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RESEARCH ARTICLE

Moment functions of higher rank on some types of hypergroups

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Abstract

We consider moment functions of higher order. In our earlier paper, we have already investigated the moment functions of higher order on groups. The main purpose of this work is to prove characterization theorems for moment functions on the multivariate polynomial hypergroups and on the Sturm–Liouville hypergroups. In the first case, the moment generating functions of higher rank are partial derivatives (taken at zero) of the composition of generating polynomials of the hypergroup and functions whose coordinates are given by the formal power series. On Sturm–Liouville hypergroups the moment functions of higher rank are restrictions of even smooth functions that also satisfy certain boundary value problems. The second characterization of moment functions of higher rank on Sturm–Liouville hypergroups is given by means of an exponential family. In this case, the moment functions of higher rank are partial derivatives of an appropriately modified exponential family again taken at zero.

Keywords Moment function · Polynomial hypergroup · Sturm–Liouville hypergroup

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1 Moment function sequences of higher rank on hypergroups

The aim of this paper is to characterize moment function sequences of higher rank on some type of hypergroups. Concerning hypergroups here we will follow the monographs [1, 11]. Some elements of hypergroup theory can be applied not only in mathematics, but also in physics. In [12] one can find the *physical* definition of a finite hypergroup by means of particle collisions. In the same paper the connection of hypergroups with Information Theory and the Second Law of Thermodynamics are presented. The later topics are discussed also in [13, Sect. 6].

Moment function sequences of higher rank on hypergroups were defined in [3] (see also [4, 5]). Here we recall the definition.

Let X be a commutative hypergroup and r a positive integer. For each multi-index α in $\mathbb{N}^r$ let $\varphi_\alpha : X \to \mathbb{C}$ be a continuous function. The family $(\varphi_\alpha)_{\alpha\in\mathbb{N}^r}$ is called a *moment function sequence of rank* r if $\varphi_0 \neq 0$, and

$$\varphi_\alpha(x * y) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \varphi_\beta(x)\varphi_{\alpha-\beta}(y) \tag{1}$$

holds for each multi-index α in $\mathbb{N}^r$ whenever x, y is in X. If the system (1) holds only for $|\alpha| \leq N$ with some given natural number N, then the family $(\varphi_\alpha)_{|\alpha|\leq N}$ is called a *finite moment function sequence of rank* r. If $r = 1$, then we simply call the family a *moment function sequence*, resp. a *finite moment function sequence*.

Clearly, for $\alpha = 0$ we have $\varphi_0(x * y) = \varphi_0(x)\varphi_0(y)$, that is, φ_0 is an exponential on the hypergroup X. We say that the moment function sequence of rank r *corresponds to the exponential* φ_0. The functions φ_α with $|\alpha| = 1$ satisfy

$$\varphi_\alpha(x * y) = \varphi_\alpha(x)\varphi_0(y) + \varphi_\alpha(y)\varphi_0(x),$$

and they are called φ_0-*sine functions*. In the case $\varphi_0 = 1$, φ_0-sine functions are called *additive functions*.

For instance, for $r = 2$ and $N = 2$ the system (1) has the following form:

$$\begin{aligned}
\varphi_{0,0}(x * y) &= \varphi_{0,0}(x)\varphi_{0,0}(y)\\
\varphi_{1,0}(x * y) &= \varphi_{1,0}(x)\varphi_{0,0}(y) + \varphi_{0,0}(x)\varphi_{1,0}(y)\\
\varphi_{0,1}(x * y) &= \varphi_{0,1}(x)\varphi_{0,0}(y) + \varphi_{0,0}(x)\varphi_{0,1}(y)\\
\varphi_{1,1}(x * y) &= \varphi_{1,1}(x)\varphi_{0,0}(y) + \varphi_{1,0}(x)\varphi_{0,1}(y) + \varphi_{0,1}(x)\varphi_{1,0}(y) +\\
&\quad + \varphi_{0,0}(x)\varphi_{1,1}(y).
\end{aligned} \tag{2}$$

