} = \langle a, b \mid a^{2^{m-1}} = 1, b^2 = 1, a^b = a^{1+2^{m-2}} \rangle.$$

The group algebras of G_i have been examined by several authors, for example V. Bódi [9]. Our results enable us to determine the derived length of FG_i over a field F of characteristic two. The claim is the following:

$$\mathrm{dl}_L(FG_i) = \begin{cases} 2, & \text{if either } i \in \{2,3\} \text{ and } m = 4 \text{ or } i \in \{1,4,5,9,10\}; \\ 4, & \text{if } \quad i \in \{15,16,18,20,24,25\} \quad \text{and} \quad m > 5; \\ 3, & \text{otherwise.} \end{cases}$$

Indeed, apart from the cases $i \in \{17, 26\}$, G_i' is cyclic. If either $i \in \{2, 3\}$ and $m = 4$ or $i \in \{1, 4, 5, 9, 10\}$ then $G_i' = C_2$ and the statement follows from e. g. Theorem 3.3.1. Furthermore, if $i \in \{15, 16, 18, 20, 24, 25\}$ and $m > 5$ then G_i has a normal subgroup with commutator subgroup of order two. Using Theorem 5.1.2 it follows that $\mathrm{dl}_L(FG_i) \leq 4$. To prove the converse inequality we can use Theorem 5.1.6. In all the other cases either G_i' is of order 4 or G_i has an abelian subgroup of index two. Then Theorem 5.1.6 guarantees the statement. Finally, $G_{17}' \cong G_{26}' \cong C_2 \times C_2$ and we can apply Proposition 2.1.2 to compute the derived length.

Chapter 6

Lie nilpotency indices of Lie nilpotent group algebras

According to [32], if FG is Lie nilpotent and G' has order p^n, then

$$t_L(FG) \leq t^L(FG) \leq p^n + 1.$$

A. Shalev in [25] began to study the question when a Lie nilpotent group algebra has the maximal upper Lie nilpotency index. The complete description of such group algebras was given by V. Bódi and E. Spinelli in [13]. In this chapter we determine the group algebras whose upper Lie nilpotency index is 'almost maximal', that is, it takes the next highest possible value, namely $p^n - p + 2$, where p^n is the order of the commutator subgroup of the basic group.

6.1 Preliminary results

Let FG be a Lie nilpotent group algebra. We consider a sequence of subgroups of G, setting

$$\mathfrak{D}_{(m)}(G) = G \cap (1 + FG^{(m)}), \qquad (m \geq 1).$$

The subgroup $\mathfrak{D}_{(m)}(G)$ is called the m-th *Lie dimension subgroup* of FG. It is possible to describe the $\mathfrak{D}_{(m)}(G)$'s in the following manner (see Theorem 2.8 of [21]):

$$\mathfrak{D}_{(m+1)}(G) = \begin{cases} G & \text{if} \quad m = 0; \\ G' & \text{if} \quad m = 1; \\ \big(\mathfrak{D}_{(m)}(G), G\big)(\mathfrak{D}_{(\lceil \frac{m}{p} \rceil+1)}(G))^p & \text{if} \quad m \geq 2, \end{cases} \tag{6.1}$$

where $\lceil \frac{m}{p} \rceil$ is the upper integer part of $\frac{m}{p}$.

By [21] there also exists an explicit expression for $\mathfrak{D}_{(m+1)}(G)$:

$$\mathfrak{D}_{(m+1)}(G) = \prod_{(j-1)p^i \geq m} \gamma_j(G)^{p^i}. \tag{6.2}$$

Evidently,

$$G = \mathfrak{D}_{(1)}(G) \supseteq \mathfrak{D}_{(2)}(G) \supseteq \cdots \supseteq \mathfrak{D}_{(m)}(G) \supseteq \cdots .$$

It is easy to check that the factor group $D_{(k)}(G)/D_{(k+1)}(G)$ is an elementary abelian p-group for any $k \geq 1$. Put

$$p^{d_{(k)}} = [\mathfrak{D}_{(k)}(G) : \mathfrak{D}_{(k+1)}(G)].$$

According to Jennings' theory [25] for the Lie dimension subgroups, we get

$$t^L(FG) = 2 + (p-1) \sum_{m \geq 1} m d_{(m+1)}, \tag{6.3}$$

and

$$\sum_{m \geq 2} d_{(m)} = n. \tag{6.4}$$

As we have already mentioned, if G' has order p^n then

$$t_L(FG) \leq t^L(FG) \leq p^n + 1.$$

The Lie nilpotent group algebras with maximal upper (and lower) Lie nilpotency indices have been described:

Proposition 6.1.1 (V. Bódi and E. Spinelli [13]). *Let G be a nilpotent group with commutator subgroup of order p^n and let* $\mathrm{char}(F) = p$. *Then $t_L(FG) = p^n + 1$ if and only if one of the following conditions holds:*

(i) G' is cyclic;

(ii) $p = 2$ and G' is the noncyclic of order 4 and $\gamma_3(G) \neq 1$.

Assume that G' has order p^n and $t^L(FG) < p^n+1$. Then, according to (6.3), the next highest possible value of $t^L(FG)$ is $p^n - p + 2$. If $t^L(FG) = p^n - p + 2$ then we shall say that FG has *almost maximal* upper Lie nilpotency index. Our goal is to determine these group algebras.

To this we need the following results:

Proposition 6.1.2 (A.K. Bhandari and I.B.S. Passi [2]). *Let FG be a Lie nilpotent group algebra of characteristic $p > 3$. Then $t_L(FG) = t^L(FG)$.*

Proposition 6.1.3 (Sahlev [26, 28]). *Let G be a nilpotent group whose commutator subgroup has order p^n and exponent p^l and let* $\mathrm{char}(F) = p$.

(i) If $d_{(m+1)} = 0$ and m is a power of p, then $\mathfrak{D}_{(m+1)}(G) = \langle 1 \rangle$.

(ii) If $d_{(m+1)} = 0$ and p^{l-1} divides m, then $\mathfrak{D}_{(m+1)}(G) = \langle 1 \rangle$.

(iii) If $p \geq 5$ and $t_L(FG) < p^n + 1$, then $t_L(FG) \leq p^{n-1} + 2p - 1$.

Proposition 6.1.4 (V. Bódi and E. Spinelli [13]). *Let G be a nilpotent group with commutator subgroup of order p^n and let* $\mathrm{char}(F) = p$. *Then $t^L(FG) = p^n + 1$ if and only if $d_{(p^i+1)} = 1$ and $d_{(j)} = 0$, where $0 \leq i \leq n-1$, $j \neq p^i + 1$ and $j > 1$.*

Let P be a finite abelian p-group, $P = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_s \rangle$, where a_i is of order p^{m_i} and $m_1 \geq m_2 \geq \cdots \geq m_s$. We call $\{a_i\}$ a basis of P. Any $g \in P$ can be written uniquely as $g = a_1^{k_1} a_2^{k_2} \cdots a_s^{k_s}$ with $0 \leq k < p^{m_i}$. We will denote k_i by $g(i)$, or by $g(a_i)$ if there are more bases of P considered.

Proposition 6.1.5 (A.A. Bódi and J. Kurdics [8]). *Let P be a finite abelian p-group presented as above and let H be a proper subgroup of P such that the order of the factor group HP^p/P^p is p^r with $r > 0$. Then the following statements hold:*

(i) *the function*

$$\nu : H \setminus P^p \to \{1, 2, \ldots, s\}, \quad \nu(h) = \min\{j \mid \gcd(h(j), p) = 1\}$$

takes r distinct values $v_1 < v_2 < \cdots < v_r$;

(ii) *there exist $b_1, b_2, \ldots, b_r \in H$ such that $\nu(b_i) = v_i, b_i(v_j) = 0$ for $j < i$, $b_i(v_i) = 1$, and if*

$$\{1, 2, \ldots, s\} = \{u_1, u_2, \ldots, u_{s-r}, v_1, v_2, \ldots, v_r\},$$

$u_1 < u_2 < \cdots < u_{s-r}$, and

$$A = \langle a_{u_1} \rangle \times \langle a_{u_2} \rangle \times \cdots \times \langle a_{u_{s-r}} \rangle,$$

called the weak complement of H in P relative to the basis $\{a_i\}$, then

$$P/A = \langle b_1 A \rangle \times \langle b_2 A \rangle \times \cdots \times \langle b_r A \rangle;$$

(iii) *weak complements of H in P, relative to any basis, are all isomorphic to each other;*

(iv) *if G is a nilpotent group of class 3 such that G' is a finite abelian p-group and the order of $\gamma_3(G)(G')^p/(G')^p$ is p^r with $r > 0$ then*

$$t_L(FG) = t^L(FG) = t_N(G') + t_N(G'/A),$$

where A is the weak complement of $\gamma_3(G)$ in G'.

6.2 Group algebras with almost maximal upper Lie nilpotency index

The description of the group algebras with almost maximal upper Lie nilpotency index will be based on the following lemmas.

