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# Novel Oscillation Theorems and Symmetric Properties of Nonlinear Delay Differential Equations of Fourth-Order with a Middle Term

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**Abstract:** The goal of this paper was to study the oscillations of a class of fourth-order nonlinear delay differential equations with a middle term. Novel oscillation theorems built on a proper Riccati-type transformation, the comparison approach, and integral-averaging conditions were developed, and several symmetric properties of the solutions are presented. For the validation of these theorems, several examples are given to highlight the core results.

**Keywords:** Riccati transformation; oscillation; fourth-order; damping term; comparison theorem

## 1. Introduction

In this study, we consider the following nonlinear delay differential equation

$$(a_2(\mu)(a_1(\mu)[u''(\mu)]^\alpha)')' + \psi(\mu, u''(\delta(\mu))) + q(\mu)f(\mu, u(g(\mu))) = 0, \quad (1)$$

where this equation is of fourth-order with a middle term such that  $\alpha \geq 1$  is a quotient of non-negative odd integers. We make the following assumptions throughout this paper:

(H<sub>1</sub>)  $a_1, a_2, q \in C([\mu_0, +\infty), [0, \infty)), I = [\mu_0, +\infty), \lim_{\mu \rightarrow +\infty} g(\mu) = \lim_{\mu \rightarrow +\infty} \delta(\mu) = \infty,$

(H<sub>2</sub>)  $p > 0$  such that  $\psi(\mu, u) \geq k_1 p(\mu) u^\alpha$ , and  $\psi(\mu, -u) = -\psi(\mu, u),$

(H<sub>3</sub>)  $f \in C(\mathbb{R}, \mathbb{R}),$  there exists a constant  $k_2 > 0$  such that  $f(\mu, u(\mu)) \geq \max\{k_2 u^\beta(\mu), k_2 u^\beta(g(\mu))\}$  and  $f(\mu, -u(\mu)) = -f(\mu, u(\mu)),$  where  $\beta > 0.$

The operators can be defined as

$$L^{[0]}u = u, \quad L^{[1]}u = u', \quad L^{[2]}u = a_1((L^{[0]}u)'' )^\alpha, \quad L^{[3]}u = a_2(L^{[2]}u)', \quad \text{and} \quad L^{[4]}u = (L^{[3]}u)'$$

The meaning of a solution to Equation (1) can be interpreted as a function  $u(\mu)$  in  $C^2[T_u, \infty)$  for which  $L^{[2]}u, L^{[4]}u$  are in  $C^1[T_u, \infty)$  and (1) holds on the interval  $[T_u, \infty)$ , such that  $T_u \geq \mu_0.$  We take into account the solutions  $u(\mu)$  in a way that  $\sup\{|u(\mu)| : \mu \geq T\} > 0$  for every  $T \geq T_u.$  We say that we have an oscillatory solution for Equation (1) when this

solution is not eventually negative as well as not eventually positive on  $[T_\mu, \infty)$ , and we call it nonoscillatory if it is eventually negative or eventually positive. In addition, if all the solutions are oscillatory, then we say that the equation is oscillatory itself.

We define

$$\begin{aligned} \pi_1(\mu_1, \mu) &= \int_{\mu_1}^{\mu} a_1^{-1/\alpha}(s)ds, & \pi_2(\mu_1, \mu) &= \int_{\mu_1}^{\mu} a_2^{-1}(s)ds, \\ \pi_3(\mu_1, \mu) &= \int_{\mu_1}^{\mu} \left( (a_1(s))^{-1} \pi_2(\mu_1, s) \right)^{1/\alpha} ds, \\ \pi_4(\mu_1, \mu) &= \int_{\mu_1}^{\mu} \int_{\mu_1}^u \left( (a_1(s))^{-1} \pi_2(\mu_1, s) \right)^{1/\alpha} ds du, \end{aligned}$$

for  $\mu_0 \leq \mu_1 \leq \mu < \infty$  and we assume that

$$\pi_1(\mu_1, \mu) = \infty, \quad \pi_2(\mu_1, \mu) = \infty \quad \text{as} \quad \mu \rightarrow \infty. \tag{2}$$

In fact, fourth-order differential equations appear in many fields and problems, such as in engineering, physics, chemical phenomena, and biological modelings, see [1–3]. The importance of these types of equations can also be seen in applications that involve problems of elasticity, structural deformation, and soil settlement, see [4,5]. Moreover, in mechanical engineering problems, the oscillatory and nonoscillatory solutions of fourth-order differential equations play an essential role, see [6–8]. Several researchers have intensively investigated the oscillation of fourth- and higher-order differential equations along with numerous approaches for establishing oscillatory criteria [9–13]. The oscillatory features of solutions have been the subject of several investigations of neutral differential equations [14–16], delay differential equations [17,18], and fractional differential equations [19,20]. Dzurina et al. in [21] studied the oscillation of the linear fourth-order delay differential equation with damping