Our purpose is to describe the general solution of the system (1). We shall do this in the following paragraphs on some special hypergroups. We note that even the system (2) is not easy to solve in general. But if we assume $\varphi_{0,0} = 1$ then we can describe the general solution of (2). Indeed, by the second and third equation we have that $\varphi_{1,0}$ and $\varphi_{0,1}$ are arbitrary additive functions on X, say $\varphi_{1,0} = a$, $\varphi_{0,1} = b$, where

$$a(x * y) = a(x) + a(y), \;\; b(x * y) = b(x) + b(y)$$

holds for each x, y in X. Let $\Phi = \varphi_{1,1} - a \cdot b$, then we have

$$\begin{aligned}\Phi(x * y) &= \varphi_{1,1}(x * y) - a(x * y)b(x * y)\\ &= \varphi_{1,1}(x) + \varphi_{1,0}(x)\varphi_{0,1}(y) + \varphi_{0,1}(x)\varphi_{1,0}(y) + \varphi_{1,1}(y)\\ &\quad -a(x)b(x) - a(x)b(y) - a(y)b(x) - a(y)b(y) = \Phi(x) + \Phi(y),\end{aligned}$$

that is, Φ is additive, say $\Phi = c$ with $c(x * y) = c(x) + c(y)$. It follows that $\varphi_{1,1} = a \cdot b + c$. Conversely, it is easy to check, that for any additive functions $a, b, c : X \to \mathbb{C}$ the functions $\varphi_{0,0} = 1$, $\varphi_{1,0} = a$, $\varphi_{0,1} = b$, $\varphi_{1,1} = c$ form a solution of the system (2).

In fact, in [4], we described moment function sequences of higher rank on commutative groups. It turns out that, if the generating exponential in a moment function sequence of higher rank on a commutative hypergroup is the identically 1 function, then the methods of the proofs from [4] can be adopted, and one concludes that such moment function sequences can be represented using Bell polynomials B_α as

$$f_\alpha(x) = B_\alpha(a(x)) \qquad (x \in X)$$

with an appropriate sequence $a = (a_\alpha)_{\alpha \in \mathbb{N}^r}$ of additive functions.

In [5] a similar description is given on polynomial hypergroups in a single variable. Further, in [7, 9] and in [8] the authors described moment function sequences of rank one on polynomial hypergroups and on Sturm–Liouville hypergroups, respectively. In this paper we focus on moment function sequences of higher rank, from which the rank one case will follow.

2 Polynomial hypergroups in several variables

The basic concepts of polynomial hypergroups in several variables can be found in [6] and also in [1]. The single variable case of moment functions on polynomial hypergroups has been considered in [8]. Here we summarize the necessary facts.

Let X be a countable set equipped with the discrete topology and let d be a positive integer. We consider a set $(Q_x)_{x \in X}$ of polynomials in d complex variables. If for any nonnegative integer n the symbol X_n denotes the set of all elements x in X for which the degree of Q_x is not greater than n, then we suppose that the polynomials Q_x with x in X_n form a basis for all polynomials of degree not greater than n. In this case for every x, y in X the product $Q_x Q_y$ admits a unique representation

$$Q_x Q_y = \sum_{w \in X} c(x, y, w) Q_w \tag{3}$$

with some complex numbers $c(x, y, w)$. A hypergroup $(X, *)$ is called a *polynomial hypergroup in d variables* or *d-dimensional polynomial hypergroup* if there exists a family of polynomials $(Q_x)_{x \in X}$ in d complex variables satisfying the above condition and such that the convolution in X is defined by

$$\delta_x * \delta_y(\{w\}) = c(x, y, w)$$

for each x, y, w in X. We say that this polynomial hypergroup is *associated with the family of polynomials* $(Q_x)_{x\in X}$. Equation (3) is called the *linearization formula*.

By the conditions on the sequence of polynomials $(Q_x)_{x\in X}$ it follows that there is exactly one element x in X for which Q_x is a nonzero constant. It is easy to see that necessarily $x = e$ is the identity of the hypergroup, and $Q_e = 1$. Clearly, X contains exactly d nonconstant linear polynomials which are linearly independent.

3 Moment functions of higher rank on multivariate polynomial hypergroups

In this section we generalize the results in [5] by characterizing moment function sequences of higher rank on multivariate polynomial hypergroups.