Lemma 6.2.1. *Let G be a nilpotent group with commutator subgroup of order p^n and let* $\mathrm{char}(F) = p$. *Then $t^L(FG) = p^n - p + 2$ if and only if one of the following conditions holds:*

(i) $p = 2, \quad n = 2 \quad$ *and* $\quad d_{(2)} = 2$;

(ii) $p = 2, \quad n > 2, \quad d_{(2^i+1)} = d_{(2^{n-1})} = 1 \quad$ *and* $\quad d_{(j)} = 0$, *where* $\quad 0 \le i \le n-2, \quad j \ne 2^i + 1, \quad j \ne 2^{n-1} \quad$ *and* $j > 1$;

(iii) $p = 3, \quad n = 2 \quad$ *and* $\quad d_{(2)} = 1 \; d_{(3)} = 1$;

(iv) $p = 3, \quad n \ge 2, \quad d_{(3^i+1)} = d_{(3^{n-1})} = 1 \quad$ *and* $\quad d_{(j)} = 0$, *where* $\quad 0 \le i \le n-2, \quad j \ne 3^i + 1, \quad j \ne 3^{n-1} \quad$ *and* $j > 1$.

Proof. If any one of the statements (i)-(iv) holds, then by (6.3), we easily get $t^L(FG) = p^n - p + 2$.

Conversely, assume that $t^L(FG) = p^n - p + 2$. If $n = 1$ then Proposition 6.1.1 states that $t^L(FG) = p^n + 1$, therefore $n \ge 2$. For $n = 2$ the equation (6.3) shows immediately that only (i) or (iii) are possible. Now, we prove that $p \le 3$. Indeed, supposing that $p \ge 5$ and G' is not cyclic, we can apply Proposition 6.1.2 and (iii) of Proposition 6.1.3 to get $t^L(FG) = t_L(FG) \le p^{n-1} + 2p - 1$. But $p^n - p + 2 > p^{n-1} + 2p - 1$, because $(p^{n-1} - 3)(p - 1) > 0$.

So, only the cases when $n > 2$ and either $p = 3$ or $p = 2$ remain. First, we shall show that $d_{(p^i+1)} > 0$ for $0 \le i \le n - 2$.

Suppose there exists $0 \le s \le n - 2$ such that $d_{(p^s+1)} = 0$. From (6.1) it follows at once that $s \ne 0$ and by (i) of Proposition 6.1.3 $\mathfrak{D}_{(p^s+1)}(G) = \langle 1 \rangle$, so $d_{(r)} = 0$ for every $r \ge p^s + 1$. Moreover, if

$d_{(q+1)} \neq 0$, then $q \leq p^s$. According to (6.3), it follows that

$$\begin{aligned} t^L(FG) &= 2+(p-1)\Big(\sum_{i=0}^{s-1} p^i + \sum_{i=0}^{s-1} p^i(d_{(p^i+1)}-1) + \sum_{q\neq p^i} qd_{(q+1)}\Big) \\ &< 2+(p-1)\Big(\sum_{i=0}^{s-1} p^i + \big(\sum_{i=0}^{s-1}(d_{(p^i+1)}-1) + \sum_{q\neq p^i} d_{(q+1)}\big)p^s\Big) \\ &= 2+(p-1)\Big(\sum_{i=0}^{s-1} p^i + (n-s)\cdot p^s\Big) \\ &< 1+p^{n-2}+(p-1)(n-(n-2))\cdot p^{n-2}. \end{aligned}$$

Hence, for $p=2$ we have $t^L(FG) < 1+3\cdot 2^{n-2} < p^n$, and if $p=3$ then $t^L(FG) < 1+5\cdot 3^{n-2} < p^n-1$, which contradicts the assumption $t^L(FG) = p^n - p + 2$.

Therefore $d_{(p^i+1)} > 0$ for $0 \leq i \leq n-2$ and by (6.4) there exists $\alpha \geq 2$ such that $d_{(\alpha)} = 1$. Then by (6.3),

$$\begin{aligned} t^L(FG) &= 2+(p-1)\sum_{i=0}^{n-2} p^i + (p-1)(\alpha-1)d_{(\alpha)} \\ &= 1+p^{n-1}+(p-1)(\alpha-1). \end{aligned}$$

Since $t^L(FG) = p^n - p + 2$, we must have $\alpha = p^{n-1}$ and the proof is done. □

Lemma 6.2.2. *Let G be a nilpotent group with commutator subgroup of order 2^n and let $\mathrm{char}(F) = 2$. If $t^L(FG) = 2^n$ then one of the following conditions holds:*

(i) G is of class 2 and G' is noncyclic of order 4;

(ii) G is of class 4 with one of the following properties:

(a) $G' \cong C_4 \times C_2$, $\gamma_3(G) \cong C_2 \times C_2$;

(b) $G' \cong C_2 \times C_2 \times C_2$.

Proof. Let $t^L(FG) = 2^n$. Then either (i) or (ii) of Lemma 6.2.1 holds. If $n = 2$ then by (6.3) and (i) of Lemma 6.2.1 we obtain that $\mathfrak{D}_{(3)}(G) = \gamma_3(G) \cdot \gamma_2(G)^2 = \langle 1 \rangle$, so the statement (i) holds.

Assume that $n \geq 3$. By (ii) of Lemma 6.2.1 we get $d_{(2^i+1)} = 1$, $d_{(2^{n-1})} = 1$ and $d_{(j)} = 0$, where $0 \leq i \leq n-2$, $j \neq 2^i + 1$, $j \neq 2^{n-1}$ and $j > 1$. The subgroup $H = \mathfrak{D}_{(2^{n-1})}(G)$ is central of order 2 and from (6.2) it follows that

$$\begin{aligned}\mathfrak{D}_{(m+1)}(G)/H &= \prod_{(j-1)2^i \geq m} \gamma_j(G)^{2^i}/H \\ &= \prod_{(j-1)2^i \geq m} \gamma_j(G/H)^{2^i} = \mathfrak{D}_{(m+1)}(G/H).\end{aligned}$$

Put $2^{\overline{d}_{(k)}} = [\mathfrak{D}_{(k)}(G/H) : \mathfrak{D}_{(k+1)}(G/H)]$ for $k \geq 1$. It is easy to check that $\overline{d}_{(2^i+1)} = 1$ and $\overline{d}_{(j)} = 0$, where $0 \leq i \leq n-2$, $j \neq 2^i + 1$ and $j > 1$.

Clearly, $\gamma_2(G/H)$ has order 2^{n-1} and $t^L(F[G/H]) = 2^{n-1}+1$. So by Propositions 6.1.1 and 6.1.4 by the group $\gamma_2(G/H)$ is either a cyclic 2-group or $C_2 \times C_2$. If $\gamma_2(G/H)$ is a cyclic 2-group then $\gamma_2(G)$ is abelian, so it is isomorphic to either $C_{2^{n-1}} \times C_2$ or C_{2^n}. If $\gamma_2(G)$ is cyclic, then by Proposition 6.1.1 we get $t^L(FG) = 2^n + 1$, so we do not need to consider this case.

Let $\gamma_2(G/H) = C_2 \times C_2$. It is easy to check that $\gamma_2(G)$ has order 8 and $\gamma_2(G)$ is one of the following groups: Q_8, D_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$. It is well-known that there is no nilpotent group G such that either $\gamma_2(G) \cong Q_8$ or $\gamma_2(G) \cong D_8$.

Assume that $\gamma_2(G) = \langle\, a, b \mid a^4 = b^2 = 1 \,\rangle \cong C_4 \times C_2$. Thus by (6.1)

$$\mathfrak{D}_{(3)}(G) = (\mathfrak{D}_{(2)}(G), G) \cdot \mathfrak{D}_{(2)}(G)^2 = \gamma_3(G) \cdot \langle a^2 \rangle.$$

Since $\mathfrak{D}_{(2)}(G)/\mathfrak{D}_{(3)}(G)$ has order 2, only one of the following cases is possible:

$$\gamma_3(G) = \langle a \rangle, \quad \gamma_3(G) = \langle ab \rangle, \quad \gamma_3(G) = \langle a^2, b \rangle,$$

$$\gamma_3(G) = \langle a^2 b \rangle, \quad \gamma_3(G) = \langle b \rangle.$$

Next we consider separately each of these cases:

Case 1.a: Let either $\gamma_3(G) = \langle a \rangle$ or $\gamma_3(G) = \langle ab \rangle$. Since $\gamma_2(G)^2 \subset \gamma_3(G)$ and $\exp\big(\gamma_{i+1}(G)/\gamma_{i+2}(G)\big)$ divides $\exp\big(\gamma_i(G)/\gamma_{i+1}(G)\big)$ for all $i \geq 1$, we have that $\gamma_k(G)^2 \subseteq \gamma_{k+1}(G)$ for every $k \geq 2$. It follows that $\gamma_2(G)^2 = \gamma_4(G)$. Moreover, $\gamma_3(G)^2 \subseteq \gamma_5(G)$. Indeed, the elements of the form (x, y), where $x \in \gamma_2(G)$ and $y \in G$ are generators of $\gamma_3(G)$, so we have to prove that $(x, y)^2 \in \gamma_5(G)$. Evidently,

$$(x^2, y) = (x, y) \cdot (x, y, x) \cdot (x, y) = (x, y)^2 \cdot (x, y, x)^{(x,y)}$$

and (x^2, y), $(x, y, x)^{(x,y)} \in \gamma_5(G)$, so $(x, y)^2 \in \gamma_5(G)$ and $\gamma_3(G)^2 \subseteq \gamma_5(G)$. Thus $\langle a^2 \rangle \subseteq \langle 1 \rangle$, a contradiction.