$$(r_3(\mu)(r_2(\mu)(r_1(\mu)y'(\mu))')')' + p(\mu)y'(\mu) + q(\mu)y(\tau(\mu)) = 0,$$

by assuming the following:  $\int_{\mu_1}^{\mu} r_i^{-1}(s)ds$  for  $i = 1, 2, 3$ . On the other hand, Said R. Grace in [22] studied the oscillation of third-order nonlinear delay differential equations with damping

$$(r_2(\mu)(r_1(\mu)y'(\mu))')' + p(\mu)y'(\mu) + q(\mu)f(y(g(\mu))) = 0,$$

under the assumption  $\int_{\mu_1}^{\mu} r_i^{-1}(s)ds$  for  $i = 1, 2$ . Vetro and Wardowski used a comparison technique with first-order differential equations together with the Kusano–Naito’s and Philos’ approaches to obtain results describing the nonoscillatory behavior of solutions [23]. It is worth to mention that oscillatory criteria play a relevant role in numerical approaches based on discrete counterparts of differential problems in the sense of discrete problems [24]. Moreover, the role of oscillations in physical processes can be seen in [25], where a revised model of the nanoparticle mass flux is used to study the thermal instability of the Rayleigh–Benard problem for a horizontal layer of nanofluid heated from below. The motion of nanoparticles is characterized by the effects of thermophoresis and Brownian diffusion.

By the motivation of the above studies, we propose here several novel oscillation criteria for Equation (1) that can be represented as a form which generalizes many differential equations, and at the same time, it is a complement to some other studies. For instance, in the study of Bartušek and Došlá [26], the authors took the case of  $\alpha = 1$ ,  $a_1 = a_2 = \delta = g = 1$  and  $\psi(\mu, u) = 1$ . In addition, when  $\psi(\mu, u)$  and  $\delta = 1$ , Grace et al. in [27] examined several novel criteria for the oscillation of fourth-order nonlinear delay differential equations.

In this study, we use the Riccati technique, which reduces the main equation into a first-order inequality, in order to obtain more efficient oscillatory properties [28]. Moreover,

by the aid of the integral-averaging condition and the comparison method, we present some sufficient conditions such that all the solutions of Equation (1) are oscillating whenever the following second-order equation

$$(a_2(\mu)z'(\mu))' + \frac{k_1 p(\mu)}{a_1(\delta(\mu))}z(\mu) = 0, \tag{3}$$

is oscillatory or nonoscillatory.

The structure of this paper is organized as follows. In Section 2, we state and prove some lemmas that we need in the rest of the paper. In Section 3, we develop several oscillation criteria for Equation (1). In Section 4, we use Ricatti’s method to deal with the oscillation criteria. Finally, we provide two examples in Section 5 for the validation of this proposed work.

### 2. Basic Lemmas

In this section, we state and prove some lemmas that are frequently used in the next sections of this paper.

**Lemma 1.** Assume that Equation (3) is nonoscillatory. If Equation (1) has a nonoscillatory solution  $u(\mu)$  on  $I$ , for  $\mu_1 \geq \mu_0$ , then there exists a  $\mu_2 \in I$  such that  $u(\mu)L^{[2]}u(\mu) > 0$  or  $u(\mu)L^{[2]}u(\mu) < 0$  for  $\mu \geq \mu_2$ .

The proof of this lemma can be seen in [27], so it is omitted here.

**Lemma 2.** If Equation (1) includes a nonoscillatory solution  $u(\mu)$  which satisfies  $u(\mu)L^{[2]}u(\mu) > 0$  in Lemma 1 for  $\mu \geq \mu_1 \geq \mu_0$ , then

$$L^{[2]}u(\mu) > \pi_2(\mu_1, \mu) L^{[3]}u(\mu), \quad \mu \geq \mu_1, \tag{4}$$

$$L^{[1]}u(\mu) > \pi_3(\mu_1, \mu) (L^{[3]}u(\mu))^{1/\alpha}, \quad \mu \geq \mu_1, \tag{5}$$

and

$$u(\mu) > \pi_4(\mu_1, \mu) (L^{[3]}u(\mu))^{1/\alpha}, \quad \mu \geq \mu_1. \tag{6}$$

**Proof.** If Equation (1) includes a nonoscillatory solution  $u$ , we assume that there will be a  $\mu_1 \geq \mu_0$  such that  $u(\mu) > 0$  and  $u(g(\mu))$  for  $\mu \geq \mu_1$ . From Equation (1), we have

$$L^{[4]}u(\mu) = -\left(\frac{k_1 p(\mu)}{a_1(\delta(\mu))}\right)L^{[2]}u(\delta(\mu)) - k_2 q(\mu)u^\beta(g(\mu)) \leq 0,$$

and  $L^{[3]}u(\mu)$  is nonincreasing on  $I$ , we get

$$L^{[2]}u(\mu) \geq \int_{\mu_1}^{\mu} (L^{[2]}u(s))' ds = \int_{\mu_1}^{\mu} (a_2(s))^{-1}L^{[3]}u(s) ds \geq \pi_2(\mu_1, \mu) L^{[3]}u(\mu).$$