Theorem 1 *Let d, r be positive integers, and X a d dimensional polynomial hypergroup generated by the family of polynomials $(Q_x)_{x\in X}$. The family of functions $\varphi_\alpha : X \to \mathbb{C}$ ($\alpha \in \mathbb{N}^r$) forms a moment function sequence of rank r on X if and only if*

$$\varphi_\alpha(x) = \partial^\alpha(Q_x \circ f)(0) \tag{4}$$

holds for all x in X and for each α in $\mathbb{N}^r$, where $f = (f_1, f_2, \ldots, f_d) : \mathbb{R}^r \to \mathbb{C}^d$ and

$$f_i(t) = \sum_{\alpha\in\mathbb{N}^r} \frac{c_{i,\alpha}}{\alpha!} t^\alpha, \;\; t \in \mathbb{R}^r$$

for $i = 1, \ldots, d$.

We note that, although here f_α is defined by a formal power series, but in formula (4) we need the coefficients only, regardless to convergence.

Proof For each natural number n, let X_n denote the set of polynomials Q_x with $\deg Q_x \le n$. Let N be a natural number, let α be in $\mathbb{N}^r$ with $|\alpha| \le N$, and let φ_α denote the function defined by (4) in terms of some $f = (f_1, f_2, \ldots, f_d) : \mathbb{R} \to \mathbb{C}^d$, where $f_i : \mathbb{R}^r \to \mathbb{C}$ is any N-times differentiable function. By the linearization formula, we have

$$(Q_x \circ f)(t)(Q_y \circ f)(t) = \sum_{w\in X} c(x, y, w)(Q_w \circ f)(t)$$

for each t in $\mathbb{R}^r$ and for all x, y in X. Applying ∂^α on both sides with respect to t and substituting $t = 0$ we have for each x, y in X

$$
\begin{aligned}
&\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\,\varphi_\beta(x)\varphi_{\alpha-\beta}(y)\\
&\quad=\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^\beta(Q_x\circ f)(0)\partial^{\alpha-\beta}(Q_y\circ f)(0)=\sum_{w\in X}c(x,y,w)\partial^\alpha(Q_w\circ f)(0)\\
&\quad=\sum_{w\in X}c(x,y,w)\varphi_\alpha(w)=\sum_{w\in X}(\delta_x*\delta_y)(w)\varphi_\alpha(w)=\varphi_\alpha(x*y),
\end{aligned}
$$

which means that the functions $\varphi_\alpha : X \to \mathbb{C}$ for $|\alpha| \le N$ form a finite moment sequence of rank r on X. As N is arbitrary, we have proved that the family φ_α with α in $\mathbb{N}^r$ given in (4) forms a moment function sequence of rank r with any complex numbers $c_{i,\alpha}$, $i = 1, 2, \ldots, d$.

To prove the converse statement we assume that the family of functions $\varphi_\alpha : X \to \mathbb{C}$ ($|\alpha| \le N$) forms a finite moment function sequence of rank r on the hypergroup X. As φ_0 is an exponential (see [11] for the form of exponentials on multivariate polynomial hypergroup), we have that $\varphi_0(x) = Q_x(\lambda)$ holds for each x in X with some λ in $\mathbb{C}^d$, where $\lambda = (c_{1,0}, c_{2,0}, \ldots, c_{d,0})$. The vectors

$$
\big(\partial_1 Q_x(\lambda), \partial_2 Q_x(\lambda), \ldots, \partial_d Q_x(\lambda)\big)
$$

for x in X_1 and $x \ne e$ are linearly independent (see [11]), consequently, for every α multi-index with $|\alpha| \le N$ the system of linear equations

$$
\varphi_\alpha(x) = \sum_{i=1}^{d} c_{i,\alpha}\partial_i Q_x(\lambda)
$$

for x in X_1 with $x \ne e$ has a unique solution $c_{i,\alpha}$ ($i = 1, 2, \ldots, d$). Then we define $f = (f_1, f_2, \ldots, f_d)$ by

$$
f_i(t) = \sum_{|\alpha|\le N}\frac{c_{i,\alpha}}{\alpha!}t^\alpha
$$

for each t in $\mathbb{R}^r$ and for $i = 1, 2, \ldots, d$. Further let

$$
\psi_\alpha(x) = \varphi_\alpha(x) - \partial^\alpha(Q_x\circ f)(0)
$$

for $|\alpha| \le N$, whenever x is in X. We show that the functions ψ_α vanish identically on X. For $\alpha = 0$ we have $\psi_0(x) = \varphi_0(x) - Q_x(f(0))$ for all x in X. However, as $f(0) = \lambda$, it follows immediately from the choice of λ that $\varphi_0(x) = Q_x(f(0))$, hence $\psi_0(x) = 0$ for each x in X.