Case 1.b: Let $\gamma_3(G) = \langle a^2 \rangle \times \langle b \rangle$ and let G be of class 3. Now, let us compute the weak complement of $\gamma_3(G)$ in $\gamma_2(G)$. It is easy to see that in the notation of Proposition 6.1.5

$$P = \gamma_2(G), \qquad H = \langle a^2, b \rangle, \qquad H \setminus P^2 = \{b, a^2 b\}$$

so $\nu(b) = \nu(a^2 b) = 2$ and the weak complement is $A = \langle a \rangle$. Since G is of class 3, by Proposition 6.1.5(iv) we have

$$t_L(FG) = t^L(FG) = t(\gamma_2(G)) + t(\gamma_2(G)/\langle a \rangle) = 7 \neq 2^n.$$

Case 1.c: Let either $\gamma_3(G) = \langle b \rangle$ or $\gamma_3(G) = \langle a^2 b \rangle$. Clearly G is of class 3. According to the notation of Proposition 6.1.5 we have that $P = \gamma_2(G)$ and either $H = \langle b \rangle$ and $H \setminus P^2 = \{b\}$ or $H = \langle a^2 b \rangle$ and $H \setminus P^2 = \{a^2 b\}$. It follows that $\nu(b) = \nu(a^2 b) = 2$ in both cases and the weak complement is $A = \langle a \rangle$. As in the case 1.b we have $t^L(FG) = 7 \neq 2^n$, a contradiction.

Now, let $\gamma_2(G) \cong C_2 \times C_2 \times C_2$. If $\gamma_3(G) \cong C_2$ then by (6.1)

$$\mathfrak{D}_{(2)}(G) = \gamma_2(G), \qquad \mathfrak{D}_{(3)}(G) = \gamma_3(G), \qquad \mathfrak{D}_{(4)}(G) = \langle 1 \rangle$$

and $d_{(2)} = 2$, $d_{(3)} = 1$, which contradicts (ii) of Lemma 6.2.1.

If $\gamma_3(G) \cong C_2 \times C_2$ and G is of class 3, then by (6.1) we have

$$\mathfrak{D}_{(2)}(G) = \gamma_2(G), \qquad \mathfrak{D}_{(3)}(G) = \gamma_3(G), \qquad \mathfrak{D}_{(4)}(G) = \langle 1 \rangle,$$

also a contradiction, because $d_{(2)} = 1$ and $d_{(3)} = 2$.

Finally, let $\gamma_2(G) = \langle a, b \mid a^{2^{n-1}} = b^2 = 1\rangle \cong C_{2^{n-1}} \times C_2$ with $n \geq 4$. According to (6.1), $\mathfrak{D}_{(2)}(G) = \gamma_2(G)$ and $\mathfrak{D}_{(3)}(G) = \gamma_3(G) \cdot \langle a^2\rangle$ hold. Since $\mathfrak{D}_{(2)}(G)/\mathfrak{D}_{(3)}(G)$ has order 2, we obtain one of the following cases:

$$\gamma_3(G) = \langle a\rangle, \qquad \gamma_3(G) = \langle ab\rangle, \qquad \gamma_3(G) = \langle b\rangle,$$
$$\gamma_3(G) = \langle a^{2^j} b\rangle, \qquad \gamma_3(G) = \langle a^{2^j}, b\rangle,$$

where $1 \leq j \leq n - 2$.

We consider each of these:

Case 2.a: Let either $\gamma_3(G) = \langle a\rangle$ or $\gamma_3(G) = \langle ab\rangle$ or $\gamma_3(G) = \langle b\rangle$. Using the arguments of the cases 1.a and 1.c above, it is easy to verify that we obtain a contradiction.

Case 2.b: Let $\gamma_3(G) = \langle a^{2^j} b\rangle$ with $1 \leq j \leq n-2$. Then by (6.1) and by (ii) of Lemma 6.2.1 we get $\mathfrak{D}_{(2)}(G) = \gamma_2(G)$, $\mathfrak{D}_{(3)}(G) = \langle a^2\rangle \times \langle b\rangle$ and $\mathfrak{D}_{(4)}(G) = (\langle a^2\rangle \times \langle b\rangle, G) \cdot \langle a^4\rangle$. So, $\mathfrak{D}_{(4)}(G)$ is either $\langle a^2\rangle$ or $\langle a^2 b\rangle$ or $\langle a^4\rangle \times \langle b\rangle$.

Suppose first that $\mathfrak{D}_{(4)}(G) = \langle a^2\rangle$. By (6.1) and (ii) of Lemma 6.2.1

$$\mathfrak{D}_{(5)}(G) = (\langle a^2\rangle, G) \cdot \mathfrak{D}_{(3)}(G)^2 = (\langle a^2\rangle, G) \cdot \langle a^4\rangle = \langle a^2\rangle.$$

This equality forces $(\langle a^2\rangle, G) = \langle a^2\rangle$ and so $\mathfrak{D}_{(k)}(G) = \langle a^2\rangle$ for each $k \geq 5$, which is impossible.

Now let $\mathfrak{D}_{(4)}(G) = \langle a^2 b\rangle$. As above,

$$\mathfrak{D}_{(4)}(G) = \mathfrak{D}_{(5)}(G) = (\langle a^2 b\rangle, G) \cdot \langle a^4\rangle = \langle a^2 b\rangle,$$

and we get $(\langle a^2 b\rangle, G) = \langle a^2 b\rangle$, which is not possible either.

Finally, let $\mathfrak{D}_{(4)}(G) = \langle a^4\rangle \times \langle b\rangle$. Suppose that there exists $k \leq 2^{n-2} + 1$, such that $\mathfrak{D}_{(k)}(G)$ is cyclic. Using the same arguments as above, we obtain that $\mathfrak{D}_{(m)}(G) \neq \langle 1\rangle$ for each m, which is impossible.

So $\mathfrak{D}_{(2^{n-2}+1)}(G) = \langle a^{2^{n-2}}\rangle \times \langle b\rangle$ and by (6.1) and by (ii) of Lemma 6.2.1

$$\begin{aligned}
\mathfrak{D}_{(2^{n-2}+2)}(G) &= (\mathfrak{D}_{(2^{n-2}+1)}(G), G) \cdot \mathfrak{D}_{(2^{n-3}+2)}(G)^2 \\
&= (\mathfrak{D}_{(2^{n-2}+1)}(G), G) = \langle \omega \mid \omega^2 = 1\rangle; \\
\mathfrak{D}_{(2^{n-2}+3)}(G) &= (\mathfrak{D}_{(2^{n-2}+2)}(G), G) = (\langle \omega\rangle, G) = \langle \omega\rangle,
\end{aligned}$$

which is not possible either.

Case 2.c: Let $\gamma_3(G) = \langle a^{2^j} \rangle \times \langle b \rangle$ with $1 \leq j \leq n-2$. It is easy to check that this case is similar to the last subcase of the case 2.b.

So the proof is complete. □

Lemma 6.2.3. *Let G be a nilpotent group with commutator subgroup of order 3^n and let* $\mathrm{char}(F) = 3$. *If $t^L(FG) = 3^n - 1$ then G is of class 3, $G' \cong C_3 \times C_3$ and $\gamma_3(G) \cong C_3$.*

Proof. Let $t^L(FG) = 3^n - 1$. Then either (iii) or (iv) of Lemma 6.2.1 holds. By (iv) of Lemma 6.2.1 it yields

$$d_{(3^i+1)} = 1, \qquad d_{(3^{n-1})} = 1, \qquad d_{(j)} = 0,$$

where $\quad 0 \leq i \leq n-2, \quad j \neq 3^i + 1, \quad j \neq 3^{n-1} \quad$ and $j > 1$.

The subgroup $H = \mathfrak{D}_{(3^{n-1})}(G)$ is central of order 3 and from (6.2), as we already proved at the beginning of the proof of Lemma 6.2.2, we have

$$\mathfrak{D}_{(m+1)}(G)/H = \mathfrak{D}_{(m+1)}(G/H).$$

It follows that $\overline{d}_{(3^i+1)} = 1$ for $0 \leq i \leq n-2$, $\overline{d}_{(j)} = 0$ for $j \neq 3^i + 1$ and $j > 1$, where $3^{\overline{d}_{(k)}} = [\mathfrak{D}_{(k)}(G/H) : \mathfrak{D}_{(k+1)}(G/H)]$.

Clearly, $\gamma_2(G/H)$ has order $= 3^{n-1}$ and $t^L(F[G/H]) = 3^{n-1} + 1$. By Propositions 6.1.1 and 6.1.4 we have that $\gamma_2(G/H)$ is a cyclic 3-group. Then $\gamma_2(G)$ is abelian, so it is isomorphic to either $C_{3^{n-1}} \times C_3$ or C_{3^n}. By Proposition 6.1.1, in the last case FG has upper and lower maximal Lie nilpotency index.