This implies that

$$u''(\mu) \geq \left(L^{[3]}u(\mu)\right)^{1/\alpha} \left((a_1(\mu))^{-1}\pi_2(\mu_1, \mu)\right)^{1/\alpha}.$$

Now, integrating twice the above statement from  $\mu_1$  to  $\mu$  and using  $L^{[3]}u(\mu) \leq 0$ , we find

$$u'(\mu) \geq \left(L^{[3]}u(\mu)\right)^{1/\alpha} \int_{\mu_1}^{\mu} \left((a_1(s))^{-1}\pi_2(\mu_1, s)\right)^{1/\alpha} ds,$$

and

$$u(\mu) \geq \left( L^{[3]}u(\mu) \right)^{1/\alpha} \int_{\mu_1}^{\mu} \int_{\mu_1}^u \left( (a_1(s))^{-1} \pi_2(\mu_1, s) \right)^{1/\alpha} ds du \quad \text{for } \mu \leq \mu_1.$$

□

**Lemma 3 ([22]).** Suppose that  $\xi \in C^1(I, \mathbb{R}^+)$ ,  $\xi(\mu) \leq \mu$ ,  $\xi'(\mu) \geq 0$ , and  $G(\mu) \in C(I, \mathbb{R}^+)$  for  $\mu \geq \mu_0$ . Assume also that  $x(\mu)$  is a bounded solution of the second-order delay differential equation

$$(a_2(\mu) x'(\mu))' - \Theta(\mu) x(\xi(\mu)) = 0. \tag{7}$$

If

$$\limsup_{\mu \rightarrow \infty} \int_{\xi(\mu)}^{\mu} \Theta(s) \pi_2(\xi(\mu), \xi(s)) ds > 1, \tag{8}$$

or

$$\limsup_{\mu \rightarrow \infty} \int_{\xi(\mu)}^{\mu} \left( (a_2(\mu))^{-1} \int_u^{\mu} \Theta(s) ds \right) du > 1, \tag{9}$$

where  $a_2(\mu)$  is as in (1), then the solutions of Equation (7) are oscillatory.

For the reader’s convenience, we define:

$$\begin{aligned} Q(\mu) &= \left( \frac{k_1 p(\mu)}{a_1(\delta(\mu))} \right) \pi_2(\mu_1, \delta(\mu)), \\ \psi(\mu) &= \exp \left( \int_{\mu_1}^{\mu} Q(s) ds \right), \\ \Theta^*(\mu) &= k_2 q(\mu) \left( \pi_4(\mu_1, g(\mu)) \right)^\beta, \\ A(\mu) &= \frac{\eta'(\mu)}{\eta(\mu)} - \frac{k_1 p(\mu)}{a_1(\delta(\mu))} \pi_2(\mu_1, \delta(\mu)), \end{aligned} \tag{10}$$

$$B(\mu) = \frac{\beta \ell_2^{\beta-\alpha} g'(\mu)}{\eta(\mu)} \left( \pi_4(\mu_1, g(\mu)) \right)^{\beta-1} \left( \pi_3(\mu_1, g(\mu)) \right)^{1/\alpha}. \tag{11}$$

### 3. Oscillation—Comparison Principle Method

Several oscillation criteria for Equation (1) are developed in this section. We start first by the following theorem.

**Theorem 1.** Assume that  $\alpha \geq \beta$  and the conditions in (2) hold, then Equation (3) is nonoscillatory. Suppose there exists a  $\xi \in C^1(I, \mathbb{R})$  such that  $g(\mu) \leq \xi(\mu) \leq \delta(\mu) \leq \mu$ ,  $\xi'(\mu) \geq 0$  for  $\mu \geq \mu_1$ , and (8) or (9) holds with

$$\Theta(\mu) = \ell_* k_2 g^\beta(\mu) q(\mu) (\pi_1(\xi(\mu), g(\mu)))^\beta - \frac{k_1 p(\mu)}{a_1(\delta(\mu))} \geq 0, \quad \mu \geq \mu_1,$$

for constant  $\ell_* > 0$ . Moreover, suppose that all the solutions of the first-order delay equation have the following

$$z'(\mu) + \psi^{1-\frac{\beta}{\alpha}}(g(\mu)) \Theta^*(\mu) z^{\frac{\beta}{\alpha}}(g(\mu)) = 0, \tag{12}$$

then all the solutions of Equation (1) are oscillatory.

**Proof.** Suppose that Equation (1) has a nonoscillatory solution  $u(\mu)$ . Assume also there exists a  $\mu \geq \mu_1$  such that  $u(\mu) > 0$  and  $u(g(\mu)) > 0$  for some  $\mu \geq \mu_0$ . From Lemma 1,  $u(\mu)$  has the condition of either  $L^{[2]}u(\mu) > 0$  or  $L^{[2]}u(\mu) < 0$  for  $\mu \geq \mu_1$ .