From the equation of the moment functions it follows by induction on $|\alpha|$ that $\varphi_\alpha(e) = 0$ for $1 \le |\alpha| \le N$, consequently, we have that $\psi_\alpha(e) = 0$ for $|\alpha| \le N$. On the other hand, for every x in X_1, the polynomial Q_x is linear, hence

$$\partial^\alpha(Q_x \circ f)(0) = \sum_{i=1}^{d} \partial_i Q_x(f(0))\partial^\alpha f_i(0) = \sum_{i=1}^{d} \partial_i Q_x(\lambda) c_{i,\alpha} = \varphi_\alpha(x)$$

holds for $1 \leq |\alpha| \leq N$, whenever $x \neq e$. This means that $\psi_\alpha(x) = 0$ for any x in X_1 and for $|\alpha| \leq N$.

Now we proceed by induction on n. Suppose that we have proved $\psi_\alpha(x) = 0$ for $|\alpha| \leq N$ and for each x in X_n, and let x be arbitrary in X_{n+1}. We know (see the proof of 3.1.2 Proposition in [1]) that Q_x has a representation in the form

$$Q_x(\lambda) = \sum_{j=1}^{s} a_j Q_{x_j}(\lambda) Q_{y_j}(\lambda) \tag{5}$$

for any λ in $\mathbb{C}^d$ with some complex numbers a_j and with some x_j in X_1 and y_j in X_n $(j = 1, 2, \ldots, s)$, where s is a positive integer. This means that

$$\delta_x = \sum_{j=1}^{s} a_j \delta_{x_j} * \delta_{y_j}$$

holds. Consequently, we have

$$\varphi_\alpha(x) = \sum_{j=1}^{s} a_j \varphi_\alpha(x_j * y_j)$$

for $|\alpha| \leq N$. On the other hand, applying ∂^α on (5) and substituting $t = 0$ we have

$$\begin{aligned}\partial^\alpha\big(Q_x \circ f\big)(0) &= \sum_{j=1}^{s} a_j \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta\big(Q_{x_j} \circ f\big)(0)\partial^{\alpha-\beta}\big(Q_{y_j} \circ f\big)(0)\\ &= \sum_{j=1}^{s} a_j \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \varphi_\beta(x_j)\varphi_{\alpha-\beta}(y_j) = \sum_{j=1}^{s} a_j \varphi_\alpha(x_j * y_j) = \varphi_\alpha(x),\end{aligned}$$

which means that $\psi_\alpha(x) = 0$ for $|\alpha| \leq N$. This completes the proof. □

4 Sturm–Liouville hypergroups

Sturm–Liouville hypergroups represent another important class of hypergroups, which arise from Sturm–Liouville boundary value problems on the nonnegative reals. In order to build up the Sturm–Liouville operator basic to the construction of hypergroups one introduces the Sturm–Liouville functions. For further details see [1] and [2]. In what follows $\mathbb{R}_0$ denotes the set of nonnegative real numbers.

The continuous function $A:\mathbb{R}_0\to\mathbb{R}$ is called a *Sturm–Liouville function* if it is positive and continuously differentiable on the positive reals. Different assumptions on A can be found in [1] which lead to the desired Sturm–Liouville problem. For a given Sturm–Liouville function A one defines the *Sturm–Liouville operator* L_A by

$$L_A f=-f''-\frac{A'}{A}f',$$

where f is a twice continuously differentiable real function on the positive reals. Using L_A one introduces the differential operator l by

$$\begin{aligned}l[u](x,y)&=(L_A)_x u(x,y)-(L_A)_y u(x,y)\\&=-\partial_1^2 u(x,y)-\frac{A'(x)}{A(x)}\partial_1 u(x,y)+\partial_2^2 u(x,y)+\frac{A'(y)}{A(y)}\partial_2 u(x,y),\end{aligned}$$

where u is twice continuously differentiable for all positive reals x, y. Here $(L_A)_x$ and $(L_A)_y$ indicates that L_A operates on functions depending on x or y, respectively.