Therefore we can assume that $\gamma_2(G) \cong C_{3^{n-1}} \times C_3$ and $n \geq 3$. Since $\exp(\gamma_2(G)) = 3^{n-1}$ and 3^{n-2} divides $2 \cdot 3^{n-2}$, by (iii) of Lemma 6.2.1 and by (ii) of Proposition 6.1.3 we obtain that $\mathfrak{D}_{(3^{n-2}+1)}(G) = \langle 1 \rangle$ and so $\mathfrak{D}_{(3^{n-1})}(G) = \langle 1 \rangle$ in contradiction with (iii) of Lemma 6.2.1. It follows that $\gamma_2(G) \cong C_3 \times C_3$.

Suppose that $\gamma_2(G) \subseteq \zeta(G)$. By (6.1) we get $\mathfrak{D}_{(3)}(G) = \langle 1 \rangle$ and so $d_{(2)} = 2$, which is in contradiction with (iii) of Lemma 6.2.1.

Finally, the case (iii) of Lemma 6.2.1 follows from the previous case. The proof is done. □

Finally we can state the main theorem of this chapter.

Theorem 6.2.4. *Let FG be a Lie nilpotent group algebra over a field F of positive characteristic p. Then FG has upper almost maximal Lie nilpotency index if and only if one of the following conditions holds:*

(i) $p = 2$, *G is of class* 2 *and G' is noncyclic of order* 4*;*

(ii) $p = 2$, *G is of class* 4, $G' = C_4 \times C_2$ *and* $\gamma_3(G) = C_2 \times C_2$*;*

(iii) $p = 2$, *G is of class* 4 *and G' is elementary abelian of order* 8*;*

(iv) $p = 3$, *G is of class* 3 *and G' is elementary abelian of order* 9*.*

Proof. Assuming that FG has upper almost maximal Lie nilpotency index, the statement is a consequence of the previous lemmas. The converse is obvious by (6.3). □

We mention that during the preparation of this thesis V. Bódi [10] proved the following statement: FG has upper almost maximal Lie nilpotency index if and only if FG has lower almost maximal Lie nilpotency index.

Summary

Introduction

The investigation of the Lie properties of group algebras as a special polinomial identity was started after the description of group algebras satisfying a polinomial identity, but the invention of the relation between the property of unit group and the associated Lie algebra of group algebras led to an extended intensity of the observation in the '80s. Under general conditions it is not easy even to decide whether an element is a unit or not, so to determine of its inverse would be extremely difficult, such as the computation of the group commutators. However, the so-called Lie commutators can be calculated without the knowledge of the elements' inverses. Considering the results connected to the series which are constructed with the help of Lie commutators we can have conclusions for the corresponding series of the group of units, for example, derived series, upper and lower central series, etc. This method was first applied by A.A. Bódi and I.I. Khripta [6]. Furthermore, Lie methods were used by C. Bagiński [1] and J. Kurdics [17, 18] for the investigation of the derived length, the nilpotency class and the Engel length of the group of units. For additional results on the Lie structure of group algebras we refer the reader to the articles [2, 7, 8, 10, 13, 19, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

In this thesis we investigate the Lie derived length and the upper Lie nilpotency index of group algebras. Before we present the new results we give a short survey of the basic concepts and notations.

Basic facts and notations

Let G be a group and F a field. By the *group algebra* of G over F, which we write as FG, we mean the set of all the formal sums $\sum_{g\in G}\alpha_g g$, where only finitely many coefficients $\alpha_g \in F$ are nonzero, and the group elements are considered to be linearly independent over F, addition is in the natural way, and multiplication is by use of the distributive laws and the calculation $g_i g_j$ $(g_i, g_j \in G)$ according to the product in G. In the special case when F is a field of characteristic $\mathrm{char}(F) = p$ and G contains an element of order p, FG is called *modular group algebra*.

Evidently, the set

$$\omega(FG) = \Big\{ \sum_{g\in G} \alpha_g g \mid \sum_{g\in G} \alpha_g = 0 \Big\}$$

is a two-sided ideal of FG, which is said to be the *augmentation ideal*. It is well-known that $\omega(FG)$ is nilpotent if and only if G is a finite p-group and $\mathrm{char}(F) = p$. Denote by $t_N(G)$ the nilpotency index of $\omega(FG)$. For example, if $G = \langle a_1\rangle \times \cdots \times \langle a_n\rangle$ and the order of a_i is p^{m_i}, then $t_N(G) = 1 + \sum_{i=1}^{n}(p^{m_i} - 1)$.

For any normal subgroup H of G the set

$$\mathfrak{I}(H) = \{(h-1)x \mid h \in H,\ x \in FG\}$$

is a two-sided ideal of FG. Evidently, $\mathfrak{I}(H) = \omega(FH)FG$.

Our group theoretical notation is mostly standard: by $\gamma_n(G)$ we mean the n-th term of the lower central series of G; by G' the commutator subgroup of G (which coincides with $\gamma_2(G)$); by $C_G(H)$ the centralizer of the subset H in G; by C_n the cyclic group of order n.

The upper integral part of a real number r is denoted by $\lceil r\rceil$.

For $x, y \in FG$ the element $[x, y] = xy - yx$ will be called the *Lie commutator* of x and y. Let us introduce in FG the new operation $[x, y] = xy - yx$. Then FG is a Lie algebra with respect to the operations $+$ and $[\ ,\ ]$, which is said to be the *associated Lie algebra* of FG. For the sequence (x_i) of elements of FG we define the *left n-normed*

Lie commutator by induction as

$$[x_1, x_2, \dots, x_n] = \big[[x_1, x_2, \dots, x_{n-1}], x_n\big].$$

Lie derived lengths of group algebras

Define the *Lie derived series* and the *strong Lie derived series* of the group algebra FG respectively, as follows: let $\delta^{[0]}(FG) = \delta^{(0)}(FG) = FG$ and

$$\delta^{[n+1]}(FG) = \big[\delta^{[n]}(FG), \delta^{[n]}(FG)\big],$$
$$\delta^{(n+1)}(FG) = \big[\delta^{(n)}(FG), \delta^{(n)}(FG)\big]FG.$$

We say that FG is *Lie solvable* if there exists $m \in \mathbb{N}$ such that $\delta^{[m]}(FG) = 0$ and the number $\mathrm{dl}_L(FG) = \min\{m \in \mathbb{N} \mid \delta^{[m]}(FG) = 0\}$ is called the *Lie derived length* of FG. Similarly, the group algebra FG is said to be *strongly Lie solvable* of derived length $\mathrm{dl}^L(FG) = m$ if $\delta^{(m)}(FG) = 0$ and $\delta^{(m-1)}(FG) \neq 0$.

According to the inclusion $\delta^{[n]}(FG) \subseteq \delta^{(n)}(FG)$, a strongly Lie solvable group algebra FG is Lie solvable too and $\mathrm{dl}_L(FG) \leq \mathrm{dl}^L(FG)$. It would be also interesting to know when the equality $\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG)$ does hold, but this question is still open.

M. Sahai [24] proved the relation

$$(*) \qquad \mathfrak{I}(G')^{2^n - 1} \subseteq \delta^{(n)}(FG) \subseteq \mathfrak{I}(G')^{2^{n-1}} \quad \text{for all} \quad n > 0,$$

from which it follows that a group algebra FG is strongly Lie solvable if and only if either G is abelian or the ideal $\mathfrak{I}(G')$ is nilpotent, that is G' is a finite p-group and $\mathrm{char}(F) = p$. The description of the Lie solvable group algebras is due to I.B.S. Passi, D.S. Passman and S.K. Sehgal [22]: a group algebra FG is Lie solvable if and only if one of the following conditions holds: (*i*) G is abelian; (*ii*) G' is a finite p-group and $\mathrm{char}(F) = p$; (*iii*) G has a subgroup of index two whose commutator subgroup is a finite 2-group and $\mathrm{char}(F) = 2$.

In general, we have very little information about the Lie derived length of group algebras. The first and, at the same time, the more significant results on this topic can be found in papers [27] and [29] of A. Shalev.

Throughout this part by FG we always mean a strongly Lie solvable group algebra.

From $(*)$ it follows immediately that

$$\lceil \log_2(t_N(G')+1)\rceil \le \mathrm{dl}^L(FG) \le \lceil \log_2(2t_N(G'))\rceil$$

and so

$$\mathrm{dl}_L(FG) \le \lceil \log_2(2t_N(G'))\rceil.$$

Applying the method used A. Shalev in the proof of Lemma 2.2 in [27] we have that if G is nilpotent of class two, then

$$\mathrm{dl}_L(FG) \le \mathrm{dl}^L(FG) = \lceil \log_2(t_N(G')+1)\rceil.$$

In particular, if G is an abelian-by-cyclic p-group with $p>2$ then

$$\mathrm{dl}_L(FG) = \lceil \log_2(t_N(G')+1)\rceil,$$

as it was stated in [29].