Assume that  $u(\mu)$  has the condition  $L^{[2]}u(\mu) > 0$ , for  $\mu \geq \mu_1$ , then one can easily see that  $L^{[3]}u(\mu) > 0$  for  $\mu \geq \mu_1$ . We can choose  $\mu_2 \geq \mu_1$  such that  $g(\mu) \geq \mu_1$  for  $\mu \geq \mu_2$ ,  $g(\mu) \rightarrow \infty$  as  $\mu \rightarrow \infty$  and we have (6),

$$u(g(\mu)) > \pi_4(\mu_1, g(\mu)) (L^{[3]}u(g(\mu)))^{1/\alpha}, \quad \mu \geq \mu_2. \tag{13}$$

By substituting (4) and (13) in Equation (1), and since  $L^{[3]}u(\mu)$  is decreasing, then

$$\left(L^{[3]}u(\mu)\right)' + \left(\frac{k_1 p(\mu)}{a_1(\delta(\mu))}\right) L^{[3]}u(\mu) \pi_2(\mu_1, \delta(\mu)) + k_2 q(\mu) \left(\pi_4(\mu_1, g(\mu))\right)^\beta \left(L^{[3]}u(g(\mu))\right)^{\beta/\alpha} \leq 0. \tag{14}$$

Taking  $\phi = L^{[3]}u$ , we have

$$\phi'(\mu) + Q(\mu)\phi(\mu) + \Theta^*(\mu)\phi^{\frac{\beta}{\alpha}}(g(\mu)) \leq 0, \tag{15}$$

or

$$\left(\psi(\mu)\phi(\mu)\right)' + \psi(\mu)\Theta^*(\mu)\phi^{\frac{\beta}{\alpha}}(g(\mu)) \leq 0, \quad \text{for } \mu \geq \mu_2. \tag{16}$$

Next, setting  $z = \psi\phi > 0$  and  $\psi(g(\mu)) \leq \phi(\mu)$ , then we have

$$z'(\mu) + \psi^{1-\frac{\beta}{\alpha}}(g(\mu))\Theta^*(\mu)z^{\frac{\beta}{\alpha}}(g(\mu)) \leq 0. \tag{17}$$

This means that (17) is positive for the above inequality. Furthermore, by [2], Corollary 2.3.5, it can be seen that (1) has a positive solution, which is clearly a contradiction.

Next, assume that  $u(\mu)$  has the condition  $L^{[2]}u(\mu) < 0$ , for  $\mu \geq \mu_1$ , then one can easily see that  $L^{[1]}u(\mu) \geq 0$ ,  $L^{[3]}u(\mu) > 0$  for  $\mu \geq \mu_3 (\geq \mu_2)$ . Using the monotonicity of  $u'(\mu)$  and the mean value property of differentiation, then there exists a  $\theta \in (0, 1)$  such that

$$u(\mu) \geq \theta \mu u'(\mu), \quad \text{for } \mu \geq \mu_3. \tag{18}$$

Set  $w(\mu) = L^{[1]}u(\mu)$ , then  $w'(\mu) = u''(\mu) < 0$ . Using (18) in Equation (1), we get

$$(a_2(\mu)(a_1(\mu)[w'(\mu)]^\alpha)')' + k_1 p(\mu)(w'(\delta(\mu)))^\alpha + k_2 (\mu\theta)^\beta q(\mu)u^\beta(g(\mu)) \leq 0,$$

and since  $(a_1(\mu)[w'(\mu)]^\alpha) < 0$ , then we have  $(a_1(\mu)[w'(\mu)]^\alpha)' > 0$  for  $\mu \geq \mu_3$ . Now, for  $v \geq u \geq \mu_3$ , we get

$$\begin{aligned} w(u) > w(u) - w(v) &= - \int_u^v a_1^{-1/\alpha}(\tau)(a_1(\tau)(w'(\tau))^\alpha)^{1/\alpha} d\tau \\ &\geq (a_1(v)(w'(v))^\alpha)^{1/\alpha} \left( \int_u^v -a_1^{-1/\alpha}(\tau) d\tau \right) \\ &= a_1^{1/\alpha}(v)(-w'(v))\pi_1(u, v). \end{aligned}$$