A hypergroup on $\mathbb{R}_0$ is called a *Sturm–Liouville hypergroup* if there exists a Sturm–Liouville function A such that, for each nonnegative even C^∞-function f on $\mathbb{R}$, the function u_f defined by

$$u_f(x,y)=\int_{\mathbb{R}_0} f\,d(\delta_x*\delta_y)\qquad (x,y>0)$$

is twice continuously differentiable and satisfies the partial differential equation

$$l[u_f]=0$$

with $\partial_2 u_f(x,0)=0$ for all positive x. Hence u_f is a solution of the Cauchy problem

$$\begin{gathered}\partial_1^2 u(x,y)+\tfrac{A'(x)}{A(x)}\,\partial_1 u(x,y)=\partial_2^2 u(x,y)+\tfrac{A'(y)}{A(y)}\,\partial_2 u(x,y),\\ \partial_2 u(x,0)=0\end{gathered}$$

for all positive x, y. From general properties of one-dimensional hypergroups given in [1], it follows that $u_f(y,0)=u_f(0,y)=f(y)$ and $\partial_1 u_f(0,y)=0$ holds, whenever y is a positive real number. In other words, u_f is the unique solution of the boundary value problem

$$\begin{aligned}&\partial_1^2 u(x,y)+\frac{A'(x)}{A(x)}\,\partial_1 u(x,y)=\partial_2^2 u(x,y)+\frac{A'(y)}{A(y)}\,\partial_2 u(x,y)\\&\partial_1 u(0,y)=0,\\&\partial_2 u(x,0)=0\qquad u(x,0)=f(x),\qquad u(0,y)=f(y)\end{aligned}\tag{6}$$

for all positive x, y. As this boundary value problem uniquely defines u_f for each f, we may consider it the *boundary value problem defining the Sturm–Liouville hypergroup*.

A systematic study of basic functional equations on Sturm–Liouville hypergroups can be found in the monograph [11]. In the sequel we shall use the following result (see Theorem 4.2. in [11], p. 62., and also [9, 10]).

Theorem 2 *Let $K = (\mathbb{R}_0, A)$ be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A. The continuous function $m : \mathbb{R}_0 \to \mathbb{C}$ is an exponential on K if and only if it is the restriction of an even C^∞-function on $\mathbb{R}$ and there exists a complex number λ such that*

$$m''(x) + \frac{A'(x)}{A(x)} m'(x) = \lambda m(x), \;\; m(0) = 1, \;\; m'(0) = 0 \tag{7}$$

holds for each $x > 0$.

5 Moment functions of higher rank on Sturm–Liouville hypergroups

Let $K = (\mathbb{R}_0, A)$ be a Sturm–Liouville hypergroup. In this section we describe all generalized moment functions defined on K.

Theorem 3 *Let $K = (\mathbb{R}_0, A)$ be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A and let r be a positive integer. The family of continuous functions $f_\alpha : \mathbb{R}_0 \to \mathbb{C}$ $(\alpha \in \mathbb{N}^r)$ forms a moment function sequence of rank r on the hypergroup K if and only if these functions are restrictions of even C^∞-functions on $\mathbb{R}$, and there are complex numbers c_α for each α in $\mathbb{N}^r$ such that*

$$f_0''(x) + \frac{A'(x)}{A(x)} f_0'(x) = c_0\, f_0(x), \qquad f_0(0) = 1, \qquad f_0'(0) = 0 \tag{8}$$

and

$$f_\alpha''(x) + \frac{A'(x)}{A(x)} f_\alpha'(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} c_\beta\, f_{\alpha-\beta}(x), \qquad f_\alpha(0) = 0, \qquad f_\alpha'(0) = 0 \tag{9}$$

holds for each positive x and for every α in $\mathbb{N}^r$.

Proof First we prove the sufficiency. If the functions $f_\alpha : \mathbb{R}_0 \to \mathbb{C}$ $(\alpha \in \mathbb{N}^r)$ satisfy the conditions (8) and (9), then f_0 is an exponential function, by Theorem 2, hence $f_0(x * y) = f_0(x) f_0(y)$ holds for all nonnegative numbers x and y. We show that equation (1) holds for all α in $\mathbb{N}^r$, that is, the function

$$h(x, y) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} f_\beta(x) f_{\alpha-\beta}(y)$$

is a solution of the differential equation in (6). The latter assertion is equivalent to the differential equation in (6) is equivalent to

$$\sum_{\beta\le\alpha}\binom{\alpha}{\beta} f''_\beta(x) f_{\alpha-\beta}(y) + \frac{A'(x)}{A(x)}\sum_{\beta\le\alpha}\binom{\alpha}{\beta} f'_\beta(x) f_{\alpha-\beta}(y)$$
$$= \sum_{\beta\le\alpha}\binom{\alpha}{\beta} f_\beta(x) f''_{\alpha-\beta}(y) + \frac{A'(y)}{A(y)}\sum_{\beta\le\alpha}\binom{\alpha}{\beta} f_\beta(x) f'_{\alpha-\beta}(y),$$