In the second chapter of this thesis we extend these results above to a larger class of groups. We obtain the following

Theorem. *Let G be a nilpotent group whose commutator subgroup is a finite p-group,* $\mathrm{char}(F)=p$ *and assume that $\gamma_3(G)\subseteq (G')^p$. Then*

$$\mathrm{dl}_L(FG) \le \mathrm{dl}^L(FG) = \lceil \log_2(t_N(G')+1)\rceil,$$

and if G is an abelian-by-cyclic p-group with $p>2$ then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2 t_N(G')+1\rceil.$$

The investigation of the third chapter was motivated by the following result of A. Shalev [27]: if G is nilpotent of class two with cyclic commutator subgroup of order p^n, then $\mathrm{dl}_L(FG) = \lceil \log_2(p^n+1)\rceil$. We generalize this result and determine both the Lie derived length and the strong Lie derived length of group algebras in the case when the commutator subgroup of the basic group is cyclic of odd order. To present our result we need the series $\left(s_l^{(m)}\right)$ whose l-th member

$$s_l^{(m)} = \begin{cases} 1 & \text{if } l = 0; \\ 2s_{l-1}^{(m)} + 1 & \text{if } s_{l-1}^{(m)} \text{ is divisible by } 2^m; \\ 2s_{l-1}^{(m)} & \text{otherwise.} \end{cases}$$

Theorem. *Let G be a group with cyclic commutator subgroup of order p^n, where p is an odd prime, and let F be a field of characteristic p.*

(i) If $G/C_G(G')$ has order p^r (that is G is nilpotent), then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2(p^n + 1) \rceil.$$

(ii) If the order of $G/C_G(G')$ is divisible by some odd prime $q \neq p$, then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2(2p^n) \rceil.$$

(iii) If $G/C_G(G')$ has order $2^m p^r$ with $m > 0$, then

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = d + 1,$$

where d is the minimal integer for which $s_d^{(m)} \geq p^n$ holds.

In the fourth chapter we study the group algebras of characteristic two. If G is a nilpotent group with (not necessary cyclic) commutator subgroup of order 2^n, we obtain the description of the group algebras FG which have the highest possible value of $\mathrm{dl}_L(FG)$, namely, $n+1$.

Theorem. *Let G be a nilpotent group with commutator subgroup of order 2^n and let F be a field of characteristic two. Then $\mathrm{dl}_L(FG) = n + 1$ if and only if one of the following conditions holds:*

(i) G' is the noncyclic group of order 4 and $\gamma_3(G) \neq 1$;

(ii) G' is cyclic of order less than 8;

(iii) G' is cyclic, $n \geq 3$ and G has nilpotency class at most n.

As a consequence, we get a necessary and sufficient condition for $\mathrm{dl}_L(FG)$ to coincide with $\mathrm{dl}^L(FG)$, provided that G' is cyclic.

Corollary. *Let G be a group with cyclic commutator subgroup of order p^n, and let F be a field of characteristic p. Then $\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG)$ if and only if one of the following conditions is satisfied:*

(i) p is odd;

(ii) $p = 2$ and $n \leq 2$;

(iii) $p = 2$, $n \geq 3$ and the nilpotency class of G is at most n.

According to $(*)$, if G is nonabelian and $\mathrm{char}(F) = p > 0$, then $\lceil \log_2(p+1) \rceil \leq \mathrm{dl}^L(FG)$. The characterization of the group algebras of minimal strong Lie derived length is also a consequence of our result.

Corollary. *Let FG be a strongly Lie solvable group algebra of characteristic $p > 0$. Then $\mathrm{dl}^L(FG) = \lceil \log_2(p+1) \rceil$ if and only if one of the following conditions holds:*

(i) $p = 2$ and G' is central elementary abelian subgroup of order 4;

(ii) G' is of order p and

a) either G' is central;

b) or $G/C_G(G')$ has order $2^m p^r$ with $m > 0, r \geq 0$, and the minimal integer d such that $s_d^{(m)} \geq p$, satisfies the inequality $2^d - 1 < p$.

In [19] F. Levin and G. Rosenberger described the group algebras with derived length two, moreover, it was also proved there that $\mathrm{dl}_L(FG) = 2$ if and only if $\mathrm{dl}^L(FG) = 2$. M. Sahai in [24] gave the complete list of the strongly Lie solvable group algebras of strong Lie derived length three for odd characteristic, and showed that the statements $\delta^{[3]}(FG) = 0$ and $\delta^{(3)}(FG) = 0$ are equivalent, provided that $\mathrm{char}(F) \geq 7$. All the other cases the question is still open. The characterization of the group algebras of Lie derived length three seems to be a difficult problem. A partial solution can be found in the fifth chapter.

Theorem. *Let G be a group with cyclic commutator subgroup of order p^n and let F be field of characteristic p. Then* $\mathrm{dl}_L(FG) = 3$ *if and only if one of the following conditions holds:*

(i) $p = 7$, $n = 1$ *and G is nilpotent;*

(ii) $p = 5$, $n = 1$ *and either* $x^g = x^{-1}$ *for all* $x \in G'$ *and* $g \notin C_G(G')$ *or G is nilpotent;*

(iii) $p = 3$, $n = 1$ *and G is not nilpotent;*

(iv) $p = 2$ *and one of the following conditions is satisfied:*

a) $n = 2$;

b) $n = 3$ *and G is of class* 4;

c) G has an abelian subgroup of index two.

We also proved the following theorems, which can give new information about the derived length in some cases.

Theorem. *Let G be a group and* $\mathrm{char}(F) = 2$. *If H is a subgroup of index two of G whose commutator subgroup H' is a finite* 2*-group, then*

$$\mathrm{dl}_L(FG) \leq \lceil \log_2 t(H') \rceil + 3.$$

Theorem. *Let G be a group with cyclic commutator subgroup of order* 2^n, $G_\beta = \{g \in G \mid x^g = x^{5^i}$ *for some* $i \in \mathbb{Z}\}$ *and let* $\mathrm{char}(F) = 2$. *Then G_β is a subgroup of index not greater than two and if G'_β has order 2^r, then*

$$r + 1 \leq \mathrm{dl}_L(FG) \leq r + 3.$$

Let G_i be a finite nonabelian 2-group of order 2^m and exponent 2^{m-2} from the list in [20]. The group algebras of this class of groups have been examined by several authors. Our results enable us to determine the derived length of FG_i over a field F of characteristic two. With the original notations of [20] we obtain that

$$\mathrm{dl}_L(FG_i) = \begin{cases} 2, & \text{if either } i \in \{2,3\} \text{ and } \ m = 4 \text{ or } i \in \{1,4,5,9,10\}; \\ 4, & \text{if } \ \ i \in \{15,16,18,20,24,25\} \ \ \text{ and } \ \ m > 5; \\ 3, & \text{otherwise.} \end{cases}$$

Lie nilpotency indices of group algebras

In the sixth chapter we study the Lie nilpotency indices of Lie nilpotent group algebras. Let $(FG)^{[1]} = FG$ and for $n > 1$ let $(FG)^{[n]}$ be the ideal of FG generated by all the Lie commutators $[x_1, \dots, x_n]$ with $x_1, \dots, x_n \in FG$. Then the ideal $(FG)^{[n]}$ is the *n-th lower Lie power* and the series

$$FG = (FG)^{[1]} =\supseteq (FG)^{[2]} \supseteq \cdots \supseteq (FG)^{[n]} \supseteq \cdots$$

is called the *lower Lie power series* of the group algebra FG.

By induction, we define the *n-th upper Lie power* $(FG)^{(n)}$ of FG as the ideal generated by all the Lie commutators $[x, y]$, where $x \in (FG)^{(n-1)}$, $y \in FG$ and $(FG)^{(1)} = FG$. The series

$$FG = (FG)^{(1)} \supseteq (FG)^{(2)} \supseteq \cdots \supseteq (FG)^{(n)} \supseteq \cdots$$

is the *upper Lie power series* of FG.

The group algebra FG is called *Lie nilpotent* if there exists n such that $(FG)^{[n]} = 0$ and the least integer of this kind is called the *Lie nilpotency index* of FG and it is denoted by $t_L(FG)$. Similarly, FG is said to be *upper Lie nilpotent* and its *upper Lie nilpotency index* is $t^L(FG) = m$ if $(FG)^{(m)} = 0$ but $(FG)^{(m-1)} \neq 0$. For the noncommutative modular group algebra FG the next theorem from A.A. Bódi and I.I. Khripta [7] is well-known: The following statements are equivalent: (i) FG is Lie nilpotent; (ii) FG is upper Lie nilpotent; (iii) G is a nilpotent group whose commutator subgroup is a finite p-group and $\mathrm{char}(F) = p$.

According to [32], if FG is Lie nilpotent and G' has order p^n, then

$$t^L(FG) \leq p^n + 1.$$

A. Shalev in [25] began to study the question when a Lie nilpotent group algebra has the maximal upper Lie nilpotency index. The complete description of such group algebras was given by V. Bódi and E. Spinelli in [13]. Joining this research we determine the group algebras whose upper Lie nilpotency index is 'almost maximal', that is, it takes the next highest possible value, namely $p^n - p + 2$, where p^n is the order of the commutator subgroup of the basic group.