Taking  $u = \xi(\mu)$  and  $v = g(\mu)$ , we obtain

$$w(g(\mu)) > \pi_1(g(\mu), \xi(\mu)) (a_1^{1/\alpha}(\xi(\mu))(-w'(\xi(\mu)))) = \pi_1(g(\mu), \xi(\mu)) x(\xi(\mu)),$$

where  $x(\mu) = a_1^{1/\alpha}(\xi(\mu))(-w'(\xi(\mu))) > 0$  for  $\mu \geq \mu_3$ . From Equation (1), we have that  $x(\mu)$  is decreasing and  $g(\mu) \leq \xi(\mu) \leq \delta(\mu) \leq \mu$ , we get

$$(a_2(\mu)z'(\mu))' + \frac{k_1 p(\mu)}{a_1(\delta(\mu))}z(\delta(\mu)) \geq k_2 (\theta g(\mu))^\beta q(\mu)\pi_1(g(\mu), \xi(\mu))z^{\frac{\beta}{\alpha}-1}(\xi(\mu))z(\xi(\mu)).$$

Since we have  $\alpha \geq \beta$  and  $z$  decreases, then there should be a constant  $\ell$  in such a way that  $z^{\frac{\beta}{\alpha}-1}(\mu) \geq \ell$  for  $\mu \geq \mu_3$ . Therefore, we obtain

$$(a_2(\mu)z'(\mu))' \geq \left( \ell k_2 (\theta g(\mu))^\beta q(\mu)\pi_1(g(\mu), \xi(\mu)) - \frac{k_1 p(\mu)}{a_1(\delta(\mu))} \right) z(\xi(\mu)).$$

Proceeding with the rest as in the proof in Lemma 3, we arrive at the required conclusion, and so the rest is omitted.  $\square$

#### 4. Oscillation—Riccati Method

This section deals with some oscillation criteria for Equation (1) by using Riccati’s method.

**Theorem 2.** Assume that  $\alpha \geq \beta$  and the conditions in (2) hold, then Equation (3) is nonoscillatory. Suppose there exist  $\eta, \xi \in C^1(I, \mathbb{R})$  such that  $g(\mu) \leq \xi(\mu) \leq \delta(\mu) \leq \mu$ ,  $\xi'(\mu) \geq 0$ , and  $\eta > 0$  for  $\mu \geq \mu_1$  with

$$\limsup_{\mu \rightarrow \infty} \int_{\mu_5}^{\mu} \left( k_2 \eta(s) q(s) - \frac{A^2(s)}{4 B(s)} \right) ds = \infty, \quad \text{for all } \mu_1 \in I, \tag{19}$$

and (8) or (9) holds with  $\Theta(\mu)$  as in Theorem 1. Then, every solution of Equation (1) is oscillatory.

**Proof.** Suppose that Equation (1) has a nonoscillatory solution  $u(\mu)$ . Assume also that there exists a  $\mu \geq \mu_1$  such that  $u(\mu) > 0$  and  $u(g(\mu)) > 0$  for some  $\mu \geq \mu_0$ . From Lemma 1,  $u(\mu)$  has the condition of either  $L^{[2]}u(\mu) > 0$  or  $L^{[2]}u(\mu) < 0$  for  $\mu \geq \mu_1$ . If the condition  $L^{[2]}u(\mu) < 0$  holds, then the proof follows from Theorem 1.

Next, if the condition  $L^{[2]}u(\mu) > 0$  holds, define

$$\omega(\mu) = \eta(\mu) \frac{L^{[3]}u(\mu)}{u^{\beta}(g(\mu))}, \quad \mu \in I, \tag{20}$$

then  $\omega(\mu) > 0$  for  $\mu \geq \mu_1$ . From (6) and  $L^{[4]}u(\mu) < 0$ , we have

$$\omega(\mu) = \eta(\mu) \frac{L^{[3]}u(\mu)}{u^{\beta}(g(\mu))} \leq \eta(\mu) \frac{L^{[3]}u(g(\mu))}{u^{\beta}(g(\mu))} \leq \eta(\mu)(\pi_4(\mu_1, g(\mu)))^{-\alpha} u^{\alpha-\beta}(g(\mu)), \tag{21}$$

for  $\mu \geq \mu_1$ . From (5) and the definition of  $L^{[2]}u(\mu)$ , we find

$$u'(g(\mu)) = L^{[1]}u(g(\mu)) \geq \pi_3(\mu_1, g(\mu))(L^{[3]}u(\delta(\mu)))^{1/\alpha} \geq \pi_3(\mu_1, g(\mu))(L^{[3]}u(g(\mu)))^{1/\alpha},$$

then,

$$\begin{aligned} \frac{u'(g(\mu))}{u(g(\mu))} &\geq \left( \frac{\pi_3(\mu_1, g(\mu))}{\eta(\delta(\mu))} \right)^{1/\alpha} \frac{\eta^{1/\alpha}(\delta(\mu))(L^{[3]}u(\mu))^{1/\alpha}}{u^{\beta/\alpha}(g(\delta(\mu)))} u^{\beta/\alpha-1}(g(\delta(\mu))) \\ &= \left( \frac{\pi_3(\mu_1, g(\mu))}{\eta(\mu)} \right)^{1/\alpha} \omega^{1/\alpha}(\mu) u^{\beta/\alpha-1}(g(\delta(\mu))). \end{aligned} \tag{22}$$