which is equivalent to

$$\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\Big(f''_\beta(x) + \frac{A'(x)}{A(x)} f'_\beta(x)\Big) f_{\alpha-\beta}(y) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\Big(f''_{\alpha-\beta}(y) + \frac{A'(y)}{A(y)} f'_{\alpha-\beta}(y)\Big) f_\beta(x)$$

that is, to

$$\sum_{\beta\le\alpha}\binom{\alpha}{\beta}\sum_{\gamma\le\beta}\binom{\beta}{\gamma} c_\gamma f_{\beta-\gamma}(x) f_{\alpha-\beta}(y) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\sum_{\gamma\le\alpha-\beta}\binom{\alpha-\beta}{\gamma} c_\gamma f_{\alpha-\beta-\gamma}(y) f_\beta(x).$$

But this equation holds true, since by choosing $\delta = \beta + \gamma$, the right hand side is equal to

$$\sum_{\delta\le\alpha}\sum_{\gamma\le\delta}^{l}\binom{\alpha}{\delta-\gamma}\binom{\alpha-(\delta-\gamma)}{\gamma} c_\gamma f_{\alpha-\delta}(y) f_{\delta-\gamma}(x),$$

which is obviously equal to the left hand side. Moreover, the boundary value conditions in (6) are also satisfied, as

$$\partial_1 h(0, y) = \sum_{\beta\le\alpha} f'_\beta(0) f_{\alpha-\beta}(y) = 0,$$

and

$$h(0, y) = \sum_{\beta\le\alpha} f_\beta(0) f_{\alpha-\beta}(y) = f_\alpha(y),$$

and similarly $\partial_2 h(x, 0) = 0$, and $h(x, 0) = f_\alpha(x)$, hence h is a solution of the boundary value problem, which implies, by uniqueness, that $h(x, y) = f_\alpha(x * y)$.

Conversely, suppose that the family of continuous functions $f_\alpha : \mathbb{R}_0 \to \mathbb{C}$ $(\alpha \in \mathbb{N}^r)$ forms a moment function sequence of rank r. Then, by definition, f_0 is an exponential and the conditions of (9) are satisfied. Now we proceed by induction, and we assume that (9) holds for the even C^∞-functions f_α, with $|\alpha| \le N$, where N is a natural number. Let $\alpha' = \alpha + (1, 0, \ldots, 0)$. We have

$$f_{\alpha'}(x * y) = \sum_{\beta \le \alpha'} \binom{\alpha'}{\beta} f_\beta(x) f_{\alpha'-\beta}(y), \tag{10}$$

and, by the definition of the hypergroup, this implies that

$$\begin{aligned} &\sum_{\beta \le \alpha'} \binom{\alpha'}{\beta} f''_\beta(x) f_{\alpha'-\beta}(y) + \frac{A'(x)}{A(x)} \sum_{\beta \le \alpha'} \binom{\alpha'}{\beta} f'_\beta(x) f_{\alpha'-\beta}(y) \\ &\quad = \sum_{\beta \le \alpha'} \binom{\alpha'}{\beta} f_\beta(x) f''_{\alpha'-\beta}(y) + \frac{A'(y)}{A(y)} \sum_{\beta \le \alpha'} \binom{\alpha'}{\beta} f_\beta(x) f'_{\alpha'-\beta}(y). \end{aligned}$$

Rearranging the terms we have

$$\begin{aligned} &\Big(f''_{\alpha'}(x) + \frac{A'(x)}{A(x)} f'_{\alpha'}(x)\Big) f_0(y) + \sum_{\beta \le \alpha} \binom{\alpha'}{\beta} \Big(f''_\beta(x) + \frac{A'(x)}{A(x)} f'_\beta(x)\Big) f_{\alpha'-\beta}(y) \\ &= \Big(f''_{\alpha'}(y) + \frac{A'(y)}{A(y)} f'_{\alpha'}(y)\Big) f_0(x) + \sum_{0<\beta \le \alpha'} \binom{\alpha'}{\beta} \Big(f''_{\alpha'-\beta}(y) + \frac{A'(y)}{A(y)} f'_{\alpha'-\beta}(y)\Big) f_\beta(x). \end{aligned}$$