Theorem. *Let FG be a Lie nilpotent group algebra over a field F of positive characteristic p. Then FG has upper almost maximal Lie nilpotency index if and only if one of the following conditions holds:*

(i) $p = 2$, *G is of class* 2 *and* $G' = C_2 \times C_2$;

(ii) $p = 2$, *G is of class* 4, $G' = C_4 \times C_2$ *and* $\gamma_3(G) = C_2 \times C_2$;

(iii) $p = 2$, *G is of class* 4 *and* $G' = C_2 \times C_2 \times C_2$;

(iv) $p = 3$, *G is of class* 3 *and* $G' = C_3 \times C_3$.

Összegzés

Bevezetés

A múlt század elején G. Frobenius munkáiban megjelent egy csoportból és testből álló érdekes algebrai konstrukció, melyet a véges csoportok reprezentációinak tanulmányozására használt. Ezt a konstrukciót E. Noether csoportalgebrának nevezte el a '30-as években. Ezt követően még évtizedekig továbbra sem önmaga, hanem reprezentációelméleti, algebrai topológiai, stb. alkalmazási lehetőségei miatt vizsgáltak csoportalgebrákat. Például csoportalgebra bevonásával igazolta ekkor W. Magnus a szabadcsoportok alsó centrálláncára vonatkozó híres tételét, melyre tisztán csoportelméleti bizonyítás csak hosszú idő eltelte után született. A 60-as évek elején azonban I. Kaplansky gyűrűelméleti problémáinak hatására a végtelen csoportokkal képzett csoportalgebrák előtérbe kerültek, velük kapcsolatban nagyon sok mély eredmény született. Néhány év alatt kialakult e terület saját problematikája, több monográfia és áttekintő dolgozat jelent meg, melyek következtében a csoport- és gyűrűelmélet határterületén kialakult a csoportalgebrák elmélete. Napjainkban a csoportalgebrák gyűrűelméleti tulajdonságainak vizsgálata a legelőrehaladottabb, de jelentős eredmények ismertek egységcsoportjaiban is. A csoportalgebrák egységcsoportja először topológiai alkalmazásai miatt került az érdeklődés középpontjába. Később, az egyszerű csoportok leírása után, mint véges p-csoportokat vizsgálták őket. A moduláris csoportalgebrák egységcsoportjának tanulmányozását S.A. Jennings kezdte el a '40-es években, de mivel szinte minden probléma megoldása újabb módszerek kidolgozását igényelte, csak lassan születtek eredmények. Az érdekesebb esetekben az egységcsoport már olyan magas rendű, hogy még napjaink számítógépeivel is hosszú időbe telik, vagy éppen lehetetlen bizonyos tulajdonságok ellenőrzése. E terület fontosabb eredményeit és nyitott kérdéseit foglalja össze A.A. Bódi [4] dolgozata.

A csoportalgebra Lie-tulajdonságainak – mint speciális polinom azonosságoknak – a vizsgálata a polinom azonosságnak eleget tevő csoportalgebrák leírása után kezdődött, de annak intenzívvé válását a csoportalgebrák egységcsoportja és asszociált Lie-algebrája közötti összefüggések felfedezése eredményezte a '80-as években. A csoportkommutátorokkal ellentétben az úgynevezett Lie-kommutátorok az elemek inverzeinek ismerete nélkül számolhatók, és a segítségükkel felépített sorozatokra vonatkozó eredmények alapján következtetések vonhatók le az egységcsoport megfelelő sorozataira. Ezt a módszert először A.A. Bódi és I.I. Khripta [6] alkalmazták a feloldható egységcsoporttal rendelkező csoportalgebrák leírására. Megmutatták, hogy a csoportalgebra egységcsoportja akkor és csak akkor feloldható csoport, ha a csoportalgebra Lie-feloldható, feltéve, hogy az alaptest p karakterisztikája nagyobb mint három, továbbá, ha az alapcsoport nem kommutatív és van végtelen rendű eleme, akkor a p-Sylow részcsoportja végtelen. Az egységcsoport feloldható hosszával C. Baginski [1]; nilpotencia osztályával I.I. Khripta, A.A. Bódi és J. Kurdics; az egységcsoport Engel-hosszával J. Kurdics foglalkozott és jutott eredményre Lie-módszereket alkalmazva (l. [6, 8, 17, 18]). A csoportalgebra egységcsoportja és asszociált Lie-algebrája közötti összhangot jól szemléltei A.A. Bódi tétele, miszerint az FG p karakteisztikájú moduláris csoportalgebra egységcsoportja akkor és csak akkor korlátos Engel-csoport, ha FG korlátos Engel-algebra. A csoportalgebra Lie-struktúrájával kapcsolatos további eredmények találhatók a [2, 7, 8, 10, 13, 19, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] dolgozatokban.

Az értekezés tárgya a csoportalgebrák Lie-feloldható hosszának és felső Lie-nilpotencia indexének a vizsgálata. Az eredmények ismertetése előtt tekinsük át a dolgozatban használt jelöléseket, definíciókat.

Alapfogalmak és jelölések

Legyen G egy csoport és F egy test. Jelölje FG az összes $\sum_{g\in G}\alpha_g g$ alakú formális összegek halmazát, ahol csak véges sok $\alpha_g \in F$ együttható nem nulla. A formális összegek skalárszorosát, összegét és szorzatát a következő képletek adják meg:

- $\beta\cdot\left(\sum_{g\in G}\alpha_g g\right) = \left(\sum_{g\in G}\alpha_g g\right)\cdot\beta = \sum_{g\in G}(\beta\alpha_g)g$ ahol $\beta\in F$;
- $\sum_{g\in G}\alpha_g g + \sum_{g\in G}\beta_g g = \sum_{g\in G}(\alpha_g+\beta_g)g$;
- $\left(\sum_{g\in G}\alpha_g g\right)\cdot\left(\sum_{g\in G}\beta_g g\right) = \sum_{g\in G}\left(\sum_{h\in G}\alpha_h\beta_{h^{-1}g}\right)g$.

Ezekkel a műveletekkel FG algebra az F test felett, melyet a G csoport F test feletti *csoportalgebrájának* nevezünk. Abban az esetben, ha az F test karakterisztikája $\mathrm{char}(F) = p$ és G tartalmaz p-rendű elemet, *moduláris csoportalgebráról* beszélünk.

Jelölje $\omega(FG)$ mindazon elemeit FG-nek, melyek $\sum_{g\in G}\alpha_g g$ alakú felírásában az α_g együtthatók összege nulla. Világos, hogy $\omega(FG)$ ideálja FG-nek, melyet a csoportalgebra *fundamentális ideáljának* nevezünk. Mint az jól ismert, $\omega(FG)$ pontosan akkor nilpotens, ha G véges p-csoport és az F test karakterisztikája p; ekkor a nilpotencia indexét $t_N(G)$-vel fogjuk jelölni. Például, ha $G = \langle a_1\rangle\times\cdots\times\langle a_n\rangle$ és a_i rendje p^{m_i}, akkor $t_N(G) = 1+\sum_{i=1}^{n}(p^{m_i}-1)$.

Legyen H normális részcsoportja a G-nek. Ekkor a

$$\{(h-1)x \mid h\in H,\ x\in FG\}$$

halmaz ideálja FG-nek, melyet $\mathfrak{I}(H)$-val fogunk jelölni. Világos, hogy $\mathfrak{I}(H) = \omega(FH)FG$.

A következő csoportelméleti jelöléseket használjuk: $\gamma_n(G)$ a G alsó centrálláncának n-edik tagja; $G' = \gamma_2(G)$ a kommutátor-részcsoportja; $C_G(H)$ pedig a G csoport H részhalmazának centralizátora G-ben; C_n az n-ed rendű ciklikus csoport. Jelölje továbbá $\lceil r\rceil$ az r valós szám felső egészrészét.

Az $[x,y] = xy-yx$ elemet, ahol $x,y\in FG$, az x és y *Lie-kommutátorának* nevezzük. Könnyen belátható, hogy ha FG-ben a szorzás helyett az $[x,y]$ műveletet tekintjük, akkor FG Lie-algebra az F test felett, melyet az FG *asszociált Lie-algebrájának* mondunk. Ha $X,Y \subseteq FG$, akkor $[X,Y]$ az $[x,y]$ Lie-kommutátorok által generált additív részcsoportot jelenti, ahol $x \in X$ és $y \in Y$. Az FG tetszőleges (x_i) sorozatára indukció segítségével értelmezzük az *n-ed rendű Lie-kommutátorokat*, úgymint

$$[x_1, x_2, \dots, x_n] = \big[[x_1, x_2, \dots, x_{n-1}], x_n\big].$$

A csoportalgebra Lie-feloldható hossza

Legyen $\delta^{[0]}(FG) = \delta^{(0)}(FG) = FG$ és ha $n \geq 0$, akkor legyen

$$\delta^{[n+1]}(FG) = \big[\delta^{[n]}(FG), \delta^{[n]}(FG)\big],$$
$$\delta^{(n+1)}(FG) = \big[\delta^{(n)}(FG), \delta^{(n)}(FG)\big]FG.$$

A $\delta^{[n]}(FG)$ sorozatot az FG *Lie-derivált sorozatának*, míg a $\delta^{(n)}(FG)$ sorozatot FG *erős Lie-derivált sorozatának* nevezzük. Azt mondjuk, hogy FG *Lie-feloldható*, ha létezik olyan m természetes szám, hogy $\delta^{[m]}(FG) = 0$, és ekkor a

$$\mathrm{dl}_L(FG) = \min\{m \in \mathbb{N} : \delta^{[m]}(FG) = 0\}$$

számot az FG *Lie-feloldható hosszának* nevezzük. Hasonlóan, ha $\delta^{(m)}(FG) = 0$ de $\delta^{(m-1)}(FG) \neq 0$, akkor FG *erősen Lie-feloldható*, melynek *erős Lie-feloldható hossza* $\mathrm{dl}^L(FG) = m$.