Moreover, since there exist a constant  $\ell_1$  and  $\mu_2 \geq \mu_1$  such that for  $L^{[3]}u(\mu) \leq L^{[3]}u(\mu_2) = \ell_1$ , then

$$\begin{aligned} L^{[2]}u(\mu) &= L^{[2]}u(\mu_2) + \int_{\mu_2}^{\mu} (L^{[2]}u(s))' ds \leq L^{[2]}u(\mu_2) + \ell_1 \int_{\mu_2}^{\mu} \frac{ds}{a_2(s)} \\ &= L^{[2]}u(\mu_2) + \ell_1 \pi_2(\mu_2, \mu) = \left[ \frac{L^{[2]}u(\mu_2)}{\pi_2(\mu_2, \mu)} + \ell_1 \right] \pi_2(\mu_2, \mu) \\ &\leq \left[ \frac{L^{[2]}u(\mu_2)}{\pi_2(\mu_2, \mu_3)} + \ell_1 \right] \pi_2(\mu_2, \mu) = \ell_1^* \pi_2(\mu_2, \mu), \end{aligned} \tag{23}$$

holds for all  $\mu \geq \mu_2$ , where  $\ell_1^* = \ell_1 + \frac{L^{[2]}u(\mu_1)}{\pi_2(\mu_2, \mu_3)}$ , which implies that

$$u'(\mu) = u'(\mu_3) + \int_{\mu_3}^{\mu} u''(s) ds \leq u'(\mu_3) + \int_{\mu_3}^{\mu} \left( \frac{\ell_1^* \pi_2(\mu_2, s)}{a_1(s)} \right)^{1/\alpha} ds = u(\mu_3) + (\ell_1^*)^{1/\alpha} \pi_3(\mu_3, \mu) = \ell_2 \pi_3(\mu_3, \mu),$$

holds for all  $\mu \geq \mu_3 (\geq \mu_2)$ , where  $\ell_2 = \frac{u(\mu_2)}{\pi_3(\mu_3, \mu_4)} + (\ell_1^*)^{1/\alpha}$ . Then,

$$u(\mu) = u(\mu_4) + \int_{\mu_4}^{\mu} u'(s) ds \leq u(\mu_4) + \int_{\mu_4}^{\mu} (\ell_2 \pi_3(\mu_3, s)) ds = u(\mu_4) + \ell_2 \pi_4(\mu_4, \mu) = \ell_2^* \pi_4(\mu_4, \mu), \tag{24}$$

holds for all  $\mu \geq \mu_4 (\geq \mu_3)$ , where  $\ell_2^* = \frac{u(\mu_4)}{\pi_4(\mu_4, \mu_1)} + \ell_2$ . Further,

$$u^{\beta/\alpha-1}(g(\mu)) \geq (\ell_2^*)^{\beta/\alpha-1} (\pi_4(\mu_4, g(\mu)))^{\beta/\alpha-1}, \quad \mu \geq \mu_4. \tag{25}$$

By using (24) and (21), we obtain

$$\omega(\mu) \leq (\ell_2^*)^{\alpha-\beta} \eta(\mu) (\pi_4(\mu_1, g(\mu)))^{-\beta}, \tag{26}$$

and hence

$$\omega^{\frac{1}{\alpha}-1}(\mu) \leq (\ell_2^*)^{(\alpha-\beta)(\frac{1}{\alpha}-1)} \eta^{\frac{1}{\alpha}-1}(\mu) (\pi_4(\mu_1, g(\mu)))^{-\beta(\frac{1}{\alpha}-1)}. \tag{27}$$

Now, differentiating (20), we get

$$\omega'(\mu) = \frac{\eta'(\mu)}{\eta(\mu)} \omega(\mu) + \frac{L^{[4]}u(\mu)}{L^{[3]}u(\mu)} \omega(\mu) - \beta g'(\mu) \frac{u'(g(\mu))}{u(g(\mu))} \omega(\mu). \tag{28}$$

Using Equations (1) and (4) in (28), we have

$$\begin{aligned} \omega'(\mu) &\leq \left[ \frac{\eta'(\mu)}{\eta(\mu)} - \frac{k_1 p(\mu)}{a_1(g(\mu))} \pi_2(\mu_4, g(\mu)) \right] \omega(\mu) - k_2 \eta(\mu) q(\mu) - \beta g'(\mu) \frac{u'(g(\mu))}{u(g(\mu))} \omega(\mu) \\ &\leq A(\mu) \omega(\mu) - k_2 \eta(\mu) q(\mu) - \beta g'(\mu) \frac{u'(g(\mu))}{u(g(\mu))} \omega(\mu). \end{aligned} \tag{29}$$