Therefore, by the induction hypothesis

$$\begin{aligned} &\Big(f''_{\alpha'}(x) + \frac{A'(x)}{A(x)} f'_{\alpha'}(x)\Big) f_0(y) + \sum_{\beta \le \alpha} \sum_{\gamma \le \beta} \binom{\alpha'}{\beta} \binom{\beta}{\gamma} c_\gamma \, f_{\beta-\gamma}(x) f_{\alpha'-\beta}(y) \\ &= \Big(f''_{\alpha'}(y) + \frac{A'(y)}{A(y)} f'_{\alpha'}(y)\Big) f_0(x) + \sum_{0<\beta \le \alpha'} \sum_{\gamma \le \alpha'-\beta} \binom{\alpha'}{\beta} \binom{\alpha'-\beta}{\gamma} c_\gamma \, f_\beta(x) f_{\alpha'-\beta-\gamma}(y). \end{aligned}$$

In other words

$$\begin{aligned} &\Big(f''_{\alpha'}(x) + \frac{A'(x)}{A(x)} f'_{\alpha'}(x)\Big) f_0(y) \\ &\quad + \sum_{\beta \le \alpha} \binom{\alpha'}{\beta} c_\beta \, f_0(x) f_{\alpha'-\beta}(y) + \sum_{\beta \le \alpha} \sum_{\gamma < \beta} \binom{\alpha'}{\beta} \binom{\beta}{\gamma} c_\gamma \, f_{\beta-\gamma}(x) f_{\alpha'-\beta}(y) \\ &= \Big(f''_{\alpha'}(y) + \frac{A'(y)}{A(y)} f'_{\alpha'}(y)\Big) f_0(x) \\ &\quad + \sum_{0<\beta \le \alpha'} \binom{\alpha'}{\beta} c_{\alpha'-\beta} \, f_\beta(x) f_0(y) + \sum_{0<\beta \le \alpha'} \sum_{\gamma < \alpha'-\beta} \binom{\alpha'}{\beta} \binom{\alpha'-\beta}{\gamma} c_\gamma \, f_\beta(x) f_{\alpha'-\beta-\gamma}(y). \end{aligned}$$

It is easy to see that the last terms on the two sides are equal. This means that

$$\Big(f''_{\alpha'}(x) + \frac{A'(x)}{A(x)} f'_{\alpha'}(x) - \sum_{0<\beta \le \alpha'} \binom{\alpha'}{\beta} c_{\alpha'-\beta} \, f_\beta(x)\Big) f_0(y)$$

$$= \Big(f''_{\alpha'}(y) + \frac{A'(y)}{A(y)} f'_{\alpha'}(y) - \sum_{\beta\le\alpha} \binom{\alpha'}{\beta} c_\beta f_{\alpha'-\beta}(y)\Big) f_0(x)$$

holds for each positive x and y, hence there exists a complex number $c_{\alpha'}$ such that

$$f''_{\alpha'}(x) + \frac{A'(x)}{A(x)} f'_{\alpha'}(x) - \sum_{0<\beta\le\alpha'} \binom{\alpha'}{\beta} c_{\alpha'-\beta}\, f_\beta(x) = c_{\alpha'} f_0(x).$$

This proves (8) for $\alpha' = \alpha + (1, 0, \ldots, 0)$. Similarly one can show it holds also for $\alpha' = \alpha + (0, 1, 0, \ldots, 0)$, ..., $\alpha' = \alpha + (0, 0, \ldots, 0, 1)$. As a consequence of (10) we also have $f_{\alpha'}(0) = 0$, and due to

$$0 = \sum_{\beta\le\alpha'} \binom{\alpha'}{\beta} f_\beta(x) f'_{\alpha'-\beta}(0) = f_0(x) f'_{\alpha'}(0)$$

we get that $f'_{\alpha'}(0) = 0$. Hence (9) holds for α' and the theorem is proved by induction. □

The next theorem uses the notion of an exponential family on a hypergroup. For the formal definition and properties see [11].