Világos, hogy $\delta^{[n]}(FG) \subseteq \delta^{(n)}(FG)$ bármely n esetén, ezért minden erősen Lie-feloldható csoportlagebra Lie-feloldható is és $\mathrm{dl}_L(FG) \leq \mathrm{dl}^L(FG)$. A kérdés, hogy mikor teljesül az egyenlőség néhány speciális esettől eltekintve nyitott.

M. Sahai [24] megmutatta, hogy minden pozitív n-re

$$(*) \qquad \mathfrak{I}(G')^{2^n-1} \subseteq \delta^{(n)}(FG) \subseteq \mathfrak{I}(G')^{2^{n-1}}$$

teljesül. Ebből következik, hogy az FG csoportalgebra akkor és csak akkor erősen Lie-feloldható, ha vagy G Abel-csoport vagy $\mathfrak{I}(G')$ nilpotens ideál, azaz G' véges p-csoport és $\mathrm{char}(F) = p$. A Lie-feloldható csoportalgebrák leírása I.B.S. Passi, D.S. Passman és S.K. Sehgal [22] nevéhez fűződik: FG pontosan akkor Lie-feloldható, ha a következő állítások egyike teljesül: (*i*) G Abel-csoport; (*ii*) G' véges p-csoport és $\mathrm{char}(F) = p$; (iii) G-nek van olyan kettő indexű részcsoportja, melynek kommutátor-részcsoportja véges 2-csoport és $\mathrm{char}(F) = 2$.

Általában nagyon kevés eredmény ismert a csoportalgebrák Lie-feloldható hosszáról. A bevezető eredmények A. Shalev [27] és [29] dolgozataiban találhatók.

Legyen a továbbiakban FG erősen Lie-feloldható csoportalgebra.

A $(*)$ tartalmazás következménye, hogy

$$\lceil \log_2(t_N(G') + 1)\rceil \leq \mathrm{dl}^L(FG) \leq \lceil \log_2(2t_N(G'))\rceil$$

és így

$$\mathrm{dl}_L(FG) \leq \lceil \log_2(2t_N(G'))\rceil.$$

A. Shalev [27] dolgozatában szereplő 2.2. lemma bizonyításának módszerét követve kapjuk, hogy ha G nilpotens másodosztályú csoport, akkor

$$\mathrm{dl}_L(FG) \leq \mathrm{dl}^L(FG) = \lceil \log_2(t_N(G') + 1)\rceil.$$

Sőt, ha p páratlan prím és a G olyan p-csoport, amely egy Abel-csoport ciklikus csoporttal való bővítése, akkor

$$\mathrm{dl}_L(FG) = \lceil \log_2(t_N(G') + 1)\rceil.$$

A második fejezetben megmutatjuk, hogy a fenti állítások a csoportalgebrák egy bővebb osztályára is érvényesek: G nilpotencia osztálya nem kell, hogy szügségképpen kettő legyen, elég ha a $\gamma_3(G) \subseteq (G')^p$ tartalmazás teljesül. Eredményünk a következő:

Tétel. *Legyen G nilpotens csoport, melynek kommutátor-részcsoportja véges p-csoport, és legyen F egy p karakterisztikájú test. Tegyük fel, hogy $\gamma_3(G) \subseteq (G')^p$. Ekkor*

$$\mathrm{dl}_L(FG) \leq \mathrm{dl}^L(FG) = \lceil \log_2(t_N(G') + 1)\rceil,$$

és speciálisan, ha p páratlan prím és a G olyan p-csoport, amely egy Abel-csoport ciklikus csoporttal való bővítése, akkor

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2 t_N(G') + 1\rceil.$$

A harmadik fejezetben azon csoportalgebrák Lie-feloldható hosszát tanulmányozzuk, melyek alapcsoportjának kommutátor-részcsoportja ciklikus. Vizsgálatainkat A. Shalev [27] következő állítása motiválta: ha G nilpotens másodosztályú csoport p^n rendű ciklikus kommutátor-részcsoporttal, akkor

$$\mathrm{dl}_L(FG) = \lceil \log_2(p^n + 1) \rceil.$$

Először megmutatjuk, hogy ha p páratlan, akkor a G nilpotencia osztályára vonatkozó feltevés nem szükséges. Ezt követően azzal az esettel foglalkozunk, mikor G nem nilpotens. Igazoljuk, hogy ekkor $\mathrm{dl}_L(FG)$ és $\mathrm{dl}^L(FG)$ értéke $\lceil \log_2(3p^n/2) \rceil$ és $\lceil \log_2(2p^n) \rceil$ között van. Mivel a $\lceil \log_2(3p^n/2) \rceil$ és $\lceil \log_2(2p^n) \rceil$ egészek különbsége legfeljebb egy, az egyenlőtlenség már „majdnem" egyértelműen meghatározza $\mathrm{dl}_L(FG)$ és $\mathrm{dl}^L(FG)$ értékeit. A pontos leírást a $G/C_G(G')$ faktorcsoport rendjének függvényében sikerült megkapni. Az eredmény ismertetéséhez szükségünk van az $(s_l^{(m)})$ sorozatra, melynek l-edik tagja

$$s_l^{(m)} = \begin{cases} 1 & \text{ha } l = 0; \\ 2s_{l-1}^{(m)} + 1 & \text{ha } s_{l-1}^{(m)} \text{ osztható } 2^m \text{ -nel;} \\ 2s_{l-1}^{(m)} & \text{egyébként.} \end{cases}$$

Tétel. *Legyen G olyan csoport, melynek kommutátor-részcsoportja p^n rendű ciklikus csoport, ahol p páratlan prím, és legyen az F egy p karakterisztikájú test.*

(i) Ha a $G/C_G(G')$ csoport rendje p^r (azaz G nilpotens) akkor

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2(p^n + 1) \rceil.$$

(ii) Ha a $G/C_G(G')$ csoport rendjének van p-től különböző páratlan prímosztója, akkor

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = \lceil \log_2(2p^n) \rceil.$$

(iii) Ha a $G/C_G(G')$ csoport rendje $2^m p^r$ és $m > 0$, akkor

$$\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG) = d + 1,$$

ahol d az a legkisebb egész szám, melyre $s_d^{(m)} \geq p^n$ teljesül.

A negyedik fejezetben a maximális Lie-feloldható hosszal rendelkező, kettő karakterisztikájú csoportalgebrákat vizsgáljuk. Igazoljuk a következő tételt:

Tétel. *Legyen a G nilpotens csoport, melynek kommutátor-részcsoportja 2^n rendű, és legyen F egy kettő karakterisztikájú test. Az FG csoportalgebra Lie-feloldható hossza pontosan akkor maximális, vagyis $n+1$, ha az alábbi állítások egyike teljesül.*

(i) *G' negyedrendű elemi Abel-csoport és $\gamma_3(G) \neq 1$;*

(ii) *G' legfeljebb negyedrendű ciklikus csoport;*

(iii) *G' ciklikus, $n \geq 3$ és G nilpotencia osztálya legfeljebb n.*

Az előző két tétel következményeként választ kapunk arra a kérdésre, hogy mikor teljesül a $\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG)$ egyenlőség, abban az esetben, ha G' ciklikus csoport.

Következmény. *Legyen G olyan csoport, melynek kommutátor-részcsoportja p^n rendű ciklikus csoport, és legyen az F egy p karakterisztikájú test. A $\mathrm{dl}_L(FG) = \mathrm{dl}^L(FG)$ egyenlőség pontosan akkor teljesül, ha igaz a következő állítások egyike.*

(i) *p páratlan;*

(ii) *$p = 2$ és $n \leq 2$;*

(iii) *$p = 2$, $n \geq 3$ és G nilpotencia osztálya legfeljebb n.*

A $(*)$ következménye, hogy az FG nemkommuatív, erősen Lie-feloldható csoportalgebrára érvényes a $\lceil \log_2(p+1) \rceil \leq \mathrm{dl}^L(FG)$ egyenlőtlenség, ha $p > 0$ az F test karakterisztikája. Eddigi eredményeinkből megkapható azoknak a csoportalgebráknak a jellemzése, melyek erős Lie-feloldható hossza minimális, azaz éppen $\lceil \log_2(p+1) \rceil$.