By using (22), (25), and (28) in (29), we have

$$\begin{aligned} \omega'(\mu) &\leq A(\mu)\omega(\mu) - k_2\eta(\mu)q(\mu) - \frac{\beta \ell_2^{\beta-\alpha} g'(\mu)}{\eta(\mu)} \left(\pi_4(\mu_1, g(\mu))\right)^{\beta-1} \left(\pi_3(\mu_1, g(\mu))\right)^{1/\alpha} \omega^2(\mu) \\ &= A(\mu)\omega(\mu) - k_2\eta(\mu)q(\mu) + B(\mu)\omega^2(\mu) \end{aligned} \tag{30}$$

$$\begin{aligned} &= -k_2\eta(\mu)q(\mu) + \left[ \sqrt{B(\mu)} \omega(\mu) - \frac{1}{2} \frac{A(\mu)}{\sqrt{B(\mu)}} \right]^2 + \frac{1}{4} \frac{A^2(\mu)}{B(\mu)} \\ &\leq -k_2\eta(\mu)q(\mu) + \frac{1}{4} \frac{A^2(\mu)}{B(\mu)}. \end{aligned} \tag{31}$$

Integrating (31) from  $\mu_5 (> \mu_4)$  to  $\mu$  gives

$$\int_{\mu_5}^{\mu} \left( k_2 \eta(s) q(s) - \frac{1}{4} \frac{A^2(s)}{B(s)} \right) ds \leq \omega(\mu_5), \tag{32}$$

which contradicts (19). □

**Corollary 1.** Assume that  $\alpha \geq \beta$  and the conditions in (2) hold, then Equation (3) is nonoscillatory. Suppose there exists  $\eta, \xi \in C^1(I, \mathbb{R})$  such that  $g(\mu) \leq \xi(\mu) \leq \delta(\mu) \leq \mu, \xi'(\mu) \geq 0$ , and  $\eta > 0$  for  $\mu \geq \mu_1$  such that the function  $A(\mu) \leq 0$ ,

$$\limsup_{\mu \rightarrow \infty} \int_{\mu_5}^{\mu} \eta(s) q(s) ds = \infty, \quad \text{for all } \mu_1 \in I, \tag{33}$$

and (8) or (9) holds with  $\Theta(\mu)$  as in Theorem 1. Then, every solution of Equation (1) is oscillatory.

Now, we check the oscillation of the solutions of Equation (1) by using Philos-type criteria.

Suppose that  $\mathbb{D}_0 = \{(\mu, s) : a \leq s < \mu < +\infty\}, \mathbb{D} = \{(\mu, s) : a \leq s \leq \mu < +\infty\}$ , the function  $H(\mu, s), H : \mathbb{D} \rightarrow \mathbb{R}$  is continuous and within the class function  $\mathfrak{R}$ , and

- (i)  $H(\mu, \mu) = 0$  for  $\mu \geq \mu_0$  and  $H(\mu, s) > 0$  for  $(\mu, s) \in \mathbb{D}_0$ ,
- (ii)  $H$  contains  $a$  and a nonpositive partial derivative which is continuous on  $\mathbb{D}_0$  subject to the second variable in a way that

$$-\frac{\partial H(\mu, s)}{\partial s} = h(\mu, s)[H(\mu, s)]^{1/2},$$

for all  $(\mu, s) \in \mathbb{D}_0$ .

**Theorem 3.** Assume that  $\alpha \geq 1$  and the conditions in (2) hold, then Equation (3) is nonoscillatory. Suppose that there exist  $\eta, \xi \in C^1(I, \mathbb{R})$  such that  $g(\mu) \leq \xi(\mu) \leq \delta(\mu) \leq \mu, \xi'(\mu) \geq 0, \eta > 0$ , and  $H(\mu, s) \in \mathfrak{R}$  for  $\mu \geq \mu_1$  with

$$\limsup_{\mu \rightarrow \infty} \frac{1}{H(\mu, \mu_5)} \int_{\mu_5}^{\mu} \left( k_2 \eta(s) q(s) H(\mu, s) - \frac{P^2(\mu, s)}{4B(s)} \right) ds = \infty \quad \text{for all } \mu_1 \in I, \tag{34}$$

where  $P(\mu, s) = h(\mu, s) - [H(\mu, s)]^{1/2}A(s)$ , and (8) or (9) holds with  $\Theta(\mu)$  as in Theorem 1. Then, every solution of Equation (1) is oscillatory.