Theorem 4 *Let $K = (\mathbb{R}_0, A)$ be the Sturm–Liouville hypergroup corresponding to the Sturm–Liouville function A with the exponential family φ and let r be a a positive integer. The family of continuous functions $f_\alpha : \mathbb{R}_0 \to \mathbb{C}$ ($\alpha \in \mathbb{N}^r$) forms a moment function sequence of rank r on the hypergroup K if and only if there are complex numbers c_α for $\alpha \in \mathbb{N}^r$ such that*

$$f_\alpha(x) = \partial^\alpha \varphi\big(x, f(t)\big)\big|_{t=0} \tag{11}$$

holds for each x in $\mathbb{R}_0$, where φ is the exponential family of the Sturm–Liouville hypergroup K and

$$f(t) = \sum_{\alpha\in\mathbb{N}^r} c_\alpha \frac{t^\alpha}{\alpha!}$$

for each t in $\mathbb{R}^r$.

As we noted in Theorem 1, although here f_α is defined by a formal power series, but in formula (11) we need the coefficients only, regardless to convergence.

Proof Let N be a natural number and let α be in $\mathbb{N}^r$ with $|\alpha| \le N$, further let φ be the exponential family of the hypergroup K, and let f be the function defined above. We shall use ∂_x, resp. ∂_t for differentiation with respect to the variable x, resp. t. If we take $\lambda = f(t)$ in (7) with $m = f_0$, and apply ∂_t^α on both sides, we obtain

$$\partial_x^2 \partial_t^\alpha \varphi\big(x, f(t)\big) + \frac{A'(x)}{A(x)} \partial_x \partial_t^\alpha \varphi\big(x, f(t)\big) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial_t^\beta f(t) \partial_t^{\alpha-\beta} \varphi\big(x, f(t)\big). \tag{12}$$

Taking $t = 0$ and we denote $f_\alpha(x) = \partial_t^\alpha \varphi\big(x, f(t)\big)\big|_{t=0}$ the following equation follows:

$$f_\alpha''(x) + \frac{A'(x)}{A(x)} f_\alpha'(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} c_\beta\, f_{\alpha-\beta}(x),$$

furthermore $f_0(0) = 1$, $f_0'(0) = 0$, and $f_\alpha(0) = 0$, $f_\alpha'(0) = 0$ in case of $\alpha > 0$. This means that all the conditions of Theorem 3 are satisfied, hence the family $(f_\alpha)_{|\alpha| \le N}$ forms a finite moment sequence of rank r. As this holds for each N, the sufficiency part of the theorem is proved.

To prove the converse we assume that the family of functions f_α forms a moment function sequence of rank r and we prove the necessity part by induction. It is obvious that the statement is true for f_0 and we suppose that we have proved $f_\alpha(x) = \partial_t^\alpha \varphi\big(x, f(t)\big)\big|_{t=0}$ for $|\alpha| \le N$, where N is a natural number. Let $|\alpha| = N$, and $\alpha' = \alpha + (1, 0, \ldots, 0)$. We consider the function

$$g(x) = f_{\alpha'}(x) - \partial_t^{\alpha'} \varphi\big(x, f(t)\big)\big|_{t=0}$$

for each positive x. Then the expression $g''(x) + \frac{A'(x)}{A(x)} g'(x)$ is equal to

$$f_{\alpha'}''(x) + \frac{A'(x)}{A(x)} f_{\alpha'}'(x) - \partial_x^2 \partial_t^{\alpha'} \varphi(x, f(t))\big|_{t=0} - \frac{A'(x)}{A(x)} \partial_x \partial_t^{\alpha'} \varphi\big(x, f(t)\big)\big|_{t=0},$$

and, using Theorem 3 and (12), we get

$$\begin{aligned} c_0 f_{\alpha'}(x) &+ \sum_{0<\beta \le \alpha'} \binom{\alpha'}{\beta} c_\beta \partial_t^{\alpha'-\beta} \varphi\big(x, f(t)\big)\big|_{t=0} \\ &- \sum_{\beta \le \alpha'} \binom{\alpha'}{\beta} c_\beta \partial_t^{\alpha'-\beta} \varphi\big(x, f(t)\big)|_{t=0} = c_0\, g(x). \end{aligned}$$

Similarly one can show it holds also for $\alpha' = \alpha + (0, 1, 0, \ldots, 0)$, …, $\alpha' = \alpha + (0, 0, \ldots, 0, 1)$. Consequently

$$g''(x) + \frac{A'(x)}{A(x)} g'(x) = c_0\, g(x), \quad g(0) = 0, \quad g'(0) = 0,$$

hence $g(x) \equiv 0$ and the proof is complete. □

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