Következmény. *Legyen FG egy erősen Lie-feloldható csoportalgebra, melynek karakterisztikája $p > 0$. A $\mathrm{dl}^L(FG) = \lceil \log_2(p+1) \rceil$ egyenlőség pontosan akkor teljesül, ha igaz az alábbi állítások egyike:*

(i) $p = 2$ *és* G' *centrális negyedrendű elemi Abel-csoport;*

(ii) G' *rendje* p *és*

> *a)* *vagy* G' *centrális;*
>
> *b)* *vagy a* $G/C_G(G')$ *fartorcsoport rendje* $2^m p^r$*, ahol* $m > 0$*,* $r \geq 0$*, és a legkisebb* d *egész szám, melyre* $s_d^{(m)} \geq p^n$*, eleget tesz a* $2^d - 1 < p$ *egyenlőtlenségnek.*

Világos, hogy $\mathrm{dl}_L(FG)$ és $\mathrm{dl}^L(FG)$ pontosan akkor egyenlő eggyel, ha G Abel-csoport. A kettő Lie- illetve erős Lie-feloldható hosszal rendelkező csoportlagebrákat F. Levin és G. Rosenberger adták meg [19] dolgozatukban. Páratlan karakterisztika esetén M. Sahai [24] cikkében leírta azokat a csoportalgebrákat, melyek erős Lie-feloldható hossza három. Sőt, azt is megmutatta, hogy a $\delta^{[3]}(FG) = 0$ és a $\delta^{(3)}(FG) = 0$ állítások ekvivalensek, ha $\mathrm{char}(F) \geq 7$. A kérdés, hogy $\mathrm{dl}_L(FG)$ mikor három minden egyéb esetben nyitott. Válasz erre a következő tétel, abban az esetben, ha az alapcsoport kommutátor-részcsoportja ciklikus.

Tétel. *Legyen* G *olyan csoport melynek kommutátor-részcsoportja* p^n *rendű ciklikus csoport és legyen az* F *egy* p *karakterisztikájú test. Az* FG *csoportalgebra Lie-feloldható hossza akkor és csak akkor három ha a következő állítások valamelyike igaz.*

(i) $p = 7$*,* $n = 1$ *és* G *nilpotens;*

(ii) $p = 5$*,* $n = 1$ *és vagy* G *nilpotens vagy minden* $x \in G'$ *és* $g \notin C_G(G')$ *esetén* $x^g = x^{-1}$*;*

(iii) $p = 3$*,* $n = 1$ *és* G *nem nilpotens;*

(iv) $p = 2$ *és teljesül az alábbi állítások egyike:*

> *a)* $n = 2$*;*
>
> *b)* $n = 3$ *és* G *nilpotencia osztálya* 4*;*
>
> *c)* G*-nek van kettő indexű Abel-részcsoportja.*

Bizonyítottuk még a következő tételeket, melyek bizonyos esetben hasznosak lehetnek a Lie-feloldható hossz meghatározására.

Tétel. *Ha G-nek van olyan H kettő indexű részcsoportja, melynek kommutátor-részcsoportja véges 2-csoport, valamint az F test karakterisztikája kettő, akkor*

$$\mathrm{dl}_L(FG) \le \lceil \log_2 t(H') \rceil + 3.$$

Tétel. *Legyen a G olyan csoport, melynek kommutátor-részcsoportja 2^n rendű ciklikus csoport, $G_\beta = \{g \in G \mid x^g = x^{5^i}$ valamely $i \in \mathbb{Z}$-re$\}$ és legyen az F egy kettő karakterisztikájú test. Ekkor G_β legfeljebb kettő indexű részcsoportja G-nek és ha a G'_β rendje 2^r, akkor*

$$r + 1 \le \mathrm{dl}_L(FG) \le r + 3.$$

A fejezet eredményeinek alkalmazásával végül meghatározzuk a 2^m rendű 2^{m-2} exponensű csoportok kettő karaktrisztikájú test feletti csoportalgebráinak Lie-feloldható hosszát. A szóban forgó csoportok leírása megtalálható a [20] dolgozatban, csoportalgebráikat már több szerző is vizsgálta, pl. V. Bódi [9]. A [20] jelölését használva eredményünk a következő:

$$\mathrm{dl}_L(FG_i) = \begin{cases} 2, & \text{ha } i \in \{2,3\} \text{ és } m = 4, \text{ vagy } i \in \{1,4,5,9,10\}; \\ 4, & \text{ha } \quad i \in \{15,16,18,20,24,25\} \quad \text{és} \quad m > 5; \\ 3, & \text{egyébként.} \end{cases}$$

A csoportalgebra Lie-nilpotencia indexe

A hatodik fejezetben egy másik Lie-tulajdonság vizsgálatára térünk át. Legyen $(FG)^{[1]} = FG$, és ha $n > 1$, akkor $(FG)^{[n]}$ az FG n-ed rendű Lie-kommutátorai által generált ideál, melyet a csoportalgebra *n-edik alsó Lie-hatványának* nevezzük. Az FG *n-edik felső Lie-hatványát* indukcióval definiáljuk: legyen $(FG)^{(1)} = FG$ és $FG^{(n)}$ az $[x, y]$ Lie-kommutátorokkal generált ideál, ahol $x \in (FG)^{(n-1)}$ és $y \in FG$.

Azt mondjuk, hogy FG *(alsó) Lie-nilpotens*, ha van olyan n természetes szám, melyre $(FG)^{[n]} = 0$. A legkisebb ilyen számot FG

alsó Lie-nilpotencia indexének nevezzük, és $t_L(FG)$-vel jelöljük. Hasonlóan, FG *felső Lie-nilpotens*, melynek *felső Lie-nilpotencia indexe* $t^L(FG) = m$, ha $(FG)^{(m)} = 0$, de $(FG)^{(m-1)} \neq 0$. A.A. Bódi és I.I. Khripta [7] igazolták, hogy az FG nemkommutatív moduláris csoportalgebrában a következő állítások ekvivalensek: (i) FG Lie-nilpotens; (ii) FG felső Lie-nilpotens; (iii) G nilpotens csoport, melynek kommutátor-részcsoportja véges p-csoport és $\mathrm{char}(F) = p$.

Világos, hogy $(FG)^{[n]} \subseteq (FG)^{(n)}$ minden n esetén, így $t_L(FG) \leq t^L(FG)$. Sőt, A.K. Bhandari és I.B.S. Passi [2] megmutatták, hogy ha $\mathrm{char}(F) > 3$, akkor $t_L(FG) = t^L(FG)$; más esetben a kérdés nyitott.

Ha FG Lie-nilpotens csoportalgebra és G' rendje p^n, akkor [32] szerint

$$t_L(FG) \leq t^L(FG) \leq p^n + 1.$$

Azoknak a Lie-nilpotens csoportalgebráknak a tanulmányozását, melyek felső Lie-nilpotencia indexe maximális (azaz $p^n + 1$), A. Shalev kezdete meg [25] dolgozatában, teljes leírásukat azonban V. Bódi és E. Spinelli [13] adták meg. Ehhez kapcsolódva jellemezzük azokat a csoportalgebrákat, melyek felső Lie-nilpotencia indexe „majdnem" maximális, azaz a következő lehetséges legnagyobb érték, konkrétan $p^n - p + 2$, ahol p^n az alapcsoport kommutátor-részcsoportjának a rendje.

Tétel. *Legyen FG Lie-nilpotens csoportalgebra a pozitív p karakterisztikájú F test felett. Az FG felső Lie-nilpotencia indexe akkor és csak akkor majdnem maximális, azaz $p^n - p + 2$, ha az alábbi állítások egyike teljesül.*

(i) $p = 2$, *G nilpotencia osztálya* 2 *és* $G' = C_2 \times C_2$;

(ii) $p = 2$, *G nilpotencia osztálya* 4, $G' = C_4 \times C_2$ *és* $\gamma_3(G) = C_2 \times C_2$;

(iii) $p = 2$, *G nilpotencia osztálya* 4 *és* $G' = C_2 \times C_2 \times C_2$;

(iv) $p = 3$, *G nilpotencia osztálya* 3 *és* $G' = C_3 \times C_3$.

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2. Bovdi, V.; Juhász, T.; Spinelli, E. Modular group algebras with almost maximal Lie nilpotency indices. Algebr. Represent. Theory 9 (2006), no. 3, 259-266.

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List of conference talks of the author

1. *The derived length of Lie soluble group algebras*, International Conference on Algebras, Modules and Group Rings, July $14-18$, 2003, Lisbon, Portugal.

2. *A csoportalgebra Lie feloldható hossza*, Országos algebra szeminárium, MTA Rényi Alfréd Matematikai Kutatóintézet, 2004. április 26., Budapest.

3. *On the derived length of Lie solvable group algebras*, Groups and Group Rings XI, June $4-11$, 2005, Bedlewo, Poland.

4. *On the derived length of Lie solvable group algebras*, Workshop on Lie algebras, their classification and applications, July $25-27$, 2005, Trento, Italy.

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On the Derived Length of Lie Solvable Group Algebras

Értekezés a doktori (PhD) fokozat megszerzése érdekében
a Matematika tudományágban

Írta: Juhász Tibor okleveles matematikus

Készült a Debreceni Egyetem Matematika- és Számítástudományok
Doktori Iskolája (Csoportalgebrák és alkalmazásaik programja)
keretében

Témavezető: Dr. Bódi Béla

A doktori szigorlati bizottság:

elnök:	Dr. Nagy Péter	....
tagok:	Dr. Pham Ngoc Ánh	....
	Dr. Király Bertalan	....

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Az értekezés bírálói:

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	Dr.	....
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