**Proof.** Suppose that Equation (1) has a nonoscillatory solution  $u(\mu)$ . Assume also that there exists a  $\mu \geq \mu_1$  such that  $u(\mu) > 0$  and  $u(g(\mu)) > 0$  for some  $\mu \geq \mu_0$ . Using the same technique as in the proof of Theorem 2, one could get the inequality (30), i.e.,

$$\omega'(\mu) \leq A(\mu)\omega(\mu) - k_2\eta(\mu)q(\mu) + B(\mu)\omega^2(\mu),$$

and hence,

$$\begin{aligned} \int_{\mu_5}^{\mu} H(\mu, s)\eta(s)q(s)ds &\leq \int_{\mu_5}^{\mu} H(\mu, s)[- \omega'(s) + A(s)\omega(s) - B(s)\omega^2(s)]ds \\ &= -H(\mu, s) [\omega(s)]_{\mu_5}^{\mu} + \int_{\mu_5}^{\mu} \left[ \frac{\partial H(\mu, s)}{\partial s} \omega(s) \right. \\ &\quad \left. + H(\mu, s) [A(s)\omega(s) - B(s)\omega^2(s)] \right] ds \\ &= H(\mu, \mu_5)\omega(\mu_5) - \int_{\mu_5}^{\mu} \left[ \omega^2(s)B(s)H(\mu, s) \right. \\ &\quad \left. + \omega(s) \left( h(\mu, s) \sqrt{H(\mu, s)} - H(\mu, s)A(s) \right) \right] ds \\ &\leq H(\mu, \mu_5)\omega(\mu_5) + \int_{\mu_5}^{\mu} \frac{P^2(\mu, s)}{4B(s)} ds, \end{aligned}$$

which shows a contradiction with (34). One can complete the proof by doing the same procedure we used in Theorem 2 and therefore, we omit it. □

### 5. Examples

Below, we provide two examples in order to show some applications of the major findings.

**Example 1.** For  $\mu \geq 1$ , consider the following fourth-order differential equation

$$u^{(iv)}(\mu) + u^{(ii)}(\mu - \pi) - 2u(\mu - 2\pi) = 0. \tag{35}$$

Here,  $a_1 = a_2 = 1, \alpha = 1, p(\mu) = 1, q(\mu) = -2, \delta(\mu) = \mu - \pi$ , and  $g(\mu) = \mu - 2\pi$ . It is easy to verify that all mentioned hypotheses of Theorem 1 hold when  $k_1 = k_2 = 1$ . So all the solutions of Equation (35) are oscillatory. Fore instance,  $u(\mu) = \sin \mu$  is one of the solutions.

**Example 2.** For  $\mu \geq 1$ , consider the following fourth-order differential equation

$$u^{(iv)}(\mu) + \frac{1}{\mu^2(\mu + 1)}u^{(ii)}(\mu) + 2u(\mu) = 0. \tag{36}$$

Here,  $a_1 = a_2 = 1, \alpha = \beta = 1, p(\mu) = 1/\mu^2(\mu + 1), q(\mu) = 2$ , and  $\delta(\mu) = g(\mu) = \mu$ . We obtain  $\pi_1(\mu_1, \mu) = \mu - 1, \pi_2(\mu_1, \mu) = \mu^2 - 1 = (\mu + 1)(\mu - 1), \pi_3(\mu_1, \mu) = \int_1^{\mu} (s^2 - 1)ds = \frac{\mu^3 - 3\mu + 2}{3}, A(s) = \frac{3}{s^2}$ , and  $B(s) = 1/4$ . Now, pick  $\eta(\mu) = 1$ , then

$$\limsup_{\mu \rightarrow \infty} \int_2^{\mu} \left( k_2 \eta(s) q(s) - \frac{A^2(s)}{4B(s)} \right) ds = \limsup_{\mu \rightarrow \infty} \int_2^{\mu} \left( k_2 \frac{4(s^3 - 3s + 2)}{3} - \frac{9}{s^4} \right) ds \rightarrow \infty \quad \text{as } \mu \rightarrow \infty,$$

and all hypotheses of Theorem 2 are satisfied, so all the solutions of Equation (36) are oscillatory.

### 6. Conclusions

In this study, using a suitable Riccati-type transformation, the integral averaging condition, and the comparison method, we offered some oscillatory properties which ensured that all the solutions of Equation (1) are oscillating under the assumption of

$$\pi_1(\mu_1, \mu) = \infty, \quad \pi_2(\mu_1, \mu) = \infty \quad \text{as } \mu \rightarrow \infty.$$

Further, we can consider the case of

$$\pi_1(\mu_1, \mu) < \infty, \quad \pi_2(\mu_1, \mu) = \infty \quad \text{as } \mu \rightarrow \infty.$$

In addition, an extension of the proposed theorems and the results we obtained might be applied on a fourth-order dynamic equation of the form

$$(a_2(\mu)(a_1(\mu)[u^{\Delta\Delta}(\mu)]^\alpha)^\Delta)^\Delta + \psi(\mu, u^{\Delta\Delta}(\delta(\mu))) + \int_c^d q(\mu, \zeta)f(\mu, u(g(\mu, \zeta)))d\zeta = 0,$$

where  $a_1, a_2, q$  are rd-continuous functions on  $T \in \mathbb{R}$  and  $g, \delta : T \rightarrow T$  are also rd-continuous functions that satisfy  $\lim_{\mu \rightarrow \infty} g(\mu, \zeta) = \lim_{\mu \rightarrow \infty} \delta(\mu) = \infty$ .

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